Study of Gaussian Channels:

Classification of One Mode Bosonic Channels,

Kraus Representation and a Result From ESD

Author:

Tanmay Singal,

Junior Research Fellow,

The Institute of Mathematical Sciences,

Taramani, Chennai

Supervisor:

Sibasish Ghosh

A thesis submitted to the

Board of Studies in Physical Science Discipline

In partial Fulfillment of Requirements

For the Degree of

Master of Science

OF

HOMI BHABA NATIONAL INSTITUTE

July 12, 2011

Bonafide Certificate

Certified that this dissertation titled "**Study of Gaussian Channels:Classification of One Mode Bosonic Channels, Kraus Representation and a Result From ESD**" is the bonafide work of Mr.Tanmay Singal who carried out the project under my supervision.

Sibasish Ghosh,

Theoretical Physics,

Institute of Mathematical Sciences,

Chennai

Date:

Acknowledgements

I would like to thank my advisor Sibasish Ghosh for his guidance through the wonderful journey that has culminated in this thesis. He has been instrumental in helping with the work and in providing advice and motivation. The fruitful discussions have helped me understand the problem and will go a long way in my eduation.

I also wish to thank Dibyakrupa Sahu for helping me with latex and Professor R. Simon for helping me with some doubts I had.

Abstract

This review starts with the basic Hilbert space structure for continuous variable systems, explaining the pre-requisites to be able to define a Gaussian state. After that a preliminary study of Gaussian channels is undertaken. The criteria for a quantum channel being Gaussian is laid out. In accordance with these criteria a classification of one-mode bosonic Gaussian channels with a single-mode environment is given. The Kraus representation for the various channels is then obtained and used to verify some of the pre-existing properties of the various channels (non-classicality breaking, entanglement breaking, extremality of the channel, etc). After this the entanglement sudden death (ESD) of a two-mode Gaussian state under the action of a Gaussian channel is studied. This channel comprises of two mutually exclusive channels, each of which is acting on one of the two-modes of the Gaussian states. The channel action comprises of interaction of the mode with a thermal bath. Both the channels, interacting with the two modes separately are at the same temperature. In the study for ESD, it is discovered that for all non-zero temperatures, all entangled-two-mode Gaussian states undergo ESD at some time or the other. For the zero temperature case S. Goyal and S. Ghosh have proved that ESD won't occur for a set kind of states. We have tried to generalize this result i.e. found another set of two mode Gaussian states which also won't undergo ESD under the channel action. It is desired to know if there are some two-mode Gaussian states which will undergo ESD at zero temperature. We find that there are some states which undergo ESD while interacting with a zero-temperature thermal bath.

Contents

1 Basic Hilbert Space Structure

Consider an infinitely dimensional quantum system (e.g. a one-dimensional simple harmonic oscillator) associated with two operators $\hat{p}$ and $\hat{q}$ having the commutation relation given by $[\hat{q}, \hat{p}] = \iota\hbar$[1].

Here, $\hat{q}$ represents the position operator and $\hat{p}$ the momentum operator.

From these two we can define two more operators given by:

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{q} + \iota\hat{p}) \tag{1}$$

and

$$\hat{a^\dagger} = \frac{1}{\sqrt{2}}(\hat{q} - \iota\hat{p}) \tag{2}$$

which will have the following commutation relation:

$$[\hat{a}, \hat{a^\dagger}] = 1 \tag{3}$$

The number operator is then given by $\hat{N} = \hat{a}\hat{a^\dagger}$.

Any quantum state is expressible in terms of a density operator, ρ with the following properties: $\hat{\rho} \geq 0$, $\hat{\rho} = \hat{\rho}^\dagger$, $Tr[\hat{\rho}] = 1$.

Hermiticity of $\hat{\rho}$ implies that it has a spectral decomposition:

$\hat{\rho} = \sum_{i=1}^{n} p_i \mid i\rangle\langle i \mid$ where $1 \geq p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$ and $\langle i \mid j\rangle = \delta_{i,j}$

Consider a state, ρ of an n-mode quantum mechanical radiation field.

Define a column vector as follows:

$$\hat{R} = \begin{pmatrix} \hat{q_1} \\ \hat{p_1} \\ \hat{q_2} \\ \hat{p_2} \\ . \\ . \\ . \\ \hat{q_n} \\ \hat{p_n} \end{pmatrix} \tag{4}$$

[1] We take $\hbar = 1$ for convenience.

where $\hat{q}_i$ and $\hat{p}_i$ denote the position and momentum operators acting on the ith mode.Thus the components of $\hat{R}$ v.i.z. $\hat{R}_1$,$\hat{R}_2$,$\hat{R}_3$,$\hat{R}_4$,...,$\hat{R}_{2n}$ denote respectively $\hat{q}_1$,$\hat{p}_1$,$\hat{q}_2$,$\hat{p}_2$,...,$\hat{p}_n$.The Hermitian conjugate of $\hat{R}$ is given by:

$$\hat{R}^\dagger = (\hat{q}_1\ \hat{p}_1\ \hat{q}_2\ \hat{p}_2\ \cdots\ \hat{q}_n\ \hat{p}_n) \tag{5}$$

The mean vector is given by:

$$\langle R\rangle = \begin{pmatrix} Tr(\hat{\rho}\hat{q}_1) \\ Tr(\hat{\rho}\hat{p}_1) \\ Tr(\hat{\rho}\hat{q}_2) \\ Tr(\hat{\rho}\hat{p}_2) \\ . \\ . \\ . \\ Tr(\hat{\rho}\hat{q}_n) \\ Tr(\hat{\rho}\hat{p}_n) \end{pmatrix} = \begin{pmatrix} \langle\hat{q}_1\rangle \\ \langle\hat{p}_1\rangle \\ \langle\hat{q}_2\rangle \\ \langle\hat{p}_2\rangle \\ . \\ . \\ . \\ \langle\hat{q}_n\rangle \\ \langle\hat{p}_n\rangle \end{pmatrix} \tag{6}$$

Consider the matrix V of correlations defined by:

$$V_{ij} = Tr(\hat{\rho}(\hat{R}_i - \langle\hat{R}_i\rangle)(\hat{R}_j - \langle\hat{R}_j\rangle)) \tag{7}$$

One can always displace the mean values of the respective position and momentum operators so as to reduce them to zero via displacement operators for any state. In other words, given a state $\hat{\rho}$, we replace the observables $\hat{R}_j$ by $\hat{R}_j - \langle\hat{R}_j\rangle \equiv \hat{R}_j - Tr(\hat{\rho}\hat{R}_j)$. From (7) we can see that this doesn't change the values taken by the second order moments in anyway. From now on we'll assume this to be true. Further we have:

$$V_{ij} = Tr(\hat{\rho}\hat{R}_i\hat{R}_j) = \frac{1}{2}[Tr(\hat{\rho}\{\hat{R}_i, R_j\} - [\hat{R}_i, \hat{R}_j])] = \gamma_{ij} + \frac{1}{2}\iota\Omega_{ij} \tag{8}$$

Here, $\{\hat{R}_i, \hat{R}_j\} = \hat{R}_i\hat{R}_j + \hat{R}_j\hat{R}_i$ is the anti-commutation relation between $\hat{R}_i$ and $\hat{R}_j$ and $[\hat{R}_i, \hat{R}_j] = \hat{R}_i\hat{R}_j + \hat{R}_j\hat{R}_i$ is the commutation relation between the two. Given that $\langle\hat{R}\rangle$ has been displaced to 0 we get:

$$\gamma_{ij} = \frac{1}{2}Tr(\hat{\rho}\{\hat{R}_i, \hat{R}_j\}) \tag{9}$$

and

$$\Omega_{ij} = -\iota Tr(\hat{\rho}[\hat{R}_i, \hat{R}_j]) \tag{10}$$

Here (γ_{ij}) is called the covariance matrix. It is a real and symmetric matrix. It basically holds the information regarding the second moments of the quantum state. In the following it's shown that (γ_{ij}) is positive semidefinite too. Let $v = \begin{pmatrix} v_1 & v_2 & ... & v_{2n} \end{pmatrix}^T \in R^{2n}$be a vector. Then:

$$v^T \gamma v = \sum_{i,j=1}^{2n} v_i \gamma_{ij} v_j = \sum_{i,j=1}^{2n} v_i v_j \langle \hat{R}_i \hat{R}_j \rangle = \langle (\sum_{i=1}^{2n} v_i \hat{R}_i)^\dagger (\sum_{j=1}^{2n} v_j \hat{R}_j) \rangle \geq 0$$

So γ is positive semi-definite.

Ω is called the commutation matrix and has the form:

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 & ... \\ -1 & 0 & 0 & 0 & ... \\ 0 & 0 & 0 & 1 & ... \\ 0 & 0 & -1 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \end{pmatrix} \tag{11}$$

The evolution of a closed quantum system is obtained by acting unitary transformation on the state.In the Heisenberg picture this transformation would amount to obtaining a new set of position and momentum operators for the respective modes as respective functions of the older ones. The functions should be such that the commutation matrix,Ω (with respect to the new position and momentum operators) would remain invariant. The set of all possible such transformations forms a group called the symplectic group, $Sp(2n, R)$. Since we are working with the $\hat{q}_i$ and $\hat{p}_i$ operators of an n-mode Hilbert space the representation of $Sp(2n, R)$ would be real i.e. the $Sp(2n, R)$ matrices which would effect all possible transformations (within a closed system) would have real entries.

The defining representation for a matrix element of $Sp(2n, R)$ matrix group is as follows:

Let $S \in Sp(2n, R)$. Then $S\Omega S^T = \Omega$

The change in the position-momentum operators is given by $\hat{R}' = S\hat{R}$.

Then the new commutation matrix would look like:

$\Omega'_{ij} = -\iota\langle[\hat{R}'_i, \hat{R}'_j]\rangle = -\iota\langle[(S\hat{R})_i, (S\hat{R})_j]\rangle = -\iota \sum_{k,l=1}^{2n} S_{ik} \langle[\hat{R}_k, \hat{R}_l]\rangle S^T_{lj} = -\iota(S\Omega S^T)_{ij} = \Omega_{ij}$.

Hence a such a transformation would not change the commutation relations between the new position and momentum operators of the respective 'new' modes.

Classical systems allow $\langle(\hat{q}_i)^2\rangle$ and$\langle(\hat{p}_i)^2\rangle$ to take any values i.e., there is no constraint that these two quantities need to satisfy to represent a possible configuration of the classical system. In quantum systems, the Heisenberg's uncertainty principle imposes constraints in the following form[2]:

$$\langle\hat{q}^2\rangle\langle\hat{p}^2\rangle - (\frac{1}{2}\langle\{\hat{q},\hat{p}\}\rangle)^2 \geq \frac{1}{4} \tag{12}$$

Hence, if the covariance matrix has the following form:

$$\gamma = \begin{pmatrix} A_{11} & A_{12} & A_{13} & ... \\ A_{21} & A_{22} & A_{23} & ... \\ A_{31} & A_{31} & A_{33} & ... \\ ... & ... & .... & ... \end{pmatrix} \tag{13}$$

where A_{ij} is a 2×2 block matrix. Then (13) implies $Det(A_{ii}) \geq \frac{1}{4}$.This basically amounts to the following:

$$\langle q_i^2\rangle\langle p_i^2\rangle - (\langle\frac{1}{2}\{q_i,p_i\}\rangle)^2 \geq \frac{1}{4} \tag{14}$$

Williamson's Theorem[3]:

Given a $2n\times 2n$ real, symmetric and positive definite matrix γ, there exists a real symplectic matrix $S \in Sp(2n,R)$ such that the transformation the latter effects upon the former is to diagonalize it in such a manner that every eigen value of the former has a degeneracy of 2 i.e.

$$S\gamma S^T = diag(\kappa_1,\kappa_1,\kappa_2,\kappa_2,...,\kappa_n,\kappa_n) \tag{15}$$

where $\kappa_i \geq 0 \; \forall \; i = 1,2,...,n$

The covariance matrix defined above are real, positive and symmetric. Hence Williamson's Theorem guarantees that there exists a symplectic transformation through which the former covariance matrix is transformed to a doubly degenerate diagonal matrix given by (15). We will call the doubly degenerate diagonal matrix so obtained the 'symplectic diagonal matrix' of a given covariance matrix. Also, the eigen spectrum $\{\kappa_1,\kappa_2,...,\kappa_n\}$ of the symplectic diagonal matrix corresponding to a given covariance matrix, are

[2] "Generalized Uncertainty Relations and Coherent and Squeezed States", D. A. Trifonov, http://arxiv.org/abs/quant-ph/0012072
[3] R. Simon, S. Chaturvedi, V. Srinivasan, J. Math. Phys. Vol. 40, 3634 (1999)

known as the latter's 'symplectic eigen spectrum'.

Consider the symplectic diagonal matrix of any covariance matrix:

$$\gamma_d = \begin{pmatrix} \kappa_1 & 0 & 0 & ... & & \\ 0 & \kappa_1 & 0 & 0... & & \\ & & . & & & \\ & & & . & & \\ & & & & . & \\ & & & & & \kappa_n & 0 \\ & & & & & 0 & \kappa_n \end{pmatrix} \tag{16}$$

For the doubly degenerate symplectic diagonal matrix it is easy to make out that the condition (14) amounts to:

$$\kappa_i \geq \frac{1}{2} \tag{17}$$

for $i = 1, 2, ..., n$.

Hence, in the doubly-degenerate symplectic diagonal covariance matrix, the product of the position and momentum variances should be greater than or equal to the Heisenberg uncertainty limit. Applying a symplectic transformation and the resulting intermingling of various modes shouldn't hurt this condition.

The covariance condition (14) can be brought down to a more elegant form[4]:

$$\gamma + \frac{\iota}{2}\Omega \geq 0 \tag{18}$$

where $\Omega = \Omega_1 \oplus \Omega_2 \oplus ...\Omega_n \oplus$ and Ω_i is the commutation matrix for the ith mode i.e. $\Omega_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Any real, positive and symmetric matrix satisfying (16) is a bonafide covariance matrix.

