Automorphisms Of Bipartite Graph Planar Algebras And Their Subfactor Planar Algebras

By

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Declaration

I, hereby declare that the investigation presented in this thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part of a degree/diploma at this or any other Institution/University.

Kajal Das

Bonafide Certificate

I, hereby certify that this dissertation titled **Automorphisms Of Bipartite Graph Planar Algebras And Their Subfactor Planar Algebras** is the bonafide work of **Kajal Das** who carried out the project under my supervision.

Prof. V. S. Sunder
Institute of Mathematical Sciences
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Date

Abstract

We study the concept of a Bipartite Graph Planar Algebra (BGPA) corresponding to a uniformly locally finite bipartite graph with a uniformly bounded spin function as well as the automorphism group of this planar algebra. This has its origins in [J2] and [B]. We show that this group is isomorphic to the semidirect product of two special types of subgroups which are easily computable from the bipartite graph. We are interested in those group actions whose fixed point subalgebras are Subfactor Planar Algebras (SPAs) because new SPAs will produce new subfactors. Finally we show that the SPA of a 'diagonal subfactor without cocycle' can be obtatined by this fixed point technique.

Acknowledgements

First and foremost I offer my most sincere gratitude to my supervisor, Prof. V. S. Sunder, who has supported me throughout my master's thesis with his patience and knowledge. He has been abundantly helpful and has assisted me in numerous ways. The discussions I had with him throughout the last year were invaluable and motivated me towards future research.

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I would also like to thank Prof. Vijay Kodiyalam for bearing with me during my presentations on Quantum Groups which has strong relation with Planar Algebra.

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Contents

Chapter 1

Introduction

The study of subfactors arose in the seminal work of Jones [J1]. The first isomorphism invariants of a subfactor which were investigated were its *index* and *principal graph*. The more tractable subfactors were those with finite depth (equivalently, the principal graph was a finite bipartite graph), in which case the former being the square of Perron-Frobenius eigenvalue of the latter.

The notion of a planar algebra arose in Jones' [J1] description of the so-called 'standard invariant of a subfactor'. The initial formulation of a planar algebra restricted itself to those planar tangles where the 'distinguished (starred) regions were always unshaded' - see [J1], [KS1]. It was soon realised that this 'parity constraint' was neither necessary nor desirable. Also, the planar algebras that arose from a subfactor had many more special properties not necessarily seen in the more general examples that the obvious/natural axiomatisation of a planar algebra demanded. Thus, it was harder to construct planar algebras which were also 'subfactor planar algebras'.

In view of the two initial subfactor invariants discussed earlier, it is not surprising that, in his quest for examples, Jones initiated the study of a class of planar algebras built out of a finite bipartite graph (in [J2]). Subsequently, this construction was re-examined by Burstein [B], allowing infinite bipartite graphs which satisfied mild 'local finiteness conditions'. His motivation was to construct a large class of planar algebras which, even if they themselves were not subfactor planar algebras, often had planar subalgebras which were! This thesis is an attempt to understand and explain Burstein's construction of what he calls a BGPA, this being an acronym for 'bipartite graph planar algebra'. Further, unlike their finite counterparts, infinite connected graphs do not possess a unique Perron-Frobenius eigenvalue (up to scaling by a positive constant); so these BGPAs have an 'spin function' or postive weight function on their vertex sets built into their definition.

Overview of the thesis:

In Chapter 2 we recall the concept of planar algebra with the original definition relaxed so as to permit starred region to be shaded. In our case each vector space and disk of a planar tangle will be doubly indexed. We then illustrate some planar tangles which are the building blocks of the operad of planar tangles and introduce the concept of *Subfactor Planar Algebra* (SPA).

In Chapter 3 we introduce *Bipartite Graph Planar Algebra* (BGPA) corresponding to a connected locally finite bipartite graph with a locally bounded *spin function*. We construct a graded

Hilbert space with the paths of the graph as orthonormal basis and embed the graded vector spaces of the BGPA inside the set of bounded operators of the graded Hilbert spaces. The above identification induces topological structures on the graded vector spaces. We then study the action of some generating planar tangles on this planar algebra and these actions happen to be continuous in a particular sense. We then try to make the BGPA into a 'good' planar algebra by introducing positive definite forms on the vector spaces and show that the existence of modulus-δ for a finite graph demands the spin function to be the Perron-Frobenius eigenvector (upto normalization) and δ to be the Perron-Frobenius eigenvalue of the graph.

In Chapter 4 we study automorphisms of BGPA. We introduce two types of automorphisms - *multiplication operator* and *graph automorphism operator* and prove that the group of automorphisms is the semidirect product of the group of graph automorphism operators and multiplication operators.

In Chapter 5 we deduce some sufficient conditions on a planar $*$-subalgebra of a BGPA to be an SPA and it becomes an important tool to decide whether the fixed point subalgera corresponding to a group action on a BGPA is SPA. We then give an example of a fixed point subfactor planar algebra.

Chapter 2

Planar algebra

Planar algebras were first introduced by Jones in [J1]. The main motivation was to study the 'standard invariants' of certain subfactors. The main bridge between planar algebras and subfactors is Jones' theorem ([J1]), which imposes planar algebra structute on the 'standard invariants' of subfactors and conversely shows that a planar algebra with some additional properties is the standard invariant of a subfactor. However, since its introduction, the definition of planar algebras has undergone some modifications, and for completeness we shall mainly follow [J1], [KS1], [JP],[Pet] with slight modifications.

2.1 Planar tangles

Planar tangles are essentially the pictorial forms of the elements of a *colored planar operad*, which is just a slight modification of *operad* and can be defined along the lines of [Ma]. Each element of the colored planar operad, i.e., each planar tangle determines a multilinear operation on the standard invariant. The definition of a planar tangle here will be a slight modification of the definition given in [J1].

2.1.1 Definition of a planar tangle

We define a set Col=$(\mathbb{N} \cup \{0\}) \times \{+, -\}$, whose members we shall call 'colors' and will denote by ordered pairs (k, ϵ). If a variable ϵ takes a value from $\{+, -\}$, then the variable $\tilde{\epsilon}$ will take the other value. If ϵ is $+$, then ϵ^n is defined as $+$ $\forall n \in \mathbb{N} \cup \{0\}$. If ϵ is $-$, then ϵ^n is defined as $+$ if n is even and $-$ if n is odd. Now we define a *planar tangle* T as a system with the following data:

1. It has a closed disk $D_0(T)$ in the complex plane and a finite collection $\{D_i(T)|i = 1, \ldots, b(T)\}$ of pairwise disjoint disks in the interior of $D_0(T)$,

2. It has a compact one dimensional submanifold T of $D_0(T) \setminus \cup_{i=1}^{b(T)} Int(D_i(T))$ with

   (a) $\partial(T) \subset \cup_{i=0}^{b(T)} \delta(D_i(T))$ and all intersections of T with each $\partial(D_i(T))$ are transversal;

   (b) $|T \cap \partial(D_i(T))| = 2k_i$ for some integers $k_i \geq 0$, for each $0 \leq i \leq b(T)$;

3. The connected components of $D_0 \backslash \left((\cup_{i=1}^{b(T)} Int(D_i(T)) \cup T \right)$ will be called *regions*. If the system satisfies the above two conditions the set of regions of T has a *chequerboard shading*. Choose a chequerboard shading of T.

4. For each disk $D_i(T)$ mark one adjacent region as $*$-region.

We say the color of $D_i(T)$ is (k_i, ϵ_i) if $|\partial(T) \cap \partial(D_i(T)| = 2k_i$ and the $*$-region of $D_i(T)$ is white (when $\epsilon_i = +$) or black (when $\epsilon_i = -$).

An example of a tangle T with 3 internal disks is illustrated below - in which the outer disk has color $(3, +)$ and the internal disks D_1, D_2 and D_3 have colors (0,-), (3,+) and (3,+) respectively:

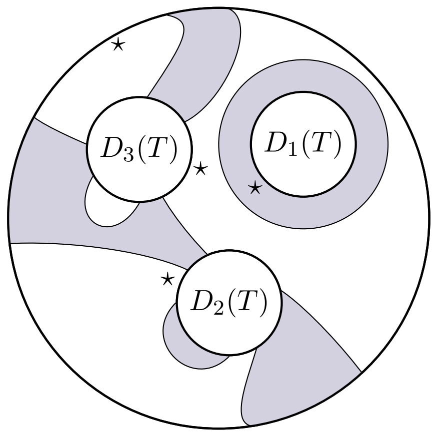


Figure 2.1: (3,+)-tangle T

The connected components of T will be called *strings* and the local maxima and minima of the y-coordinate function of each string will be called *singular points* if they do not occur at the boundary of the disks.

Two tangles T and S will be called *equivalent* if $b(T) = b(S)$ and there exists an orientation preserving diffeomorphism of the complex plane which maps each $D_i(T)$ onto $D_i(S)$, the submanifold T onto S, and preserves the $*$-regions of each disk. Finally, an equivalence class under above equivalence is called a (k_0, ϵ_0)-*tangle*.

2.1.2 Composition of two tangles

There is a natural way to 'compose' two tangles. Suppose T is a $(k_0(T), \epsilon_0(T))$-tangle, with $b(T)(\geq 1)$ internal disks of colors $(k_j(T), \epsilon_j(T))$ and S is a (k_i, ϵ_i)-tangle for some $i \in \{1, \ldots, b(T)\}$ with $b(S)$ internal disks of colors $(k_r(S), \epsilon_r(S)$, then the compostion of S and T will be obtained by gluing the boundaries of $D_0(S)$ and $D_i(T)$ taking care to match $*$-regions and colors of the adjacent regions and matching and smoothing the strings at $\partial D_0(S)$ and finally deleting $\partial D_0(S)$; the resulting tangle T_1 will be denoted by $T \circ_i S$. The numbering of the internal disks of T_1 is given by

$$D_j(T_1) = \begin{cases} D_j(T) & \text{if } 1 \leq j < i \text{ ;} \\ D_{j-i+1}(S) & \text{if } i \leq j \leq i + b(S) - 1; \\ D_{j-b(S)+1}(T) & \text{if } i + b(S) \leq j \leq b(T) + b(S) - 1. \end{cases}$$

We illustrate this by the following example in Figure 2.2:

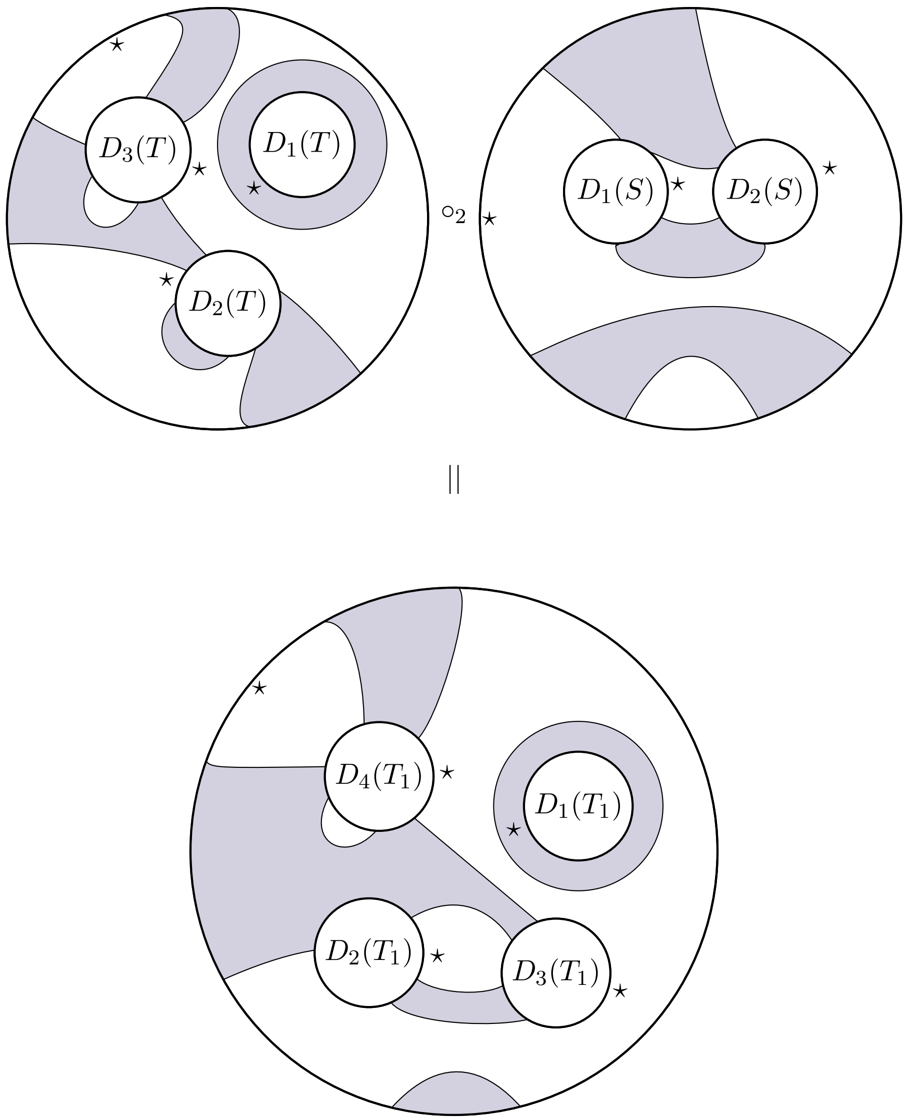


Figure 2.2: Composition of two tangles

2.1.3 Standard form of a planar tangle

Suppose a planar tangle T is arranged so that

1. all the disks are horizontal rectangles with all strings (except loops) emanating from the top or bottom edge of the rectangles;

2. for each disk the number of strings attached to the top and bottom edge are same;

3. the left edge of each rectangle is adjacent to the $*$-region of the same;

4. the total number of singular points of the strings is finite;

5. we can partition the outer disk into a finite number of horizontal strips having the internal disks and the singular points in different strips.

This is called a "Standard form representative" of a tangle. A standard form representative of the above tangle T is shown below:

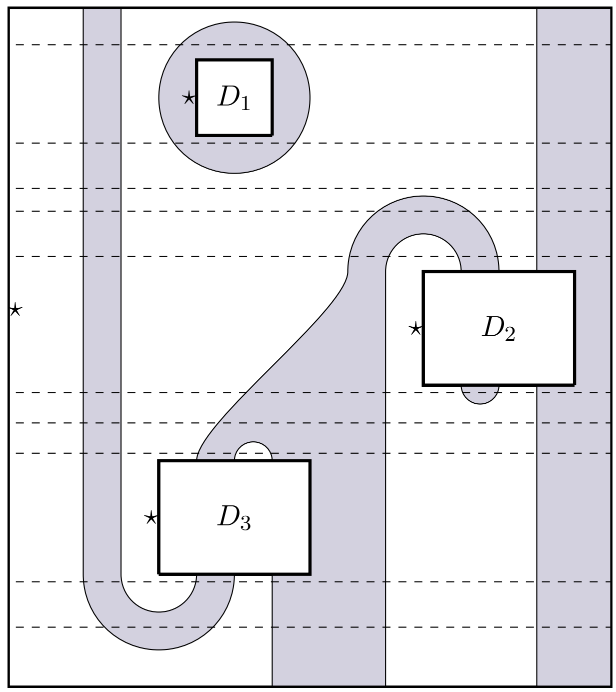
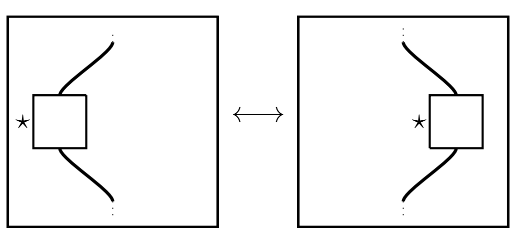


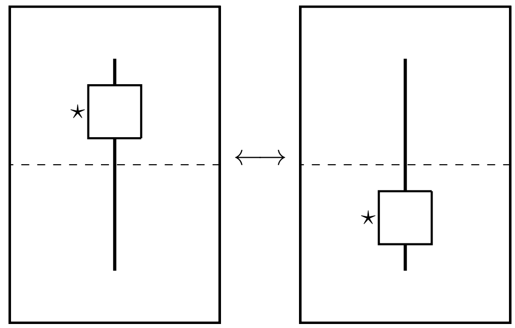
Figure 2.3: Standard form of the tangle T

Note that there could be several standard forms of a tangle T. However it is a fact from [J1], [BDG1] that one standard form of all such T can be transformed into another by a finite sequence of moves of the following five types:

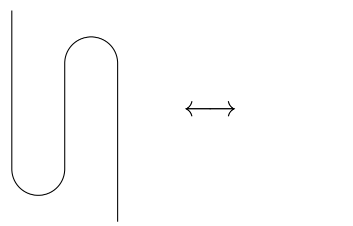
I. *Horizontal sliding of the internal disks*:



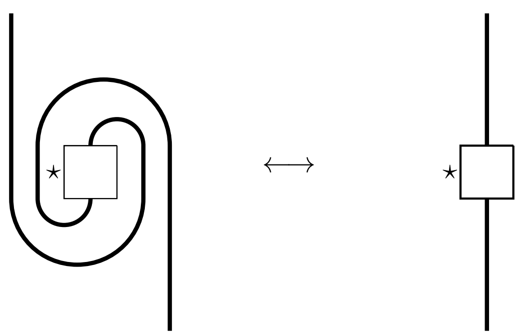
II. *Vertical sliding of the internal disks*:



III. *Vertical Wiggling of the strings*:



IV. 360° *rotation of an internal disk*:

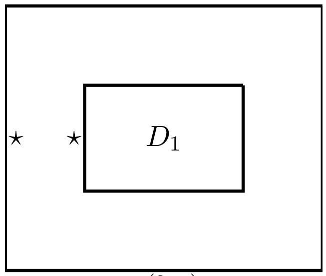


V. *isotopy in a segment of a string such that at each time the curve in the isotopoy does not have any singular point.*

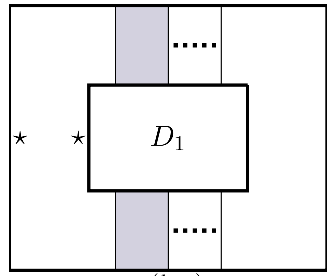
2.1.4 Some important tangles

In this sub-section we define a set of important tangles in the operad of colored tangles. Some of them form a generating set in the collection of all tangles, i.e., any tangle can be obtained by composing finitely many tangles from the set of generating tangles.

1. *Identity tangles:*



$\mathcal{I}_{(0,+)}^{(0,+)}$

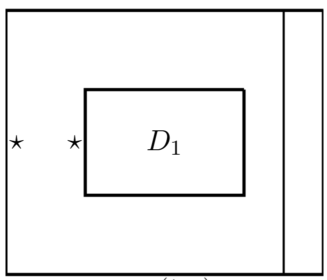


$\mathcal{I}_{(k,+)}^{(k,+)}$

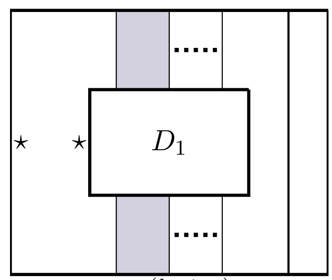
Similarly, $\mathcal{I}_{(0,-)}^{(0,-)}$ and $\mathcal{I}_{(k,-)}^{(k,-)}$ are defined by reverse shadings respectively.

2. *Inclusion tangles:*

   (a) *Right Inclusion tangles:*



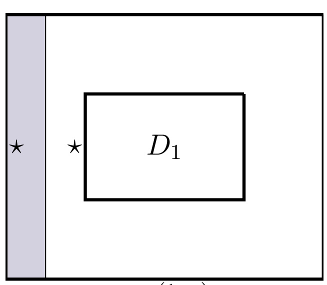
$\mathcal{RI}_{(0,+)}^{(1,+)}$



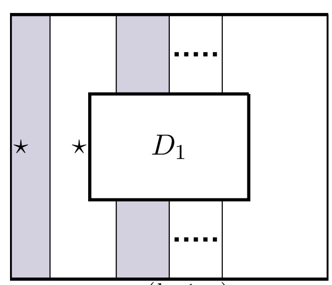
$\mathcal{RI}_{(k,+)}^{(k+1,+)}$

Similarly, $\mathcal{RI}_{(0,-)}^{(1,-)}$ and $\mathcal{RI}_{(k,-)}^{(k+1,-)}$ are defined by reverse shadings respectively.

