GEOMETRY OF TENSOR TRIANGULATED CATEGORIES

By

Umesh Vanktesh Dubey

MATH10200604010

The Institute of Mathematical Sciences
Chennai 600113

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Declaration

I hereby declare that the investigation presented in this thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part of a degree/diploma at this or any other Institution/University.

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Abstract

Given a quasi-projective scheme X with an action of a finite group G we consider the tensor triangulated category $\mathcal{D}^G(X)$. We relate the spectrum of this category, as defined by P. Balmer, with the spectrum of the category of all perfect complexes over the scheme X/G. Similarly, we consider the category of perfect complexes $\mathcal{D}^{per}(X)$ over a split super-scheme X. We give isomorphism of the spectrum of $\mathcal{D}^{per}(X)$ with the spectrum of $\mathcal{D}^{per}(X^0)$. Here X^0 denotes the even part of the super-scheme X ; it is a scheme in the usual sense.
The computation of these two spectrums gives examples of two distinct categories with isomorphic Balmer spectrums. Our result also shows the limitations of the geometric notion spectrum beyond the category of schemes. We suggest some possible generalisations of Balmer's notion of spectrum.

Contents

List of Publications

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Chapter 1

Introduction

This work is motivated by Balmer's construction of the triangular spectrum. We have tried to understand geometric objects coming from a derived category of coherent sheaves and some triangulated categories. The study of geometric objects coming from triangulated category is interesting for many reasons. One reason is to extend the study of schemes from commutative algebras to non-commutative algebras. Another reason is to capture some geometry which is not visible via scheme structure; the formulation of homological mirror symmetry by Kontsevich is one such example. There are other examples outside algebraic geometry which make the study of such geometric association interesting.

The notion of triangulated categories was first axiomatized by Grothendieck and Verdier in order to develop Serre duality in a relative setting. It was an idea of Grothendieck to extract more homological information from the totality of complexes. Verdier and Illusie had developed triangulated categories, like the derived category of coherent sheaves and perfect complexes respectively, for better understanding of dualities.

Now triangulated categories have found many applications in other branches of mathematics outside algebraic geometry.

In algebraic geometry triangulated categories arose as the derived category of coherent sheaves and also as the category of perfect complexes on schemes; which was later realized as compact objects in the derived category of quasi-coherent sheaves over quasi compact and separated schemes by Neeman[33]. In modular representation theory triangulated categories appear in the form of the stable module category. The introduction to Balmer[3] contains more examples and motivation to study such abstract objects.

Gabriel[15] and later Rosenberg[37] proved that abelian category of quasi-coherent sheaves completely determines the underlying variety. Mukai[29] had given examples of two non-isomorphic varieties, which have equivalent derived categories of coherent sheaves. So the derived category of a coherent sheaves cannot reconstruct the underlying variety for these cases. However Bondal and Orlov[9] proved it does determine the scheme whenever the canonical sheaf or the anti-canonical sheaf is ample. Still it is an interesting question to find an optimal class of varieties with such reconstruction results.

Since the triangulated structure alone is not enough for reconstruction Balmer[3] used tensor structure to reconstruct a smooth scheme from its bounded derived category of coherent sheaves. In fact, Balmer used the category of perfect complexes for quasi compact and quasi separated schemes to reconstruct the underlying schemes. The category of perfect complexes with tensor structure contains enough information about the scheme even in singular case. Balmer first constructs a topological space, which he called the triangular spectrum, associated with any essentially small tensor triangulated category. Balmer proved this topological space gives the universal support data on a triangulated category. Construction of the triangular spectrum depends on a classification of certain thick subcategories. Balmer used the classification of thick subcategories given by Thomason[39], which was motivated by results of Neeman[31] and Hopkins[21] for the affine case, to relate the triangular spectrum with the underlying scheme. Further, using the localisation result of Thomason[40], Balmer defined a sheaf of local triangulated categories. By considering endomorphisms of this sheaf of triangulated categories, Balmer constructs a sheaf of rings which reconstructs the scheme structure. Balmer firstly gave the reconstruction of quasi compact and quasi separated schemes using atomic subcategories, (see [2]). Later Balmer generalized the notion of prime ideal from commutative algebra to this abstract setting and demonstrated the usefulness of this concept in this generality, (see [3]). Balmer proved his reconstruction theorem under the assumption that the space is topologically Noetherian which was later removed by Krause et. al.[11]. Balmer proved later that this construction always gives a locally ringed space. He also gave an example coming from topology where this locally ringed space fails to be a scheme.

Using his definition of triangular spectrum Balmer applied many techniques from algebraic geometry to modular representation theory like gluing and the Picard group[5]. One question that naturally arises is how good is Spec as an invariant of the tensor triangulated category? It turns out that there do exist pairs of tensor triangulated categories which have isomorphic Specs (isomorphic as ringed spaces). We give two such examples. This motivates the need for some other finer geometric object attached to tensor triangulated categories. We shall compute the triangular spectrum in an equivariant setting, and for some superschemes by relating it with already known triangular spectrums. This computation is the starting point of this work and occupies a large part of it. More precisely, the first example consists of smooth quasi projective scheme, say X, with action of a finite group G as an automorphism. Hence we get the finite map $\pi : X \to Y := X/G$ which will give an exact functor

$$\pi^* : \mathcal{D}^{per}(X/G) \to \mathcal{D}^G(X).$$

We prove the following theorem.

Theorem 1.0.1. *Assume that the scheme X is smooth quasi projective and G is a finite group acting on X. The induced map*

$$\mathrm{Spec}(\pi^*) : \mathrm{Spec}(\mathcal{D}^G(X)) \to \mathrm{Spec}(\mathcal{D}^b(X/G))$$

is an isomorphism of locally ringed spaces.

Here Spec denotes the construction due to Balmer[3]. The proof involves a computation using some results from representation theory. The second example is a computation of the Balmer spectrum for a split superscheme X. Superschemes, defined by Manin and Deligne (see for example [27]), are also an important object of study in modern algebraic geometry, specially due to applications in physics. We consider the triangulated category $\mathcal{D}^{per}(X)$ of "perfect complexes" (the definition being modified appropriately in the super setting) on this superscheme.

Theorem 1.0.2. *Let X be a split superscheme. Let $X_0 = (X, \mathcal{O}_{X,0})$ be the 0-th part of this superscheme. X_0 is by definition a scheme. Then we have an isomorphism of locally ringed spaces*

$$f : X_0 \to \mathrm{Spec}(\mathcal{D}^{per}(X)).$$

The proof of homeomorphism adapts the classification of thick tensor ideals due to Thomason[39] as demonstrated by Balmer[3]. Again, following Balmer[3] we use the generalized localization theorem of [Theorem 2.1, Neeman[33]] to finish the proof.

We also tried to explore some ways to strengthen the geometric association of Balmer so that we can recover the tensor triangulated category. This is the problem of *categorical reconstruction*, of realizing the tensor triangulated category which we started with, as the tensor triangulated category canonically associated with a geometric object.

Overview of thesis

Now we shall give the content of each chapter for the convenience of the reader.

The **first chapter** recalls various preliminaries which are well known. We shall start with the definition of triangulated categories and exact functors between them. We shall also recall the definition of a derived category which is an important example of a triangulated category. Then we state the existence of derived functors and various relations between them. We recall the definitions of various functors in the theory of schemes. Next, by relating the sheaves of modules over superschemes with usual schemes, we give generalizations of many definitions and properties of perfect complexes for the super scheme case.

The **second chapter** recalls the definition of spectrum of a tensor triangulated category defined by Balmer. We recall various properties and general results from the papers of Balmer. We state the reconstruction result of Balmer as extended to non-Noetherian case by Krause[11]. We recall the definition of support data and the universal property of Balmer spectrum which is used crucially for the reconstruction. We state the functoriality result for the Balmer spectrum under non-unital functors (but the proof is not very different).

The **third chapter** contains new results which relate the spectrum of the bounded derived category of equivariant sheaves over a smooth quasi projective scheme with the spectrum of perfect complexes over the orbit space. First, we recall some basics on equivariant sheaves and we then prove the main theorem by dividing it into three cases - trivial action, free action and the general case. We also give the proof for curves as an interesting and important example.

The **fourth chapter** computes the spectrum for the tensor triangulated category of a split superscheme. In fact, this computation follows the steps laid down by Balmer for usual schemes. We get a relation between the spectrum of the underlying even scheme with the spectrum of a superscheme. We also recall the result of Neeman which is a generalization of the localization result of Thomason. The classification of radical thick tensor ideals is given by relating them with usual schemes. Here we use the generalization of the category of perfect complexes given earlier.

The **fifth chapter** contains some of our suggestions for the enrichment of Balmer spectrum. In the first section we give two ways of defining generalized spaces using the underlying topological space of the Balmer spectrum. In second section we give a functor of points approach to the Balmer spectrum.

Chapter 2

Background material and basic tools

In this chapter we recall some basic definitions and useful results. We briefly describe the contents of this chapter. We start with definition of triangulated category in first section and we then recall some basic results. Later, in various subsections we recall the definition of an exact functor between two triangulated categories, the definition of subcategories and localization. These notions are used by Balmer to define his spectrum. We also briefly mention the relation between a thick subcategory and the Verdier localization. The second section describes the derived category, which is an important example of a triangulated category. We recall Grothendieck's result on the existence of a derived functor. In various subsections we describe derived categories of schemes, superschemes and G-schemes. At the end we relate the category of perfect complexes over a superscheme with the category of perfect complexes over usual scheme. A reference to each result is given. We mostly follow the references [23][20][16][19] [40][42][22][26].

2.1 Triangulated category

The notion of a triangulated category was defined by Grothendieck and Verdier in order to extend Serre duality. We shall follow[20][16][23] for basic definitions and some basic properties. Also, we use auto-equivalence in place of automorphism for the translation functor in the definition. For simplicity, we shall not mention the various natural isomorphisms.

Definition 2.1.1. An additive category $\mathcal{T}$ with an additive auto-equivalence $S : \mathcal{T} \to \mathcal{T}; A \mapsto A[1]$, *suspension* or *translation*, is called a *suspended category.*

Here we have used $A[1]$ in place of $S(A)$ and in general we shall use $A[i]$ for $S^i(A)$. A *triangle* in a suspended category is a sextuple (A, B, C, f, g, h) of three objects and three morphisms as follows,

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1].$$

A morphism between two triangles (A, B, C, f, g, h) and (A', B', C', f', g', h') is defined as a commutative diagram,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[r, "h"] \arrow[d] & A[1] \arrow[d] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & A'[1].
\end{tikzcd}$$

Definition 2.1.2. A suspended category with a triangulation is called triangulated category. Here triangulation on a suspended category is defined as a collection of triangles $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$, called *distinguished triangles*, s.t. following axioms are satisfied,

(TR1) Given objects A and B, and a morphism $f : A \to B$ in $\mathcal{T}$

(i) The triangle $A \xrightarrow{id} A \to 0 \to A[1]$ is a distinguished triangle.

(ii) There exists a distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ containing the morphism f.

(iii) Every triangle isomorphic to a *distinguished triangle* is also a *distinguished triangle*.

(TR2) A triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is distinguished iff the triangle $B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$ is distinguished.

(TR3) Following commutative diagram can be completed to a morphism of triangles

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[r, "h"] \arrow[d, dashed] & A[1] \arrow[d] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & A'[1]
\end{tikzcd}$$

(TR4) *(Verdier Axiom)* Given three distinguished triangles

$$\begin{cases} A \xrightarrow{f} B \to C' \to A[1] \\ B \xrightarrow{g} C \to A' \to B[1] \\ A \xrightarrow{g \circ f} C \to B' \to A[1] \end{cases}$$

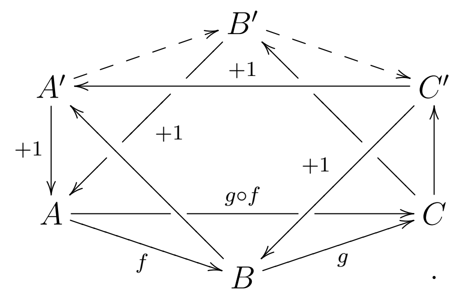
there exists a distinguished triangle

$$C' \to B' \to A' \to C'[1]$$

s.t. following diagram is commutative,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal, "id_A"'] & B \arrow[r] \arrow[d, "g"] & C' \arrow[r] \arrow[d] & A[1] \arrow[d, "id_{A[1]}"] \\
A \arrow[r, "g \circ f"] \arrow[d, "f"'] & C \arrow[r] \arrow[d, equal, "id_C"] & B' \arrow[r] \arrow[d] & A[1] \arrow[d, "f[1]"] \\
B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, equal, "id_{A'}"] & B[1] \arrow[d] \\
C' \arrow[r] & B' \arrow[r] & A' \arrow[r] & C'[1]
\end{tikzcd}$$

Originally Verdier called it *Octahedral Axiom* due to following description of this axiom,



All non cyclic squares and non cyclic triangles are commutative and cyclic triangles are distinguished triangles in above octahedron.

Remark 2.1.3.

1. A category that satisfies all axioms except the last Verdier axiom (TR4) is called a *pre-triangulated category*.
2. As shown by May[28] some of the conditions are redundant; e.g. in above definition (TR3) and necessary part of (TR2) follows from the remaining axioms.

(While writing the sets of homomorphisms, we will avoid writing the category as a subscript with Hom(,) to avoid cluttering the notation.)

There exist many triangulated structures on a suspended category. One such example is obtained by just changing sign in a distinguished triangle as follows. We say $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is an anti-distinguished triangle if the triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{-h} A[1]$ is a distinguished triangle. The collection of all such anti-distinguished triangles gives another triangulation of the category $\mathcal{T}$, see [Def 1.5.9, Kashiwara[23]]. Given a triangulated category $\mathcal{T}$, we have following well known results,

Proposition 2.1.4. *1. For any distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$, the composite $g \circ f$ is zero.*

2. Given a distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$, the triangle $A \xrightarrow{-f} B \xrightarrow{-g} C \xrightarrow{-h} A[1]$ is also distinguished.

3. The collection of distinguished triangles is closed under taking finite direct sums.

4. There is a canonical triangulation on the opposite category of $\mathcal{T}$ i.e. dual triangulation.

There are some useful results which can be proved using these results and definitions, see Hartshorne[20] or Manin[16].

Proposition 2.1.5. *1. Every distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ gives a long exact sequences of abelian groups as follows,*

$$\to \mathrm{Hom}(D, A) \to \mathrm{Hom}(D, B) \to \mathrm{Hom}(D, C) \to \mathrm{Hom}(D, A[1]) \to .$$

Similarly we have an exact sequence using the other functor,

$$\to \mathrm{Hom}(C, D) \to \mathrm{Hom}(B, D) \to \mathrm{Hom}(A, D) \to \mathrm{Hom}(C[-1], D) \to .$$

2. (five lemma) Given a commutative square with the vertical maps as isomorphisms, there exists a third vertical morphism using [TR3]. This vertical morphism is an isomorphism,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\wr"] & B \arrow[r, "g"] \arrow[d, "\wr"] & C \arrow[r, "h"] \arrow[d, dashed] & A[1] \arrow[d, "\wr"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & A'[1]
\end{tikzcd}$$

3. Given a distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$, the object C is referred as a cone of the morphism f. A cone of any morphism is unique up to (non-unique) isomorphism.

The last statement of previous proposition says that a cone of any morphism is not unique in general however a lemma proved by Deligne et. al.[7] in this context is quite useful.

Lemma 2.1.6 ([7])**.** *Given two distinguished triangles and a map $v : B \to B'$,*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, dashed, "u"] & B \arrow[r, "g"] \arrow[d, "v"] & C \arrow[r, "h"] \arrow[d, dashed, "w"] & A[1] \arrow[d, dashed, "u[1]"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & A'[1] \end{tikzcd}$$

following conditions are equivalent,

1. $g'vf = 0$
2. *there exists a map u s.t. first box in diagram is commutative.*
3. *there exists a map w s.t. second box in diagram is commutative.*
4. (u, v, w) *is a map of triangles.*

If one of the above conditions is satisfied and also $\mathrm{Hom}(A, C'[-1]) = 0$ *then morphism u(resp. w) in 2.(resp 3.) is unique.*

Orlov[Lemma 3.1.2, [35]] gave a generalization of this result for uniqueness of the total object (or convolution) of a Postnikov tower.

2.1.1 Exact functor

Recall, that a functor $F : \mathcal{C} \to \mathcal{C}'$ between additive categories is called an *additive functor* if the maps,

$$F(A, B) : \mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B)),$$

are group homomorphisms for any pair of objects A and B of $\mathcal{C}$. The functor F is called *fully faithful* if, in addition further $F(A, B)$ is an isomorphism for each pair of objects A and B. An *essentially surjective* functor is defined as a functor F s.t. for every object A' of $\mathcal{C}'$ there exists an object A in $\mathcal{C}$ with $A' \simeq F(A)$. An additive functor is an *isomorphism* if there exists an additive functor $G : \mathcal{C}' \to \mathcal{C}$ s.t. $G \circ F = Id_{\mathcal{C}}$ and $F \circ G = Id_{\mathcal{C}'}$. Recall that a natural transformation η between two functors F and G is by definition a map $\eta_A : F(A) \to G(A)$ for each object A s.t. for each morphisms $f : A \to B$ following diagram is commutative,

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta_B"] & G(B). \end{tikzcd}$$

We shall denote the collection of all natural transformations between two functors F and G by $Nat(F, G)$. A natural transformation is called a *natural isomorphism* if every η_A is an isomorphism. An additive functor is called an *equivalence* if there exist an additive functor G and natural isomorphisms

$$\eta : F \circ G \xrightarrow{\sim} Id_{\mathcal{C}'} \text{ and } \mu : G \circ F \xrightarrow{\sim} Id_{\mathcal{C}}.$$

We use the following well known result to check whether a given functor is an equivalence of categories (see [Prop. 1.16, A. Vistoli [14]])

Lemma 2.1.7. *A functor F is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

We define a cohomological functor as follows, see [20],

Definition 2.1.8. An additive functor $H : \mathcal{T} \to \mathcal{A}$ from a triangulated category $\mathcal{T}$ to an abelian category $\mathcal{A}$ is called a *cohomological functor*, if for any distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ the following sequence is exact,

$$\cdots \to H(A) \xrightarrow{H(f)} H(B) \xrightarrow{H(g)} H(C) \xrightarrow{H(h)} H(A[1]) \to \cdots .$$

We analogously define a contravariant cohomological functor by reversing the arrows in the long exact sequence.

An example of a cohomological functor (resp. contravariant cohomological functor) is given by the functor $\mathrm{Hom}(A, \_)$ (resp. $\mathrm{Hom}(\_, A)$) for an object A in the triangulated category $\mathcal{T}$, see 2.1.5. We denote by $\mathcal{A}\mathfrak{b}$ the category of all abelian groups. Since by definition the triangulated category $\mathcal{T}$ is an additive category, we have canonical additive functors given by homomorphisms as follows,

$$\mathrm{Hom}(A, \_) : \mathcal{T} \to \mathcal{A}\mathfrak{b}; C \mapsto \mathrm{Hom}(A, C)$$
$$\mathrm{Hom}(\_, B) : \mathcal{T}^\circ \to \mathcal{A}\mathfrak{b}; C \mapsto \mathrm{Hom}(C, B).$$

Following Manin[16], we use the notation

$$\mathrm{Hom}^i(A, B) := \mathrm{Hom}(A[-i], B) = \mathrm{Hom}(A, B[i]) \text{ for each } i \in \mathbb{Z}.$$

An additive covariant functor from an additive category to the abelian category $\mathcal{A}\mathfrak{b}$ is called *representable* if it is naturally isomorphic to a functor of the form $\mathrm{Hom}(A, \_)$. The object A is called the representing object of the functor and it is unique up to unique isomorphism. The following well known lemma gives an embedding of any category as a full subcategory of category of all contravariant functors on it.

Lemma 2.1.9 (Yoneda). *The functor taking an object A to the representable functor $\mathrm{Hom}(A, \_)$ is a fully faithful functor from an additive category $\mathcal{T}$ to $Fun(\mathcal{T}^\circ, \mathcal{A}\mathfrak{b})$. Here $Fun(\mathcal{T}^\circ, \mathcal{A}\mathfrak{b})$ is the category of all contravariant additive functors from $\mathcal{T}$ to $\mathcal{A}\mathfrak{b}$.*

Remark 2.1.10. The lemma is also valid for all categories by replacing $\mathcal{A}\mathfrak{b}$ by $\mathcal{S}\mathfrak{et}$, the category of sets.

Definition 2.1.11. An additive functor $F : \mathcal{T} \to \mathcal{T}'$ between two triangulated categories is called *exact* or *triangulated* if it commutes with translation, i.e. $F(A[1]) \simeq F(A)[1]$ for each object A of $\mathcal{T}$, *and* preserves distinguished triangles, i.e. the triangle

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \xrightarrow{F(h)} F(A)[1]$$

is distinguished for every distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$.

An exact functor is called an isomorphism (resp. equivalence) of triangulated categories if it is an isomorphism (resp. equivalence) as an additive functor.

We recall some definitions and properties of an adjoint functor. A functor $F : \mathcal{C} \to \mathcal{C}'$ is called *left adjoint* of a functor $G : \mathcal{C}' \to \mathcal{C}$ or a functor G is called *right adjoint* of a functor F if the following conditions are satisfied, i.e. there exist functorial isomorphisms,

$$\mathrm{Hom}(F(A), A') \simeq \mathrm{Hom}(A, G(A')) \text{ for all } A \in ob(\mathcal{C}) \text{ and } A' \in \mathcal{C}'.$$

If F is a left adjoint of a functor G then there exist adjunction natural transformations as follows,

$$\sigma : Id_{\mathcal{C}} \to G \circ F; A \mapsto G \circ F(A) \text{ and } \rho : F \circ G \to Id_{\mathcal{C}'}; F \circ G(A') \mapsto A'.$$

These maps are given by the image of identity elements via the adjunction isomorphisms. We recall now some properties of adjoint functors, see Kashiwara[23] for details.

Proposition 2.1.12. *Suppose $F : \mathcal{T} \to \mathcal{T}'$ is a left adjoint of a functor $G : \mathcal{T}' \to \mathcal{T}$.*

1. *The adjoint functor G can be defined using a representing object of the functor $\mathrm{Hom}(F(\_), A') : \mathcal{T}'^{\circ} \to \mathcal{A}\mathfrak{b}$ for each object $A' \in ob(\mathcal{T}')$.*

2. *If F is an exact functor between two triangulated categories then G is also an exact functor.*

3. *The adjoint functor of a functor is unique up to isomorphism.*

Note that the first assertion relates the existence of an adjoint functor with the representability of certain functors.

2.1.2 Triangulated subcategory and localization

A triangulated subcategory of a triangulated category $\mathcal{T}$ is defined as a full additive subcategory which is preserved under the suspension functor and is a triangulated category with induced triangulation from the category $\mathcal{T}$. Here the induced triangulation is defined as the collection of distinguished triangles of $\mathcal{T}$ with all three objects of triangle from the subcategory. In other words, we can define *triangulated subcategory* as a full subcategory with a triangulated structure s.t the inclusion functor is an exact functor of triangulated categories.

Definition 2.1.13. 1. A triangulated subcategory is called *thick* if it is closed under direct summands of its objects. In other words, a subcategory is called *thick* if the *kernel* and *cokernel* of any projector[1] is an object of the subcategory.

2. A triangulated subcategory is called a *triangulated ideal* if whenever two objects of a distinguished triangle are in the subcategory then the third object is also in the subcategory.

3. A strictly full subcategory is called *left (resp. right) admissible subcategory* if the inclusion of the subcategory has left (resp. right) adjoint functor.

4. A *left (resp. right) orthogonal subcategory* of a triangulated subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is defined as a full subcategory consisting of objects $A \in \mathcal{T}$ s.t. $\mathrm{Hom}(A, B) = 0$ (resp. $\mathrm{Hom}(B, A) = 0$) for every object $B \in \mathcal{S}$.

[1]any endomorphism with $p^2 = p$.

The definition of an orthogonal subcategory of a given category was introduced by Verdier[41]. He also related it with the quotient by subcategories. Later we shall describe the definition of quotient in detail. Using [TR1] (i) and the result 2.1.4 we can see that any triangulated ideal is a thick subcategory.

Localization is another important construction which is used to produce new examples of triangulated categories from old ones. It has universal properties similar to localizations in commutative algebra. Localization is used to invert certain classes of morphisms and this can be done in general by formally inverting the arrows. However, we are interested in some more structure on the collection of homomorphisms. The first problem with the localization of a class of morphisms in an additive category is to get an additive category structure on the localized category. The collection of homomorphisms between two objects might not be a set in general after localization, see [10.3.6, Weibel[42]] for more discussion on this. As mentioned in Weibel[42], this set theoretic difficulty can be avoided by fixing some Grothendieck universe where this collection is small or by taking a locally small multiplicative collection. We shall now define the self dual notion of a multiplicative family which ensures the existence of triangulated structure after the localization, see [41][20][16][23][42].

Definition 2.1.14. A collection $\mathcal{S}$ of morphisms in a triangulated category $\mathcal{T}$ is said to be a *multiplicative family compatible with the triangulation* if following conditions are satisfied,

SM1 The collection $\mathcal{S}$ is closed under composition, i.e. $f \circ g \in \mathcal{S}$ whenever $f, g \in \mathcal{S}$, and contains the identity morphism for every objects, i.e. $id_A \in \mathcal{S}$ for every object A in $\mathcal{T}$.

SM2 (Ore condition) Given two morphisms f and s with $s \in \mathcal{S}$ there exists a map $s' \in \mathcal{S}$ and a map f' s.t. following diagram is commutative,

$$\begin{tikzcd} A' \arrow[r, dashed, "f'"] \arrow[d, dashed, "s'"'] & B' \arrow[d, "s"] \\ A \arrow[r, "f"] & B. \end{tikzcd}$$

Similarly we have a dual diagram with arrow reversed i.e. given two morphisms $f' \in \mathcal{T}$ and $s' \in \mathcal{S}$ following diagram is commutative,

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "s'"'] & B' \arrow[d, dashed, "s"] \\ A \arrow[r, dashed, "f"] & B. \end{tikzcd}$$

SM3 Following two statements are equivalent for two morphisms $f, g : A \rightrightarrows B$,

1. there exists a morphism $s : A' \to A$ in $\mathcal{S}$ s.t. $f \circ s = g \circ s$.
2. there exists a morphism $t : B \to B'$ in $\mathcal{S}$ s.t. $t \circ f = t \circ g$.

SM4 (Saturatedness) The map $f \circ g \in \mathcal{S}$ and $g \circ h \in \mathcal{S}$ if and only if $g \in \mathcal{S}$.

SM5 The family $\mathcal{S}$ is closed under the action of the translation functor i.e. $s[\pm 1] \in \mathcal{S}$ for every $s \in \mathcal{S}$.

SM6 The following commutative diagram, with $s_1, s_2 \in \mathcal{S}$, can be completed to a morphism of a distinguished triangles,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "s_1"'] & B \arrow[r, "g"] \arrow[d, "s_2"'] & C \arrow[r, "h"] \arrow[d, dashed, "s_3"'] & A[1] \arrow[d, "s_1[1]"'] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & A'[1]
\end{tikzcd}$$

where $s_3 \in \mathcal{S}$.

If the family satisfies only axioms **[Sm1]**-**[SM3]** then it is called a *multiplicative family* and it is called a *saturated multiplicative family* if, in addition, the axiom [**SM4**] is satisfied, see [41]. We recall the explicit construction of localization, see [III.2.8 lemma [16]] for proof,

Proposition 2.1.15. *An additive category $\mathcal{C}$ and a multiplicative set $\mathcal{S}$ of morphisms in $\mathcal{C}$ define a new category $\mathcal{C}[\mathcal{S}^{-1}]$. The objects of $\mathcal{C}[\mathcal{S}^{-1}]$ are the same as objects of $\mathcal{C}$ and the morphisms between two objects in $\mathcal{C}[\mathcal{S}^{-1}]$ are given by equivalence classes,*

$$\mathrm{Hom}_{\mathcal{C}[\mathcal{S}^{-1}]}(A, B) := \{A \stackrel{s}{\leftarrow} A' \stackrel{f}{\to} B / s \in \mathcal{S} \textit{ and } f \in \mathrm{Hom}_{\mathcal{C}}(A', A)\}/\sim .$$

Here the pair of morphisms $A \stackrel{s}{\leftarrow} A' \stackrel{f}{\to} B$ is equivalent to $A \stackrel{t}{\leftarrow} A'' \stackrel{g}{\to} B$ if there exist a morphism $r : A''' \to A'$ in $\mathcal{S}$ and a morphism $h : A''' \to A''$ s.t. $s \circ r = t \circ h$ and $f \circ r = g \circ h$. Given two morphisms $A \stackrel{s}{\leftarrow} A' \stackrel{f}{\to} B$ and $B \stackrel{t}{\leftarrow} B' \stackrel{g}{\to} C$ there exist morphisms $t' : A'' \to A'$ in $\mathcal{S}$ and $f' : A'' \to B'$ using [**SM2**].

$$\begin{tikzcd}
 & & A''' \arrow[dl, "r"'] \arrow[dr, "h"] & & \\
 & A' \arrow[dl, "s"'] \arrow[drrr, "f" near end] & & A'' \arrow[dlll, "t"' near end] \arrow[dr, "g"] & \\
A & & & & B
\end{tikzcd}$$

We define the composition as $A \xleftarrow{s \circ t'} A'' \xrightarrow{g \circ f'} C$. There is a canonical functor $Q : \mathcal{C} \to \mathcal{C}[\mathcal{S}^{-1}]$ given by identity on objects and the additive map $Q(A, B)$: $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}[\mathcal{S}^{-1}]}(A, B)$ given by $(f : A \to B) \mapsto (A \xleftarrow{id_A} A \xrightarrow{f} B)$. This functor sends each morphism in $\mathcal{S}$ to an isomorphism.

The collections of morphisms in the localization category can also be defined as a filtered limit (or inductive limit) of certain functors which we will briefly indicate here. Given an object A of an additive category $\mathcal{C}$ we can define a new category $\mathcal{S}/A$. The objects of $\mathcal{S}/A$ consist of arrows of the form $s : A' \to A$ with $s \in \mathcal{S}$ and morphisms consist of a map $t : A'' \to A'$ with following commutative diagram

$$\begin{tikzcd} A'' \arrow[r, "t"] \arrow[dr, "s''"'] & A' \arrow[d, "s'"] \\ & A. \end{tikzcd}$$

The axioms **[Sm1]-[SM3]** ensures that the category $(\mathcal{S}/A)^\circ$ is a filtered category, see Verdier[41]. Hence the morphism can be defined as a filtered limit

$$\mathrm{Hom}_{\mathcal{C}[\mathcal{S}^{-1}]}(A, B) := \lim_{A' \in (\mathcal{S}/A)^\circ} \mathrm{Hom}_{\mathcal{C}}(A', B).$$

For an additive category $\mathcal{C}$ and a multiplicative set (or locally small multiplicative family) $\mathcal{S}$ the localization has the following universal property,

Proposition 2.1.16. *An additive functor $F : \mathcal{C} \to \mathcal{D}$ which takes any morphism from the multiplicative set $\mathcal{S}$ to an isomorphism factors uniquely (up to isomorphism) through the functor Q. In other words, there exists a unique additive functor $G : \mathcal{C}[\mathcal{S}^{-1}] \to \mathcal{D}$ s.t. following diagram is commutative,*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Q"] \arrow[dr, "F"'] & \mathcal{C}[\mathcal{S}^{-1}] \arrow[d, dashed, "G"] \\ & \mathcal{D}. \end{tikzcd}$$

The proof of this universal property can be found in Verdier[41].

Now if we take a multiplicative set compatible with the triangulation then the localization of a triangulated category is again a triangulated category and the natural functor Q is an exact functor, see [IV.2.2 Theorem, Manin[16]].

In fact, Verdier also related this localization with the quotient of the triangulated category by certain subcategories. For this we need the saturatedness of the multiplicative set. To any saturated multiplicative set $\mathcal{S}$ we can associate the thick subcategory $\varphi(\mathcal{S})$ which is the full subcategory generated by all objects C which fit in some distinguished triangle

$$A \xrightarrow{s} B \to C \to A[1]$$

where s is a morphism contained in the saturated multiplicative set $\mathcal{S}$. Similarly we can associate a saturated multiplicative set $\psi(\mathfrak{q})$ to any thick subcategory $\mathfrak{q}$ by taking the collection of all morphism which fit in a distinguished triangle

$$A \xrightarrow{f} B \to C \to A[1]$$

with C as an object of the thick subcategory $\mathfrak{q}$. We have following result relating these two collections, see [41] [16],

Proposition 2.1.17. *The two maps defined above give an order[2] preserving bijection between following two collections,*

$$\{\ \textit{saturated multiplicative sets}\ \} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \{\ \textit{thick subcategories}\}.$$

If we take the sub-collection of saturated multiplicative sets compatible with the triangulation in a triangulated category then the same association gives a bijection of this collection with the collection of thick triangulated ideals, see [Cor 2.2.11, Verdier[41]]. We define the quotient by any thick triangulated ideal $\mathfrak{q}$ in a triangulated category $\mathcal{T}$ as a localization category $\mathcal{T}[\psi(\mathfrak{q})^{-1}]$ and we shall denote this quotient by $\mathcal{T}/\mathfrak{q}$.