Instead of expressing the covariance matrix γ as second order moments of the $\hat{q}$ and $\hat{p}$ operators we can use the$\hat{a}$ and $\hat{a}^\dagger$ operators to express second order moments.

Define the column vector $\hat{R}^{(c)}$ as follows:

[4]R. Simon, N. Mukunda, B. Dutta, Phys. Rev. A Vol. 49,1567 (1994)

$$\hat{R}^{(c)} \equiv \begin{pmatrix} \hat{a_1} \\ \hat{a_1}^\dagger \\ \hat{a_2} \\ \hat{a_2}^\dagger \\ . \\ . \\ . \\ \hat{a_n} \\ \hat{a_n}^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \iota & 0 & 0 & ... & 0 & 0 \\ 1 & -\iota & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & \iota & ... & 0 & 0 \\ 0 & 0 & 1 & -\iota & ... & 0 & 0 \\ . & & & & & & \\ 0 & 0 & 0 & 0 & ... & 1 & \iota \\ 0 & 0 & 0 & 0 & ... & 1 & -\iota \end{pmatrix} \begin{pmatrix} \hat{q_1} \\ \hat{p_1} \\ \hat{q_2} \\ \hat{p_2} \\ . \\ . \\ . \\ \hat{q_n} \\ \hat{p_n} \end{pmatrix} \tag{19}$$

For the n-mode system, the complex covariance matrix $\gamma^{(c)}$ is hence defined by taking the mean of each matrix element in the following operator-matrix:

$$\hat{R}^{(c)}(\hat{R}^{(c)})^\dagger = \begin{pmatrix} \hat{a_1} \\ \hat{a_1}^\dagger \\ \hat{a_2} \\ \hat{a_2}^\dagger \\ . \\ . \\ . \\ \hat{a_n} \\ \hat{a_n}^\dagger \end{pmatrix} \begin{pmatrix} \hat{a_1}^\dagger \hat{a_1} \hat{a_2}^\dagger \hat{a_2} ... \hat{a_n}^\dagger \hat{a_n} \end{pmatrix} \tag{20}$$

Hence, the complex covariance matrix can be obtained as:

$$\gamma^{(c)} = \Lambda \gamma \Lambda^T \tag{21}$$

where Λ is the matrix on the left on the RHS in (19).

2 Gaussian States

To get to the definition of Gaussian states, we go through the underlying mathematical structure briefly:

A little bit about phase space variables in Quantum Mechanics: In QM, for a single particle endowed with a position and momentum operator to describe it, the domain of the phase space is R^2. One can equivalently describe the system using the annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators defined in (1) and (2) respectively. For the latter mode of description, then the phase space variables comprises of the complex plane. Phase space variables play a different role in quantum mechanics than what they do in classical mechanics. In classical mechanics a point in the phase space assigns a unique value to the various degrees of freedom needed to specify the configuration of the corresponding classical system completely. In comparison in quantum mechanics, no system can have unique values assigned to all the various degrees of freedom due to Heisenberg's uncertainty principle. Thus in QM, phase space variables don't adopt the same role they do in CM. In QM they are used for what is known as the 'Weyl Correspondence' which assigns a bijective function on the phase space of the system to Hermitian operators (see (23). A bit of this correspondence will be explained in the material to come.

Any operator acting on the single-mode Hilbert Space introduced, whether bounded or unbounded, can be expanded as a function of a series of monomials made up of the creation and annihilation operators (and identity operator corresponding to all the zero order terms in the expansion). This can be proved as follows:

Consider any operator, $\hat{F}$ acting on the Hilbert Space. Expand it in the Fock-State basis:[5]

$$
\begin{aligned}
\hat{F} &= \sum_{n,m=0}^{\infty} |n\rangle\langle n|\hat{F}|m\rangle\langle m| \\
&= \sum_{n,m=0}^{\infty} \langle n|\hat{F}|m\rangle \frac{1}{\sqrt{n!}\sqrt{m!}} (\hat{a}^\dagger)^n |0\rangle\langle 0| \hat{a}^m
\end{aligned}
$$

Now $|0\rangle\langle 0| = \exp(-\hat{a}^\dagger \hat{a})$.

A very important class of unitary operators for continuous variable systems are the displacement operators, $\hat{D}(\alpha)$. For a single mode system these are defined as follows:

$$\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^* \hat{a}}, \forall\ \alpha \in R \tag{22}$$

These operators are functions of the phase-space variables.Also, they possess a completeness property, as in, any operator with a finite Hilbert-Schmidt norm can be expanded as a convolution of these displacement operators. This is in accord with the statement made before that any operator can be expanded as a series comprising of annihilation and creation operators.

[5]The Fock state basis is the eigenbasis of the number operator.

$$\hat{F} = \int f(\alpha)\hat{D}(-\alpha)d^2\alpha \tag{23}$$

where $f(\alpha) = Tr[\hat{D}(\alpha)\hat{F}]$ and $\sqrt{\int |f(\alpha)|^2 d^2\alpha} = \sqrt{Tr(\hat{F}^\dagger\hat{F})}$.

We see that when $\hat{F}$ has a finite Hilbert Schimdt norm[6] then the function $f(\alpha)$ will be square-integrable.

Now using the Baker-Campell Hausdorff formula we obtain:,

$$D(\alpha) = e^{-|\alpha|^2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}} = e^{|\alpha|^2}e^{-\alpha^*\hat{a}}e^{\alpha\hat{a}^\dagger} \tag{24}$$

Upon expanding the exponentials in (24) we see that there is a difference in the ordering of the creation and annihilation operators in the various expressions corresponding to the same displacement operator. On the right most of (24), the annihilation operators are set to the left which leaves the creation operators on the right. This kind of ordering is known as anti-normal ordering. In the centre of (24) we see that the creation operators are always on the left and hence the annihilation operators are always on the right. Expanding the expression on the RHS of (22) we see that the annihilation and creation operators are ordered symmetrically i.e. in the expansion, for any monomial of $\hat{a}$ and $\hat{a}^\dagger$ of a particular ordering there is another monomial in which the creation and annihilation operators are swapped, rendering the entire expansion symmetrical with respect to ordering. This kind of ordering in operators is known as symmetrical ordering (or Weyl ordering). Since the creation and annihilation operators don't commute (see (3), to compensate for the different ordering schemes adopted, the expression $e^(\pm |\alpha|^2)$ appears as a coefficient for the normal and anti-normal ordering.

The ordering of the annihilation and creation operators is generalized as follows:

Define ordered displacement operators as:

$$D(\alpha, s) = e^{s|\alpha|^2}\hat{D}(\alpha) \tag{25}$$

The parameter s is called the order parameter.

When s = 1, we get the normally ordered displacement operator:

$$D(\alpha, 1) = e^{|\alpha|^2}D(\alpha) = e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}} \tag{26}$$

As in, all the monomials appearing in the expansion of $D(\alpha, 1)$ are normally ordered.

[6]The Hilbert Schmidt norm for a bounded operator is defined as $\sqrt{Tr(\hat{A}^\dagger\hat{A})}$.

When s = -1, we get the anti-normally ordered displacement operators:

$$(\alpha, -1) = e^{-|\alpha|^2} D(\alpha) = e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \tag{27}$$

where all the monomials in the expansion are anti-normally ordered.

When s = 0, we get the usual displacement operator in the sense it was defined in (22). The corresponding monomials are symmetrically ordered.One can saturate all different kinds of ordering by varying the order-parameter s within and upon the unit circle in the complex plane[7].

Now,

$$\hat{F} = \int Tr[\hat{F}\hat{D}(\alpha)]\hat{D}(\alpha) d^2\alpha = \int Tr[\hat{F}\hat{D}(\alpha, -s)]\hat{D}(\alpha, s) d^2\alpha \tag{28}$$

In (28) we obtain an example of what is known as the Weyl Correspondence. Here an operator $\hat{F}$ is expanded as a convolution of the s-ordered displacement operator, $\hat{D}(\alpha)$. Let $f(\alpha, s) \equiv Tr(\hat{F}\hat{D}(\alpha, -s)) = Tr(\hat{F}\hat{D}(\alpha)e^{-s|\alpha|^2})$. Hence we have an operator to function mapping as follows:

$$\hat{F} \longrightarrow f(\alpha, s) \tag{29}$$

When s is varied below zero, due to the presence of the term $e^(- s|\alpha|^2)$ in $f(\alpha, s)$, the function $f(\alpha, s)$ may not remain square integrable. This is the reason why not all operators of finite Hilbert-Schimdt norm are expandable as a convolution of anti-normally ordered displacement operator.

This can easily be generalized to the multi-mode case where the corresponding displacement operators are merely tensor products of the displacement operators of individual modes.

All density operators defined on the Hilbert Space have a finite Hilbert Schmidt norm. Hence these are expandable as the convolution integral as done in (28). When $\hat{F}$ is a density operator, then the convoluting function is called the state's "characteristic function", $\chi(\alpha)$. This function is called so because it generates moments for the various monomials of the creation and annihilation operators. The usage of different ordering parameter gives one the moments of correspondingly ordered bosonic operators.

The s-ordered-characteristic function is given as:

$$\chi(\alpha, s) = Tr(\hat{\rho}\hat{D}(\alpha, s)) \tag{30}$$

Now, differentiating the displacement operators by the phase-space variables, α and α^* and setting the

[7]See "Ordered Expansions in Boson Amplitude Operators" by K. Cahill and R Glauber, Physical Review Vol. 177, 1857

latter to 0,

$$\{(\hat{a}^\dagger)^n \hat{a}^m\}_s \equiv \frac{\partial^{n+m}\hat{D}(\alpha,s)}{\partial\alpha^n\partial(\alpha^*)^m}|_{|\alpha|=0} \tag{31}$$

Thus we get,

$$\begin{aligned}\frac{\partial^{n+m}\chi(\alpha,s)}{\partial\alpha^n\partial(\alpha^*)^m}|_{|\alpha|=0} =& Tr(\hat{\rho}\{\frac{\partial^{n+m}\hat{D}(\alpha,s)}{\partial\alpha^n\partial(\alpha^*)^m}|_{|\alpha|=0}\}) \\ &= Tr(\hat{\rho}\{(\hat{a}^\dagger)^n\hat{a}^m\}_s) \\ &= \langle\{(\hat{a}^\dagger)^n\hat{a}^m\}_s\rangle\end{aligned}$$

Instead of working with characteristic functions of varying order, we can work with their fourier transforms as well.

The fourier transforms of normally ordered characteristic function is called the Sudarshan Glauber function and is denoted by $P(\xi)$ where ξ denotes the phase space variables as usual.

$$P(\xi) = \int d^2\alpha e^{\xi\alpha^* - \xi^*\alpha}\chi(\alpha,1) \tag{32}$$

Glauber stated a criterion which says that a system is considered classical if the Sudarshan Glauber function, $P(\alpha)$ is non-negative for every point α on the phase space. Equivalent to this is the statement that the covariance matrix defined in (9) satisfies $\gamma - \frac{1}{2}I_{2n} \geq 0$.[8]

The fourier transform of the symmetrically ordered characteristic function is the called the Wigner Distribution function, $W(\xi)$.

$$W(\xi) = \int d^2\alpha e^{\xi\alpha^* - \xi^*\alpha}\chi(\alpha,0) \tag{33}$$

The fourier transform of the anti-normally ordered characteristic function is the called the Husimi function, $Q(\xi)$.

$$Q(\xi) = \int d^2\alpha e^{\xi\alpha^* - \xi^*\alpha}\chi(\alpha,-1) \tag{34}$$

After havin defined the necessary pre-requisites we now move on to what a Gaussian state is.

Gaussian states are those quantum states whose symmetrically ordered characteristic function are Gaus-

[8]See equation (18) in "Inseparability Criterion for Continuous Variable Systems" by Lu-Ming Duan, H. Giedke, J. I. Cirac and Zoller, Phys Rev. Lett. Vol. 84, 2722.

sian functions in the phase space variables.

$$\chi(\alpha, 0) = \exp(-\frac{1}{\sqrt{2}}\alpha^\dagger \gamma^{(c)}\alpha + \iota\langle \hat{R}^{(c)}\rangle \alpha) \tag{35}$$

These are an important class of states, commonly encountered in nature and easy to produce in the laboratory.

3 Entanglement Criteria of Two Mode Gaussian States

Consider a two mode Gaussian state. A Gaussian state is completely defined by its covariance matrix γ and the mean vector $\langle \hat{R}\rangle$. Let one of the modes be denoted by 1 and the other by 2. The necessary and sufficient condition for γ to be a bonafide covariance matrix is given by (18). Consider any bonafide covariance matrix, γ of a two-mode system. This matrix can be brought into a canonical form using local symplectic transformations on both the modes - 1 and 2 in the following way:

$$\gamma_0 = (S_1 \oplus S_2)\gamma({S_1}^T \oplus {S_2}^T) \tag{36}$$

where γ_0 is the following form:

$$\gamma_0 = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & d \\ c & 0 & b & 0 \\ 0 & d & 0 & b \end{pmatrix} \tag{37}$$

This can be done in the following way. Let:

$$\gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \tag{38}$$

$A(B)$ submatrix is the covariance matrix of subsystem 1(2). Hence using Williamson's theorem one can diagonalize this matrix using $S'_{1(2)}$ to turn it into a multiple of I_2. Let's suppose the cross-covariance matrix C changes from C to C'. The cross-covariance matrix C' contains information about the correlations between the two systems. Consider the singular value decomposition of C':

$$C' = O_1 C_0 O_2 \tag{39}$$

where C_0 is a diagonal matrix with and $O_{1(2)} \in SO(2)$. Upon choosing the following:

$$S_{1(2)} = S'_{1(2)} O^{-1}_{1(2)} \tag{40}$$

we get the covariance matrix in the canonical form (36) where $a \geq \frac{1}{2}$, $b \geq \frac{1}{2}$ (see (17).

