

# CONTROLLED ENTANGLEMENT DYNAMICS IN OPEN QUANTUM SYSTEMS

By  
Sandeep K Goyal

THE INSTITUTE OF MATHEMATICAL SCIENCES, CHENNAI.

A thesis submitted to the  
Board of Studies in Physical Sciences

In partial fulfillment of the requirements

For the Degree of

DOCTOR OF PHILOSOPHY

*of*

HOMI BHABHA NATIONAL INSTITUTE



February 15, 2012

# Homi Bhabha National Institute

## Recommendations of the Viva Voce Board

As members of the Viva Voce Board, we recommend that the dissertation prepared by **Sandeep K Goyal** entitled “Controlled entanglement dynamics in open quantum systems” may be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

----- **Date :**  
Chairman : Chair of committee

----- **Date :**  
Convener : Conv of Committee

----- **Date :**  
Member : Member 1 of committee

----- **Date :**  
Member : Member 2 of committee

Final approval and acceptance of this dissertation is contingent upon the candidate’s submission of the final copies of the dissertation to HBNI.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it may be accepted as fulfilling the dissertation requirement.

----- **Date :**  
Guide : Sibasish Ghosh

## DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and the work has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution or University.

Sandeep K Goyal

## ACKNOWLEDGEMENTS

First and foremost, I would like to thank my supervisor Dr. Sibasish Ghosh for his inspirational help and guidance through the rigorous but wonderful journey that has culminated in this thesis. He has been there through thick and thin, guiding me flawlessly through the rough waters of writing this thesis. I am also deeply indebted to Prof. Simon, whose masterful teaching simplified the mysteries of the quantum world and inspired me to take up this field of research. Subtle remarks made by him along the way have translated into profound and insightful statements. Profs. Howard Carmichael and Thomas Konrad have taught me invaluable tricks of the trade, for which I remain ever grateful. I count myself lucky to have had the opportunity to learn from such excellent teachers. Collaborating with Dr. Subhashish Banerjee has enriched me as a quantum physicist, for which I thank him.

Special thanks go to Prof G. Baskaran, Prof. R. Shankar, Dr. Ronojoy Adhikari and Dr. S.R. Hassan for tips, suggestions, criticisms and analytical remarks. I have benefitted greatly from these interactions.

Discussions with Dr. Sujit Kumar Choudhary and Dr. Chandrasekhar Madiah have been extremely fruitful in my research. I have learnt a lot from them. My collaboration with Dr. Madiah has been particularly enjoyable and stimulating, and I thank him for his help with both the analytical and numerical aspects of the problems.

Further discussions with fellow researchers Rajeev Singh, J. Solomon Ivan, Krishnakumar Sabapathy, Mayukh Nilay Khan, Rajarshi Pal, Umesh Dubey, Ajay Singh Thakur and Shashank Pandey have been illuminating. Rajeev Singh in particular has gone through the calculations and numerics with a sieve and has been a helpful critic and proofreader. Somdeb Ghose has helped in proofreading parts of this thesis.

My fellow students at The Institute of Mathematical Sciences have made me feel at home, away from home. Rajeev Singh, Prem Prakash Pandey, Neeldhara Misra, Somdeb Ghose, Soumyajit Pramanick, Madhushree Basu, A.B. Belliappa, Anoop Varghese, Rohan Poojary and P.V. Sriluckshmy have all provided enough moral support and fun to last a lifetime. Last, but definitely not the least, I would

like to thank my family for their love and support through all this.

## Abstract

The study of entanglement has gained prominence in recent years due to the advent of fields such as Quantum Optics and Quantum Information Theory — advances that have harnessed such counter-intuitive quantum phenomena into elements of everyday life, improving it in the process. Ideal quantum systems, “closed” to the outside world, remain quantum forever and thus manage to retain entanglement. Real quantum systems, however, are “open” to the environment and are therefore susceptible to the phenomenon of decoherence. The resultant loss of entanglement is a major hindrance to the effectiveness of quantum information tasks. In this thesis we have studied the evolution of entanglement in various types of open quantum systems (OQS) coupled in various ways to local baths. We have also studied existing ways and means of controlling the decay of entanglement and have proposed a new method of doing so.

We have studied the evolution of entanglement in OQS undergoing Markovian dynamics by using the Lindblad master equation as well as the method of quantum trajectories. We have analyzed the onset of the phenomenon of *entanglement sudden death* in finite as well as infinite dimensional OQS, connected either locally to a thermal bath or its squeezed variant, or via a quantum non-demolition-type (QND-type) interaction to a local thermal bath. We have found that the QND-type system-bath interaction works best to conserve entanglement in finite dimensional systems, whereas a squeezed thermal bath causes entanglement sudden death even at zero temperature.

We have also studied some well-known methods of controlling decoherence in open systems with respect to their ability of preserving entanglement. Some of these procedures include coupling the system to a thermal bath of photonic crystals where the photonic band gap suppresses decoherence, modulation of the system-bath frequency in an attempt to contain decoherence, using the method of resonance fluorescence where an external field modulates the transition frequency of the two-qubit systems with each qubit being a two-state atomic system to contain decoherence, and using high-frequency radio waves to decouple the system and bath dynamically and thus reducing decoherence. Our study has revealed a rather startling fact — some of these decoherence control procedures actually result in an elevated rate of loss and a resultant quicker death of entanglement. This is a

surprising and extremely counter-intuitive result.

Finally, we have proposed a new method of shielding the entanglement in the system from dissipative environmental effects. In this procedure, an ancillary system in a fixed initial state is allowed to interact with the main system through exchange interactions. We have, through numerical computation, showed that an ancilla in the ground state extends the lifespan of entanglement in the main system. Increasing the size of the ancilla, that is, increasing the value of  $n$  in an  $n$ -qubit ancilla slows down entanglement loss for a two-qubit system connected locally to a thermal bath.





# Contents

<b>I</b>	<b>General</b>	<b>1</b>
<b>1</b>	<b>Introduction</b>	<b>3</b>
1.1	Why should one study dynamics of open quantum system? . . . . .	4
1.2	How one can study the dynamics of open systems . . . . .	5
1.3	Outline of the thesis . . . . .	6
<b>2</b>	<b>Entanglement</b>	<b>9</b>
2.1	Quantum states and Density operator . . . . .	10
2.2	Definition of entanglement . . . . .	12
2.3	Separability Criteria . . . . .	19
2.3.1	Peres-Horodecki criteria . . . . .	19
2.3.2	Reduction Criteria . . . . .	21
2.4	Measure of entanglement . . . . .	22
2.4.1	von-Neumann entropy . . . . .	24
2.4.2	Reñyi entropy . . . . .	24
2.4.3	Entanglement of formation (EoF) . . . . .	25
2.4.4	Concurrence . . . . .	26
2.4.5	G-concurrence . . . . .	28
2.4.6	Logarithmic negativity . . . . .	29
<b>3</b>	<b>Quantum operations and Master equation</b>	<b>31</b>
3.1	Operator sum representation (Kraus operators) . . . . .	34
3.2	Quantum dynamical semigroup . . . . .	36
3.2.1	Master equation: quantum dynamical semigroup approach . . . . .	38
3.3	Heisenberg and Interaction Picture . . . . .	41
3.3.1	Microscopic derivation of the Markovian master equation . . . . .	43
3.3.2	Lindblad form of the Master Equation . . . . .	47
<b>4</b>	<b>Quantum trajectories</b>	<b>53</b>
4.1	Perturbative expansion for the density operator . . . . .	53
4.2	Unravelling the Lindblad master equation . . . . .	55

## Contents

4.3	Monte Carlo simulation . . . . .	57
<b>5</b>	<b>Channels and states</b>	<b>59</b>
5.1	Channel-State Duality . . . . .	59
5.2	Factorization law of entanglement decay . . . . .	60
5.3	Entanglement breaking channels . . . . .	63
5.4	Example of qubit channels . . . . .	64
5.4.1	Depolarizing channels . . . . .	64
5.4.2	Amplitude damping channel . . . . .	65
5.4.3	Phase damping channel . . . . .	66
<b>6</b>	<b>Non-Markovian Dynamics</b>	<b>67</b>
6.1	The Nakajima-Zwanzig technique . . . . .	67
6.2	Time convolutionless technique . . . . .	69
6.3	Spontaneous decay of two-level atom: Exact master equation . . . . .	70
6.4	Post-Markovian processes . . . . .	73
<b>II</b>	<b>Applications</b>	<b>75</b>
<b>7</b>	<b>Entanglement dynamics: quantum trajectory approach</b>	<b>77</b>
<b>8</b>	<b>Entanglement dynamics in finite dimensional systems</b>	<b>81</b>
8.1	Master equation for single-qubit system . . . . .	82
8.2	Evolution of entanglement in two-qubit system . . . . .	84
8.3	Factorization law for entanglement decay for $d \otimes d$ systems . . . . .	88
8.4	ESD in $n$ -qubit system . . . . .	89
8.5	Squeezed thermal bath . . . . .	90
8.6	Quantum non demolition (QND) interaction . . . . .	93
8.7	ESD and rebirth of entanglement: non-Markovian evolution . . . . .	94
<b>9</b>	<b>Evolution of entanglement and quantum correlations in the system of harmonic oscillators</b>	<b>99</b>
9.1	Master equation for a single mode system . . . . .	101
9.2	Coherent state representation . . . . .	102
9.3	Covariance matrix . . . . .	103

9.4	ESD of two-mode Gaussian state . . . . .	105
9.5	Squeezed thermal bath . . . . .	107
9.6	ESD for $n$ -mode Gaussian states . . . . .	108
<b>10</b>	<b>Effect of control procedures on entanglement dynamics</b>	<b>111</b>
10.1	Evolution of entanglement in the presence of photonic crystals . . .	112
10.2	Frequency modulation . . . . .	115
10.3	Resonance fluorescence . . . . .	118
10.4	Dynamic decoupling and the effect on ESD . . . . .	125
10.4.1	Bang-Bang decoupling when the bath consists of harmonic oscillators . . . . .	125
10.4.2	Josephson Junction qubit . . . . .	128
10.5	Summing up . . . . .	142
<b>11</b>	<b>Assisted suppression of decoherence</b>	<b>149</b>
11.1	Dynamics of the system plus ancilla in the presence of thermal bath	149
11.2	Dynamics in terms of Kraus operators . . . . .	151
11.3	Results . . . . .	152
<b>12</b>	<b>Conclusion</b>	<b>157</b>