2.2 Derived category

In this section we shall give some examples of triangulated categories. First, we give the definition and certain properties of the derived category of an abelian category which gives an important class of examples of triangulated categories, see [16][23] for more details.

Recall that an abelian category is defined as an additive category with existence of kernel and cokernel of each morphism and a natural isomorphism, say $\bar{f}$, for each morphism $f : A \to B$ which fits in following commutative diagram,

$$\begin{tikzcd} A \arrow[r] \arrow[rrr, bend right, "f"'] & A/ker(f) \arrow[r, "\bar{f}"] & Im(f) \arrow[r, hook, two heads] & B. \end{tikzcd}$$

[2]Orders on both sides are given by inclusion.

Given an additive category $\mathcal{A}$ we can define the category of complexes, $\mathcal{C}^{\#}(\mathcal{A})$ for $\# \in \{+,-,b,\emptyset\}$, which is again an additive category. The morphism $f^{\cdot}$ between two complexes $A^{\cdot}$ and $B^{\cdot}$ is defined as following commutative diagram,

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d^{i-1}"] \arrow[d, "f^{i-1}"'] & A^{i} \arrow[r, "d^{i}"] \arrow[d, "f^{i}"] & A^{i+1} \arrow[r, "d^{i+1}"] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d^{i-1}"] & B^{i} \arrow[r, "d^{i}"] & B^{i+1} \arrow[r, "d^{i+1}"] & \cdots .
\end{tikzcd}$$

If the category $\mathcal{A}$ is an abelian category then the category $\mathcal{C}^{\#}(\mathcal{A})$ for $\# \in \{+,-,b,\emptyset\}$ is also an abelian category. The homotopy between two morphisms is defined as a collection of morphisms $h^i : A^i \to B^{i-1}$ for each $i \in \mathbb{Z}$

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d^{i-1}"] \arrow[dl, "h^{i-1}"'] & A^{i} \arrow[r, "d^{i}"] \arrow[dl, "h^{i}"'] & A^{i+1} \arrow[r, "d^{i+1}"] \arrow[dl, "h^{i+1}"'] & \cdots \arrow[dl, "h^{i+2}"'] \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d^{i-1}"] & B^{i} \arrow[r, "d^{i}"] & B^{i+1} \arrow[r, "d^{i+1}"] & \cdots .
\end{tikzcd}$$

s.t. $f^i - g^i = h^{i+1} \circ d^i + d^{i-1} \circ h^i$ holds. We say that two morphisms $f^{\cdot}$ and $g^{\cdot}$ are equivalent, denoted as $f^{\cdot} \sim g^{\cdot}$, if $f^{\cdot}$ is homotopic to $g^{\cdot}$. We can easily check that this defines an equivalence relation on group $\mathrm{Hom}_{\mathcal{C}^{\#}(\mathcal{A})}(A^{\cdot}, B^{\cdot})$ for $\# \in \{+,-,b,\emptyset\}$. We define the new category $\mathcal{K}^{\#}(\mathcal{A})$ by taking the same objects as $\mathcal{C}^{\#}(\mathcal{A})$ and morphisms given by quotient

$$\mathrm{Hom}_{\mathcal{K}^{\#}(\mathcal{A})}(A^{\cdot}, B^{\cdot}) := \mathrm{Hom}_{\mathcal{C}^{\#}(\mathcal{A})}(A^{\cdot}, B^{\cdot})/\sim .$$

We can also define the homotopy category by taking the quotient of morphisms by the subgroup containing all morphisms which are homotopic to the zero map. Since the category $\mathcal{A}$ is an additive category, the quotient category will have structure of an additive category. Hence $\mathcal{K}^{\#}(\mathcal{A})$ for $\# \in \{+,-,b,\emptyset\}$ is an additive category. The category $\mathcal{K}^{\#}(\mathcal{A})$ is called the homotopy category associated with a category $\mathcal{A}$. We shall denote the homotopy category without any boundedness assumption on complexes by $\mathcal{K}(\mathcal{A})$ in place of $\mathcal{K}^{\emptyset}(\mathcal{A})$ for simplicity. It is easy to see that other three categories $\mathcal{K}^b(\mathcal{A}), \mathcal{K}^+(\mathcal{A})$ and $\mathcal{K}^-(\mathcal{A})$ are also full additive subcategories of $\mathcal{K}(\mathcal{A})$. The homotopy category associated to an abelian category may not be an abelian category, see [Ex IV.1.1 , Manin[16]]. We shall now give a translation functor and a collection of distinguished triangles on the category $\mathcal{K}(\mathcal{A})$ to get a triangulated category structure. The translation functor is defined as $(A^{\cdot}[k])^i := A^{i+k}$ and $d^i_{(A^{\cdot}[k])} := (-1)^k d^{i+k}_{A^{\cdot}}$ for each object $A^{\cdot}$. The map between morphisms is given as follows

$$T^k(A^{\cdot}, B^{\cdot}) : \mathrm{Hom}(A^{\cdot}, B^{\cdot}) \to \mathrm{Hom}(A^{\cdot}[k], B^{\cdot}[k]); f \mapsto f[k]$$

where $f[k]^i := f^{i+k}$ for each $i \in \mathbb{Z}$. To define a distinguished triangle we need definition of the cone of a morphism. The cone of a morphism $f : A^\cdot \to B^\cdot$ is defined as the following complex

$$C(f) := A^\cdot[1] \oplus B. \text{ and } d^i_{C(F)} := \begin{pmatrix} d^i_{A^\cdot[1]} & 0 \\ f[1]^i & d^i_{B^\cdot} \end{pmatrix} = \begin{pmatrix} -d^{i+1}_{A^\cdot} & 0 \\ f^{i+1} & d^i_{B^\cdot} \end{pmatrix}.$$

It is easy to see that there are maps $g : B^\cdot \to C(f)$ and $h : C(f) \to A^\cdot[1]$ given by canonical inclusion and projection respectively. Hence we get a triangle for each morphism in category of complexes as follows,

$$A^\cdot \xrightarrow{f} B^\cdot \xrightarrow{g} C(f) \xrightarrow{h} A^\cdot[1].$$

This will give a triangle in the homotopy category of any additive category. Now define any triangle isomorphic to above triangle as a distinguished triangle. The homotopy category $\mathcal{K}(\mathcal{A})$ with the above defined translation is a suspended category and moreover there is a following result.

Proposition 2.2.1 (Prop. 1.4.4, Kashiwara[23])**.** *The category $\mathcal{K}(\mathcal{A})$ with translation functor T and the collection of distinguished triangles defined as above is a triangulated category.*

If we take the additive full subcategories $\mathcal{K}^b(\mathcal{A}), \mathcal{K}^+(\mathcal{A})$ and $\mathcal{K}^-(\mathcal{A})$ then the cone construction of a morphism in any of these subcategories is again in the same subcategory. Hence these full subcategories, with the induced translation functor and the induced collection of triangles, are full triangulated subcategories. We assume that the category $\mathcal{A}$ is an abelian category. Hence we can define the cohomological functors $H^i : \mathcal{K}(\mathcal{A}) \to \mathcal{A}$ for each $i \in \mathbb{Z}$. Recall that the cohomology of a complex $A^\cdot$ is defined as a subquotient of $A^\cdot$. If we take $Z^i(A^\cdot) := Ker(d^i_{A^\cdot})$ and $B^i(A^\cdot) := Im(d^{i+1}_{A^\cdot})$ then the cohomology of a complex $A^\cdot$ is given by $\mathcal{H}^i(A^\cdot) := Z^i(A^\cdot)/B^i(A^\cdot)$. Using these cohomological functors we have following definition,

Definition 2.2.2. A morphism $f : A^\cdot \to B^\cdot$ between two complexes is said to be a *quasi-isomorphism* if the induced morphisms $\mathcal{H}^i(f)$ are isomorphisms for each $i \in \mathbb{Z}$. We say that $A^\cdot$ is *quasi-isomorphic* to $B^\cdot$.

The collection of all quasi-isomorphisms is denoted as Qis. Note that the definition of quasi isomorphism is not symmetric i.e. there exists an example of quasi isomorphism which does not have inverse quasi isomorphism. More precisely if we consider the category $\mathcal{K}(\mathbb{Z} - mod)$ then the map

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\ & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

is a quasi isomorphism but there does not exists a map in other direction. Now to define the derived category we need to invert the collection of quasi isomorphisms.

Proposition 2.2.3 (Prop 4.1, Hartshorne[20])**.** *The collection Qis is a saturated multiplicative family compatible with the triangulation.*

Using the bijection between the collection of saturated multiplicative families and thick subcategories, we get the thick subcategory $\mathcal{N}$, also referred as a null system in [def 1.6.6 Kashiwara[23]]. The full subcategory $\mathcal{N}$ is given by all objects $A^\cdot$ s.t. $\mathcal{H}^i(A^\cdot) = 0$ for each $i \in \mathbb{Z}$ i.e it contains all acyclic complexes. We define the derived category as the localization of quasi isomorphisms or the quotient by the null system.

Definition 2.2.4 (Derived category)**.** The derived category $\mathcal{D}(\mathcal{A})$ associated to any abelian category $\mathcal{A}$ is defined as a triangulated category

$$\mathcal{D}(\mathcal{A}) := \mathcal{K}(\mathcal{A})[Qis^{-1}] = \mathcal{K}(\mathcal{A})/\mathcal{N}.$$

Similarly we can also define $\mathcal{D}(\mathcal{A})^{\pm}$ and $\mathcal{D}(\mathcal{A})^b$ as the localization of the collection of induced quasi isomorphisms. These triangulated subcategories can be realized as full subcategories of $\mathcal{D}(\mathcal{A})$ due to following result,

Proposition 2.2.5 (Prop. 2.3.5, [41] or Prop. 1.6.5, [23])**.** *Suppose $\mathcal{T}$ is a triangulated category and $\mathcal{T}'$ is a full triangulated subcategory. We shall denote by $\mathcal{S}$ a multiplicative family and by $\mathcal{S}'$ the induced family $\mathcal{T}' \cap \mathcal{S}$. Further assume that either of these two conditions is satisfied.*

1. *Given a morphism $s : A \to B$ with $B \in Ob(\mathcal{T}')$ and $s \in \mathcal{S}$ there exists a morphism $s' : B' \to A$ in multiplicative family $\mathcal{S}$ s.t. $B' \in Ob(\mathcal{T}')$ and the composition $s \circ s' \in \mathcal{S}'$.*

2. *Similar condition as above with arrow reversed.*

Then following assertions are true,

1. *The family $\mathcal{S}'$ is a multiplicative family. If $\mathcal{S}$ is a saturated multiplicative family (resp. compatible with the triangulation) then the family $\mathcal{S}'$ is also a saturated multiplicative family (resp. compatible with the triangulation).*

2. *The induced functor $\bar{i} : \mathcal{T}'[\mathcal{S}'^{-1}] \to \mathcal{T}[\mathcal{S}^{-1}]$ given by the canonical inclusion functor $i : \mathcal{T}' \to \mathcal{T}$ is fully faithful and $\mathcal{T}'[\mathcal{S}'^{-1}]$ is equivalent to full triangulated subcategory of $\mathcal{T}[\mathcal{S}^{-1}]$.*

Now using above result it is enough to observe that the multiplicative family Qis satisfies the hypothesis, see [Prop. 1.7.2 , [23]]. Also we can see that these full subcategories of $\mathcal{D}(\mathcal{A})$ satisfies $\mathcal{D}^b(\mathcal{A}) = \mathcal{D}^+(\mathcal{A}) \cap \mathcal{D}^-(\mathcal{A})$.

Digression

We shall extend above construction of derived category associated to an abelian category. Suppose $\mathcal{A}$ is an additive category with a fixed embedding in an abelian category. It is always possible to choose one such embedding due to Yoneda's lemma which gives an embedding of $\mathcal{A}$ inside the abelian category $Fun(\mathcal{A}^\circ, \mathcal{A}\mathfrak{b})$. Let us denote by $\mathcal{B}$ a fixed abelian category containing the additive category $\mathcal{A}$ as a full subcategory. Using this embedding we can get the embedding of the category of complexes $\mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$. Since $\mathcal{A}$ is a full subcategory of $\mathcal{B}$ therefore the inclusion functor will give a fully faithful functor $I : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$ which is an exact functor of triangulated categories. Since the composition of an exact functor with a cohomological functor is again a cohomological functor therefore the embedding I gives a cohomological functors

$$\mathcal{H}^i : \mathcal{K}(\mathcal{A}) \to \mathcal{B}; A^\cdot \mapsto \mathcal{H}^i(A^\cdot) := \mathcal{H}^i(I(A^\cdot)) \text{ for each } i \in \mathbb{Z}.$$

We can define the quasi isomorphism between two complexes as a map $f : A^\cdot \to B^\cdot$ s.t. the maps $\mathcal{H}^i(f) : \mathcal{H}^i(A^\cdot) \to \mathcal{H}^i(B^\cdot)$ are isomorphisms in $\mathcal{B}$ for each $i \in \mathbb{Z}$. This collection of quasi isomorphisms is denoted as Qis as before. We have following result similar to the abelian category case.

Proposition 2.2.6. *The collection Qis is a saturated multiplicative family compatible with triangulation of triangulated category $\mathcal{K}(\mathcal{A})$.*

We shall define the derived category associated to an additive category $\mathcal{A}$ as the localization $\mathcal{D}(\mathcal{A}) := \mathcal{K}[Qis^{-1}]$. Note that this construction depends on the choice of fully faithful embedding $\mathcal{A} \to \mathcal{B}$. Hence, for definiteness, we use the embedding of $\mathcal{A}$ in $Fun(\mathcal{A}^\circ, \mathcal{A}\mathfrak{b})$ to define the quasi isomorphisms.

2.3 Derived functors

We have already defined an exact functor between two triangulated categories. There are certain exact functors between derived categories which approximate the functors at the level of abelian categories. These are called (left or right) derived functors associated to a functor. We shall recall the definition of derived functors and some criterion for existence of derived functors. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two abelian categories and $F : \mathcal{A}_1 \to \mathcal{A}_2$ be an additive functor. Recall that the functor F is called left (resp. right) exact if for each short exact sequence $0 \to A' \to A \to A'' \to 0$ following sequence is exact,

$$0 \to F(A') \to F(A) \to F(A'')(\text{ resp. } F(A') \to F(A) \to F(A'') \to 0).$$

We say an additive functor is exact if it is both left and right exact. If we have any additive functor $F : \mathcal{A}_1 \to \mathcal{A}_2$ then we can extend this functor canonically to the category of complexes. Hence we get an additive functor $\mathcal{C}(F) : \mathcal{C}(\mathcal{A}_1) \to \mathcal{C}(\mathcal{A}_2)$ which takes any complex $A^\cdot$ to a complex $F(A^\cdot)$. A morphism f between two complexes $A^\cdot$ and $B^\cdot$ is homotopic to 0 if there exists a map (not morphism of complexes) $h : A^\cdot \to B^\cdot[-1]$ s.t. $f = d \circ h + h \circ d$. Hence we have $F(f) = d \circ F(h) + F(h) \circ d$ and therefore the morphism $F(f)$ is homotopic to 0. This will give the functor $\mathcal{K}(F) : \mathcal{K}(\mathcal{A}_1) \to \mathcal{K}(\mathcal{A}_2)$ which will be an exact functor between these triangulated categories. Recall the canonical localization functor is denoted as $Q_{\mathcal{A}_1} : \mathcal{K}^\#(\mathcal{A}_1) \to \mathcal{D}^\#(\mathcal{A}_1)$ which is an exact functor. We shall fix the notation $\# \in \{+, -, b, \emptyset\}$ for boundedness conditions as earlier.

Definition 2.3.1 (Derived functor)**.** The right (resp. left) derived functor associated to an additive functor $F : \mathcal{A}_1 \to \mathcal{A}_2$ is a pair consist of an exact functor $RF^\# : \mathcal{D}^\#(\mathcal{A}_1) \to \mathcal{D}^\#(\mathcal{A}_2)$ (resp. $LF^\# : \mathcal{D}^\#(\mathcal{A}_1) \to \mathcal{D}^\#(\mathcal{A}_2)$)and a natural transformation $\varepsilon_F : Q_{\mathcal{A}_2} \circ \mathcal{K}(F) \to RF^\# \circ Q_{\mathcal{A}_1}$ (resp. $\xi_F : LF^\# \circ Q_{\mathcal{A}_1} \to Q_{\mathcal{A}_2} \circ \mathcal{K}(F)$),

$$\begin{tikzcd} \mathcal{K}^\#(\mathcal{A}_1) \arrow[r, "\mathcal{K}(F)"] \arrow[d, "Q_{\mathcal{A}_1}"'] & \mathcal{K}^\#(\mathcal{A}_2) \arrow[d, "Q_{\mathcal{A}_2}"] \arrow[dl, dashed, "\varepsilon_F"'] \\ \mathcal{D}^\#(\mathcal{A}_1) \arrow[r, "RF^\#"] & \mathcal{D}^\#(\mathcal{A}_2) \end{tikzcd}$$

(resp. similar diagram with ξ_F in reverse direction), s.t. for any exact functor $G : \mathcal{D}^\#(\mathcal{A}_1) \to \mathcal{D}^\#(\mathcal{A}_2)$ and a natural transformation $\varepsilon : Q_{\mathcal{A}_2} \circ \mathcal{K}(F) \to G \circ Q_{\mathcal{A}_1}$ (resp. $\xi : LF^\# \circ Q_{\mathcal{A}_1} \to Q_{\mathcal{A}_2} \circ \mathcal{K}(F)$) there exists a unique natural transformation $\eta : RF^\# \to G$ (resp. $\eta' : G \to LF^\#$) s.t. $(\eta \circ Q_{\mathcal{A}_1}) \circ \varepsilon_F = \varepsilon$(resp. $\xi_F \circ (\eta \circ Q_{\mathcal{A}_1}) = \xi$). We can also define $RF^\#$ (resp. $LF^\#$)as an additive functor s.t. following map is a bijection for each exact functor G,

$$Nat(RF^\#, G) \to Nat(Q_{\mathcal{A}_2} \circ \mathcal{K}(F), G \circ Q_{\mathcal{A}_1})$$

$$(\text{ resp. } Nat(G, LF^\#) \to Nat(G \circ Q_{\mathcal{A}_1}, Q_{\mathcal{A}_2} \circ \mathcal{K}(F))).$$

Remark 2.3.2.

1. If $RF^\#$ exists then it is unique up to unique isomorphism.
2. If we take any full triangulated subcategory of $\mathcal{K}(\mathcal{A}_1)$ which satisfies the hypothesis of prop.2.2.5 in place of $\mathcal{K}^\#(\mathcal{A}_1)$ then we have similar definition of derived functors.
3. If we compose $RF^\#$ (resp. $LF^\#$) with cohomological functor $\mathcal{H}^i$ we get the classical ith right (resp. left) derived functors. We denote it by $R^i(F)^\#$ (resp. $L^i(F)^\#$).

Now we describe some cases where derived functors exists. Firstly to get an isomorphism of derived functor with the satellite functors we assume semi-exactness on the functor, see [2.3, Grothendieck[19]] for more details. Also the construction of left derived functors is similar to right derived functors, so we briefly recall the construction for right derived functors. We need following definition of adapted class to a given left exact (covariant) functor $F : \mathcal{A}_1 \to \mathcal{A}_2$, [II.6.3 , Manin[16]].

Definition 2.3.3 (F-adapted class)**.** A collection of objects $\mathcal{R}$, stable under finite direct sum, is said to be *adapted* to a left exact functor F if following conditions are satisfied,

1. Every object of $\mathcal{A}_1$ is embedded in some object of the collection $\mathcal{R}$.
2. If $A \to A'' \to 0$ is an exact sequence of objects from the collection $\mathcal{R}$ then the sequence $F(A) \to F(A'') \to 0$ is also an exact sequence.
3. Given a short exact sequence $0 \to A' \to A \to A'' \to 0$, if objects A' and A are contained in the collection $\mathcal{R}$, then the object A'' is also contained in the collection $\mathcal{R}$.

The category $\mathcal{A}_1$ with a collection of objects $\mathcal{R}$ which satisfies the condition (i) above is called a category with sufficient (or enough) $\mathcal{R}$-objects. The conditions (ii) and (iii) are equivalent to saying that the functor $\mathcal{K}^+(F)$ restricted to $\mathcal{K}^+(\mathcal{R})$ preserves acyclic objects. Therefore the functor $Q \circ \mathcal{K}^+(F)$ restricted to the subcategory $\mathcal{K}^+(\mathcal{R})$ takes an acyclic complex to the zero object. Hence using the universal property, it factors through localization of quasi isomorphisms inside $\mathcal{K}^+(\mathcal{R})$. Now using prop.2.2.5 we get the embedding of $\mathcal{K}^+(\mathcal{R})[Qis^{-1}]$ inside $\mathcal{D}^+(\mathcal{A}_1)$. Since $\mathcal{A}_1$ has enough $\mathcal{R}$-objects this functor is also an essentially surjective functor. Hence it gives an equivalence of categories $\mathcal{K}^+(\mathcal{R})[Qis^{-1}]$ and $\mathcal{D}^+(\mathcal{A}_1)$. We have following commutative diagram,

$$\begin{tikzcd}
\mathcal{K}^+(\mathcal{R}) \arrow[r, hook] \arrow[dd, "Q"] & \mathcal{K}^+(\mathcal{A}_1) \arrow[r, "\mathcal{K}^+(F)"] \arrow[d, "Q"] & \mathcal{K}^+(\mathcal{A}_2) \arrow[dd, "Q"] \\
& \mathcal{D}^+(\mathcal{A}_1) \arrow[dl, dashed, bend right] \arrow[dr, dashed, "RF^+"] & \\
\mathcal{K}^+(\mathcal{R})[Qis^{-1}] \arrow[ur, "\sim"'] \arrow[rr] & & \mathcal{D}^+(\mathcal{A}_2).
\end{tikzcd}$$

The derived functor RF^+ is defined as a composition of two maps as shown in above diagram and $R^iF^+(A)$ represents the i-th cohomology of this complex. A priori, this definition of derived functor depends on the choice of

F-adapted class and a choice of equivalence coming from it, but using the universal property of derived functors, we get isomorphisms between all such constructions.

If an abelian category has sufficiently many injective objects then any left exact functor admits a right derived functor (or collection of all injective objects is adapted to every left exact functors), see[Theorem III.6.12 ,[16]]. Later we shall see many examples of derived functors. In our case we always consider an abelian category with enough injective objects. As we mentioned before, by changing the arrow we can define an adapted class for a right exact functor. We can also see that the collection of projective objects will be adapted to every right exact functor. The lack of interesting abelian categories with enough projective objects is the main motivation behind the definition of adapted class of objects. Hence given a right exact functor $G : \mathcal{A}_1 \to \mathcal{A}_2$ with enough G-adapted (or projective) objects there exists a left derived functor $LG^- : \mathcal{D}^-(\mathcal{A}_1) \to \mathcal{D}^-(\mathcal{A}_2)$. If the category $\mathcal{A}_1$ has enough injective (resp. projective) objects then the canonical inclusion of $\mathcal{K}^+(\mathcal{R})$ (resp. $\mathcal{K}^-(\mathcal{R})$) is an equivalence of categories, see [Prop. I.4.7, [20]]. The classical definition of left (resp. right) derived functors was given by considering first an injective (resp. projective) resolution and taking cohomology after applying the functor term by term. This is same as choosing a quasi-inverse of the canonical equivalence given by the inclusion of $\mathcal{K}^+(\mathcal{R})$ (or $\mathcal{K}^-(\mathcal{R})$). If we have a bi-functor $F : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}_3$ with a semi exactness in each variable then the existence of an adapted class gives the derived functor. We shall assume for simplicity that F is left exact in both variables and $\mathcal{A}_2$ has enough injectives (resp. $\mathcal{A}_1$ has enough projective objects). We have following result on the existence of right derived functor similar to [lemma I.6.2, [20]].

Proposition 2.3.4. *Suppose $I^{\cdot} \in \mathcal{K}^+(\mathcal{A}_2)$ is a complex of injective objects and $A_1^{\cdot} \in \mathcal{K}(\mathcal{A}_1)$ is any complex of objects. Assume that if either $I^{\cdot}$ is acyclic or $A_1^{\cdot}$ is acyclic then $F(A_1^{\cdot}, I^{\cdot})$ is acyclic. Then the derived functor $RF : \mathcal{D}(\mathcal{A}_1)\times\mathcal{D}^+(\mathcal{A}_2) \to \mathcal{D}(\mathcal{A}_3)$ exists. An analogous assumption for the case of enough projective objects gives the derived functor $LF : \mathcal{D}^-(\mathcal{A}_1)\times\mathcal{D}(\mathcal{A}_2) \to \mathcal{D}(\mathcal{A}_3)$.*

A justification similar to [lemma I.6.3 , [20]] can be used here for the ambiguous notation for derived bi-functor. An important example of a bi-functor comes from the functor given by Hom. If an abelian category has enough injectives then we have a derived functor $R\operatorname{Hom}(\_,\_) : \mathcal{D}(\mathcal{A})^{\circ} \times \mathcal{D}(\mathcal{A})^+ \to \mathcal{A}(\mathcal{A}\mathfrak{b})$. We also define the bi-functor

$$Ext^i(A_1, A_2) := \operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(A_1, A_2[i]) = \operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(A_1[-i], A_2)$$

which we call Yoneda extensions. There is a relation between the derived functors of Hom and the Yoneda extensions defined as above, see [Theorem 6.4, [20]].

Proposition 2.3.5. *There is a functorial isomorphism*

$$Ext^i(A_1, A_2) \simeq \mathcal{H}^i(R\,\mathrm{Hom}^+(A_1, A_2))$$

for each $A_1 \in Ob(\mathcal{D}(\mathcal{A}))$ *and* $A_2 \in Ob(\mathcal{D}^+(\mathcal{A}))$ *.*

Given two left exact functors $F_1 : \mathcal{A}_1 \to \mathcal{A}_2$ and $F_2 : \mathcal{A}_2 \to \mathcal{A}_3$, the following result gives a relation between the derived functor of the composition of two functors, and the composition of their derived functors.

Proposition 2.3.6 (Thm 2.4.1 ,[19])**.** *Suppose* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *are adapted class w.r.t. functor* F_1 *and* F_2 *respectively. Further, suppose that the image of the adapted class* $\mathcal{R}_1$ *is contained in the adapted class of* F_2 *i.e.* $F(\mathcal{R}_1) \subseteq \mathcal{R}_2$*. Then the right derived functors* RF_1^+, RF_2^+ *and* $R(F_2 \circ F_1)^+$ *exist and the natural map* $R(F_2 \circ F_1)^+ \to RF_2^+ \circ RF_1^+$ *is an isomorphism.*

Similar statements can be made in the case of right exact functors defined on categories with enough projective objects. Classically, the composite of two derived functors was related to derived functor of the composite via a spectral sequence. If we have two functors with all hypothesis as in above proposition then there exists a spectral sequence, called Grothendieck spectral sequence, as follows

$$E_2^{p,q} = R^pG(R^qF(A^\cdot)) \Rightarrow R^n(G \circ F)(A^\cdot).$$

This spectral sequence is functorial in $A^\cdot$, see [Theorem III.7.7, [16]] for a proof and more details.

2.4 Examples

2.4.1 Derived category of a commutative ring

We specialize to some particular cases of abelian categories to get the derived categories which are used later. Let R be a ring with unity. We denote by $\mathcal{M}\mathfrak{od}(R)$ the category of all left modules over R. We know that $\mathcal{M}\mathfrak{od}(R)$ is an abelian category. Now we can define the category of complexes $\mathcal{C}(\mathcal{M}\mathfrak{od}(R))$ and denote it by $\mathcal{C}(R)$. Similarly, we define homotopy category of complexes of R modules which is denoted as $\mathcal{K}(R)$. Using the localization of quasi isomorphisms we get the derived category of R modules and denoted as $\mathcal{D}(R)$.

A boundedness conditions on complexes gives full subcategories $\mathcal{D}^{\pm}(R)$ and $\mathcal{D}^{b}(R)$. Since every left R module is embedded in an injective object, we have a quasi isomorphism of any object in $\mathcal{D}^{+}(R)$ with a bounded below complex of injective objects. Hence, using the result [Prop. 4.7 ,[20]] there exists an equivalence of triangulated categories $\mathcal{D}^{+}(R)$ and $\mathcal{K}^{+}(\mathcal{I})$ where $\mathcal{I}$ is the additive category of all R-injective objects. Since there exists a bounded above free resolution for any module M, there are enough projective objects. Hence we have a canonical equivalence between $\mathcal{K}^{-}(R)$ and $\mathcal{D}^{-}(R)$ given by the natural inclusion. Suppose $f : R \to S$ is a ring homomorphism. We can define functor

$$f^{*} : \mathcal{M}\mathfrak{od}(R) \to \mathcal{M}\mathfrak{od}(S); A \mapsto S \otimes_{R} A.$$

Note that the ring homomorphism f gives a left as well as right R module structure on S and hence the above pull back functor is well defined right exact functor. Since there are enough flat objects which are adapted to this functor we get a left derived functor Lf^{*}. Also, the ring homomorphism gives a left R module structure on every left S module, we have push forward or forgetful functor

$$f_{*} : \mathcal{M}\mathfrak{od}(S) \to \mathcal{M}\mathfrak{od}(R); B \mapsto_{R} B.$$

Since this functor is an exact functor, we get a derived functor f_{*}. The functor Lf^{*} is a left adjoint of the functor f_{*}. In this thesis we are mainly interested in rings with some commutativity constraints. So, in later sections we shall restrict to commutative and super commutative rings. Since these categories are equivalent to categories coming from the more general notion of scheme and superschemes, we shall postpone various relations among these functors to later sections. For the case of commutative rings, the classical functors

$$Ext(A_1, A_2) : \mathcal{D}_{R}^{\circ} \times \mathcal{D}_{R}^{+} \to \mathcal{D}_{\mathbb{Z}} \text{ and } Tor(A_1, A_2) : \mathcal{D}_{R}^{-} \times \mathcal{D}_{R}^{-} \to \mathcal{D}_{\mathbb{Z}}^{-}$$

are defined as derived functors associated with the bi-functors

$$\mathrm{Hom}(\_,\_) : \mathcal{M}\mathfrak{od}(R)^{\circ} \times \mathcal{M}\mathfrak{od}(R) \to \mathcal{A}\mathfrak{b} \text{ and } \otimes : \mathcal{M}\mathfrak{od}(R) \times \mathcal{M}\mathfrak{od}(R) \to \mathcal{A}\mathfrak{b}$$

respectively. For the case of arbitrary rings we have to replace the category of left module by right module in first coordinate of tensor functor. We shall use the notation $\mathcal{D}_{R}$ for the triangulated category $K^{b}(R - Proj)$ i.e. the bounded homotopy category associated to the category of finitely generated projective modules over commutative ring R.

2.4.2 Derived category of a scheme

A ringed space is a pair consist of a topological space X and a sheaf of ring $\mathcal{O}_X$ which is called the structure sheaf. We denote by $\mathcal{M}\mathfrak{od}(X)$ the category of all sheaves of left modules of $\mathcal{O}_X$. We shall not discuss general ringed spaces here. We start with a scheme[3] in this section which is an important example of a locally ringed space. Firstly, recall that a ringed space is called locally ringed space if stalk of the structure sheaf at each point is a local ring. Further a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ between two locally ringed spaces is local i.e. induced morphism $f_x^{\#} : \mathcal{O}_{Y,f(X)} \to \mathcal{O}_{X,x}$ for each $x \in X$ satisfies $(f_x^{\#})^{-1}(m_{f(x)}) = m_x$ wheres $m_{f(x)}$ and m_x are maximal ideals. We restrict to the class of locally ringed spaces coming from schemes. Recall that for a scheme there is an abelian category $\mathcal{M}\mathfrak{od}(X)$ containing all sheaves of modules . We shall denote by $\mathcal{Q}\mathfrak{coh}(X)$, the full subcategory containing all quasi coherent sheaves. The category of coherent sheaves is denoted as $\mathcal{C}\mathfrak{oh}(X)$. As in the case of rings the category of complexes and homotopy category of complexes are denoted as $\mathcal{C}(X)$ and $\mathcal{K}(X)$ respectively. The categories $\mathcal{D}_{qc}(X)$ and $\mathcal{D}_c(X)$ (or $\mathcal{D}_X$ for simplicity) represent the derived category of quasi coherent sheaves and coherent sheaves respectively. We have similar notations as earlier for various boundedness restrictions on complexes. Since there exist enough injectives in the abelian category $\mathcal{Q}\mathfrak{coh}(X)$, by using [Theorem III.6.12 ,[16]] we get derived functors of some well known functors. We shall recall some of these functors which we need later.