The necessary and sufficient condition for 4×4 matrix of the form of (36) to be a covariance matrix is then given by:[9]:

$$4(ab - c^2)(ab - d^2) \geq a^2 + b^2 + 2cd - \frac{1}{4} \tag{41}$$

To obtain the conditions above, one needs to find the eigen values of the matrix $\gamma_0 + \frac{\iota}{2}\Omega$ and apply the condition that the obtained eigenvalues be non-negative

(41) has been obtained for a canonical form of the covariance matrix (36). Any 4×4 covariance matrix can be brought to this canonical form by the method described above. Upon expanding the terms in (41) one obtains:

$$a^2b^2 + (\frac{1}{4} - cd)^2 - ab(c^2 + d^2) - \frac{1}{4}(a^2 + b^2) \geq 0 \tag{42}$$

where we see that:$a^2 = DetA$,$b^2 = DetB$,$cd = DetC$ and $ab(c^2 + d^2) = Tr(A\Omega C\Omega B\Omega C^T\Omega)$ when the 2×2 matrices A,B and C are given by (36)

All these quantities ($DetA$,$DetB$,$DetC$ and $Tr(A\Omega C\Omega B\Omega C^T\Omega)$) are invariant under local symplectic transformations i.e. invariant under any transformation $S \in Sp(2, R) \otimes Sp(2, R)$. Now, the eigen values of a matrix don't change sign under an $Sp(2, R) \otimes Sp(2, R)$ transformation[10] Thus $\gamma_0 \geq 0 \Leftrightarrow \gamma \geq 0$. Thus (41) (or (42)) can also be written as:

$$detAdetB + (\frac{1}{4} - detC)^2 - Tr(AJCJBJC^TJ) \geq \frac{1}{4}(detA + detB) \tag{43}$$

This gives the necessary and sufficient criteria for any 4×4 matrix to be a bonafide covariance matrix for some two-mode state.

Peres gave a necessary criteria for the separability of bi-partite density matrices[11]. The necessary criteria states that if the partial transpose of the density matrix with respect to any subsystem is not a bona-fide density matrix, then the system (represented by the density matrix) is necessarily entangled. Hence

[9] "Peres-Horodecki Separability Criterion for Continuous Variable Systems" R. Simon, Phys Rev. Lett. Vol. 84, 2726

[10] Let A be a positive semi-definite $n \times n$ matrix i.e. $v^T Av \geq 0, \forall v \in R^n$. Hence $v^T(SAS^T)v \geq 0 \forall v \in R^n$.

[11] "Separability Criteria for Density Matrices" A. Peres, Phys. Rev. Lett. Vol. 77, 1413

positivity of the partial transpose is a necessary condition for the separability of the state.

For the case of 2-mode gaussian states this was shown to be a sufficient condition by Simon(see footnote (9)). There is a physical interpretation to partial transposition in the case of continuous systems. That is that density-matrix transposition implies time reversal i.e. transposition implies $\hat{p} \longrightarrow -\hat{p}$. Thus in case of partial transposition, the momentum operator of the sub-system being transposed will change signature. In our case let system-2 be transposed. Then resulting change covariance matrix γ (38) is given by:

$$\gamma \longrightarrow diag(1,1,1,-1)\gamma diag(1,1,1,-1) \tag{44}$$

The only term that undergoes any change in (44) is $DetC$ which undergoes a change of signature. Hence the condition of separability of the covariance matrix is as follows:

$$detAdetB + (\frac{1}{4} - |detC|)^2 - Tr(AJCJBJC^T J) \geq \frac{1}{4}(detA + detB) \tag{45}$$

We can see from (45) in case $DetC \geq 0$, the state is necessarily separable. For a two-mode Gaussian state to be entangled (43) would have to be satisfied but not (45).

4 Gaussian Channels

A quantum channel Φ is a completely positive trace preserving map from the set of operators on one Hilbert space to the set of operators on another Hilbert space i.e. $\Phi\colon \mathcal{B}(\mathcal{H_A}) \longrightarrow \mathcal{B}(\mathcal{H_B})$ (where $\mathcal{B}(\mathcal{H})$ denotes the set of all bounded operators acting on the Hilbert Space H and H_A and H_B denote two Hilbert Space) is a quantum channel if[12]:

1. Trace Preserving Condition:-

$$Tr(\hat{F}_A) = Tr(\Phi(\hat{F}_A))\text{for all } \hat{F}_A \in \mathcal{B} \tag{46}$$

2. Positivity Condition:-

$$\hat{F}_A \geq 0 \Rightarrow \Phi(\hat{F}_A) \geq 0 \tag{47}$$

3. Completely Positivity Preserving Condition:- If H_C denotes a third Hilbert space of, say, dimension n

[12]"Quantum Information and Quantum Computation", Nielsen and Chuang, 8.2.4 "Axiomatic Approach to Quantum Operations", page 366

and $\hat{F} \in \mathcal{B}(\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{C}})$ with $\hat{F} \geq 0$. Then:

$$(\Phi \otimes I_n)(\hat{F}) \geq 0 \tag{48}$$

$\forall$ positive integer n and $I_n : \mathcal{B}(\mathcal{H}_{\mathcal{C}}) \longrightarrow \mathcal{B}(\mathcal{H}_{\mathcal{C}})$ is the identity map. A channel can be given a mathematical form by means of what is called a Kraus Represenation which is based on the interaction between the system (called Principal System) and a modelling environment. Since it's the principal system we are interested in there is a certain amount of flexibility with regards to the environment used to obtain the Kraus representation for a channel. A Kraus Representation is a set of operators acting on the input operator to give the output operators. This can be depicted as follows:

Let $\hat{\rho}_s$ be in initial state of the principal system. We assume an environment coupled to the principal system and that the state $\hat{\rho}_{in}$ of the entire system is initially in a product state with some initial pure state of the environment, $|1\rangle_E\langle 1|$:

$$\hat{\rho}_{in} = \hat{\rho}_s \otimes |1\rangle_E\langle 1| \tag{49}$$

Let the joint initial state undergo a unitary evolution, $\hat{U}$ under a Hamiltonian defined on the joint Hilbert space of the principal system and the environment to reach a final state, $\hat{\rho}_{out}$.

$$\hat{\rho}_{out} = \hat{U}\hat{\rho}_{in}\hat{U}^\dagger \tag{50}$$

Consider an ordered orthonormal basis for the environment system featuring $|1\rangle_E$ as a basis-vector, $\{|i\rangle_E\}$. Tracing out the environment system with respect to this basis we get the desired output state for the channel $\hat{\rho'_s}$:

$$\hat{\rho}'_s = \sum_{i=1}^{n} \langle i|\hat{U}|1\rangle \hat{\rho}_s \langle 1|\hat{U}^\dagger|i\rangle = \sum_{i=1}^{n} \hat{A}_i \hat{\rho}_s \hat{A}_i^\dagger \tag{51}$$

where the set of operators $\{\hat{A}_i\}$ are called the Kraus Operators and these operators satisfy the following trace-preserving condition:

$$\sum_{i=1}^{n} \hat{A}_i^\dagger \hat{A}_i = I_n, \text{ the } n \times n \text{ identity matrix.} \tag{52}$$

Given that we have found a set of Kraus Operators which satisfy (52), the map induced by the set of Kraus Operators is completely positive and trace-preserving.

Let our principle system comprise of n_s modes and the environment comprise of n_E modes.Consider $S \in Sp(2(n_s + n_E), R)$. Let it act on the position and momentum vector of the system$\oplus$environment.

$$\hat{R}' = S\hat{R} \tag{53}$$

where $\hat{R} = \hat{R}_s \oplus \hat{R}_E$ and $\hat{R}$ is given by (4) for the principal system and the environment.

$Sp(2(n_s + n_E), R)$ possess a unitary representation[13] which acts on the tensor product Hilbert Space of the principal system and environment. The unitary operators of this representation are generated by quadratic Hermitian operators. Let $\hat{U}$ be the unitary operator acting on the Hilbert, $H_S \otimes H_E$ space of the entire system which corresponds to the symplectic transformation S in its defining representation matrix form. Equation (53) can also be written as:

$$\hat{R}'_i = \hat{U}\hat{R}_i\hat{U}^\dagger \tag{54}$$

where $\hat{R}_i$ is the ith component of the position-momentum vector $\hat{R}$.

From (54), we can see that the symplectic transformation, S on the joint system can considered equivalent to performing a unitary transformation, U on the Hilbert space $H_S \otimes H_E$ joint system. We are limiting our discussion to the unitary transformations generated by quadratic hamiltonians.

One can see that due to (50) the covariance matrix, $\gamma = \gamma_s \oplus \gamma_E$ of the system+environment will undergo the following transformation:

$$\gamma' = S\gamma S^T \tag{55}$$

Hence through this transformation $\gamma_s \longrightarrow \gamma'_s$ where γ'_s represents the new covariance matrix of the principal system. It can be easily checked that γ'_s satisfies the conditions for covariance (18) if γ_s does.

Consider the two mode case - with one mode principal system and one mode environment. In the following we work in the Heisenberg picture and real variables:

Let the symmetrically ordered characteristic function of the joint state (i.e. one mode principal system and one-mode environment) of the system be $\chi(z, 0) = \chi_W(z)$. Here $z \in R^4$ represents the real phase-space variables. The phase space variables for the principal system will be denoted by $z_s = (x_s, y_s)$ and those for the environment will be denoted as $z_E = (x_E, y_E)$. Hence we get that:

[13]"Real symplectic groups in quantum mechanics and optics", A. Dutta, N. Mukunda and R. Simon, Pramana Journal of Physics, Vol. 45, pp. 471.

$$z = z_s \oplus z_E \tag{56}$$

$$\chi_W(z_s \oplus z_E) = Tr[(\hat{\rho}_s \otimes |1\rangle_E\langle 1|)\hat{D}(z_s \oplus z_E)] \tag{57}$$

After (54) is performed one again evaluates the characteristic function:

$$\chi'_W(z_s \oplus z_E) = Tr[(\rho_s \otimes |1\rangle_E\langle 1|)\exp(\iota \hat{R}^T S^T z)] \tag{58}$$

We divide the matrix of the symplectic transformation S into 2×2 blocks as follows:

$$S = \begin{pmatrix} X & Y \\ X' & Y' \end{pmatrix} \tag{59}$$

Hence (58) becomes:

$$\chi'_W(z_s \oplus z_E) = Tr(\rho_s \otimes |1\rangle_E\langle 1| \exp(\iota(X\hat{R}_s + Y\hat{R}_E)^T z_s + \iota(X'\hat{R}_s + Y'\hat{R}_E)^T z_E)) \tag{60}$$

We aren't interested in the characteristic function of the entire system but only the principal system so we set the phase space variables $z_E = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Hence the characteristic function of the principal system is now given by:

$$\chi'^s_W(z_s) = Tr(\rho_s \otimes |1\rangle_E\langle 1| exp(\iota(X\hat{R}_s + Y\hat{R}_E)^T z_s)) = \chi^s_W(X^T z)\chi^E_W(Y^T z_s) \tag{61}$$

Hence the final characteristic function is a product of the initial system characteristic function and the environment state characteristic function.

We are interested in a class of channels called Gaussian Channels wherein Gaussian states are tranformed ONLY into Gaussian states by the channel. From (61) it's understood that this is possible only if the environment used is also in a Gaussian state.

Hence the conditions for a channel being a Gaussian channel have been obtained as :

1. The unitary interaction between the system and environment should have a quadratic generator corresponding to some symplectic transformation.

2. The environment should also have to be in the Gaussian state initially.

The displacement operator of the principal system, $\hat{D}(z_s)$ undergoes a transformation as follows:

$$\hat{D}(z_s) \longrightarrow \hat{D}(X^T z_s)\chi_W^E(Y^T z_s) = \hat{D}(X^T z_s)f(z) \tag{62}$$

The function $f(z_s)$ is the extra noise that is added as a result of the interaction. Since the environment has a Gaussian characteristic function, $f(z_s)$ is Gaussian in nature. No arbitrary function can act as extra noise because a certain set of completely-positivity conditions need to be satisfied for this. These completely-positivity conditions are as follows[14]:

Since f(z)[15] is gaussian in nature, it will be of the form $f(z) = exp(-\frac{1}{2}\alpha(z,z))$ where α is in a quadratic form.

If α_E is the covariance matrix for the environment then

$$\alpha(z_1, z_2) \equiv z_1^T Y^T \alpha_E Y z_2 \tag{63}$$

For any finite set of z belonging to the phase-space of the principal system construct the matrix M as follows:

$$M_{ij} \equiv \alpha(z_i, z_j) - \frac{\iota}{2}\Delta(z_i, z_j) + \frac{\iota}{2}\Delta(X^T z_i, X^T z_j) \tag{64}$$

where $\Delta(z_i, z_j) = x_j y_i - x_i y_j$ is the symplectic product of two phase space vectors. The positivity of the above matrix for any set of phase space vectors, $z = \begin{pmatrix} x \\ y \end{pmatrix}$ guarantees the complete positivity of the Gaussian Channel. This test can be simplified to quite a degree.

Let's construct the matrix M taking n phase-space vectors $\{z_i\}_{i=1}^{\infty}$. We get:

$$M_{ij} = z_i^T(\alpha - \frac{\iota}{2}(detX - 1)\Omega)z_j = z_i^T \beta z_j \tag{65}$$

where $\beta \equiv \alpha - \frac{\iota}{2}(detX - 1)\Omega$[16]. Let η be an n-dimensional complex vector. If M is to be positive then $\eta^\dagger M \eta \geq 0$. Testing this:

$$\eta^\dagger M \eta = (\sum_{i=1}^{n} \eta_i^* z_i)\beta(\sum_{j=1}^{n} \eta_j z_j) \tag{66}$$

[14] A.S. Holevo,One-mode quantum Gaussian Channels(2006) quant-ph/0607051

[15] For the sake of convenience from now on, unless otherwise specified, z_s will be written as z to represent the phase space variables for the principal system.