# List of Figures

2.1	Convex structure of the set of separable density operators embedded in the convex set of density operators. The pure separable states of the system lie at the points where the boundaries of the two sets touch. . . . .	15
2.2	Diagrammatic representation of positive maps . . . . .	18
2.3	A diagrammatic representation for PPT criteria. . . . .	22
2.4	von-Neumann entropy for a set of canonical states $(\sqrt{p} 00\rangle + \sqrt{1-p} 11\rangle)$ for two-qubit system. All pure entangled states of two qubits can be achieved from these states by local unitary transformation by varying $p$ in $[0, 1]$ . . . . .	25
5.1	Schematic diagram for channel state duality. . . . .	61
7.1	The contour plot for entanglement dynamics. On $x$ -axis we have $\Gamma_A$ and time is on $y$ -axis. The amount of entanglement depends on the color of the contour. It increases from blue to red. . . . .	79
8.1	In this plot we have plotted the evolution of entanglement for different initial states. . . . .	85
8.2	Evolution of entanglement for different bath temperatures. This plot shows that the time to ESD decreases as we increase the temperature. . . . .	86
8.3	Time for ESD for different values of bath temperature . . . . .	87
8.4	In this plot we have plotted the entanglement evolution for the non-Markovian dynamics. Here we have concurrence on the $y$ -axis and $\gamma_0 t$ on the $x$ -axis. The parameter $\lambda$ is set to be $0.1\gamma_0$ and the initial state considered is $( 01\rangle +  10\rangle)/\sqrt{2}$ . One can clearly see the phenomenon of ESD and rebirth of entanglement. . . . .	96
10.1	Evolution of entanglement in photonic band gap crystals at zero temperature. . . . .	119

List of Figures

10.2 In this plot we have time of ESD, i.e, the time at which a maximally entangled initial state loses all its entanglement and become separable when exposed to a bath, against the frequency of modulation  $\nu$ . In this case we have kept the value of  $m$  to be the first zero of the Bessel function  $J_0$ , i.e,  $m = 2.4048$ . Also, at  $\nu = 0$  the value of  $t_{ESD}$  is 1.4. The value of the other parameters are:  $\kappa = 0.1$ ,  $\Delta = 0.1$ ,  $C_0^{+-} = C_0^{-+} = 0.1$ . . . . . 120

10.3 Entanglement (concurrence) of  $M_{rf}$  as a function of time for different values of Rabi frequency  $\Omega$ , varying from 0 to 0.5: 0 (the last curve on the right hand side) corresponding to pure damping and 0.5 (first curve on the left hand side) corresponding to the underdamped case, i.e., it covers both the overdamped as well as the underdamped cases. In the inset one can see that as we increase the  $\Omega$  the  $t_{ESD}$  seem to converge at  $t = 17.0$ . Here  $\gamma = 0.1$ . . . . . 123

10.4 The plot for the evolution of coherence for a single qubit when the initial state of the qubit is  $|\psi\rangle = |0\rangle$  in the presence of thermal bath. We can see that the coherence increases as the Rabi frequency  $\Omega$  is increased. . . . . 124

10.5 Sketch of the pulse sequence used in bang-bang decoupling procedure. 127

10.6 Schematic diagram for Josephson-junction charge qubit . . . . . 129

10.7 Contour of the entanglement after time  $t = 5$  for all values of  $E_j$  and  $\epsilon$  (Josephson junction Hamiltonian parameters) The temperature in this case is equal to 0,  $\gamma = 1$ ,  $\kappa = v/\gamma = 0.45$ . . . . . 136

10.8 Evolution of entanglement for the pure dephasing case, i.e,  $E_j = 0$ . We can see that whereas entanglement decay is exponential for the  $1/f$  noise, it slows down for a bath of harmonic oscillator (inset) at zero temperature. . . . . 138

10.9 Evolution of entanglement for  $1/f$  (telegraph) noise at zero temperature. Here different curves represent the evolution of entanglement for different values of  $E_j$  (from 0 to 1), with the curve at the bottom corresponding to  $E_j = 0$  and that at the top to  $E_j = 1$ , while  $\epsilon$  is fixed and equal to 1. . . . . 139

10.10 Evolution of entanglement with respect to time, for different coupling strengths and temperature  $T = 0$ ,  $E_j = 1 = \epsilon$ . . . . . 140

10.11	Effect of bang-bang decoupling on entanglement, in the weak coupling regime. If we compare this plot with FIG. (10.10(a)), with the curves corresponding to the same values of $\kappa$ , we see that the bang-bang decoupling causes entanglement to disappear faster in time for a fixed value of the coupling strength. Here the parameters are same as in FIG. (10.10(a)) and the pulse strength is $U = 50\pi$ with time for which the pulse was activated is $\tau_p = 0.01$ . In the inset we have the evolution of entanglement for very small range (0.01 to 0.1) of coupling $\kappa$ . The thickest curve is the one corresponding to $\kappa = 0.38$ . This curve is important in the sense that it has the largest $t_{ESD}$ . . .	142
10.12	Entanglement evolution for the coupling parameter range $0.3 < \kappa < 0.4$ , with the uppermost curve corresponding to $\kappa = 0.3$ and the lowest (bottom) curve corresponding to 0.4. One can see from this plot the formation and disappearance of the kink. . . . .	143
10.13	Effect of bang-bang decoupling on entanglement in the strong coupling region. Here again we can see the effect of bang-bang decoupling on the entanglement if we compare this plot with FIG.(10.10(b)). Here the parameters are same as in FIG. (10.10(b)) and the pulse strength is $U = 50\pi$ with the pulse duration $\tau = 0.01$ . . . . .	144
10.14	Plot for the evolution of coherence in the case of Telegraph noise in weak coupling region, i.e, $\kappa < 1$ . Here all the parameters has the value same as in Fig. (10.12) and (10.11) and $\kappa = 0.38$ . . . . .	144
10.15	Plot for the evolution of coherence in the case of Telegraph noise in strong coupling region, i.e, $\kappa \geq 1$ . Here the value of all the parameters are same as in Fig. (10.13) and $\kappa = 5.38$ . . . . .	145
10.16	$t_{ESD}$ is plotted as a function of coupling strength $\kappa$ . Here we can see that there is a clear distinction between the strong and the weak coupling region. As we increase $\kappa$ the $t_{ESD}$ tends to freeze and asymptotic value of $t_{ESD}$ is around 10. The parameters used are as in the previous plots. . . . .	145
10.17	$t_{ESD}$ is plotted as a function of $E_j$ . This shows that as we go away from pure dephasing the $t_{ESD}$ increases. The parameters used are same as in the previous plots. . . . .	146

List of Figures

11.1 Schematic representation of system plus bath plus ancilla. . . . . 151

11.2 Here we are plotting the coherence (real part of  $\langle\sigma_+\rangle$ ) for a single qubit. The blue curve shows the evolution of coherence in the absence of ancilla whereas the green curve is in the presence of ancilla. One can see that the green curve is always above the blue one and hence the presence of ancilla arrest the decay of coherence. Here the initial state of the system qubit is  $(|0\rangle + |1\rangle)/\sqrt{2}$  and ancilla is in ground state  $(|0\rangle)$ . . . . . 153

11.3 Here we are plotting the coefficient of  $\sigma_z$  for a single qubit for the same setup as in Fig. (11.2). Again the improvement due to the presence of ancilla is obvious from the plot. . . . . 154

11.4 The effect of two-qubit ancilla on the evolution of entanglement when the initial state of the system  $A_1 + B_1$  is maximally entangled state  $(|00\rangle + |11\rangle)/\sqrt{2}$ . The state of ancilla is  $|\chi\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle$ . The average number of photons in the thermal cavity is  $N = 0.3$ . . . . . 154

11.5 The effect of the increasing number of qubits in ancilla on the evolution of entanglement when the initial state of the system  $A_1 + B_1$  is maximally entangled state  $(|00\rangle + |11\rangle)/\sqrt{2}$ . All the qubits in ancilla are in the ground state  $|0\rangle$ . . . . . 155



# List of Tables

10.1 Summary of the results. . . . .	148
--------------------------------------	-----

Part I  
General



# 1

## Introduction

Quantum physics often presents us with problems of bewildering complexity, where multiple systems interact among each other in multifarious ways. Solving such systems is often impossible, and hence we, as physicists, tend to simplify matter — both to increase the probability of solvability of relevant problems as well as to make life easier. One such approach is to look at certain small systems of a finite number of degrees of freedom, which interact with larger systems of many more degrees of freedom — ideally reservoirs with essentially an infinite number of degrees of freedom. These smaller systems are *open*, as opposed to *closed* systems that do not interact with their surroundings, say by exchanging energy in a random manner. For example, consider an atom in vacuum which is in a superposition of its ground and excited states, the system being a two-level one. At a random point of time, the atom releases a photon to its surrounding and collapses to its ground state. Here the two-level single-atomic system is open, as it has exchanged energy with its environment — here the vacuum — by emitting a photon.

The evolution of every closed quantum system is governed by a unitary transformation. If the system is open, then of course its evolution is not unitary. However, if the definition of the word ‘system’ be extended to mean the system-bath composite, then these two together evolve in a unitary fashion. System-bath interactions give rise to correlations between the states of the system and the bath, which results in the destruction of quantum coherence or of interference in the system. This is the phenomenon of decoherence, and it is primarily responsible for the decay of entanglement — one of the most important resources of quantum mechanics and quantum information theory. Entanglement and its evolution in open quantum

systems in the presence of decoherence make for fascinating study.

In this chapter, we will discuss the *hows* and the *whys* of this study to find out both the utility of studying such systems and phenomena, and the techniques we use to do the same.

## 1.1 Why should one study dynamics of open quantum system?

Quantum systems exhibit exotic phenomena that are far removed from our daily experiences. Yet they are there and are primarily responsible for the electronics and information age that we live in. Computers, the backbone of our present day and age, would not have been possible if quantum mechanics did not work. We do not directly perceive quantum effects in our daily life, which of course is perfectly natural — and a matter of great relief, some might say. Interference, coherence, entanglement: all these are essential quantum phenomena, and they occur in quantum regimes. An ideal closed quantum system will exhibit these phenomena forever, but no real system can be perfectly insulated from its surroundings, and are inevitably open. More often than not, it is possible to quietly ignore these weak interactions and treat the system as closed. However, there are times when this is not possible — for instance, if the system is too *leaky*, that is, the system-surroundings interactions are too strong, then decoherence will start making an impact. After a sufficient amount of time, open systems no longer remain quantum, and thus do not exhibit phenomena like interference or entanglement, to name a few. Also, the rate at which the transition from quantum to classical takes place depends on the size of the system. As a result, macroscopic systems like the ones we encounter in daily life almost never exhibit quantum phenomena.

The transfer and processing of information is one of the most significant aspects of the modern information age, and this has become possible in much efficient ways due to quantum information processing and communication (QIPC) protocols [23, 20, 54, 15, 22, 16, 24, 21, 27, 189]. As a result, it is now possible to carry out teleportation with almost full efficiency [187, 188, 111, 32, 31, 75, 24]. However, open systems preclude the use of such protocols for indefinite times due to the loss of coherence. For example, in order to carry out a one hundred percent efficient

teleportation, one requires a bipartite maximally entangled pure state that serves as a communication channel between two parties located far apart. However, since this channel can never be a purely closed one, decoherence will play its part and reduce the purity of the state by turning the pure state into a mixed state. Such mixed states result in a drop of efficiency of the channel, and, if decoherence acts long enough, the channel becomes essentially useless for efficient teleportation.