Since there are enough flat objects, therefore the bi-functor given by usual tensor product gives a derived bi-functor, see [Page 95, [20]],

$$\otimes^L : \mathcal{D}^-(\mathcal{M}\mathfrak{od}(X)) \times \mathcal{D}^-(\mathcal{M}\mathfrak{od}(X)) \to \mathcal{D}^-(\mathcal{M}\mathfrak{od}(X)).$$

Further this functor restricted to full thick subcategories $\mathcal{D}^-_{qc}(X)$ and $\mathcal{D}^-_c(X)$ gives an induced tensor derived bi-functor. The classical Tor functor is a restriction of hypertor functor, $Tor_i(\mathcal{F}_1, \mathcal{F}_2) := \mathcal{H}^{-i}(\mathcal{F}_1 \otimes^L \mathcal{F}_2)$ for each $i \in \mathbb{Z}$, on the abelian category. Similarly there is another bi-functor called internal Hom and denoted as $\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)$ for each pair of sheaves $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{M}\mathfrak{od}(X)$. This functor is left exact in both coordinates. Since there are enough injective objects, we get following right derived bi-functor

$$R\mathcal{H}om : \mathcal{D}^+(\mathcal{M}\mathfrak{od}(X)) \times \mathcal{D}^+(\mathcal{M}\mathfrak{od}(X)) \to \mathcal{D}^+(\mathcal{M}\mathfrak{od}(X)).$$

As above this functor will also induce the derived bi-functor on the full subcategories $\mathcal{D}^+_{qc}(X)$ and $\mathcal{D}^+_c(X)$.

[3] it is always Noetherian unless explicitly stated otherwise.

Recall, the global section functor is defined as

$$\Gamma : \mathcal{M}\mathfrak{od}(R) \to \mathcal{A}\mathfrak{b}; \mathcal{F} \mapsto \Gamma(\mathcal{F}).$$

Here $\Gamma(\mathcal{F})$ represents the set of all global sections of the sheaf of modules $\mathcal{F}$. It is a left exact functor. The right derived functor of Γ is denoted as $H^i(X, \mathcal{F}) := R\Gamma^i(\mathcal{F})$. If we consider schemes over some field k, then the functor Γ takes values in the category of vector spaces.

More generally, we can define the push forward functor which gives the global section functor as particular case. Suppose $f : X \to Y$ is a morphism of schemes. There is a left exact functor

$$f_* : \mathcal{M}\mathfrak{od}(X) \to \mathcal{M}\mathfrak{od}(Y); \mathcal{F} \mapsto f_*\mathcal{F}.$$

Recall that $f_*\mathcal{F}$ is the sheaf associated to the presheaf that associates $f_*\mathcal{F}(V) := \mathcal{F}(f^{-1}(V))$ to each open set V in Y. This will give a right derived functor

$$Rf_* : \mathcal{D}^+(\mathcal{M}\mathfrak{od}(X)) \to \mathcal{D}^+(\mathcal{M}\mathfrak{od}(Y)).$$

We have two full thick subcategories $\mathcal{Q}\mathfrak{coh}(X)$ and $\mathcal{C}\mathfrak{oh}(X)$ of the abelian category $\mathcal{M}\mathfrak{od}(X)$. Note that there are examples of injective objects in $\mathcal{Q}\mathfrak{coh}(X)$ which are not injective when considered as an object of $\mathcal{M}\mathfrak{od}(X)$ in general, see [Appendix B, [40]] for more details.

We shall always work with Noetherian schemes unless explicitly stated otherwise. In this case the derived functor defined on $\mathcal{D}^+_{qc}(X)$ and the restriction of the derived functor Rf_* on $\mathcal{D}^+(\mathcal{M}\mathfrak{od}(X))$ coincide. Further if we consider a proper map of schemes then we get a restriction of the above derived functor to the derived category of coherent sheaves,

$$Rf_* : \mathcal{D}^+_c(X) \to \mathcal{D}^+_c(Y).$$

Now using the result of Grothendieck,[Theorem 3.6.5, [19]], we get a restriction of push forward functor to the bounded derived subcategory of schemes. We have following functors for a proper morphism $f : X \to Y$

$$\begin{tikzcd}
\mathcal{D}^b_{qc}(X) \arrow[r, "Rf_*"] & \mathcal{D}^b_{qc}(Y) \\
\mathcal{D}^b_c(X) \arrow[r, "Rf_*"] \arrow[u, hook] & \mathcal{D}^b_c(Y) \arrow[u, hook]
\end{tikzcd}$$

If we consider the scheme over some field then the map $f : X \to spec(k)$ will give $R^i f_*(\mathcal{F}) = H^i(X, \mathcal{F})$ for each $i \in \mathbb{Z}$. Also, if we consider a map $f : X \to Y$ of two k-schemes then $\Gamma \circ f_* = \Gamma$. Now using Grothendieck spectral sequence for these functors we get the Leray spectral sequence as follows,

$$E_2^{p,q} = H^p(X, R^q f_*(\mathcal{F})) \Rightarrow H^n(X, \mathcal{F}).$$

There are other spectral sequences which come from the Grothendieck spectral sequence, see [Page 74,[22]] for some more cases of Leray spectral sequence. Next we shall recall the right adjoint of the push forward functor.

Let $f : X \to Y$ be a morphism between two schemes. Recall, the inverse image functor is defined as,

$$f^{-1} : \mathcal{M}\mathfrak{od}(Y) \to \mathcal{M}\mathfrak{od}(X, f^{-1}(\mathcal{O}_Y); \mathcal{G} \mapsto f^{-1}\mathcal{G}.$$

Here the $f^{-1}(\mathcal{O}_Y)$ module $f^{-1}\mathcal{G}$ is defined as a sheaf associated to the presheaf that associates $f^{-1}\mathcal{G}(U) := \varinjlim_{f(U)\subseteq V} \mathcal{G}(V)$ to each open set U in X. This is an exact functor. Now using the tensor functor $\_ \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X : \mathcal{M}\mathfrak{od}(Y) \to \mathcal{M}\mathfrak{od}(X)$, we can define the functor $f^* := (\_ \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \circ f^{-1}$. Since tensor is a right exact functor therefore the functor f^* is right exact. Since there are enough flat objects, we get the left derived functor

$$Lf^* : \mathcal{D}^-(\mathcal{M}\mathfrak{od}(Y)) \to \mathcal{D}^-(\mathcal{M}\mathfrak{od}(X))$$

which is same as composition of functors $(\_ \otimes^L_{f^{-1}Y} \mathcal{O}_X) \circ f^{-1}$. If f is a flat morphism then this functor is exact and it will extend to the derived category. It is denoted as f^* for a flat map f. It restricts to the thick subcategories $\mathcal{D}^-_{qc}(Y)$ and $\mathcal{D}^-_c(Y)$. We use the same notation for these restrictions. Moreover $Lf^* : \mathcal{D}^-_c(Y) \to \mathcal{D}^-_c(X)$ is a left adjoint of the functor $Rf_* : \mathcal{D}^+(\mathcal{M}\mathfrak{od}(X)) \to \mathcal{D}^+(\mathcal{M}\mathfrak{od}(Y))$ for a morphism f between two schemes of finite Krull dimension, see [Cor 5.11,[20]]. These functors have many compatible identities. We now recall two very useful formulas.

Proposition 2.4.1. *Let $f : X \to Y$ be a morphism between two schemes of finite Krull dimension.*

1. *(Projection formula)[Prop. 5.6, [20]] If f is a quasi-compact morphism then there exist functorial isomorphisms*

   $$Rf_*(\mathcal{F}^\cdot) \otimes^L \mathcal{G}^\cdot \xrightarrow{\sim} Rf_*(\mathcal{F}^\cdot \otimes^L Lf^*(\mathcal{G}^\cdot))$$

   for each $\mathcal{F}^\cdot \in \mathcal{D}^-(X)$ and $\mathcal{G}^\cdot \in \mathcal{D}^-_{qc}(Y)$.

2. *(Flat base change)[Prop. 5.12, [20]] Let $u : Y' \to Y$ be a flat map and $X' := X \times_Y Y'$ be a fiber product with the following Cartesian diagram*

   $$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"] & Y. \end{tikzcd}$$

If f is a morphism of finite type then there exist functorial isomorphisms

$$u^* \circ Rf_*(\mathcal{F}^\cdot) \xrightarrow{\sim} Rg_* \circ v^*(\mathcal{F}^\cdot) \text{ for each } \mathcal{F}^\cdot \in \mathcal{D}_{qc}(X).$$

2.4.3 Derived category of G-scheme

Throughout this section, let k be a field and G be a finite group whose order is coprime to the characteristic of k. Let X be a smooth quasi-projective variety over k, with an action of the finite group G i.e. there is a group homomorphism from G to the automorphism group of algebraic variety X. We say G acts freely on X if $gx \neq x$ for any $x \in X$ and any $g \in G$ with $g \neq e$. Recall following general result proved in Mumford's book [page 66, [30]] on the existence of a group quotient,

Theorem 2.4.2. *Let X be an algebraic variety and G a finite group of automorphisms of X. Suppose that for any $x \in X$, the orbit Gx of x is contained in an affine open subset of X. Then there is a pair (Y, π) where Y is a variety and $\pi : X \to Y$ a morphism, satisfying:*

1. *as a topological space, (Y, π) is the quotient of X for the G-action; and*
2. *if $\pi_*(\mathcal{O}_X)^G$ denotes the subsheaf of G-invariants of $\pi_*(\mathcal{O}_X)$ for the action of G on $\pi_*(\mathcal{O}_X)$ deduced from 1, the natural homomorphism $\mathcal{O}_Y \to \pi_*(\mathcal{O}_X)^G$ is an isomorphism.*

The pair (Y, π) is determined up to an isomorphism by these conditions. The morphism π is finite, surjective and separable. Y is affine if X is affine.

If further G acts freely on X, π is an étale morphism.

In the remark after the proof [page 69, [30]], Mumford further showed that quasi projective varieties always satisfy the hypothesis of the theorem. We denote this quotient space (if it exists) by X/G.

Definition 2.4.3. 1. For a variety X with a G action, and $H \subset G$ a subgroup, let X^H be the subvariety of fixed points of H.

2. A G-invariant component is defined to be a minimal G-invariant subvariety of X with reduced structure such that its dimension is equal to $\dim X$.

Proposition 2.4.4. *With the notation in the above paragraph,*

1. *X^H is a closed subvariety.*

2. *If $H_1 \subseteq H_2$ are subgroups then we have a reverse inclusion $X^{H_2} \subseteq X^{H_1}$*

3. *If Z is any G-invariant component of X then there exists an open subset of Z with free action of G/H for unique subgroup H.*

4. *If Z is any G-invariant subvariety of X then there exists the set of subgroups H_i for $i = 1, \dots, r$ and open subsets $W_i, i = 1, \dots, r$ of Z such that G/H_i acts freely on W_i for $i = 1, \dots, r$. Here r is the number of G-invariant components of Z. Also note that the open subsets W_i are pairwise disjoint, and* $\dim(Z \setminus \cup_i W_i) < \dim Z$.

Proof of 1. Since $X^H = \cap_{h \in H} X^h$ where X^h is the fixed points of the automorphism corresponding to h under the action. It is enough to prove that the invariant set X^h of any automorphism h of a variety is a closed subset. X is a quasi-projective Noetherian variety and hence separated. Therefore, the diagonal and the graph of any automorphism will be a closed subset of $X \times X$. The intersection of the graph of an automorphism with the diagonal will be a closed subset of the diagonal. Hence the invariant of the automorphism h will be closed in X.
Proof of 2. It clearly follows from the formulae $X^{H_i} = \cap_{h \in H_i} X^h$.
Proof of 3. For any algebraic subset there exists the subgroup H such that G/H acts faithfully(or effectively) onto it. Without loss of generality we assume that G acts faithfully on Y. For a faithful action Y^H is a proper subset of Y for any nontrivial normal subgroup H of G. Define the open subset of Y as

$$U = Y - (\cup_{H \trianglelefteq G} Y^H)$$

where the union on the right side is over all nontrivial normal subgroups. It is now easy to see that G acts freely on open set U.
Proof of 4. Using *3.*, it is enough to prove that any algebraic subset can be uniquely written as union of G-invariant components of Y, and an algebraic subset of dimension strictly less than $\dim Y$. Since Y is Noetherian, it will be finite union of irreducible closed subsets. Take the finite set S of generic points of irreducible subsets of Y, which have the same dimension as Y. Now the action of G on Y induces an action on the finite set S; since an automorphism of Y will take any irreducible subset to another irreducible subset of the same dimension. Thus S can be uniquely written as a disjoint union of G-invariant subsets. By taking union of closure of the generic points in each invariant subset, we get the G-invariant components of Y. Clearly, any nonempty intersection of U_i and U_j for $i \neq j$ will give a proper G-invariant component, and this will contradict the minimality. $\square$

Since the scheme is quasi projective there exists an orbit space, see [30], which we denoted as X/G. As G is a finite group we get a finite map $\pi : X \to X/G$ which is a perfect morphism (see [Def. 2.5.2, [40]]. Recall that a G equivariant or G linearized sheaf is defined as follows

Definition 2.4.5. A G-sheaf (or G-equivariant sheaf or an equivariant sheaf with respect to the group G) on X is a sheaf $\mathcal{F}$ together with isomorphisms $\rho_g : \mathcal{F} \to g^*\mathcal{F}$ for all $g \in G$ such that following diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\rho_h"] \arrow[drr, "\rho_{gh}"'] & h^*\mathcal{F} \arrow[r, "h^*\rho_g"] & h^*g^*\mathcal{F} \arrow[d, equal] \\ & & (gh)^*\mathcal{F} \end{tikzcd}$$

is commutative for any pair $g, h \in G$. A G-sheaf is a pair $(\mathcal{F}, \rho)$.

We shall now look at some properties of G-sheaves (definition 2.4.5). The G-sheaves form a category $\mathfrak{Qcoh}^G(X)$ as follows. Given two G-sheaves $(\mathcal{F}, \rho)$ and $(\mathcal{G}, \psi)$, the group of morphisms of $\mathcal{O}_X$-modules $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$ gets a G action, where $g \in G$ acts on θ to give $\psi_g^{-1} \circ g^*\theta \circ \rho_g$. $\mathrm{Hom}_{\mathfrak{Qcoh}^G(X)}((\mathcal{F}, \rho), (\mathcal{G}, \psi))$ is defined to be the group of G-invariant morphisms in $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$.

$\mathfrak{Qcoh}^G(X)$ is an abelian category. Define $\mathfrak{Coh}^G(X)$ to be the abelian subcategory of $\mathfrak{Qcoh}^G(X)$ consisting of objects $(\mathcal{F}, \rho)$, for which $\mathcal{F}$ is coherent. In Tohoku paper of Grothendieck [19] it was proved that $\mathfrak{Qcoh}^G(X)$ has enough injectives. Also, for finite G and quasi-projective X, there is an ample invertible G-sheaf, allowing G-equivariant locally free resolutions (see [19], [10]). Therefore derived functors of various functors like π_*, π^* and $\otimes$ will always exist, in a similar fashion as in the non-equivariant case, and for simplicity we shall write π_*, π^* and $\otimes$ for $\mathbf{R}\pi_*$, $\mathbf{R}\pi^*$ and $\otimes^{\mathbf{L}}$ respectively.

Definition 2.4.6. Let $\mathcal{D}^G(X)$ be the bounded derived category of $\mathfrak{Coh}^G(X)$.

Remark 2.4.7.

1. As in the case of $\mathcal{D}^b(X)$, we have a symmetric monoidal structure on $\mathcal{D}^G(X)$ given by the (left) derived functor of the tensor structure on $\mathfrak{Coh}^G(X)$. Also $\mathcal{D}^G(X)$ has a natural structure of a k-linear category. We shall use this fact later.

2. Note that here and elsewhere (for example theorem 1.0.1 and its special cases mentioned later), we assume X to be smooth to make the definition of $\mathcal{D}^G(X)$ meaningful. It might be possible to remove the assumption that X is smooth. But we will not consider that question here.

Given an algebraic variety X with an action of a finite group G we have a natural morphism $\pi : X \to X/G$ which further gives a functor $\pi_* : \mathcal{Coh}^G(X) \to \mathcal{Coh}^G(X/G)$ and by taking G-invariant part of image we can define a functor $\pi_*^G : \mathcal{Coh}^G(X) \to \mathcal{Coh}(X/G)$ i.e. $\pi_*^G(\mathcal{F}, \rho) = (\pi_*(\mathcal{F}, \rho))^G$ for all $(\mathcal{F}, \rho) \in \mathcal{Coh}^G(X)$. We have following result when G acts freely on X (see Mumford's book [30] for the proof).

Proposition 2.4.8. *Let $\pi : X \to X/G$ be a natural morphism given by free action of the finite group G on X. The map $\pi^* : \mathcal{Coh}(X/G) \to \mathcal{Coh}^G(X)$ is an equivalence of abelian categories with the quasi-inverse π_*^G. Further locally free sheaves corresponds to locally free sheaves of the same rank.*

Now we can extend above equivalence to get a tensor equivalence π^* between tensor triangulated categories $\mathcal{D}^b(X/G)$ and $\mathcal{D}^G(X)$. In general these two categories are not equivalent. Next we prove that there exists a canonical decomposition, similar to the canonical decomposition of a finite dimensional representation of G.

Suppose X is a smooth quasi projective variety over a field k, with the structure morphism $\eta : X \to \mathrm{Spec}(k)$. The category of all coherent sheaves on affine variety $\mathrm{Spec}(k)$ can be identified with category of all finite dimensional vector spaces and the category of all G-equivariant sheaves can be identified with finite dimensional k-linear G representations. By using properties of the pullback functor η^*, which is exact, we can prove the following basic results; see [10] for details and some similar results.

Lemma 2.4.9. *1. $\eta^*(k) = \mathcal{O}_X$.*

2. $\eta^*(V_1 \otimes V_2) = \eta^*(V_1) \otimes \eta^*(V_2)$.

3. $\eta^*(V^*) = (\eta^*(V))^*$.

4. If $X = \mathrm{Spec}(R)$ is an affine variety then we have following relation between functors:

(a) $\eta^*(V) \otimes \widetilde{M} = \widetilde{V \otimes M}$.

(b) $\pi_\lambda(\widetilde{M}) = \widetilde{\pi_\lambda(M)}$

Let $(\mathcal{G}, \lambda)$ be an object in $\mathcal{Coh}^G(X)$. We shall denote $\eta^*(V) \otimes \mathcal{G}$ by $V \otimes \mathcal{G}$ for simplicity. For the trivial action of G on X, the association of a G-sheaf $\mathcal{G}$ on X to its G-invariant subsheaf is functorial. More precisely, the exact functor

$$(\_\_)^G : \mathcal{Coh}^G X \to \mathcal{Coh}^G X$$

induces an exact functor

$$(\_\_)^G : \mathcal{D}^G(X) \to \mathcal{D}^G(X).$$

Note that the action of G on an object in the image of this functor is trivial. Thus the image of $(\_\_)^G$ lies in $\mathcal{D}^b(X)$, where $\mathcal{D}^b(X)$ is considered as a subcategory of $\mathcal{D}^G(X)$ consisting of objects with trivial G-action (see [first paragraph of section 4.4, [10]]). For a vector space V over k with an action of G, define the exact functor

$$\mathrm{Hom}_G(V, \_\_) = (\eta^* V \otimes \_\_)^G : \mathcal{D}^G(X) \to \mathcal{D}^G(X).$$

Notice that each object contained in the image of the functor $\mathrm{Hom}_G(V, \_)$ are trivial G-sheaves. Thus the image of $\mathrm{Hom}_G(V, \_\_)$ lies in $\mathcal{D}^b(X)$. Let V_λ be an irreducible representation of the group G. We have the evaluation map from $V_\lambda \otimes V_\lambda^*$ to k. We can pullback the usual evaluation map from the representation category to the bounded derived category of G-equivariant sheaves. Thus we have the following morphism,

$$\eta^*(ev_{V_\lambda}) \otimes id : V_\lambda \otimes V_\lambda^* \otimes \mathcal{F} \to \mathcal{F}.$$

Now by using the fact that the G-invariant part of a G-module V is a direct summand of $V^* \otimes V$, and the map $\eta^*(ev) \otimes id$ we get the following map, which we denote by $ev_{\mathcal{F}}$,

$$ev_{\mathcal{F}} : \oplus_\lambda V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, \mathcal{F}^\cdot) \to \mathcal{F}^\cdot.$$

We have following lemma which is used later to prove canonical decomposition.

Lemma 2.4.10. *The association sending $\mathcal{F}^\cdot$ to $\oplus_\lambda V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, \mathcal{F}^\cdot)$ gives an exact functor from $\mathcal{D}^G(X)$ to itself. Further, the objectwise morphism $ev_{\mathcal{F}}$ induces a natural transformation between this functor and the identity functor.*

Proof. Since the association $\mathrm{Hom}_G(V, \_)$ is a functor, it is easy to see thet the association taking $\mathcal{F}^\cdot$ to $\oplus_\lambda V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, \mathcal{F}^\cdot)$ is functorial. Consider a morphism $f : \mathcal{F}_1^\cdot \to \mathcal{F}_2^\cdot$ in $\mathcal{D}^G(X)$. Now the naturality of morphism ev follows from the commutativity of following diagrams,

$$\begin{tikzcd}
\bigoplus\limits_\lambda V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, \mathcal{F}_1^\cdot) \arrow[r] \arrow[d] & \bigoplus\limits_\lambda V_\lambda \otimes V_\lambda^* \otimes \mathcal{F}_1^\cdot \arrow[r] \arrow[d] & \mathcal{F}_1^\cdot \arrow[d, "f"] \\
\bigoplus\limits_\lambda V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, \mathcal{F}_2^\cdot) \arrow[r] & \bigoplus\limits_\lambda V_\lambda \otimes V_\lambda^* \otimes \mathcal{F}_2^\cdot \arrow[r] & \mathcal{F}_2^\cdot
\end{tikzcd}$$

Here $f : \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism compatible with the action of the finite group G and therefore gives commutativity of the left square. $\square$

We recall a general result about G actions.

Lemma 2.4.11. *Suppose M is a k-linear G-representation (need not be finite dimensional) for finite group G. The following canonical evaluation map is an isomorphism*

$$ev : \oplus_\lambda V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, M) \to M.$$

Proof. See [Proposition 4.1.15, [18]] □

Definition 2.4.12. We define *amplitude length* to be the integral function

$$\mathrm{ampl} : \mathcal{D}^G(X) \to \mathbb{Z}; \mathcal{F}^\cdot \mapsto |\{i \in \mathbb{Z}/\mathcal{H}^i(\mathcal{F}^\cdot) \neq 0\}|$$

that is, it is the number of non-zero hypercohomologies $\mathcal{H}^i$ of a bounded complex.

We prove the canonical decomposition of any object using pullback and reduction to affine case.

Proposition 2.4.13. *Suppose X is an algebraic set (need not be a smooth variety) over a field K with trivial action of a finite group G whose order is coprime to* $\mathrm{char}(K)$. *Let $\mathcal{F}^\cdot$ be a bounded complex of G-equivariant coherent sheaves i.e. ampl$(\mathcal{F}^\cdot) < \infty$. There exists a direct sum decomposition of $\mathcal{F}^\cdot$ as follows,*

$$\mathcal{F}^\cdot = \oplus_\lambda V_\lambda \otimes \mathcal{F}_\lambda$$

where V_λ are finite dimensional irreducible representations of G and $\mathcal{F}_\lambda = (V_\lambda^* \otimes \mathcal{F}^\cdot)^G =:$ $\mathrm{Hom}_G(V_\lambda, \mathcal{F}^\cdot)$. Here the complexes $\mathcal{F}_\lambda$ are trivial G-equivariant sheaves or usual sheaves.

Proof. We shall divide proof into two steps. In the first step we prove the case of coherent sheaf concentrated in degree zero (which we refer as a pure sheaf). In the second step we prove isomorphism of the map ev using the first step.
Step 1. Let $\mathcal{F}^\cdot$ be a complex with a coherent sheaf concentrated at zero, say $\mathcal{F}$. We can assume that the variety X is affine as it is enough to prove isomorphism on any affine cover. Hence we can assume that $\mathcal{F} = \widetilde{M}$. Thus we reduce the problem to proving that the following map is a bijection.

$$ev : \oplus_\lambda V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, M) \to M.$$

This map is an equivariant morphism, see [Page 184, [18]] for more discussions on this. It is enough to prove that the map ev is bijection as a k-linear morphism but this follows from the lemma 2.4.11.

Step 2. Since *ev* is a natural transformation, the full subcategory of $\mathcal{D}^G(X)$, on which *ev* is a natural isomorphism, is thick. By **Step 1.**, it contains shifts of sheaves and hence must be the whole of $\mathcal{D}^G(X)$.

Hence using these two steps we have the canonical decomposition as stated and further it is easy to observe that $\mathcal{F}_\lambda$ are trivial as G-sheaves i.e. all ρ_g are identity, see definition 2.4.5. □

We shall use proposition 2.4.13 in the following form.

Corollary 2.4.14. *Let X be a smooth algebraic variety defined over k with a G action. Let $U \subset X$ be a (possibly singular) G-invariant, locally closed subset, with $i_U : U \to X$ being the inclusion. Suppose H is a subgroup of G with the property that it acts trivially on U. Then for any object $(\mathcal{G}, \rho) \in \mathcal{D}^G(X)$ we have the canonical decomposition,*

$$(\mathcal{F}, \rho) = \oplus_\lambda W_\lambda \otimes (\mathcal{F}, \rho)_\lambda$$

where $\mathcal{F} = i_U^* \mathcal{G}$ and $(\mathcal{F}, \rho)_\lambda = (W_\lambda^* \otimes (\mathcal{F}, \rho))^H$ and W_λ is a finite dimensional irreducible representation of the subgroup H, and sum is over all finite dimensional irreducible representation of H. The subgroup H acts trivially on $(\mathcal{F}, \rho)_\lambda$ and this will induce the natural action of the group G/H on $(\mathcal{F}, \rho)_\lambda$.

Proof. Note that $\mathcal{G}$ has finite amplitude length, and hence so does $\mathcal{F}$. Thus the above proposition applies to U. □

Note that if G acts trivially on X then we can take $H = G$ and as a particular case we shall get the canonical decomposition,

$$(\mathcal{F}, \rho) = \oplus_\lambda V_\lambda \otimes (\mathcal{F}, \rho)_\lambda$$

where $(\mathcal{F}, \rho)_\lambda = (V_\lambda^* \otimes (\mathcal{F}, \rho))^G$ and V_λ is a finite dimensional irreducible representation of the group G.

2.4.4 Derived category of a superscheme

In this section first we shall recall the basic definition of superscheme and some properties of it. We shall relate various notion for some superschemes with the usual scheme with certain diagram.

Superalgebra

An associative $\mathbb{Z}/2\mathbb{Z}$-grading ring is an associative ring R with direct sum decomposition $R = R^0 \oplus R^1$ as an additive group so that multiplication preserves the grading i.e. $R^iR^j \subseteq R^{i+j}$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$. There exists a parity function which takes values in ring $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ for every homogeneous element of R i.e. if $r \in R^i$ then parity denoted $\bar{r} = i$. Now we restrict to following important class of rings,

Definition 2.4.15. An associative $\mathbb{Z}/2\mathbb{Z}$ graded ring with unity, $R = R^0 \oplus R^1$ is called supercommutative if the supercommutator of a ring R is zero i.e. $[r_1, r_2] := r_1r_2 - (-1)^{\tilde{r_1}\tilde{r_2}}r_2r_1 = 0$ for all $r_1, r_2 \in R$. Further ring is called k-superalgebra if R is supercommutative k-algebra with $k \subseteq R^0$.We shall assume that $2 \in R$ is invertible. This will ensure that the elements with parity 1 are nilpotents.

As usual we can define an abelian category of left modules over any k-superalgebra R, say $\mathcal{M}\mathfrak{od}(R)$. An object of this category is a $\mathbb{Z}/2\mathbb{Z}$- graded abelian group with a left R-module structure which is compatible with the grading i.e. $R^iM^j \subseteq M^{i+j}$ for all $i, j = 0, 1$. Morphism between these objects is a graded morphism compatible with the action of R. Similarly there exists a parity function defined for each homogeneous element of a module M denoted as above. We can define the parity change functor Π : $\mathcal{M}\mathfrak{od}(R) \to \mathcal{M}\mathfrak{od}(R); M \mapsto \Pi M$ with $\mathbb{Z}/2\mathbb{Z}$ grading given by $(\Pi M)^0 = M^1$ and $(\Pi M)^1 = M^0$. There exist an exact faithful functor from $\mathcal{M}\mathfrak{od}(R)$ as follows,

$$ff : \mathcal{M}\mathfrak{od}(R) \to \mathcal{M}\mathfrak{od}(R^0) \times \mathcal{M}\mathfrak{od}(R^0).$$

A canonical right module structure on left R modules is given by $mr := (-1)^{\bar{m}\bar{r}}rm$. Now using this structure we can define tensor product of two left R- modules M_1 and M_2 as quotient of $M_1 \otimes_{R^0} M_2$ with submodule generated by homogeneous elements

$$\{r_1m_1 \otimes m_2 - (-1)^{\bar{m}_1}m_1 \otimes r_1m_2 | r_1 \in R^1, m_i \in M^i\}$$

Here $M_1 \otimes_{R^0} M_2$ is defined as a tensor product of two $\mathbb{Z}/2\mathbb{Z}$ graded modules over a commutative ring R^0. The tensor product $M_1 \otimes_R M_2$ is then a $\mathbb{Z}/2\mathbb{Z}$ graded module with $\overline{m \otimes n} = \bar{m}+\bar{n}$. A commutativity constraint is similar to the case of tensor product of supervector spaces. Another important notion in commutative algebra is localization. It is easy to define localization of rings and modules if multiplicative set is contained in the center of a ring. For super commutative ring we can define localization at any homogeneous prime ideal. It is easy to observe that given a R module M and a prime ideal $\mathfrak{p}$, the localization $M_{\mathfrak{p}} = 0$ iff $({}_{R^0}M)_{\mathfrak{p}} = 0$ (or $({}_{(R/J)}M)_{\mathfrak{p}} = 0$ where $J := R{\cdot}R^1$. We can also prove the following version of Nakayama's lemma for superrings.

Proposition 2.4.16 (Nakayama's lemma)**.** *Suppose a finitely generated R module M satisfies $IM = M$ for the homogeneous ideal I given by the intersection of all maximal homogeneous ideals then $M = 0$.*

Proof. The proof is similar to the case of rings. Firstly observe that R_1 has all nilpotent elements. Hence for any element $a := a_0 + a_1$ with $a_i \in R_i$ the element $1 - a$ is unit iff $1 - a_0$ is unit in $R_0 \cap I$. But $R_0 \cap I$ is the Jacobson radical of R_0 and therefore $1 - a_0$ is unit. Rest of the proof is similar to the commutative case, [see Prop. 2.6, [1]]. □

Now using Nakayama's lemma we get following result whose proof is similar to the commutative case.

Corollary 2.4.17. *Suppose* $(R, \mathfrak{m})$ *is a local superring. Let* M, M_1 *and* M_2 *be finitely generated* R *modules.*

1. *A finitely generated module* $M = 0$ *if and only if* $M \otimes R/\mathfrak{m} = 0$.

2. $M_1 \otimes M_2 = 0$ *if and only if* $M_1 = 0$ *or* $M_2 = 0$.

Split Superscheme

Given any topological space X we can define a super ringed space by attaching a sheaf of superrings on X. We shall denote a sheaf of superrings with $\mathbb{Z}/2\mathbb{Z}$ grading as $\mathcal{O}_X = \mathcal{O}_{X,0} \oplus \mathcal{O}_{X,1}$. Similarly we can define sheaf of modules and parity change functor Π over such a ringed space as before. We have following definition,

Definition 2.4.18. A ringed space $(X, \mathcal{O}_X)$ is called a *superspace* if the ring $\mathcal{O}_X(U)$ associated to any open subset U is supercommutative and each stalk is local ring. A *superspace* is called *superscheme* if in addition, the ringed space $(X, \mathcal{O}_{X,0})$ is a scheme and $\mathcal{O}_{X,1}$ is a coherent sheaf over $\mathcal{O}_{X,0}$.