[16] $\alpha = Y^T \alpha_E Y$ where α_E is the initial covariance matrix of the environment system

Now, $\sum_{i=1}^{n} \eta_i z_i$ can be considered an arbitrary vector of the phase space. Hence we can see that iff $\beta \geq 0$ then $M \geq 0$.

5 Classification of One-Mode Gaussian Channel (canonical forms)

We study the classification of one-mode Bosonic Gaussian Channels which result from the interaction with a single mode environment. This classification is based on properties (i.e. the determinant, rank) of the 2×2 sub-matrices X, Y, X' and Y' of the symplectic matrix (see (59) which acts on the joint system of principal system and environment. [17]. We are interested to obtain only the canonical forms of these Gaussian Channels and work towards that end.

The conditions imposed upon X, Y, X' and Y' so that they be sub-matrices of a symplectic matrix $S \in Sp(4, R)$:

$$\begin{pmatrix} X & Y \\ X' & Y' \end{pmatrix} \begin{pmatrix} \Omega & 0 \\ 0 & \Omega \end{pmatrix} \begin{pmatrix} X^T & X'^T \\ Y^T & Y'^T \end{pmatrix}$$

$$= \left(\begin{array}{cc} X\Omega X^T + Y\Omega Y^T & X\Omega X'^T + Y\Omega Y'^T \\ X'\Omega X^T + Y'\Omega Y^T & X'\Omega X'^T + Y'\Omega Y'^T \end{array} \right) = \left(\begin{array}{cc} \Omega & 0 \\ 0 & \Omega \end{array} \right) \tag{67}$$

(5) imposes 3 conditions on the sub-matrices X, Y, X' and Y'.

$$X\Omega X^T + Y\Omega Y^T = \Omega \tag{68}$$

$$X'\Omega X'^T + Y'\Omega Y'^T = \Omega \tag{69}$$

$$X\Omega X'^T + Y\Omega Y'^T = 0 \tag{70}$$

For any 2×2 matrix (not necessarily symplectic) P:

$$P\Omega P^T = \left(\begin{array}{cc} 0 & detP \\ -detP & 0 \end{array} \right) \tag{71}$$

From (71) we can re-write the condition (68) as:

$$detX + detY = 1 \tag{72}$$

[17] F Cariso, V. Giovannetti, A. S. Holevo, New Journal of Physics 8 (310) (2006)

A similar deduction can be made for X' and Y' matrices too.

We briefly summarize the channel-action for this case, additionally explaining how one obtains the canonical forms (or simplest forms) for a type of one-mode Gaussian channel[18].

We are given a one-mode Gaussian state. This can be ANY one-mode Gaussian state and there are no restrictions on this. Before the channel is applied the position and momentum operators of the principal system are $\hat{q}_s$ and $\hat{p}_s$ respectively and that for the environment are $\hat{q}_E$ and $\hat{p}_E$. The symmetrically ordered displacement operators of the joint system are given by $\hat{D}(z) = \exp(\iota \hat{R}^T z)$ where $\hat{R}$ and z are given by (4) and (56). The characteristic funtion corresponding to the joint system $\hat{\rho}_{in} = \hat{\rho}_s \otimes \hat{\rho_E}$ is hence given (30) whilst keeping the order parameter, $s = 0$. Since the joint initial state (i.e. before any channel action) is a product state of the principal system and the environment, the characteristic function of the joint initial system is the product of the respective individual characteristic functions of the principal system and the environment i.e.

$$\chi^{in}{}_W(z) = \chi^{s}{}_W(z_s) \chi^{E}{}_W(z_E) \tag{73}$$

If α_s and α_E are the respective covariance matrices of the principal system and the environment before the channel action, then the characteristic function of the joint system is given by:

$$\chi^{in}{}_W(z) = \exp(-\frac{1}{2} z_s^T \alpha_s z_s) \exp(-\frac{1}{2} z_E^T \alpha_E z_E) \tag{74}$$

We work in the Heisenberg picture, hence the change brought about by the channel action will be on the position and momentum operators of the joint system i.e. $\hat{R}$ and not $\hat{\rho}_{in}$.

The channel action is summarized as follows: Let $\hat{R}'$ denote the column comprising of the transformated position and momentum operators of joint system i.e. the ones after the channel action. Then $\hat{R}'$ is related with $\hat{R}$ by (53). In detail, this looks like the following:

[18] From this point upto the point before classification of various channels, z represents the phase-space variables of the joint system i.e. the principal system and the environment whereas z_s represents the phase space variables of the principle system only

$$
\begin{pmatrix} \hat{q}'_s \\ \hat{p}'_s \\ \hat{q}'_E \\ \hat{p}'_E \end{pmatrix} = \begin{pmatrix} X & Y \\ X' & Y' \end{pmatrix} \begin{pmatrix} \hat{q}_s \\ \hat{p}_s \\ \hat{q}_E \\ \hat{p}_E \end{pmatrix}
= \begin{pmatrix} X \begin{pmatrix} \hat{q}_s \\ \hat{p}_s \end{pmatrix} + Y \begin{pmatrix} \hat{q}_E \\ \hat{p}_E \end{pmatrix} \\ X' \begin{pmatrix} \hat{q}_s \\ \hat{p}_s \end{pmatrix} + Y' \begin{pmatrix} \hat{q}_E \\ \hat{p}_E \end{pmatrix} \end{pmatrix} \tag{75}
$$

Hence we have obtained the transformed position and momentum operators for the joint system. The new position and momentum operators will now be used in the displacement operators i.e. the displacement operators are changed to:

$$
\begin{aligned}
\hat{D}'(z) &= \exp[\iota \begin{pmatrix} {}^{(\hat{q}_s,\hat{p}_s)}X^T + {}^{(\hat{q}_E,\hat{p}_E)}\,Y^T & ,^{(\hat{q}_s,\hat{p}_s)}\,X'^T + {}^{(\hat{q}_E,\hat{p}_E)}\,Y'^T \end{pmatrix} \begin{pmatrix} z_s \\ z_E \end{pmatrix}] \\
&= \exp[\iota^{(\hat{q}_s,\hat{p}_s)}(X^T z_s + X'^T z_E) + \iota^{(\hat{q}_E,\hat{p}_E)}(Y^T z_s + Y'^T z_E)]
\end{aligned} \tag{76}
$$

The characteristic function for the joint system changes as follows:

$$
\chi'^{in}{}_W(z) = \chi^s{}_W(X^T z_s + X'^T z_E)\chi^E{}_W(Y^T z_s + Y'^T z_E)) \tag{77}
$$

After the channel action suppose we want to pick out the characteristic function of the principal system only, we set the phase space variables of the environment system to zero. Upon so doing we obtain:

$$
\begin{aligned}
\chi'^s{}_W(z_s) &= \chi^s{}_W(X^T z_s)\chi^E{}_W(Y^T z_s) \\
&= \exp[-\frac{1}{2} z_s^T (X\alpha_s X^T) z_s] \exp[-\frac{1}{2} z_s^T (Y\alpha_E Y^T) z_s]
\end{aligned} \tag{78}
$$

Our channel is basically identified by two things: the symplectic matrix which enacts the transformation on the joint system, S and the environment sytem (determined by the covariance matrix α_E). Specifying both of these uniquely identifies the Gaussian channels. To bring the Gaussian channel in a canonical form, what we do is to apply symplectic transformations on the principal system before and after the channel has been applied to it. The enactment of these is to be considered as part of the whole channel. We see that by these symplectic transformations, every one-mode Gaussian channel (for a one-mode environment) can be brought to one canonical form or the other.

This is done in the following way:

Perform an $Sp(2,R)$ transformation, T_1 on the $\hat{q}_s$ and $\hat{p}_s$ operators before the channel action.

$$T_1 \begin{pmatrix} \hat{q}_s \\ \hat{p}_s \end{pmatrix} \Longleftrightarrow T_1 \alpha_s T_1^T \tag{79}$$

(79) tells us basically that instead of applying S to the joint system comprising of the principal system (as it was given to us) and the environment, we change the principle system by applying a local symplectic transformation on it and apply S on this joint system.

Next, perform an $Sp(2,R)$ transformation, T_2 on the $\hat{q'}_s$ and $\hat{p'}_s$ operators after the channel-action.

$$T_2 \begin{pmatrix} \hat{q'}_s \\ \hat{p'}_s \end{pmatrix} = T_2 (X \begin{pmatrix} \hat{q}_s \\ \hat{p}_s \end{pmatrix} + Y \begin{pmatrix} \hat{q}_E \\ \hat{p}_E \end{pmatrix}) \Longleftrightarrow T_2^T z_s \tag{80}$$

Hence we can see that performing T_1 on the principal system after the channel action is alternately equivalent to transforming the principal system phase space variables by T_1^T. (80) basically tells us that we change the outpute state.

Using both of these new features in the pre-existing channel (78) becomes:

$$\begin{aligned} \chi'^s{}_W(z_s) &= \exp[-\frac{1}{2} z_s^T T_2 (X T_1 \alpha_s T_1^T X^T) T_2^T z_s] \exp[-\frac{1}{2} z_s^T (T_2 Y \alpha_E Y^T T_2^T) z_s] \\ &= \chi_W^s (T_1 X^T T_2^T z_s) \chi_W^E (Y^T T_2^T z_s) \end{aligned} \tag{81}$$

Having specified the procedure to obtain the canonical forms for a Gaussian state, we now proceed to the classification.

Case A1:$X = 0_2$ and hence according to (72),$detY = 1$

So $Y \in Sp(2,R)$ i.e. $Y \Omega Y^T = \Omega$. Since $detX = 0$ the completely-positive condition: $\alpha - \frac{\iota}{2}\Omega \geq 0$ is trivially satisfied since α is a covariance matrix for the environment system. The change in the characteristic function for the system thus is[19]:

$$\chi_W^s(z) \longrightarrow \chi_W'^s(z) = \chi_W^E(z) = \exp(-\frac{1}{2}z^T T_2 Y \alpha_E Y^T T_2^T z) \tag{82}$$

Now since $Y \in Sp(2,R)$, $\alpha = Y\alpha_E Y^T$ is also a covariance matrix for the environment. Since we want the simplest form of the channel, we choose T_2 such that in accordance with Williamson's theorem $T_2\alpha T_2^T$ is symplectic diagonal. In that case $T_2\alpha T_2^T = (N_0 + \frac{1}{2})I_2$. Hence we get:

$$\chi_W'^s(z) = exp(-\frac{1}{2}(N_0 + \frac{1}{2})(x^2 + y^2)) \tag{83}$$

Since $X = 0$, $Y' = 0$ and $X' \in Sp(2,R)$.

Hence our symplectic matrix in canonical form looks like:

$$S = \begin{pmatrix} 0 & I_2 \\ Q & 0 \end{pmatrix} \tag{84}$$

where $Q \in Sp(2,R)$ and where the environment system is a thermal Gaussian state with the mean number N_0. We still have a degree of freedom over what P can be but for the simplest form we choose it to be I_2. A1 is called the completely depolarizing channel. All information of the input is lost after the application of this channel.

Case A2: X is a rank 1 matrix. $detY = 1$.

The completely-positive conditions are the same as before and is again trivially satisfied. The only difference is in the canonical form adopted by the channel. One can apply local symplectic transformations before and after the application of the channel to bring the symplectic matrix in the following canonical form using (68),(69) and (70):

$$S = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_{11} & c_{12} & 0 & -c_{12} \\ c_{21} & c_{22} & 0 & -c_{22} \end{pmatrix} \tag{85}$$

where the environment system is again a thermal system with the mean number N_0.

We still have degree of freedom to choose what X' and Y' should be. $X' = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in Sp(2,R)$.

[19]For convenience, $z = z_s$ represents the phase space variabls of the principal system again from here onwards

Our characteristic functions changes as follows:

$$\chi'^{s}_{W}(z) = \chi^{s}_{W}(x,0) exp(-\frac{1}{2}(N_0 + \frac{1}{2})(x^2 + y^2)) \tag{86}$$

Both A1 and A2 are referred to as Singular channels because X matrix is rank 0 and 1 respectively.

Case B1: $detX = 1$ and $Y = 0$. The positivity condition is just $\alpha \geq 0$ which we know holds. Applying suitable local symplectic transformations before and after the application of the channel the symplectic transformation comes to the form:

$$S = \begin{pmatrix} I_2 & 0 \\ 0 & Q \end{pmatrix} \tag{87}$$

where $Q \in Sp(2, r)$.

This is the ideal channel which doesn't change the state of the system after running it through the channel. It adds no noise to the input state. The system is not effected by any environment in this case.

Case B2: $detX = 1$ and Y is rank 1. The positivity condition is again $\alpha \geq 0$ which holds. Using local symplectic transformations before and after the application of the channel gives us:

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -c_{11} & 0 & c_{11} & c_{12} \\ -c_{21} & 0 & c_{21} & c_{22} \end{pmatrix} \tag{88}$$

Of course, $Y' \in Sp(2, R)$.

The characteristic function change is:

$$\chi'^{s}_{W}(z) = \chi^{s}_{W}(z) exp(-\frac{1}{2} N_c y^2) \tag{89}$$

We have the degree of freedom of choosing $N_c \geq 0$. Since the noise is embedded in only one quadrature this is known as the Single Quadrature Channel.

Case C:

$detX = k^2 \geq 0$. Then $detY = 1 - k^2$. The completely-positivity conditions is $\alpha - \frac{\iota}{2}|k^2 - 1|\Omega \geq 0$. Hence we get that $\frac{\alpha}{|k^2-1|}$ should be a bonafide covariance matrix for the environmental system. Again by local symplectic transformations we can choose a canonical form for S:

C1: When $0 \leq k^2 \leq 1$

(68),(69) and (70) impose the condition: $detX' = 1 - k^2$ and $detY' = k^2$. Any X' and Y' which satisfy these are a suitable candidate for the respective lower sub-matrices of S and we have the freedom to choose X' and Y' upto the said conditions being satisfied.

$$S = \begin{pmatrix} k & 0 & \sqrt{1-k^2} & 0 \\ 0 & k & 0 & \sqrt{1-k^2} \\ \sqrt{1-k^2} & 0 & -k & 0 \\ 0 & \sqrt{1-k^2} & 0 & -k \end{pmatrix} \tag{90}$$

where the environment system is again a thermal state with mean number N_0.