Entanglement is, of course, one of the most important resources in quantum mechanics and quantum information theory and is indispensable for a large number of quantum protocols and information tasks. For example, the efficiency of teleportation and super-dense coding protocols depends on the amount of entanglement in the channel. In order to ensure the success of such tasks, the preservation of entanglement in communication channels becomes crucial. As a result, it is imperative to not only have knowledge about the evolution of entanglement in open quantum systems, but also to devise new ways and means of controlling the decay of entanglement in the dissipative presence of decoherence. The next section gives a short overview of some of the methods used to carry out such a study.

## 1.2 How one can study the dynamics of open systems

In general, the final state of a quantum system undergoing dynamical evolution is related to its initial state through a linear transformation. For closed systems, this transformation is also unitary and is described by Schrödinger evolution. This is not possible when the quantum system is open, since the evolution is then non-unitary. As a result, solving the dynamics of open quantum systems is a difficult affair. However, one way around this is to consider an open system to be a part of a larger system-bath combine which is closed to the outside. In such a scenario, it is possible to extract the dynamics of the (open) system from that of the combine by averaging out the effects of the bath on the system, using the method of partial traces. Applying various subsequent approximations, one can thus derive an equation of motion of the evolution of the system alone. This is the general quantum master equation and it appears in several forms which include the Nakajima-Zwanzig equation [33], quantum Langevin equation [71, 70, 19, 77] and

Lindblad master equation [115, 33]. In this thesis, we shall discuss the Lindblad master equation and use it to analyse the dynamics of both finite as well as infinite dimensional quantum systems.

The master equation is an analytical method of studying the dynamics of quantum systems. In addition to this, a number of numerical methods have also been devised over the years to this effect. The quantum trajectory technique [43, 42], the Monte Carlo wave function method [61, 62, 78, 46, 123] and the stochastic Schrödinger equation [30] are some of the well known examples. We will discuss the quantum trajectory technique in some detail.

### 1.3 Outline of the thesis

In this thesis, we study the evolution of entanglement in open quantum systems (OQS) undergoing dissipative interactions with the environment. Our focus will primarily be on two classes of problems. In the first, we will compare how entanglement dynamics is affected by changing either the dimensionality of the system, or the nature of the dissipative interaction, or both. To this end, we will consider finite as well as infinite dimensional systems, and thermal, squeezed thermal and quantum non-demolition baths each of which act locally. In particular, we will look for a complete loss of entanglement in finite time – a phenomenon known as *entanglement sudden death* (ESD) – in these setups. In the second class of problems, we will explore various schemes of controlling the decay of coherence in OQS and the effect of those schemes on entanglement dynamics. Some such schemes, such as resonance fluorescence and dynamical decoupling, are known to suppress decoherence in OQS but fail in the case of entanglement.

In our analysis, *we will restrict ourselves to Markovian dynamics and will consequently use the Lindblad form of the master equation for studying entanglement dynamics*. We will use the factorization law of entanglement decay [107] to characterize the dynamics of arbitrary two-qubit pure states in terms of the dynamics of two-qubit maximally entangled states. To simplify our analysis, we will make use of the correspondence between quantum maps and positive operators by invoking the Choi-Jamiolkowski isomorphism [101, 50], also known as channel-state duality. We will represent the action of a bath on the system by means of Kraus operators.

In addition to the master equation analysis we will also study some particular systems using the quantum trajectory technique. This is a numerical technique that uses state vectors instead of density operators to analyse the piecewise continuous coherent part and the stochastic quantum jump part of dynamical evolution.

The thesis is organized as follows: in the first part of the thesis we briefly review the basic concepts and techniques used in the study. In the second chapter of this part we discuss different measures of entanglement. The third chapter is devoted to the quantum dynamical semigroup approach to Lindblad master equation and give a microscopic derivation of the master equation. In the fourth chapter we discuss quantum trajectory method – a numerical technique for solving open system dynamics. The fifth chapter deals with various types of special channels as well as channel-state duality and the factorization law for entanglement decay. To give a complete description of the master equation approach to the dynamics of open quantum systems, we describe briefly some issues on non-Markovian master equation in chapter six although we will not use it in our future discussions.

The second part of the thesis deals mainly with the application of the concepts mentioned above. In chapter seven we discuss the entanglement dynamics from the quantum trajectory point of view. In chapters eight and nine, we discuss entanglement dynamics in the presence of three types of baths – thermal, squeezed thermal and quantum non-demolition. In chapter eight, we make use of the *factorization law for entanglement decay* [107] to fully characterize entanglement dynamics in finite dimensional systems by observing the action of the bath on maximally entangled states. In chapter nine, we study the evolution of two-mode Gaussian states for the system consisting of coupled harmonic oscillators in the presence of local thermal and squeezed thermal baths [87]. Our main goal is to observe the transition from entanglement to separability in this system. In chapter ten we discuss the effects of certain control procedures on entanglement dynamics [88], for example, bath of photonic crystals, frequency modulation, resonance fluorescence, dynamical decoupling. Finally, in chapter eleven, we provide a technique of shielding entanglement from environmental effects [85]. We conclude the thesis in chapter twelve with a list of further problems that may be investigated in future.





# 2

## Entanglement

The wonderful counter-intuitive phenomenon of entanglement is what sets quantum mechanics apart from classical physics. Entanglement seemingly manages to violate jointly the notion of locality and the element of physical reality – an effect that even the great Albert Einstein called “spooky action at a distance”. Together with fellow researchers Podolsky and Rosen, Einstein came up with the famous EPR paradox which challenged the completeness of quantum theory as a description of physical reality [65]. The local hidden-variables theory [28, 29] espoused by Bohm and others attempted to add elements of reality — the so-called “hidden variables” — to quantum mechanics in order to explain entanglement without invoking action at a distance. However, in 1964, Bell’s theorem [17] put an end to the possibility of the existence of local hidden variables and showed that the paradox is a generic feature of the real world. He showed that entanglement is what makes quantum mechanics such a fascinating subject, and makes it different from the classical theories.

Physically, entanglement is a manifestation of the superposition principle of quantum mechanics. Mathematically, it comes about from the tensor product structure of the Hilbert space for a composite quantum system. The principle of superposition — which states that superposition of two or more states of a quantum system is a valid new state — is present in classical optics as well. However, in general [157] we do not see equivalent of entanglement in classical optics due to the lack of a tensor product structure.

We start with the basic concept of quantum state vectors and density operators in the first section. We shall discuss the concept of entanglement in more detail in

the following sections which include definition of entanglement, separability criteria and the measure of entanglement.

## 2.1 Quantum states and Density operator

A state in quantum mechanics is represented by a *state vector*  $|\psi\rangle$  which is a vector in a  $d$ -dimensional Hilbert space  $\mathcal{H}$  corresponding to the quantum system  $\mathcal{S}$ . The state vector  $|\psi\rangle$  can be written as linear combination of orthonormal state vectors  $\{|i\rangle\}_{i=1}^d$  as:

$$|\psi\rangle = \sum_{i=1}^d \alpha_i |i\rangle. \quad (2.1)$$

The coefficients  $\alpha_i$ , which are complex numbers, represent probability amplitudes, while their modulus square of each element represent probabilities of finding the system in state  $|i\rangle$ . Being genuine probability, the moduli square satisfy the normalization condition

$$\sum_i |\alpha_i|^2 = 1. \quad (2.2)$$

A more general way of representing the state of the quantum system is through *density operators*. The density operator corresponding to the state  $|\psi\rangle$  is:

$$\rho = |\psi\rangle\langle\psi|, \quad (2.3)$$

$$= \sum_{i,j} \alpha_i \alpha_j^* |i\rangle\langle j|. \quad (2.4)$$

A density operator  $\rho$  is, by definition, a positive semi-definite Hermitian operator with unit trace:

$$\begin{aligned} \rho &= \rho^\dagger \\ \langle\phi|\rho|\phi\rangle &\geq 0 \quad \forall |\phi\rangle \in \mathcal{H} \\ \text{tr}\rho &= 1 \end{aligned} \quad (2.5)$$

A state which is the statistical average (classical mixture) of two or more quan-

tum states (as opposed to superposition) cannot be represented by a state vector, and necessitates the use of density matrices. Consider system whose state is a mixture of two states  $|\psi_1\rangle$  and  $|\psi_2\rangle$  with probabilities  $p_1$  and  $p_2 = 1 - p_1$ . If we are suppose to guess a pure state for this system we might choose:

$$|\Psi_\theta\rangle = \sqrt{p_1}|\psi_1\rangle + e^{i\theta}\sqrt{p_2}|\psi_2\rangle, \quad (2.6)$$

where  $\theta$  is arbitrary. The expectation value of an observable  $O$  in this state will be:

$$\langle O \rangle = \langle \Psi_\theta | O | \Psi_\theta \rangle, \quad (2.7)$$

$$= \text{tr}(O |\Psi_\theta\rangle \langle \Psi_\theta|). \quad (2.8)$$

The expectation value of the observable  $O$  depends on the choice of  $\theta$ . To get rid of this  $\theta$  dependence we take the average of the expectation value over all  $\theta$  which results in:

$$\langle O \rangle = p_1 \langle \psi_1 | O | \psi_1 \rangle + p_2 \langle \psi_2 | O | \psi_2 \rangle. \quad (2.9)$$

Note that no pure state can give rise to this expectation value. If we define an operator  $\rho$  for this system such that

$$\rho = p_1 |\psi_1\rangle \langle \psi_1| + p_2 |\psi_2\rangle \langle \psi_2| \quad (2.10)$$

then the expectation value of the observable  $O$  can be written as:

$$\langle O \rangle = \text{tr}(O\rho). \quad (2.11)$$

The operator  $\rho$  represents the density operator for the system.

Thus, the density operator of the statistical average of a number of states is the convex sum of the density operators of the individual states. It is easily seen that the new density operator also satisfies all the conditions (2.5). Though we have a unique density operator  $\rho$  for a given distribution of states, the reverse is not true. For a fixed  $\rho$  we can have infinitely many decompositions representing different ensembles.

To see that consider the spectral decomposition of  $\rho$ :

$$\rho = \sum \lambda_i |\psi_i\rangle\langle\psi_i| \quad (2.12)$$

where  $\lambda_i \geq 0$ ,  $\sum_i \lambda_i = 1$  and  $\langle\psi_i|\psi_j\rangle = \delta_{ij}$ . Consider a unitary operator  $W$ . We define a new set of non-normalized vectors  $\{|\phi_k\rangle\}$  related to the set  $\{|\psi_i\rangle\}$ :

$$|\phi_k\rangle = \sum_i W_{ki} \sqrt{\lambda_i} |\psi_i\rangle. \quad (2.13)$$

We see that  $\rho$  can be written as  $\rho = \sum_k |\phi_k\rangle\langle\phi_k|$ . Hence each and every unitary matrix  $W$  gives rise to a different decomposition. Thus, there are infinitely many decompositions.

We end this section by listing a few properties of density operators:

- The set of density operators is a convex set, i.e, the convex sum of two or more density operators is again a density operator.
- Rank one density operators or one dimensional projectors are called *pure states*, other density operators are called *mixed states*.
- Any density matrix can be written as

$$\rho = \frac{1}{d} (\mathcal{I} + \Lambda), \quad (2.14)$$

where  $d$  is the dimension of the system,  $\mathcal{I}$  is the  $d \times d$  identity operator and  $\Lambda$  is some traceless Hermitian matrix.