A superscheme X is called quasi compact and quasi separated if $(X, \mathcal{O}_{X,0})$ is quasi compact and quasi separated. Similarly a superscheme is (topologically) Noetherian if $(X, \mathcal{O}_{X,0})$ is (topologically) Noetherian. We shall use these notion in later chapters to borrow results developed by Grothendieck. We say that a superscheme is *affine* if the even part of structure sheaf $(X, \mathcal{O}_{X,0})$ is affine. It is easy to see that any affine superscheme gives a super commutative ring. Equivalently an affine superscheme associated to any super commutative ring can be defined in a manner similar to usual affine schemes. Note that in the definition of superscheme the odd part is a coherent sheaf of modules over the even part. Therefore if the even part of a superscheme is Noetherian then we shall get a left (or two sided) Noetherian superscheme. Given a superscheme $(X, \mathcal{O}_X)$ we can define sheaf of ideal[page 83, [26]] $J_X := \mathcal{O}_X \cdot \mathcal{O}_{X,1}$. Define $GrX := \oplus_{i \geq 0} J_X^i / J_X^{i+1}$ where $J_X^0 := \mathcal{O}_X$ and we denote the first term of GrX as $Gr_0X = \mathcal{O}_X/J_X$. Now using these notations we can define structure sheaves of *even* scheme and *reduced* scheme associated to the superscheme X as follows,

$$\mathcal{O}_{X_{rd}} := Gr_0X \text{ and } \mathcal{O}_{X_{red}} := \mathcal{O}_X/\sqrt{J_X}.$$

Here J_X/J_X^2 is a locally free sheaf of finite rank $0|d$ for some d over $\mathcal{O}_{X_{rd}}$. And GrX is a Grassmann algebra over $\mathcal{O}_{X_{rd}}$ of locally free sheaf J_X/J_X^2. Following particular class of superschemes are defined in [page 85, Manin[26]].

Definition 2.4.19. A superscheme $(X, \mathcal{O}_X)$ is called *split* if the graded sheaf GrX with mod 2 grading is isomorphic as a locally superringed sheaf to the structure sheaf $\mathcal{O}_X$.

Manin has also given a way to construct such a split superscheme. If we take purely even scheme $(X, \mathcal{O}_X)$ and a locally free sheaf $\mathcal{V}$ over $\mathcal{O}_X$ then we can define symmetric algebra of odd locally free sheaf $\Pi\mathcal{V}$, which is denoted $S(\Pi\mathcal{V})$, then $(X, S(\Pi\mathcal{V}))$ is a split superscheme. An important example is given by projective superscheme $\mathbb{P}^{m|n}$ where the locally free sheaf $\mathcal{V}$ is $\mathcal{O}(-1)^n$. An example of a nonsplit superscheme given in [page 86, Manin [26]] is Grassmann superscheme $G(1|1, \mathbb{C}^{2|2})$ which is also an example of a superprojective scheme.

We can define an abelian category of sheaf of left modules over $\mathcal{O}_X$, denoted $\mathcal{M}\mathfrak{od}^s(X)$ or $\mathcal{M}\mathfrak{od}(\mathcal{O}_X)$. As above we have a natural right module structure given by the Koszul sign rule. When $(X, \mathcal{O}_X)$ is affine superscheme given by super ring R then we can define the sheaf of module associated to any R-module M as in the commutative case. Hence we can define quasi-coherent and coherent sheaves over any superscheme. Therefore we shall get two abelian subcategories namely category of all quasi-coherent sheaves and coherent sheaves. We denote them by $\mathcal{Q}\mathfrak{coh}(\mathcal{O}_X)$ and $\mathcal{C}\mathfrak{oh}(\mathcal{O}_X)$ respectively. Now similar to affine case we have a forgetful functor as follows,

$$ff : \mathcal{M}\mathfrak{od}(\mathcal{O}_X) \to \mathcal{M}\mathfrak{od}(\mathcal{O}_{X,0}) \times \mathcal{M}\mathfrak{od}(\mathcal{O}_{X,0}).$$

It is an exact faithful functor. we can easily see that

$$\begin{aligned} \mathcal{Q}\mathfrak{coh}(\mathcal{O}_X) &= ff^{-1}(\mathcal{Q}\mathfrak{coh}(\mathcal{O}_{X,0}) \times \mathcal{Q}\mathfrak{coh}(\mathcal{O}_{X,0})) \\ \mathcal{C}\mathfrak{oh}(\mathcal{O}_X) &= ff^{-1}(\mathcal{C}\mathfrak{oh}(\mathcal{O}_{X,0}) \times \mathcal{C}\mathfrak{oh}(\mathcal{O}_{X,0})). \end{aligned}$$

One can also define locally free sheaves on superscheme.

Definition 2.4.20. A sheaf $\mathcal{F}$ on a superscheme X is said to be *locally free* of rank $m|n$ if it is locally isomorphic to $(\mathcal{O}_X)^{\oplus m} \oplus (\Pi\mathcal{O}_X)^{\oplus n}$.

We can define the tensor product of two sheaves of modules over superscheme similar to usual scheme. We shall use the canonical identification of sheaf of left and right modules by Koszul sign rule. Define tensor product of two sheaves of modules $\mathcal{F}_1$ and $\mathcal{F}_2$ as the sheaf associated to pre sheaf given by

$$U \mapsto (\mathcal{F}_1 \otimes \mathcal{F}_2)(U) := \mathcal{F}_1(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}_2(U).$$

Note that with this definition of tensor structure the commutative constraint is given by sign rule i.e.

$$\begin{aligned}
&\mathcal{F} \otimes \mathcal{G} \cong \mathcal{G} \otimes \mathcal{F} \text{ where the isomorphism is given by,}\\
&f \otimes g \mapsto -g \otimes f \text{ if both } \mathcal{F} \text{ and } \mathcal{G} \text{ are odd,}\\
&f \otimes g \mapsto g \otimes f \text{ otherwise,}
\end{aligned}$$

where f and g are sections on some open set U.

Now we can prove some easy properties of this tensor product by just reducing to affine case,

Lemma 2.4.21. *Suppose $(X, \mathcal{O}_X)$ is a split superscheme and $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules. Then we have*

1. $(\Pi\mathcal{F}) \otimes \mathcal{G} = \mathcal{F} \otimes (\Pi\mathcal{G}) = \Pi(\mathcal{F} \otimes \mathcal{G})$
2. *$\mathcal{F} \otimes \mathcal{O}_{X_{rd}}$ has trivial action of J_X and hence it is a $\mathcal{O}_{X_{rd}}$-module.*

Given a split superscheme $(X, \mathcal{O}_X = S^{\cdot}(\Pi\mathcal{V}) = \Pi\Lambda^{\cdot}(\mathcal{V}))$ there is one more forgetful functor as follows,

$$ff : \mathcal{M}\mathfrak{od}(\mathcal{O}_X) \to \mathcal{M}\mathfrak{od}(\mathcal{O}_{X_{rd}}) \times \mathcal{M}\mathfrak{od}(\mathcal{O}_{X_{rd}}).$$

This functor is defined using the obvious inclusion of $\mathcal{O}_{X_{rd}}$ inside $\mathcal{O}_X$ which comes from the definition of split superscheme. Note also that the Grassmann algebra constructed from locally free sheaf $\mathcal{V}$ gives a locally free sheaf of $\mathcal{O}_{X_{rd}}$ module. Therefore structure sheaf $\mathcal{O}_X$ is locally free sheaf as a $\mathcal{O}_{X_{rd}}$ module.

Similar to usual scheme we can take $\mathcal{D}(X) := \mathcal{D}(\mathcal{M}\mathfrak{od}(X))$ the derived category of abelian category $\mathcal{M}\mathfrak{od}(X)$. There are various triangulated subcategories like $\mathcal{D}^{\sharp}(qc/X) := \mathcal{D}^{\sharp}(\mathcal{Q}\mathfrak{coh}(X))$ and $\mathcal{D}^{\sharp}(coh/X) := \mathcal{D}^{\sharp}(\mathcal{C}\mathfrak{oh}(X))$ where $\sharp = +, -, b$ or $\emptyset$. For convenience we shall denote by $\mathcal{D}^{\sharp}(X^0) := \mathcal{D}^{\sharp}(\mathcal{M}\mathfrak{od}(\mathcal{O}_X^0))$ (resp. $\mathcal{D}^{\sharp}(X_{rd}) := \mathcal{D}^{\sharp}(\mathcal{M}\mathfrak{od}(\mathcal{O}_{X_{rd}}))$) the derived category of modules over purely even scheme $(X, \mathcal{O}_X^0)$ (resp. $X_{rd} = (X, Gr_0 X)$). $\mathcal{D}_{qc}(X)$ (resp. $\mathcal{D}_{coh}(X)$) will denote the full subcategory of $\mathcal{D}(X)$ containing all complexes of $\mathcal{O}_X$-moudles with quasi-coherent (resp. coherent) cohomology sheaves.

We need definitions of derived functors and various relations between them for unbounded complexes of modules over superschemes. superschemes. To extend various functors to unbounded complexes we need notion of K-injective (K-projective) resolutions, see [38]. Following definition was given in [38].

Definition 2.4.22. An unbounded complex $A^{\cdot}$ of an abelian category is called K-injective (resp. K-projective) if for every acyclic complex $S^{\cdot}$, the complex $\mathrm{Hom}^{\cdot}(S^{\cdot}, A^{\cdot})$ (resp. $\mathrm{Hom}^{\cdot}(A^{\cdot}, S^{\cdot})$) is acyclic.

It is proved in the same paper, that an abelian category for which inverse (resp. direct) limit exists, and which has enough injectives (resp. projectives) admits a K-injective (resp. K-projective) resolution for any unbounded complex , see [cor 3.9 (resp. cor. 3.5), [38]]. Similar to the scheme case the abelian category $\mathfrak{Qcoh}(X)$ of all quasi coherent shaves over superscheme has arbitrary small coproducts. Therefore we can extnd various functors to unbounded derived category as demonstrated by Spaltenstein, [see sec. 6, [38]]. Moreover the abelian category $\mathfrak{Qcoh}(X)$ will have K-flat resolution for every unbounded complex and hence derived functor of tensor product functor can be extended to unbounded derived category and various relation among these functors can be extended from bounded derived category case to unbounded derived category, see [38] for more details.

The following criterion based on Nakayama's lemma will be used later.

Proposition 2.4.23. *Suppose $(R, \mathfrak{m})$ is a local superring. Suppose $M^{\cdot}, M_1^{\cdot}$ and $M_2^{\cdot}$ are bounded complexes of finitely generated R-modules.*

1. *$M^{\cdot}$ is acyclic iff $M^{\cdot} \otimes R/\mathfrak{m}$ is acyclic.*
2. *$M_1^{\cdot} \otimes M_2^{\cdot}$ is acyclic iff $M_1^{\cdot}$ or $M_2^{\cdot}$ is acyclic.*

Proof. The proof of (i) is similar to the proof of Thomason [lemma 3.3 (a), [39]]. Indeed, using spectral sequence mentioned in the proof of Thomason [lemma 3.3 (a), [39]] the proof reduces to the case of finitely generated modules which follows from the above result 2.4.17.
The proof of (ii) follows from the proof of (i) using following natural isomorphism

$$(M_1^{\cdot} \otimes M_2^{\cdot}) \otimes R/\mathfrak{m} \simeq (M_1^{\cdot} \otimes R/\mathfrak{m}) \otimes_{R/\mathfrak{m}} (M_2^{\cdot} \otimes R/\mathfrak{m}).$$

□

2.4.5 Perfect complexes over schemes & superschemes

We now define another important triangulated subcategory of $\mathcal{D}_{qc}(X)$ for schemes and superschemes. We refer to the notes of Thomason[40] for details. Throughout this section our schemes and superschemes are assumed to be quasi compact and quasi separated. We first recall some basic definitions and results for the case of schemes. We shall later try to extend some of these definitions and results for superschemes.

Definition 2.4.24.

1. A complex $\mathcal{F}^\cdot$ is *strictly pseudo coherent* if it is quasi isomorphic to a bounded above complex of locally free coherent sheaves (or algebraic vector bundle).
2. A *pseudo coherent complex* is a complex of $\mathcal{O}_X$ modules which is locally quasi isomorphic to strict pseudo coherent complexes.
3. A complex $\mathcal{F}^\cdot$ of sheaves of modules over scheme $(X, \mathcal{O}_X)$ is called *strictly perfect* if it is bounded below complex of strict pseudo coherent complex i.e. $\mathcal{F}^\cdot$ is quasi isomorphic to bounded complex of locally free coherent sheaf of $\mathcal{O}_X$ module.
4. A complex $\mathcal{F}^\cdot$ is called *perfect* if it is locally quasi isomorphic to strict perfect complex.

There are some other equivalent ways to characterize the pseudo coherence, see [lemma 2.2.5, [40]]. Following result gives an interesting equivalent way of defining the pseudo coherence on Noetherian schemes.

Proposition 2.4.25 (example 2.2.8, [40])**.** *A complex of $\mathcal{O}_X$ modules, $\mathcal{F}^\cdot$, on a Noetherian scheme X is pseudo coherent iff $\mathcal{F}^\cdot$ is bounded above and the cohomologies $\mathcal{H}^i(\mathcal{F}^\cdot)$ are coherent $\mathcal{O}_X$ modules.*

Now we need following definition of Tor-amplitude for characterization of perfectness.

Definition 2.4.26 (Def. 2.2.11,[40])**.** A complex of $\mathcal{O}_X$ modules $\mathcal{F}^\cdot$ has finite Tor-amplitude if there exist integers $a \leq b$ s.t. for each $\mathcal{O}_X$ module $\mathcal{G}$ the cohomologies $\mathcal{H}^k(\mathcal{F}^\cdot \otimes^L_{\mathcal{O}_X} \mathcal{G})$ are zero unless $a \leq k \leq b$. we say that $\mathcal{F}^\cdot$ is locally of finite Tor-amplitude if above condition holds for restriction of complex $\mathcal{F}^\cdot$ on each open subset of X.

We have perfectness in terms of Tor-amplitude as follows.

Proposition 2.4.27 (Prop. 2.2.12, [40])**.** *A complex $\mathcal{F}^\cdot$ of $\mathcal{O}_X$ modules is perfect iff it is pseudo coherent and has locally finite Tor-amplitude.*

Since perfect complexes have quasi coherent cohomologies we can realize the category of all perfect complexes inside the derived category of quasi coherent sheaves. We shall denote the triangulated subcategory of all perfect complexes as $\mathcal{D}^{per}(X) \subseteq \mathcal{D}_{qc}(X)$. Now we recall an important class of schemes where some global characterizations are possible for perfect complexes.

Definition 2.4.28. A *scheme with ample family of line bundles*, is a scheme which is quasi compact and quasi separated, and which has family of line bundles $\{\mathcal{L}_\alpha\}$ s.t. for any quasi coherent sheaf $\mathcal{F}$, the evaluation map

$$ev : \oplus_{\alpha,n\geq 1}\Gamma(X, \mathcal{F} \otimes \mathcal{L}_\alpha^{\otimes n}) \otimes \mathcal{L}_\alpha^{\otimes(-n)} \to \mathcal{F}$$

is an epimorphism.

Example 2.4.29 (Example 2.1.2,[40]).

1. Affine schemes.
2. Quasi projective scheme over affine scheme.
3. Separated regular Noetherian scheme.
4. Any affine or quasi projective map of schemes, $f : X \to Y$, with an ample family of line bundles on Y gives an ample family of line bundles on X.

There is a global characterization of perfect complexes on such schemes.

Theorem 2.4.30 (Theorem 2.4.3, [40])**.** *Suppose X is a schemes with ample family of line bundles and $\mathcal{F}^\cdot$ is a complex with quasi coherent cohomologies. Then following are equivalent*

1. *$\mathcal{F}^\cdot$ is a perfect complex.*
2. *$\mathcal{F}^\cdot$ is quasi isomorphic to a strict perfect complex. with direct sums*

Next we shall recall the existence of various functors which restrict to perfect complexes and some compatibilities of these functor similar to schemes. Suppose $f : X \to Y$ is a morphism of schemes. There exists an exact functor,

$$Lf^* : \mathcal{D}^-(\mathcal{M}\mathfrak{od}(Y)) \to \mathcal{D}^-(\mathcal{M}\mathfrak{od}(X)),$$

which preserves pseudo coherent and perfect complexes. Hence it induces the functor

$$Lf^* : \mathcal{D}^{per}(Y) \to \mathcal{D}^{per}(X).$$

If the morphism f is of finite Tor-dimension, i.e. $\mathcal{O}_X$ is of finite Tor-dimension as a sheaf of modules over sheaf of ring $f^{-1}(\mathcal{O}_Y)$ over X, then it induces a functor $Lf^* : \mathcal{D}^b(\mathcal{M}\mathfrak{od}(Y)) \to \mathcal{D}^b(\mathcal{M}\mathfrak{od}(X))$ which preserves pseudo coherence and hence perfectness.

If $\mathcal{F}^\cdot$ and $\mathcal{G}^\cdot$ are pseudo coherent then $\mathcal{F}^\cdot \otimes^L \mathcal{G}^\cdot$ is also pseudo coherent. Now if $\mathcal{F}^\cdot$ is perfect complex and $\mathcal{G}^\cdot$ is cohomologically bounded and pseudo coherent then $\mathcal{F}^\cdot \otimes \mathcal{G}^\cdot$ is cohomologically bounded and pseudo coherent. If scheme X is quasi compact then we have induced functor

$$\otimes^L : \mathcal{D}^{per}(X) \times \mathcal{D}^{per}(X) \to \mathcal{D}^{per}(X); (\mathcal{F}^\cdot, \mathcal{G}^\cdot) \mapsto \mathcal{F}^\cdot \otimes \mathcal{G}^\cdot.$$

Now we shall describe the induced derived inverse image functor for perfect complexes. We first recall some definitions.

Definition 2.4.31. Suppose $f : X \to Y$ is a morphism of locally finite type between two schemes. The morphism f is said to be *pseudo coherent* iff for each point $x \in X$ there exists a open set $x \in U \subseteq X$ and an open set $V \subseteq Y$ s.t. the restriction $f : U \to V$ factors as $f = g \circ i$, where $i : U \to Z$ is a closed immersion with sheaf $i_*\mathcal{O}_U$ a pseudo coherent sheaf over Z and $g : Z \to V$ is a smooth map. The morphism f is said to be *perfect* iff f is pseudo coherent and locally of finite Tor-dimension.

Example 2.4.32 (2.5.3 , [40]).

1. If Y is Noetherian then $f : X \to Y$ is a pseudo coherent morphism. If Y is not Noetherian then f need not be perfect morphism.
2. Any smooth morphism is a perfect morphism.
3. Any regular closed immersion is a perfect morphism.
4. Any locally complete intersection morphism is a perfect morphism.

Now following result gives the conditions for the existence of the derived functor of the push forward.

Theorem 2.4.33 (2.5.4, [40])**.** *Suppose $f : X \to Y$ is a proper morphism. Suppose either f is a projective morphism or Y is a locally Noetherian scheme. Suppose $\mathcal{F}^\cdot$ is a pseudo coherent (resp. perfect) complex. Then if f is pseudo coherent (resp. perfect), $Rf_*\mathcal{F}^\cdot$ is a pseudo coherent (resp. perfect) complex.*

In particular if $f : X \to Y$ is a proper morphism between two Noetherian schemes then there is a induced functor

$$Rf_* : \mathcal{D}^{per}(X) \to \mathcal{D}^{per}(Y).$$

We shall recall some induced compatibilities between these functors for category of perfect complexes like projection formula and base change formula.

Theorem 2.4.34 (2.5.5, [40])**.** *Suppose $f : X \to Y$ is a quasi compact and quasi separated morphism with Y a quasi compact scheme. Suppose $\mathcal{F}^\cdot$ and $\mathcal{G}^\cdot$ are cohomologically bounded complex with quasi coherent cohomologies. Either $\mathcal{F}^\cdot$ or $\mathcal{G}^\cdot$ has finite Tor amplitude then the canonical map is a quasi isomorphism in $\mathcal{D}(\mathcal{M}\mathfrak{od}(Y))$*

$$Rf_*(\mathcal{F}^\cdot) \otimes^L_{\mathcal{O}_Y} \mathcal{G}^\cdot \xrightarrow{\sim} Rf_*(\mathcal{F}^\cdot \otimes^L_{\mathcal{O}_X} Lf^*\mathcal{G}^\cdot).$$

In particular for perfect complexes the above projection formula is valid.

Theorem 2.4.35 (2.5.6, [40])**.** *Suppose $f : X \to Y$ is a quasi compact and quasi separated morphism with Y a quasi compact scheme. Consider following Cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y. \end{tikzcd}$$

Suppose f and g are Tor independent over Y, i.e. for each $x \in X$ and $y' \in Y'$ with $f(x) = y = g(y')$ we have $\mathrm{Tor}^p_{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x}, \mathcal{O}_{y',Y'}) = 0$ for each $p \geq 1$. Let $\mathcal{F}^\cdot$ be a cohomologically bounded complex over X, with quasi coherent cohomology. Either $\mathcal{F}^\cdot$ has finite Tor amplitude over the sheaf of ring $f^{-1}(\mathcal{O}_Y)$ on the space X, or the map g has finite Tor dimension. Then there exists a canonical base change quasi isomorphism

$$Lg^*Rf_*\mathcal{F}^\cdot \xrightarrow{\sim} Rf'_*Lg'^*\mathcal{F}^\cdot.$$

In particular if we take f a proper morphism, and g a flat morphism, then base change quasi isomorphism will be in $\mathcal{D}^{per}(X)$ for perfect complexes.

Now we shall extend some of these results for the case of superschemes or split superschemes. Recall a quasi coherent sheaf of modules over superscheme is defined similar to scheme case as sheaf of module which is locally isomorphic to sheaf coming from module over super ring. We shall denote the derived category of all quasi coherent sheaves over X as $\mathcal{D}_{qc}(X)$. Now we define the full subcategory of $\mathcal{D}_{qc}(X)$ containing all perfect complexes. Here perfect complex over superscheme comes from the following definition.

Definition 2.4.36 (Perfect complexes)**.** Given a complex $\mathcal{F}^\cdot$ of quasi coherent sheaves of modules over superscheme $(X, \mathcal{O}_X)$ is called *strictly perfect* if $\mathcal{F}^\cdot$ is quasi isomorphic to a bounded complex of locally free coherent sheaf of $\mathcal{O}_X$ modules. A complex $\mathcal{F}^\cdot$ is called *perfect* if it is locally quasi isomorphic to a bounded complex of locally free coherent sheaves.

We shall denote the triangulated subcategory of all perfect complexes as $\mathcal{D}^{per}(X) \subseteq \mathcal{D}_{qc}(X)$. Similar to scheme case we can extend various functors at the level of these triangulated categories. Hence we can prove $\mathcal{D}^{per}(X)$ is a tensor triangulated category with tensor given by derived functor of usual tensor product defined as above. We need to recall a few more results which might be proved in a way similar to the commutative case. First we need a definition.

Definition 2.4.37. An object t in a triangulated category $\mathcal{T}$, which is closed under the formation of arbitrary small coproducts, is said to be *compact* if $Hom(t, \_)$ respects coproducts. In a triangulated category $\mathcal{T}$, the full subcategory of all compact objects is denoted as $\mathcal{T}^c$.

Now we shall use the following results.

1. The category of perfect complexes over affine schemes is equivalent to the category of projective modules over the respective superalgebras.

2. (Compare [corollary 3.3.5, [8]].) If X is an affine superscheme, Then the obvious functor $\mathcal{D}(qc/X) \to \mathcal{D}_{qc}(X)$ has a quasi-inverse $R\Gamma(X, \_)$.

3. (Compare [equation 3.4, page 12, [8]].) Suppose X is a superscheme and suppose $X = U_1 \cup U_2$ where U_1 and U_2 are open and suppose $U_{12} := U_1 \cap U_2$. Let j_1, j_2 and j_{12} be the inclusions of U_1, U_2 and U_{12} in X respectively. Suppose A is a K-injective complex on X and E be another object in $\mathcal{D}(X)$. Then we have a distinguished triangle

   $$\mathcal{RH}om(E, A) \to \mathcal{RH}om(j_1^*E, j_1^*A) \oplus \mathcal{RH}om(j_2^*E, j_2^*A) \to \mathcal{RH}om(j_{12}^*E, j_{12}^*A) \xrightarrow{+}$$

4. (Compare [proposition 3.3.1, [8]].) (Reduction principle) If P is a property satified by superschemes, and if

   (a) P is true of affine schemes; and
   (b) If P holds for U_1, U_2 and U_{12}, then it is true for X,

   then P holds for all quasi-compact and quasi-separated superschemes.

5. (Compare Lemma 3.3.6, [8].) If X is an affine superscheme, then the category of compact objects in $\mathcal{D}(X)$ is the category of perfect complexes.

Later we shall prove using the results of Neeman that the above defined category of perfect complexes is same as all compact objects in cocomplete category $\mathcal{D}_{qc}(X)$.

The following forgetful functors relate the category of perfect complex over superscheme with schemes. Here we can extend the forgetful functor defined earlier using exactness,

$$ff : \mathcal{D}^{\sharp}(X) \to \mathcal{D}^{\sharp}(X^0) \times \mathcal{D}^{\sharp}(X^0) \text{ and } ff : \mathcal{D}^{\sharp}_{qc}(X) \to \mathcal{D}^{\sharp}_{qc}(X^0) \times \mathcal{D}^{\sharp}_{qc}(X^0).$$

Here $\sharp \in \{+, -, b, \emptyset\}$. We have similar forgetful functors for the case of coherent sheaves. If we restrict to split superscheme then we also have forgetful functor for the case of locally free sheaves (or vector bundles). Hence for a split superschemes we have following forgetful functor for the triangulated subcategory of perfect complexes,

$$ff : \mathcal{D}^{per}(X) \to \mathcal{D}^{per}(X_{rd}) \times \mathcal{D}^{per}(X_{rd})$$

Note that this functor need not be a tensor functor.

If we have morphism $f : X \to Y$ between two superschemes then we have pull back and push forward functors Lf^* and respectively Rf_*. Various relations between these functors also hold for superschemes. We had also defined the symmetric tensor structure for sheaves of modules over superschemes. It is easy to see from that definition that it will induce a functor for quasi coherent and coherent sheaves.

Chapter 3

Generalities on Balmer spectrum

In this chapter, we shall recall some basic constructions and properties of spectrum. In the first section we briefly recall the definition of symmetric monoidal structures. We also give some definitions of tensor functors which Balmer uses for functoriality. In the second section we give the definition of the prime spectrum and its topological properties. Then we give a variant of the functoriality result of Balmer. We also briefly recall the proof of reconstruction using classifying support data and the localisation theorem of Thomason and Troubagh. There is a generalization of Balmer's reconstruction theorem for non-Noetherian case, see[11], but we shall not give a proof here and refer to original paper for details. We shall state the result for the non-Noetherian case.

3.1 Tensor triangulated category

We shall recall the definition of tensor compatible symmetric monoidal structure on a triangulated category. A symmetric monoidal structure on an additive category $\mathcal{D}$ is a bifunctor

$$\otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$$

with compatible associative and commutative constraints and a unit object which satisfies pentagon and hexagon axioms, see [Chap. XI,[25]] (or [Part II 1.1 ,[24]]) for details. A symmetric monoidal structure $(\mathcal{D}, \otimes, \mathbb{1}, \alpha, \tau, \mu_l, \mu_r)$ on $\mathcal{D}$ is denoted as $(\mathcal{D}, \otimes, \mathbb{1})$ for simplicity, following Balmer[3], or simply $\mathcal{D}$ if there is no danger of confusion.

Definition 3.1.1. A *tensor triangulated category* is a triple $(\mathcal{D}, \otimes, \mathbb{1})$ consisting of a triangulated category with symmetric monoidal structure which preserves translation, i.e. $A[1] \otimes B = (A \otimes B)[1]$, and it is exact in each variable. The unit is denoted by $\mathbb{1}$ (or Id).

Now we shall recall the definition of some functors which are used later to prove the functoriality of the spectrum.

Definition 3.1.2. **(a)** An additive functor, $F : \mathcal{D}_1 \to \mathcal{D}_2$, is called *exact (or triangulated)* if it commutes with translation and takes a distinguished triangle to a distinguished triangle.

(b) An exact functor, $F : \mathcal{D}_1 \to \mathcal{D}_2$, is called a *tensor functor* if there exists a natural isomorphism $\eta(a, b) : F(a) \otimes F(b) \to F(a \otimes b)$ for objects a and b of $\mathcal{D}_1$.

(c) A tensor functor, $F : \mathcal{D}_1 \to \mathcal{D}_2$, is called *dominant* if $\langle F(\mathcal{D}_1) \rangle = \mathcal{D}_2$ i.e. the smallest thick tensor ideal generated by the image of the functor F is $\mathcal{D}_2$.

Note that every unital tensor functor is a dominant tensor functor.

3.2 Triangular spectrum

In this section we shall recall some definitions and results from Balmer's papers [2] [3]. Suppose $(\mathcal{D}, \otimes, \mathbb{1})$ is an essentially small tensor triangulated category, which we shall denote by $\mathcal{D}$ for simplicity.

3.2.1 Basic set up

We shall first recall some basic definitions needed to define triangular spectrum. We shall also give some properties of the underlying topological space of a triangular spectrum.

Definition 3.2.1. A *thick tensor ideal* $\mathcal{A}$ of $\mathcal{D}$ is a full sub category containing zero object and satisfying the following conditions:

(a) $\mathcal{A}$ is *triangulated*: if any two terms of a distinguished triangle are in $\mathcal{A}$ then third term is also in $\mathcal{A}$. In particular direct sum of any two objects of $\mathcal{A}$ is again in $\mathcal{A}$ and this we refer as an **additivity**. This property applied to the distinguished triangles $A[-1] \to 0 \to A \xrightarrow[+1]{=} A$ and $A \to 0 \to A[1] \xrightarrow[+1]{=} A[1]$ shows that $\mathcal{A}$ is closed under translations.

(b) $\mathcal{A}$ is *thick*: If $A \oplus B \in \mathcal{A}$ then $A \in \mathcal{A}$.

(c) $\mathcal{A}$ is *tensor ideal*: if A or $B \in \mathcal{A}$ then $A \otimes B \in \mathcal{A}$.

If $\mathcal{E}$ is any collection of objects of $\mathcal{D}$ then we shall denote by $\langle \mathcal{E} \rangle$ the smallest thick tensor ideal generated by this collection in $\mathcal{D}$.

Now we shall give an explicit description of a thick tensor ideal generated by some collection $\mathcal{E}$ in a tensor triangulated category. We first use some definitions from Bondal[8] here. Recall $add(\mathcal{E})$ was defined as an additive category generated by $\mathcal{E}$ and closed under taking shifts inside $\mathcal{D}$. Similarly define $ideal(\mathcal{E})$ as a full sub category generated by objects of the form $\oplus_i A_i \otimes X_i$ for each $A_i \in \mathcal{D}$ and $X_i \in \mathcal{E}$. Since $\mathbb{1}[k] \otimes X$ is contained in $ideal(\mathcal{E})$, it is closed under taking finite direct sum, shifts and tensoring with any object of $\mathcal{D}$. Recall that there is an operation on subcategories i.e. $\mathcal{A} \star \mathcal{B}$, and defined as the the full sub category generated by objects X which fit in a distinguished triangle of the form

$$A \to X \to B \to A[1] \text{ with } A \in \mathcal{A} \text{ and } B \in \mathcal{B}.$$

As observed in [section 2.2, Bondal et. al. [8]], if $\mathcal{A}$ and $\mathcal{B}$ are closed under shifts and direct sums then $\mathcal{A} \star \mathcal{B}$ is also closed under shifts and direct sums. Similarly we can see that if $\mathcal{A}$ and $\mathcal{B}$ are tensor ideal then $\mathcal{A} \star \mathcal{B}$ is also a tensor ideal. Take $smd(\mathcal{A})$ to be the full subcategory generated by all direct summands of objects of $\mathcal{A}$. Now combining these two operations we can define a new operation on subcategories as follows,

$$\mathcal{A} \diamond \mathcal{B} := smd(\mathcal{A} \star \mathcal{B}).$$

Using this operation we can define the full subcategories $\langle \mathcal{E} \rangle^n$ for each non-negative integer as

$$\langle \mathcal{E} \rangle^n := \langle \mathcal{E} \rangle^{n-1} \diamond \langle \mathcal{E} \rangle^0 \text{ where } \langle \mathcal{E} \rangle^0 := smd(ideal(\mathcal{E})).$$

Now we can see following description of ideal generated by a collection $\mathcal{E}$,

Lemma 3.2.2. *$\langle \mathcal{E} \rangle = \cup_{n \geq 0} \langle \mathcal{E} \rangle^n$.*

Proof of the above lemma follows from the fact that right hand side subcategory is a thick tensor ideal and contains every thick tensor ideal containing the collection $\mathcal{E}$.