   (b) *Left Inclusion tangles:*



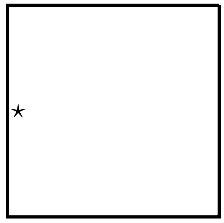
$\mathcal{LI}_{(0,+)}^{(1,-)}$



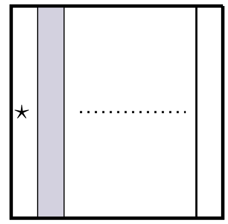
$\mathcal{LI}_{(k,+)}^{(k+1,-)}$

Similarly, $\mathcal{LI}_{(0,-)}^{(1,+)}$ and $\mathcal{LI}_{(k,-)}^{(k+1,+)}$ are defined by reverse shadings respectively.

3. *Unit tangles:*



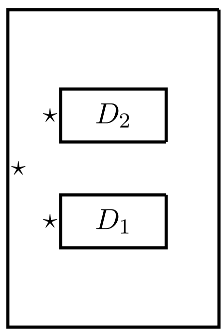
$1^{(0,+)}$



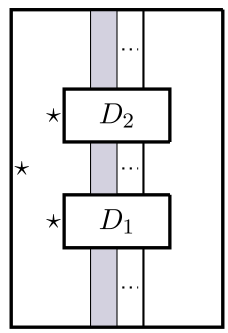
$1^{(k,+)}$

Similarly, $1^{(0,-)}$ and $1^{(k,-)}$ are defined by reverse shadings respectively.

4. *Multiplication tangles:*



$\mathcal{M}_{(0,+)}$

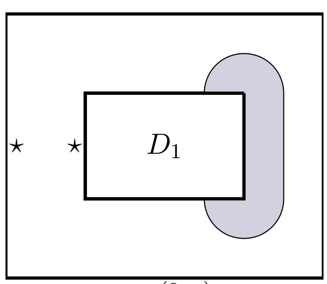


$\mathcal{M}_{(k,+)}$

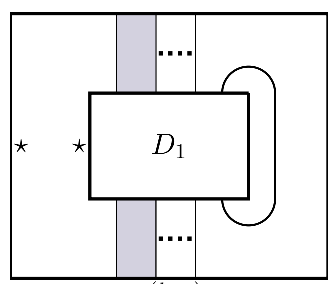
Similarly, $\mathcal{M}_{(0,-)}$ and $\mathcal{M}_{(k,-)}$ are defined by reverse shadings respectively.

5. *Conditional Expectation tangles:*

   (a) *Right Expectation tangles:*



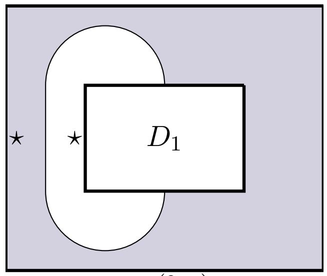
$\mathcal{RE}^{(0,+)}_{(1,+)}$



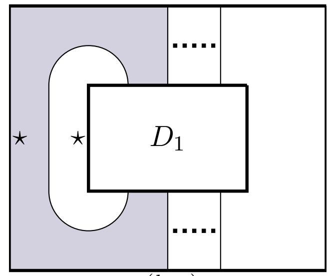
$\mathcal{RE}^{(k,+)}_{(k+1,+)}$

Similarly, $\mathcal{RE}^{(0,-)}_{(1,-)}$ and $\mathcal{RE}^{(k,-)}_{(k+1,-)}$ are defined by reverse shadings respectively.

   (b) *Left Expectation tangles:*



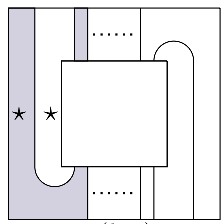
$\mathcal{LE}^{(0,-)}_{(1,+)}$



$\mathcal{LE}^{(k,-)}_{(k+1,+)}$

Similarly, $\mathcal{LE}^{(0,+)}_{(1,-)}$ and $\mathcal{LE}^{(k,+)}_{(k+1,-)}$ are defined by reverse shadings respectively.

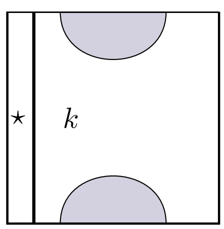
6. *Half rotation tangles:*



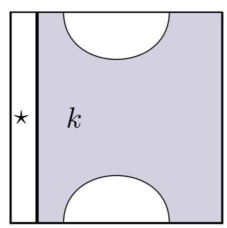
$\mathcal{R}^{(k,-)}_{(k,+)}$

Similarly, $\mathcal{R}^{(k,+)}_{(k,-)}$ is defined by reverse shading.

7. *Temperley-Lieb tangles:*



$\mathcal{TL}^{(k,+)}$(k is even)



$\mathcal{TL}^{(k,-)}$(k is odd)

Similarly, $\mathcal{TL}^{(k,-)}$ is defined by reverse shading. For the notational convenience, $\mathcal{TL}^{(0,+)}$ and $\mathcal{TL}^{(0,-)}$ will be denoted by $\mathcal{TL}^{+}$ and $\mathcal{TL}^{-}$ respectively.

Now we are ready to give the list of generating tangles.

Theorem 2.1.1. *[KS1]Let $\mathcal{T}$ be a collection of colored tangles containing*

$$\mathcal{G} := \left\{1^{(0,\pm)}, \mathcal{TL}^{+}, \mathcal{TL}^{-}\right\} \cup \left\{\mathcal{M}_{(k,\pm)}, \mathcal{RI}^{(k,\pm)}_{(k+1,\pm)}, \mathcal{LI}^{(k,\mp)}_{(k+1,\pm)}, \mathcal{RE}^{(k,\pm)}_{(k+1,\pm)}, \mathcal{LE}^{(k,\pm)}_{(k+1,\mp)} : k \geq 0\right\}, \quad (2.1)$$

and suppose $\mathcal{T}$ is closed under composition of tangles, whenever it makes sense. Then $\mathcal{T}$ contains all tangles.

Definition 2.1.2. *For any tangle T we define the 'adjoint' of T, denoted by T^*, as the reflection of T about any line in the plane such that the shadings and the $*$-regions are preserved by this reflection.*

2.2 Planar algebras

A *planar algebra* is a collection of vector spaces $P = \{P_{(k,\epsilon)} : (k,\epsilon) \in Col\}$ which admits an action of each tangle, i.e., if T is a (k_0,ϵ_0)-tangle with b internal disks of colors (k_i,ϵ_i) then there is a linear map (called *tangle map*)

$$Z_T^P : \begin{cases} \otimes_{i=1}^{b} P_{(k_i,\epsilon_i)} \to P_{(k_0,\epsilon_0)} & \text{if } b(T) > 0\ ; \\ \mathbb{C} \to P_{(k_0,\epsilon_0)} & \text{if } b(T) = 0. \end{cases}$$

such that the set of tangle maps satisfy the following conditions:

1. *Compatibility with the composition of tangles:* Suppose T and S are as in § 2.1.2, then

   (i) if $b(S) > 0$ the following diagram commutes:

$$\begin{tikzcd}
(\otimes_{j=1}^{i-1} P_{(k_j(T),\epsilon_j(T))}) \otimes (\otimes_{r=1}^{b(S)} P_{(k_r(S),\epsilon_r(S))}) \otimes (\otimes_{j=i+1}^{(b(T)} P_{(k_j(T),\epsilon_j(T))}) \arrow[d, "id \otimes Z_S^P \otimes id"'] \arrow[dr, "Z^P_{T\circ_i S}"] & \\
\otimes_{j=1}^{b(T)} P_{(k_j(T),\epsilon_j(T))} \arrow[r, "Z_T^P"'] & P_{(k_0(T),\epsilon_0(T))}
\end{tikzcd}$$

   (ii) if $b(S) = 0$ this diagram commutes:

$$\begin{tikzcd}
\otimes_{j\neq i} P_{(k_j(T),\epsilon_j(T)} \arrow[d, "\cong"'] \arrow[dr, "Z^P_{T\circ_i S}"] & \\
(\otimes_{j=1}^{i-1} P_{(k_j(T),\epsilon_j(T))}) \otimes \mathbb{C} \otimes (\otimes_{j=i+1}^{(b(T)} P_{(k_j(T),\epsilon_j(T))}) \arrow[d, "id \otimes Z_S^P \otimes id"'] & P_{(k_0(T),\epsilon_0(T))} \\
\otimes_{j=1}^{b(T)} P_{(k_j(T),\epsilon_j(T))} \arrow[ur, "Z_T^P"'] &
\end{tikzcd}$$

2. *Independence of the ordering of the internal disks:* For a tangle T with b internal disks of colors (k_j,ϵ_j), and for a permutation $\sigma \in S_b$, let us write U_σ for the map

$$\begin{aligned} U_\sigma : \otimes_{j=1}^{b} P_{(k_{\sigma(j)},\epsilon_{\sigma(j)})} &\to \otimes_{j=1}^{b} P_{(k_j,\epsilon_j)} \\ \otimes_{j=1}^{b} x_{\sigma(j)} &\mapsto \otimes_{j=1}^{b} x_j \end{aligned}$$

Let us define $\sigma^{-1}(T)$ to be the tangle which differs from T only in the numbering of its internal disks, this numbering being given by $D_i(\sigma^{-1}(T)) = D_{\sigma(i)}(T), 1 \leq i \leq b$. Then $Z_T^P \circ U_\sigma = Z^P_{\sigma^{-1}(T)}$.

3. *Non-degeneracy Condition:* P must satisfy $Z^P_{\mathcal{I}^{(k,\epsilon)}_{(k,\epsilon)}} = id_{P_{(k,\epsilon)}}\ \forall (k,\epsilon) \in Col$.

Remark 2.2.1. *For each $(k,\epsilon) \in Col$, $P_{(k,\epsilon)}$ is a unital asociative algebra with respect to multiplication given by $x_1x_2 := Z^P_{M_{(k,\epsilon)}}(x_1 \otimes x_2)$ for all $x_1, x_2 \in P_{(k,\epsilon)}$, and multiplicative identity being the element $1_{(k,\epsilon)} := Z^P_{1^{(k,\epsilon)}}(1) \in P_{(k,\epsilon)}$.*

Definition 2.2.2. *Let $P = \{P_{(k,\epsilon)} : (k,\epsilon) \in Col\}$ and $Q = \{Q_{(k,\epsilon)} : (k,\epsilon) \in Col\}$ be two planar algebras. A* planar algebra morphism *from P to Q is a collection $\phi = \{\phi_{(k,\epsilon)} : (k,\epsilon) \in Col\}$ of linear maps $\phi_{(k,\epsilon)} : P_{(k,\epsilon)} \to Q_{(k,\epsilon)}$ which commutes with all the tangle maps, i.e., if T is a (k_0,ϵ_0)-tangle with b internal disks of colors (k_i,ϵ_i), then*

$$\phi_{k_0} \circ Z_T^P = \begin{cases} Z_T^Q \circ (\otimes_{i=1}^b \phi_{(k_i,\epsilon_i)}) & \text{if } b > 0; \text{ and} \\ Z_T^Q & \text{if } b = 0. \end{cases} \tag{2.2}$$

Further, the morphism ϕ is said to be a planar algebra isomorphism *if the maps ϕ_k are all linear isomorphisms. By an* automorphism of a planar algebra *P we shall mean a '$*$' preserving planar isomorphism from P into itself and the set of all automorphisms of a planar algebra P will be denoted by $Aut(P)$.*

Remark 2.2.3. *[VP] Let P and Q be planar algebras as above, and let $\{\phi_{(k,\epsilon)} : P_{(k,\epsilon)} \to Q_{(k,\epsilon)}, (k,\epsilon) \in Col\}$ be a collection of linear maps. If $\mathcal{T}$ is a the set of those tangles T for which equation 2.2 holds, then $\mathcal{T}$ is closed under composition of tangles. Thus, by Theorem 2.1.1, in order to verify whether such a collection is a planar algebra morphism or not, we just need to check equation 2.2 for the set of generating tangles.*

Definition 2.2.4. *Let G be a group and $P = \{P_{(k,\epsilon)} : (k,\epsilon) \in Col\}$ be a planar algebra. We say G* acts on P *if there exists a group homomorphism from G into $Aut(P)$.*

Remarks 2.2.5. *1. Suppose $\phi : G \to Aut(P)$ is a group homomorphism and $\phi(g) = \{\phi_{(k,\epsilon)}(g) : P_{(k,\epsilon)} \to P_{(k,\epsilon)} | (k,\epsilon) \in Col\}\ \forall g \in G$. Then for notational convenience, we write gx for the element $\phi_{(k,\epsilon)}(g)(x)$, for all $g \in G$, $x \in P_{(k,\epsilon)}$ and $(k,\epsilon) \in Col$. Defining*

$$P^G_{(k,\epsilon)} := \{x \in P_{(k,\epsilon)} : gx = x\, \forall g \in G\}, (k,\epsilon) \in Col,$$

we note that $P^G := \{P^G_{(k,\epsilon)} : (k,\epsilon) \in Col\}$ is a planar subalgebra of P.

2. When the planar algebra P is clear from the context, the tangle map Z_T^P will be denoted by Z_T only. Also, deleting the superscript and subscript, we only use $Z_{\mathcal{LI}}$ instead of $Z_{\mathcal{LI}^{k+1,\tilde{\epsilon}}_{(k,\epsilon)}}$ when the domain V_k^ϵ is clear from the context, and we follow the same type of notation for the right embedding tangles and conditional expectation tangles. The map $Z_{\mathcal{LI}^{(k+n,\tilde{\epsilon})}_{(k+n-1,\epsilon)}} \circ \cdots \circ Z_{\mathcal{LI}^{(k+2,\epsilon)}_{(k+1,\tilde{\epsilon})}} \circ Z_{\mathcal{LI}^{(k+1,\tilde{\epsilon})}_{(k,\epsilon)}}$ will be denoted by $[Z_{\mathcal{LI}}]^n$, where n is odd, and we follow the same type of notation when n is even and for the right embedding tangles and expectation tangles.

2.3 Subfactor planar algebras

First we define some 'good' properties of a planar algebra, and then a planar algebra having those 'good' properties is defined as a 'subfactor planar algebra'.

Some good properties of planar algebras:

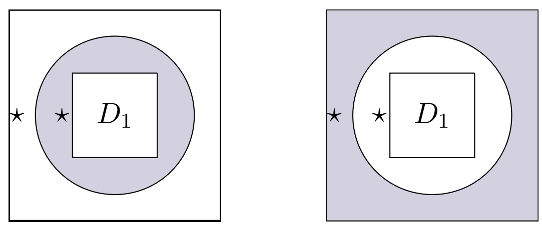
1. A planar algebra is said to be *connected* (respectively, *irreducible*) if dim $P_{(0,\pm)} = 1$ (respectively, dim $P_{(1,\pm)} = 1$).

   Remark: If P is a connected planar algebra, then there is a unique algebra isomorphism between $P_{(0,\pm)}$ and $\mathbb{C}$. By this isomorphism we identify $P_{(0,\pm)}$ with $\mathbb{C}$.

2. A connected planar algebra P is said to have *modulus* δ if there is a scalar δ such that

   $$Z^P_{T^\pm_\mp}(1_{(0,\pm)}) = \delta 1_{(0,\mp)}$$

   where the tangles $T^\pm_\mp$ are illustrated below:

   T^+_- T^-_+

   (a) *Remark:* A planar algebra P having modulus δ is equivalent to saying that a contractible loop comes out as the constant δ, i.e., if T is a tangle with a contractible loop and T' is the same tangle except that loop, then $Z^P_{T'} = \delta Z^P_T$.

   (b) *Remark:* If a planar algebra has non-zero modulus, then inclusion tangles give injective maps.

3. A planar algebra P is said to be *spherical* if $Z^P_{\mathcal{LE}^{(0,-)}_{(1,+)}} = Z^P_{\mathcal{RE}^{(0,+)}_{(1,+)}}$, where $P_{(0,\pm)}$ are identified with $\mathbb{C}$ as above.

Subfactor Planar Algebra (SPA): A planar algebra is said to be a *subfactor planar algebra* if

1. P is connected, spherical, has a positive modulus;

2. each $P_{(k,\epsilon)}$ is a finite dimensional C^*-algebra;

3. for any (k_0, ϵ_0)-tangle T with b internal disks of colors (k_i, ϵ_i),

   $$[Z^P_T(x_1 \otimes \cdots \otimes x_b)]^* = Z^P_{T^*}(x_1^* \otimes \cdots \otimes x_b^*) \tag{2.3}$$

   $\forall x_i \in P_{(k_i,\epsilon_i)}, 1 \leq i \leq b$ ('$*$ property' of P);

4. the scalar sesquilinear form on each $P_{(k,\epsilon)}$ given by $\langle x, y\rangle = [Z^P_{\mathcal{RE}}]^n(y^*x)$, where $x, y \in P_{(k,\epsilon)}$ and $P_{(0,\epsilon)}$ is identified with $\mathbb{C}$, is a positive definite form ('positive definiteness' of P).

Remark 2.3.1. *[DGG] Let $P = \{P_{(k,\epsilon)} : (k,\epsilon) \in Col\}$ and $Q = \{Q_{(k,\epsilon)} : (k,\epsilon) \in Col\}$ be two planar algebras with non-zero moduli and each of $P_{(k,\epsilon)}$ and $Q_{(k,\epsilon)}$ is finite dimensional. Also, let $\phi := \{\phi_{(k,\epsilon)} : P_{(k,\epsilon)} \to Q_{(k,\epsilon)}, (k,\epsilon) \in Col\}$ be a collection of linear maps. Then ϕ is a planar algebra morphism if it commutes with the actions of the following set of tangles:*

$$\left\{\mathcal{M}_{(k,+)}, \mathcal{RI}^{(k+1,+)}_{(k,+)}, \mathcal{LI}^{(k+1,+)}_{(k,-)}, \mathcal{LE}^{(k,-)}_{(k+1,+)}, \mathcal{TL}^{(k,+)} : k \geq 0\right\}.$$

The importance of planar algebras in subfactor theory lies in Jones' theorem. Now we state Jones' theorem.

Theorem 2.3.2. *[J1] Let $N \subset M$ be a finite index extremal II_1 subfactor with $[M : N] = \delta^{1/2} < \infty$. Let $N(:= M_{-1}) \subset M(:= M_0) \subset^{e_1} M_1 \subset^{e_2} M_2 \subset \cdots$ be its associated tower of basic construction. Then the 'standard invariant' of the subfactor $N \subset M$ is given by the following grid of relative commutants:*

$$\begin{array}{ccccccccc} \mathbb{C} = & N' \cap M_{-1} & \hookrightarrow & N' \cap M_0 & \hookrightarrow & N' \cap M_1 & \hookrightarrow & N' \cap M_2 & \hookrightarrow \cdots \\ & & & \uparrow & & \uparrow & & \uparrow & \\ & & \mathbb{C} = & M' \cap M_0 & \hookrightarrow & M' \cap M_1 & \hookrightarrow & M' \cap M_2 & \hookrightarrow \cdots \end{array}$$

Define $P^+_n := N' \cap M_{n-1}$ and $P^-_n := M' \cap M_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then there is a unique subfactor planar algebra structure on $P^{N\subset M} := P = \{P^\epsilon_n : (n,\epsilon) \in Col\}$ satisfying the following properties:

1. *$Z^P_{\mathcal{TL}^{(n,+)}}(1) = \delta e_{n+1}$, $Z^P_{\mathcal{TL}^{(n,-)}}(1) = \delta e_{n+2}$ for all $n \geq 0$;*
2. *$Z^P_{\mathcal{RI}^{(n+1,+)}_{(n,+)}}$, $Z^P_{\mathcal{RI}^{(n+1,-)}_{(n,-)}}$ and $Z^P_{\mathcal{LI}^{(n+1,+)}_{(n,-)}}$ are the inclusions $N' \cap M_{n-1} \hookrightarrow N' \cap M_n$, $M' \cap M_n \hookrightarrow M' \cap M_{n+1}$ and $M' \cap M_n \hookrightarrow N' \cap M_n$ respectively for all $n \geq 0$;*
3. *$Z^P_{\mathcal{RE}^{(n,+)}_{(n+1,+)}} = \delta \mathbb{E}^{N'\cap M_n}_{N'\cap M_{n-1}}$ for all $n \geq 1$ and $Z^P_{\mathcal{RE}^{(n,-)}_{(n+1,-)}} = \delta \mathbb{E}^{M'\cap M_{n+1}}_{M'\cap M_n}$ for all $n \geq 1$;*
4. *$Z^P_{\mathcal{RE}^{(0,+)}_{(1,+)}}(x) = \delta tr_{M_0}(x)$ for all $x \in N' \cap M_0$ and $Z^P_{\mathcal{RE}^{(0,-)}_{(1,-)}}(x) = \delta tr_{M_1}(x)$ for all $x \in M' \cap M_1$;*
5. *$Z^P_{\mathcal{LE}^{(n,-)}_{(n+1,+)}} = \delta \mathbb{E}^{N'\cap M_n}_{M'\cap M_n}$ for all $n \geq 1$ and $Z^P_{\mathcal{LE}^{(0,-)}_{(1,+)}}(x) = \delta tr_{M_0}(x)$ for all $x \in N' M_0$.*

where $\{e_n : n \in \mathbb{N}\}$ and $\{\mathbb{E}^{N'\cap M_n}_{N'\cap M_{n-1}}, \mathbb{E}^{M'\cap M_{n+1}}_{M'\cap M_n}, \mathbb{E}^{N'\cap M_n}_{M'\cap M_n} : n \in \mathbb{N}\}$ are the sets of Jones' projections and conditional expectations associated to the subfactor $N \subset M$ respectively.