## 2.2 Definition of entanglement

Let  $|\psi\rangle$  be a state of system  $S$ .  $|\psi\rangle$  belongs to a Hilbert space  $\mathcal{H}_S$  associated to  $S$ . Should the system  $S$  be composed of two subsystems  $A$  and  $B$ , then this Hilbert space can be decomposed as:  $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$ . In such systems we can define entanglement as:

**Definition 1 (Entanglement in pure state:)** A pure bipartite (or multipartite) state  $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$  is called **product** or **separable** if

$$|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\chi\rangle_B \quad (2.15)$$

and **entangled** if not. Here  $|\phi\rangle_A \in \mathcal{H}_A$  and  $|\chi\rangle_B \in \mathcal{H}_B$ .

To get a better understanding of the definition consider an arbitrary bipartite state

$$|\psi\rangle = \sum_{i,j} \alpha_{ij} |i\rangle_A \otimes |j\rangle_B, \quad (2.16)$$

where  $\{|i\rangle_A\}$  is a complete orthonormal basis of  $\mathcal{H}_A$  and  $\{|j\rangle_B\}$  is a complete orthonormal basis of  $\mathcal{H}_B$  and  $\sum_{i,j} |\alpha_{ij}|^2 = 1$ . If  $\alpha$  is the matrix of coefficients such that  $\alpha = [\alpha_{ij}]$ , then there exist two unitary matrices  $w$  and  $x$  such that

$$\alpha = w\gamma x^T \quad (2.17)$$

where  $\gamma = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_d)$  is a diagonal matrix with non-negative  $\gamma_j$ 's. This is called the *singular value decomposition* [95]. Substituting Eq.(2.17) into Eq.(2.16) results in

$$|\psi\rangle_S = \sum_{ij} \sum_{km} w_{ik} \gamma_k x_{jm} \delta_{km} |i\rangle_A \otimes |j\rangle_B \quad (2.18)$$

$$= \sum_k \gamma_k |\chi_k\rangle_A \otimes |\eta_k\rangle_B, \quad (2.19)$$

where

$$|\chi_k\rangle_A = \sum_i w_{ik} |i\rangle_A \quad (2.20)$$

$$|\eta_k\rangle_B = \sum_j x_{jk} |j\rangle_B. \quad (2.21)$$

Eq.(2.19) is known as the *Schmidt decomposition* [95, 130]. The *Schmidt rank* is given by the number of nonzero  $\gamma_k$ . Since  $\{|i\rangle\}$  and  $\{|j\rangle\}$  form an orthonormal basis, the states  $\{|\chi_i\rangle\}$  and  $\{|\eta\rangle\}$  also form an orthonormal basis for the subsystem

$A$  and  $B$  respectively. The Schmidt rank is an indicator of whether a state is separable or not: states having a Schmidt rank greater than unity are not separable. The maximum value that the rank can take is given by  $\min\{\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)\}$ .

If the state  $|\psi\rangle$  of Eq. (2.16) is separable then the reduced density operator corresponding to any subsystem must also be a pure state. Consider the reduced density operator for the subsystem  $A$ :

$$\rho_A = \text{tr}_B(|\psi\rangle\langle\psi|) \quad (2.22)$$

$$\rho_A = \sum_{im} \sum_k \alpha_{ik} \alpha_{mk}^* |i\rangle_A \langle m| \quad (2.23)$$

$$= \sum_k \gamma_k^2 |\chi_k\rangle_A \langle \chi_k|. \quad (2.24)$$

If the state  $|\psi\rangle$  is separable then its Schmidt rank is unity, which means  $\rho_A = |\chi_k\rangle\langle\chi_k|$  for one and only one  $k$ . This proves that the reduced density operator of a subsystem of a pure separable system is itself pure. Conversely, the purity, or lack thereof, of any such reduced density operator indicates the absence, or presence of entanglement respectively.

The question of separability in states which are statistical mixtures of pure states, i.e, mixed states requires deeper understanding of the space of positive operators acting on the Hilbert space  $\mathcal{H}$ . We define the entanglement in mixed states as:

**Definition 2 (Entanglement in mixed states)** *If a mixed bipartite state  $\rho$  can be written as convex combination of product states, i.e,*

$$\rho = \sum_k p_k \rho_k^{(A)} \otimes \rho_k^{(B)}, \quad (2.25)$$

*then it is called **separable**. Otherwise, it is called **entangled**. Here  $\rho_k^{(A)}$ 's are density operators of system  $A$  while  $\rho_k^{(B)}$  are density operators of system  $B$  and  $\sum_k p_k = 1$  with  $p_k \geq 0$ .*

As changing the basis of a subsystem should not change its physical relation to the rest of the world, entanglement should not change under local unitary trans-

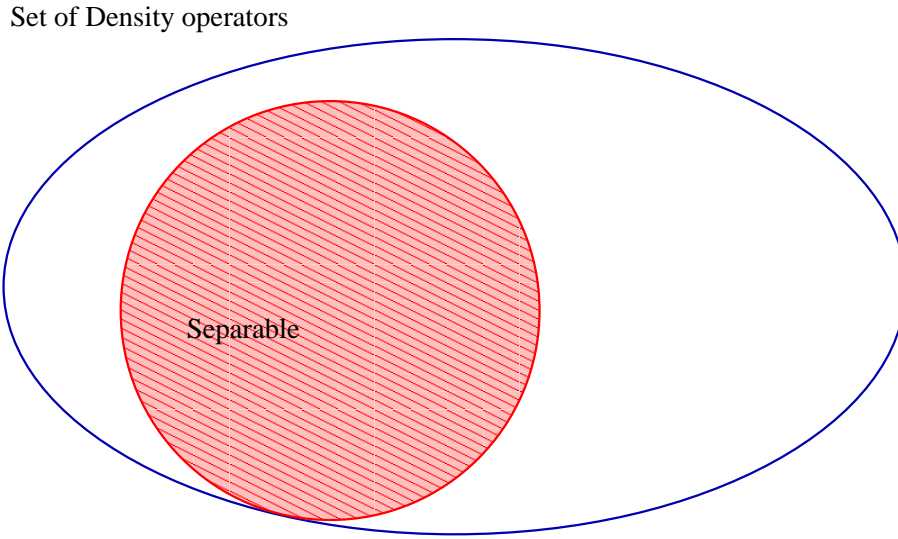


Figure 2.1: Convex structure of the set of separable density operators embedded in the convex set of density operators. The pure separable states of the system lie at the points where the boundaries of the two sets touch.

form, i.e.,

$$E(\rho) = E(\rho') \quad (2.26)$$

where  $\rho' = (U_1 \otimes U_2)\rho(U_1 \otimes U_2)^\dagger$ ,  $U_1, U_2$  being unitary operators.

Note that separable states are closed under convex sum, i.e, if  $\rho_1$  and  $\rho_2$  are both separable and  $q$  is a positive number between 0 and 1 then  $\rho$  is given by

$$\rho = q\rho_1 + (1 - q)\rho_2 \quad (2.27)$$

is also separable. As we remarked earlier, the full set of density operators is also a convex set (see Fig. (2.1)). We have already discussed that there exist infinitely many decompositions for a given density operator. We thus need to exhaust all the decompositions before establishing that a density operator is entangled. Naturally, this is an immense task and requires huge resources. Therefore, we need to look for alternative ways to detect entanglement.

Entanglement in mixed states can be understood better by the concept of entanglement witness and positive but not completely positive maps [96].



## Chapter 2. Entanglement

Entanglement witness, as is obvious from the name, are observables that detect entanglement in a given state. For every entangled state there exists an entanglement witness and vice versa. The mathematical definition of the entanglement witness follows from a theorem by Horodecki [99].

**Theorem 1** *A bipartite density matrix  $\rho$  is entangled if and only if there exist a Hermitian operator  $W$  such that*

- $\text{tr}(W\rho) < 0$
- $\text{tr}(W\sigma) \geq 0, \forall$  separable  $\sigma$ .

The proof of the above theorem follows from the Hahn-Banach (HB) theorem [64]. Take a compact convex set  $\mathcal{S}$  and a point  $T$  outside this set. The HB theorem then states that there always exists both a continuous function  $f$  and a real number  $\xi$  such that, for all pairs  $\{s \in \mathcal{S}, T\}$  we have

$$f(s) > \xi > f(T). \quad (2.28)$$

We can choose this function to be  $\text{tr}(W \cdot)$  and  $\xi = 0$  such that,

$$\text{tr}(\rho W) < 0 \quad (2.29)$$

for a given entangled state  $\rho$  and

$$\text{tr}(\sigma W) \geq 0 \quad (2.30)$$

for all separable state  $\sigma$ . Thus proving the validity of the theorem. *This operator  $W$  is the **Entanglement witness**.* Since the expectation value of any positive operator is always a positive number, the operator  $W$  is a non-positive operator. But it is positive for all separable states.

A linear map is a function that maps a set of operators acting on a certain vector space to another set of operators acting on a different (or same) vector space. *Positive maps* are those that map every positive operator to positive operator. Thus we can formally give a definition of positive maps.

**Definition 3** *Consider a linear map  $\mathcal{L}$  between the space of operators  $\mathcal{B}(\mathcal{H}_1)$  acting*

on Hilbert space  $\mathcal{H}_1$  and the space of operators  $\mathcal{B}(\mathcal{H}_2)$  acting on Hilbert space  $\mathcal{H}_2$

$$\mathcal{L} : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2). \quad (2.31)$$

- Map  $\mathcal{L}$  is **positive** if  $\mathcal{L}(A) \geq 0$ ,  $\forall A \geq 0$  and  $A \in \mathcal{B}(\mathcal{H}_1)$ , i.e, the map  $\mathcal{L}$  maps positive operators in  $\mathcal{B}(\mathcal{H}_1)$  to positive operators in  $\mathcal{B}(\mathcal{H}_2)$ .
- The map  $\mathcal{L}$  is **completely positive** (CP) if the map

$$\mathcal{M} = \mathcal{L} \otimes \mathcal{I}_n : \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_n) \rightarrow \mathcal{B}(\mathcal{H}_2) \otimes \mathcal{B}(\mathcal{H}_n)$$

is positive for all  $n \in \mathcal{N}$ . Here  $\mathcal{I}_n$  denotes the identity map on the space  $\mathcal{B}(\mathcal{H}_n)$  of dimension  $n$ . Completely positive maps form a subset of positive maps.

- The map  $\mathcal{L}$  is **unital** if  $\mathcal{L}(\mathcal{I}) = \mathcal{I}$  and
- $\mathcal{L}$  is **trace preserving** if  $\text{tr}\mathcal{L}(A) = \text{tr}A$  for all  $A$ .

Completely positive maps are mathematical equivalents of physical processes. Since a physical process is simply a mapping of the system from one state to another irrespective of any other system interacting with it, all physical processes can be represented by completely positive maps — there is a one-to-one analogy between the two. On the other hand, there exist positive maps which are not completely positive and do not represent physical processes. They are not useless however, and can be utilized (mathematically) to distinguish entangled states from separable ones.