Definition 3.2.3. A *prime ideal* of $\mathcal{D}$ is a proper thick tensor ideal $\mathcal{P} \subsetneq \mathcal{D}$ such that $A \otimes B \in \mathcal{P}$implies that either $A \in \mathcal{P}$ or $B \in \mathcal{P}$. The *triangular spectrum* of $\mathcal{D}$ is defined as set of all prime ideals, i.e.

$$\mathrm{Spc}(\mathcal{D}) = \{\mathcal{P} \mid \mathcal{P} \text{ is a prime ideal of } \mathcal{D}\}.$$

The Zariski topology on this set is defined as follows: closed sets are of the form

$$Z(\mathcal{S}) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{D}) \mid \mathcal{S} \cap \mathcal{P} = \varnothing\},$$

where $\mathcal{S}$ is a family of objects of $\mathcal{D}$; or equivalently we can define the open subsets to be of the form

$$U(\mathcal{S}) := \mathrm{Spc}(\mathcal{D}) \backslash Z(\mathcal{S}).$$

In particular, we shall denote by

$$\mathrm{supp}(A) := Z(\{A\}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{D}) \mid A \notin \mathcal{P}\},$$

the basic closed sets and similarly $U(\{A\})$ denotes the basic open sets.

A collection of objects $\mathcal{S} \subset \mathcal{D}$ is called a *tensor multiplicative family* of objects if $1 \in \mathcal{S}$ and if $A, B \in \mathcal{S}$ then $A \otimes B \in \mathcal{S}$.

We shall recall here the following lemma [Lemma 2.2 in Balmer's paper[3]] which we need later,

Lemma 3.2.4. *Let $\mathcal{D}$ be a non-zero tensor triangulated category and $\mathcal{I} \subset \mathcal{D}$ be a thick tensor ideal. Suppose $\mathcal{S} \subset \mathcal{D}$ is a tensor multiplicative family of objects s.t. $\mathcal{S} \cap \mathcal{I} = \varnothing$ Then there exists a prime ideal $\mathcal{P} \in \mathrm{Spc}(\mathcal{D})$ such that $\mathcal{I} \subset \mathcal{P}$ and $\mathcal{P} \cap \mathcal{S} = \varnothing$.*

Now we shall collect some topological properties of spectrum proved in Balmer[3].

Proposition 3.2.5 (section 2, [3])**.** **(a)** *The collection of open subsets*

$$\{U(A) | \textit{ for each } A \in \mathcal{D}\}$$

is a basis of the topology on the $\mathrm{Spc}(\mathcal{D})$*. Hence the topological space* $\mathrm{Spc}(\mathcal{D})$ *is a quasi separated.*

(b) *The closure of any point $\mathcal{P} \in \mathrm{Spc}(\mathcal{D})$ is given by*

$$\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{D}) | \mathcal{Q} \subseteq \mathcal{P}\}.$$

Hence $\overline{\{\mathcal{P}_1\}} = \overline{\{\mathcal{P}_2\}} \Rightarrow \mathcal{P}_1 = \mathcal{P}_2$.

(c) *There exists a minimal prime ideal if $\mathcal{D}$ is nonzero. Moreover if $\mathcal{P} \in$* $\mathrm{Spc}(\mathcal{D})$ *then there exists a minimal element $\mathcal{P}' \subseteq \mathcal{P}$. Hence every nonempty closed subset has a closed point.*

(d) *An open set U is quasi compact if and only if there exists an object $A \in \mathcal{D}$ s.t. $U = U(A)$. In particular,* $\mathrm{Spc}(\mathcal{D})$ *is quasi compact.*

(e) *The topological space* $\mathrm{Spc}(\mathcal{D})$ *is Noetherian if and only if every closed subset is a support of some object of* $\mathcal{D}$.

(f) *Every nonempty irreducible closed subset of* $\mathrm{Spc}(\mathcal{D})$ *has a unique generic point.*

Proof. See Balmer[3] $\square$

3.2.2 Functoriality

We shall now recall the functoriality of Spc on all essentially small tensor triangulated category with a morphism given by an unital tensor functors, see [3] for details. Since it is not difficult to see that it is also true for an essentially small tensor triangulated categories with morphism given by a dominant tensor functor therefore we state this slight variant. We have following result.

Proposition 3.2.6. *Given a dominant tensor functor* $F : \mathcal{D}_1 \to \mathcal{D}_2$*, the map* $\mathrm{Spc}(F) : \mathrm{Spc}(\mathcal{D}_2) \to \mathrm{Spc}(\mathcal{D}_1)$ *defined as* $\mathcal{P} \mapsto F^{-1}(\mathcal{P})$ *is well defined, continuous and for all objects* $a \in \mathcal{D}_1$*, we have* $\mathrm{Spc}(F)^{-1}(\mathrm{supp}(a)) = \mathrm{supp}(F(a))$ *in* $\mathrm{Spc}(\mathcal{D}_2)$.

This defines a contravariant functor $\mathrm{Spc}(-)$ *from the category of essentially small tensor triangulated categories with dominant tensor functors as morphisms to the category of topological spaces. So if* F, G *are two dominant tensor functors then* $\mathrm{Spc}(G \circ F) = \mathrm{Spc}(F) \circ \mathrm{Spc}(G)$.

Proof. (Similar to [proposition 3.6, Balmer [3]]) $\square$

Corollary 3.2.7. **(i)** *If a unital tensor functor* $F : \mathcal{D}_1 \to \mathcal{D}_2$ *is essentially surjective then* $\mathrm{Spc}(F) : \mathrm{Spc}(\mathcal{D}_2) \to \mathrm{Spc}(\mathcal{D}_1)$ *is injective.*

(ii) *If a tensor functor* $F : \mathcal{D}_1 \to \mathcal{D}_2$ *is an equivalence then every quasi-inverse functor of* F *is a dominant tensor functor. And also* $\mathrm{Spc}(F)$ *is a homeomorphism.*

Proof. **(i)** [Cor 3.8, [3]]
(ii) First observe that the continuous map $\mathrm{Spc}(F)$ given by a dominant tensor functor is independent of natural isomorphism defining the tensor functor (recall definition 3.1.2). Now using functoriality of above proposition we have an homeomorphism whenever a quasi-inverse of F is an dominant tensor functor. Suppose G is a quasi-inverse of F. Since $G \circ F \simeq Id$, the exact functor G is dominant. Suppose $\eta : F \circ G \to Id$ and $\mu : G \circ F \to Id$

are natural isomorphisms. Now we get a required natural isomorphism by composing as follows,

$$G(a)\otimes G(b) \xrightarrow{\mu^{-1}} GF(G(a)\otimes G(b)) = G(FG(a)\otimes FG(b)) \xrightarrow{G(\eta_a\otimes\eta_b)} G(a\otimes b).$$

Here we used a fact that $G(\eta_a \otimes \eta_b)$ gives a natural transformation. $\square$

We shall recall the results which describe the image of $\mathrm{Spc}(F)$ for a given unital tensor functor $F : \mathcal{D}_1 \to \mathcal{D}_2$.

Proposition 3.2.8. **(i)** *Suppose* $\mathcal{S} := \{A \in \mathcal{D}_1 | \langle F(A)\rangle = \mathcal{D}_2\}$ *is a collection of objects whose image is dense. The closure of the image of* $\mathrm{Spc}(F)$ *is*

$$\overline{Im(\mathrm{Spc}(F))} = Z(\mathcal{S}).$$

(ii) *Let* $Q : \mathcal{D} \to \mathcal{D}/\mathcal{I}$ *be a quotient functor. The map* $\mathrm{Spc}(Q)$ *induces a homeomorphism between* $\mathrm{Spc}(\mathcal{D}/\mathcal{I})$ *and the subspace* $\{\mathcal{P} \in \mathrm{Spc}(\mathcal{D}) | \mathcal{I} \subseteq \mathcal{P}\}$ *of those primes containing* $\mathcal{I}$*.*

Proof. **(i)** [Prop. 3.9, [3]]
(ii) [Prop. 3.11, [3]] $\square$

One has the notion of the idempotent completion of an additive category. If we have a triangulated category $\mathcal{D}$ then there is the triangulated category structure on idempotent completion $\tilde{\mathcal{D}}$ s.t. the canonical functor $i : \mathcal{D} \to \tilde{\mathcal{D}}$ is an exact functor, see [Theorem 1.5 ,[6]]. Following result relates the spectrum of tensor triangulated category with its idempotent completion.

Proposition 3.2.9 (Prop. 3.13 and Cor. 3.14, [3])**.** *Let* $\mathcal{D}$ *be a full tensor triangulated subcategory of tensor triangulated category* $\mathcal{D}'$ *with same unit. Suppose* $\mathcal{D}$ *is cofinal in* $\mathcal{D}'$*, i.e. for each object* A' *of* $\mathcal{D}'$ *there exists an object* A *in* $\mathcal{D}'$ *s.t.* $A \oplus A' \in \mathcal{D}$*. Then the map* $\mathcal{Q} \mapsto \mathcal{Q} \cap \mathcal{D}$ *defines a homeomorphism* $\mathrm{Spc}(\mathcal{D}') \xrightarrow{\sim} \mathrm{Spc}(\mathcal{D})$*. In particular if* $i : \mathcal{D} \to \tilde{\mathcal{D}}$ *is an idempotent completion then* $\mathrm{Spc}(i) : \mathrm{Spc}(\tilde{\mathcal{D}}) \xrightarrow{\sim} \mathrm{Spc}(\mathcal{D})$ *is a homeomorphism.*

3.2.3 Support data

The association of topological space to a tensor triangulated category described above is a good space to see the support of each object in a sense of Grothendieckian philosophy. As Balmer proves that this association is universal w.r.t. some properties. Now we shall recall these properties which are known as *support data*.

Definition 3.2.10. *Support data* on a tensor triangulated category $\mathcal{D}$ is a pair (X, σ) where X is a topological space and σ is an assignment of closed subset of X to each object $A \in \mathcal{D}$ s.t. following conditions are satisfied:

(SD 1) $\sigma(0) = \emptyset$ and $\sigma(\mathbb{1}) = \mathrm{Spc}(\mathcal{D})$.

(SD 2) $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$.

(SD 3) $\sigma(A[1]) = \sigma(A)$.

(SD 4) $\sigma(A) \subseteq \sigma(B) \cup \sigma(C)$ for each distinguished triangle $A \to B \to C \to A[1]$.

(SD 5) $\sigma(A \otimes B) = \sigma(A) \cap \sigma(B)$.

The morphism $f : (X, \sigma) \to (Y, \tau)$ between two support data on a tensor triangulated category $\mathcal{D}$ is defined as a continuous map $f : X \to Y$ such that $\sigma(A) = f^{-1}(\tau(A))$ for each object $A \in \mathcal{D}$. The morphism f is called an isomorphism if and only if $f : X \to Y$ is a homeomorphism.

Example 3.2.11. Let X be a Noetherian scheme and $\mathcal{F}^{\cdot}$ is any bounded complex of quasi coherent sheaves on X. We can define the homological support of $\mathcal{F}^{\cdot}$ as follows,

$$\mathrm{supph}(\mathcal{F}^{\cdot}) := \cup_{i \in \mathbb{Z}} \mathrm{supp}(\mathcal{H}^i(\mathcal{F}^{\cdot})).$$

We shall generalize this example of a support data to the case of superscheme in later chapter.

The pair $(\mathrm{Spc}(\mathcal{D}), supp)$ has following universal property:

Theorem 3.2.12 (Theorem 3.2, [3]). *The pair* $(\mathrm{Spc}(\mathcal{D}), supp)$ *is the final support data on a tensor triangulated category* $\mathcal{D}$*. In other words, If* (X, σ) *is a support data on a tensor triangulated category* $\mathcal{D}$ *satisfying conditions* $(SD1-5)$ *then there exists a unique morphism of support data* $f : X \to \mathrm{Spc}(\mathcal{D})$*. Explicitly, the map* f *is defined as* $f(x) = \{A \in \mathcal{D} | x \notin \sigma(A)\} \in \mathrm{Spc}(\mathcal{D})$ *for each* $x \in X$*.*

Remark 3.2.13. The collection $\{a \in \mathcal{D} | x \notin \sigma(A)\}$ is a prime ideal for each $x \in X$.

Recall that a subset $Y \subseteq X$ of a topological space X is specialization closed if it is a union of closed subsets or equivalently if $y \in Y$ then $\overline{\{y\}} \subseteq Y$. Now we shall define the classifying support data on a tensor triangulated category.

Definition 3.2.14. Support data (X, σ) on tensor triangulated category $\mathcal{D}$ is called classifying support data if following two conditions are satisfied:

(a) The topological space X is Noetherian and any non empty irreducible closed subset $Z \subseteq X$ has a unique generic point i.e. there exists an element x s.t. $\overline{\{x\}} = Z$.

(b) We have a bijection

$$\theta : \{Y \subseteq X | Y \text{specialization closed}\} \xrightarrow{\sim} \{\mathcal{I} \subseteq \mathcal{D} | \mathcal{I} \text{ radical thick } \otimes\text{-ideal}\}$$

defined by $Y \mapsto \{A \in \mathcal{D} | \sigma(A) \subseteq Y\}$, with inverse $\mathcal{I} \mapsto \sigma(\mathcal{I}) := \cup_{A \in \mathcal{I}} \sigma(A)$.

The following result gives an identification of any classifying support data with the spectrum.

Theorem 3.2.15 (Theorem 5.2 , [3]). *Suppose* (X, σ) *is a classifying support data on* $\mathcal{D}$*. Then the canonical map* $f : X \to \mathrm{Spc}(\mathcal{D})$ *,defined above, is a homeomorphism.*

The pair (X, supph) for a Noetherian scheme X gives an example of classifying support data. We'll give another example of classifying support data in the chapter on superschemes.

3.2.4 Balmer reconstruction

Now we shall recall the definition of a structure sheaf defined on $\mathrm{Spc}(\mathcal{D})$ as in [section 6, Balmer [3]].

Definition 3.2.16. For any open set $U \subset \mathrm{Spc}(\mathcal{D})$, let $Z := \mathrm{Spc}(\mathcal{D}) \setminus U$ be a closed complement and let $\mathcal{D}_Z$ be the thick tensor ideal of $\mathcal{D}$ supported on Z. We denote by $\mathcal{O}_{\mathcal{D}}$ the sheafification of following presheaf of rings: $U \mapsto \mathrm{End}(\mathbb{1}_U)$ where $\mathbb{1}_U \in \frac{\mathcal{D}}{\mathcal{D}_Z}$ is the image of the unit $\mathbb{1}$ of $\mathcal{D}$ via the localization map. The restriction maps are defined using localization maps in the obvious way. The sheaf of commutative ring $\mathcal{O}_{\mathcal{D}}$ makes the topological space $\mathrm{Spc}(\mathcal{D})$ a ringed space, which we shall denote by $\mathrm{Spec}(\mathcal{D}) := (\mathrm{Spc}(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$.

Remark 3.2.17. Using the functoriality of Spc and the definition of structure sheaf $\mathcal{O}_{\mathcal{D}}$ we get the functoriality of Spec.

Now functoriality and computation of spectrum gives the Balmer reconstruction. The following theorem was proved in Balmer[3] which computes the spectrum for certain tensor triangulated categories.

Theorem 3.2.18 (Theorem 6.3, Balmer[3]). *For* X *a topologically Noetherian scheme,*

$$\mathrm{Spec}(\mathcal{D}^{per}(X)) \simeq X.$$

Since the homological support data on a quasi compact and quasi separated scheme gives a classifying support data therefore we get a homeomorphism defined as in Theorem 3.2.15. This homeomorphism gives an isomorphism of structure sheaves using Thomason localization theorem [Theorem 2.13, Balmer[2]]. The construction of spectrum given in [theorem 6.3, Balmer [3]] was extended by Buan, Krause and Solberg [theorem 8.5 [11]] from topologically noetherian schemes to more general quasi-compact, quasi-separated schemes.

Theorem 3.2.19 (theorem 54, Balmer [4])**.** *Let X be a quasi-compact and quasi-separated scheme. Suppose $\mathcal{D}^{per}(X)$ dentoes the tensor triangulated category of perfect complexes. Then*

$$\mathrm{Spec}(\mathcal{D}^{per}(X)) \simeq X$$

as ringed spaces.

Chapter 4

Balmer spectrum for smooth G-schemes

This chapter contains our[13] computation of Balmer spectrum for smooth G-schemes. The G-scheme here just means the scheme with an action of a finite group G. We define the category of G-equivariant sheaves and its derived category which we refer here the derived category of a G-scheme. We use here results of Mumford[30] for the existence of Orbit space and restrict to smooth quasi projective schemes. Such schemes are special as they have ample family of line bundles, so perfectness of complexes behave well under this assumption. We do not follow Balmer's proof for scheme case here. Rather we prove the isomorphism of spectrum for derived category of G-scheme X with derived category of orbit scheme X/G. The results of this chapter also shows the strong restriction coming from keeping extra data of tensor as the Mackay correspondence can't arise as tensor exact functors. This follows from our result and Balmer reconstruction.

Throughout this chapter, G is a finite group and k is a field whose characteristic is coprime to the order of G. The varieties we consider will be defined over this field k.

4.1 Some special cases

We shall first do some particular cases before going to the general case. The general case will be done in the next section 4.2. Note that the next section 4.2 does not depend on this one. This section is only there to provide a motivation for the result and to demonstrate some easy proofs in simple cases.

4.1.1 Case 1: G-scheme with trivial action

In this example we shall compute Balmer's triangular spectrum for equivariant sheaves over some quasi-projective varieties with G-action. We first give two proofs for the case of a finite group action on a single point $\mathrm{Spec}(k)$ where k is a field with characteristic coprime to the order of group G.

Let G and k be as above. As usual $\mathcal{R}\mathfrak{ep}(G)$ is the category of all finite dimensional k linear representation of a group G. We can define a strict symmetric monoidal structure on this category using the usual tensor product of representations i.e. if V_1 and V_2 are two representations of G then $V_1 \otimes V_2$ is the tensor product as k vector spaces with diagonal action. We shall denote the bounded derived category of abelian category $\mathcal{R}\mathfrak{ep}(G)$ (resp. $\mathcal{R}\mathfrak{ep}(\{0\})$, for the trivial group $\{0\}$) as $\mathcal{D}_{k[G]}$ (resp. $\mathcal{D}_k$). We can extend the above tensor product of representations to get a symmetric tensor triangulated structure on $\mathcal{D}_{k[G]}$.

Proposition 4.1.1. $\mathrm{Spec}(\mathcal{D}_{k[G]}) \cong \mathrm{Spec}(\mathcal{D}_k) \cong \mathrm{Spec}(k)$.

We give two proofs of this proposition. The second proof generalizes to the final general case. The reason for including the first proof is purely to demonstrate another method of seeing the above statement, and has no other implication.

First proof. Since $\mathcal{R}\mathfrak{ep}(\{0\})$ is a semisimple abelian category with k as its unit it is easy to see that $\mathrm{Spec}(\mathcal{D}_k) \cong \mathrm{Spec}(k)$ as a variety. Therefore it is enough to prove the first isomorphism. The unit object of $\mathcal{D}_{k[G]}$ is k with endomorphism ring isomorphic to k so it remains to say that the trivial ideal, i.e. ideal with only zero object, is the only prime ideal. To prove this observe that for any nonzero finite dimensional representation of G, say W, we have the representation $W^* \otimes W \simeq \mathrm{End}_k(W, W)$, see Proposition 10.30 of [12], containing G invariant element given by identity endomorphism. Now this G invariant element will give trivial representation as an summand of $W^* \otimes W$. So if any prime ideal contains any any non-zero representation then using thickness we will get unit object inside prime ideal which is absurd. □

As mentioned earlier, for the sake of generalisation we shall give another proof of proposition 4.1.1.

Second proof of Proposition 4.1.1. Consider the two exact tensor functors $F : \mathcal{D}_{k[G]} \to \mathcal{D}_k$ and $G : \mathcal{D}_k \to \mathcal{D}_{k[G]}$ where F is the forgetful functor and G comes from the augmentation map of the group algebra $k[G]$ i.e. sending each complex of vector space to a complex of $k[G]$ module with the trivial action of a group G. Note that $F \circ G = Id$ and hence $\mathrm{Spec}(G) \circ \mathrm{Spec}(F) = \mathrm{Id}$. Hence the following lemma will complete the proof of the proposition. □

Lemma 4.1.2. $\mathrm{Spec}(F) \circ \mathrm{Spec}(G) = \mathrm{Id}$.

Proof. Let $\mathcal{P} \in \mathrm{Spec}(\mathcal{D}_{k[G]})$ be a prime ideal. We want to prove that $(G \circ F)^{-1}(\mathcal{P}) = \mathcal{P}$. If $V \in \mathcal{M}\mathfrak{od}(k[G])$ is any $k[G]$-module, then we have the canonical decomposition,

$$V = \bigoplus_{\lambda} V_\lambda \otimes (V_\lambda^* \otimes V)^G$$

where V_λ is an irreducible representation of a group G. Further $(V_\lambda^* \otimes V)^G$ is a direct summand of $(V_\lambda^* \otimes V)$ as is seen using the projector $\frac{1}{|G|}\sum_{g \in G} \rho_g$ where ρ_g comes from the action of a group G on $(V_\lambda^* \otimes V)$. Since any complex in $\mathcal{D}_{k[G]}$ is isomorphic to the direct sum of translates of the cohomology of that complex, to prove above assertion its enough to prove that $(G \circ F)^{-1}(\mathcal{P} \cap \mathcal{M}\mathfrak{od}(k[G])) = \mathcal{P} \cap \mathcal{M}\mathfrak{od}(k[G])$. Observe that,

$$\begin{aligned}
V \in (\mathcal{P} \cap \mathcal{M}\mathfrak{od}(k[G])) \;&\Leftrightarrow\; (V_\lambda \otimes V)^G \in (\mathcal{P} \cap \mathcal{M}\mathfrak{od}(k[G]) \\
&\qquad \text{using thickness and additivity} \\
&\Leftrightarrow\; (V_\lambda \otimes V)^G \in (G \circ F)^{-1}(\mathcal{P} \cap \mathcal{M}\mathfrak{od}(k[G])) \\
&\qquad \text{As } (G \circ F)(W) = W \text{ if } G \text{ acts trivially on } W \\
&\Leftrightarrow\; V \in (G \circ F)^{-1}(\mathcal{P} \cap \mathcal{M}\mathfrak{od}(k[G])) \\
&\qquad \text{using thickness and additivity.}
\end{aligned}$$

The above observation completes the proof of the lemma, and hence of proposition 4.1.1. □

Now, we shall extend the above example. Let X be a smooth variety considered as a space with the trivial action of a finite group G. Recall the definitions and some properties of a G-sheaves from the preliminary section 2.4.3. Let $\mathcal{C}\mathfrak{oh}(X)$ (resp. $\mathcal{C}\mathfrak{oh}^G(X)$) be the abelian category of all coherent sheaves (resp. coherent G-sheaves) over X. We have two functors F and G similar to the previous example defined as follows,

$$\begin{aligned}
F : \mathcal{C}\mathfrak{oh}^G(X) &\to \mathcal{C}\mathfrak{oh}(X) \quad \& \quad & G : \mathcal{C}\mathfrak{oh}(X) &\to \mathcal{C}\mathfrak{oh}^G(X) \\
(\mathcal{F}, \rho) &\mapsto \mathcal{F} & \mathcal{F} &\mapsto (\mathcal{F}, id)
\end{aligned}$$

Note that the functor F (respectively G) is a faithful (respectively fully faithful) exact functor. Thus we get two exact derived functors of the above two functors, $F : \mathcal{D}^G(X) \to \mathcal{D}^b(X)$ and $G : \mathcal{D}^b(X) \to \mathcal{D}^G(X)$ which by abuse of notation are denoted by the same symbols.

Recall that $\mathcal{D}^G(X)$ and $\mathcal{D}^b(X)$ are a tensor triangulated categories which makes the functors F and G unital tensor functors and hence using the functorial property of "Spec" we shall get two morphisms

$$\mathrm{Spec}(F) : \mathrm{Spec}(\mathcal{D}^b(X)) \to \mathrm{Spec}(\mathcal{D}^G(X)) \text{ and}$$
$$\mathrm{Spec}(G) : \mathrm{Spec}(\mathcal{D}^G(X)) \to \mathrm{Spec}(\mathcal{D}^b(X)).$$

Proposition 4.1.3. $\mathrm{Spec}(\mathcal{D}^G(X)) \cong \mathrm{Spec}(\mathcal{D}^b(X)) \cong X$.

Proof. Here, the second isomorphism was proved by Balmer [3] which enables him to reconstruct the variety from its associated tensor triangulated category of coherent sheaves. We shall use the idea of previous example to prove the first isomorphism.

Since $F \circ G = Id$, functoriality of the "Spec" will give $\mathrm{Spec}(G) \circ \mathrm{Spec}(F) = \mathrm{Id}$. Now it remains to prove that $\mathrm{Spec}(F) \circ \mathrm{Spec}(G) = \mathrm{Id}$. Note that every object $(\mathcal{F}, \rho) \in \mathcal{D}^G(X)$ has the canonical decomposition as follows,

$$(\mathcal{F}, \rho) = \bigoplus_{\lambda} V_\lambda \otimes (\mathcal{F}, \rho)_\lambda$$

where $(\mathcal{F}, \rho)_\lambda = (V_\lambda^* \otimes (\mathcal{F}, \rho))^G$ and V_λ is a finite dimensional irreducible representation of the group G, see section 2.4.3 for proof. Also note that $(\mathcal{F}, \rho)_\lambda$ is an ordinary sheaf with the trivial action of a group G and also using similar projector as above, i.e. $\frac{1}{|G|}\sum\limits_{g \in G} \rho_g$, we can prove that $(\mathcal{F}, \rho)_\lambda$ is an direct summand of the sheaf $(V_\lambda^* \otimes (\mathcal{F}, \rho))$. Now we use the following lemma.

Lemma 4.1.4. $\mathrm{Spec}(F) \circ \mathrm{Spec}(G) = \mathrm{Id}$.

Proof. Let $\mathcal{P} \in \mathrm{Spec}(\mathcal{D}^G(X))$ be a prime ideal. We want to prove that $(G \circ F)^{-1}(\mathcal{P}) = \mathcal{P}$. Now using the canonical decomposition of each objects of the triangulated category $\mathcal{D}^G(X)$. we have,

$$\begin{aligned}
(\mathcal{F}, \rho) \in \mathcal{P} \quad &\Leftrightarrow \quad (\mathcal{F}, \rho)_\lambda \in \mathcal{P} \text{ using thickness, additivity and projector} \\
&\Leftrightarrow \quad (\mathcal{F}, \rho)_\lambda \in (G \circ F)^{-1}(\mathcal{P}) \\
&\qquad \text{Since } (G \circ F)(\mathcal{F}, id) = (\mathcal{F}, id) \text{ if } G \text{ acts trivially i.e. } \rho = id \\
&\Leftrightarrow \quad (\mathcal{F}, \rho) \in (G \circ F)^{-1}(\mathcal{P}) \\
&\qquad \text{ using thickness, additivity and projector.}
\end{aligned}$$

Hence the above observation completes the proof of lemma. □

Now, using the above lemma, it follows that $\mathrm{Spec}(F)$ is an isomorphism between $\mathrm{Spec}(\mathcal{D}^G(X))$ and $\mathrm{Spec}(\mathcal{D}^b(X))$. □

Remark 4.1.5. The proof for the case of trivial action on smooth varieties does not need the assumption of quasi-projectivity on the variety X. However, the condition of quasi-projectivity is necessary for the general case, to ensure the existence of the quotient variety.

4.1.2 Case 2: G-scheme with free action

Now we shall consider the case where a finite group G acts freely on X. We refer to section 2.4.3 for the definitions. Recall that, we have a canonical map $\pi : X \to Y := X/G$ which is a G-equivariant map with the trivial action of G on Y. Now we can also define two functors associated with π: $\pi^* : \mathcal{C}\mathfrak{oh}(Y) \to \mathcal{C}\mathfrak{oh}^G(X)$ and $\pi_*^G : \mathcal{C}\mathfrak{oh}^G(X) \to \mathcal{C}\mathfrak{oh}(Y)$ where $\pi_*^G =$ G-equivariant part of π_*. We had also seen in 2.4.3 that π^* is a tensor functor in general; and when G acts freely it is also an equivalence of categories with π_*^G as its quasi-inverse. Hence we shall get an equivalence of the tensor triangulated categories $\mathcal{D}^b(Y)$ and $\mathcal{D}^G(X)$. Since an equivalence gives an isomorphism of "Spec", (cf. section 2.4.3), therefore we get an isomorphism $\mathrm{Spec}(\pi^*) : \mathrm{Spec}(\mathcal{D}^G(X)) \to \mathrm{Spec}(\mathcal{D}^b(Y))$ with its inverse given by $\mathrm{Spec}(\pi_*^G)$. In fact using case 2 and this argument, we can give slightly more general statement as follows.

Corollary 4.1.6. *Suppose finite group G acts freely on a quasi-projective variety X modulo some normal subgroup H. In other words, the subgroup H acts trivially, and the induced action of the quotient group G/H is free. Then*

$$\mathrm{Spec}(\mathcal{D}^G(X)) \cong \mathrm{Spec}(\mathcal{D}^b(Y)) \cong Y$$

where $Y := X/G$ as before.

Proof. As mentioned above, the proof goes in similar lines as in case 2, using a more general canonical decomposition of objects of $\mathcal{D}^G(X)$:

$$(\mathcal{F}, \rho) = \bigoplus_{\lambda} W_\lambda \otimes (\mathcal{F}, \rho)_\lambda$$

where $(\mathcal{F}, \rho)_\lambda = (W_\lambda^* \otimes (\mathcal{F}, \rho))^H$, W_λ is a finite dimensional irreducible representation of the group H, and the group G/H acts naturally on $(\mathcal{F}, \rho)_\lambda$. See corollary 2.4.14 for the proof. $\square$

4.2 The general case

In this section we shall consider the more general situation of a finite group G acting on a smooth quasi-projective variety X and we further assume that the group G acts faithfully. As is assumed throughout the section, the order of G and the characteristic of the base field k are coprime to each other. Define $\pi : X \to Y := X/G$ as above an G-equivariant map. Here the action of G on Y is trivial. Note that for a finite group, the quotient space always exists 2.4.3.

We shall first prove some basic results which we need later for the proof. We give a distinguished triangle for any complex of G-equivariant coherent sheaf $\mathcal{F}$ over X in following result.

Proposition 4.2.1. *Let G, k, X and Y be as above.*

1. *Suppose V is a G-invariant open subset of Y with the induced action of G on V, which is trivial. Then for $\mathcal{G}$ in $\mathcal{D}^G(Y)$,*

$$i_V^*(\mathcal{G}^G) = (i_V^*\mathcal{G})^G$$

2. *Suppose G acts faithfully on X. If $\mathcal{F} \in \mathcal{D}^G(X)$ with $\mathrm{supph}(\mathcal{F}) = X$ then we have a distinguished triangle*

$$\pi^*\pi_*^G(\mathcal{F}) \to \mathcal{F} \to \mathcal{F}_1$$

with $\mathrm{supph}(\mathcal{F}_1) \subsetneq \mathrm{supph}(\mathcal{F})$. Same is true if we have faithful action of G on $\mathrm{supph}(\mathcal{F}) \subsetneq X$.

Proof of 1. It follows from the definition of G-equivariant functions.
Proof of 2. Since G acts faithfully on X we can use proposition 2.4.4 to get an open subset $U \subseteq X$ with free action of the group G . We shall use induction on amplitude length, $\mathrm{ampl}(\mathcal{F})$. When $\mathrm{ampl}(\mathcal{F}) = 1$ then $\mathcal{F}$ is a shift of a coherent sheaf so enough to prove for coherent sheaf. Now using the fact that $\mathrm{supph}(\mathcal{F}) = X$ we have $i_U^*(\mathcal{F}) \neq 0$. There is a natural morphism coming from adjunction and inclusion of G-invariant part, say $\eta : \pi^*\pi_*^G(\mathcal{F}) \to \mathcal{F}$. Using flat base change and part 1 of 4.2.1 we get an isomorphism $i_U^*\pi^*\pi_*^G(\mathcal{F}) \simeq \pi^*\pi_*^G(i_U^*\mathcal{F})$. Now this will give an isomorphism, as G act freely on U, i.e. $i_U^*(\eta) : i_U^*\pi^*\pi_*^G(\mathcal{F}) \to i_U^*\mathcal{F}$ is an isomorphism. Hence cone of the map η will have support outside an open set U. This completes the first step of induction.