Characteristic function change:

$$\chi_W'^s(z) = \chi_W^s(kz)exp(-\frac{1}{2}(1-k^2)(N_0 + \frac{1}{2})(x^2 + y^2)) \tag{91}$$

C1 is the beam-splitter channel with additional Gaussian noise due to interaction with a thermal bath. This can be seen from the following effect on the canonical variables of the principal system:

$$\begin{pmatrix} q_s \\ p_s \end{pmatrix} = \begin{pmatrix} q\sqrt{1-k^2} - kq_E \\ p\sqrt{1-k^2} - kp_E \end{pmatrix} \tag{92}$$

We see that the effect of the channel amounts to an $SO(2)$ between the position operators of the principal system and the environment and the SAME rotation between the corresponding momentum operators of the principal system and environment.

This channel is also known as the attenuator channel since the mean number $\langle N \rangle$ of the output is lower than the input. This is shown as follows:

Let the covariance matrix of the system be α_s before the application of $C1(k)$. Then $\langle N \rangle = Tr(\alpha_s) - 1$. After $C1(k)$ has been applied, $\langle N \rangle = k^2(Tr(\alpha_s) - 1)$[20]. Since $0 \leq k \leq 1$, we see that the mean number reduces as a consequence of this channel.

C2: When $k^2 \geq 1$

(68),(69) and (70) impose the condition: $detX' = 1 - k^2$ and $detY' = k^2$. Any X' and Y' which satisfy these are a suitable candidate for the respective lower sub-matrices of S and we have the freedom to choose

[20] This is for the case where $N_0 = 0$. Hence the thermal environment is in a vacuum state.

X' and Y' upto the said conditions being satisfied.

$$S = \begin{pmatrix} k & 0 & \sqrt{k^2-1} & 0 \\ 0 & k & 0 & -\sqrt{k^2-1} \\ \sqrt{k^2-1} & 0 & k & 0 \\ 0 & -\sqrt{k^2-1} & 0 & k \end{pmatrix} \tag{93}$$

where the environment system is again a thermal state with mean number N_0.

Characteristic function changes as:

$$\chi'^s_W(z) = \chi^s_W(kz)exp(-\frac{1}{2}(k^2-1)(N_0+\frac{1}{2})(x^2+y^2)) \tag{94}$$

This is the amplification channel. As in the mean number $\langle N \rangle$ increases since it goes from $Tr(\alpha)-1$ to $k^2(Tr(\alpha)-1)$. Since $k \geq 1$ this time, the mean number increases.

Case D:

$detX = -k^2$ and hence $detY = k^2+1$

The completely positive condition is that $\frac{\alpha}{1+k^2} + \frac{\iota}{2}\Omega \geq 0$. Hence $\frac{\alpha}{k^2+1}$ has to be a bonafide covariance matrix.

(68),(69) and (70) impose the condition: $detX' = 1+k^2$ and $detY' = -k^2$. Any X' and Y' which satisfy these are a suitable candidate for the respective lower sub-matrices of S and we have the freedom to choose X' and Y' upto the said conditions being satisfied. Using local symplectic transformations the canonical form of the symplectic transformation effecting the channel

$$S = \begin{pmatrix} k & 0 & \sqrt{k^2+1} & 0 \\ 0 & -k & 0 & \sqrt{k^2+1} \\ \sqrt{k^2+1} & 0 & k & 0 \\ 0 & \sqrt{k^2+1} & 0 & -k \end{pmatrix} \tag{95}$$

where the environment system is again a thermal state with mean number N_0.

Characteristic function change:

$$\chi'^s_W(\xi) = \chi^s_W(k\begin{pmatrix} x \\ -y \end{pmatrix})exp(-\frac{1}{2}(k^2+1)(N_0+\frac{1}{2})(x^2+y^2)) \tag{96}$$

This channel is known as the phase-conjugation channel or the transposition channel. The reason for this is that undergoing this channel means that the operator gets transposed with additional Gaussian noise required to maintain the channel completely positive.

Here we have obtained the canonical forms of Gaussian Channels. While A and B channels don't have any channel parameter to identify them, C and D channels require two parameters to identify them completely. One is k which gives the degree of attenuation/amplification and the other N_0 which specifies the temperature of the environment which one is working with. The quantum limited channels (channels with just the minimum amount of noise to keep the channel completely positive) are exempted from this particular parameter.

5.1 Kraus Representation: One-Mode Bosonic Gaussian Channels

One can obtain the Kraus representation for one-mode bosonic gaussian channels[21]. The Kraus representation for all the channels are obtained for a 0 temperature thermal environment.But later using the semi-group property we can see that the concatenation of different quantum limited channels give us the Kraus representation for noisy channels (at higher temperatures) too.

Let $\hat{\rho}_s$ and $\hat{\rho}_E = |0\rangle\langle 0|$ be the respective initial states of the system and environment. Let $\hat{\rho}'_s$ be the state of the system after the channel has acted upon it. Let $\hat{U}$ be the unitary evolution of the joint system during the channel action.As in (52) trace the environment out. This is done in the Fock state basis here:

$$\hat{\rho}'_s = Tr_E(\hat{U}\hat{\rho}_s \otimes |0\rangle_E\langle 0|\hat{U}^\dagger) \tag{97}$$
$$= \sum_{l=0}^{\infty}\langle l_E|\hat{U}|0_E\rangle\hat{\rho}_s\langle 0_E|\hat{U}^\dagger|l_E\rangle = \sum_{l=0}^{\infty}\hat{W}_l\hat{\rho}_s\hat{W}_l^\dagger$$

where $\hat{W}_l$ represent the Kraus Operators. Consider expanding $\hat{W}_l$ in the Fock state basis[22]:

$$\hat{W}_l = \sum_{n_1,m_1}^{\infty} C_{n_1 0}^{m_1 l}|m_1\rangle\langle n_1| \tag{98}$$

Now

$$C_{n_1 n_2}^{m_1 m_2} = \langle m_1 m_2|\hat{U}|n_1 n_2\rangle = \int_{-\infty}^{\infty} dx_1 dx_2 \langle m_1 m_2|x_1 x_2\rangle\langle x_1 x_2 \hat{U}|n_1 n_2\rangle \tag{99}$$

We see that the matrices in (84),(85),(87),(88),(90),(93) and (95) $\in GL(2,R)\otimes GL(2,R)$ subgroup

[21] J Solomon Ivan, R. Simon, K. Sabapathy http://arxiv.org/abs/1012.4266

[22] Since the fock state basis will be utilized here, it makes more sense to work with the complex representation featuring $\hat{a}$ and $\hat{a}^\dagger$ operators than the $\hat{p}$ and $\hat{q}$ operators

of $Sp(4, R)$ where the position operators of the principal and environment system intermingle amongst themselves and same so for the momentum operators. All the symplectic transformations can, thus, be brought to the form:

$$S = M \oplus (M^{-1}) \tag{100}$$

where M acts on the position operators of both modes only and (M^{-1}) acts on the momentum operators of both modes only.

Since $\hat{U}$ corresponds to a symplectic transformation on the canonical variables, $\langle x_1 x_2|\hat{U}$ can be easily evaluated:

$$\langle x_1, x_2|\hat{U} = \langle (M^{-1}(x_1, x_2)^T)_1, (M^{-1}(x_1, x_2)^T)_2| = \langle x_1', x_2'| \tag{101}$$

Hence in (99) we get:

$$C^{m_1 m_2}_{n_1 n_2} = \int_{-\infty}^{\infty} dx_1 dx_2 \psi^*_{m1}(x_1)\psi^*_{m2}(x_1)\psi_{n1}(x_1')\psi_{n2}(x_2') \tag{102}$$

where $\psi_{n2}(x)$ is the wavefunction of the n-Fock state.

Utilizing the generating function for the wave function of Fock states in the equation above we get:

$$C^{m_1,m_1}_{n_1,n_2} = \frac{1}{\sqrt{n_1!n_2!m_1!m_2!}} \frac{\partial^{m_1}}{\partial \eta_1^{m_1}} \frac{\partial^{m_2}}{\partial \eta_2^{m_2}} \frac{\partial^{n_1}}{\partial z_1^{n_1}} \frac{\partial^{n_2}}{\partial z_2^{n_2}} F(z_1, z_2, \eta_1, \eta_2)|_{z_1,z_2,\eta_1,\eta_2=0} \tag{103}$$

where $F(z_1, z_2, \eta_1, \eta_2)$ is the generating function and is given by

$$F(z_1, z_2, \eta_1, \eta_2) = \frac{1}{\pi} \int_{-\infty}^{\infty} dx_1 dx_2 \exp\{-\frac{1}{2}[(x_1 - \eta_1\sqrt{2})^2 + (x_2 - \eta_2\sqrt{2})^2 \tag{104}$$

$$+ (x_1' - z_1\sqrt{2})^2 + (x_2' - z_2\sqrt{2})^2 - \eta_1^2 - \eta_2^2 - z_1^2 - z_2^2]\} \tag{105}$$

5.2 Singular Case:A

5.2.1 A1

In the case of A1 we don't need to use the technique given above, the Kraus operators can be deduced directly as:

$$\hat{W}_l = |0\rangle\langle l| \tag{106}$$

Then:

$$A1(\hat{\rho}_s) = \sum_{i=0}^{\infty} |0\rangle\langle l|\hat{\rho}_s|l\rangle\langle 0| = |0\rangle\langle 0| \tag{107}$$

Some properties of the channel are that it's non-classicality breaking[23] and hence entanglement breaking[24]. Also it has a fixed point at $\hat{\rho}_s = |0\rangle\langle 0|$.

All operators of the form $\hat{W}_l^\dagger \hat{W}_l'$ are mutually orthogonal. This proves that this operator is a an extremal channel.

5.2.2 A2

The symplectic matrix is given by:

$$S = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \tag{108}$$

Here $M = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $M^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$.

We can use the generating function given by equation (103) but in place of the fock state in the place of m_2, we expand using the position eigen-kets $|q\rangle$:

$$C_{n_1,n_2}^{m_1,q} = \langle m_1, q|\hat{U}|n_1, m_1\rangle \tag{109}$$

where q represents the position eigenvalue.

The Kraus operators come to be:

$$\hat{V}_q = \sum_{m_1,n_1}^{\infty} C_{n_1,0}^{m_1,q} |m_1\rangle\langle n_1| = |\frac{q}{\sqrt{2}})\langle q| \tag{110}$$

where $|\frac{q}{\sqrt{2}})$ represents the coherent state $|\alpha\rangle$ with $\alpha = \frac{q}{\sqrt{2}}$.

[23] A channel being non-classicality breaking implies that for whatever input to the channel, the output will necessarily be classical

[24] A channel being entanglement breaking implies that for whatever input state the output of channel will necessarily be separable.

These Kraus operators can be easily verified by checking consistency with (86)[25].

The trace preserving property of these Kraus operators is easily seen to be satisfied. Given that this channel has a representation of rank one Kraus operators it is non-classicality breaking and hence entanglement breaking[26].

This channel has no fixed points. Now, the set of all operators $\{\hat{V}_q^{\dagger}\hat{V}_q'\}$ is a mutually orthogonal set which means that the given channel is an extremal channel.

5.3 B

5.3.1 B1

This is the identity channel i.e. the channel doesn't change the state of the system at all. This is obviously not non-classicality breaking, not entanglement breaking either and every point is a fixed point. The Kraus operators are given by identity matrices. This channel is obviously not an extremal channel.

5.3.2 B2

This is the identity channel along with some Gaussian noise in one quadrature. Hence this is not a quantum-limited channel. Just as in the case of A2 channel we can find the Kraus operators by using the position eigenkets instead of the fock-state eigenkets. The Kraus operators are given by:

$$\hat{Z}_q = \frac{1}{(N_c\pi)^{\frac{1}{4}}}\exp[-\frac{q^2}{2N_c}]\hat{D}(\frac{q}{\sqrt{2}}) \tag{111}$$

This channel is neither entanglement breaking nor non-classicality breaking nor is it extremal

5.4 C

5.4.1 C1 Attenuator Channel

The generating function (104) in this case becomes:

$$F(z_1, z_2, \eta_1, \eta_2) = \exp[\eta_2(\sqrt{1-k^2}z_1 + kz_2) + \eta_1(kz_1 - \sqrt{1-k^2}z_2] \tag{112}$$

Utilizing the generator function ,we get the Kraus operators to be:

$$\hat{B}_l(k) = \sum_{m=0}^{\infty}\sqrt{{}^{m+l}C_l}(\sqrt{1-k^2})^l k^m|m\rangle\langle m+l|, l = 0, 1, 2... \tag{113}$$

[25] $N_0 = 0$ in this case
[26] http://arxiv.org/abs/0802.0235 Entanglement Breaking Channels in Infinite Dimensions-A.S. Holevo (2008)

One can easily verify these to be genuine Kraus operators representing the channel by testing whether they will satisfy equation (91) or not. By (91) we see that:

$$\chi'_N \xi = \chi_N k \xi \tag{114}$$

Hence the normally ordered characteristic function merely undergoes a scaling due to the effect of the channel as per (64). Now the outer product of coherent vector undergoes the following transformation:

$$|\alpha\rangle\langle\alpha| \longrightarrow \hat{B}_l(k)|\alpha\rangle\langle\alpha|\hat{B}_l \dagger (k) = |k\alpha\rangle\langle\alpha| \tag{115}$$

Thus the Sudarshan-Glauber P function undergoes the following transformation due to this:

$$P(\alpha) \longrightarrow \frac{1}{k^2}P(\frac{\alpha}{k}) \tag{116}$$

which is consistent with (115). Thus the Kraus operators obtained have been verified.