**Theorem 2** A state  $\rho_{AB}$  is separable iff for all positive maps  $\mathcal{L}$  on  $\mathcal{B}(\mathcal{H}_B)$ , the operator  $(\mathcal{I}_A \otimes \mathcal{L}_B)(\rho_{AB}) \geq 0$ .

The proof of the above theorem is evident considering the separable decomposition for  $\rho_{AB}$ :

$$\rho_{AB} = \sum_k p_k \rho_{A_k} \otimes \rho_{B_k}. \quad (2.32)$$

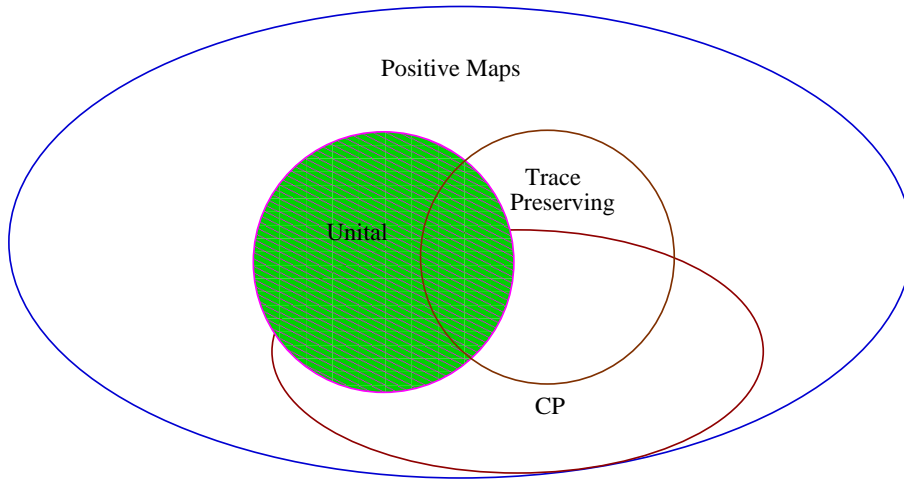


Figure 2.2: Diagrammatic representation of positive maps

A one-sided action of  $\mathcal{L}$  gives us

$$(\mathcal{I} \otimes \mathcal{L})(\rho_{AB}) = \sum_k p_k \rho_{A_k} \otimes \mathcal{L}(\rho_{B_k}). \quad (2.33)$$

Since  $\mathcal{L}(\rho_{B_k})$ s are positive, the operator on the right hand side is positive. Therefore,  $(\mathcal{I} \otimes \mathcal{L})(\rho_{AB}) \geq 0$  for positive  $\mathcal{L}$  is a necessary condition for  $\rho$  to be separable. That this condition is also sufficient was shown by Horodecki *et al* [99].

The transpose operation is one example of a positive map since transposition does not change the eigenvalues of a matrix. However, it is not a completely positive map. Any positive map of the type  $\mathcal{L} : \mathcal{B}(\mathcal{C}_2) \rightarrow \mathcal{B}(\mathcal{C}_2)$  or of the type  $\mathcal{L} : \mathcal{B}(\mathcal{C}_2) \rightarrow \mathcal{B}(\mathcal{C}_3)$  can be written as

$$\mathcal{L} = CP_1 + CP_2 o T, \quad (2.34)$$

where  $CP_i$  are completely positive maps and  $T$  is a transposition map. The maps which can be written in such form are called *decomposable maps*. Not all positive maps in higher dimensions are decomposable. Graphical structure of positive maps can be seen in Fig.(2.2)

## 2.3 Separability Criteria

The definition of separability is straightforward [175]. In practice however it is difficult to find out if there exists a separable decomposition for a given density operator, given that there are infinitely many possible decompositions for a density operator. As a result, operational criteria for testing the separability of states were sought and found (at least for some quantum states) [99, 158]. For pure states this was achieved by testing whether the Schmidt rank was unity or not — unity indicating separability — the main advantage of this being the ease of computation of the Schmidt rank [130].

However, for mixed states, the task of determining separability becomes much more challenging. Our understanding of positive maps and entanglement witnesses helps us discuss a few possible separability criteria for the most general density operator which are discussed in detail in following subsections.

### 2.3.1 Peres-Horodecki criteria

Consider a bipartite density operator  $\rho$ . If  $\rho$  is separable we can write it as:

$$\rho = \sum_k p_k \rho_{A_k} \otimes \rho_{B_k}. \quad (2.35)$$

Performing transposition operation on a subsystem (say  $B$ ) results in

$$\tilde{\rho} = \sum_k p_k \rho_{A_k} \otimes \rho_{B_k}^T. \quad (2.36)$$

Since  $\rho_{B_k}$  and  $\rho_{B_k}^T$  are both positive and unit trace operators,  $\tilde{\rho}$  represents a valid density operator. Therefore, performing transposition operation on a subsystem, i.e, partial transposition, maps the set of separable states to itself. But this is not true in general for non-separable density operators. For example, consider the following one parameter family of density operators called Werner states [175]

$$\rho_p = (1 - p) \frac{\mathcal{I}_4}{4} + p |\psi^-\rangle \langle \psi^-|, \quad (2.37)$$

## Chapter 2. Entanglement

where  $0 \leq p \leq 1$  and  $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ . The matrix form of  $\rho_p$  in  $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$  basis is:

$$\rho_p = \begin{pmatrix} \frac{1-p}{4} & 0 & 0 & 0 \\ 0 & \frac{1+p}{4} & -\frac{p}{2} & 0 \\ 0 & -\frac{p}{2} & \frac{1+p}{4} & 0 \\ 0 & 0 & 0 & \frac{1-p}{4} \end{pmatrix} \quad (2.38)$$

with eigenvalues  $(1-p)/4$ ,  $(1-p)/4$ ,  $(1-p)/4$ ,  $(1+3p)/4$ . The partial transposition of  $\rho_p$  results in

$$\rho_p^{T_B} = \begin{pmatrix} \frac{1-p}{4} & 0 & 0 & -\frac{p}{2} \\ 0 & \frac{1+p}{4} & 0 & 0 \\ 0 & 0 & \frac{1+p}{4} & 0 \\ -\frac{p}{2} & 0 & 0 & \frac{1-p}{4} \end{pmatrix} \quad (2.39)$$

with eigenvalues  $(1+p)/4$ ,  $(1+p)/4$ ,  $(1+p)/4$ ,  $(1-3p)/4$ . We can see that when  $1/3 < p \leq 1$ , the partial transposition of  $\rho_p$  results in a non-positive operator. Therefore, we arrive at a necessary condition for separable states given by the following theorem:

**Theorem 3** *The partial transposition of a separable state is always positive, i.e, if a density operator does not remain positive on partial transposition, it is necessarily an entangled state [137].*

Since all the positive maps are decomposable in  $2 \otimes 2$  and  $2 \otimes 3$  systems, positivity under partial transposition becomes a sufficiency condition as well.

**Theorem 4** *A  $2 \otimes 2$  and  $2 \otimes 3$  dimensional quantum system is separable if and only if it is positive under partial transposition (PPT) [99].*

This is the Peres-Horodecki criteria. For higher dimensional system PPT is just a necessary condition but not sufficient.

To see the action of PPT diagrammatically consider the four Bell state  $|\Phi^\pm\rangle$  and  $|\Psi^\pm\rangle$  where

$$|\Phi^+\rangle\langle\Phi^+| = \frac{1}{4}(\mathcal{I} + \sigma_z \otimes \sigma_z + \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y),$$

$$\begin{aligned}
|\Phi^-\rangle\langle\Phi^-| &= \frac{1}{4}(\mathcal{I} + \sigma_z \otimes \sigma_z - \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y), \\
|\Psi^+\rangle\langle\Psi^+| &= \frac{1}{4}(\mathcal{I} - \sigma_z \otimes \sigma_z - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y), \\
|\Psi^-\rangle\langle\Psi^-| &= \frac{1}{4}(\mathcal{I} - \sigma_z \otimes \sigma_z + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y).
\end{aligned}$$

We can write

$$\begin{aligned}
|\Phi^+\rangle &= (1, -1, 1), \\
|\Phi^-\rangle &= (-1, 1, 1), \\
|\Psi^+\rangle &= (-1, -1, -1), \\
|\Psi^-\rangle &= (1, 1, -1)
\end{aligned}$$

by the coefficient of the  $\sigma_i \otimes \sigma_i$  matrices. So we can represent them as the four vertices on a cube as shown in Fig. (2.3). These four points forms a tetrahedron. All the states inside the tetrahedron are mixed states which are convex combination of the four Bell states.

Note that transposition takes  $\sigma_y$  to  $-\sigma_y$ , other two sigma matrices remain unchanged. In geometric terms it is equivalent to taking reflection about  $xz$  plane. On taking partial transposition the tetrahedron in Fig. (2.3) will flip about  $xz$  plane. The overlap between the old tetrahedron and the new one is an octahedron which represents the set of PPT states and hence separable states. All other states which are inside the tetrahedron but outside the octahedron are entangled states.

### 2.3.2 Reduction Criteria

The reduction criteria [98] makes use of the positive map defined as:

$$\Lambda(A) = (\text{tr}A)\mathcal{I} - A. \quad (2.40)$$

It is easy to see that if  $A \geq 0$ , the operator  $\Lambda(A)$  is also positive. One sided operation of  $\Lambda$  on a bipartite state  $\rho$  gives

$$\rho_A \otimes \mathcal{I} - \rho \geq 0 \quad (2.41)$$

$$\mathcal{I} \otimes \rho_B - \rho \geq 0, \quad (2.42)$$

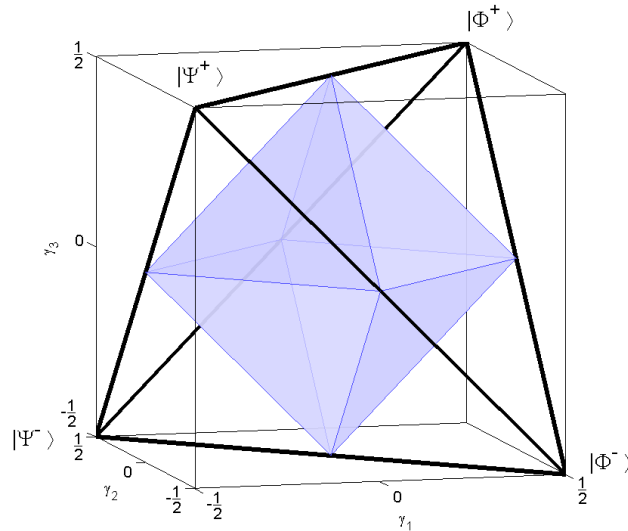


Figure 2.3: A diagrammatic representation for PPT criteria.

where  $\rho_{A,B} = \text{tr}_{B,A}(\rho)$ . Therefore, we have

**Theorem 5** *A separable state  $\rho$  must satisfy Eq. (2.41) and Eq. (2.42).*

This criteria is necessary and sufficient for  $2 \times 2$  and  $2 \times 3$  quantum systems. However, it is weaker than PPT criteria in higher dimensions.