Now assume that for all $\mathcal{F}$ with $\mathrm{ampl}(\mathcal{G}) \leq (n-1)$ we have such a distinguished triangle. Now consider $\mathcal{F}$ with $\mathrm{ampl}(\mathcal{F}) = n$ with highest cohomology in degree n. We have usual truncation distinguished triangle $\tau^{\leq(n-1)}(\mathcal{F}) \to \mathcal{F} \to \mathcal{H}^n(\mathcal{F})[-n]$. Using exactness of i_U^* and argument similar to first step of induction we have a following commutative diagram (we have used same notation η for different sheaves),

$$\begin{tikzcd}
i_U^*\pi^*\pi_*^G\tau^{\leq(n-1)}(\mathcal{F}) \arrow[r] \arrow[d, "i_U^*(\eta)"] & i_U^*\pi^*\pi_*^G\mathcal{F} \arrow[r] \arrow[d, "i_U^*(\eta)"] & i_U^*\pi^*\pi_*^G\mathcal{H}^n(\mathcal{F})[-n] \arrow[d, "i_U^*(\eta)"] \\
i_U^*\tau^{\leq(n-1)}(\mathcal{F}) \arrow[r] & i_U^*\mathcal{F} \arrow[r] & i_U^*\mathcal{H}^n(\mathcal{F})[-n]
\end{tikzcd}$$

Since both the extreme vertical arrows are isomorphism using induction hypothesis, we have isomorphism of the middle $i_U^*(\eta)$. Therefore cone of the map η will have proper support. $\square$

Following crucial lemma is used later to show injectivity of the map induced by spectrum.

Lemma 4.2.2. *Let $\pi : X \to Y$ be the quotient map as before.*

1. *Given $\mathcal{F} \in \mathcal{D}^G(X)$ we have* $\operatorname{supph}(\pi_*\mathcal{F}) = \pi(\operatorname{supph}\mathcal{F})$.
2. *There exists a tower of distinguished triangles for each object $\mathcal{F}$ in $\mathcal{D}^G(X)$,*

$$\begin{tikzcd}[column sep=small]
\mathcal{F} = \mathcal{F}_0 \arrow[rr] & & \mathcal{F}_1 \arrow[dl, dashed] & \cdots & \mathcal{F}_{m-1} \arrow[rr] & & \mathcal{F}_m \arrow[dl, dashed] \arrow[dr, equal] & \\
& \mathcal{G}_1 \arrow[ul] & & \cdots & & \mathcal{G}_{m-1} \arrow[ul] & & \mathcal{G}_m
\end{tikzcd}$$

where $\mathcal{G}_i = \bigoplus_{\lambda_i} W_{\lambda_i} \otimes \pi^*\pi_*^{G/H_i}(\mathcal{F}_{\lambda_i})$ *with the sum being over the irreducible representations of the corresponding H_i's,* $\operatorname{supph}(\mathcal{F}_m) \subsetneq \ldots \subsetneq \operatorname{supph}(\mathcal{F})$.

Furthermore

$$\operatorname{supph}(\pi_*^{G/H_j}(\mathcal{F}_{\lambda_j})) \subseteq \operatorname{supph}(\pi_*(\mathcal{F}_{\lambda_j})) = \pi(\operatorname{supph}(\mathcal{F}_{\lambda_j})).$$

Proof of 1. Consider $\mathcal{F} \in \mathcal{D}^G(X)$ a complex of G-sheaves. We have the special case of the Grothendieck-Leray spectral sequence [Pg. 74 (3.4) [22]] as follows,

$$E_2^{p,q} = R^p\pi_*(\mathcal{H}^q(\mathcal{F})) \Rightarrow R^{p+q}\pi_*(\mathcal{F}).$$

Since $R^p\pi_* = 0$ for each $p > 0$ the above spectral sequence will degenerate and we get that $\pi_*(\mathcal{H}^i(\mathcal{F})) = \mathcal{H}^i(\pi_*(\mathcal{F}))$. Here as before $\mathcal{H}^i(\mathcal{F})$ represents the i-th cohomology sheaf of the complex $\mathcal{F}$. Now this will give the equality,

$$\operatorname{supph}(\pi_*(\mathcal{F})) = \cup_i \operatorname{supp}(\mathcal{H}^i(\pi_*\mathcal{F})) = \cup_i \operatorname{supp}(\pi_*\mathcal{H}^i(\mathcal{F})).$$

Suppose we prove the assertion for pure sheaves, i.e. complexes of sheaves concentrated on degree 0, then following observation will complete the proof.

$$\operatorname{supph}(\pi_*\mathcal{F}) = \cup_i \operatorname{supp}(\pi_*\mathcal{H}^i(\mathcal{F})) = \cup_i \pi(\operatorname{supp}(\mathcal{H}^i(\mathcal{F}))) = \pi(\operatorname{supph}(\mathcal{F})).$$

Now it remains to prove the assertion for pure sheaves. We shall denote by $\mathcal{F}_U$ the restriction of the sheaf $\mathcal{F}$ on the open set U of X. Suppose V_j is an open affine cover of Y and $U_j := \pi^{-1}(V_j)$ is the affine open cover of X. We shall denote the restriction of the map π on U_j with same notation π. Now using the flat base change we have $\pi_*(\mathcal{F}_{U_j}) = (\pi_*\mathcal{F})_{V_j}$ for any sheaf $\mathcal{F}$ on X. Suppose the above assertion is true for affine case then following observations will complete the proof.

$$\pi(\operatorname{supp}(\mathcal{F})) = \pi(\cup_j(\operatorname{supp}(\mathcal{F}) \cap U_j)) = \cup_j \pi(\operatorname{supp} \mathcal{F}_{U_j}) = \cup_j \operatorname{supp}(\pi_*\mathcal{F}_{U_j})$$

$$= \cup_j \operatorname{supp}((\pi_*\mathcal{F})_{V_j}) = \cup_j \operatorname{supp}(\pi_*\mathcal{F}) \cap V_j = \operatorname{supp}(\pi_*\mathcal{F}).$$

It remains to prove the assertion for pure sheaves on affine varieties. Suppose $\pi : \operatorname{Spec} B \to \operatorname{Spec} A$ is a quotient map for the action of G on $\operatorname{Spec} B$, and $\tilde{N}$ is a pure G-equivariant sheaf on $\operatorname{Spec}(B)$, corresponding to the B-module N. Since A and B are noetherian rings, this reduces to following fact.

$$V(\operatorname{ann}({}_A N)) = \pi(V(\operatorname{ann}(N)).$$

Here $\operatorname{ann}(N)$ denotes the annihilator ideal and $V(\operatorname{ann}(N))$ denotes the closed set given by all prime ideal containing the ideal $\operatorname{ann}(N)$. Let $\bar{\pi} : A \to B$ be the algebra map corresponding to π.

Now to show $V(\operatorname{ann}({}_A N)) = \pi(V(\operatorname{ann}(N))$ it is enough to prove that

$$\bar{\pi}^{-1}(\operatorname{ann}(N)) = \operatorname{ann}({}_A N).$$

This follows as $x \in \bar{\pi}^{-1}(\operatorname{ann}(N))$ iff $\bar{\pi}(x)N = 0$. This is equivalent to $x({}_A N) = 0$ which in turn holds iff $x \in \operatorname{ann}({}_A N)$. This concludes the proof of 1.

Proof of 2. To prove the first part we use induction on the dimension of the homological support of $\mathcal{F}$. Note that the homological support is invariant under the action of G. If dimension is zero then it will be set of G-invariant points and we shall get the direct sums of skyscrapers on these points. If we have free action of G/H for some subgroup H then we have the canonical decomposition by 2.4.14. This proves that the induction starts.

For the induction step, assume that for all $\mathcal{G}$ with $\dim \operatorname{supph}(\mathcal{G}) \leq n-1$, we have a tower as in the statement of the lemma. Now consider $\mathcal{F}$ with $\dim \operatorname{supph}(\mathcal{F}) = n$. Here $\operatorname{supph}(\mathcal{F})$ is a union of G-invariant components and using the proposition 2.4.4 we get subsets U_i, open in $\operatorname{supph}(\mathcal{F})$ for $i = 1, \ldots, r$ and subgroups H_i for $i = 1, \ldots, r$. As observed before, these U_i are mutually disjoint and there is a free action of group G/H_i on U_i for $i = 1, \ldots, r$. Consider the open subset $U_1 \subset \operatorname{supph}(\mathcal{F})$. Let i_{U_1} be the inclusion of U_1 in X. By 2.4.14, we can decompose $i^*_{U_1}(\mathcal{F})$ as

$$i^*_{U_1}(\mathcal{F}) = \bigoplus_{\lambda} W_\lambda \otimes \mathcal{F}_\lambda$$

where each W_λ is an irreducible representation of subgroup H_1, and the $\mathcal{F}_\lambda$'s are G/H_1-sheaves over the open subset U_1. Using adjunction and 2.4.8, we get a canonical isomorphism, $\eta_\lambda : \pi^*\pi_*^{G/H_1}(\mathcal{F}_\lambda) \to \mathcal{F}_\lambda$ in $\mathcal{D}^G(U_1)$. Putting these together, we get an isomorphism

$$\underset{\lambda}{\oplus} W_\lambda \otimes \pi^*\pi_*^{G/H_1}(\mathcal{F}_\lambda) \xrightarrow{\sim} i_{U_1}^*(\mathcal{F}) = \underset{\lambda}{\oplus} W_\lambda \otimes \mathcal{F}_\lambda. \tag{4.1}$$

Let $\mathcal{F}_{\lambda_1} = (i_{U_1})_*\mathcal{F}_\lambda$. Then, $\mathcal{F}_\lambda \cong i_{U_1}^*\mathcal{F}_{\lambda_1}$, since the adjunction $i_{U_1}^* i_{U_1 *}\mathcal{F}_{\lambda_1} \to \mathcal{F}_{\lambda_1}$ induces an isomorphism on stalks, as U_1 is open in $\mathrm{supph}(\mathcal{F}_{\lambda_1})$. Also, since U_1 is open in $\mathrm{supph}(\mathcal{F})$, there exists an open subset $\tilde{U_1} \subset X$ such that $\tilde{U_1} \cap \mathrm{supph}(\mathcal{F}) = U_1$. Let $\bar{U_1} = \tilde{U_1} \cup (X \setminus \mathrm{supph}(\mathcal{F})$. Now we shall prove that

$$\pi^*\pi_*^{G/H_1} i_{\bar{U_1}}^*(\mathcal{F}_{\lambda_1}) \cong i_{\bar{U_1}}^*\pi^*\pi_*^{G/H_1}(\mathcal{F}_{\lambda_1}).$$

This follows from flat base change and some functorial properties, by considering the diagram,

$$\begin{tikzcd} \bar{U_1} \arrow[r, hook, "i_{\bar{U_1}}"] \arrow[d, "\pi"] & X \arrow[d, "\pi"] \\ V_1 \arrow[r, hook, "i_{V_1}"] & Y \end{tikzcd}$$

and from the following sequence of canonical isomorphisms,

$$\begin{aligned} i_{\bar{U_1}}^*(\pi^*\pi_*^{G/H_1}(\mathcal{F}_{\lambda_1})) &\cong \pi^* i_{V_1}^*(\pi_*(\mathcal{F}_{\lambda_1}))^{G/H_1} \cong \pi^*(i_{V_1}^*\pi_*(\mathcal{F}_{\lambda_1}))^{G/H_1} \\ &\cong \pi^*(\pi_* i_{\bar{U_1}}^*(\mathcal{F}_{\lambda_1}))^{G/H_1} = \pi^*\pi_*^{G/H_1} i_{\bar{U_1}}^*(\mathcal{F}_{\lambda_1}). \end{aligned}$$

The isomorphism, proved in the previous paragraph, and equation (4.1) implies that the map $i_{U_1}^*(\tilde{\eta})$ is an isomorphism by looking at stalks, where $\tilde{\eta}$ is the map $\oplus_\lambda W_\lambda \otimes \pi^*\pi_*^{G/H_1}(\mathcal{F}_{\lambda_1}) \to \mathcal{F}$ coming from the appropriate adjunction maps. We shall denote $\oplus_\lambda W_\lambda \otimes \pi^*\pi_*^{G/H_1}(\mathcal{F}_{\lambda_1})$ by $\mathcal{G}_1$. Now using 1. of 4.2.2 we get that $\mathrm{supph}(\pi_*^{G/H_1}(\mathcal{F}_{\lambda_1})) \subseteq \mathrm{supph}(\pi_*(\mathcal{F}_{\lambda_1})) = \pi(\mathrm{supph}(\mathcal{F}_{\lambda_1}))$.

From the above discussion, the cone of the map $\tilde{\eta}$, say $\mathcal{F}_1$, will have the property that $i_{U_1}^*(\mathcal{F}_1) = 0$ and hence $\mathrm{supph}(\mathcal{F}_1) \subseteq (\mathrm{supph}(\mathcal{F}) \setminus U_1) \subsetneq \mathrm{supph}(\mathcal{F})$. Now we can proceed similarly with $\mathcal{F}_1$ whose support has less number of G-invariant components than $\mathcal{F}$ and hence in finitely many steps (in less than r steps) the dimension of homological support will drop. Hence we shall get $\mathcal{F}_i$ and $\mathcal{G}_i$ for $i = 1, \ldots, s$ with the stated restrictions on supports. The dimension of $\mathrm{supph}(\mathcal{F}_s) \leq n - 1$ and that concludes the induction step. □

We shall now prove the main result of this section in several steps for simplicity.

Proposition 4.2.3. *The morphism of locally ringed spaces*

$$\operatorname{Spec}\pi^* : \operatorname{Spec}\mathcal{D}^G(X) \to \operatorname{Spec}\mathcal{D}^{per}(Y)$$

is an isomorphism. Since by Balmer [theorem 54, [4]] $\operatorname{Spec}\mathcal{D}^{per}(Y) \cong Y$ *as schemes,*

$$\operatorname{Spec}\mathcal{D}^G(X) \cong Y$$

as schemes.

Here again as before the second isomorphism is a particular case of the more general reconstruction result of Balmer [2] [3]. Hence we shall just prove the first isomorphism. We know there are two exact functors $\pi^* : \mathcal{D}^{per}(Y) \to \mathcal{D}^G(X)$ and $\pi_* : \mathcal{D}^G(X) \to \mathcal{D}^{per}(Y)$. We also know that the map π^* is an unital tensor functor and hence it will give the map $\operatorname{Spec}(\pi^*) : \operatorname{Spec}(\mathcal{D}^G(X)) \to \operatorname{Spec}(\mathcal{D}^{per}(Y))$. Note that π_* need not be a tensor functor. We shall prove that $\operatorname{Spec}(\pi^*)$ is a closed bijection and induces an isomorphism for the structure sheaves.

To simplify the proof we will break it in several steps. The first two steps will prove that $\operatorname{Spec}(\pi^*)$ gives a bijection of sets on the underlying topological spaces of the two Specs in question. The next step will show that the underlying topological spaces are homeomorphic. Then finally in step 4 we prove that the Specs of the tensor triangulated categories under consideration, are isomorphic as ringed spaces.

Step 1: $\operatorname{Spec}(\pi^*)$ **is onto.**

Suppose $\mathfrak{q} \in \operatorname{Spec}(\mathcal{D}^{per}(Y))$ is a prime ideal then we want to construct an prime ideal $\mathfrak{p}$ in $\operatorname{Spec}(\mathcal{D}^G(X))$ such that $\mathfrak{q} = (\pi^*)^{-1}(\mathfrak{p})$. Recall that $\langle \pi^*(\mathfrak{q}) \rangle$ denotes the thick tensor ideal generated by the image of $\mathfrak{q}$ via functor π^* in a tensor triangulated category $\mathcal{D}^G(X)$. We have a following lemma which uses the explicit description of thick tensor ideal $\langle \pi^*(\mathfrak{q}) \rangle$.

Lemma 4.2.4. $\pi_*(\langle \pi^*(\mathfrak{q}) \rangle) \subseteq \mathfrak{q}$.

Proof. To prove this lemma, we use lemma 3.2.2 i.e.

$$\langle \pi^*(\mathfrak{q}) \rangle = \cup_{n \geq 0} \langle \pi^*(\mathfrak{q}) \rangle^n$$

where $\langle \pi^*(\mathfrak{q}) \rangle^n$ constructed inductively by taking $\langle \pi^*(\mathfrak{q}) \rangle^0$ as the summands of tensor ideal generated by $\pi^*(\mathfrak{q})$ and $\langle \pi^*(\mathfrak{q}) \rangle^n$ to be the thick tensor ideal containing cone of morphism between any two objects of $\langle \pi^*(\mathfrak{q}) \rangle^{(n-1)}$ and $\langle \pi^*(\mathfrak{q}) \rangle^0$. Here cone of a morphism refers to the third object of any distinguished triangle having this morphism as a base or equivalently we can use $\diamond$ operation. The above equality follows from the lemma 3.2.2 proved earlier.

We shall use induction on n in the above explicit description. For $n = 0$, given $\mathcal{F} \in \mathfrak{q}$,

$$\pi_*(\pi^*(\mathcal{F}) \otimes \mathcal{G}) = \mathcal{F} \otimes \pi_*(\mathcal{G}) \in \mathfrak{q},$$

and hence $\pi_*(\langle \pi^*(\mathfrak{q})\rangle^0) \subseteq \mathfrak{q}$ using thickness of $\mathfrak{q}$.

Using induction suppose we know that $\pi_*(\langle \pi^*(\mathfrak{q})\rangle^{(n-1)}) \subseteq \mathfrak{q}$. Since π_* is an exact functor, it follows that the image under π_* of a cone of any morphism is a cone of π_* of the morphism. Hence using the triangulated ideal property and thickness of $\mathfrak{q}$ it follows that $\pi_*(\langle \pi^*(\mathfrak{q})\rangle^n) \subseteq \mathfrak{q}$. Therefore we have $\pi_*(\langle \pi^*(\mathfrak{q})\rangle) = \pi_*(\cup_{n\geq 0}\langle \pi^*(\mathfrak{q})\rangle^n) \subseteq \mathfrak{q}$. □

Lemma 4.2.5. *$\pi^*(\mathcal{D}^{per}(Y) \setminus \mathfrak{q}) \cap \langle \pi^*(\mathfrak{q})\rangle = \varnothing$.*

Proof. To prove this by contradiction, suppose that there exists an object $\mathcal{G} \in (\mathcal{D}^{per}(Y) \setminus \mathfrak{q})$ such that $\pi^*(\mathcal{G}) \in \langle \pi^*(\mathfrak{q})\rangle$. Then using the above lemma $\pi_*(\pi^*\mathcal{G}) \in \mathfrak{q}$. On the other hand, the projection formula implies $\pi_*(\pi^*\mathcal{G}) = \mathcal{G} \otimes \pi_*(\mathcal{O}_X)$, which we saw is in $\mathfrak{q}$.

Using the primality of $\mathfrak{q}$ it follows that $\pi_*(\mathcal{O}_X) \in \mathfrak{q}$. Now $(\pi_*(\mathcal{O}_X))^G = \mathcal{O}_Y$ is a direct summand of $\pi_*(\mathcal{O}_X)$ by the canonical decomposition of a G-sheaves on Y. Hence $\mathcal{O}_Y$ is an object of $\mathfrak{q}$; which is absurd. □

To complete Step 1, we apply Balmer's result 3.2.4 to get an prime ideal $\mathfrak{p}$, such that $\pi^*(\mathcal{D}^{per}(Y) \setminus \mathfrak{q}) \cap \mathfrak{p} = \varnothing$ and $\langle \pi^*(\mathfrak{q})\rangle \subseteq \mathfrak{p}$. Hence we shall get $\mathfrak{q} = (\pi^*)^{-1}(\mathfrak{p})$ which proves the surjectivity of the map $\mathrm{Spec}(\pi^*)$.

Step 2: Injectivity of $\mathrm{Spec}(\pi^*)$

First we shall give proof of injectivity for the case of a *smooth projective curve* as it is simpler and shows the key idea for the general case. First, we have following basic result for the case of a smooth projective curve which simplifies the proof in this case. First we recall some basic definitions.

Definition 4.2.6 (Def. 13.1.18, [23])**.** An abelian category $\mathcal{A}$ is called *hereditary* abelian category if

$$\operatorname{Hom} \mathcal{D}^b(\mathcal{A})(A, B[i]) = 0 \text{ for each } i \geq 2.$$

In other words, an abelian category $\mathcal{A}$ is called *hereditary* if the homological dimension of $\mathcal{A}$ is 0 or 1

Proposition 4.2.7. *1. Any object of $\mathcal{D}^b(\mathcal{A})$,for a hereditary abelian category $\mathcal{A}$, is non-canonically isomorphic to the direct sum of its cohomologies with shifts. In particular, this is true for $\mathcal{A} = \mathfrak{Coh}(X)$ where X is a smooth projective curve.*

2. *Every coherent sheaf over smooth projective curve X is a direct sum of a coherent skyscraper sheaves and a locally free coherent sheaves.*

Proof. We shall briefly indicate the proof of these well known results.
Proof of (i) See [cor. 13.1.20, [23]] or [cor. 3.15, [22]] for the proof of first part. See [prop. 3.13, [22]] for the proof that the homological dimension of $\mathcal{C}\mathfrak{oh}(X)$ is same as dimension of X for smooth projective scheme X. In particular, $\mathcal{C}\mathfrak{oh}(X)$ is hereditary abelian category for smooth projective curve X.
Proof of (ii) Since torsion free sheaves on smooth curve are locally free we get that $\mathcal{F}/\mathcal{F}_{tor}$ is locally free. The sheaf $\mathcal{F}_{tor}$ is supported on finite points and hence sum of finite coherent sheaves supported on point which is coherent skyscraper sheaf. Now it is enough to prove that $\mathcal{F} = \mathcal{F}_{tor} \oplus \mathcal{F}/\mathcal{F}_{tor}$. But this follows from the vanishing of the following torsion sheaf,

$$H^1(X, (\mathcal{F}/\check{\mathcal{F}}_{tor}) \otimes \mathcal{F}_{tor}) = 0.$$

□

Using above result we prove the following proposition.

Proposition 4.2.8. *The map* $\mathrm{Spec}(\pi^*) : \mathrm{Spec}(\mathcal{D}^G(X)) \to \mathrm{Spec}(\mathcal{D}^b(Y))$ *is an injective map between smooth projective curves X and Y.*

Proof. Suppose not, let $\mathfrak{p}_1, \mathfrak{p}_2$ be two distinct points of $\mathrm{Spec}(\mathcal{D}^G(X))$ mapping to the same point $\mathfrak{q}_y$ where y is given by the identification of $\mathrm{Spec}(\mathcal{D}^b(Y))$ with Y. Let $\mathcal{F}$ be an element of $\mathfrak{p}_1$ and using the above proposition (4.2.7) we can assume that it is a pure sheaf. We have the following lemma which gives a restriction on the homological support of such elements.

Lemma 4.2.9. $\mathrm{supp}(\mathcal{F}) \subseteq (X \setminus \pi^{-1}(y))$.

First let us complete the proof of the proposition assuming this lemma. From the lemma it follows that $\mathrm{supp}(\mathcal{F})$ is a proper subset of X with a G-action. Therefore $\mathrm{supp}(\mathcal{F})$ is a finite set of points and using thickness further we can assume that it is a single orbit. Suppose H is a stabiliser of this orbit. Then G/H will act freely on $\mathrm{supp}(\mathcal{F})$. Now we have the decomposition,

$$\mathcal{F} = \oplus_\lambda W_\lambda \otimes \mathcal{F}_\lambda \simeq \oplus_\lambda W_\lambda \otimes \pi^*\pi_*^{G/H}(\mathcal{F}_\lambda)$$

where W_λ is an irreducible representation of H. Therefore $\mathcal{F} \in \mathfrak{p}_1 \cap \mathfrak{p}_2$, since using a projector $\mathcal{F}_\lambda = (W_\lambda^* \otimes \mathcal{F})^G \simeq \pi^*\pi_*^{G/H}(\mathcal{F}_\lambda) \in \mathfrak{p}_1 \cap \mathfrak{p}_2$, and hence $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$. Using similar arguments we can prove $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$. This is a contradiction as $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are distinct points.

This proves the proposition assuming the lemma. Next we prove the lemma. □

Proof of lemma. We prove it by contradiction. Assume $\mathrm{supp}(\mathcal{F}) \cap \pi^{-1}(y) \neq \varnothing$. If y is a closed point then we can assume that $\mathrm{supp}(\mathcal{F}) = \pi^{-1}(y)$ since we can always tensor with the object $\mathcal{O}_{\pi^{-1}(y)}$ which will give an object of $\mathfrak{p}_1$. And if H is a stabiliser of this finite G set then we shall have the usual decomposition $\mathcal{F} = \oplus_\lambda W_\lambda \otimes \pi^* \pi_*^{G/H}(\mathcal{F}_\lambda)$. Since $\mathfrak{p}_1$ is prime ideal and $W_\lambda \otimes \mathcal{O}_X \notin \mathfrak{p}_1$, see first proof of 4.1.1. This will prove that $\pi^*\pi_*^{G/H}(\mathcal{F}_\lambda) \in \mathfrak{p}_1$. Using our assumption that $\mathrm{Spec}(\pi^*)(\mathfrak{p}_1) = \mathfrak{q}_y$ we get an object, $\pi_*^{G/H}(\mathcal{F}_\lambda)$, of $\mathfrak{q}_y$ supported on y which is a contradiction.

Similarly, if y is a generic point of X then using the above proposition 4.2.7 we can assume that $\mathcal{F}$ is a G-equivariant vector bundle. In the paper [36] the hyperelliptic curve was considered as $\mathbb{Z}/2\mathbb{Z}$-space and any vector bundle with lift of this involution was embedded in a short exact sequence with an equivariant vector bundle lifted from projective line and skyscrapers on Weierstrass points. Therefore the short exact sequence (4.7) from paper [36] gives approximation of equivariant vector bundles with vector bundle lifted from below and suggests the approach for general varieties. Now using a short exact sequence, similar to short exact sequence (4.7) from paper[36],

$$0 \to \pi^*\pi_*^G(\mathcal{F}) \to \mathcal{F} \to \mathcal{F}' \to 0$$

with $\mathcal{F}'$ supported on a points we can prove $\mathcal{F}' \in \mathfrak{p}_1$. Hence $\pi^*\pi_*^G(\mathcal{F}) \in \mathfrak{p}_1$. Now using our assumption $\pi_*^G(\mathcal{F}) \in \mathfrak{q}_y$ which is a contradiction as $\pi_*^G(\mathcal{F})$ is a vector bundle. □

This finishes the case of curves.

Now we shall start with the proof of more general situation. First we prove a technical lemma.

Lemma 4.2.10. *Let $\pi : X \to Y$ be the quotient map as before. Let $\mathfrak{p}$ be a prime ideal in $\mathcal{D}^G(X)$ and suppose that $(\pi^*)^{-1}(\mathfrak{p}) = \mathfrak{q}_y$. Here, y is the point in Y corresponding to $\mathfrak{q}_y$ in* $\mathrm{Spec}(\mathcal{D}^{per}(Y)) \cong Y$.

1. *Let $\mathcal{F} \in \mathcal{D}^G(X)$ be such that its homological support is contained in $(X \setminus \pi^{-1}(y))$. Then, it is an object of $\mathfrak{p}$.*

2. *Let $\mathcal{F}$ be an object of $\mathfrak{p}$. Then* $\mathrm{supph}(\mathcal{F}) \subseteq (X \setminus \pi^{-1}(y))$.

Proof of 1. Using 2. of 4.2.2, there is a tower whose lower terms $\mathcal{G}_i := \bigoplus_{\lambda_i} W_{\lambda_i} \otimes \pi^*\pi_*^{G/H_i}(\mathcal{F}_{\lambda_i})$ have support contained in the subset $X - \pi^{-1}(y)$. Since $\mathrm{supph}(W_{\lambda_i} \otimes \mathcal{O}_X) = X$, we have $\mathrm{supph}(\pi^*\pi_*^{G/H_i}(\mathcal{F}_{\lambda_i})) \subseteq X - \pi^{-1}(y)$. Using 1. of 4.2.2, the support of $\pi_*^{G/H_i}(\mathcal{F}_{\lambda_i})$ will be in $Y - y$ and hence $\pi_*^{G/H_i}(\mathcal{F}_{\lambda_i}) \in \mathfrak{q}_y$. We know $\pi^*(\mathfrak{q}_y) \subseteq \mathfrak{p}$ where $\mathrm{Spec}(\pi^*)(\mathfrak{p}) = \mathfrak{q}_y$ is given. This will prove

$\pi^*\pi_*^{G/H_i}(\mathcal{F}_{\lambda_i}) \in \mathfrak{p}$ and hence $\mathcal{G}_i \in \mathfrak{p}$. Now using the tower and the definition of a triangulated ideal, $\mathcal{F}$ is contained in $\mathfrak{p}$.
Proof of 2. Suppose $\mathrm{supph}(\mathcal{F}) \cap \pi^{-1}(y) \neq \varnothing$ and hence we get $\mathcal{F}' = \mathcal{F} \otimes \mathcal{O}_{\pi^{-1}(\bar{y})} \in \mathfrak{p}$. Observe that $\mathrm{supph}(\mathcal{F}') = \pi^{-1}(\bar{y}) = \overline{\pi^{-1}(y)}$. Now applying the same procedure as in 2. of 4.2.2, we shall get a distinguished triangle

$$\bigoplus_{\lambda} W_\lambda \otimes \pi^*\pi_*^{G/H}(\mathcal{F}'_\lambda) \to \mathcal{F}' \to \mathcal{F}'' \to$$

with $\mathrm{supph}(\mathcal{F}'') \subsetneq \mathrm{supph}(\mathcal{F}')$. Since the G-invariant subset $\mathrm{supph}(\mathcal{F}'')$ is a proper subset of $\overline{\pi^{-1}(y)}$ therefore $\mathrm{supph}(\mathcal{F}'') \cap \pi^{-1}(y) = \varnothing$. Using 1. above, we get that $\mathcal{F}'' \in \mathfrak{p}$. Hence using triangulated ideal property the third object of distinguished triangle will be in $\mathfrak{p}$ i.e. $\bigoplus_{\lambda} W_\lambda \otimes \pi^*\pi_*^{G/H}(\mathcal{F}'_\lambda) \in \mathfrak{p}$. But this gives $\pi^*\pi_*^{G/H}(\mathcal{F}'_\lambda) \in \mathfrak{p}$ with $\mathrm{supph}(\pi_*^{G/H}(\mathcal{F}'_\lambda)) \subseteq \bar{y}$ as $W_\lambda \otimes \mathcal{O}_X$ is not in any $\mathfrak{p}$ because $W_\lambda^* \otimes W_\lambda \otimes \mathcal{O}_X$ contains the $\mathcal{O}_X$ as direct summand, see Proposition 10.30 of [12]. And, at least for one λ, say λ_0, we have $\mathrm{supph}(\pi_*^{G/H}(\mathcal{F}'_\lambda)) = \bar{y}$ which gives $\pi_*^{G/H}(\mathcal{F}'_{\lambda_0}) \notin \mathfrak{q}_y$. This is a contradiction as $\pi^*(\mathcal{D}^{per}(Y)) \cap \mathfrak{p} = \pi^*(\mathfrak{q}_y)$. □

We are now ready to give a proof of the main result which gives the injectivity of the map $\mathrm{Spec}(\pi^*)$.

Proposition 4.2.11. *Suppose X is a smooth quasi-projective varieties of dimension n.Then the map* $\mathrm{Spec}(\pi^*) : \mathrm{Spec}(\mathcal{D}^G(X)) \to \mathrm{Spec}(\mathcal{D}^{per}(Y))$ *is injective.*

Proof. We prove this proposition by contradiction. Let $\mathfrak{p}_1, \mathfrak{p}_2$ be two distinct points of $\mathrm{Spec}(D^G(X))$ which maps to the same point $\mathfrak{q}_y$ i.e. $(\pi^*)^{-1}(\mathfrak{p}_1) = (\pi^*)^{-1}(\mathfrak{p}_2) = \mathfrak{q}_y$. Let $\mathcal{F} \in \mathfrak{p}_1$ be an complex of G-equivariant sheaves. Now we use the above lemma.