Conversely, for any subfactor planar algebra P, Jones via Popa [Po1] shows that there exists a finite index extremal II_1 subfactor $N \subset M$ of which P is the standard invariant.

So, Jones' theorem not only helps in the classification of subfactors, but also gives a way to construct new subfactors, and the construction of new subfactors is important in subfactor theory. The construction of subfactors from SPAs was re-established recently by Guionnet-Jones-Shlyakhtenko ([GJS]), Jones-Shlyakhtenko-Walker ([JSW]) and Kodiyalam-Sunder ([KS2]).

Chapter 3

Bipartite Graph Planar Algebra

Bipartite Graph Planar Algebra (BGPA) was first introduced by Jones for finite connected bipartite graphs in [J2] for constructing new SPAs and Burstein extended this idea for 'uniformly locally finite connected bipartite graphs' in [B]. Also, the recent developments show that BGPA is an important object in planar algebra. Peters in [Pet] found the planar algebra of the Haagerup subfactor inside the planar algebra of the bipartite graph of its principal graph, and Bigelow, Morrison, Peters, and Snyder using this technique gave the first construction of the extended Haagerup subfactor [BMPS]. Both of these results stemmed from the supposition that a finite depth, subfactor planar algebra is embedded in the graph planar algebra of its principal graph, which was shown by Jones and Penneys in [JP]. However, in this chapter we shall mainly follow the formalisms given in [J2], [VP] and [B].

3.1 Definitions, notations and assumptions

Let Γ be a uniformly locally finite connected bipartite graph, i.e., each vertex of Γ is an endpoint of at most N edges for some $N \in \mathbb{N}$. Suppose V^+, V^- and E be the sets of positive vertices, negative vertices and edges respectively. Let μ be a function from $V^+ \sqcup V^-$ to $(0, \infty)$ satisfying the following *uniform boundedness condition*: there is some $M > 0$ such that for any two adjacent vertices v and w, we have $\mu(v)/\mu(w) < M$. The function μ is called a *'spin function'* on Γ.

For any $(k, \epsilon) \in Col$, define l_k^ϵ as the set of all loops of length $2k$ starting at a vertex in V^ϵ, p_k^ϵ as the set of all paths of length k starting at a vertex in V^ϵ and V_k^ϵ as the vector space of all bounded functions from l_k^ϵ to $\mathbb{C}$. For any path p in Γ, $s(p)$ and $t(p)$ will denote the starting and terminal point of p respectively. For any path p and loop L, $\tilde{p}$ and $\tilde{L}$ will denote the reverse of p and L respectively. Concatenation of two paths p and q with $t(p) = s(q)$ will be denoted by $c(p, q)$. If a loop of length $2k$ is the concatenation of two k-length paths π and $\tilde{\lambda}$, we denote the loop by the ordered pair (π, λ).

Let T be a (k_0, ϵ_0)-tangle with $b(T)$ internal disks $\{D_i | i = 1, \ldots, b(T)\}$ such that (k_i, ϵ_i) is the color of D_i.

Definition 3.1.1. *A state σ of a tangle T is a function*

$$\sigma : \{\textit{regions of } T\} \sqcup \{\textit{strings of } T\} \to V^+ \sqcup V^- \sqcup E$$

such that

1. $\sigma(\{\textit{unshaded regions}\}) \subset V^{+}, \sigma(\{\textit{shaded regions}\}) \subset V^{-}$;
2. $\sigma(\{\textit{strings}\}) \subset E$;
3. *if a string s lies in the closure of two regions r_1 and r_2, then $\sigma(s)$ is an edge between the vertices $\sigma(r_1)$ and $\sigma(r_2)$.*

Remarks 3.1.2. *1. We observe that a state σ on a tangle T induces a unique loop at each disk of T in the following way.*
Let D be a disk of T of color (k, ϵ) with $*$-region r_1. We start from a point of $\partial(D) \cap R_1$, traverse along the boundary of D in anticlockwise direction and come back to the starting point. Suppose the regions and strings we come across while traversing are in the following order:

$$\begin{tikzcd}
& r_{2k} \arrow[dl, "s_{2k}"'] & r_{k+2} \arrow[l, dotted] & \\
r_1 \arrow[dr, "s_1"'] & & & r_{k+1} \arrow[ul, "s_{k+1}"'] \\
& r_2 \arrow[r, dotted] & r_k \arrow[ur, "s_k"'] &
\end{tikzcd}$$

where r_i and s_i are the adjacent regions and strings respectively. Then the loop induced at D is:

$$\begin{tikzcd}
& \sigma(r_{2k}) \arrow[dl, "\sigma(s_{2k})"'] & \sigma(r_{k+2}) \arrow[l, dotted] & \\
\sigma(r_1) \arrow[dr, "\sigma(s_1)"'] & & & \sigma(r_{k+1}) \arrow[ul, "\sigma(s_{k+1})"'] \\
& \sigma(r_2) \arrow[r, dotted] & \sigma(r_k) \arrow[ur, "\sigma(s_k)"'] &
\end{tikzcd}$$

and the loop will be denoted by $\sigma(D)$.

2. If T is a (k, ϵ)-tangle and L is a loop of length $2k$ and based at a vertex in V^{ϵ} , then the number of states σ on T such that $\sigma(D) = L$ is bounded by $N^{|S(T)|}$, where $|S(T)|$ denotes the number of strings in T .

3. We isotope the tangle T to a standard form representative and for each singular point α of T we define $\mu_{\alpha} := \mu[\sigma(\textit{inner region at } \alpha)]/\mu[\sigma(\textit{outer region at } \alpha)]$

For a given spin function μ *and a tangle* T*,*

$$c(T,\sigma) := \prod_{\textit{singular points } \alpha \textit{ of } T} \mu_\alpha$$

is independent of the standard form representatives of T *because the value of* $c(T,\sigma)$ *is invariant under each of the five moves discussed in § 2.1.3.*

4. Suppose T *and* S *be two tangles as in § 2.1.2. Let* σ *and* σ' *be two states on* T *and* S *respectively such that* $\sigma(D_i(T))=\sigma'(D_0(S))$*. Then* $\sigma \circ_i \sigma'$ *denotes the state on* $T \circ_i S$ *defined by glueing* σ *and* σ' *in natural way. We observe that there is a 1-1 correspondence between*

$$\{(\sigma,\sigma')|\sigma \textit{ and } \sigma' \textit{ are states on } T \textit{ and } S \textit{ respectively with } \sigma(D_i(T))=\sigma'(D_0(S))\}$$

and $\{$*states on* $T \circ_i S\}$ *by* $(\sigma,\sigma') \mapsto \sigma \circ_i \sigma'$*. We also note that* $c(T \circ_i S, \sigma \circ_i \sigma') = c(T,\sigma)c(S,\sigma')$*.*

5. For $A \in V_k^\epsilon$*, we define* $\|A\|_\infty = \sup\{|A(L)| : L \in l_k^\epsilon\}$*.*

3.2 Planar algebra structure on V_Γ

We take one of the standard form representatives of the tangle T. Let $|\alpha(T)|$ be the total number of singular points in that standard form representative of T and $|S(T)|$ be the total number of strings in T. Now we define the linear map $Z_T^{V_\Gamma}$ as follows:

Case1: Suppose $b(T) > 0$. We take $x_i \in V_{k_i}^{\epsilon_i}$ $\forall i = 1, \ldots, b(T)$. Then

$$\left| \sum_{\text{states } \sigma \text{ on } T \ \ni \ \sigma(T) = L} \prod_{i=1}^{b(T)} x_i(\sigma(D))c(T,\sigma) \right|$$

$$\leq \quad M^{|\alpha(T)|}N^{|S(T)|}(Max_i\{\|x_i\|_\infty\})^{b(T)},$$

$\forall\, L \in l_{k_0}^{\epsilon_0}$. Therefore, $Z_T^{V_\Gamma} : \otimes_{i=1}^{b(T)} V_{k_i}^{\epsilon_i} \to V_{k_0}^{\epsilon_0}$, given by

$$\begin{aligned} &[Z_T^{V_\Gamma}(x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_b(T))](L) \\ = &\sum_{\text{states } \sigma \text{ on } T \ \ni \ \sigma(T) = L} \prod_{i=1}^{b(T)} x_i(\sigma(D))c(T,\sigma), \end{aligned}$$

where $L \in l_{k_0}^{\epsilon_0}$, is defined as the action of T on V_Γ. By the fifth remark made in Remarks 3.1.2, we observe that $c(T,\sigma)$ is independent of the standard form of T. Therefore, $Z_T^{V_\Gamma}$ is a well-defined map.

Case2: Suppose $b(T) = 0$. Then

$$\left|\sum_{\text{states } \sigma \text{ on } T \ \ni \ \sigma(T)=L} c(T,\sigma)\right| \leq N^{|S(T)|}M^{|\alpha(T)|},$$

$\forall\, L \in l_{k_0}^{\epsilon_0}$. Therefore, $Z_T^{V_\Gamma} : \mathbb{C} \to V_{k_0}^{\epsilon_0}$, given by

$$[Z_T^{V_\Gamma}(1)](L) = \sum_{\text{states } \sigma \text{ on } T \ \ni \ \sigma(T)=L} c(T,\sigma),$$

where $L \in l_{k_0}^{\epsilon_0}$, is defined as the action of T on V_Γ. By a similar argument like the first case, $Z_T^{V_\Gamma}$ is a well-defined map.

Theorem 3.2.1. *$\{V_k^\epsilon : (k,\epsilon) \in \ Col\}$ forms a planar algebra under above tangle actions.*

Proof. The 'Independence of the ordering of the internal disks' and 'Non-degeneracy Condition' immediately follow from the definition of $Z_T^{V_\Gamma}$. Now we prove the first condition given in Definition 2.2. Suppose T and S be two tangles as given there. We first consider the case when S has at least one internal disk. Take $x_j \in V_{k_j}^{\epsilon_j}\ \forall\ j = 1,\ldots,i-1$ and $i,\ldots,b(T)$ and $y_r \in V_{k_r}^{\epsilon_r}\ \forall\ r = 1,\ldots,b(S)$. We shall prove that

$$Z_T^{V_\Gamma}(\otimes_{j=1}^{i-1}x_j \otimes (Z_S^{V_\Gamma}(\otimes_{r=1}^{b(S)}y_r)) \otimes_{j=i+1}^{b(T)} x_j) = Z_{T\circ_i S}^{V_\Gamma}(\otimes_{j=1}^{i-1}x_j \otimes_{r=1}^{b(S)} y_r \otimes_{j=i+1}^{b(T)} x_j).$$

Let L be a loop in $l_{k_0(T)}^{\epsilon_0(T)}$. Then we have

$$\begin{aligned}
& Z_T^{V_\Gamma}(\otimes_{j=1}^{i-1}x_j \otimes (Z_S^{V_\Gamma}(\otimes_{r=1}^{b(S)}y_r)) \otimes_{j=i+1}^{b(T)} x_j)(L) \\
= & \sum_{\substack{\text{states } \sigma \text{ on } T\circ_i S \\ \ni\ \sigma(D_0(T\circ_i S)) = L}} \prod_{j=1;j\neq i}^{b(T)} x_j(\sigma(D_j(T))) \prod_{r=1}^{b(S)} y_r(\sigma(D_r(S)))c(T\circ_i S,\sigma) \\
= & \sum_{\substack{\text{states } \sigma \text{ on } T \\ \ni\ \sigma(D_0(T)) = L}} \left(\prod_{j=1;j\neq i}^{b(T)} x_j(\sigma(D_j(T))c(T,\sigma))\right)\left(\sum_{\substack{\text{states } \sigma' \text{ on } S \\ \ni\ \sigma'(D_0(S)) = \sigma(D_i(T))}} \prod_{r=1}^{b(S)} y_r(\sigma(D_r(S)))c(S,\sigma')\right) \\
= & Z_T^{V_\Gamma}(\otimes_{j=1}^{i-1}x_j \otimes (Z_S^{V_\Gamma}(\otimes_{r=1}^{b(S)}y_r)) \otimes_{j=i+1}^{b(T)} x_j)(L)
\end{aligned}$$

In a similar way we can check the 'Compatibility with the composition of tangles' when S has no internal disks. ◻

Definition 3.2.2. *Define $* : V_k^\epsilon \to V_k^\epsilon$ by $A^*(L) = \overline{A(\tilde{L})}$ for all $A \in V_k^\epsilon, L \in l_k^\epsilon$. We later prove in Theorem 3.5.2 that V_Γ has $*$ property.*

3.3 V_k^ϵ as a von Neumann algebra

Suppose for each $(k,\epsilon) \in$ Col, $H_{(k,\epsilon)}$ is the Hilbert space with an orthonormal basis $\{x_p\}$ labelled by the paths $p \in p_k^\epsilon$. Let $H^v_{(k,\epsilon)}$ be the closed linear span of $\{x_p : s(p) = v\}$ in $H_{(k,\epsilon)}$ for $v \in V^\epsilon$, and $H^{vw}_{(k,\epsilon)}$ be the closed linear span of $\{x_p : s(p) = v, t(p) = w\}$ in $H_{(k,\epsilon)}$ for $v \in V^\epsilon$ and $w \in V^+ \sqcup V^-$. Observe that $H_{(k,\epsilon)} = \bigoplus\limits_v H^v_{(k,\epsilon)} = \bigoplus\limits_{v,w} H^{vw}_{(k,\epsilon)}$. Let us define two types of projections s_v and t_w in $\mathcal{B}(H_{(k,\epsilon)})$ for all $(k,\epsilon) \in Col$, which we shall often use later. For any two vertices v and w, we define s_v as the projection onto the closed linear span of $\{x_p : p \in p_k^\epsilon, s(p) = v, (k,\epsilon) \in Col\}$ and t_w as the projection onto the closed linear span of $\{x_p : p \in p_k^\epsilon, t(p) = w, (k,\epsilon) \in Col\}$.

Now we define 'concatenation' and 'reverse' operators on $\{H_{(k,\epsilon)} : (k,\epsilon) \in Col\}$.

Definition 3.3.1. *For all $(k,\epsilon), (k',\epsilon') \in Col$, we define 'concatenation' operator $c : H_{(k,\epsilon)} \otimes H_{(k',\epsilon')} \to H_{(k+k',\epsilon)}$ by linearly and continuously extending the following map:*

$$c(x_p \otimes x_q) = \begin{cases} x_{c(p,q)} & \text{if } t(p) = s(q) \ ; \\ 0 & \text{otherwise.} \end{cases}$$

$\forall\ p \in p_k^\epsilon, q \in p_{k'}^{\epsilon'}$. We define 'reverse' operator $rev : H_1^\epsilon \to H_1^{\tilde{\epsilon}}$ by continuously and anti-linearly extending $x_e \mapsto x_{\tilde{e}}$ $\forall e \in p_1^\epsilon$.

Remark 3.3.2. *$c : H_{(k,\epsilon)} \otimes H_{(k',\epsilon')} \to H_{(k+k',\epsilon)}$ is a partial isometry onto $H_{(k+k',\epsilon)}$ with initial space as the closed linear span of $\{x_p \otimes x_q : p \in p_k^\epsilon, q \in p_k^{\epsilon'}, t(p) = s(q)\}$ and 'rev' is an anti-unitary operator.*

Next we prove a lemma related to 'concatenation' and 'reverse' operators.

Lemma 3.3.3. *Suppose v and w are two adjacent vertices in Γ with $v \in V^\epsilon$. Let $\{e_i : i = 1, \ldots, n(v,w)\}$ be the set edges between v and w, and $\{a_i : i = 1, \ldots, n(v,w)\}$ be an orthonormal basis of $H^{vw}_{(1,\epsilon)}$. Then*

$$\sum_{i=1}^{n(v,w)} c(a_i \otimes rev(a_i)) = \sum_{i=1}^{n(v,w)} c(x_{e_i} \otimes x_{\tilde{e_i}}).$$

Proof. First observe that $\{c(x_{e_i} \otimes x_{\tilde{e_j}}) : i,j = 1, \ldots, n(v,w)\}$ is an orthonormal basis of $c(H^{vw}_{(1,\epsilon)} \otimes H^{wv}_{(1,\tilde{\epsilon})})$. Therefore, it suffices to prove that the inner product of $\sum\limits_{i=1}^{n(v,w)} c(a_i \otimes rev(a_i))$ with $c(x_{e_i} \otimes x_{\tilde{e_j}})$ is δ_i^j $\forall i,j = 1, \ldots, n(v,w)$. Let $U = ((u_{ij}))$ be the unitary matrix such that $a_i = \sum\limits_j u_{ij} x_{e_j})$ $\forall i$. Then one simple computation gives that the inner product of $\sum\limits_{i=1}^{n(v,w)} c(a_i \otimes rev(a_i))$ with $c(x_{e_i} \otimes x_{\tilde{e_j}})$ is equal to the inner product of ith column and jth column of U, and which is equal to δ_i^j. $\square$

Now we identifty V_k^ϵ with a type I von Neumann subalgebra of $\mathcal{B}(H_{(k,\epsilon)})$ in the following manner: We take $A \in V_k^\epsilon$ and it will correspond to an operator $\hat{A}$ in $\mathcal{B}(H_{(k,\epsilon)})$. Since $H^{vw}_{(k,\epsilon)}$ is finite dimensional for all $v \in V^\epsilon$ and $w \in V^+ \sqcup V^-$, the matrix $(A(\pi,\lambda))$ defines a unique operator on $H^{vw}_{(k,\epsilon)}$ with respect to the basis $\{x_\lambda : \lambda \in p^\epsilon_k\}$. Since the dimension of $H^{vw}_{(k,\epsilon)}$ is bounded by N^k for all $v \in V^\epsilon$ and $w \in V^+ \sqcup V^-$, the operator norm of $\hat{A}$ on $H^{vw}_{(k,\epsilon)}$ is bounded by $N^k \|A\|_\infty$ $\forall v \in V^\epsilon$, $w \in V^+ \sqcup V^-$. Therefore, $\hat{A}$ extends to a continuous linear operator on $H_{(k,\epsilon)}$ with operator norm $\|A\|_{op} \leq N^k \|A\|_\infty$.