## 2.4 Measure of entanglement

After learning whether a given state is entangled or not, one can ask *how much entangled is an entangled state?* A number of papers has been written on this problem [179, 69, 34, 66, 122, 141, 168, 181]. But the problem still remains open for systems which are bigger than two-qubit systems. We have a number of measures and we are going to discuss some of these in detail. But before going to these measures, it is a good idea to discuss general properties of a bona-fide measure of entanglement.

A measure of entanglement is like a measure for distance between two points. Just that we are given only one point (a mixed state) and we have to find how far this point is from the closest separable state. Like the measure of distance,

entanglement measure must also satisfy a number of properties. But before going into the details we need to set our grounds. We assume a bipartite system with Hilbert space  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$  and  $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = d$ . Following are the properties an ideal entanglement measure must satisfy—

1. An entanglement measure is a function  $E$  which assigns to each state  $\rho \in \mathcal{B}(\mathcal{H})$  a non-negative real number  $E(\rho)$  where  $E(\rho) = 0$  implies that the state is separable.
2. (**Monotonicity under LOCC**) Entanglement is a non-local property of a quantum system, therefore,  $E$  should not increase under any local operation accompanied by classical communication schemes.
3. (**Local unitary invariance**)  $E$  should be preserved under local unitary transformation, i.e,

$$E(\rho) = E((u \otimes v)(\rho)(u^\dagger \otimes v^\dagger)), \quad (2.43)$$

for unitary operators  $u$  and  $v$ .

4. (**Concavity**) Convex sum of two entangled density operators can result in a separable density operator, therefore, the entanglement measure should be a concave function, i.e,

$$E(\lambda\rho_1 + (1 - \lambda)\rho_2) \leq \lambda E(\rho_1) + (1 - \lambda)E(\rho_2), \quad (2.44)$$

where  $0 \leq \lambda \leq 1$ .

5. (**Continuity**) In the limit of vanishing (by any suitable measure) distance between two states  $\rho_1$  and  $\rho_2$ , ( $\|\rho_1 - \rho_2\| \rightarrow 0$ ) the difference between their entanglement should tend to zero ( $E(\rho_1) - E(\rho_2) \rightarrow 0$ ).
6. (**Additivity**) For two states  $\rho_1$  and  $\rho_2$

$$E(\rho_1 \otimes \rho_2) = E(\rho_1) + E(\rho_2). \quad (2.45)$$

Any map which maps density operators to scalars can be considered as a measure of entanglement if it satisfies all the conditions discussed above. In the following



subsection we will discuss a number of entanglement measures both for pure and for mixed states.

### 2.4.1 von-Neumann entropy

Unlike classical statistical system, even if the state of a bipartite quantum system is pure, the individual subsystem might be in mixed state. The lack of the information about individual subsystems is caused by a non-local property, namely, entanglement. Therefore, by measuring the amount of the lack of information (measure of impurity) in one of the subsystems it is possible to deduce the amount of entanglement. There are a number of ways to measure the impurity in a quantum system and von-Neumann entropy is one of them. For a density operator  $\rho$ , the von-Neumann entropy is defined as [130]:

$$S(\rho) = -\text{tr} \rho \log \rho = -\sum_k q_k \log q_k, \quad (2.46)$$

where  $q_k$  are the eigenvalues of  $\rho$ . von-Neumann entropy satisfies all the properties listed in previous section, i.e, it is positive, it is monotonic under LOCC, it is continuous and additive. FIG.(2.4) confirm the concavity property of the entropy. Therefore, it is a valid entanglement measure for pure states.

### 2.4.2 Rényi entropy

Rényi entropy, generalization of Shannon entropy, is a family of functionals for quantifying the uncertainty or randomness of a system. In a quantum state it measures the impurity of the state. The Rényi entropy of order  $\alpha$  can be defined as [147, 72]:

$$H_\alpha(\rho) = \frac{1}{1-\alpha} \log(\text{tr} \rho^\alpha). \quad (2.47)$$

where  $\alpha$  can take any value between 0 and  $\infty$  except  $\alpha = 1$ . In the limit  $\alpha \rightarrow 1$  the Rényi entropy converges to von-Neumann entropy —

$$H_{\alpha \rightarrow 1}(\rho) = S(\rho). \quad (2.48)$$

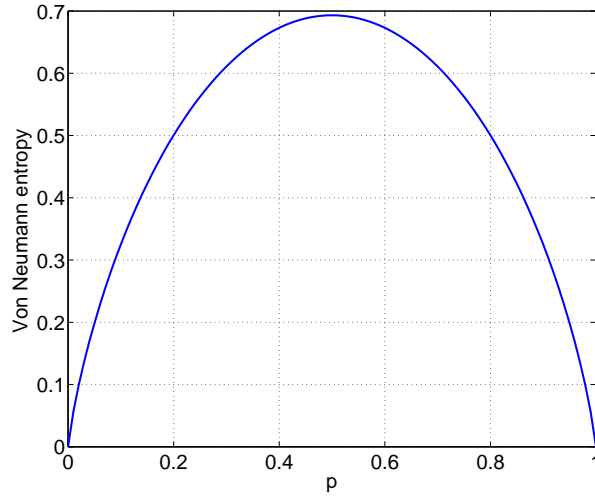


Figure 2.4: von-Neumann entropy for a set of canonical states  $(\sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle)$  for two-qubit system. All pure entangled states of two qubits can be achieved from these states by local unitary transformation by varying  $p$  in  $[0, 1]$ .

Like von-Neumann entropy, Rényi entropy of the reduced density matrix also satisfies all the conditions for a valid entanglement measure.

### 2.4.3 Entanglement of formation (EoF)

The entanglement of formation is the minimal convex extension of the von-Neumann entropy of the reduced density operator to mixed states. This measure gives us the minimum amount of entanglement required to construct a given bipartite density operator.

Every state can be decomposed as a convex combination of pure states projections as:

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|, \quad (2.49)$$

where  $p_k \geq 0$  such that  $\sum_k p_k = 1$  and  $|\psi_k\rangle$  need not be orthogonal. We call this particular decomposition an *ensemble* and represent it by  $\{p_k, |\psi_k\rangle\}$ , then the definition of EoF is [49, 162]

**Definition 4** *EoF* is defined as the minimized average von-Neumann entropy of the reduced density operators of the pure states  $|\psi_k\rangle$ , minimization is realized over all possible decompositions of  $\rho$ , i.e.,

$$EoF(\rho) = \min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k E(|\psi_k\rangle\langle\psi_k|), \quad (2.50)$$

where  $E(|\psi_k\rangle\langle\psi_k|)$  is the von-Neumann measure of entanglement for pure state  $|\psi_k\rangle$ .

Since the definition of EoF involves optimization, it is in general difficult to calculate EoF.

#### 2.4.4 Concurrence

Concurrence is a measure of entanglement in two-qubit systems. This measure was introduced by Bennett, Divincenzo, Smolin and Wootters in 1996 [25] and generalized by Wootters, Hill and others later [91, 179, 181, 180]. To understand this measure, consider a two-qubit pure product state, i.e.,

$$|\psi\rangle = |\chi\rangle \otimes |\eta\rangle, \quad (2.51)$$

where

$$|\chi\rangle = \alpha_1|0\rangle + \beta_1|1\rangle \quad (2.52)$$

$$|\eta\rangle = \alpha_2|0\rangle + \beta_2|1\rangle. \quad (2.53)$$

We have

$$|\chi_\perp\rangle = \beta_1|0\rangle - \alpha_1^*|1\rangle \quad (2.54)$$

$$|\eta_\perp\rangle = \beta_2|0\rangle - \alpha_2^*|1\rangle \quad (2.55)$$

which are orthogonal to  $|\chi\rangle$  and  $|\eta\rangle$  respectively. The state

$$|\tilde{\psi}\rangle = |\chi_\perp\rangle \otimes |\eta_\perp\rangle = (\sigma_2 \otimes \sigma_2)\mathcal{C}|\psi\rangle \quad (2.56)$$

is orthogonal to  $|\psi\rangle$  where  $\mathcal{C}$  is complex conjugation operation in the standard basis. Now consider a general pure two-qubit state  $|\varphi\rangle = \sum_{ij} \alpha_{ij} |i\rangle \otimes |j\rangle$ . We can write

$$|\tilde{\varphi}\rangle = \sum_{ij} \alpha_{ij}^* (\sigma_2 \otimes \sigma_2) |i\rangle \otimes |j\rangle. \quad (2.57)$$

The quantity  $|\langle \tilde{\varphi} | \varphi \rangle|$  will be zero for a pure product state otherwise it will have value between zero and one. In that sense this quantity can serve as the measure of entanglement. To calculate this quantity consider the coefficient matrix  $\alpha = [\alpha_{ij}]$  for the state  $|\varphi\rangle$ . Similarly the coefficient matrix for  $|\tilde{\varphi}\rangle$  in the basis of  $|i\rangle$  and  $|j\rangle$  can be written as:

$$\tilde{\alpha} = -\sigma_2 \alpha^* \sigma_2 \quad (2.58)$$

and hence the quantity  $|\langle \tilde{\varphi} | \varphi \rangle|$  can be written as  $|\text{tr}(-\sigma_2 \alpha^* \sigma_2 \alpha)|$ . If we write the state  $|\varphi\rangle$  in the “magic basis”, i.e.,

$$|\varphi\rangle = \sum_i \gamma_i |e_i\rangle \quad (2.59)$$

where

$$\begin{aligned} |e_1\rangle &= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle), \\ |e_2\rangle &= \frac{i}{\sqrt{2}} (|00\rangle - |11\rangle), \\ |e_3\rangle &= \frac{i}{\sqrt{2}} (|01\rangle + |10\rangle), \\ |e_4\rangle &= \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle), \end{aligned}$$

the quantity  $|\langle \tilde{\varphi} | \varphi \rangle|$  turns out to be

$$C(|\varphi\rangle) = \left| \sum_i \gamma_i^2 \right| \quad (2.60)$$

which we call *concurrence* of  $|\varphi\rangle$ .

## Chapter 2. Entanglement

The state  $|\tilde{\varphi}\rangle$  represents a kind of spin-flipped state which is compared with the original state which gives a measure of entanglement. For a two-qubit density operator  $\rho$  the spin-flip operation is given by:

$$\tilde{\rho} = (\sigma_2 \otimes \sigma_2)\rho^*(\sigma_2 \otimes \sigma_2), \quad (2.61)$$

where  $\rho^*$  is the complex conjugation in the standard basis. We define a quantity  $R^2 = \tilde{\rho}\rho$ . The concurrence is then given by

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \quad (2.62)$$

where the  $\lambda_i$  are the square root of the eigenvalues of the matrix  $R^2$  in descending order.