Using 2., we have $\mathrm{supph}(\mathcal{F}) \subseteq (X - \pi^{-1}(y))$. Therefore using 1., and the fact that $(\pi^*)^{-1}(\mathfrak{p}_2) = \mathfrak{q}_y$, we get that $\mathcal{F} \in \mathfrak{p}_1 \cap \mathfrak{p}_2$. Hence $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$. Similarly, $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$ implying that $\mathfrak{p}_1 = \mathfrak{p}_2$. This contradicts the assumption that $\mathfrak{p}_1 \neq \mathfrak{p}_2$, and hence proves the proposition. □

Step 3: $\mathrm{Spec}(\pi^*)$ **is closed and hence is a homeomorphism.**

Here we need bijection of the above step to prove closedness of the map $\mathrm{Spec}(\pi^*)$. We shall use the fact that $W \otimes \mathcal{O}_X \notin \mathfrak{p}$ for any finite dimensional representation and any prime ideal $\mathfrak{p}$. This follows from the fact that the representation on $W^* \otimes W \otimes \mathcal{O}_X$, coming from $W \otimes \mathcal{O}_X$, has the trivial representation as a direct summand, see Proposition 10.30 of [12]. Since supp($\mathcal{F}$), $\mathcal{F} \in \mathcal{D}^G(X)$, are the basic closed sets therefore it is enough to prove that their image under the map Spec(π^*) are closed. Now to prove this we shall use the description given in lemma 4.2.2 for any object of $\mathcal{D}^G(X)$. Writing $\mathcal{G}_{\lambda_j} = \pi_*^{G/H_j}(\mathcal{F}_{\lambda_j})$ for simplicity, we have the following lemma.

Lemma 4.2.12. $\mathrm{Spec}(\pi^*)(\mathrm{supp}(\mathcal{F})) = \bigcup_j \bigcup_{\lambda_j} \mathrm{supp}(\mathcal{G}_{\lambda_j})$.

Proof. Given $\mathcal{F} \in \mathfrak{p}$ we have $\mathcal{G}_{\lambda_j}$'s as in lemma 4.2.2. Now,

$$\begin{array}{lrcl} & \mathcal{F} \in \mathfrak{p} & \Leftrightarrow & W_{\lambda_j} \otimes \pi^*(\mathcal{G}_{\lambda_j}) \in \mathfrak{p} \qquad \forall j, \lambda_j \\ & & \Leftrightarrow & \pi^*(\mathcal{G}_{\lambda_j}) \in \mathfrak{p}, \quad \text{since } W_{\lambda_j} \otimes \mathcal{O}_X \notin \mathfrak{p}. \\ \text{Therefore} & \mathcal{F} \notin \mathfrak{p} & \Leftrightarrow & \exists\, \lambda_j \text{ such that } \pi^*(\mathcal{G}_{\lambda_j}) \notin \mathfrak{p}. \end{array}$$

Let $\mathfrak{p} \in \mathrm{supp}(\mathcal{F})$ and hence by the definition $\mathcal{F} \notin \mathfrak{p}$. Now using the above observation there exists a λ_j such that $\pi^*(\mathcal{G}_{\lambda_j}) \notin \mathfrak{p}$ i.e. $\mathcal{G}_{\lambda_j} \notin (\pi^*)^{-1}(\mathfrak{p}) = \mathrm{Spec}(\pi^*)(\mathfrak{p})$ and hence $\mathrm{Spec}(\pi^*)(\mathfrak{p}) \in \mathrm{supp}(\mathcal{G}_{\lambda_j})$. Therefore

$$\mathrm{Spec}(\pi^*)(\mathrm{supp}(\mathcal{F})) \subseteq \cup_j \cup_{\lambda_j} \mathrm{supp}(\mathcal{G}_{\lambda_j}).$$

Conversely suppose $\mathfrak{q} \in \cup_j \cup_{\lambda_j} \mathrm{supp}(\mathcal{G}_{\lambda_j})$ and hence $\mathfrak{q} \in \mathrm{supp}(\mathcal{G}_{\lambda_j})$ for some λ_j. Therefore by definition $\mathcal{G}_{\lambda_j} \notin \mathfrak{q}$ but using the bijection of the map Spec(π^*) we have $\mathcal{G}_{\lambda_j} \notin (\pi^*)^{-1}(\mathfrak{p}) = \mathfrak{q}$ for some $\mathfrak{p}$. Now it follows that $\pi^*(\mathcal{G}_{\lambda_j}) \notin \mathfrak{p}$ and once again using the above observation we have $\mathcal{F} \notin \mathfrak{p}$ i.e. $\mathfrak{p} \in \mathrm{supp}(\mathcal{F})$. Hence we have $\cup_j \cup_{\lambda_j} \mathrm{supp}(\mathcal{G}_{\lambda_j}) \subseteq \mathrm{Spec}(\pi^*)(\mathrm{supp}(\mathcal{F}))$. $\square$

Since union in right hand side of above lemma is finite it follows that the image of supp($\mathcal{F}$) under the map Spec(π^*) is closed for all $\mathcal{F} \in \mathcal{D}^G(X)$. Hence the map Spec($\pi^*$) is a closed map and therefore it is a homeomorphism.

Remark 4.2.13.

1. The classification of thick tensor ideals in $\mathcal{D}^G(X)$ is given by Thomason subsets of X/G. More precisely, the thick tensor ideals in $\mathcal{D}^G(X)$ are generated by objects, whose images have as their support a Thomason subset of Y. Thus the bijection can be restated as a bijection between thick tensor ideals of $\mathcal{D}^G(X)$ and G-invariant Thomason subsets of X.

2. In a fashion, similar to [theorem 4.1, [39]], one can use the classification of thick tensor ideals of $\mathcal{D}^G(X)$ to give a classification of strictly full tensor ideals of $\mathcal{D}^G(X)$.

Remark 4.2.14. Given an unital tensor functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ one can ask the conditions on F which gives the homeomorphism of the map Spc(F). We can give slightly abstract conditions using the above proofs. If F satisfies the following two conditions,

(i) For each object A of $\mathcal{D}_2$ there exists a following tower

$$\begin{tikzcd}
A \arrow[rr] & & A_1 \arrow[dl, dashed] & \cdots & A_{n-1} \arrow[rr] & & A_n \arrow[dl, dashed] \arrow[dr, equal] & \\
& B_1 \arrow[ul] & & \cdots & & B_n \arrow[ul] & & B_{n+1}
\end{tikzcd}$$

s.t. the lower level objects $B_j \in add(Im(F) \otimes gen(\mathcal{D}_2))$. Here $gen(\mathcal{D}_2)$ represents the collection of objects which generates $\mathcal{D}_2$ as thick tensor ideal and *add* of some collection of objects represents the full additive category generated by the collection. Further, for each prime ideal $\mathfrak{p} \in \mathrm{Spc}(\mathcal{D}_2)$ following condition is satisfied,

$$A \in \mathfrak{p} \Leftrightarrow B_j \in \mathfrak{p}.$$

(ii) $F(\mathcal{D}_1 - \mathfrak{q}) \cap \langle F(\mathfrak{q}) \rangle = \emptyset$. (Or there exists a prime ideal containing $F(\mathfrak{q})$ and not containing object $F(A)$ for each objects A of $\mathcal{D}_1 - \mathfrak{q}$.)

then the continuous map $\mathrm{Spc}(F)$ is an homeomorphism.

Step 4: $\mathrm{Spec}(\pi^*)$ **is an isomorphism.**

In this step we shall prove that the above homeomorphism $\mathrm{spec}(\pi^*)$ is, in fact, an isomorphism. We begin by proving the following lemma which we shall use later.

Lemma 4.2.15. *There exist a natural transformation $\eta : \pi^* \pi_*^G \to Id$ (resp. $\mu : Id \to \pi_*^G \pi^*$) such that $\eta(\mathcal{O}_X) = id$ (resp. $\mu(\mathcal{O}_Y) = id$) where $\pi^* \pi_*^G(\mathcal{O}_X) = \mathcal{O}_X$ (resp. $\pi_*^G \pi^*(\mathcal{O}_Y) = \mathcal{O}_Y$).*

Proof. We shall prove the existence of η, as μ can be found using similar arguments. Since the functor π^* is a left adjoint of the functor π_* we have a natural transformation $\eta' : \pi^* \pi_* \to \mathrm{Id}$ given by the adjunction property. We also have a natural transformation given by inclusion of G-invariant part of sheaves on Y, say I. Now composing with the functors π^* and π_* we get another natural transformation which composed with η' gives the η i.e. $\eta := \eta' \circ (\pi^* \cdot I \cdot \pi_*)$. Now to prove $\eta(\mathcal{O}_X) = \mathrm{Id}$ we can assume that X is an affine variety. Suppose $\tilde{A}$ is a structure sheaf of X and $\tilde{B}$ is the structure sheaf of Y. Since π^* is a unital tensor functor, $\pi^*(\mathcal{O}_Y) = \mathcal{O}_X$. This implies $\mathbf{R}^i \pi^* = 0$ for $i > 0$. Similarly, using the Leray spectral sequence one can deduce $\mathbf{R}^i \pi_* = 0$ for $i > 0$. Thus we get a morphism $\pi^* \pi_*^G(\tilde{A}) \to \tilde{A}$, in place of its derived functors. Now clearly the multiplication map $A \otimes_B ({}_B A)^G \to A$ is just inverse of the natural identification map of A with $A \otimes_B ({}_B A)^G$. Hence the map $\eta(\mathcal{O}_X) : \tilde{A} \to \tilde{A}$ is an identity map. Similarly we can prove that $\mu(\mathcal{O}_Y) = \mathrm{Id}$.

$\square$

Recall the definitions of structure sheaves and associated map of the sheaves given by the unital tensor functor of underlying tensor triangulated categories 3.2 i.e. given an unital functor $\pi^* : \mathcal{D}^{per}(Y) \to \mathcal{D}^G(X)$ the morphism $\mathrm{Spec}(\pi^*)$ induces a map of the structure sheaves, $\mathrm{Spec}(\pi^*)^\# : \mathcal{O}_Y \to \mathcal{O}_X$. We shall prove that this map is an isomorphism by observing that $\mathrm{Spec}(\pi^*)^\#(V)$ is an isomorphism for every open set $V \subseteq \mathrm{Spec}(\mathcal{D}^{per}(Y))$. If we take $U = \pi^{-1}(V)$, $Z = Y \setminus V$ and $Z' = X \setminus U$ then we have a functor $\pi_V^* : \frac{\mathcal{D}^{per}(Y)}{\mathcal{D}^{per}_Z(Y)} \to \frac{\mathcal{D}^G(X)}{\mathcal{D}^G_{Z'}(X)}$ which will induce a map $\mathrm{Spec}(\pi^*)^\#(V) := \pi_V^* : \mathrm{End}_{\frac{\mathcal{D}^{per}(Y)}{\mathcal{D}^{per}_Z(Y)}}(\mathcal{O}_Y) \to \mathrm{End}_{\frac{\mathcal{D}^G(X)}{\mathcal{D}^G_{Z'}(X)}}(\mathcal{O}_X)$.

Lemma 4.2.16. *The map* $\pi_V^* : \mathrm{End}_{\frac{\mathcal{D}^{per}(Y)}{\mathcal{D}^{per}_Z(Y)}}(\mathcal{O}_Y) \to \mathrm{End}_{\frac{\mathcal{D}^G(X)}{\mathcal{D}^G_{Z'}(X)}}(\mathcal{O}_X)$ *is surjective.*

Proof. Suppose $[\mathcal{O}_Y \xleftarrow{s} \mathcal{G} \xrightarrow{a} \mathcal{O}_Y]$ is an element of $\mathrm{End}_{\frac{\mathcal{D}^{per}(Y)}{\mathcal{D}^{per}_Z(Y)}}(\mathcal{O}_Y)$ then the map π^* will send it to an element

$$[\mathcal{O}_X \xleftarrow{\pi^*(s)} \pi^*(\mathcal{G}) \xrightarrow{\pi^*(a)} \mathcal{O}_X] \in \mathrm{End}_{\frac{\mathcal{D}^G(X)}{\mathcal{D}^G_{Z'}(X)}}(\mathcal{O}_X).$$

It is now enough to prove that this map is a bijection.

Let $[\mathcal{O}_X \xleftarrow{t} \mathcal{F} \xrightarrow{b} \mathcal{O}_X] \in \mathrm{End}_{\frac{\mathcal{D}^G(X)}{\mathcal{D}^G_{Z'}(X)}}(\mathcal{O}_X)$ be a given element then using the functor π_*^G we shall get an element $[\mathcal{O}_Y \xleftarrow{\pi_*^G(t)} \pi_*^G(\mathcal{F}) \xrightarrow{\pi_*^G(b)} \mathcal{O}_Y] \in \mathrm{End}_{\frac{\mathcal{D}^{per}(Y)}{\mathcal{D}^{per}_Z(Y)}}(\mathcal{O}_Y)$ as $\mathrm{supph}(C(\pi_*^G(t))) \subseteq Z$ using the flat base change and the canonical isomorphism,

$$i_V^* \pi_*^G(\mathcal{F}) \simeq (i_V^* \pi_*(\mathcal{F}))^G \simeq \pi_*^G(i_U^* \mathcal{F}) \xrightarrow{i_U^*(t)} \pi_*^G(i_U^* \mathcal{O}_X) \simeq \mathcal{O}_V.$$

Now we want to prove that

$$[\mathcal{O}_X \xleftarrow{t} \mathcal{F} \xrightarrow{b} \mathcal{O}_X] = [\mathcal{O}_X \xleftarrow{\pi^*\pi_*^G(t)} \pi^*\pi_*^G(\mathcal{F}) \xrightarrow{\pi^*\pi_*^G(b)} \mathcal{O}_X].$$

Using the lemma 4.2.15, we have a natural map $\eta(\mathcal{F}) : \pi^*\pi_*^G(\mathcal{F}) \to \mathcal{F}$, so to prove the assertion it is now enough to check that $t \circ \eta(\mathcal{F}) = \pi^*\pi_*^G(t)$, $b \circ \eta(\mathcal{F}) = \pi^*\pi_*^G(b)$ and the cone of $\eta(\mathcal{F})$ is supported on Z' that is $C(\eta(\mathcal{F})) \in$

$\mathcal{D}^G_{Z'}(X)$. Here the first two assertions follows from the following commutative diagrams which are a consequence of lemma 4.2.15.

$$\begin{tikzcd} \pi^*\pi^G_*(\mathcal{F}) \arrow[r, "\eta(\mathcal{F})"] \arrow[d, "\pi^*\pi^G_*(t)"'] & \mathcal{F} \arrow[d, "t"] \\ \mathcal{O}_X \arrow[r, equal, "\eta(\mathcal{O}_X)"] & \mathcal{O}_X \end{tikzcd} \qquad \begin{tikzcd} \pi^*\pi^G_*(\mathcal{F}) \arrow[r, "\eta(\mathcal{F})"] \arrow[d, "\pi^*\pi^G_*(b)"'] & \mathcal{F} \arrow[d, "b"] \\ \mathcal{O}_X \arrow[r, equal, "\eta(\mathcal{O}_X)"] & \mathcal{O}_X \end{tikzcd}$$

Now the last assertion $C(\eta(\mathcal{F})) \in \mathcal{D}^G_{Z'}(X)$ is equivalent to $i^*_U C(\eta(\mathcal{F})) \simeq 0$ in $\mathcal{D}^G(U)$ but as the functor i^*_U is exact this assertion is same as $C(i^*_U\eta(\mathcal{F})) \simeq 0$. Since a cone of an isomorphism is zero it is enough to check that the map $i^*_U\eta(\mathcal{F})$ is an isomorphism. And this follows from the following commutative diagram.

$$\begin{tikzcd} i^*_U\pi^*\pi^G_*(\mathcal{F}) \arrow[r, "i^*_U\eta(\mathcal{F})"] \arrow[d, "\wr"] & i^*_U\mathcal{F} \arrow[d, equal] \\ \pi^*\pi^G_*(i^*_U\mathcal{F}) \arrow[r, "\eta(i^*_U\mathcal{F})"] \arrow[d, "\pi^*\pi^G_* i^*_U(t)"', "\wr"] & i^*_U\mathcal{F} \arrow[d, "i^*_U(t)"', "\wr"] \\ \pi^*\pi^G_*(\mathcal{O}_U) \arrow[r, equal, "\eta(\mathcal{O}_U)"] & \mathcal{O}_U \end{tikzcd}$$

In above diagram we had used the same notations π and η for its restriction on open subsets. Here the top left vertical isomorphism comes from the flat base change formula and using the following canonical isomorphism.

$$i^*_U\pi^*\pi^G_*(\mathcal{F}) \simeq \pi^* i^*_V(\pi_*(\mathcal{F}))^G \simeq \pi^*(i^*_V\pi_*(\mathcal{F}))^G \simeq \pi^*(\pi_* i^*_U(\mathcal{F}))^G = \pi^*\pi^G_*(i^*_U\mathcal{F}).$$

This proves that π^*_V is surjective. $\square$

Lemma 4.2.17. *π^*_V is injective.*

Proof. Let $[\mathcal{O}_Y \xleftarrow{s} \mathcal{G} \xrightarrow{a} \mathcal{O}_Y] \in \mathrm{End}_{\frac{\mathcal{D}^{per}(Y)}{\mathcal{D}^{per}_Z(Y)}}(\mathcal{O}_Y)$ maps to zero in $\mathrm{End}_{\frac{\mathcal{D}^G(X)}{\mathcal{D}^G_{Z'}(X)}}(\mathcal{O}_X)$ i.e. $[\mathcal{O}_X \xleftarrow{\pi^*(s)} \pi^*(\mathcal{G}) \xrightarrow{\pi^*(a)} \mathcal{O}_X] = 0$ which is equivalent to the existence of $\mathcal{F}$ and a map $t : \mathcal{F} \to \pi^*\mathcal{G}$ with $\mathrm{supph}(C(t)) \subseteq Z'$ such that $\pi^*(a) \circ t = 0$. Now the map $\pi^G_*(t) : \pi^G_*(\mathcal{F}) \to \pi^G_*\pi^*(\mathcal{G})$ gives $\pi^G_*\pi^*(a) \circ \pi^G_*(t) = 0$ and as proved earlier we know that $\mathrm{supph}(C(\pi^G_*(t))) \subseteq Z$ whenever $\mathrm{supph}(C(t)) \subseteq Z'$. Hence the element $[\mathcal{O}_Y \xleftarrow{\pi^G_*\pi^*(s)} \pi^G_*\pi^*(\mathcal{G}) \xrightarrow{\pi^G_*\pi^*(a)} \mathcal{O}_Y] = 0$ in $End_{\frac{\mathcal{D}^{per}(Y)}{\mathcal{D}^{per}_Z(Y)}}(\mathcal{O}_Y)$. We shall prove that $[\mathcal{O}_Y \xleftarrow{s} \mathcal{G} \xrightarrow{a} \mathcal{O}_Y] = [\mathcal{O}_Y \xleftarrow{\pi^G_*\pi^*(s)} \pi^G_*\pi^*(\mathcal{G}) \xrightarrow{\pi^G_*\pi^*(a)} \mathcal{O}_Y]$ as an elements of $\mathrm{End}_{\frac{\mathcal{D}^{per}(Y)}{\mathcal{D}^{per}_Z(Y)}}(\mathcal{O}_Y)$. Now using lemma

4.2.15 we have a map $\mu(\mathcal{G}) : \mathcal{G} \to \pi_*^G\pi^*(\mathcal{G})$ which gives the following commutative diagrams as before using lemma 4.2.15,

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\mu(\mathcal{G})"] \arrow[d, "s"'] & \pi_*^G\pi^*(\mathcal{G}) \arrow[d, "\pi_*^G\pi^*(s)"] \\
\mathcal{O}_Y \arrow[r, equal, "\mu(\mathcal{O}_Y)"] & \mathcal{O}_Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{G} \arrow[r, "\mu(\mathcal{G})"] \arrow[d, "a"'] & \pi_*^G\pi^*(\mathcal{G}) \arrow[d, "\pi_*^G\pi^*(a)"] \\
\mathcal{O}_Y \arrow[r, equal, "\mu(\mathcal{O}_Y)"] & \mathcal{O}_Y
\end{tikzcd}$$

Therefore it remains to prove that $i_V^*C(\mu(\mathcal{G})) = 0$ but as before this is equivalent to proving $C(i_V^*\mu(\mathcal{G})) = 0$ since the functor i_V^* is an exact functor. Again using the fact that a cone of an isomorphism is zero it is enough to prove that $i_V^*\mu(\mathcal{G})$ is an isomorphism. This clearly follows from the following commutative diagrams,

$$\begin{tikzcd}
i_V^*\mathcal{G} \arrow[r, "i_V^*\mu(\mathcal{G})"] \arrow[d, equal] & i_V^*\pi_*^G\pi^*(\mathcal{G}) \arrow[d, "\wr"'] \\
i_V^*\mathcal{G} \arrow[r, "\mu(i_V^*\mathcal{G})"] \arrow[d, "i_V^*(s)"', "\wr"] & \pi_*^G\pi^*(i_V^*\mathcal{G}) \arrow[d, "\wr"', "\pi_*^G\pi^*i_V^*(s)"] \\
\mathcal{O}_V \arrow[r, equal, "\mu(\mathcal{O}_V)"] & \pi_*^G\pi^*(\mathcal{O}_V).
\end{tikzcd}$$

Here again as earlier the top right vertical isomorphism comes from the flat base change and the following sequence of natural isomorphisms.

$$i_V^*\pi_*^G\pi^*(\mathcal{G}) \simeq i_V^*(\pi_*\pi^*\mathcal{G})^G \simeq (i_V^*\pi_*\pi^*\mathcal{G})^G \simeq \pi_*^G i_U^*\pi^*\mathcal{G} \simeq \pi_*^G\pi^*(i_V^*\mathcal{G}).$$

This proves injectivity of the map π_V^*. $\square$

From the above two lemmas it follows that π_V^* is an isomorphism and hence $\mathrm{Spec}(\pi^*)$ is an isomorphism of the varieties $\mathrm{Spec}(\mathcal{D}^{per}(Y))$ and $\mathrm{Spec}(\mathcal{D}^G(X))$.

Chapter 5

Balmer spectrum for split superschemes

This chapter gives computation of the Balmer spectrum similar to the proof of Balmer. We first recall some facts in the case of schemes as given by Neeman[33]. We then define support data for the case of superschemes and prove that this support data is classifying support data. Then we use Balmer's result to conclude the result. Finally we prove various assumptions which we need to apply Neeman's version of the localization theorem. (We need a split superscheme to prove cocompleteness.)

5.1 Support data

We refer to an earlier chapter for the definition and some properties of superschemes. As in the case of schemes we can define the *support* of a quasi-coherent sheaf as a subset of X containing all super prime ideals where the stalk of the sheaf is nonzero. Since non-triviality of the stalk at any point $\mathfrak{p}$ is a local property we can check it in an affine open set containing $\mathfrak{p}$. Now from the earlier observation $\mathcal{F}_{\mathfrak{p}} = 0$ iff $\mathcal{F}^0_{\mathfrak{p}} = 0 = \mathcal{F}^1_{\mathfrak{p}}$ as stalks of a sheaves of $\mathcal{O}_{X_{rd}}$ modules $\mathcal{F}^0$ and $\mathcal{F}^1$. Therefore for a quasi coherent sheaf $\mathcal{F}$ we have $\mathrm{supp}(\mathcal{F}) = \mathrm{supp}(ff(\mathcal{F})) = \mathrm{supp}(\mathcal{F}^0) \cup \mathrm{supp}(\mathcal{F}^1)$. Now the assignment of support can be extended to the derived category as follows,

$$\mathrm{supph}(\mathcal{F}^{\cdot}) := \cup_{i \in \mathbb{Z}} \mathrm{supp}(\mathcal{H}^i(\mathcal{F}^{\cdot})).$$

This association can be restricted to the thick subcategory $\mathcal{D}^{per}(X)$ for quasi compact and quasi separated scheme X. As the forgetful functor is an exact functor we have the following relation between supports as in the case of sheaves,

$$\operatorname{supph}(\mathcal{F}^\cdot) = \operatorname{supph}(ff(\mathcal{F}^\cdot)) = \operatorname{supph}(\mathcal{F}^0) \cup \operatorname{supph}(\mathcal{F}^1)$$

Above observation gives following result similar to the result of Thomason [lemma 3.3(c), [39]].

Lemma 5.1.1. *Suppose X is a quasi-compact and quasi-separated superscheme and $\mathcal{F}^\cdot \in \mathcal{D}^{per}(X)$. Then the subset* $\operatorname{supph}(\mathcal{F}^\cdot)$ *is closed and* $X \setminus \operatorname{supph}(\mathcal{F}^\cdot)$ *is a quasi-compact subset of X.*

Using this property of supports we can prove the following result,

Lemma 5.1.2. *The pair* $(X, \operatorname{supph})$ *defined as above gives a support data on the triangulated category $\mathcal{D}^{per}(X)$.*

Proof. Since the forgetful functor is an exact functor and we have the equality $(\mathcal{F}^\cdot) = \operatorname{supph}(ff(\mathcal{F}^\cdot))$ therefore the support data properties (SD 1)-(SD 4) of [definition 3.1, [3]] are easy to prove. We shall just prove (SD 5) here, which states that $U(\mathcal{F}_1 \otimes \mathcal{F}_2) = U(\mathcal{F}_1) \cup U(\mathcal{F}_2)$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are perfect complexes and $U(\mathcal{F}_i) = X \setminus \operatorname{supph} \mathcal{F}_i$. This is equivalent to the statement that for every $x \in X$, $(\mathcal{F}_1 \otimes \mathcal{F}_2)_x$ is acyclic if and only if either $(\mathcal{F}_1)_x$ or $(\mathcal{F}_2)_x$ is acyclic. Since checking nontriviality of the stalk is a local question, we can assume that X is an affine superscheme. First we observe that any perfect complex $\mathcal{F}^\cdot$ is a strict perfect complex and hence a bounded complex of fintely generated projective modules. Hence by taking local superring $R = \mathcal{O}_{X,x}$, and observing that $(\mathcal{F}_1 \otimes \mathcal{F}_2)_x \cong (\mathcal{F}_1)_x \otimes (\mathcal{F}_2)_x$, the proof follows from the result 2.4.23(2). □

Definition 5.1.3. A subset $Z \subset X$ is said to be *Thomason* if $Z = \cup_\alpha Z_\alpha$ where each Z_α is closed and $X \setminus Z_\alpha$ is quasi-compact.

Note that if X is noetherian, the Thomason subsets match with specialization closed subsets.

Consider a split superscheme $(X, \mathcal{O}_X)$. Note that, the inclusion $i : X_{rd} \to X$ given by the surjection $i^\# : \mathcal{O}_X \to \mathcal{O}_{X_{rd}}$. $i^\#$ splits to give a projection $p : X \to X_{rd}$ with $p \circ i = id_{X_{rd}}$. Let $i_*, p^* : \mathcal{D}_{qc}(X_{rd}) \to \mathcal{D}_{qc}(X)$ and $i^* : \mathcal{D}_{qc}(X) \to \mathcal{D}_{qc}(X_{rd})$ be the induced derived functors.

Proposition 5.1.4. *For an ideal $\mathcal{E}$ in $\mathcal{D}_{qc}(X_{rd})$, and for an $\mathcal{O}_{X_{rd}}$ module, $\mathcal{G}$*

1. *$i_*(i^*\mathcal{G}) = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{rd}}$. Further* $\operatorname{supph} i^*\mathcal{G} = \operatorname{supph} \mathcal{G} \subset X$.
2. *$i_*(\langle i^*\mathcal{E} \rangle) \subset \mathcal{E}$.*

3. i_* is dominant, that is the thick tensor ideal generated by the image of i_* is $\mathcal{D}_{qc}(X)$.

Proof. The proof of *1* is clear from the definition. *Proof of 2.* We shall use the definition of $\langle \mathcal{E} \rangle$ given in lemma 3.2.2. Since i_* is an exact functor, it is enough to prove that $i_*(ideal(i^*\mathcal{E})) \subset \mathcal{E}$. Thus, it is enough to see that for $\mathcal{A} \in \mathcal{D}_{qc}(X_{rd})$ and $\mathcal{J} \in \mathcal{E}$,

$$\begin{aligned} i_*(\mathcal{A} \otimes_{\mathcal{O}_{X_{rd}}} i^*\mathcal{J}) &= i_*(i^*p^*\mathcal{A} \otimes_{\mathcal{O}_{X_{rd}}} i^*\mathcal{J}) = i_*i^*(p^*\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{J}) \\ &= (p^*\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{J}) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{rd}} = \mathcal{J} \otimes_{\mathcal{O}_X} (p^*\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{rd}}) \\ &\in \langle \mathcal{E} \rangle. \end{aligned}$$

Proof of 3. Since $(X, \mathcal{O}_X)$ is a split superscheme, we have identification of $\mathcal{O}_X$ with GrX. The sheaf GrX is an exterior algebra over purely odd locally free sheaf $\Pi\mathcal{V} := J_X/J_X^2$ and each subquotient J_X^i/J_X^{i+1} can be identified with $\Pi^i\Lambda^i\mathcal{V}$. Hence each subquotient is purely odd or purely even locally free sheaves. The $\mathbb{Z}$-grading on sheaf GrX gives a filtration for structure sheaf $\mathcal{O}_X$ and hence we have following tower for structure sheaf $\mathcal{O}_X$,

$$\begin{tikzcd} \mathcal{O}_X \arrow[dr] & & J_X \arrow[ll] & \cdots & J_X^{n-1} \arrow[dr] & & J_X^n \arrow[ll] \arrow[dr, equal] & \\ & \mathcal{O}_{X_{rd}} \arrow[ur, dashed] & & \cdots & & \Pi^{n-1}\Lambda^{n-1}\mathcal{V} \arrow[ur, dashed] & & \Pi^n\Lambda^n\mathcal{V}. \end{tikzcd}$$

In above tower, each of the terms in the lower row is complex of either purely odd or purely even sheaves. And using property of tensor proved in 2.4.21, we have $\Pi^i\Lambda^i\mathcal{V} = (\Pi^i\mathcal{O}_{X_{rd}}) \otimes \Lambda^i\mathcal{V}$. Therefore the ideal generated by the image of the functor i_* contains the all the terms in the lower row of the above tower and hence i_* is a dominant functor. □

We shall denote the functor i_* by $\mathbf{i_{rd}}$ from now on. We shall now prove that above support data is in fact classifying support data as defined in Balmer[3]. We need following classification (see [3]) of thick tensor subcategories of $\mathcal{D}^{per}(X)$ which we prove by relating it with the case of schemes.

Proposition 5.1.5. *Given a quasi-compact and quasi-separated split superscheme $(X, \mathcal{O}_X)$ we have a bijection,*

$$\theta : \{Y \subset X | Y \textit{ Thomason subset } \} \xrightarrow{\sim} \{\mathcal{I} \subset \mathcal{D}^{per}(X) | \mathcal{I} \textit{ radical thick } \otimes\textit{-ideal}\}$$

defined by $Y \mapsto \{\mathcal{F}^\cdot \in \mathcal{D}^{per}(X) | \operatorname{supph}(\mathcal{F}^\cdot) \subset Y\}$, with inverse, say η, $\mathcal{I} \mapsto$ $\operatorname{supph}(\mathcal{I}) := \cup_{\mathcal{F}^\cdot \in \mathcal{I}} \operatorname{supph}(\mathcal{F}^\cdot)$.

Proof. Using support data properties (SD 1) - (SD 5) [definition 3.1, Balmer [3]] we can prove that $\theta(Y)$ is a radical thick tensor ideal and hence the map θ is well defined. To prove that $\eta(\mathcal{I})$ is a Thomason subset, it is enough to prove that for any $y \in \eta(\mathcal{I})$ there is a closed set containing this point. By definition y is in the homological support of some object $\mathcal{F}^\cdot \in \mathcal{I}$. Hence $y \in \mathrm{supph}(ff(\mathcal{F}^\cdot))$ which is a closed subset.

It is easy to check that $\eta \circ \theta(Y) \subseteq Y$ and $\mathcal{I} \subseteq \theta \circ \eta(\mathcal{I})$. To prove that $Y \subseteq \eta\circ\theta(Y)$ it is enough to show that for any closed subset Z there exists an object with support Z. But there exists an $\mathcal{O}_{X_{rd}}$ perfect sheaf with support Z and hence via the natural map $\mathcal{O}_X \to \mathcal{O}_{X_{rd}}$ we get a perfect sheaf with support Z.