Theorem 3.3.4. *The map $V^\epsilon_k \to \mathcal{B}(H_{(k,\epsilon)})$ defined by $A \mapsto \hat{A}$ is a one-one $*$-algebra morphism and the image of V^ϵ_k in $\mathcal{B}(H_{(k,\epsilon)})$ is a type I von Neumann algebra.*

Proof. Let $A, B \in V^\epsilon_k$. Consider two basis elements x_π and x_λ of $H_{(k,\epsilon)}$. then

$$\begin{aligned}
\langle \widehat{AB}(x_\lambda), x_\pi \rangle &= \begin{cases} AB((\pi,\lambda)) & \text{if } s(\pi) = s(\lambda) \text{ and } t(\pi) = t(\lambda), \\ 0 & \text{otherwise;} \end{cases} \\
&= \begin{cases} \displaystyle\sum_{\substack{\delta \in p^\epsilon_k \ni s(\delta) = s(\pi), \\ t(\delta) = t(\pi)}} A((\pi,\delta))B((\delta,\lambda)) & \text{if } s(\pi) = s(\lambda) \text{ and } t(\pi) = t(\lambda), \\ 0 & \text{otherwise;} \end{cases} \\
&= \langle \hat{A}\hat{B}(x_\lambda), x_\pi \rangle
\end{aligned}$$

$$\text{and } \langle \widehat{A^*}(x_\lambda), x_\pi \rangle = A^*((\pi,\lambda)) = \overline{A((\lambda,\pi))} = \overline{\langle \hat{A}(x_\pi), x_\lambda \rangle} = \langle \hat{A}^*(x_\lambda), x_\pi \rangle.$$

Therefore, $A \mapsto \hat{A}$ is a $*$-algebra morphism. The injectivity of the map directly follows from definition. Now to prove that the image of V^ϵ_k in $\mathcal{B}(H_{(k,\epsilon)})$ is a von Neumann algebra it suffices to show that the image is equal to the commutant of $\{P_{H^{vw}_{(k,\epsilon)}} : v \in V^\epsilon,\, w \in V^+ \sqcup V^-\}$, because the commutant of a $*$-closed subset of $\mathcal{B}(H_{(k,\epsilon)})$ is a von Neumann algebra. Clearly $\hat{A}(H^{vw}_{(k,\epsilon)}) \subseteq H^{vw}_{(k,\epsilon)}$ $\forall\, v \in V^\epsilon$, $w \in V^+ \sqcup V^-$ and $A \in V^\epsilon_k$. Therefore, $\hat{A} \in \{P_{H^{vw}_{(k,\epsilon)}} : v \in V^\epsilon,\, w \in V^+ \sqcup V^-\}'$ $\forall\, A \in V^\epsilon_k$. Suppose $B \in \{P_{H^{vw}_{(k,\epsilon)}} : v \in V^\epsilon,\, w \in V^+ \sqcup V^-\}$. Then $|\langle B(x_\lambda), x_\pi \rangle| \leq \|B\|_{op}$ $\forall\, x_\pi, x_\lambda$. Therefore, for $A \in V^\epsilon_k$ defined by $A(L) := \langle B(x_\lambda), x_\pi \rangle$, where $L = (\pi,\lambda) \in l^\epsilon_k$, we have $\hat{A} = B$. So the image of V^ϵ_k is $\{P_{H^{vw}_{(k,\epsilon)}} : v \in V^\epsilon,\, w \in V^+ \sqcup V^-\}'$. Now from the facts that $H_{(k,\epsilon)} = \bigoplus_{v,w} H^{vw}_{(k,\epsilon)}$ and the image of V^ϵ_k is equal to $\{P_{H^{vw}_{(k,\epsilon)}} : v \in V^\epsilon,\, w \in V^+ \sqcup V^-\}'$, we can conclude that the image of V^ϵ_k is isomorphic to $\bigoplus_{vw} \mathcal{B}(H^{vw}_{(k,\epsilon)})$, which implies that the image of V^ϵ_k is a type I von Neumann algebra.

□

Remark 3.3.5. *From now onwards, the image of V^ϵ_k in $\mathcal{B}(H_{(k,\epsilon)})$, under the map $\widehat{\ }$, also will be denoted by V^ϵ_k.*

3.4 Actions of some tangles on V_Γ

In this section we study the actions of inclusion tangles, expectation tangles and Temperley-Lieb tangles on V_Γ and discuss some properties of them.

1. *Actions of $\mathcal{LI}$ and $\mathcal{RI}$ on V_Γ:* Suppose $A \in V^\epsilon_k$. Then $Z_{\mathcal{LI}}(A) \in V^{\tilde{\epsilon}}_{k+1}$ is given as follows:

$$[Z_{\mathcal{LI}}(A)](L) = \begin{cases} A((\pi,\lambda)) & \text{if } L = (c(e,\pi), c(e,\lambda)) \text{ for some edge e and paths } \pi,\lambda \\ & \text{ with } s(\pi)=s(\lambda)=t(e), t(\pi)=t(\lambda) \text{ and } s(e) \in V^{\tilde{\epsilon}}, \\ 0 & \text{otherwise;} \end{cases}$$

where L is a loop in $l^{\tilde{\epsilon}}_{k+1}$. If we look at $Z_{\mathcal{LI}}(A)$ as an operator on $H_{(k+1,\tilde{\epsilon})}$, then it is given by:

$$[Z_{\mathcal{LI}}(A)](x_p) = \sum_{\pi \ni s(\pi)=s(\lambda), t(\pi)=t(\lambda)} A((\pi,\lambda)) c(x_e, x_\pi), \tag{3.1}$$

where $e \in p^{\tilde{\epsilon}}_1, \lambda \in p^{\epsilon}_k$ with $s(\lambda) = t(e)$ and $p = c(e,\lambda)$.

Now we observe that $H_{(k+1,\tilde{\epsilon})}$ is the direct sum of subspaces $H^{(e,v)}_{(k+1,\tilde{\epsilon})}$ generated by $\{c(x_e, x_\lambda) : \lambda \in p^{\epsilon}_k, s(\lambda) = t(e) = v\}$, where $v \in V^{\epsilon}$ and $e \in p^{\tilde{\epsilon}}_1$ with $t(e) = v$. $Z_{\mathcal{LI}}(A)$ maps each of these subspaces into itself and the action of $Z_{\mathcal{LI}}(A)$ on $H^{(e,v)}_{k+1}$ is same as the action of A on $H^{v}_{(k,\epsilon)}$ after identifying $H^{v}_{(k,\epsilon)}$ with $H^{(e,v)}_{(k+1,\tilde{\epsilon})}$ by $x_\lambda \mapsto c(x_e, x_\lambda)$, $\forall \lambda \in p^{\epsilon}_k$ with $s(\lambda) = v$. Therefore, $\|Z_{\mathcal{LI}}(A)\|_{op} = \|A\|_{op}\ \forall\ A \in V^{\epsilon}_k$.

Similarly, $Z_{\mathcal{RI}} : V^{\epsilon}_k \to V^{\epsilon}_{k+1}$ is given by

$$[Z_{\mathcal{RI}}(A)](x_p) = \sum_{\pi \ni s(\pi)=s(\lambda), t(\pi)=t(\lambda)} A((\pi,\lambda)) c(x_\pi, x_e),$$

where $A \in V^{\epsilon}_k$, e is an edge, $\lambda \in p^{\epsilon}_k$ with $t(\lambda) = s(e)$ and $p = c(\lambda, e)$, and $\|Z_{\mathcal{RI}}(A)\|_{op} = \|A\|_{op}$ $\forall A \in V^{\epsilon}_k$. It is clearly seen that both $Z_{\mathcal{LI}}$ and $Z_{\mathcal{RI}}$ are injective.

2. *Actions of $\mathcal{LE}$ and $\mathcal{RE}$ on V_Γ:* For $A \in V^{\epsilon}_k$, $Z_{\mathcal{LE}}(A) \in V^{\tilde{\epsilon}}_{(k-1)}$ is given as follows:

$$[Z_{\mathcal{LE}}(A)](L) = \sum_{e \in p^{\epsilon}_1 \ni t(e)=s(\pi)=s(\lambda)} A(c(e,\pi), c(e,\lambda)) \left(\mu(s(e)/\mu(t(e))\right)^2,$$

where $L = (\pi,\lambda)$ is a loop in $l^{\tilde{\epsilon}}_{k-1}$. If we look at $Z_{\mathcal{LE}}(A)$ as an operator on $H_{(k-1,\tilde{\epsilon})}$, then it is given by

$$Z_{\mathcal{LE}}(A)(x_\lambda) = \sum_{\substack{\pi \ni s(\pi)=s(\lambda), \\ t(\pi)=t(\lambda)}} [\sum_{\substack{e \in p^{\epsilon}_1 \ni \\ t(e)=s(\pi)=s(\lambda)}} A(c(e,\pi), c(e,\lambda)) \left(\mu(s(e)/\mu(t(e))\right)^2]\ \ x_\pi, \tag{3.2}$$

where $\lambda \in p^{\tilde{\epsilon}}_{k-1}$. From the above expression of $Z_{\mathcal{LE}}(A)(L)$, we obtain $|Z_{\mathcal{LE}}(A)(L)| \le NM^2\|A\|_\infty\ \forall L \in l^{\tilde{\epsilon}}_{k-1}$. Therefore, $\|Z_{\mathcal{LE}}(A)\|_{op} \le N^k M^2 \|A\|_\infty \le N^k M^2 \|A\|_{op}\ \forall A \in V^{\epsilon}_k$. Similarly, for $A \in V^{\epsilon}_k$, $Z_{\mathcal{RE}}(A)$ as an operator on H^{ϵ}_{k-1} is given by:

$$Z_{\mathcal{RE}}(A)(x_\lambda) = \sum_{\substack{\pi \ni s(\pi)=s(\lambda), \\ t(\pi)=t(\lambda)}} [\sum_{\substack{edge\ e \ni \\ s(e)=t(\pi)=t(\lambda)}} A(c(\pi,e), c(x_\lambda, x_e)) \left(\mu(t(e)/\mu(s(e))\right)^2]\ \ x_\pi,$$

where $\lambda \in p^{\epsilon}_k$, and $\|Z_{\mathcal{RE}}(A)\|_{op} \le N^k M^2 \|A\|_{op}\ \forall A \in V^{\epsilon}_k$.

3. *Action of $\mathcal{TL}^\epsilon$ tangles on V_Γ:* $Z_{\mathcal{TL}^\epsilon}(1) \in V_2^\epsilon$ is given by:

$$Z_{\mathcal{TL}^\epsilon}(1)(L) = \begin{cases} \mu(t(e_1))\mu(t(e_2))/\mu^2(s(e_1)) & \text{if } L = (c(e_1, \tilde{e_1}), c(e_2, \tilde{e_2})), \text{ where } e_1, e_2 \\ & \text{are two edges with } s(e_1) = s(e_2) \in V^\epsilon; \\ 0 & \text{otherwise.} \end{cases}$$

For each vertex $v \in V^\epsilon$, let

$$y_v = \sum_{f \in p_1^\epsilon \ni s(f) = v} \mu(t(f)) c(x_f, x_{\tilde{f}}) \text{ and } \delta_v = \sum_{f \in p_1^\epsilon \ni s(f) = v} \mu^2(t(f))/\mu^2(v).$$

Then $Z_{\mathcal{TL}^\epsilon}(1)$ as an operator on $H^v_{(2,\epsilon)}$, where $v \in V^\epsilon$, is given by:

$$Z_{\mathcal{TL}^\epsilon}(1)(x_p) = \begin{cases} \mu(t(e))/(\mu^2(v)) y_v & \text{if } p = c(e, \tilde{e}) \text{ for some } e \in p_1^\epsilon \text{ with } s(e) = v; \\ 0 & \text{otherwise.} \end{cases}$$

Since $Z_{\mathcal{TL}^\epsilon}(1)(L) = Z_{\mathcal{TL}^\epsilon}(1)(\tilde{L})$ for all $L \in l_2^\epsilon$, $Z_{\mathcal{TL}^\epsilon}(1)$ is a selfadjoint operator. Observe that

$$\begin{aligned} Z_{\mathcal{TL}^\epsilon}(1)(y_v) &= \sum_{f \in p_1^\epsilon \ni s(f) = v} \mu(t(f)) Z_{\mathcal{TL}^\epsilon}(1)(c(x_f, x_{\tilde{f}})) \\ &= \sum_{f \in p_1^\epsilon \ni s(f) = v} (\mu^2(t(f))/\mu^2(v)) y_v = \delta_v y_v. \end{aligned}$$

If δ_v is independent of v, then the range of $Z_{\mathcal{TL}^\epsilon}(1)$ is the closed linear span of $\{y_v : v \in V^\epsilon\}$ and $Z_{\mathcal{TL}^\epsilon}(1)$ being a self adjoint operator is zero off the closed linear span of $\{y_v : v \in V^\epsilon\}$.

We now introduce the concept of 'partition function' from [J2] and prove a lemma which will help establish the 'sphericality' of a planar subalgebra of V_Γ.

Lemma 3.4.1. *Let Γ be a finite connected bipartite graph with a spin function μ. Define a 'partition function' $Z^\epsilon : V_0^\epsilon \to \mathbb{C}$ by mapping the indicator function 1_v to $\mu^4(v)$ $\forall v \in V^\epsilon$. Then*

$$Z^+ \circ Z_{\mathcal{RE}^{(0,+)}_{(1,+)}} = Z^- \circ Z_{\mathcal{LE}^{(0,-)}_{(1,+)}}.$$

Proof. Let $A \in V_1^+$. Observe that

$$[Z_{\mathcal{RE}}(A)](v) = \sum_{e \in p_1^+ \ni s(e) = v} A((e,e))[\mu^2(t(e))/\mu^2(v)]$$

$\forall v \in V^+$. So we have

$$Z_{\mathcal{RE}}(A) = \sum_{v \in V^+} [\sum_{e \in p_1^+ \ni s(e) = v} A((e,e))[\mu^2(t(e))/\mu^2(v)]]\, 1_v,$$

which implies

$$
\begin{aligned}
[Z^+ \circ Z_{\mathcal{RE}}](A) &= \sum_{v \in V^+} \sum_{e \in p_1^+ \ni s(e)=v} A((e,e))\mu^2(t(e))\mu^2(v) \\
&= \sum_{e \in p_1^+} A((e,e))\mu^2(t(e))\mu^2(s(e)).
\end{aligned}
$$

Similarly, we have

$$
Z_{\mathcal{LE}}(A) = \sum_{v \in V^-} [\sum_{e \in p_1^+ \ni t(e)=v} A((e,e))[\mu^2(s(e))/\mu^2(v)]]\, 1_v,
$$

which implies

$$
\begin{aligned}
[Z^- \circ Z_{\mathcal{LE}}](A) &= \sum_{v \in V^-} \sum_{e \in p_1^+ \ni t(e)=v} A((e,e))\mu^2(s(e))\mu^2(v) \\
&= \sum_{e \in p_1^+} A((e,e))\mu^2(t(e))\mu^2(s(e)) \\
&= [Z^+ \circ Z_{\mathcal{RE}}](A).
\end{aligned}
$$

□

Now we discuss a special type of continuity of the tangle-maps which is close to strong continuity. This concept is relevent in this context because the continuity of Z_{RC} in this sense helps establish the positive-definiteness of a sesquilinear form on V_k^ϵ.

Theorem 3.4.2. *Let T be a tangle with $b(\geq 1)$ internal discs. Suppose $A_i^j \xrightarrow{SOT} A^j\ \forall j = 1, \ldots, b$ with $\sup\limits_{i,j} \|A_i^j\| < \infty$. Then*

$$
Z_T\left(A_i^1 \otimes A_i^2 \cdots \otimes A_i^b\right) \xrightarrow{SOT} Z_T\left(A^1 \otimes A^2 \cdots \otimes A^b\right).
$$

Moreover, $\sup\limits_i \|Z_T\left(A_i^1 \otimes A_i^2 \cdots \otimes A_i^b\right)\| < \infty$.

Proof. From Theorem 2.1.1, we can conclude that Z_T can be obtained by composing $Z_{\mathcal{LI}}$, $Z_{\mathcal{RI}}$, $Z_{\mathcal{LE}}$, $Z_{\mathcal{RE}}$, $Z_{\mathcal{M}_{(k,\epsilon)}}$ and Z_{TL^ϵ} in the way discussed in 'Compatibility with the composition of tangles' in the definition of planar algebra. A little thought gives that it suffices to prove the theorem for $Z_{\mathcal{LI}}$, $Z_{\mathcal{RI}}$, $Z_{\mathcal{LE}}$, $Z_{\mathcal{RE}}$ and $Z_{\mathcal{M}_{(k,\epsilon)}}$.

Consider V_k^ϵ for some $(k,\epsilon) \in Col$. Let $A_i \xrightarrow{SOT} A$ in V_k^ϵ with $\sup\limits_i \|A_i\|_{op} < \infty$. Therefore, $A_i(L) \to A(L)$ for all $L \in l_k^\epsilon$. So from the expression in equation (3.1), we have $Z_{\mathcal{LI}}(A_i)(x_p) \to Z_{\mathcal{LI}}(A)(x_p)$ for all $p \in p_{k+1}^{\tilde{\epsilon}}$. Now from $\sup\limits_i \|A_i\|_{op} < \infty$ and $\|Z_{\mathcal{LI}}(A_i)\|_{op} = \|A_i\|_{op}$, we get $\sup\limits_i \|Z_{\mathcal{LI}}(A_i)\|_{op} < \infty$. Therefore, $Z_{\mathcal{LI}}(A_i) \xrightarrow{SOT} Z_{\mathcal{LI}}(A)$. So the theorem is true for $Z_{\mathcal{LI}}$. In a similar way, from the expression in equation (3.2), we have $Z_{\mathcal{LE}}(A_i)(x_p) \to Z_{\mathcal{LE}}(A)(x_p)\ \forall p \in p_{k-1}^{\tilde{\epsilon}}$. Now from $\sup\limits_i \|A_i\|_{op} < \infty$ and $\|Z_{\mathcal{LE}}(A_i)\|_{op} \leq N^k M^2 \|A_i\|_{op}$, we get $\sup\limits_i \|Z_{\mathcal{LI}}(A_i)\|_{op} < \infty$. Therefore, $Z_{\mathcal{LE}}(A_i) \xrightarrow{SOT} Z_{\mathcal{LE}}(A)$. So the theorem is true for $Z_{\mathcal{LE}}$. Similarly, we can prove that the theorem is true for $Z_{\mathcal{RI}}$ and $Z_{\mathcal{RE}}$.

Now consider $A_i \xrightarrow{SOT} A$ and $B_i \xrightarrow{SOT} B$ in V_k^ϵ with $\sup_i \|A_i\|_{op} < \infty$ and $\sup_i \|B_i\|_{op} < \infty$. It is clear that $\sup_i \|A_iB_i\|_{op} < \infty$. Now for $\xi \in H_{(k,\epsilon)}$, we have

$$\begin{array}{rcl} \|(A_iB_i)(\xi) - (AB)(\xi)\| & = & \|(A_iB_i)(\xi) - (A_iB)(\xi) + (A_iB)(\xi) - (AB)(\xi)\| \\ & \leq & \|A_i\|\|B_i(\xi) - B(\xi)\| + \|(A_i - A)(B(\xi))\|, \end{array}$$

which implies that $A_iB_i \xrightarrow{SOT} AB$. □

Remark 3.4.3. *'$*$'-map is also continuous in the above sense. Consider $A_i \xrightarrow{SOT} A$ in V_k^ϵ with $\sup_i \|A_i\|_{op} < \infty$. Therefore, $A_i(L) \to A(L)$ $\forall L \in l_k^\epsilon$. A_i^*, which implies $A_i^*(L) = \overline{A_i(\tilde{L})} \to \overline{A(\tilde{L})} = A^*(L)$ $\forall L \in l_k^\epsilon$. A_i^*. Now from the equation*

$$B^*(x_p) = \sum_{\substack{q \in p_k^\epsilon \\ \ni s(q)=s(p), t(q)=t(p)}} B^*((q,p))x_q,$$

$\forall B \in V_k^\epsilon$ and $p \in p_k^\epsilon$, we obtain $A_i^*(x_p) \to A^*(x_p)$ $\forall p \in p_k^\epsilon$. Also, we have $\sup_i \|A_i^*\| < \infty$. Therefore, $A_i^* \xrightarrow{SOT} A^*$.

3.5 Some good properties of V_Γ

The BGPAs are almost never of subfactor type, because their vector spaces are too large. In fact, in most cases V_0^ϵ is not one dimensional. However, they possess several of the necessary properties required for SPAs, which are inherited by planar subalgebras. We may therefore try to find SPAs by looking at small planar subalgebras of BGPAs.

Theorem 3.5.1. *For each $(k, \epsilon) \in Col$, $\langle , \rangle : V_k^\epsilon \times V_k^\epsilon \to V_0^\epsilon$ given by $\langle x, y\rangle = [Z_{\mathcal{RE}}]^k(y^*x)$ $\forall x, y \in V_k^\epsilon$, is a positive definite V_0^ϵ-valued sesquilinear map on V_k^ϵ.*

Proof. Clearly $\langle , \rangle$ is a sesquilinear form on V_k^ϵ. Let $A \in V_k^\epsilon$. We prove that $\langle A, A\rangle$ is a positive operator in V_0^ϵ. For each $v \in V^\epsilon$ and $w \in V^+ \sqcup V^-$, let us define $A_{vw} \in V_k^\epsilon$ as the restriction of A on $H_{(k,\epsilon)}^{vw}$ and zero on $[H_{(k,\epsilon)}^{vw}]^\perp$. Then clearly $A = \sum_{v,w} A_{vw}$ in SOT and $\langle A_{vw}, A_{v'w'}\rangle = 0$ if $(v, w) \neq (v', w')$. Let for $p, q \in p_k^\epsilon$ with $s(p) = s(q) = v$ and $t(p) = t(q) = w$, E_{pq}^{vw} denote a rank one partial isometry which sends the basis element x_q into x_p. We observe that $[Z_{\mathcal{RE}}]^k(E_{pq}^{vw}) = 1_v$ if $p = q$ and 0 otherwise.