Another thing to note is that since the Sudarshan-Glauber function also merely undergoes scaling due to the channel, any non-classical P function will remain non-classical. Hence we have also verified that the channel is not a non-classicality breaking channel.

One can compare the cumulants of the symmetrically ordered characteristic functions to check what the fixed points of the channel are. Upon doing so one obtains that apart from the vacuum state $|0\rangle\langle 0|$, no other state will remain a fixed point for the channel.

The beamsplitter channel is not entanglement breaking. This is verified when we see that all the Kraus operators are of ∞ dimension. It's not possible to find any finite rank operator using any isometry. Hence consistency of this fact is added by the Kraus operators as well.

The set of all operators of the type $\{\hat{B}_l^\dagger \hat{B}_{l'}\}$ are linearly independent. Hence the channel is extremal.

5.4.2 C2 Amplifier Channel

The generating function is given by the following:

$$F(z_1, z_2, \eta_1, \eta_2) = \frac{1}{k}\exp[\frac{1}{k}(\eta_1 z_1 + \eta_2 z_2) + \sqrt{1-k^{-2}}(\eta_1\eta_2 - z_1 z_2)] \tag{117}$$

Using this we obtain the Kraus operators:

$$\hat{A}_l = \frac{1}{k}\sum_{m=0}^{\infty}\sqrt{{}^{m+l}C_l}(\sqrt{1-k^{-2}})^l\frac{1}{k^m}|m+l\rangle\langle m|, l = 0,1,2... \tag{118}$$

One can verify the action of the Kraus operators by comparing it with (94). (94) tell us that the Husimi function undergoes the transformation:

$$\chi_A(\xi) \longrightarrow \chi_A(k\xi) \tag{119}$$

The utilizing the Kraus operators obtained we get the transformation in the Husimi function as:

$$\langle\alpha|\hat{\rho}_s|\alpha\rangle \longrightarrow \frac{1}{k^2}Q(\frac{\alpha}{k}) \tag{120}$$

(119) and (120) are consistent with each other. This verifies the Kraus operators obtained. Going by the transformation in the normally ordered characteristic function:

$$\chi_N(\xi) \longrightarrow \chi_N(k\xi)\exp(-(k^2-1)|\xi|^2) \tag{121}$$

The fourier transform of the above suggests that the P function transforms to a convolution of the P function with a Gaussian function (corresponding to the latter factor on the RHS of above equation). Hence the P function will remain point-wise non-negative where it earlier was. This concludes to the fact that the amplifier channel cannot be non-classicality generating.

Comparing the cumulants of any input state with its output we find that there's no fixed point for the channel. Also one cannot construct an isometric finite rank set of Kraus operators for this channel which goes to show that this is not an entanglement breaking channel.

The amplifier channel is an extremal channel.

5.5 D

The generating function goes as:

$$F(z_1,z_2,\eta_1,\eta_2) = \frac{1}{\sqrt{1+k^2}}e^{(\sqrt{1+k^{-2}})^{-1}(\eta_1\eta_2 - z_1z_2)+(\sqrt{1+k^2})^{-1}(\eta_1 z_2+\eta_2 z_1)} \tag{122}$$

From there we get the Kraus operators to be as:

$$\hat{T}_l(k) = \sqrt{1+k^2}^{-1}\sum_{n=0}^{l}\sqrt{1+k^2}^{-n}\sqrt{1+k^{-2}}^{l-n}\sqrt{{}^lC_n}|l-n\rangle\langle n|, l = 0, 1, 2... \tag{123}$$

To check the consistency of the Kraus operators with the phase conjugation channel the effect of both on the operator $|n\rangle\langle m|$ is compared. After a tedious amount of algebra one finds that both are equal proving verifying the Kraus operators to be indeed the ones for the phase conjugation channel.

Now using (96) we get that the normally ordered characteristic function changes as:

$$\chi_N(\xi) \longrightarrow \chi_A(-k\xi^*) \tag{124}$$

So the normally ordered characteristic function changes to become a scaled anti-normally ordered characteristic function. The latter is known to ALWAYS be classicaly because it amounts to $\langle\alpha|\hat{\rho}|\alpha\rangle$ which is always positive. Hence the phase-conjugation channel is a classicality breaking channel and is also an entanglement breaking channel therefore.

On comparing the cumulants of the input and output states one finds that only a subset of the thermal state are left invariant under the action of the channel. If $\hat{\rho}_s = (1-x)\sum_{n=0}^{\infty} x^n|n\rangle\langle n|$ is any thermal state with $x = e^{-\beta E}$ then for $x = k^2$ the channel will remain invariant.

The channel is an extremal channel because the set of all operators of the form $\{\hat{T}_l(k)^\dagger\hat{T}_{l'}(k)\}$ are linearly independent.

This finishes off the Kraus representation for the one-mode bosonic gaussian channels[27]

5.6 Semi-group Property of One-Mode Bosonic Gaussian Channels

Using (83),(86),(91),(94) and (96) we can establish a semi-group property of the various Gaussian Bosonic channels. The reason B2 (i.e. (89) doesn't feature in this list is because it is the only channel among the entire set (apart from the identity channel) which can't be reproduced as a concatenation of other bosonic gaussian channels.

The semi-group property of various channels in their canonical forms are as follows:

[27]There is one channel remaining. It is an extension of the B2 channel in that it adds Gaussian noise to both the quadratures rather than just one. But this channel requires a two mode environment to materialize and we are only dealing with a one-mode environment here.

$D(k_2;0)D(k_1;0) = C1(k_2k_1;2k_2^2(1+k_1^2))$ if $k_2k_1 \leq 1$

$D(k_2;0)D(k_1;0) = C2(k_2k_1;2(1+k_2^2))$ if $k_2k_1 \geq 1$

$D(k_2;0)C1(k_1;0) = D(k_1k_2;2k_2^2(1-k_1^2))$

$D(k_2;0)C2(k_1;0) = D(k_1k_2;0)$

$D(k_2;0)A2(0) = A2(2k_2^2)$

$C1(k_2;0)D(k_2;0) = D(k_2k_1;0)$

$C1(k_2;0)C1(k_1;0) = C1(k_1k_2;0)$

$C1(k_2;0)C2(k_1;0) = C1(k_2k_1;2k_2^2(k_1^2-1))$ if $k_2k_1 \leq 1$

$C1(k_2;0)C2(k_1;0) = C2(k_2k_1;2(1-k_2^2))$ if $k_2k_1 \geq 1$

$C1(k_2;0)A2(0) = A2(0)$

$C2(k_2;0)D(k_1;0) = D(k_2k_1;2(k_2^2-1))$

$C2(k_2;0)C1(k_1;0) = C1(k_2k_1;2(k_2^2-1))$ if $k_2k_1 \leq 1$

$C2(k_2;0)C1(k_1;0) = C2(k_2k_1;2k_2^2(1-k_1^2))$ if $k_2k_1 \leq 1$

$C2(k_2;0)C2(k_1;0) = C2(k_1k_2;0)$

$C2(k_2;0)A2(0) = A2(2(k_2^2-1))$

$A2(0)D(k_1;0) = A2(\sqrt{k_1^2+2}-1)$

$A2(0)C1(k_1;0) = A2(\sqrt{2-k_1^2}-1)$

$A2(0)C2(k_1;0) = A2(k_1-1)$

$A2(0)A2(0) = A2(\sqrt{2}-1)$

6 Entanglement Sudden Death

Quantum entanglement is one of the most useful resources in quantum information theory. An entangled quantum system usually undergoes an asymptotic transition to classicality i.e. decoherence. Certain entangled systems are shown to undergo a complete loss of entanglement between sub-systems in a finite amount of time. Such a phenomenon is called Entanglement Sudden Death. "Sudden“ because unlike decoherence time, ESD time is exponentially smaller.

Here we will deal with the problem of entanglement sudden death of pair of simple harmonic oscillators set in a thermal bath. The basic problem is to know how much time it takes for the sub-systems to disentangle from each other under the influence of interaction with the environment, if it undergoes entanglement at all.

6.1 Master Equation for a Single Mode System

Consider a single harmonic oscillator placed in a thermal bath at temperature T. Let the density matrix of the system (i.e. the SHO) at time t be represented by $\hat{\rho}(t)$. The time evolution of this state interacting with the thermal bath is given by the following master equation. The joint state of the system and environment is given by $\hat{\rho}\otimes\hat{\rho}_{th}$ where the subscript 'th' stands for thermal. This evolution is considered to be Markovian in nature.[28]:

$$\begin{aligned}\frac{d}{dt}\rho_s(t) =& - i\omega_0 \left[a^\dagger a, \rho_s(t)\right] + \gamma_0 \left(N+1\right) \\ &\times \left(a\rho_s(t)a^\dagger - \frac{1}{2}a^\dagger a\rho_s(t) - \frac{1}{2}\rho_s(t)a^\dagger a\right) \\ &+ \gamma_0(N) \left(a^\dagger\rho_s(t)a - \frac{1}{2}a^\dagger a\rho_s(t) - \frac{1}{2}\rho_s(t)a^\dagger a\right). \end{aligned} \tag{125}$$

where γ_0 is the damping rate. The first time on the RHS with the Hamiltonian $\omega_0\hat{a}^\dagger\hat{a}$ describes the free evolution of the system whereas the latter terms describe dissipation. $N = e^{\beta\omega_0} - 1$ is the mean number of the mode.

6.2 Coherent State Representation

Translating (125) into the coherent state representation the equation of evolution of the Sudarshan-Glauber function is given by:

$$\begin{aligned}&\frac{\partial}{\partial t}P(\alpha,\alpha^*,t) \\ &\qquad = -[(-\iota\omega_0 - \frac{\gamma_0}{2})\frac{\partial}{\partial\alpha}\alpha + (\iota\omega_0 - \frac{\gamma_0}{2})\frac{\partial}{\partial\alpha^*}\alpha^*] \\ &\qquad P(\alpha,\alpha^*,t) + \gamma_0 N\frac{\partial^2}{\partial\alpha\partial\alpha^*}P(\alpha,\alpha^*,t)\end{aligned}$$

[28]"Quantum-to-classical transition and entanglement suddent death in Gaussian states under local heat-bath-dynamics" S.K. Goyal, S. Ghosh, Phys. Rev. A 82, 042337

An ansatz of the form $P(\alpha,\alpha^*,t)=\frac{1}{\pi\sigma^2(t)}e^{-\frac{|\alpha-\beta(t)|^2}{\sigma^2(t)}}$ with the initial condition: $P(\alpha,\alpha^*,0)=\delta^2(\alpha-\alpha_0)$ and where $\beta(t)=\int d^2\alpha P(\alpha,\alpha^*,t)$ and $\sigma^2(t)=N(1-e^{-\gamma_0 t})$. We ignore the terms invovling ω_0 which denote the free-evolution of the system, since we're only interested in the dissipation of the system. The linearity of the evolution map ensures that any state $\hat{\rho}=\int P(\lambda,\lambda^*,0)|\lambda\rangle\langle\lambda|d^2\lambda$ will evolve to $\hat{\rho}(t)=\int P(\alpha,\alpha^*,t)|\alpha\rangle\langle\alpha|d^2\alpha$ where

$$P(\alpha,\alpha^*,t)=\int P(\lambda,\lambda^*,0)e^{-\frac{|\alpha-\beta(t)|^2}{\sigma^2(t)}}d^2\beta \text{ with } \beta=\lambda e^{-\frac{\gamma_0 t}{2}} \tag{126}$$

6.3 Covariance Matrix Evolution

From (126) we obtain the symmetrically ordered characteristic function of the state of the simple harmonic oscillator as:

$$\chi'_W(\xi,t)=\exp(-\frac{1}{2}\xi^\dagger(V(0)e^{-\gamma_0 t}+(1+2N)(1-e^{-\gamma_0 t})I)\xi) \tag{127}$$

where $V(0)$ denotes the covariance matrix before the interaction [29].

Hence the time evolution of the covariance matrix is given by:

$$V(t)=V(0)e^{-\gamma_0 t}+(1+2N)(1-e^{-\gamma_0 t})I \tag{128}$$

We need to guarantee that $V(t)$ is a bona-fide covariance marix for all times. To this end we observe that a covariance matrix evolves under the action of a general Gaussian channel as:

$$V_f=AVA^T+B \tag{129}$$

where B is a positive operator[30]. Here $A=e^{-\frac{\gamma_0 t}{2}}I$ and $B=(1+2N)(1-e^{-\gamma_0 t})I\geq 0$. Hence $V(t)$ is a bonafide covariance matrix for all times. Thus change in system brought about by the interaction with the environment can be assumed as the result of the action of a quantum Gaussian channel on the system.

6.4 ESD of a Two Mode Gaussian State

Consider a two mode Gaussian state. Each mode is coupled to a separate thermal bath and undergoes dissipative evolution with respect to that bath. We want to infer about the entanglement properties of

[29] To avoid confusing with the damping rate γ_0, covariance matrices are represented with V from now on.
[30] T. Heinosaari, A. S. Holevo and M.M. Wolf, Quantum Inf. Comp. 10, 0619 (2010)

the two mode Gaussian state upon the application of two separate (but equivalent[31]) thermal baths on the repsective modes. Our initial gaussian state is thus taken to be entangled. The time evolution of covariance matrix for the gaussian state is given by:

$$V(t) = V(0)e^{-\gamma_0 t} + (2N+1)(1-e^{-\gamma_0 t})I \tag{130}$$

Any covariance matrix V needs to satisfy the aforementioned property (see paragraph below (32) to be classical[32] :

$$V - I \geq 0 \tag{131}$$

Classicality of state subsumes entanglement i.e. no entangled state will be classical.[33]If V(t) becomes classical at some point of time i.e. if V(t) satisfied (131) at some finite time we know that it has undergone ESD. From (130) we see that asymptotically the joint state will reach a thermal state i.e. it's covariance matrix V will be proportional to the Identity matrix. Hence, asymptotically it will be disentangled anyway. But our point here is to establish whether the state becomes disentangled at finite time after the interaction with the environment has commenced.