Finally to prove that $\theta \circ \eta(\mathcal{I}) \subseteq \mathcal{I}$ it is enough to prove that for any $\mathcal{F}^\cdot \in \theta \circ \eta(\mathcal{I})$ the object $\mathcal{F}^\cdot \in \mathcal{I}$. Now following the proof of theorem 3.15 of Thomason [39] we reduce to proving that if $\mathrm{supph}(\mathcal{F}^\cdot) \subseteq \mathrm{supph}(\mathcal{G}^\cdot)$ for some object $\mathcal{G}^\cdot \in \mathcal{I}$, then $\mathcal{F}^\cdot \in \mathcal{I}$. By 5.1.4(1) we have that $\mathrm{supph}\, i^*\mathcal{F}^\cdot \subseteq \mathrm{supph}\, i^*\mathcal{G}^\cdot$. Now by [Thomason [39]] $i^*\mathcal{F}^\cdot \in \langle i^*\mathcal{G}^\cdot \rangle$. Therefore by 5.1.4(2) $\mathbf{i_{rd}} i^*\mathcal{F}^\cdot \in \langle \mathcal{G}^\cdot \rangle$. Again using 5.1.4(1), $\mathcal{F}^\cdot \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{rd}} \in \langle \mathcal{G}^\cdot \rangle \subset \mathcal{I}$. $\mathcal{O}_{X_{rd}}$ does not belong to any prime ideal since $\mathbf{i_{rd}}$ is dominant. Thus, using the fact that $\mathcal{I}$ is intersection of all primes containing it, $\mathcal{F}^\cdot \in \mathcal{I}$. □

With this result it follows that (X, supph) is a classifying support data on the tensor triangulated category $\mathcal{D}^{per}(X)$ for a quasi-compact and quasi-separated split superscheme X, see [definition 6.9, [**?**]] (and also [def. 5.1, Balmer [3]] for the simpler noetherian case) for definitions. The following corollary is a restatement of the first part of Theorem 8.5 of [**?**].

Corollary 5.1.6. *The canonical map* $f : X \to \mathrm{Spc}(\mathcal{D}^{per}(X))$ *given by* $x \mapsto \{\mathcal{F}^\cdot \in \mathcal{D}^{per}(X) | x \notin \mathrm{supph}(\mathcal{F}^\cdot)\}$ *is a homeomorphism.*

Remark 5.1.7. One can use the classification of thick tensor ideals of the category of perfect complexes over quasi-compact and quasi-separated schemes to give a classification of strictly full tensor ideals, in the same way as in [theorem 4.1, [39]].

5.2 Localization theorem and spectrum for a split superscheme

We shall prove a localization theorem (similar to that proved by Thomason) for split superschemes by using the generalisation of Thomason's result proved by Neeman[32]. First we recall some notation. Given a closed subset Z of X we can define the full triangulated subcategory $\mathcal{D}_{qc,Z}(X) \subseteq \mathcal{D}_{qc}(X)$

consisting of all objects with homological support contained in Z. Suppose U is the open complement of closed subset Z. There is a canonical restriction functor $j^* : \mathcal{D}_{qc}(X) \to \mathcal{D}_{qc}(U)$ and clearly it will be the trivial functor on the thick subategory $\mathcal{D}_{qc,Z}(X)$.

We have following result whose proof is similar to the case of schemes,

Proposition 5.2.1. *The canonical functor induced from the functor j^*, which by abuse of notation we call $j^* : \mathcal{D}_{qc}(X)/\mathcal{D}_{qc,Z}(X) \xrightarrow{\sim} \mathcal{D}_{qc}(U)$ is an equivalence.*

Proof. Using K-injective resolution we can derive j_* to unbounded derived category and we can prove, in a way similar to the scheme case, that it gives the inverse to the functor j^*. □

Definition 5.2.2. Suppose $\mathcal{T}$ is a triangulated category which is closed under formation of arbitrary small coproducts. $\mathcal{T}$ is said to be *compactly generated* if there exists a set T of compact objects (definition 2.4.37) such that $\mathcal{T}$ is a smallest triangulated subcategory containing T which is closed under coproducts and distinguished triangles. Equivalently, $\mathcal{T}$ is compactly generated iff $T^{\perp} := \{x \in \mathcal{T} | Hom_{\mathcal{T}}(t,x) = 0 \text{ for all } t \in T\} = 0$. The set of compact objects T is called *generating set* if further T is closed under suspension or translation.

An example of such triangulated category can be given using derived category of left R-modules and category of quasi-coherent sheaves over superschemes. A result [remark 1.2.2, [34]] of Neeman says that distinguished triangles are preserved under coproducts i.e. in a cocomplete triangulated category coproduct of distinguished triangle is distinguished. Now we shall recall the theorem 2.1 of Neeman[33] which is proved in great generality and is a slight strengthening of theorem 2.1 of Neeman[32].

Theorem 5.2.3 (Neeman[32][33])**.** *Let $\mathcal{S}$ be a compactly generated triangulated category. Let R be a set of compact objects of $\mathcal{S}$ closed under suspension. Let $\mathcal{R}$ be the smallest full subcategory of $\mathcal{S}$ containing R and closed with respect to coproducts and triangles. Let $\mathcal{T}$ be the Verdier quotient $\mathcal{S}/\mathcal{R}$. Then we know :*

1. *The category $\mathcal{R}$ is compactly generated, with R as a generating set.*
2. *If R happens to be a generating set for all of $\mathcal{S}$, then $\mathcal{R} = \mathcal{S}$.*
3. *If $R \subset \mathcal{R}$ is closed under the formation of triangles and direct summands, then it is all of $\mathcal{R}^c$. In any case $\mathcal{R}^c = \mathcal{R} \cap \mathcal{S}^c$.*

4. *The induced functor $F : \mathcal{S}^c/\mathcal{R}^c \to \mathcal{T}^c$ is fully faithful and every object of $\mathcal{T}^c$ is isomorphic to direct summand of image of the functor F. In particular, if $\mathcal{T}^c$ is an idempotent complete then we get an equivalence from idempotent completion $\widetilde{\mathcal{S}^c/\mathcal{R}^c}$ to the triangulated category $\mathcal{T}^c$.*

In our particular situation we take $\mathcal{S} := \mathcal{D}_{qc}(X), \mathcal{R} := \mathcal{D}_{qc,Z}(X)$ and as we proved above in 5.2.1 the quotient will be $\mathcal{T} := \mathcal{D}_{qc}(U)$. We shall now prove following result which will provide all hypothesis required for the application of Neeman's theorem.

Proposition 5.2.4. *The following statements are true for any split superscheme $(X, \mathcal{O}_X)$*

1. *The triangulated category $\mathcal{D}_{qc}(X)$ is closed under the formation of arbitrary small coproducts.*

2. *The triangulated category $\mathcal{D}_{qc}(X)$ is a compactly generated category.*

3. *$\mathcal{D}_{qc,Z}(X)^c \simeq \mathcal{D}_Z^{per}(X)$ for any closed subset Z of X.*

Proof. Proof of 1. This is similar to the scheme case, as in example 1.3 of Neeman[33]. *Proof of 2.* Suppose $T \subset \mathcal{D}_{qc}(X)$ denotes the set of objects obtained by taking the image of all perfect complexes of $\mathcal{O}_{X_{rd}}$ under the functors i_{rd} and Π applied in that order. Let $\mathcal{F} \in \mathcal{D}_{qc}(X)$. Since every unbounded complex of $\mathcal{O}_X$-modules over a superscheme X has K-flat resolution, we can assume that $\mathcal{F}$ is a K-flat. Now using the tower in the proof of 5.1.4 of structure sheaf $\mathcal{O}_X$ we have following tower for $\mathcal{F} \in \mathcal{D}_{qc}(X)$,

$$\begin{tikzcd}
\mathcal{F} \arrow[dr] & & \mathcal{G}^1 \arrow[ll] & \cdots & \mathcal{G}_{n-1} \arrow[dr] & & \mathcal{G}_n \arrow[ll] \arrow[dr, equal] & \\
& \mathcal{F}_1 \arrow[ur, dashed] & & \cdots & & \mathcal{F}_{n-1} \arrow[ur, dashed] & & \mathcal{F}_n
\end{tikzcd} .$$

The base of above tower,$\mathcal{F}_i := \mathcal{F} \otimes_{\mathcal{O}_X} \Pi^i\Lambda^i(\mathcal{V}) \in Im(\mathbf{i_{rd}})$, is generated by objects of the set T. Hence every object $\mathcal{F} \in \mathcal{D}_{qc}(X)$ is generated by the set T. It is now enough to prove that all objects of the set T are compact in $\mathcal{D}_{qc}(X)$. Since Π commutes with coproducts it is enough to prove compactness of the image of the functor $\mathbf{i_{rd}}$ restricted to compact objects. Let $\mathcal{S}$ be image of a $\mathcal{O}_{X_{rd}}$ perfect complex. We want to prove that $\mathrm{Hom}(\mathcal{S}, \_)$ commutes with small coproducts, that is,

$$\mathrm{Hom}(\mathcal{S}, \bigoplus_{\alpha\in\Lambda} \mathcal{F}_\alpha) \simeq \bigoplus_{\alpha\in\Lambda} \mathrm{Hom}(\mathcal{S}, \mathcal{F}_\alpha).$$

Considering above tower for each $\mathcal{F}_\alpha$ we get coproduct of tower as above. Using remark that small coproducts preserve distinguished triangles, [remark 1.2.2, [34]], we get tower of distinguished triangles for $\bigoplus_{\alpha\in\Lambda}\mathcal{F}_\alpha$. If we denote by $\mathcal{F}_{\alpha,i}$ the lower terms of the corresponding towers then we have following isomorphism using functor $\mathbf{i_{rd}}$

$$\mathrm{Hom}(\mathcal{S}, \bigoplus_{\alpha\in\Lambda}\mathcal{F}_{\alpha,i}) \simeq \bigoplus_{\alpha\in\Lambda}\mathrm{Hom}(\mathcal{S}, \mathcal{F}_{\alpha,i}).$$

Using *dévissage* the proof follows from long exact sequence associated to $\mathrm{Hom}(\mathcal{S}, \_)$ and five lemma. *Proof of 3.* It is enough to prove that all perfect complexes are compact objects. Indeed, the full subcategory of perfect complexes is closed under triangles and direct summands as in the case of schemes. Hence by taking R to be all perfect complexes the above result of Neeman proves that all compact objects are perfect complexes. Now to prove that every perfect complex is a compact object we have to first observe the following,

$$(H^0(\mathcal{RH}om(\mathcal{F}, \mathcal{G})))^0 = Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

Here $\mathcal{RH}om(\mathcal{F}, \mathcal{G})$ is the (internal) homomorphism between $\mathcal{F}$ and $\mathcal{G}$. Rest of the proof is similar to the proof given in example 1.13 of Neeman[33]. □

Using the above result it is easy to deduce following corollary,

Corollary 5.2.5. *Given a split superscheme $(X, \mathcal{O}_X)$ we have an equivalence of tensor triangulated categories,* $\widetilde{j^*} : \widetilde{\mathcal{D}^{per}(X)/\mathcal{D}^{per}_Z(X)} \xrightarrow{\sim} \mathcal{D}^{per}(U)$.

Proof. In the set up of theorem 5.2.3, suppose $\mathcal{R} = \mathcal{D}_{qc,Z}(X)$, $\mathcal{S} = \mathcal{D}_{qc}(X)$ and $\mathcal{T} = \mathcal{D}_{qc}(U)$. Then the propositions 5.2.1 and 5.2.4 imply that the conditions for theorem 5.2.3 are satisfied. Therefore, by 5.2.3(4), $\widetilde{j^*}$ is an equivalence. This proves the result as $\widetilde{j^*}$ is a tensor functor. □

As in Balmer[3] we shall use this localization result to give a relation between structure sheaves. Balmer[3] has defined structure sheaf of $\mathrm{Spc}(\mathcal{K})$ for any tensor triangulated category $\mathcal{K}$ as a sheaf associated to the presheaf given by $U \mapsto End_{\mathcal{K}/\mathcal{K}_Z}(1_U)$ where U is an open set and $1_U \in (\mathcal{K}/\mathcal{K}_Z)$ is the image of tensor unit $1 \in \mathcal{K}$. Define $\mathrm{Spec}(\mathcal{D}^{per}(X)) := (\mathrm{Spc}(\mathcal{D}^{per}(X)), \mathcal{O}_{\mathcal{D}^{per}(X)})$ the locally ringed space associated to the tensor triangulated category $\mathcal{D}^{per}(X)$. Now the homeomorphism f defined in 5.1.6 above for a split superscheme gives a map of locally ringed spaces, $f : (X \simeq X^0, \mathcal{O}_{X^0}) \to \mathrm{Spec}(\mathcal{D}^{per}(X))$. Here the map of structure sheaves comes from the identification given in corollary 5.2.5. We have the following result similar to Theorem 6.3 of Balmer[3].

Theorem 5.2.6. *Suppose X is a quasi-compact and quasi-separated split superscheme. The map $f : X^0 \simeq \mathrm{Spec}(\mathcal{D}^{per}(X))$ defined as above is an isomorphism of locally ringed spaces.*

Proof. Using the homeomorphism f it is enough to prove isomorphism of structure sheaves. Hence we can assume that the superscheme is affine. Now using the remark 8.2 of Balmer[2] and localization theorem 5.2.5 we can prove that the induced map of sheaves is an isomorphism. $\square$

Chapter 6

Generalized spaces

This chapter contains our proposal for a broader notion of geometric spaces associated with a tensor triangulated category. Since the Balmer construction of spectrum does not always admit categorical reconstruction therefore we propose these enriched geometric spaces in order to get finer geometric invariants associated to a tensor triangulated categories. In the first section we give some definitions of these enriched geometric spaces. We also compute these for some classes of examples which are studied in earlier chapters of this thesis. We define a categorical reconstruction similar to Balmer's geometric reconstruction. Using our geometric spaces we can prove the categorical reconstruction for some classes of examples which includes the category of perfect complexes over quasi compact and quasi separated schemes. In the second section we give a functor of points approach to the Balmer spectrum. We define a functor from the category of rings to the category of sets for small tensor triangulated category. We also give a definition of a category fibered in groupoids which gives a possible fibered category approach to the Balmer spectrum.

6.1 Definitions of generalized spaces

We shall first define the notion of geometric reconstruction and then we give the definition of categorical reconstruction. Let us denote by $\mathcal{TT}$ the category of all essentially small unital tensor triangulated categories with morphisms given by a unital tensor functors. Also denote the category of all locally ringed spaces by $\mathcal{LRS}$. Balmer constructed a functor between these two categories as follows

$$\mathrm{Spec} : \mathcal{TT} \to \mathcal{LRS} \text{ which takes } \mathcal{D} \mapsto \mathrm{Spec}(\mathcal{D}).$$

There is a functor in other direction defined on the full subcategory of all schemes.

$$\text{Perf} : \mathcal{S}ch \to \mathcal{TT}; X \mapsto \mathcal{D}^{per}(X).$$

Now Balmer's result can be restated as $\text{Spec} \circ \text{Perf} = Id$ when restricted to the full subcategory of quasi compact and quasi separated schemes. The existence of such functors with equality on a full subcategory of geometric origin is an example of geometric reconstruction. Balmer's geometric reconstruction is a generalization of the geometric reconstruction result of Bondal-Orlov [9]. We can ask a similar question of geometric reconstruction for other objects of $\mathcal{LRS}$ which come from geometry like G-schemes and superschemes. Let us denote by $\mathcal{C}$ the category with objects given by pairs consist of a topological space X and a tensor additive category $\mathcal{A}$ with a faithful functor to the category of diagrams of sheaves over X. We propose following definition of *geometric reconstruction*.

Definition 6.1.1 (Geometric reconstruction)**.** A *geometric reconstruction* is defined as a quadruple $(\mathcal{TT}', \mathcal{C}', F, G)$ consist of a subcategory $\mathcal{C}'$ of the category $\mathcal{C}$, a subcategory $\mathcal{TT}'$ of the category $\mathcal{TT}$ and functors $F : \mathcal{TT}' \to \mathcal{C}', G : \mathcal{C}' \to \mathcal{TT}'$ with the relation $F \circ G = Id$ when restricted to some full subcategory of $\mathcal{C}'$ coming from geometry.

Similarly, we propose a definition of *categorical reconstruction*.

Definition 6.1.2 (Categorical reconstruction)**.** A *categorical reconstruction* is defined as a quadruple $(\mathcal{TT}', \mathcal{C}', F, G)$ consist of a subcategory $\mathcal{C}'$ of the category $\mathcal{C}$, a subcategory $\mathcal{TT}'$ of the category $\mathcal{TT}$ and two functors $F : \mathcal{TT}' \to \mathcal{C}', G : \mathcal{C}' \to \mathcal{TT}'$ with the relation $G \circ F = Id$ when restricted to some full subcategory of $\mathcal{TT}'$.

These definitions are used here to understand the various constructions of geometric association which originates from tensor triangulated categories. We would like to understand the strength of functors like Spec in terms of these general definitions. We also give some more examples which fit into these formulations and strengthens the Spec construction. Note that, the category $\mathcal{LRS}$ can be realized in many ways as a full subcategory of the category $\mathcal{C}$. One way is to take the underlying topological space and the tensor abelian category of sheaves of modules. If we take full subcategory of all schemes then we can take the tensor abelian category to be the category of coherent sheaves. Hence the construction of Bondal-Orlov and Balmer can be realized as a particular example of geometric reconstruction. Observe that if we take the full subcategory of $\mathcal{C}$ generated by all smooth projective schemes with canonical (or anti-canonical) sheaf ample then the geometric reconstruction of Bondal-Orlov will become particular case of Balmer geometric reconstruction.

Let X be a smooth quasi projective scheme with an action of a finite group G. The quotient of X by G exists and is denoted as Y. There is a finite map $\pi : X \to Y$ mapping each orbit to a point in the orbit space Y. Consider the additive category $\mathcal{A} := \pi_*(Vb^G(X))$ where the category $Vb^G(X)$ denotes the category of all G-equivariant vector bundles on X. We can push the tensor structure from the additive category $Vb^G(X)$ as follows,

$$\pi_*(\mathcal{F}) \boxtimes \pi_*(\mathcal{G}) := \pi_*(\mathcal{F} \otimes \mathcal{G}).$$

In particular, if we take the trivial action of the group G then the additive category $\mathcal{A}$ can be realized as a diagram (or quiver) of vector bundles on X. Here the diagram associated with group G is given by one vertex and loops parametrize by elements of the group G which satisfies some relations coming from group operations. We had seen before that the Balmer reconstruction applied to the category $\mathcal{D}^G(X)$ with trivial action of group G on X gives back only X. To generalize it slightly define the functor, say F, from the category $\mathcal{D}^G(X)$ to be a pair (Spec($\mathcal{D}^G(X), \mathcal{C}\mathfrak{oh}^G(X)$) i.e. Balmer's construction with the standard t-structure. We can take $\mathcal{D}^b$ to be the functor sending pair $(X, \mathcal{A})$ to tensor triangulated category $\mathcal{D}^b(\mathcal{A})$ in other direction. Take the subcategory $\mathcal{TT}'$ to be all tensor triangulated categories $\mathcal{D}^G(X)$ associated to smooth variety X and finite group G with trivial action on X. And take subcategory $\mathcal{C}'$ to be category of all pairs containing smooth variety X and abelian category $\mathcal{C}\mathfrak{oh}^G(X)$. With these notations its clear we have both geometric as well as categorical reconstruction. In this particular example the pair $(X, \mathcal{C}\mathfrak{oh}^G(X))$ can be seen as the generalized space associated with the triangulated category $\mathcal{D}^G(X)$. Hence as a **first attempt** to define the finer geometric object associated with a tensor triangulated category $\mathcal{D}$, we have following definition of generalized space,

Definition 6.1.3 (Generalized space)**.** A *generalized space associated to an essentially small tensor triangulated category $\mathcal{D}$ is a pair consist of the topological space* Spc($\mathcal{D}$) *and a tensor abelian category $\mathcal{A}$ realized as a tensor abelian category of sheaves of diagrams over* Spc($\mathcal{D}$) *which generates the category $\mathcal{D}$ i.e. $\mathcal{D} \simeq \mathcal{D}^b(\mathcal{A})$ as a tensor triangulated category.*

So if we have categorical reconstruction in a sense of definition above then we get a generalized space associated to a tensor triangulated category. Therefore we have found at least one class of examples containing category of schemes where such generalized space do exist. One way to get the categorical reconstruction in some cases is to take a tensor compatible t-structure which is realized as sheaves of diagram over $\mathrm{Spc}(\mathcal{D})$. Here a tensor compatible t-structure is defined as a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ with following properties,

$$\mathcal{D}^{\leq n} \otimes \mathcal{D}^{\leq m} \subseteq \mathcal{D}^{\leq (n+m)}.$$

Now as an **second attempt** to get the finer geometric invariant of tensor triangulated category, we have following definition,

Definition 6.1.4 (Generalized space)**.** A generalized space associated to an essentially small tensor triangulated category $\mathcal{D}$ is a pair consist of topological space $\mathrm{Spc}(\mathcal{D})$ and a tensor additive category $\mathcal{A}$ adapted to tensor structure, realized as a category of sheaves of diagram over $\mathrm{Spc}(\mathcal{D})$ and an equivalence $\mathcal{D} \simeq K^b(\mathcal{A})$ as a tensor triangulated category.

In example where tensor triangulated category is $\mathcal{D}^G(X)$ with trivial action of group G on smooth variety X, we have $(\mathrm{Spc}(\mathcal{D}^G(X)), Vb^G(X))$ as a generalized space associated to the category $\mathcal{D}^G(X)$. Also it is immediate from the definition that using functor K^b in place of $\mathcal{D}^b$ gives the categorical reconstruction.

6.2 Functor of points approach to spectrum

In this section we shall give functor of points approach to the Balmer spectrum for small categories. We shall first define the set valued contra-variant functor from the category of all finitely generated commutative rings, $\mathcal{R}\mathfrak{ing}$. We shall give some known examples coming from our earlier works.

6.2.1 Categorical spectrum functor

We first give the definition of Balmer spectrum functor. Balmer defined a locally ringed space associated to an essentially small tensor triangulated categories. We can associate a functor

$$h_{\mathrm{Spec}(\mathcal{D})} : \mathcal{R}\mathfrak{ing} \to \mathcal{S}\mathfrak{et}$$

which associates to a ring R the set $\mathrm{Hom}_{\mathcal{LRS}}(\mathrm{Spec}(R), \mathrm{Spec}(\mathcal{D}))$. This gives a presheaf on the fpqc site $\mathcal{R}\mathfrak{ing}$. We shall call this functor the Balmer spectrum functor. We can now define the other set valued functor which will be like a categorical representable functor of a tensor triangulated category $\mathcal{D}$. We'll call it the categorical spectrum functor. To get a set valued functor we shall restrict to small tensor triangulated categories.

Definition 6.2.1. The categorical spectrum functor $h_{\mathcal{D}} : \mathcal{R}\mathfrak{ing} \to \mathcal{S}\mathfrak{et}$ associated to a small tensor triangulated category $\mathcal{D}$ is defined as $h_{\mathcal{D}}(R) := \mathrm{Hom}^{\otimes}(\mathcal{D}, \mathcal{D}_R)/\sim$. Here $\mathrm{Hom}^{\otimes}(\mathcal{D}, \mathcal{D}_R)$ represents the collection of all covariant tensor exact functors. We say two functors F_1 and F_2 are equivalent, denoted as $F_1 \sim F_2$, if there exists a natural isomorphism between them.

We prove following lemma which gives well-definedness of the above association.

Lemma 6.2.2. *The functor $h_{\mathcal{D}}$ is well-defined and hence gives a presheaf on the category of all schemes.*

Proof. It is enough to prove that the functor $h_{\mathcal{D}}$ is well defined as it will give set valued presheaf on the category $\mathcal{R}\mathfrak{ing}^{\circ}$. This will give presheaf on the category of all affine schemes. As usual this will determine the presheaf on the category of all schemes, see [14] for more details. Any ring homomorphism $f : R \to S$ will induce the tensor exact functor $f^* : \mathcal{D}_R \to \mathcal{D}_S$ given by $M^{\cdot} \mapsto M^{\cdot} \otimes_R S$. This will induce the following map between morphisms,

$$h_{\mathcal{D}}(R, S) : \mathrm{Hom}(R, S) \to \mathrm{Hom}(h_{\mathcal{D}}(R), h_{\mathcal{D}}(S)); f \mapsto (f^* \circ \_).$$

Here the set map $f^* \circ \_ : h_{\mathcal{D}}(R) \to h_{\mathcal{D}}(S)$ is defined as $F \mapsto f^* \circ F$. Observe that a natural isomorphism $\eta : F_1 \to F_2$ will induce the natural isomorphism $f^*(\eta) : f^* \circ F_1 \to f^* \circ F_2$. Hence this map is well defined and functorial. This completes the proof. $\square$

Recall, for every scheme X there is a presheaf associated to it defined by $R \mapsto \chi(R) := \mathrm{Hom}(\mathrm{Spec}\, R, X)$, here $\chi(R)$ is the R valued points of the scheme X, see [14]. Grothendieck proved following result,

Theorem 6.2.3 (Grothendieck)**.** *The presheaf χ associated to any scheme X is a sheaf on the fpqc Grothendieck site $\mathcal{R}\mathfrak{ing}$.*

If the Balmer spectrum functor is represented by a scheme then it will be an fpqc sheaf on Grothendieck site $\mathcal{R}\mathfrak{ing}$. Our results show that for the case of G-scheme and superscheme the Balmer spectrum functor is represented by a scheme. Similarly we can ask for the fpqc sheaf property for the categorical spectrum functor. We shall prove in the case of $\mathcal{D}_R$ that the categorical spectrum functor is isomorphic to the Balmer spectrum functor and hence it will be an fpqc sheaf. Since the Balmer construction is functorial we can get a morphism between both pre-sheaves. Define a set map for each ring R as follows,

$$\eta_R : h_{\mathcal{D}} \to h_{\mathrm{Spec}(\mathcal{D})}; F \mapsto \mathrm{Spec}(F).$$

Proposition 6.2.4. *The association η_R defines a well defined natural transformation between these two functors.*

Proof. Using the result [Corollary 3.7, Balmer[3]] we get that two equivalent functors gives the same image and hence η is well defined. If $f : R \to S$ is any morphism then we can get a functor $f^* : \mathcal{D}_R \to \mathcal{D}_S$. Now to prove that η is natural transformation, it is enough to prove that following diagram is commutative,

$$\begin{tikzcd} h_{\mathcal{D}}(R) \arrow[r, "\eta_R"] \arrow[d, "h_{\mathcal{D}}(f)"] & h_{\mathrm{Spec}(\mathcal{D})}(R) \arrow[d, "h_{\mathrm{Spec}(\mathcal{D})}(f)"] \\ h_{\mathcal{D}}(S) \arrow[r, "\eta_S"] & h_{\mathrm{Spec}(\mathcal{D})}(S) \end{tikzcd} .$$

But this follows from the functoriality and the fact that for any ring morphism $f : R \to S$ we have $\mathrm{Spec}(f) = \mathrm{Spec}(f^*)$ which will be proved in next lemma. $\square$

Lemma 6.2.5. *If $f : R \to S$ is any ring morphism then* $\mathrm{Spec}(f) = \mathrm{Spec}(f^*)$*. Here the* Spec *on left is usual spectrum map of affine schemes and the* Spec *on right is spectrum of Balmer. Moreover there is a isomorphism between* $\mathrm{Hom}(R, S)$ *and* $h_{\mathcal{D}_R}(S) = \mathrm{Hom}(\mathcal{D}_R, \mathcal{D}_S)/\sim$*.*

Proof. Since for affine schemes any morphism is completely determined by induced map between global sections of structure sheaves. Therefore it is enough to prove that the map induced by $\mathrm{Spec}(f^*)$ on global sections is the same as the ring homomorphism f. But by definition the global sections of the structure sheaf are the same as the endomorphism ring of the tensor unit. Hence it is enough to prove that the ring homomorphism induced by the functor f^* is f. But this is clear as f^* is nothing but functor given by tensoring with the R module S via the ring map f.

We shall prove the bijection between ring morphisms and unital tensor functors upto isomorphisms. First observe that any tensor functor $F : \mathcal{D}_R \to \mathcal{D}_S$ gives a ring homomorphism between endomorphisms of units R and S as follows,

$$F(R, R) : R \simeq \mathrm{Hom}(R, R) \to \mathrm{Hom}(S, S) \simeq S; \alpha \mapsto \alpha \otimes_R id_S.$$

We'll prove that given any functor F is completely determined by the ring map $F(R, R)$. This will give the required bijection. If we can define the additive functor from the category of all projective R-modules to projective S-modules then it will uniquely determine the additive functor from $\mathcal{D}_R := K^b(Proj - R)$ to $\mathcal{D}_S$. Since the functor F is unital and additive therefore it will send any R^n to S^n for each positive integer n. We know the map

$f : R \simeq \mathrm{Hom}(R, R) \to \mathrm{Hom}(S, S)$. This will give the map $F(R^n, R^m)$ for each m and n using additivity. Now we can realize every projective module via a projector $\pi : R^n \to R^n$ i.e. $\pi^2 = \pi$. A morphism between any two projector is same as following commutative diagram,

$$\begin{tikzcd} R^n \arrow[r, "\pi"] \arrow[d, "\alpha"] & R^n \arrow[d, "\alpha"] \\ R^m \arrow[r, "\pi'"] & R^m \end{tikzcd} .$$

Hence it gives the uniquely defined additive functor up to isomorphism for each ring morphism f. □

It follows from above lemma that the categorical spectrum functor and the Balmer spectrum functor coincide for every $\mathcal{D}_R$ where R is a commutative Noetherian ring. In case of $\mathcal{D}_X$ for quasi compact and quasi separated schemes it will be interesting to know more information about natural transformation η. Since for the case of smooth quasi-projective G-schemes and split superschemes we know that Balmer spectrum functor is an fpqc sheaf. Therefore η will induce a map from the fpqc sheafification of the categorical spectrum functor and the Balmer spectrum functor. So, it will be interesting to know the relation between Balmer spectrum functor and this sheafified categorical spectrum functor which we hope to pursue in future.

Now we can associate a functor with values in groupoids. Define the category $\mathrm{Hom}(\mathcal{D}, \mathcal{D}')$ for small tensor triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ to be a groupoid with objects given by collection of all unital tensor exact functors from $\mathcal{D}$ to $\mathcal{D}'$ and morphisms given by natural isomorphisms. Using this category we can form a subcategory, for a fixed $\mathcal{D}$, of the category of groupoids. An object of this category is the collection of all functors from $\mathcal{D}$ to $\mathcal{D}'$ for every tensor triangulated category $\mathcal{D}'$ and 1-morphisms are given by functors from $\mathcal{D}'$ to $\mathcal{D}''$ which will give map from $\mathrm{Hom}(\mathcal{D}, \mathcal{D}')$ to $\mathrm{Hom}(\mathcal{D}, \mathcal{D}'')$. The 2-morphisms are given by natural isomorphisms between any two functors from $\mathcal{D}'$ to $\mathcal{D}''$. Define the functor $\mathfrak{h}_{\mathcal{D}}(R) := \mathrm{Hom}(\mathcal{D}, \mathcal{D}_R)$ from the category $\mathcal{R}\mathfrak{ing}$ to the category of all groupoids. We can think the category $\mathcal{R}\mathfrak{ing}$ as 2-category by taking identity as a 2-morphism, in a sense of [Appendix B, Gomez[17]]. Hence we get following result.

Proposition 6.2.6. *The functor $\mathfrak{h}_{\mathcal{D}} : \mathcal{R}\mathfrak{ing} \to \mathcal{G}\mathfrak{roupoid}$ between 2-categories is a 2-functor. Hence $\mathfrak{h}_{\mathcal{D}}$ is a presheaf of groupoids.*

Proof. As defined above any object R goes to the category $\mathrm{Hom}(\mathcal{D}, \mathcal{D}_R)$. Any ring morphism f (or 1-morphism) in 2-category $\mathcal{R}\mathfrak{ing}$ goes to functor

$$f^* : \mathrm{Hom}(\mathcal{D}, \mathcal{D}_R) \to \mathrm{Hom}(\mathcal{D}, \mathcal{D}_S); F \mapsto f^* \circ F.$$

Any 2-morphism in $\mathcal{R}\mathfrak{ing}$ is just the identity and hence it goes to the identity natural isomorphism. If $f : R \to S$ and $g : S \to T$ are any two ring morphisms then there is a canonical isomorphism between two functors $(g \circ f)^*$ and $(g^* \circ f^*)$ defined from $\mathcal{M}\mathfrak{od}(R)$ to $\mathcal{M}\mathfrak{od}(S)$. This canonical isomorphism will extend to an isomorphism of $(g \circ f)^{*}$[1] and $g^* \circ f^*$ defined from $\mathcal{D}_R$ to $\mathcal{D}_S$. Now with this notation it is easy to see that the functor $\mathfrak{h}_{\mathcal{D}}$ satisfies other properties of 2-functors, see [Appendix B, Gomez[17]] for the definitions. $\square$

Hence using the above presheaf we shall get a category fibered in groupoids over $\mathcal{R}\mathfrak{ing}$. It is interesting to know whether $\mathfrak{h}_{\mathcal{D}}$ is pre-stack (or stack) on the fpqc site $\mathcal{R}\mathfrak{ing}$. We would like to pursue the relation between the 2-functor $\mathfrak{h}_{\mathcal{D}}$ and Balmer spectrum functor in future.

[1] Here we used the same notation for the left derived functor of the pullback functor as before.

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