For an explicit calculation of the concurrence consider the Werner state  $\rho_p$ . The operator  $\tilde{\rho}_p = (\sigma_2 \otimes \sigma_2)\rho_p^*(\sigma_2 \otimes \sigma_2)$  can be written as:

$$\tilde{\rho}_p = \begin{pmatrix} \frac{1-p}{4} & 0 & 0 & 0 \\ 0 & \frac{1+p}{4} & -\frac{p}{2} & 0 \\ 0 & -\frac{p}{2} & \frac{1+p}{4} & 0 \\ 0 & 0 & 0 & \frac{1-p}{4} \end{pmatrix} \quad (2.63)$$

which is same as  $\rho_p$ . Thus  $R^2 = \rho_p^2$  and the eigenvalues of  $R^2$  will be square of  $(1-p)/4$ ,  $(1-p)/4$ ,  $(1-p)/4$ ,  $(1+3p)/4$ , the eigenvalues of  $\rho_p$ . We can calculate the concurrence from here as:

$$\begin{aligned} C(\rho_p) &= \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \\ &= \frac{3p-1}{2}. \end{aligned} \quad (2.64)$$

### 2.4.5 G-concurrence

G-concurrence for a pure state is defined as the geometric mean of the Schmidt coefficient of the pure state [84]. For a bipartite state  $|\psi\rangle$  it can be written as:

$$G(|\psi\rangle) = d(\lambda_1\lambda_2\cdots\lambda_d)^{1/d} \quad (2.65)$$

where  $d$  is the dimension of each subsystem. The state  $|\psi\rangle$  can be written as

$$|\psi\rangle = \sum_{ij} A_{ij} |ij\rangle. \quad (2.66)$$

The G-concurrence in terms of the coefficient matrix  $A$  can be written as:

$$G(|\psi\rangle) = d[\det(AA^\dagger)]^{1/d} \quad (2.67)$$

### 2.4.6 Logarithmic negativity

The logarithmic negativity is an entanglement measure which relies on the negativity of an entangled state under partial transposition. This measure is easy to calculate and as good as any other measure for two-qubit systems. The negativity for a bipartite state can be defined as [139]

$$\mathcal{N}(\rho) = \frac{\|\rho^{\Gamma_A}\|_1 - 1}{2}, \quad (2.68)$$

where  $\Gamma_A$  stands for partial transposition with respect to subsystem  $A$  and  $\|\cdot\|_1$  is trace norm of an operator and can be written as

$$\|\xi\|_1 = \text{tr} \left( \sqrt{\xi^\dagger \xi} \right), \quad (2.69)$$

i.e, the sum of the singular values of  $\xi$ . All those states which have negative eigenvalues after taking partial transposition will have non-zero negativity.

Logarithmic negativity  $LN$  [139] can be defined as

$$LN(\rho) = \log_2 (2\mathcal{N}(\rho) + 1). \quad (2.70)$$

It is easy to see that negativity is first order approximation of logarithmic negativity.

To see the calculation of negativity and logarithmic negativity explicitly, con-

sider again the Werner state  $\rho_p$ . For this state

$$\|\rho_p^{\Gamma_A}\|_1 = \left| \frac{1+p}{4} \right| + \left| \frac{1+p}{4} \right| + \left| \frac{1+p}{4} \right| + \left| \frac{1-3p}{4} \right|, \quad (2.71)$$

$$= 3 \left| \frac{1+p}{4} \right| + \left| \frac{1-3p}{4} \right|. \quad (2.72)$$

For  $p > 1/3$  the term  $1 - 3p$  is negative, therefore,

$$\|\rho_p^{\Gamma_A}\|_1 = 3 \frac{1+p}{4} + \frac{3p-1}{4} \quad (2.73)$$

$$= \frac{3p+1}{2} \quad (2.74)$$

and thus the negativity  $\mathcal{N}(\rho_p) = (3p - 1)/4$  and the logarithmic negativity is  $LN(\rho_p) = \log_2((3p + 1)/2)$ .

This measure (i.e, logarithmic negativity) of entanglement is good for two-qubit systems, since PPT is necessary and sufficient condition for separability in this case. But for higher dimensional systems, there are states which are entangled and are PPT. Such states are called *bound entangled states* or *non-distillable entangled states*. For such states logarithmic negativity fails as a bona-fide measure of entanglement.

# 3

## Quantum operations and Master equation

Quantum operations (QO) or dynamical maps are completely positive linear maps from a set of density operators to another set of density operators. One example of a QO is time evolution, which can either be unitary or nonunitary. Evolution through a noisy channel is another example. Measurement and the projections are also QO's. QO's were first discussed as a general stochastic transformation for a density matrix by Sudarshan *et al.* [161]. Since then a lot of work has been done on this subject. [159, 176, 117, 73, 134].

Any linear map  $\varphi : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$  which preserves the following basic properties of quantum states is a QO.

### 1. Hermiticity :

$$\varphi(A^\dagger) = \varphi(A)^\dagger, \quad (3.1)$$

i.e, hermitian operators are mapped to hermitian operators.

### 2. Positivity:

$$\varphi(A) \geq 0 \quad \forall A \geq 0, \quad (3.2)$$

i.e, all the positive operators are mapped to positive operators. To represent physical operation, a map  $\varphi$  needs to be completely positive, i.e,

$$(\mathcal{I}_n \otimes \varphi) Z \geq 0 \quad \forall Z \geq 0, \quad (3.3)$$



where  $\mathcal{I}_n$  is an identity map acting on  $n \times n$  operators and  $Z \in \mathcal{B}(\mathcal{H}_n \otimes \mathcal{H}_1)$ .

**3. Trace condition:**

$$\text{tr}\varphi(A) = 1 \quad \forall \text{tr}A = 1, \quad (3.4)$$

Any linear map can be represented in matrix form. Consider a map  $\Lambda$  represented by the matrix  $V$  connecting the initial density operator  $\rho_S(0)$  of a quantum system  $S$  to the density operator  $\rho_S(t)$  at time  $t$ . We can write:

$$\rho_{S_{ij}}(t) = \sum_{kl} V_{ij,kl}(t) \rho_{S_{kl}}(0), \quad (3.5)$$

For  $V$  to be a valid QO it needs to satisfy conditions ((3.1), (3.2), (3.4)) which, in this context, are the following:

***Hermiticity:***

$$\begin{aligned} \rho_{S_{ij}}(t) &= \rho_{S_{ji}}^*(t) \\ \Rightarrow \sum_{kl} V_{ij,kl}(t) \rho_{S_{kl}}(0) &= \sum_{kl} V_{ji,kl}^*(t) \rho_{S_{kl}}^*(0) \\ &= \sum_{kl} V_{ji,kl}^*(t) \rho_{S_{lk}}(0) \\ \Rightarrow V_{ij,kl}(t) &= V_{ji,lk}^*(t). \end{aligned} \quad (3.6)$$

If we define an operator  $M$  such that

$$M_{ik,jl} = V_{ij,kl}, \quad (3.7)$$

then Eq.(3.6) is simply

$$M_{ij,kl} = M_{kl,ij}^*. \quad (3.8)$$

This implies that for  $V$  to represent a Hermitian map,  $M$  needs to be a Hermitian operator.

***Positivity:***

Positivity of a density operator is:

$$\sum_{ij} x_i^* \rho_{ij}(t) x_j \geq 0 \quad \forall |x\rangle = [x_1, x_2, \dots, x_d]^T \in \mathcal{H}, \quad (3.9)$$

which is

$$\sum_{ijkl} x_i^* \rho_{kl}(0) x_j M_{ik,jl} \geq 0 \quad \forall |x\rangle \in \mathcal{H}. \quad (3.10)$$

If the spectral decomposition of the density operator is

$$\begin{aligned} \rho(0) &= \sum_m \lambda_m |\psi^m\rangle \langle \psi^m| \\ \rho_{kl}(0) &= \sum_m \lambda_m \psi_k^m \psi_l^{*m}, \end{aligned} \quad (3.11)$$

then from Eq. (3.10):

$$\begin{aligned} &\sum_{ij,kl,m} \lambda_m \psi_k^m \psi_l^{*m} x_i^* x_j M_{ik,jl} \geq 0 \\ \Rightarrow &\sum_m \lambda_m \sum_{ij,kl} x_i^* \psi_k^m M_{ik,jl} \psi_l^{*m} x_j \geq 0 \\ &\Rightarrow \langle x \psi^{m*} | M | x \psi^{m*} \rangle \geq 0, \end{aligned} \quad (3.12)$$

where  $\sqrt{\lambda_m} |x \psi^{m*}\rangle = |x\rangle \otimes |\psi^{m*}\rangle$ . Condition Eq. (3.12) implies that  $M$  needs to be positive on product states. For complete positivity  $M$  needs to be a positive operator.

### ***Trace condition:***

Since the trace of a density operator is always unity, there is a further constraint on the dynamical map. We know that

$$\begin{aligned} \text{tr}(\rho_S(t)) &= \sum_i \rho_{Sii}(t) \\ &= \sum_{kl,i} V_{ii,kl} \rho_{kl}(0) = 1 \end{aligned} \quad (3.13)$$

for all density operators  $\rho_s(0)$ . This gives us a condition on  $V$

$$\sum_i V_{ii,kl} = \delta_{kl}. \quad (3.14)$$

This constraint ensures that the trace of a density operator is always preserved. In this fashion we have derived all the constraints on the matrix representation  $V$  of a dynamical map  $\Lambda$ . In the following sections we will discuss operator sum representation and Kraus operators as alternate way of studying the dynamics of an open quantum system.

### 3.1 Operator sum representation (Kraus operators)

A dynamical map  $\Lambda$  can be represented using the matrix  $V$  (see Eq. (3.5)) or alternatively, the matrix  $M$  (see Eq. (3.7), where  $M$  needs to be a positive operator). The action of a map  $\Lambda$  can be fully characterized by the following theorem:

**Theorem 6** *Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be Hilbert spaces of dimension  $d_1$  and  $d_2$  respectively. Let  $\Lambda$  be a quantum operation taking the density operators acting on  $\mathcal{H}_1$  to those acting on  $\mathcal{H}_2$ . Then there are operators  $\{A_i\}_{1 \leq i \leq d_1 d_2}$  acting from  $\mathcal{H}_1$  to  $\mathcal{H}_2$  such that*

$$\Lambda(\rho) = \sum_i A_i \rho A_i^\dagger, \quad (3.15)$$

where  $\sum_i A_i^\dagger A_i \leq \mathcal{I}$ . Conversely, any map  $\Lambda$  of this form is a quantum operation provided  $\sum_i A_i^\dagger A_i \leq \mathcal{I}$  the identity operator acting on  $\mathcal{H}_1$ .

The representation in (3.15) is called *operator sum representation* and the operators  $\{A_i\}$  are called Kraus operators. For a given quantum operation  $\Lambda$ , the set  $\{A_i\}$  of Kraus operators is not unique. The following theorem states that all such sets of Kraus operators which represent the same quantum operation are related by a unitary transformation.