Equation (131) affords a convenient test on whether a state will undergo transition to classicality in finite time or not. All one needs to do is to test whether evolution according to (130) implies that (131) is satisfied at some time or not. (131) cannot be satisfied at $t = 0$ since if it was our initial state would be classical and hence couldn't be entangled. Thus, the smallest eigen value of $V(0) < 1$ at $t = 0$.

From (130) the time evolution of eigenvalues of $V(t)$ is given by:

$$n(t) = n(0)e^{-\gamma_0 t} + (1+2N)(1-e^{-\gamma_0 t}) \tag{132}$$

We see that for $T > 0$ i.e. for $N > 0$ all the eigenvalues of covariance matrix $V(t)$ will have to be greater than 1 at some finite time guaranteeing that (131) will be satisfied by the covariance matrix after some finite time. From this we conclude that the gaussian state WILL undergo ESD when both modes are coupled to non-zero temperature thermal baths of the same temperature. We want to obtain an upper limit to ESD for this system. For this we follow the time evolution of the smallest eigenvalue of covariance matrix. Depending upon the degree of entanglement between both the modes, this smallest eigenvalue can

[31] As in, both the baths are at the same temperature

[32] Here, the covariance matrix is defined as $2\times$ the covariance matrix defined in (9). i.e., $\gamma_{ij} = Tr(\hat{\rho}\{\hat{R}_i, \hat{R}_j\})$. This is why the $\frac{1}{2}$ factor in front of the identity matrix in the classicality condition vanishes in (131)

[33] The converse is not true. A non-entangled state can still be non-classical.

be made to be as small as possible[34]

Let t_c be the time the smallest eigenvalue $n_{min}(t)$ equals 1 according to the evolution of (132). From there we get that:

$$t_c = -\frac{1}{\gamma_0} ln[\frac{N}{N + \frac{1}{2} - n_{min}(0)}] \tag{133}$$

Thus t_c gives us the time the state will become classical. This doesn't mean that the state would undergo ESD at the SAME time, but it definitely guarantees that the system has undergone ESD before or at time t_c. Thus t_c gives an upper bound to the ESD for the system. This result only holds for temperature $T > 0$. For when $T \longrightarrow 0$ then $N \longrightarrow 0$ and using both the limits in (133) we get that $t_c \longrightarrow \infty$. Thus for zero temperature baths, the system will attain classicality asymptotically only, i.e. it won't attain classicality in any finite time. This however doesn't negate ESD at any finite time, because the system can still be non-classical without being entangled. Hence, we need to adopt a different approach to determining whether ESD happens or not for zero temperature baths.

For a special kind of symmetric states, who covariance matrices are given by the following equation:

$$V(0) = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & -c \\ c & 0 & a & 0 \\ 0 & -c & 0 & a \end{pmatrix} \tag{134}$$

Sandeep Goyal and Sibasish Ghosh have proved that ESD won't occur at any finite time(see footnote 28 on page 38 for the reference).

It is desirable to know whether this result holds in general and if it doesn't hold in general, then to what extend it can be generalized.

6.5 Commutativity of Channel-Action with Local Symplectic Transformations

At zero temperature, we already know that a certain class of two-mode Gaussian states whose covariance matrices are of the form given by (134) won't undergo ESD upon the application of the channel to the state. We want to know if this result holds in general. If it doesn't, then it's desired to know how much we can extend this result. To this end we take up the following exercise. The reason of taking up this exercise will be given at the end of the section:

[34]The covariance matrix is always a positive matrix. Hence the eigenvalues of the covariance matrix will always be greater than zero, but can be made to come as close to zero as possible.

It is desirable to know whether the action of the channel (i.e. the interaction between state and environment) commutes with symplectic transformations acting on the one-mode-system or not. In other words, if $\hat{\rho}(0)$ is the initial state of the system and $\hat{\rho}'(0)$ is the state obtained by performing a local unitary transformation $\hat{U}$ on the system, how are the two states related after time evolution under the action of the channel? Is it possible to obtain $\hat{\rho}(t > 0)$ from $\hat{\rho}'(t > 0)$ through the inverse transformation $\hat{U}^{-1}$? If this can't be done using the inverse of the same transformation $\hat{U}$ initially applied, then any other transformation? We generalize the question in the following way:

Let us invent a terminology: two states $\hat{\rho}$ and $\hat{\rho}'$ are said to be equivalent upto local symplectic transformations if there exists a unitary transformation generated by quadratic hermitian operators $\hat{U}$ such that $\hat{\rho}' = \hat{U}\hat{\rho}$. Equivalence is a transitive relation. Consider any set of all such equivalent states. Take any two states from this set. Let them undergo evolution under the said channel. After the evolution - are the states still equivalent? If not, is there any subset of local unitary transformations underwhich they remain equivalent after the channel action?

Since we are concerned with gaussian states here, we can work with the covariance matrix evolution under the said channel. Let $V(0)$ and $V'(0)$ be two covariance matrices associated with equivalent states $\hat{\rho}(0)$ and $\hat{\rho}'(0)$ respectively. In other words, there exists a symplectic transformation $S \in Sp(2, R)$ such that:

$$V'(0) = SV(0)S^T \tag{135}$$

Now, if a covariance matrix V undergoes a symplectic transformation S, then the matrix $V\Omega$ undergoes a similarity transformation i.e. if $V' = SVS^T$ then

$$V'\Omega = SVS^T\Omega = SVS^T(S^T)^{-1}\Omega S^{-1} = SV\Omega S^{-1} \tag{136}$$

Hence, the eigen-spectrum of $V\Omega$ remains invariant no matter what symplectic transformation S we perform on a covariance matrix V. To test whether two covariance matrices $V(t > 0)$ and $V'(t > 0)$ are equivalent it is sufficient to compare the eigenspectrum of $V(t > 0)\Omega$ and $V'(t > 0)\Omega$. Since tracing a matrix is equivalent to addition of all its eigenvalues, equality of the traces of $V(t > 0)\Omega$ and $V'(0)\Omega$ would give a "necessary" condition for equivalence of V and V'. But there is a problem here. If $\{\kappa_1, \kappa_2, ..., \kappa_n\}$ is the symplectic eigenspectrum of V, then the eigenvalues of $V\Omega$ are $\{\iota\kappa_i, -\iota\kappa_i\}_{i=1}^n$. Thus $Tr(V\Omega) = 0$. Hence instead of $V\Omega$ we work with $V\Omega V\Omega$ which will also have an invariant eigen spectrum under symplectic transformations on the covariance matrx V. The corresponding eigenvalues of $V\Omega V\Omega$ are $\{-\kappa_1^2, -\kappa_1^2, -\kappa_2^2, -\kappa_2^2, ..., -\kappa_n^2, -\kappa_n^2\}$.

The covariance matrix evolution is given by (130). Just for convenience we write it down once again:

$$V(t) = V(0)e^{-\gamma_0 t} + (2N+1)(1-e^{-\gamma_0 t})I \tag{137}$$

and

$$V'(t) = V'(0)e^{-\gamma_0 t} + (2N+1)(1-e^{-\gamma_0 t})I \tag{138}$$

Let's apply a local symplectic transformation $S' \in Sp(2,R)$ on $V'(t)$ to get $V''(t)$.

$$V''(t) = V''(0)e^{-\gamma_0 t} + (2N+1)(1-e^{-\gamma_0 t})S'S'^T \tag{139}$$

S' lends flexibility in testing whether the traces of $V\Omega V\Omega$ and $V''\Omega V''\Omega$ can be made equal by varying $S' \in Sp(2,R)$ or not.

Now,

$$\begin{aligned} Tr[V(t)\Omega V(t)\Omega] =& Tr[(V(0)\Omega)^2]e^{-\gamma_0 t} - 2(2N+1)^2(1-e^{-\gamma_0 t})^2 \\ & - 2(2N+1)(1-e^{-\gamma_0 t})e^{-\gamma_0 t}Tr[V(0)] \end{aligned}$$

and

$$\begin{aligned} Tr[V''(t)\Omega V''(t)\Omega] &= Tr[(V''(0)\Omega)^2]e^{-\gamma_0 t} - 2(2N+1)^2(1-e^{-\gamma_0 t})^2 \\ &+ 2(2N+1)(1-e^{-\gamma_0 t})e^{-\gamma_0 t}Tr[V''(0)\Omega S'S'^T\Omega] \\ &= Tr[(V(0)\Omega)^2]e^{-\gamma_0 t} - 2(2N+1)^2(1-e^{-\gamma_0 t})^2 \\ &+ 2(2N+1)(1-e^{-\gamma_0 t})e^{-\gamma_0 t}Tr[S'V'(0)S'^T\Omega S'S'^T\Omega] \\ &= Tr[(V(0)\Omega)^2]e^{-\gamma_0 t} - 2(2N+1)^2(1-e^{-\gamma_0 t})^2 \\ &- 2(2N+1)(1-e^{-\gamma_0 t})e^{-\gamma_0 t}Tr[S^TSV(0)] \end{aligned}$$

Subtracting $Tr[V(t)\Omega V(t)\Omega]$ from $Tr[V''(t)\Omega V''(t)\Omega]$ we get:

$$Tr[V''(t)\Omega V''(t)\Omega]-Tr[V(t)\Omega V(t)\Omega]$$
$$= -2(2N+1)(1-e^{-\gamma_0 t})e^{-\gamma_0 t}Tr[V(0)(S^TS-I)]$$

Thus we can see from the above equation that the necessary condition for equivalence of $V(t)$ and $V'(t)$ is satisfied only when $S^TS = I$. This holds true only for the $SO(2)$ subgroup of $Sp(2,R)$.

In fact, that $SO(2)$ commutes with the channel can be seen from (139). If one chooses $S' = S^T$, one obtains that $V''(t) = V(t)$.

Thus, under the channel action the outputs of any two input states, equivalent upto unitary transformations, are inequivalent unless the respective covariance matrices of the inputs are related by an $SO(2)$ transformations. Thus among all symplectic transformations only $SO(2)$ transformations commute with the channel.

Now, for the aim of this exercise:

Consider any two-mode Gaussian state $\hat{\rho(0)}$ (identified by its covariance matrix, $V(0)$) that doesn't undergo ESD after the channel (the channel mentioned in section 6.4 i.e. the one which acts on both the modes of the two-mode state) is applied to it at zero temperature. Now take the covariance matrix, $V(0)$ of the two-mode Gaussian state, $\hat{\rho(0)}$ (before the channel was applied to it) and perform an $SO(2) \oplus SO(2)$ transformation on this covariance matrix to obtain a new two-mode covariance matrix, $V'(0)$. This new covariance matrix, $V'(0)$ corresponds to a new Gaussian state, $\hat{\rho'(0)}$ which is equivalent to the former upto a local-symplectic transformation which in this case is rotation. Let us apply the channel to this new Gaussian state, $\hat{\rho'(0)}$.

By the exercise we've just performed in this section, we know that $V(t>0)$ and $V'(t>0)$ are both equivalent (upto an $SO(2) \oplus SO(2)$ transformation) for all times. i.e. one can obtain $V(t)$ from $V'(t)$, at any time, by performing the inverse of the transformation that was initially performed on the former to obtain the latter before the channel was applied. Now the condition for a two-mode Gaussian state to be entangled is that (45) shoudn't be satisfied. This condition is invariant under local-symplectic transformations. Hence, knowing that a certain Gaussian state (in our case, represented by covariance matrix $V(t>0)$) won't satisfy (45) for some time t, means that any other Gaussian state obtained from the former by local unitary transformations should have its covariance matrix (in our case, $V'(t>0)$) also not satisfy (45). This means, that if $\hat{\rho(t)}$ will remain entangled at all finite times (i.e. won't undergo ESD) then so will $\hat{\rho'(t)}$ also remain entangled.

Hence, the exercise we've just performed helps us to generalize the existing result i.e. it helps us find new Gaussian states which don't undergo ESD from the ones we already know.

6.6 Extending the ESD Result for Different States at Zero Temperature

Whether or not a two-mode Gaussian state is entangled is verified by looking at its covariance matrix. The covariance matrix should not satisfy (45). Having already obtained a set of states which don't undergo ESD, we now want to ask: Can one find instances of ESD occurring for certain two-mode Gaussian states under the effect of the channel (for T=0)?

What one needs to do to answer this question is to look for a 4×4 real, symmetric and positive-semidefinite matrix of the form (38) which:

1. Satisfies (43) but 2. Doesn't satisfy (45).

(Satisfying (43) implies that the matrix is a bona-fide covariance matrix and that there exists a Gaussian state corresponding to this covariance matrix. Dissatisfying (45) implies that the corresponding Gaussian state is entangled.)

Having found such a matrix, apply the channel and see if the matrix undergoing evolution according to (130) will satisfy (45) at any finite time.

For the sake of comparison, we reproduce all the necessary and sufficient conditions for a general, 4×4 real and symmetric two-mode matrix (taken of the form of (38)) to be a bonafide two-mode covariance matrix representing an entangled Gaussian state[35]:

Sufficient Condition for covariance[36]:-

$$detAdetB + (1 - detC)^2 - Tr(AJCJBJC^TJ) \geq detA + detB \tag{140}$$

(129) guarantees that under the application of the channel, this condition is still satisfied.

As explained in Section 3, $DetC \geq 0$ necessarily corresponds to a separable state. Hence we take $DetC < 0$.