Suppose $A = \sum_{v,w} A_{vw}$ and $A_{vw} = \sum_{p,q} \lambda_{pq}^{vw} E_{pq}^{vw}$ as above. Therefore, using Theorem 3.4.2 and Remark 3.4.3, we have

$$\langle A, A\rangle = \sum_{v,w} \langle A_{vw}, A_{vw}\rangle = \sum_{v,w} \Big[\sum_{\substack{p,q \ni s(p)=s(q)=v, \\ t(p)=t(q)=w}} |\lambda_{pq}^{vw}|^2 \Big] \; 1_v,$$

and hence $\langle A, A\rangle$ is a positive operator. Also $\langle A, A\rangle = 0$ implies $\lambda_{pq}^{vw} = 0$ $\forall p, q, v, w$, i.e., $A = 0$. Therefore, $\langle , \rangle$ is a positive definite sesquilinear form. □

Theorem 3.5.2. *'$*$' commutes with the tangle maps, i.e., for any (k_0, ϵ_0)-tangle T with b internal disks of colors (k_i, ϵ_i);*

$$[Z_T^{V_\Gamma}(x_1 \otimes \cdots \otimes x_b)]^* = Z_{T^*}^{V_\Gamma}(x_1^* \otimes \cdots \otimes x_b^*), \tag{3.3}$$

$\forall x_i \in V_{k_i}^{\epsilon_i}, 1 \leq i \leq b$;

Proof. First we observe that there is a natural 1-1 correspondence between the states on T and T^* by mapping a state σ on T to the state σ' on T^*, which is the composition of σ with the reflection map from T^* to T. We also observe that $\sigma(D_i(T)) = \sigma'(\widetilde{D_i(T^*)})$. Therefore, for $L \in l_{k_0}^{\epsilon_0}$ we have

$$\begin{aligned}
& [Z_T^{V_\Gamma}(x_1 \otimes \cdots \otimes x_b)]^*(L) \\
= & \sum_{\substack{\text{states } \sigma \text{ on } T \\ \ni\ \sigma(T) = \tilde{L}}} \prod_{i=1}^{b} \overline{x_i(\sigma(D_i(T)))} c(T, \sigma) \\
= & \sum_{\substack{\text{states } \sigma' \text{ on } T^* \\ \ni\ \sigma'(D_i(T)) = L}} \prod_{i=1}^{b} x_i^*(\sigma'(\widetilde{D_i(T^*)})) c(T, \sigma) \\
= & [Z_{T^*}^{V_\Gamma}(x_1^* \otimes \cdots \otimes x_b^*)](L).
\end{aligned}$$

$\square$

For a uniformly locally finite connected bipartite graph Γ with possibly infinite vertices, the adjacency matrix can be seen as a linear map on $\mathbb{C}^{|V|}$, where $|V|$ is the total number of vertices of Γ, by matrix multiplication. We now prove a necessary and sufficient for having modulus δ of V_Γ in terms of the adjacency matrix of Γ.

Theorem 3.5.3. *For $\delta > 0$ the following are equivalent:*
(i) $\delta_v = \delta\, \forall v \in V^+ \sqcup V^-$;
(ii) $[Z_{\mathcal{TL}^\epsilon}(1)]^2 = \delta Z_{\mathcal{TL}^\epsilon}(1)$;
(iii) V_Γ has modulus δ;
(iv) δ is an eigenvalue of the adjacency matrix of Γ with eigenvector $(\mu^2(v))_{v \in V^+ \sqcup V^-}$.

Proof. $(1 \Rightarrow 2)$ $[Z_{\mathcal{TL}^\epsilon}(1)]^2(y_v) = \delta^2 y_v = \delta Z_{\mathcal{TL}^\epsilon}(1)(y_v)$ and both sides are zero off the closed linear span of $\{y_v : v \in V^\epsilon\}$.

$(2 \Rightarrow 1)$ $[Z_{\mathcal{TL}^\epsilon}(1)]^2(y_v) = \delta Z_{\mathcal{TL}^\epsilon}(1)(y_v)$, for $v \in V^\epsilon$, which implies $\delta_v^2 y_v = \delta \delta_v y_v$, which further implies $\delta_v = \delta$.

$(1 \Leftrightarrow 3)$ $[Z_{T_\epsilon^{\tilde{\epsilon}}}(1)](v) = \delta_v v$, for all $v \in V^{\tilde{\epsilon}}$. Therefore, $Z_{T_\epsilon^{\tilde{\epsilon}}}(1) = \delta 1 \Leftrightarrow \delta_v = \delta\, \forall v \in V^+ \sqcup V^-$.

$(1 \Leftrightarrow 4)$ It follows from the observation that $\delta_v = \sum_{w \ni n(v,w) \neq 0} n(v,w) \mu^2(w)/\mu^2(v)\ \forall v$. $\square$

Remark 3.5.4. *Suppose the adjacency matrix of a uniformly locally finite bipartite graph Γ has an eigenvector $(x(v))_{v \in V^+ \sqcup V^-} \in \mathbb{C}^{|V^+ \sqcup V^-|}$, where $x(v) > 0\ \forall v$, with eigenvalue $\delta > 0$. Then $((x(v))^{1/2})_{v \in V^+ \sqcup V^-}$ gives a uniformly bounded spin function on Γ. The vector $((x(v))^{1/2})_{v \in V^+ \sqcup V^-}$*

will be called a modulus-δ spin function on Γ. For a finite connected bipartite graph Γ, there is only one modulus-δ spin function on Γ (up to normalization), where δ is the Perron-Frobenius eigenvalue of Γ and the entries of the spin function are the square roots of the entries of the Perron-Frobenius eigen vector. A uniformly locally finite connected bipartite graph may have many modulus-δ spin functions with many values of δ.

Chapter 4

Automorphisms of a BGPA

In this chapter we mainly study the automorphism group of a BGPA. We express any automorphism of a BGPA as the product of two types of automorphisms - *'graph automorphism operator'* and *'multiplication operator'*. These two types of automorphisms are easily computable from the given graph. As in Chapter 3, we assume that Γ is a uniformly locally finite connected bipartite graph. We also assume that μ is a modulus-δ spin function on Γ. We shall use all the notations of Chapter 3 related to the graph Γ and planar algebra V_Γ.

4.1 Operators on $H_{(k,\epsilon)}$ commuting with 'concatenation' and 'reverse' operators

In this section we derive a sufficient condition for an invertible graded linear map of $\{V_k^\epsilon : (k,\epsilon) \in Col\}$, given by $\{Ad(U_k^\epsilon) : U_k^\epsilon$ is unitary in $\mathcal{B}(H_{(k,\epsilon)})$ and $(k,\epsilon) \in Col\}$, to be an automorphism of V_Γ. In that sufficient condition we demand that the unitaries $\{U_k^\epsilon : (k,\epsilon) \in Col\}$ commute with the 'concatenation' operator and 'reverse' operator. This sufficient condition will help prove that the 'graph automorphism operators' and 'multiplication operators' are automorphisms of V_Γ.

Lemma 4.1.1. *Let Γ be a bipartite graph as before. Let $U = \{U_{(k,\epsilon)} : (k,\epsilon) \in Col\}$ be a collection of unitaries in $\mathcal{B}(H_{(k,\epsilon)})$ which commute with the 'concatenation' operator, i.e., $c \circ (U_{(k,\epsilon)} \otimes U_{(k',\epsilon')}) = U_{(k+k',\epsilon)} \circ c$ for all $(k,\epsilon), (k',\epsilon') \in Col$. Then the action of U on $H_{(0,+)}$ and $H_{(0,-)}$ is given by a graph automorphism, i.e., there exists a graph automorphism κ such that the restriction of $U_{(0,+)}$ on $\{x_v : v \in V^+\}$ and $U_{(0,-)}$ on $\{x_v : v \in V^-\}$ are given by κ. Moreover, $Ad(U_{(k,\epsilon)})(V_k^\epsilon) \subseteq V_k^\epsilon$, $\forall (k,\epsilon) \in Col$.*

Proof. Let x_v be a standard basis element in $H_{(0,+)}$. We know that $c(x_v \otimes x_v) = x_v$. Therefore, from $c \circ (U_{(0,+)} \otimes U_{(0,+)}) = U_{(0,+)} \circ c$ we have $c(U_{(0,+)}(x_v) \otimes U_{(0,+)}(x_v)) = U_{(0,+)}(x_v)$. Now if we have $y = \sum_{v \in V^+} \lambda_v x_v$ in $H_{(0,+)}$ with $c(y \otimes y) = y$, then it implies $\sum_v \lambda_v^2 x_v = \sum_v \lambda_v x_v$ which further implies $\lambda_v = 0$ or 1. Therefore, $U(x_v) = \sum_{w \in S} x_w$ for some finite subset S of V^+. But unitarity of U implies $U(x_v) = x_w$ for some $w \in V^+$ and therefore the restriction of U on $\{x_v : v \in V^+\}$ gives a permutation of $\{x_v : v \in V^+\}$. Similarly, the restriction of U on $\{x_v : v \in V^-\}$ gives a permutation on $\{x_v : v \in V^-\}$.

Now we prove that the permutation κ of $V^+ \sqcup V^-$ coming from the restriction of U extends to a graph automorphism of Γ. It suffices to prove that $n(\kappa(v), \kappa(w)) = n(v, w)$ for all $v \in V^+, w \in V^-$; where for any two vertices $v \in V^+, w \in V^-$, $n(v, w)$ denotes the number of edges between v and w. For any path p with $s(p) = v \in V^\epsilon$ and $t(p) = w$, we have $c(U(x_v) \otimes c(U(x_p) \otimes U(x_w))) = U(x_p)$, that means $c(x_{\kappa(v)} \otimes c(U(x_p) \otimes x_{\kappa(w)})) = U(x_p)$, which implies that $U(H^{vw}_{(k,\epsilon)}) \subseteq H^{\kappa(v)\kappa(w)}_{(k,\epsilon)}$. Now we prove that $U^* = \{U^*_{(k,\epsilon)} : (k, \epsilon) \in \text{Col}\}$ also commutes with the 'concatenation' operator. From the unitarity of U, we have $c \circ (U \otimes U) \circ (U^* \otimes U^*) = UU^*c$, which implies $U \circ c \circ (U^* \otimes U^*) = UU^*c$ because U commutes with 'concatenation' operator, which further implies $c \circ (U^* \otimes U^*) = U^*c$. Since U^* commutes with 'concatenation' operator, applying the above argument to U^*, we have $U^*(H^{(\kappa(v),\kappa(w))}_{(k,\epsilon)}) \subseteq H^{vw}_{(k,\epsilon)}$. Therefore, $U(H^{vw}_{(k,\epsilon)}) = H^{\kappa(v)\kappa(w)}_{(k,\epsilon)}$, which implies dim $U(H^{vw}_{(1,+)}) =$ dim $H^{(\kappa(v),\kappa(w))}_{(1,+)}$, that means $n(\kappa(v), \kappa(w)) = n(v, w)$.

We observe that $U_{(k,\epsilon)}(s_v t_w)U^*_{(k,\epsilon)} = s_{\kappa(v)}t_{\kappa(w)}$. Therefore, if $A \in V^\epsilon_k = \{s_v t_w : v \in V^\epsilon \text{ and } w \in V^+ \sqcup V^-\}'$, then for $v \in V^\epsilon$ and $w \in V^+ \sqcup V^-$ we have

$$s_v t_w (U^*AU) = U^* s_{\kappa(v)} t_{\kappa(w)} AU = U^* A s_{\kappa(v)} t_{\kappa(w)} U = U^*AU(s_v t_w).$$

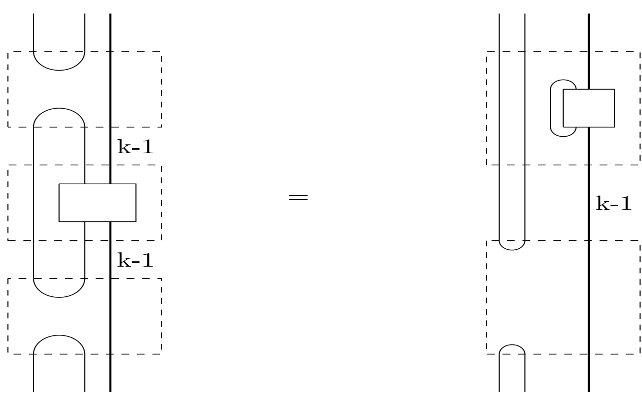
Therefore, $U^*AU \in \{s_v t_w : v \in V^\epsilon \text{ and } w \in V^+ \sqcup V^-\}'$ which implies $Ad(U_{(k,\epsilon)})(V^\epsilon_k) \subseteq V^\epsilon_k \, \forall (k, \epsilon) \in$ Col. □

Lemma 4.1.2. *If $\omega = \{\omega_{(k,\epsilon)} | \omega_{(k,\epsilon)} : V^\epsilon_k \to V^\epsilon_k$ is unital algebra map and $(k, \epsilon) \in$ Col$\}$ commutes with the tangle maps $Z_{\mathcal{RI}}, Z_{\mathcal{LI}}$ and fixes $Z_{TL^\epsilon}(1)$, then it also commutes with $Z_{\mathcal{RE}}$ and $Z_{\mathcal{LE}}$.*

Proof. We first show that

$$[Z_{\mathcal{RI}}]^{k-1}(Z_{TL^\epsilon}(1))Z_{\mathcal{LI}}(A)[Z_{\mathcal{RI}}]^{k-1}(Z_{TL^\epsilon}(1)) = [Z_{\mathcal{RI}}]^{k-1}(Z_{TL^\epsilon}(1))[Z_{\mathcal{LI}}]^2(Z_{\mathcal{LE}}(A))$$

for all $A \in V^\epsilon_k$, which follows from the following tangle-equality:



Similarly, we can prove that

$$[Z_{\mathcal{LI}}]^{k-1}(Z_{TL^\epsilon}(1))Z_{\mathcal{RI}}(A)[Z_{\mathcal{LI}}]^{k-1}(Z_{TL^\epsilon}(1)) = [Z_{\mathcal{LI}}]^{k-1}(Z_{TL^\epsilon}(1))[Z_{\mathcal{RI}}]^2(Z_{\mathcal{RE}}(A)).$$

for all $A \in V^\epsilon_k$. Now we have

$$
\begin{array}{rl}
& [Z_{\mathcal{RI}}]^{k-1}(Z_{TL^\epsilon}(1))[Z_{\mathcal{LI}}]^2(Z_{\mathcal{LE}}(\omega(A))) \\
= & [Z_{\mathcal{RI}}]^{k-1}(Z_{TL^\epsilon}(1))Z_{\mathcal{LI}}(\omega(A))[Z_{\mathcal{RI}}]^{k-1}(Z_{TL^\epsilon}(1)) \\
= & \omega([Z_{\mathcal{RI}}]^{k-1}(Z_{TL^\epsilon}(1))Z_{\mathcal{LI}}(A)[Z_{\mathcal{RI}}]^{k-1}(Z_{TL^\epsilon}(1))) \\
= & \omega([Z_{\mathcal{RI}}]^{k-1}(Z_{TL^\epsilon}(1))[Z_{\mathcal{LI}}]^2(Z_{\mathcal{LE}}(A))) \\
= & [Z_{\mathcal{RI}}]^{k-1}(Z_{TL^\epsilon}(1))[Z_{\mathcal{LI}}]^2(\omega(Z_{\mathcal{LE}}(A)),
\end{array}
$$

which implies $[Z_{\mathcal{RI}}]^{k-1}(Z_{TL^\epsilon}(1))[Z_{\mathcal{LI}}]^2\left(Z_{\mathcal{LE}}(\omega(A)) - \omega(Z_{\mathcal{LE}}(A))\right) = 0$ for all $A \in V_k^\epsilon$. We observe that $Z_{\mathcal{LE}}\left([Z_{\mathcal{RI}}]^{k-1}(Z_{TL^\epsilon}(1))[Z_{\mathcal{LI}}]^2(B)\right) = Z_{\mathcal{LI}}(B)$ for all $B \in V_k^\epsilon$ by an easy tangle-equality. Therefore, $[Z_{\mathcal{RI}}]^{k-1}(Z_{TL^\epsilon}(1))[Z_{\mathcal{LI}}]^2(B) = 0$ implies

$$
Z_{\mathcal{LI}}(B) = Z_{\mathcal{LE}}\left([Z_{\mathcal{RI}}]^{k-1}(Z_{TL^\epsilon}(1))[Z_{\mathcal{LI}}]^2(B)\right) = 0,
$$

which further implies $B = 0$, since $Z_{\mathcal{LI}}$ is injective. Therefore, $Z_{\mathcal{LE}}(\omega(A) = \omega(Z_{\mathcal{LE}}(A))$ for all $A \in V_k^\epsilon$. Similarly, we can prove that $Z_{\mathcal{RE}}$ commutes with ω. $\square$

Theorem 4.1.3. *Let Γ be a uniformly locally finite connected bipartite graph as before with a modulus-δ spin function μ. Let U be a collection of unitary operators as in Lemma 4.1.1 which commutes with the 'concatenation' operator. Assume further that the restriction of U to $H_{(1,\epsilon)}$ commutes with the 'reverse' operator, and the vertex permutation κ induced by U scales the spin function, i.e., $\sigma(\kappa(v)) = \lambda\sigma(v)$ for all vertices v and some scalar $\lambda > 0$. Then $Ad(U) := \{Ad(U_k^\epsilon) : V_k^\epsilon \to V_k^\epsilon | (k,\epsilon) \in Col\}$ is an automorphism of V_Γ.*

Proof. From Lemma 4.1.1, we know that $Ad(U_{(k,\epsilon)})(V_k^\epsilon) \subseteq V_k^\epsilon$ for all $(k,\epsilon) \in$ Col. Now from Theorem 2.2.3, Theorem 2.1.1 and Lemma 4.1.2, we conclude that to prove that $Ad(U)$ is a planar algebra morphism it suffices to prove that $Ad(U)$ commutes with the multiplication tangles, unit tangles, embedding tangles and Temperley-Lieb tangles. Since $U_{(k,\epsilon)}$ is unitary, $Ad(U_{(k,\epsilon)})$ is a unital $*$-algebra map.

Let us take $A \in V_k^\epsilon$ and any basis element $x_p = x_{c(e,\lambda)}$ in $H_{k+1}^{\tilde{\epsilon}}$, where $e \in p_1^{\tilde{\epsilon}}$ and $\lambda \in p_k^\epsilon$ with $t(e) = s(\lambda)$. Then we have

$$
\begin{array}{rcl}
U^* Z_{\mathcal{LI}}(A)U(x_{c(e,\lambda)}) & = & U^* Z_{\mathcal{LI}}(A)c(U(x_e) \otimes U(x_\lambda)) \\
& = & U^* c(U(x_e) \otimes AU(x_\lambda)) \\
& = & c(x_e \otimes U^* AU(x_\lambda)) \\
& = & Z_{\mathcal{LI}}(U^* AU)(x_{c(e,\lambda)}).
\end{array}
$$

So $Ad(U)$ commutes with left embedding. By a similar argument it commutes with right embedding as well.