Sufficient Condition for Entanglement:

For time, t = 0, when channel has not been applied on the state:-

[35] The condition of covariance automatically satisfies the condition of positivity of the matrix, hence this need not be separateyl checked.
[36] The missing factors is explained by footnote 30

$$detA detB + (1 - |detC|)^2 - Tr(AJCJBJC^T J) - detA + detB < 0 \tag{141}$$

For time, $t > 0$, when the channel is being applied on the state:-

$$x^2(\mu x^2 + \nu x + \zeta) < 0 \tag{142}$$

where

$$x = e^{-\gamma_0 t}\text{,gives the time dependence of the covariance matrix}$$

$$\begin{aligned}\mu =& 1 - Tr(A + B) + detA + detB + TrATrB - TrAdetB - TrBdetA + detAdetB + (detC)^2 \\ &- Tr(AJCJBJC^T J) - Tr(JAJCC^T) - Tr(JBJC^T C) - Tr(CC^T)\end{aligned}$$

$$\begin{aligned}\nu =& 3Tr(A + B) - 2TrATrB - 2(detA + detB) + detATrB + detBTrA - 4 \\ &+ Tr(AJCJBJC^T J) + Tr(JAJCC^T) + Tr(JBJC^T C) + 2Tr(CC^T) \text{ and}\end{aligned}$$

$$\zeta = (2 - TrA)(2 - TrB) - 2|detC| - Tr(CC^T)$$

We can see that x goes from 1 to 0 as time t flows from 0 to ∞.

So having found a 4×4 matrix of the form (38) which satisfies (140) and (141), we need to find a value of time, t when (142) won't be satisfied, to establish that the state will indeed undergo ESD.

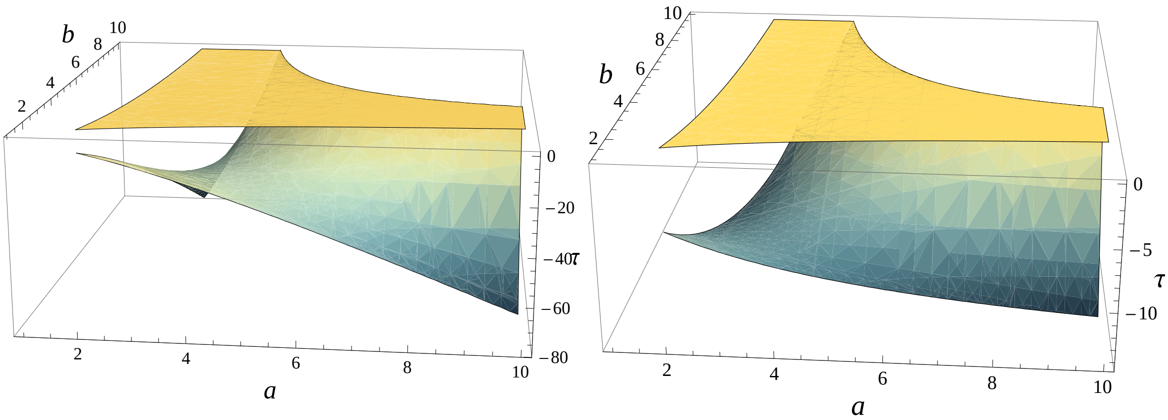
As can be seen the LHS of (142) is a rather complicated expression and it would involve too many parameters to verify generally whether two-mode Gaussian states undergo ESD or not, even numerically. For this reason, we choose to simplify the complexity of our problem by taking the covariance matrix to be in a canonical form in which the number of parameters involved is lower.

6.6.1 States That Won't Undergo ESD

S. Ghosh and S. Goyal have found a certain set of two-mode Gaussian states which won't undergo ESD under the channel. These states have a covariance matrix of the form given by (134). These are symmetric two-mode Gaussian states in which $|c| = |d|$ (see (37). Can this result be extended to non-symmetric two-mode Gaussian states too? Do the states with covariance matrix of the form (143) with $|c| = |d|$ undergo ESD?

$$V(0) = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & -c \\ c & 0 & b & 0 \\ 0 & -c & 0 & b \end{pmatrix} \tag{143}$$

We plot τ against a and b fixing $c = 0.84 \times \sqrt{ab}$ and different values of x.



(a) x=1

(b) x=0.002

Figure 1: ESD doesn't take place when $|c| = |d|$.

Again, any point on the τ surface represents a bona-fide two-mode Gaussian state. If the point lies below the $z = 0$ surface then it's entangled. If it lies above the $z = 0$ plane, then it is separable. We can see that for the case where $c = 0.84\sqrt{ab}$, all the points on the τ surface lie below the $z = 0$ plane for $x = 1$ and for $x = 0.002$. One can vary all the parameters (i.e. $a, b, c,$) and compare again for different values of x's. We still find that all the points on the τ surface lie below the $z = 0$ surface, hence showing us that no two-mode Gaussian state with a covariance matrix of the form given by (143) will undergo ESD.

6.6.2 Certain Two-Mode Gaussian States Undergo ESD at T=0

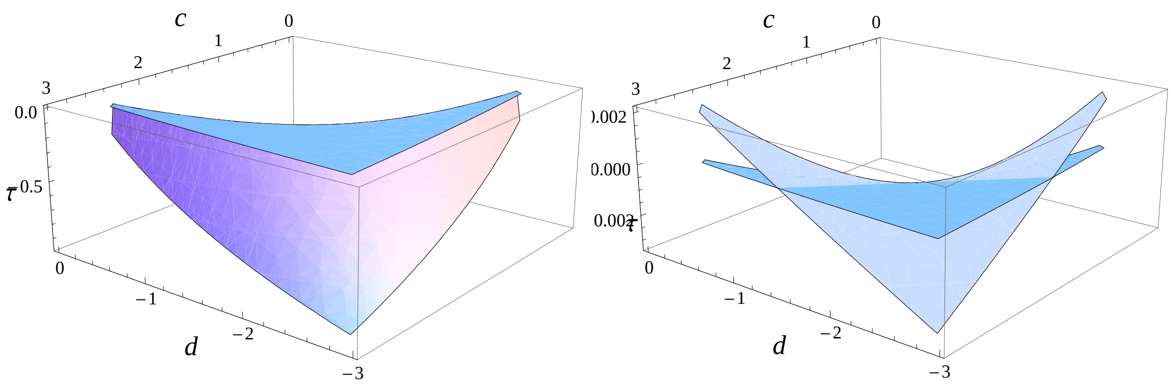
We start out with a symmetrical Gaussian state whose covariance matrix is in the canonical form given by:

$$V(0) = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & d \\ c & 0 & a & 0 \\ 0 & d & 0 & a \end{pmatrix} \tag{144}$$

Comparing (38) with (144), we find that (17) implies $a \geq 1$.

Consider the expression on the LHS of (141) i.e. $x^2(\mu x^2 + \nu x + \zeta)$. It is sufficient to see what happens to the expression $\tau = \mu x^2 + \nu x + \zeta$ i.e. we can make out from this expression itself whether the state will be entangled (i.e. $\tau < 0$) or not ($\tau \geq 0$). To establish the existence of states that do undergo ESD, we plot τ against c amd d for specified values of a and x.

In the following series of plots, we start at $x = 1$ i.e. $t = 0$ and increase to $x \to 0$ i.e. $t \to \infty$.



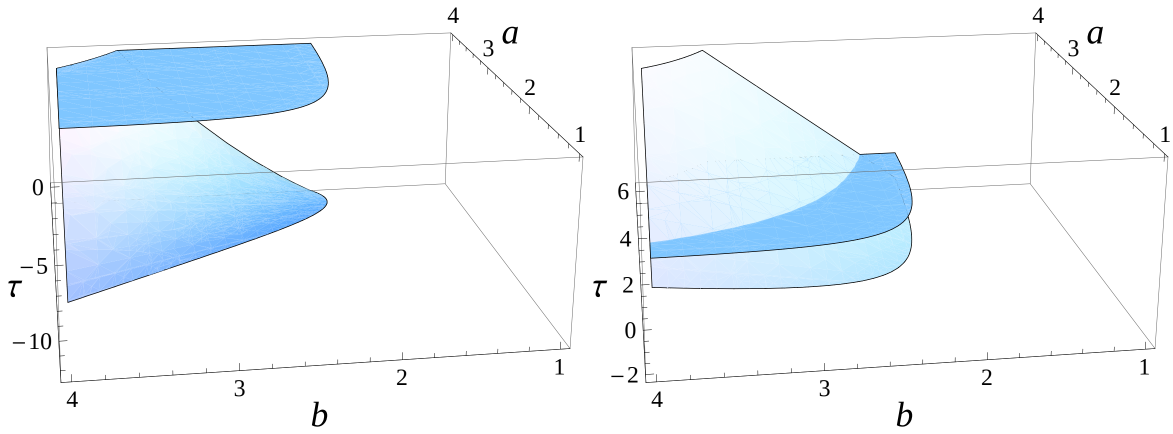
(a) x=1

(b) x=0.002

Figure 2: Some two-mode Gaussian states whose covariance matrix $V(0)$ is of the form (144 undergo ESD.)

τ(z-axis) is plotted against d (x-axis) and c (y-axis) with a = 2.975 and two different values of x, specifying different times. Each point on the τ surface represents a two-mode Gaussian state whose covariance matrix is of the form (144) with a = 2.975 and c and d being specified by the co-ordinates on the x-y plane. In accordance with the requirement that $detC < 0$ for a two-mode Gaussian entangled state, we have $d < 0$ and $c \geq 0$. Also, the $z = 0$ plane has also been shown. In fig. 2(a) x=1, all points on the τ surface are below the $z = 0$ plane, meaning that all states are entangled. In fig. 2(b), x=0.002[37] and here a portion of the τ surface lies above the $z = 0$ plane indicating that these the states corresponding to these points have have undergone ESD to become separable now.Hence, ESD does take place for some

[37]One may wonder why we have taken x=0.002 instead of taking x=0. The reason for this is that x=0 IS the limit $t \to \infty$. In this limit the decoherence effect would have completely taken place, rendering the two-mode Gaussian state separable (to be specific, it turns into a two-mode vacuum state for the zero-temperature environment). We want to observe whether ESD takes place or not, not the effect due to decoherence. Hence we do not take the limiting value for x, but let x tend to the limit and see if there is any ESD before decoherence.



h

(a) x=1

(b) x=0.002

Figure 3: Some two-mode Gaussian states whose covariance matrix $V(0)$ is of the form (144) undergo ESD. Here c=0.402 and d=-0.88. At x=0, all the states are entangled whereas at x=0.002, many states have become separable.

Gaussian states whose covariance matrices are of the form (144). Given the value of γ_0, the actual time of ESD can be predicted for such states. One can see that as $|c| - |d|$ increases, the states undergo ESD earlier. Along the line $|c| = |d|$, no states undergo ESD, which is consistent with the earlier findings of S. Ghosh and S. Goyal.

Similarly, we can see that some of the two-mode Gaussian states with covariance matrix of the form (37) will undergo ESD (see fig 3(a) and fig 3(b)).

Conclusion

In this work, we started with the basic Hilbert Space structure for continuous variable quantum systems used in Quantum Information Theory. We describe what Gaussian states are and give the necessary and sufficient condition for the separability of two-mode Gaussian states. Furtheron we describe what Gaussian Channels are and give a classification of one-mode Bosonic Gaussian channels with a single-mode environment along with the associated Kraus representation. After that we studied of a two-mode Gaussian state under the action of a channel. The channel consists of two exclusive thermal bath environments, both at the same temperature and each acting individually on one mode of the two-mode Gaussian state. It is found that for all non-zero temperatures, all two-mode Gaussian states will undergo ESD. We tried to extend this result to the zero temperature. Also we found that the channel doesn't commute with local-symplectic transformations $Sp(2,R) \otimes Sp(2,R)$ acting on the states other than the transformations from the $SO(2) \otimes SO(2)$ subgroup of $Sp(2,R) \otimes Sp(2,R)$. Since we already know some states which don't undergo ESD, we can obtain a new class of states which won't undergo ESD from this i.e. the amount of entanglement between the two mode of the two-mode Gaussian states doesn't change under local unitary transformations. Since the channel-action commutes only with the $SO(2)\otimes SO(2)$ transformations, two states equivalent under an $SO(2) \otimes SO(2)$ transformation before the channel-action will also remain equivalent after the channel action. Hence, the amount of entanglement between the two-modes for both such states is the same. Hence if one doesn't undergo ESD, the other also wouldn't. We also try to extend the result (i.e. finding states which will undergo ESD and which won't at zero temperature) by choosing different forms of the covariance matrix. It's found that extending this result for the general case, even numerically, is extremely difficult owing to the large number of parameters involved. Hence we take a canonical form of the covariance matrix. We obtain a class of covariance matrices which don't undergo ESD and also another class which does undergo ESD. Despite this exercise we have not been able to generalize this result for all two-mode Gaussian states. Work on that is still going on.

References

[1] Quantum Computation and Quantum Information, Neilsen and Chuang

[2] "Generalized Uncertainty Relations and Coherent and Squeezed States", D. A. Trifonov, http://arxiv.org/abs/quant-ph/0012072

[3] Simon, Chaturvedi, Srinivasan J. Math. Phys. Vol. 40, No. 7 (1999)

[4] Simon, Mukunda, Dutta, Phys. Rev. A Vol. 49, 1567(1994)

[5] R. Simon, Phys. Rev. Lett. Vol. 84, 2726

[6] "Separability Criteria for Density Matrices", A. Peres, Phys. Rev. Lett. Vol. 77, 1413

[7] "One-Mode Quantum Gaussian Channels", A. S. Holevo, quant-phy/0607051

[8] F. Cariso, V. Giovannetti, A. S. Holevo, New Journal of Physics 8(310) (2006)

[9] J. Solomon Ivan, R. Simon, K. Sabapathy, http://arxiv.org/abs/1012.4266

[10] "Entanglement Breaking Channels in Infinite Dimensions", A. S. Holevo, http://arxiv.org/abs/0802.0235

[11] Quantum-to-Classical Transition And Entanglement Sudden Death in Gaussian States Under Local Heat Bath Dynamics", S. K. Goyal, S. Gosh, Phys. Rev. A. 82, 042337

[12] T. Heinosaari, A. S. Holevo and M. M. Wolf, Quantum Inf. Comp. 10, 0619(2010)