Now we prove that $Ad(U)$ commutes with Temperley-Lieb tangles, i.e., it fixes $Z_{TL^\epsilon}(1)$. Let us define $y_{vw} = \sum_{f \in p_1^\epsilon \ni s(f)=v, t(f)=w} c(x_f \otimes rev(x_f))$, for all adjacent vertices v and w. Then $y_v = \sum_{f \in p_1^\epsilon \ni s(f)=v} \mu(t(f)) x_{c(f,\tilde{f})}$, as defined earlier, can also be written as $y_v = \sum_{w \ni n(v,w) \neq 0} \mu(w) y_{vw}$ for all $v \in V^\epsilon$. $\{x_f : f \in p_1^\epsilon, s(f) = v, t(f) = w\}$ is an orthonormal basis of $H_{(1,\epsilon)}^{vw}$. Therefore, $\{U(x_f) : s(f) = v, t(f) = w\}$ is an orthonormal basis of $H_{(1,\epsilon)}^{\kappa(v)\kappa(w)}$. Therefore, by Lemma 3.3.3

$$
\sum_{f \ni s(f)=v, t(f)=w} c(U(x_f) \otimes rev(U(x_f))) = y_{\kappa(v)\kappa(w)}.
$$

Since U commutes with the 'concatenation' operator and the 'reverse' operator, we have

$$U(y_{vw}) = \sum_{f \ni s(f)=v, t(f)=w} c(U(x_f) \otimes rev(U(x_f))) = y_{\kappa(v)\kappa(w)}.$$

Therefore, for all $v \in V^\epsilon$, we obtain

$$\begin{aligned} U(y_v) &= \sum_{w \ni n(v,w) \neq 0} \mu(w) y_{\kappa(v)\kappa(w)} \\ &= \lambda^{-1}[\sum_{w \ni n(\kappa(v),w) \neq 0} \mu(\kappa(w)) y_{\kappa(v)w}] = \lambda^{-1} y_{\kappa(v)}. \end{aligned}$$

Also, for all $v \in V^\epsilon$, we have

$$\begin{aligned} \delta_{\kappa(v)} &= \sum_{f \ni s(f)=\kappa(v)} \mu^2(t(f))/\mu^2(\kappa(v)) \\ &= \sum_{f \ni s(f)=v} \mu^2(\kappa(t(f)))/\mu^2(\kappa(v)) \\ &= \sum_{f \ni s(f)=v} \lambda^2 \mu^2(t(f))/\lambda^2 \mu^2(v) \\ &= \sum_{f \ni s(f)=v} \mu^2(t(f))/\mu^2(v) = \delta_v. \end{aligned}$$

Therefore, for all $v \in V^\epsilon$, we have

$$U^* Z_{TL^\epsilon} U(y_v) = U^* Z_{TL^\epsilon}(\lambda^{-1} y_{\kappa(v)}) = \lambda^{-1} \delta_{\kappa(v)} U^*(y_{\kappa(v)}) = \delta_{\kappa(v)} y_v = \delta_v y_v = Z_{TL^\epsilon}(y_v),$$

and both $U^* Z_{TL^\epsilon} U$ and Z_{TL^ϵ} are zero off the closed linear span of $\{y_v : v \in V_\epsilon\}$. So $Ad(U)$ fixes $Z_{TL^\epsilon}(1)$.

Let $U^* = \{(U_k^\epsilon)^* : (k,\epsilon) \in Col$ and $Ad(U^*) = \{Ad((U_k^\epsilon)^*) : (k,\epsilon) \in Col\}$. By a similar argument given in the proof of Lemma 4.1.1, we can prove that U^* commutes with the 'concatenation' operator. Now $rev \circ U_1^\epsilon = U_1^{\tilde{\epsilon}} \circ rev$ implies that $(U_1^{\tilde{\epsilon}})^* \circ rev = rev \circ (U_1^\epsilon)^*$, i.e., the restriction of U^* to $H_{(1,\epsilon)}$ commutes with the 'reverse' operator. Therefore, $Ad(U^*)$ is a planar algebra morphism and $Ad(U)$ is an automorphism of V_Γ with $Ad(U^*)$ as the inverse of $Ad(U)$. $\square$

4.2 Graph automorphism operators and multiplication operators

In this section we define the 'graph automorphism' operator and 'multiplication' operator and discuss some properties of these operators.

Definition 4.2.1. *Let Γ be a bipartite graph as before with a modulus-δ spin function μ. Let κ be a permutation of the vertices of Γ such that $n(\kappa(v),\kappa(w)) = n(v,w)$ for all vertices v and w, $\kappa(v) \in V^\epsilon$ for $v \in V^\epsilon$ and $\mu(\kappa(v)) = \lambda\mu(v)$ for all $v \in V^\epsilon$ and for some $\lambda > 0$. Label each of the m edges between two adjacent vertices by $\{1, \ldots, m\}$. We can extend κ to the edges of Γ by mapping the ith edge between (v,w) to the ith edge between $(\kappa(v),\kappa(w))$. Now extend κ to the paths of Γ by sending a path $p = c(e_1, \cdots, e_k)$ of length k to the path $\kappa(p) = c(\kappa(e_1), \cdots, \kappa(e_k))$, where e_i's are the edges of Γ, which gives rise to a unitary operator U_k^ϵ on $H_{(k,\epsilon)}$ for all $(k,\epsilon) \in Col$. Let*

$U = \{U_k^\epsilon : (k,\epsilon) \in Col\}$. Clearly U satisfies all the hypotheses of Theorem 4.1.3. Therefore, by Theorem 4.1.3,

$$Ad(U) := \{Ad(U_k^\epsilon) : V_k^\epsilon \to V_k^\epsilon | (k,\epsilon) \in Col\}$$

is an automorphism of V_Γ. $Ad(U)$ is said to be the graph automorphism operator associated to κ. The set of 'graph automorphism' operators of V_Γ will be denoted by G_Γ.

Definition 4.2.2. *Let O^+ be a unitary operator in V_1^+. Let $O^- = rev \circ O^+ \circ rev$. Let K be the closed linear span of $\{\otimes_{i=1}^k x_{e_i} : t(e_i) = s(e_{i+1})$ for $i = 1, \ldots, k-1\}$ in $(H_{(1,\epsilon)} \otimes H_{(1,\tilde{\epsilon})} \otimes H_{(1,\epsilon)} \otimes \cdots k$ times$)$. Let c be the 'concatenation' operator from $(H_{(1,\epsilon)} \otimes H_{(1,\tilde{\epsilon})} \otimes H_{(1,\epsilon)} \otimes \cdots k$ times$)$ to $H_{(k,\epsilon)}$. Observe that c is an onto partial isometry with initial subspace K and $(O^\epsilon \otimes O^{\tilde{\epsilon}} \otimes O^\epsilon \cdots k$ times $)$ maps the range of c^* isometrically onto the initial subspace K of c. Therefore, for all $(k,\epsilon) \in Col$,*

$$U_k^\epsilon := c \circ (O^\epsilon \otimes O^{\tilde{\epsilon}} \otimes O^\epsilon \cdots k \text{ times }) \circ c^*$$

is an unitary operator on $H_{(k,\epsilon)}$. Let $U = \{U_k^\epsilon : (k,\epsilon)\}$. Clearly U satisfies all the hypotheses of Theorem 4.1.3. Therefore, by Theorem 4.1.3,

$$Ad(U) := \{Ad(U_k^\epsilon) : V_k^\epsilon \to V_k^\epsilon | (k,\epsilon) \in Col\}$$

is an automorphism of V_Γ. $Ad(U)$ is said to be the 'multiplication' operator corresponding to the unitary O^+ in V_1^+. If O^+ acts on each basis element by scalar multiplication, then the corresponding multiplication operator is said to be a 'scalar multiplication' operator. The set of 'multiplication' operators of V_Γ will be denoted by M_Γ.

We discuss the properties of 'graph automorphism' and 'multiplication' operators in the following lemmas.

Lemma 4.2.3. *G_Γ and M_Γ are subgroups of $Aut(V_\Gamma)$.*

Proof. Let α and β be two 'graph automorphism' operators. Suppose κ and γ are the graph automophisms associated to α and β respectively. Then it is easy to see that $\alpha \circ \beta$ is given by the graph automorphism $\kappa \circ \gamma$ and α^{-1} is given by the graph automorphism κ^{-1}. Therefore, G_Γ is a subgroup of $Aut(V_\Gamma)$.

Now suppose that α and β are two 'multiplication' operators. Let $\{\alpha_{(k,\epsilon)} : (k,\epsilon) \in Col\}$ be the unitary operators associated α and $\{\beta_{(k,\epsilon)} : (k,\epsilon) \in Col\}$ be the unitary operators associated to β. From the definition of 'multiplication operators', we have

$$\begin{aligned} & \alpha_{(k,\epsilon)} = c \circ (\alpha_{(1,\epsilon)} \otimes \alpha_{(1,\tilde{\epsilon})} \otimes \alpha_{(1,\epsilon)} \cdots \text{k times }) \circ c^* \\ \text{and} \quad & \beta_{(k,\epsilon)} = c \circ (\beta_{(1,\epsilon)} \otimes \beta_{(1,\tilde{\epsilon})} \otimes \beta_{(1,\epsilon)} \cdots \text{k times }) \circ c^*. \end{aligned}$$

Therefore, $\alpha_{(k,\epsilon)}\beta_{(k,\epsilon)}$ is given by the follwing expression:

$$c \circ \big((\alpha_{(1,\epsilon)}\beta_{(1,\epsilon)}) \otimes (\alpha_{(1,\tilde{\epsilon})}\beta_{(1,\tilde{\epsilon})}) \otimes (\alpha_{(1,\epsilon)}\beta_{(1,\epsilon)}) \cdots \text{k times } \big) \circ c^*.$$

Also observe that $rev \circ (\alpha_{(1,+)}\beta_{(1,+)}) \circ rev = \alpha_{(1,-)}\beta_{(1,-)}$. Therefore, $\alpha\beta$ is a 'multiplication' operator given by the unitary $\alpha_{(1,+)}\beta_{(1,+)}$ in V_1^+. It is easy to check that α^{-1} is also a 'multiplication' operator given by the unitary $(\alpha_{(1,+)})^*$ in V_1^+. Therefore, M_Γ is a subgroup of $Aut(V_\Gamma)$.

∎

Lemma 4.2.4. *M_Γ is closed under the conjugation action of the elements of G_Γ and $M_\Gamma \cap G_\Gamma = 1$.*

Proof. Let $\{\alpha_{(k,\epsilon)} : (k,\epsilon) \in Col\}$ be the unitary operators associated to a 'graph automorphism' operator α and $\{\beta_{(k,\epsilon)} : (k,\epsilon) \in Col\}$ be the unitary operators associated to a 'multiplication' operator β. Therefore, the unitaries associated to $\alpha^{-1}\beta\alpha$ are given by $\{\alpha^{-1}_{(k,\epsilon)}\beta_{(k,\epsilon)}\alpha_{(k,\epsilon)} : (k,\epsilon) \in Col\}$. Observe that

$$\alpha_{(k,\epsilon)} = c \circ (\alpha_{(1,\epsilon)} \otimes \alpha_{(1,\tilde{\epsilon})} \otimes \alpha_{(1,\epsilon)} \cdots \text{k times }) \circ c^*.$$

And also by the definition of 'multiplication' operator,

$$\beta_{(k,\epsilon)} = c \circ (\beta_{(1,\epsilon)} \otimes \beta_{(1,\tilde{\epsilon})} \otimes \beta_{(1,\epsilon)} \cdots \text{k times }) \circ c^*.$$

So $\alpha^{-1}_{(k,\epsilon)}\beta_{(k,\epsilon)}\alpha_{(k,\epsilon)}$ is given by the following expression:

$$c \circ \Big((\alpha^{-1}_{(1,\epsilon)}\beta_{(1,\epsilon)}\alpha_{(1,\epsilon)}) \otimes (\alpha^{-1}_{(1,\tilde{\epsilon})}\beta_{(1,\tilde{\epsilon})}\alpha_{(1,\tilde{\epsilon})}) \otimes (\alpha^{-1}_{(1,\epsilon)}\beta_{(1,\epsilon)}\alpha_{(1,\epsilon)}) \cdots \text{k times } \Big) \circ c^*.$$

We define $O^+ := \alpha^{-1}_{(1,+)}\beta_{(1,+)}\alpha_{(1,+)}$ in V_1^+. Observe that

$$\begin{aligned} & \alpha^{-1}_{(1,-)}\beta_{(1,-)}\alpha_{(1,-)} \\ = \; & \Big(rev \circ \alpha^{-1}_{(1,+)} \circ rev\Big) \circ \big(rev \circ \beta_{(1,+)} \circ rev\big) \circ \big(rev \circ \alpha_{(1,+)} \circ rev\big) \\ = \; & rev \circ \Big(\alpha^{-1}_{(1,+)} \circ \beta_{(1,+)} \circ \alpha_{(1,+)}\Big) \circ rev \\ = \; & rev \circ O^+ \circ rev. \end{aligned}$$

Therefore, $\alpha^{-1}\beta\alpha$ is a multiplication operator associated to the unitary O^+ in V_1^+. Now we prove that $G_\Gamma \cap M_\Gamma = 1$. Suppose $\alpha \in G_\Gamma \cap M_\Gamma$. Let $\{\alpha_{(k,\epsilon)} : (k,\epsilon) \in Col\}$ be the unitary operators associated to α. Since α is a multiplication operator, $\alpha_{(1,+)}(H^{vw}_{(1,+)}) = H^{vw}_{(1,+)}$ $\forall v \in V^+$ and $w \in V^-$. Therefore, the permuation on the set of vertices induced by α is trivial, which implies $\alpha_{(1,+)} = Id_{H_{(1,+)}}$, which further implies $\alpha = 1$.

□

4.3 $Aut(V_\Gamma)$ as the semidirect product of M_Γ and G_Γ

In this section we prove the main theorem of the thesis that any automorphism of V_Γ is the composition of a 'graph automorphism operator' and a 'multiplication operator'. We prove it in Theorem 4.3.5 after a couple of lemmas.

Lemma 4.3.1. *Let α be an automorphism of V_Γ. Then there is a graph automorphism operator β such that α and β agree on V_0^ϵ.*

Proof. For $v \in V^\epsilon$, s_v is a minimal projection in V_0^ϵ. Since the restriction of α on V_0^ϵ is an automorphism of V_0^ϵ, $\alpha(s_v)$ is a minimal projection in V_0^ϵ. Now observe that any minimal projection in V_0^ϵ is equal to s_w for some $w \in V^\epsilon$. Therefore, there is a permutation κ of the vertices of Γ preserving the parity such that $\alpha(s_v) = s_{\kappa(v)}$ $\forall v \in V^+ \sqcup V^-$.

Let $v \in V^+$ and $w \in V^-$. Now $Z_{\mathcal{RI}}(s_v)Z_{\mathcal{LI}}(s_w)$ is a projection onto the closed linear span of the edges between v and w. Observe that this is either zero or a minimal central projection in V_1^+. Since α commutes with $Z_{\mathcal{LI}}$ and $Z_{\mathcal{RI}}$, we have

$$\alpha(Z_{\mathcal{RI}}(s_v)Z_{\mathcal{LI}}(s_w)V_1^+) = Z_{\mathcal{RI}}(s_{\kappa(v)})Z_{\mathcal{LI}}(s_{\kappa(w)})V_1^+.$$

The dimension of $Z_{\mathcal{RI}}(s_v)Z_{\mathcal{LI}}(s_w)V_1^+$ is the square of $n(v,w)$. Since α is an automorphism, the dimension is preserved by α. Therefore, $n(v,w) = n(\kappa(v),\kappa(w))$.
Now we prove that κ scales the spin function. Let $v \in V^\epsilon$. Then

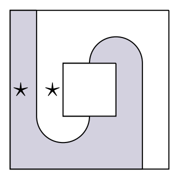
$$T_\epsilon^{\tilde{\epsilon}}(s_v) = \sum_{w \ni n(v,w)\neq 0} n(v,w)[\mu^2(v)/\mu^2(w)]s_w.$$

Since α commutes with this tangle, we have

$$\sum_{w \ni n(v,w)\neq 0} n(v,w)[\mu^2(v)/\mu^2(w)]s_{\kappa(w)} = \sum_{w \ni n(\kappa(v),w)\neq 0} n(\kappa(v),w)[\mu^2(\kappa(v))/\mu^2(w)]s_w,$$

which implies $\mu(v)/\mu(w) = \mu(\kappa(v))/\mu(\kappa(w))$ for the adjacent vertices v and w. By induction we can prove it for any two vertices v and w. Therefore, $\mu(v)/\mu(\kappa(v)) = \mu(w)/\mu(\kappa(w))$ for any two vertices v and w, which means $\mu(\kappa(v)) = \lambda\mu(v)$ $\forall$ vertices v in Γ and for some $\lambda > 0$. Suppose κ extends to the graph automorphism operator β. Then α and β agree on V_0^ϵ. $\square$

Lemma 4.3.2. *Consider the map $Z_T : V_1^+ \to V_1^-$ given by the following tangle T.*



T

Then $Z_T(A) = rev \circ A^* \circ rev$, for all $A \in V_1^+$. Moreover, we have $Z_T(AB) = Z_T(B)Z_T(A)$, $Z_T(A^*) = [Z_T(A)]^*$ $\forall A, B \in V_1^+$ and Z_T is invertible with inverse map $Z_{T^{-1}}^{V_\Gamma}$, where T^{-1} is the tangle T with reverse shading.

Proof. We observe that $Z_T(A)((e,f)) = A((\tilde{f},\tilde{e}))$, $\forall A \in V_1^+$ and $L = (e,f) \in l_2^-$. Therefore,

$$Z_T(A)(x_f) = \sum_{\substack{e \in p_1^- \ni s(e)=s(f),\\ t(e)=t(f)}} A((\tilde{f},\tilde{e}))x_e,$$

$\forall A \in V_1^+$ and $f \in p_1^-$. On the other hand we have

$$\begin{aligned} [rev \circ A^* \circ rev](x_f) = rev \circ A^*(x_{\tilde{f}}) &= rev\left(\sum_{\substack{e \in p_1^+ \ni s(e)=s(\tilde{f}),\\ t(e)=t(\tilde{f})}} A^*((e,\tilde{f}))x_e\right) \\ &= \sum_{\substack{e \in p_1^- \ni s(e)=s(f),\\ t(e)=t(f)}} A((\tilde{f},\tilde{e}))x_e, \end{aligned}$$

$\forall f \in p_1$. Therefore, $Z_T(A) = rev \circ A^* \circ rev$, for all $A \in V_1^+$.

Now by the first part of this lemma and the anti-unitarity of 'rev', we get $Z_T(AB) = rev \circ (AB)^* \circ rev = (rev \circ B^* \circ rev) \circ (rev \circ A^* \circ rev) = Z_T(B)Z_T(A)$ and $[Z_T(A)]^* = (rev \circ A \circ rev)^* = rev \circ A^* \circ rev = Z_T(A^*)$. $\square$

Lemma 4.3.3. *Let α be an automorphism of V_Γ which acts trivially on V_0^ϵ. Then there is a multiplication operator β such that α and β agree on V_1^ϵ.*

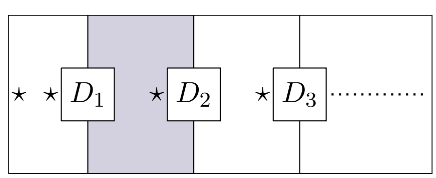
Proof. We know that the center of V_1^+ is the σ-weak closure of the subspace generated by $\{Z_{\mathcal{RI}}(s_v) Z_{\mathcal{LI}}(t_w) : v \in V^+, w \in V^-$ with $n(v,w) \neq 0\}$. Since α acts trivially on V_0^ϵ, it fixes all elements of the form $Z_{\mathcal{RI}}(s_v)Z_{\mathcal{LI}}(t_w)$, where $v \in V^+, w \in V^-$ with $n(v,w) \neq 0$, which means it fixes the center V_1^+ elementwise. Now V_1^+ being a type I von Neumann algebra, α acts as an inner automorphism on V_1^+. Therefore, there exists an unitary O^+ in V_1^+ such that $Z_T(A) = (O^+)^* A O^+$ $\forall A \in V_1^+$, where T is the as in Lemma 4.3.2. We shall prove that α on V_1^- is given by $Ad(O^-)$, where $O^- = rev \circ O \circ rev$. From Lemma 4.3.2, we obtain $O^- = Z_T((O^+)^*)$. Now using the properties of Z_T discussed in Lemma 4.3.2 and the fact that α commutes with Z_T, we obtain

$$\begin{array}{rcl} \alpha(A) & = & \alpha[Z_T Z_{T^{-1}}(A)] \\ & = & Z_T \alpha[Z_{T^{-1}}(A)] \\ & = & Z_T[(O^+)^* Z_{T^{-1}}(A) O^+] \\ & = & Z_T(O^+) \ A \ Z_T((O^+)^*) \\ & = & [Z_T((O^+)^*)]^* \ A \ Z_T((O^+)^*) \\ & = & (O^-)^* A O^-. \end{array}$$

$\forall A \in V_1^-$. Let β be the multiplication operator corresponding to the unitary O^+ in V_1^+. Then α and β agree on V_1^ϵ. $\square$

Lemma 4.3.4. *Let α be an automorphism of V_Γ which acts trivially on V_1^ϵ. Then α is a scalar multiplication operator.*

Proof. If p and q are two paths in p_k^ϵ with same end points, let E_{pq} denote the rank one partial isometry in V_k^ϵ which maps the basis element x_q to x_p. For a loop $l = (p,q)$, by E_l we mean E_{pq}. Let $p = c(e_1, \cdots, e_k) \in p_k^+$. We observe that $Z_T(\otimes_{i=1}^k E_{e_i e_i}) = E_{pp}$, where T is the tangle as given below:



Since α commutes with Z_T and acts trivially on V_1^ϵ, it also fixes E_{pp} $\forall p \in p_k^+$. Similarly, we can prove that α fixes E_{pp} $\forall p \in p_k^-$. Now consider the rank one partial isometry E_{pq} corresponding to the loop $l = (p,q) \in l_k^\epsilon$. We have $E_{pp}E_{pq}E_{qq} = E_{pq}$, which implies $E_{pp}\alpha(E_{pq})E_{qq} = \alpha(E_{pq})$ because α fixes E_{pp} and E_{qq}. Therefore, α sends E_{pq} to a scalar multiple of itself. Let us denote this scalar by $\rho(l)$.

Claim: (i) $|\rho(l)| = 1$ for all loops l in Γ;
(ii) $\rho(l) = 1$ on the loops of length 2;
(iii) $\rho(l)$ is independent of the base point of l;

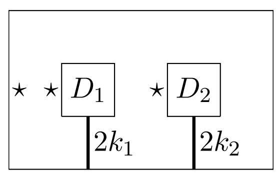
(iv) $\rho(l_1 * l_2) = \rho(l_1)\rho(l_2)$, where $l_1 * l_2$ denotes the concatenation of two loops l_1 and l_2 based at the same point;
(v) Suppose l' is a loop in Γ that goes first from a vertex v_0 to v_1 by a path p, then across a loop l based at v_1, then back to v_0 by the same path p. Then $\rho(l) = \rho(l')$.

Proof. (i) $\alpha(E_l) = \rho(l)E_l$, where $l = (p, q)$. Since α is a $*$-algebra morphism and it fixes the projection E_{pp}, applying α on both sides of the equation $E^*_{pq}E_{pq} = E_{pp}$ we obtain $|\rho(l)| = 1$.

(ii) It follows from the fact that α acts trivially on V_1^ϵ $\forall \epsilon$.

(iii) Let $l = (p, q) \in l_k^\epsilon$. Also let l' be the same loop l with the base point as the next vertex of the base point of l. Observe that $Z_{\mathcal{R}_{(k,\epsilon)}^{(k,\bar{\epsilon})}}(E_l) = E_{l'}$. Since α commutes with $Z_{\mathcal{R}_{(k,\epsilon)}^{(k,\bar{\epsilon})}}$, we have $\rho(l) = \rho(l')$. Therefore, $\rho(l)$ is independent of the base point of l.

(iv) Let $l_1 * l_2$ be the concatenation of two loops l_1 and l_2 based at the same point and of lengths $2k_1$ and $2k_2$ respectively. Observe that $Z_T(E_{l_1} \otimes E_{l_2}) = E_{l_1 * l_2}$, where T is the tangle given as follows:



Since α commutes with α, applying α on both sides of $Z_T(E_{l_1} \otimes E_{l_2}) = E_{l_1 * l_2}$, we obtain $\rho(l_1 * l_2) = \rho(l_1)\rho(l_2)$.

(v) It easily follows from (ii), (iii) and (iv). $\square$

Let Y be a maximal tree in Γ. Fix a vertex $v_0 \in V^+$ in the interior of Y. Let $S := \{e_\alpha : \alpha \in \Lambda\}$ be the set of edges in Γ whose interiors are in $\Gamma - Y$ and whose starting points are in V^+. Each edge e_α determines a loop l_α in Γ that goes first from v_0 to $s(e_\alpha)$ by a path in Y, then across e_α, then back to v_0 by a path in Y. We know that $\pi_1(\Gamma)$ is a free group with basis the classes $[l_\alpha]$ corresponding to the edges e_α. Define a unitary operator O^+ in V_1^+ as follows:

$$O^+(x_{e_\alpha}) = \begin{cases} \rho(l_\alpha)x_{e_\alpha} & \text{if } e_\alpha \in S; \\ x_{e_\alpha} & \text{otherwise.} \end{cases}$$

Let $U = \{U_k^\epsilon : (k, \epsilon) \in Col\}$ be the collection of unitary operators corresponding to O^+ obtained by the procedure discussed in Definition 4.2.2 and $Ad(U) = \{Ad(U_k^\epsilon) : (k, \epsilon) \in Col\}$ be the corresponding scalar multiplication operator. Now we shall prove that $Ad(U)(E_{l_\alpha}) = \rho(l_\alpha)E_{l_\alpha}$ $\forall \alpha \in \Lambda$. Let $l_\alpha = (p_\alpha, q_\alpha)$, where $\alpha \in \Lambda$.

Case 1. Suppose the edge e_α is in the path p_α. Then $U(x_{p_\alpha}) = \rho(l_\alpha)x_{p_\alpha}$ and $U^*(x_{q_\alpha}) = x_{q_\alpha}$, which implies $U^*E_{l_\alpha}U(x_p) = \rho(l_\alpha)x_q$ and is zero on the other basis elements. Therefore, $Ad(U)(E_{l_\alpha}) = \rho(l_\alpha)E_{l_\alpha}$.

Case 2. Suppose the edge e_α is in the path q_α. Then $U(x_{p_\alpha}) = x_{p_\alpha}$ and $U^*(q_\alpha) = \rho(l_\alpha)q_\alpha$, which implies $U^*E_{l_\alpha}U(x_p) = \rho(l_\alpha)x_q$ and is zero on the other basis elements. Therefore, $Ad(U)(E_{l_\alpha}) =$

$\rho(l_\alpha)E_{l_\alpha}$.

So from the previous claim and from the fact that $Ad(U)(E_{l_\alpha}) = \rho(l_\alpha)E_{l_\alpha}\ \forall \alpha \in \Lambda$, we conclude that $Ad(U)(E_l) = \alpha(E_l)$ for all loops l in Γ. Since V_k^ϵ is the σ-weak closure of $\{E_l : l \in l_k^\epsilon\}$ for all $(k,\epsilon) \in Col$ and each automorphism of V_Γ is σ-weakly continuous on V_k^ϵ for all $(k,\epsilon) \in Col$, we have $Ad(U) = \alpha$. Therefore, α is a scalar multiplication operator.

$\square$

Theorem 4.3.5. *Consider the planar algebra V_Γ with Γ as before. Let α be an automorphism of V_Γ. Then $\alpha = \beta\gamma$ for some $\beta \in G_\Gamma$ and $\gamma \in M_\Gamma$. Moreover, $Aut(V_\Gamma) = M_\Gamma \rtimes G_\Gamma$.*

Proof. From Lemma 4.3.1, we know that there exists $\beta \in G_\Gamma$ such that α and β agree on $V_0^\epsilon\ \forall\ \epsilon$. Therefore, $\beta^{-1}\alpha$ acts trivially on $V_0^\epsilon\ \forall\ \epsilon$. By Lemma 4.3.3, there exists $\gamma_1 \in M_\Gamma$ such that $\beta^{-1}\alpha$ and γ_1 agree on $V_1^\epsilon\ \forall\ \epsilon$. Therefore, $\gamma_1^{-1}\beta^{-1}\alpha$ acts trivially on $V_1^\epsilon\ \forall\ \epsilon$. So by Lemma 4.3.4, $\gamma_1^{-1}\beta^{-1}\alpha$ is a scalar multiplication operator, say γ_2. Therefore, $\alpha = \beta\gamma_1\gamma_2$ with $\beta \in G_\Gamma$ and $\gamma_1\gamma_2 \in M_\Gamma$.

From Lemma 4.2.4, we know that M_Γ is closed under the conjugation action of G_Γ and $M_\Gamma \cap G_\Gamma = 1$. Also from the first part of the theorem we obtain $G_\Gamma M_\Gamma = Aut(V_\Gamma)$. Therefore, $Aut(V_\Gamma) = M_\Gamma \rtimes G_\Gamma$. $\square$

Chapter 5

Subfactor planar algebras of BGPAs

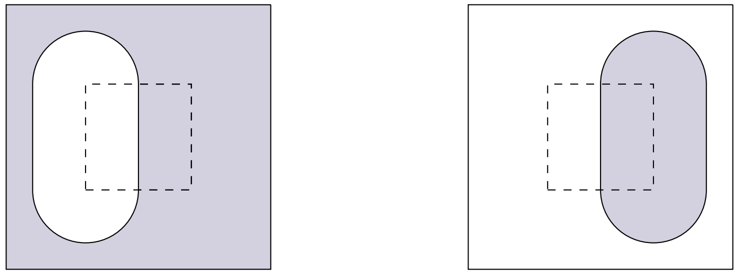
In this chapter we describe a tool for constructing a wide range of subfactor planar algebras obtained as fixed point subalgebras of BGPAs under group actions. It happens that the fixed point set of a BGPA under a group action is a planar $*$-subalgebra, but it fails to be an SPA in general. So in the first section we derive some sufficient conditions for a planar $*$-subalgebra to be an SPA.

5.1 Sufficient conditions for a planar $*$-subalgebra of a BGPA to be an SPA

In this section we first derive a sufficient condition for connected finite bipartite graphs, which just demands that the planar $*$-subalgebra X has 1-dimensinal intersection with V_0^ϵ and the spin function μ is a modulus-δ spin function. For a unifomly locally finite connected bipartite graph we need that the planar $*$-subalgebra X is spherical, it has 1-dimensional intersection with V_0^ϵ and the spin function μ is a modulus-δ spin function. In this chapter we follow all the notations used in the previous chapters.

Theorem 5.1.1. *Let V_Γ be a BGPA corresponding to a connected finite bipartite graph Γ. Let μ be a modulus-δ spin function on Γ. Suppose X is a planar $*$-subalgebra of V_Γ such that $dim\, X \cap V_0^\epsilon = 1$ for all ϵ. Then X is a subfactor planar algebra.*

Proof. From Theorem 3.5.1, we know that the scalar sesquilinear form defined by $\langle x, y\rangle = [Z^P_{\mathcal{RE}}]^n(y^*x)$, where $x, y \in P_{(k,\epsilon)}$, is positive definite, and from Theorem 3.5.3, we know that X has $*$ property. Since Γ is a finite graph, clearly $dim\, X \cap V_k^\epsilon$ is finite dimensional for all $(k, \epsilon) \in Col$. So it remains only to show that X is spherical. Let $A \in V_1^+ \cap X$. Since $dim\, X \cap V_0^\epsilon = 1$ for all ϵ, $Z^X_{\mathcal{LE}}(A) = \lambda_l \sum_{v\in V^-} 1_v$ and $Z^X_{\mathcal{RE}}(A) = \lambda_r \sum_{v\in V^+} 1_v$, for some $\lambda_l, \lambda_r \in \mathbb{C}$. So $Z^- \circ Z^X_{\mathcal{LE}}(A) = \lambda_l \sum_{v\in V^-} \mu^4(v)$ and $Z^+ \circ Z^X_{\mathcal{RE}}(A) = \lambda_l \sum_{v\in V^+} \mu^4(v)$. By Lemma 3.4.1, $Z^- \circ Z^X_{\mathcal{LE}}(A) = Z^+ \circ Z^X_{\mathcal{RE}}(A)$. Therefore, $\lambda_l/\lambda_r = [\sum_{v\in V^+} \mu^4(v)]/[\sum_{v\in V^-} \mu^4(v)]$, and which is independent of A. Since μ is a modulus-δ spin function, if we take $B = Z^X_{1(1,+)}(1)$, we have $Z^X_{\mathcal{LE}}(B) = \delta \sum_{v\in V^-} 1_v$ and $Z^X_{\mathcal{RE}}(B) = \delta \sum_{v\in V^+} 1_v$ by the following pictures.

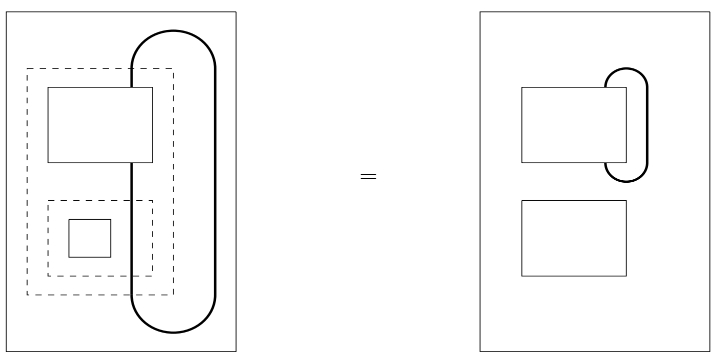


$\mathcal{LC} \circ 1^{(1,+)}$ $\qquad$ $\mathcal{RC} \circ 1^{(1,+)}$

Now $Z^- \circ Z^X_{\mathcal{LE}}(B) = Z^+ \circ Z^X_{\mathcal{RE}}(B)$ gives $[\sum_{v \in V^+} \mu^4(v)]/[\sum_{v \in V^-} \mu^4(v)] = \delta/\delta = 1$. Therefore, $\lambda_l = \lambda_r$, that means $Z^X_{\mathcal{LE}}(A) = Z^X_{\mathcal{RE}}(A)$. So X is spherical. $\square$

Theorem 5.1.2. *Let V_Γ be a BGPA, where Γ is a uniformly locally finite connected bipartite graph. Let X be a planar $*$-subalgebra of V_Γ with $dim\, X \cap V^\epsilon_0 = 1\ \forall \epsilon$. Then $X \cap V^\epsilon_k$ is finite dimensional $\forall (k,\epsilon) \in Col$. Moreover, if we assume that X is spherical and the spin function μ is a modulus-δ spin function, then X is a subfactor planar algebra.*

Proof. Fix a $(k,\epsilon) \in Col$ and a vertex $v \in V^\epsilon$. Consider the minimal projection $s_v \in V^\epsilon_0$. Observe that $y := [\mathcal{RI}]^k(s_v)$ is a projection on $H^v_{(k,\epsilon)}$. Define a map $f : X \cap V^\epsilon_k \to yV^\epsilon_k$ by $x \mapsto yx$ $\forall x \in X \cap V^\epsilon_k$. Since $H^v_{(k,\epsilon)}$ is finite dimensinal, yV^ϵ_k is also a finite dimensional subspace of V^ϵ_k. Now to prove that $X \cap V^\epsilon_k$ is finite dimensional, it suffices to prove that the linear map f is injective. Suppose $x \in X \cap V^\epsilon_k$ such that $yx = 0$. Since x fixes the subspace $H^v_{(k,\epsilon)}$, it will commute with y. Therefore, $yx^*x = 0$, which implies $[\mathcal{RE}]^k(yx^*x) = 0$. Now observe from the following tangle-equality that $[\mathcal{RE}]^k(yz) = s_v[\mathcal{RE}]^k(z)\, \forall z \in V^\epsilon_k$.



Therefore, $[\mathcal{RE}]^k(yx^*x) = 0$ implies $s_v[\mathcal{RE}]^k(x^*x) = 0$. Since $dim\, X \cap V^\epsilon_0 = 1$, $[\mathcal{RE}]^k(x^*x)$ is a scalar multiple of identity in V^ϵ_0. So $s_v[\mathcal{RE}]^k(x^*x) = 0$ implies $\langle x, x\rangle = [\mathcal{RE}]^k(x^*x) = 0$. Now from Theorem 3.5.1, we know that $\langle , \rangle$ is a positive definite form. Threfore, $x = 0$, which implies f is injective. So $X \cap V^\epsilon_k$ is finite dimensional $\forall\, (k,\epsilon) \in Col$.

Theorem 3.5.1 gives that the scalar sesquilinear form defined by $\langle x, y\rangle = [Z^P_{\mathcal{RE}}]^n(y^*x)$, where $x, y \in P_{(k,\epsilon)}$, is positive definite and from Theorem 3.5.2 we obtain that X has $*$ property. Now the assumption that the spin function μ is a modulus-δ spin function ensures that V_Γ has a positive modulus (by Theorem 3.5.3). So further assuming the 'sphericality' property of X, we have X is a subfactor planar algebra. $\square$

5.2 Fixed point subfactor planar algebras

Let a group G act on a BGPA V_Γ by the group homomorphism $\alpha : G \to Aut(V_\Gamma)$. A little thought gives that V_Γ^G is a planar *-subalgebra of V_Γ. We take a modulus-δ spin function on Γ so that V_Γ has a positive modulus. Now V_Γ^G will be an SPA if we further assume the 'connectedness' of V_Γ^G, when Γ is finite, and both the 'connectedness' and 'sphericality' of V_Γ^G, when Γ is infinite. In the following remarks we describe two sufficient conditions for the 'connectedness' and 'sphericality' of V_Γ^G which are easy to verify.

Remark 5.2.1. *Suppose, for all $g \in G$, $\alpha(g) = \beta(g)\gamma(g)$, where $\beta(g)$ is a 'graph automorphism' operator and $\gamma(g)$ is a 'multiplication operator'. Now observe that V_Γ^G is connected if and only if $\{\beta(g) : g \in G\}$ acts on V^+ and V^- transitively.*

Remark 5.2.2. *If V_Γ has modulus δ, we know that $Z_{\mathcal{LE}^{(0,-)}_{(1,+)}}(1) = \delta 1$ and $Z_{\mathcal{RE}^{(0,+)}_{(1,+)}}(1) = \delta 1$. So if we have the 'irreducibility' of V_Γ^G, then V_Γ^G automatically becomes spherical.*

Now we are ready to give the definition of 'fixed point subfactor planar algebra'.

Definition 5.2.3. *Let V_Γ be a BGPA and G be a subgroup of $Aut(V_\Gamma)$ with V_Γ^G an SPA. Then V_Γ is said to be a 'fixed point subfactor planar algebra'.*

5.3 An Example of Fixed Point SPA

In this section we present a detailed proof of the fact that the SPA of a 'diagonal subfactor without cocyle' can be obtained as a fixed point subfactor planar algebra, what was remarked in [B] as an example of a fixed point SPA. To do this, we will follow the description of the tower of basic construction of a 'diagonal subfactor' as given in [BDG1].

Let N be a II_1 factor and G is a finitely generated subgroup of $Out(N)$. Let $\{g_i : i \in I\}$ generate G, where I is a finite set. We assume that $g_1 = 1$. Let $\alpha : G \to Aut(N)$ be a lift of G in $Aut(N)$ such that $\alpha(1) = 1$. Set $\alpha_i = \alpha(g_i)$ for all $i \in I$. Let $M_{I^n}(\mathbb{C})$ denote the matrix algebra over $\mathbb{C}$ with rows and columns indexed by the elements of I^n for all non-negative integers n. Define $M = N \otimes M_I(\mathbb{C})$. Now define the subfactor $N \hookrightarrow M$ by sending $x \in N$ to $\sum_{i\in I} \alpha_i(x) \otimes E_{i,i}$ for all $x \in N$. This is called a 'diagonal subfactor'. If the lift α is a group homomorphism, we call it a 'diagonal subfactor without cocyle'. We shall be interested in diagonal subfactors without cocyle, so from now onwards the lift α is a group homomorphism. For $\underline{i} = (i_1, i_2, \cdots, i_n) \in I^n$, the automorphism $\alpha_{i_1}^{-1}\alpha_{i_2}\cdots\alpha_{i_n}^{(-1)^n}$ will be denoted by $alt_\alpha(\underline{i})$.

Computation of the tower of basic construction of the subfactor $N \hookrightarrow M$:

Now we compute the tower of basic construction of the subfactor $N \hookrightarrow M$ from [BDG1] and [Pi-Po]. Define $M_n = N \otimes M_{I^{n+1}}(\mathbb{C})$ for all $n \geq -1$. Suppose $M_{n-1} = N \otimes M_{I^n}(\mathbb{C})$ is included in

$M_n = N \otimes M_{I^{n+1}}(\mathbb{C})$ in the following way:

$$x \otimes E_{\underline{i},\underline{j}} \mapsto \sum_{k\in I} \alpha_k^{(-1)^n}(x) \otimes E_{(\underline{i},k),(\underline{j},k)}$$

where $x \in N$, $\underline{i}, \underline{j} \in I^{n+1}$ and $n \geq 0$. Therefore, N is included in M_n by the following map:

$$y \mapsto \sum_{\underline{i}\in I^{n+1}} alt_\alpha^{-1}(\underline{i})(y) \otimes E_{\underline{i},\underline{i}}$$

for all $y \in N$. Let tr_N be the unique normal faithful tracial state on N. Define tr_{M_n} on M_n by

$$tr_{M_n}(\sum_{\underline{i},\underline{j}\in I^{n+1}} x_{\underline{i},\underline{j}} \otimes E_{\underline{i},\underline{j}}) = |I|^{-1} \sum_{\underline{i},\underline{i}\in I^{n+1}} tr_N(x_{\underline{i},\underline{i}})$$

where $x_{\underline{i},\underline{j}} \in N$ and $n \geq 0$. tr_{M_n} is the unique faithful normal tracial state on M_n. Observe that the inclusion maps preserve the traces. Then the unique trace preserving conditional expectation $\mathbb{E}_{M_{n-1}}^{M_n}$ from M_n to M_{n-1} is given by

$$x \otimes E_{(\underline{i},k),(\underline{j},l)} \mapsto \delta_{k,l}|I|^{-1}\alpha_k^{(-1)^{n-1}}(x) \otimes E_{\underline{i},\underline{j}}$$

where $x \in N$ and $n \geq 1$. Consider the orthogonal projections

$$e_n = |I|^{-1} \sum_{\underline{k}\in I^{n-1}, i,j\in I} 1_N \otimes E_{(\underline{k},i,i),(\underline{k},j,j)} \in M_n$$

for all $n \geq 1$. We observe the following:

(i) $[M_n : M_{n-1}] = |I|^2$ for all $n \geq 0$;

(ii) each M_n is a II_1 factor;

(iii) $[e_n, y] = 0$ for all $y \in N$ and $n \geq 1$;

(iV) $\mathbb{E}_{M_{n-1}}^{M_n}(e_n) = |I|^{-2}\, 1_{M_n}$ for all $n \geq 1$.

Therefore, using Proposition 1.2 in [Pi-Po] we have the following Proposition:

Proposition 5.3.1. *With the running notations,*

$$N \hookrightarrow M \hookrightarrow M_1 \hookrightarrow \cdots \hookrightarrow M_n \hookrightarrow \cdots$$

is the Jones tower of II_1 factors associated to the subfactor $N \hookrightarrow M$ where the Jones projections and conditional expectations are given by $\{e_n : n \geq 1\}$ and $\{\mathbb{E}_{M_{n-1}}^{M_n} : n \geq 1\}$ respectively.

Computation of the relative commutants of the subfactor $N \hookrightarrow M$:

We now compute the relative commutants associated to the subfactor $N \hookrightarrow M$. Let $\sum_{\underline{i},\underline{j} \in I^{n+1}} x_{\underline{i},\underline{j}} \otimes E_{\underline{i},\underline{j}} \in N' \cap M_n$, where $n \geq 0$. Then it will commute with $\sum_{\underline{k} \in I^{n+1}} alt_\alpha^{-1}(\underline{k})(y) \otimes E_{\underline{k},\underline{k}}$ for all $y \in N$, which implies $x_{\underline{i},\underline{j}} \left(alt_\alpha^{-1}(\underline{j}) alt_\alpha(\underline{i})\right)(y) = y x_{\underline{i},\underline{j}}$ for all $y \in N$ and $\underline{i}, \underline{j} \in I^{n+1}$. It implies that $x_{\underline{i},\underline{j}} = 0$ if $alt_\alpha(\underline{i}) \neq alt_\alpha(\underline{j})$ and $x_{\underline{i},\underline{j}}$ is a scalar if $alt_\alpha(\underline{i}) = alt_\alpha(\underline{j})$. So for all $n \geq 0$, $N' \cap M_n$ is the span of

$$\{1_N \otimes E_{\underline{i},\underline{j}} | \underline{i}, \underline{j} \in I^{n+1} \text{ and } alt_\alpha(\underline{i}) = alt_\alpha(\underline{j})\}.$$

Now for $n \geq 0$ identify $M_n = N \otimes M_{I^{n+1}}(\mathbb{C})$ with $(N \otimes M_I(\mathbb{C})) \otimes M_{I^n}(\mathbb{C}) = M \otimes M_{I^n}(\mathbb{C})$ by

$$\sum_{\underline{i},\underline{j} \in I^n, k,l \in I} x_{(k,\underline{i}),(l,\underline{j})} \otimes E_{(k,\underline{i}),(l,\underline{j})} \mapsto \sum_{\underline{i},\underline{j} \in I^n, k,l \in I} (x_{(k,\underline{i}),(l,\underline{j})} \otimes E_{k,l}) \otimes E_{\underline{i},\underline{j}}.$$

For all α_k, $k \in I$, the automorphism α_k of N induces an automorphism β_k of M by the following way:

$$\beta_k(\sum_{i,j \in I} x_{i,j} \otimes E_{i,j}) = \sum_{i,j \in I} \alpha_k^{-1}(x_{i,j}) \otimes E_{i,j}.$$

Let $alt_\beta(\underline{i}) = \beta_{i_1}^{-1} \beta_{i_2} \cdots \beta_{i_n}^{(-1)^n}$, where $\underline{i} = (i_1, i_2, \cdots, i_n) \in I^n$. Now observe that the inclusion $M_{n-1} \hookrightarrow M_n$ is also given by the following map:

$$x \otimes E_{\underline{i},\underline{j}} \mapsto \sum_{k \in I} \beta_k^{(-1)^{n-1}}(x) \otimes E_{(\underline{i},k),(\underline{j},k)}$$

where $x \in M$, $\underline{i}, \underline{j} \in I^{n-1}$ and $n \geq 1$. Also, $e_n \in M_n$ is given by

$$e_n = |I|^{-1} \sum_{\underline{k} \in I^{n-2}, i,j \in I} 1_M \otimes E_{(\underline{k},i,i),(\underline{k},j,j)} \in M_n$$

for all $n \geq 2$ and the conditional expectation $\mathbb{E}_{M_{n-1}}^{M_n}$ from M_n to M_{n-1} is given by

$$x \otimes E_{(\underline{i},k),(\underline{j},l)} \mapsto \delta_{k,l} |I|^{-1} \alpha_k^{(-1)^n}(x) \otimes E_{\underline{i},\underline{j}}$$

where $x \in N$, $\underline{i}, \underline{j} \in I^{n-1}$, $k, l \in I$ and $n \geq 2$. We also observe that the inclusion $M \hookrightarrow M_n$ is given by the following map:

$$y \mapsto \sum_{\underline{i} \in I^n} alt_\beta^{-1}(\underline{i})(y) \otimes E_{\underline{i},\underline{i}}$$

for all $y \in M$. By a similar argument in the previous paragraph we can prove that $M' \cap M_n$ is given by the span of

$$\{1_M \otimes E_{\underline{i},\underline{j}} | \underline{i}, \underline{j} \in I^n \text{ and } alt_\beta(\underline{i}) = alt_\beta(\underline{j})\}.$$

Now observe that $1_M \otimes E_{\underline{i},\underline{j}}$, where $\underline{i}, \underline{j} \in I^n$ and $alt_\beta(\underline{i}) = alt_\beta(\underline{j})$, can also be written as $\sum_{k \in I} 1_N \otimes E_{(k,\underline{i}),(k,\underline{j})}$, where $alt_\alpha((k,\underline{i})) = alt_\alpha((k,\underline{j}))$. Therefore, the inclusion $M' \cap M_n \hookrightarrow N' \cap M_n$ is given by:

$$1_M \otimes E_{\underline{i},\underline{j}} \hookrightarrow \sum_{k \in I} 1_N \otimes E_{(k,\underline{i}),(k,\underline{j})}$$

for all $n \geq 0$ and the unique trace preserving conditional expectation $\mathbb{E}^{N'\cap M_n}_{M'\cap M_n}$ from $N' \cap M_n$ to $M' \cap M_n$ is given by:

$$1_N \otimes E_{(k,\underline{i}),(l,\underline{j})} \mapsto \delta_{k,l} |I|^{-1} 1_M \otimes E_{\underline{i},\underline{j}}.$$

for all $n \geq 1$.

From now onwards we shall look $N' \cap M_n$ as a subalgebra of $N \otimes M_{I^{n+1}}(\mathbb{C})$ for all $n \geq -1$ and $M' \cap M_n$ as a subalgebra of $M \otimes M_{I^n}(\mathbb{C})$ for all $n \geq 0$. The standard invariant of the subfactor $N \hookrightarrow M$ is given by the following grid of relative commutants:

$$\begin{tikzcd}
\mathbb{C} = N' \cap M_{-1} \arrow[r, hook] & N' \cap M_0 \arrow[r, hook] & N' \cap M_1 \arrow[r, hook] & N' \cap M_2 \arrow[r, hook] & \cdots \\
\mathbb{C} = & M' \cap M_0 \arrow[r, hook] \arrow[u, hook] & M' \cap M_1 \arrow[r, hook] \arrow[u, hook] & M' \cap M_2 \arrow[r, hook] \arrow[u, hook] & \cdots
\end{tikzcd}$$

Now we prove the extremality of this subfactor.

Proposition 5.3.2. *With the running notations, $N \hookrightarrow M$ is an extremal subfactor.*

Proof. From [Po2], we observe that $N \hookrightarrow M$ is extremal if and only if

$$[pMp : Np] = tr_M(p)^2 [M : N] \tag{5.1}$$

for all projections p in $N' \cap M$. Now observe that if two projectios p and q are unitarily equivalent and p satisfies equation (5.1), then q also satisfies the equation. Since $N' \cap M = \{1_N \otimes x : x \in M_I(\mathbb{C})\}$, any projection in $N' \cap M$ is of the form $1_N \otimes p$ for some projection p in $M_I(\mathbb{C})$. But any projection p in $M_I(\mathbb{C})$ is unitarily equivalent to $q := \sum_{i \in I'} E_{i,i}$ for some $I' \subseteq I$. Therefore, it suffices to prove equation (5.1) only for $1_N \otimes q$, but that also holds because $[qMq : Nq] = |I'|^2$ and $tr_M(q) = |I'|/|I|$. ☐

Define $P^+_n = N' \cap M_{n-1}$ and $P^-_n = M' \cap M_n$ for all $n \geq 0$. Therefore, by 2.3.2 there is a unique subfactor planar algebra structure on $P^{N \hookrightarrow M} := P = \{P^\epsilon_n : (n, \epsilon) \in Col\}$ satisfying the following properties:

1. $Z^P_{\mathcal{TL}^{(n,+)}}(1) = |I| e_{n+1}$, $Z^P_{\mathcal{TL}^{(n,-)}}(1) = |I| e_{n+2}$ for all $n \geq 0$;

2. $Z^P_{\mathcal{RI}^{(n+1,+)}_{(n,+)}}$, $Z^P_{\mathcal{RI}^{(n+1,-)}_{(n,-)}}$ and $Z^P_{\mathcal{LI}^{(n+1,+)}_{(n,-)}}$ are the inclusions $N' \cap M_{n-1} \hookrightarrow N' \cap M_n$, $M' \cap M_n \hookrightarrow M' \cap M_{n+1}$ and $M' \cap M_n \hookrightarrow N' \cap M_n$ respectively for all $n \geq 0$;

3. $Z^P_{\mathcal{RE}^{(n,+)}_{(n+1,+)}} = |I| \mathbb{E}^{N'\cap M_n}_{N'\cap M_{n-1}}$ for all $n \geq 1$ and $Z^P_{\mathcal{RE}^{(n,-)}_{(n+1,-)}} = |I| \mathbb{E}^{M'\cap M_{n+1}}_{M'\cap M_n}$ for all $n \geq 1$;

4. $Z^P_{\mathcal{RE}^{(0,+)}_{(1,+)}}(x) = |I| tr_{M_0}(x)$ for all $x \in N' \cap M_0$ and $Z^P_{\mathcal{RE}^{(0,-)}_{(1,-)}}(x) = |I| tr_{M_1}(x)$ for all $x \in M' \cap M_1$;

5. $Z^P_{\mathcal{LE}^{(n,-)}_{(n+1,+)}} = |I| \mathbb{E}^{N'\cap M_n}_{M'\cap M_n}$ for all $n \geq 1$ and $Z^P_{\mathcal{LE}^{(0,-)}_{(1,+)}}(x) = |I| tr_{M_0}(x)$ for all $x \in N' \cap M_0$.

Construction of a fixed point SPA V_Γ^G:

Now we construct a uniformly locally finite connected bipartite graph Γ. Let $\{v_g^+ : g \in G\}$ and $\{v_g^- : g \in G\}$ be the set of positive and negative vertices, respectively, corresponding to each element of G. Two vertces v_g^+ and v_h^- will be connected if and only if there exists $i \in I$ such that $g = hg_i^{-1}$. Define a modulus-δ spin function μ by giving the weight 1 to each vertex. Let V_Γ be the BGPA corresponding to the graph G. Observe that the modulus of V_Γ is $|I|$. Now define a graph automorphism by $h.v_g^\epsilon := v_{hg}^\epsilon$ for all $g, h \in G$ and $\epsilon \in \{+,-\}$. Observe that the spin function is invariant under this graph automorphism, so it induces an automorphism of V_Γ. Therefore, there is a group action of G on V_Γ. For all $\underline{i} \in I^n$, we define $alt_{(G,+)}(\underline{i}) = g_{i_1}^{-1} g_{i_2} \cdots g_{i_n}^{(-1)^n}$ and $alt_{(G,-)}(\underline{i}) = g_{i_1} g_{i_2}^{-1} \cdots g_{i_n}^{(-1)^{n-1}}$. It is an immediate observation that $alt_\alpha(\underline{i}) = alt_\alpha(\underline{j})$ iff $alt_{(G,+)}(\underline{i}) = alt_{(G,+)}(\underline{j})$ and $alt_\beta(\underline{i}) = alt_\beta(\underline{j})$ iff $alt_{(G,-)}(\underline{i}) = alt_{(G,-)}(\underline{j})$. Now observe that for each positive vertex v_g^+ of Γ, there is a one-one correspondence between $\{(alt_\alpha(\underline{i}), alt_\alpha(\underline{j})) | \underline{i}, \underline{j} \in I^n \text{ and } alt_\alpha(\underline{i}) = alt_\alpha(\underline{j})\}$ and the set of all loops of length $2n$ based at v_g^+ by mapping $(alt_\alpha(\underline{i}), alt_\alpha(\underline{j}))$ to the loop $L_{v_g^+}^{(alt_{(G,+)}(\underline{i}), alt_{(G,+)}(\underline{j}))}$ defined as below:

$$v_g^+ \to v_{gg_{i_1}^{-1}}^- \to v_{gg_{i_1}^{-1} g_{i_2}}^+ \to \cdots \to v_{g\, alt_{(G,+)}(\underline{i})}^{\epsilon^n} \to v_{g\, alt_{(G,+)}(\underline{i})\, g_{i_n}^{(-1)^{n-1}}}^{\epsilon^{n+1}} \to \cdots \to v_{g\, alt_{(G,+)}(\underline{i})\, alt_{(G,+)}^{-1}(\underline{j})}^{\epsilon^{2n}} = v_g^+.$$

Similarly, for each negative vertex v_g^- of Γ, there is a one-one correspondence between $\{(alt_\beta(\underline{i}), alt_\beta(\underline{j})) | \underline{i}, \underline{j} \in I^n \text{ and } alt_\beta(\underline{i}) = alt_\beta(\underline{j})\}$ and the set of all loops of length $2n$ based at v_g^- by mapping $(alt_\beta(\underline{i}), alt_\beta(\underline{j}))$ to the loop $L_{v_g^-}^{(alt_{(G,-)}(\underline{i}), alt_{(G,-)}(\underline{j}))}$ defined as below:

$$v_g^- \to v_{gg_{i_1}}^- \to v_{gg_{i_1} g_{i_2}^{-1}}^{-1} \to \cdots \to v_{g\, alt_{(G,-)}(\underline{i})}^{\epsilon^n} \to v_{g\, alt_{(G,-)}(\underline{i})\, g_{i_n}^{(-1)^n}}^{\epsilon^{n+1}} \to \cdots \to v_{g\, alt_{(G,-)}(\underline{i})\, alt_{(G,-)}^{-1}(\underline{j})}^{\epsilon^{2n}} = v_g^-.$$

Observe that

$$\left\{ \sum_{g \in G} L_{v_g^+}^{\left(alt_{(G,+)}(\underline{i}), alt_{(G,+)}(\underline{j})\right)} \text{ (SOT) } : \underline{i}, \underline{j} \in I^n \text{ and } alt_{(G,+)}(\underline{i}) = alt_{(G,+)}(\underline{j}) \right\}$$

is a basis of the $(n,+)$ colored vector space of V_Γ^G for all $n \geq 0$ and

$$\left\{ \sum_{g \in G} L_{v_g^-}^{\left(alt_{(G,-)}(\underline{i}), alt_{(G,-)}(\underline{j})\right)} \text{ (SOT) } : \underline{i}, \underline{j} \in I^n \text{ and } alt_{(G,-)}(\underline{i}) = alt_{(G,-)}(\underline{j}) \right\}$$

is a basis of the $(n,-)$ colored vector space of V_Γ^G colored for all $n \geq 0$. Now we prove that V_Γ^G is a subfactor planar algebra.

Proposition 5.3.3. *With the running notations, V_Γ^G is a subfactor planar algebra.*

Proof. We observe that V_Γ^G has finite dimensinal intersection with V_n^ϵ for all $(n, \epsilon) \in Col$. Therefore, it suffices to prove the sphericality of V_Γ^G. Observe that

$$Z_{\mathcal{LE}_{(1,+)}^{(0,-)}}^{V_\Gamma} \left(\sum_{g \in G} L_{v_g^+}^{\left(alt_{(G,+)}(\underline{i}), alt_{(G,+)}(\underline{j})\right)} \right) = 1_{V_0^-}$$

for all $\underline{i}, \underline{j} \in I^n$ with $alt_{(G,-)}(\underline{i}) = alt_{(G,-)}(\underline{j})$ and

$$Z^{V_\Gamma}_{\mathcal{RE}^{(0,+)}_{(1,+)}}\left(\sum_{g\in G} L^{\left(alt_{(G,+)}(\underline{i}),alt_{(G,+)}(\underline{j})\right)}_{v_g^+}\right) = 1_{V_0^+}$$

for all $\underline{i}, \underline{j} \in I^n$ with $alt_{(G,+)}(\underline{i}) = alt_{(G,+)}(\underline{j})$. Therefore, V_Γ^G is a subfactor planar algebra. $\square$

Isomorphism between $P^{N\hookrightarrow M}$ and V_Γ^G:

Now we are ready to give the isomorphism between the SPA corresponding to a diagonal subfactor without cocyle and a fixed point SPA.

Theorem 5.3.4. *With the running notations, we have the subfactor planar algebra P corresponding to the diagonal subfactor $N \hookrightarrow M$ (without cocycle) is isomorphic to the fixed point SPA V_Γ^G.*

Proof. First define a vector space isomorphism $\phi_{(n,+)} : P_n^+ \to V_n^+$ by

$$1_N \otimes E_{\underline{i},\underline{j}} \mapsto \sum_{g\in G} L^{\left(alt_{(G,+)}(\underline{i}),alt_{(G,+)}(\underline{j})\right)}_{v_g^+}$$

for all $\underline{i}, \underline{j} \in I^n$ with $alt_\alpha(\underline{i}) = alt_\alpha(\underline{j})$ and $n \geq 0$, and a vector space isomorphism $\phi_{(n,-)} : P_n^- \to V_n^-$ by

$$1_M \otimes E_{\underline{i},\underline{j}} \mapsto \sum_{g\in G} L^{\left(alt_{(G,-)}(\underline{i}),alt_{(G,-)}(\underline{j})\right)}_{v_g^-}$$

for all $\underline{i}, \underline{j} \in I^n$ with $alt_\beta(\underline{i}) = alt_\beta(\underline{j})$ and $n \geq 0$. Now it is easy to check that $\{\phi_{(n,\epsilon)} : (n,\epsilon) \in Col\}$ respects the actions of the tangles - $1^{(n,\epsilon)}$, $\mathcal{M}_{(n,\epsilon)}$, $\mathcal{RI}^{(n+1,\epsilon)}_{(n,\epsilon)}$, $\mathcal{RE}^{(n,\epsilon)}_{(n+1,\epsilon)}$, $\mathcal{TL}^{(n,\epsilon)}$ for all $(n,\epsilon) \in Col$ and $\mathcal{LI}^{(n+1,+)}_{(n,-)}$, $\mathcal{LE}^{(n,-)}_{(n+1,+)}$ for all $n \geq 0$. Therefore, by 2.3.1 the subfactor planar algebra P is isomorphic to the fixed point subfactor planar algebra V_Γ^G. $\square$

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