**Theorem 7** Let  $\Lambda$  be a quantum operation from Hilbert space  $\mathcal{H}_1$  to  $\mathcal{H}_2$  and let the sets  $\{A_i\}_{i=1}^m$  and  $\{B_i\}_{i=1}^n$  be two sets of Kraus operators representing the action of  $\Lambda$ . Then there is an isometry  $U = [u_{ij}]_{1 \leq i \leq n, 1 \leq j \leq m}$  such that

$$B_i = \sum_j u_{ij} A_j. \quad (3.16)$$

For a map  $\Lambda$  (and hence for  $V$  and  $M$ ), it is straightforward to calculate a set of Kraus operators which represent the action of  $\Lambda$ . To do this consider the spectral decomposition for  $M$

$$\begin{aligned} M &= \sum_m \gamma_m |m\rangle \langle m| \\ &= \sum_m |a_m\rangle \langle a_m| \end{aligned} \quad (3.17)$$

where  $|a_m\rangle = \sqrt{\gamma_m} |m\rangle$ . This gives us

$$\begin{aligned} M_{ij,kl} &= \sum_m a_{ij}^m a_{lk}^{m*} \text{ where } |a_m\rangle = \sum_{ij} a_{ij}^m |ij\rangle \\ &= \sum_m (a^m \otimes a^{m*})_{ik,jl} \text{ where } a^m = [a_{ij}^m]_{i,j=1}^d \\ &= V_{ik,jl}. \end{aligned}$$

We can rewrite the Eq. (3.5) in terms of  $a$ 's as:

$$\begin{aligned} \rho_{S_{ij}}(t) &= \sum_{kl} V_{ij,kl} \rho_{S_{kl}}(0) \\ &= \sum_{kl,m} a_{ik}^m a_{jl}^{m*} \rho_{S_{kl}}(0) \\ \rho_S(t) &= \sum_m a^m \rho_S(0) a^{m\dagger}. \end{aligned} \quad (3.18)$$

Hence the matrices  $a^m$  are the Kraus operators. Using this set of Kraus operators it is trivial to calculate all other sets of Kraus operators. This is done by taking the unitary mixing of the Kraus operators  $a^m$ . This gives us the Kraus operators

$b_k$  which are obtained using

$$b_k = \sum_m u_{km} a^m. \quad (3.19)$$

Here  $U = [u_{km}]$  is a unitary operator (in general  $U$  needs to be isometry operator, i.e,  $U^\dagger U = \mathcal{I}$  whereas  $UU^\dagger \neq \mathcal{I}$ ), and  $\{b_k\}$ 's are Kraus operators representing the same evolution as the  $\{a^m\}$ 's. In this fashion we can calculate the entire set of Kraus operators for a given evolution.

## 3.2 Quantum dynamical semigroup

The evolution of an open system is a physical process which is mathematically analogous to a map from a set of density operators to itself. If  $\rho(0)$  be the initial density operator of the open quantum system (OQS), then the density matrix at time  $t$ ,  $\rho(t)$ , is obtained by

$$\Lambda(t)\rho(0) = \rho(t), \quad (3.20)$$

where  $\Lambda$  is the evolution map. This map  $\Lambda$  maps the set of density operator  $S(\mathcal{H}_S)$  acting on the Hilbert space  $\mathcal{H}_S$  to itself, i.e,

$$\Lambda : S(\mathcal{H}_S) \rightarrow S(\mathcal{H}_S). \quad (3.21)$$

Such a map which describes the change in the state of the system over time is called a *dynamical map*.

The total Hamiltonian of the system and bath can be written as:

$$H = H_S + H_B + H_I, \quad (3.22)$$

where  $H_S$  is free Hamiltonian of the system,  $H_B$  is the free Hamiltonian of the bath and  $H_I$  is the interaction between the system and the bath. The evolution of the system-bath combine is governed by the unitary operator

$$U(t) = \exp(-iHt). \quad (3.23)$$

For simplicity, let us assume that initially, the system and the bath are uncorrelated, therefore the state of the system and bath is taken as:

$$\rho(0) = \rho_S(0) \otimes \rho_B \quad (3.24)$$

which evolves to

$$\rho(t) = U(t)(\rho_S(0) \otimes \rho_B)U^\dagger(t). \quad (3.25)$$

Since we are interested in the state of the system alone, we can obtain it by tracing out the bath degree of freedom from  $\rho(t)$ . We get

$$\rho_S(t) = \text{tr}_B [U(t)(\rho_S(0) \otimes \rho_B)U^\dagger(t)], \quad (3.26)$$

which implies that

$$\rho_S(t) = \Lambda(t)\rho_S(0) = \text{tr}_B [U(t)(\rho_S(0) \otimes \rho_B)U^\dagger(t)]. \quad (3.27)$$

We wish to get a clearer picture of the dynamical map  $\Lambda(t)$ . To this end, we first write down the spectral decomposition of  $\rho_B$ :

$$\rho_B = \sum_m \lambda_m |m\rangle\langle m|, \quad (3.28)$$

where  $\{\lambda_m\}$  are real positive numbers satisfying  $\sum_m \lambda_m = 1$ . Substituting Eq.(3.28) into Eq.(3.27) we get.

$$\begin{aligned} \Lambda(t)\rho_S(0) &= \text{tr}_B \left[ U(t)(\rho_S(0) \otimes \sum_m \lambda_m |m\rangle\langle m|)U^\dagger(t) \right] \\ &= \sum_n \langle n| \left[ U(t)(\rho_S(0) \otimes \sum_m \lambda_m |m\rangle\langle m|)U^\dagger(t) \right] |n\rangle \\ &= \sum_n \sum_m \lambda_m \langle n|U(t)|m\rangle \rho_S(0) \langle m|U^\dagger(t)|n\rangle. \end{aligned} \quad (3.29)$$

Eq.(3.29) can be rewritten as

$$\Lambda(t)\rho_S(0) = \sum_n \sum_m W_{nm}(t)\rho_S(0)W_{nm}^\dagger(t) \quad (3.30)$$

where  $W_{nm}(t) = \sqrt{\lambda_m}\langle n|U(t)|m\rangle$  are the Kraus operators acting on  $\rho_S(0)$ . Since  $\sum_{nm} W_{nm}^\dagger W_{nm} = \mathcal{I}$ ,  $\text{tr}(\Lambda(t)\rho_S(0)) = \text{tr}\rho_S(t) = 1$ . This explicitly confirms that  $\Lambda(t)$  is a trace preserving map.  $\Lambda(t)$  gives rise to a one parameter family of dynamical maps, since time  $t$  appears as a parameter and where  $\Lambda(0)$  is the identity. If there is no memory effect in the system, i.e, if the bath correlation function decays much faster than the system response time, then the one parameter family of dynamical maps will satisfy the dynamical semigroup property, i.e,

$$\Lambda(t_2)\Lambda(t_1) = \Lambda(t_2 + t_1). \quad (3.31)$$

This loss of memory effect is what gives rise to Markovian characteristics.

The non-unitary nature of the evolution of open quantum systems necessitates the use of a master equation to describe it. A master equation is simply an equation of motion of the density operator of the relevant system. Here we shall be primarily concerned with Markovian master equations [83, 178, 33] which are easier to deal with, thanks to the quantum dynamical semigroup property [82, 156] they possess.

In the next section we will discuss Lindblad master equation from quantum dynamical semigroup point of view followed by microscopic derivation for the same.

### 3.2.1 Master equation: quantum dynamical semigroup approach

Consider a dynamical map  $\Lambda$  acting on the state of a  $d$ -dimensional quantum system  $S$ . The evolution of a state  $\rho_S(0)$  of system  $S$  is then given by

$$\rho_S(t) = \Lambda(t)\rho_S(0). \quad (3.32)$$

Markovian dynamics implies that  $\Lambda(t_2)\Lambda(t_1) = \Lambda(t_2 + t_1)$ , and this allows us to

write down the further evolution of the state  $\rho_S(t)$  for an infinitesimal time  $\epsilon$ :

$$\rho_S(t + \epsilon) = \Lambda(\epsilon)\rho_S(t). \quad (3.33)$$

The time derivative of  $\rho_S(t)$  can then be written as

$$\begin{aligned} \frac{d}{dt}\rho_S(t) &= \lim_{\epsilon \rightarrow 0} \frac{\rho_S(t + \epsilon) - \rho_S(t)}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\Lambda(\epsilon) - \mathcal{I}}{\epsilon} \rho_S(t) \\ &= \mathcal{L}\rho_S(t) \end{aligned} \quad (3.34)$$

Thus  $\mathcal{L}$  is the generator of the semigroup  $\Lambda(t)$ . We begin with the action of  $\Lambda$  on the operator  $\rho_S(t)$ .

$$\Lambda(\epsilon)\rho_S(t) = \sum_{nm} W_{nm}(\epsilon)\rho_S(t)W_{nm}^\dagger(\epsilon) \quad (3.35)$$

where  $W_{nm}$  are  $d \times d$  Kraus operators defined in Sec. 3.2. We can write the Kraus operators in a complete basis of orthonormal operators  $\{A_i\}$ , i.e.,

$$W_{nm}(\epsilon) = \sum_i a_{nm}^i(\epsilon)A_i \quad (3.36)$$

where  $\text{tr}(A_i^\dagger A_j) = \delta_{ij}$  and  $a_{nm}^i(\epsilon) = \text{tr}[A_i^\dagger W_{nm}(\epsilon)]$ . We can rewrite Eq.(11.8) as

$$\begin{aligned} \Lambda(\epsilon)\rho_S(t) &= \sum_{nm} \sum_{ij} a_{nm}^i(\epsilon)a_{nm}^{j*}(\epsilon)A_i\rho_S(0)A_j^\dagger \\ &= \sum_{i,j=0}^{d^2-1} b_{ij}(\epsilon)A_i\rho_S(t)A_j^\dagger, \end{aligned} \quad (3.37)$$

where  $b_{ij}(\epsilon) = \sum_{nm} a_{nm}^i(\epsilon)a_{nm}^{j*}(\epsilon)$ . We can choose  $A_0$  to be  $\mathcal{I}/\sqrt{d}$  without loss of generality. This will make the other operators  $\{A_i\}_{i \neq 0}$  traceless and allow us to substitute Eq.(3.37) into Eq.(11.10) to get

$$\frac{d}{dt}\rho_S(t) = c_{00}\rho_S(t) + B\rho_S(t) + \rho_S(t)B^\dagger + \sum_{ij \neq 0} c_{ij}A_i\rho_S(t)A_j^\dagger \quad (3.38)$$



### Chapter 3. Quantum operations and Master equation

where

$$\begin{aligned} c_{00} &= \lim_{\epsilon \rightarrow 0} \frac{b_{00} - d}{d\epsilon} \\ c_{ij} &= \lim_{\epsilon \rightarrow 0} \frac{b_{ij}}{\epsilon} \\ B &= \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{d}\epsilon} \sum_i b_{i0}(\epsilon) A_i. \end{aligned}$$

Combining the first two terms of Eq.(3.38), gives

$$\frac{d}{dt}\rho_s(t) = -i[H, \rho_s(t)] + \{G, \rho_s(t)\} + \sum_{ij} c_{ij} A_i \rho_s(t) A_j^\dagger \quad (3.39)$$

where

$$H = \frac{1}{2i}[B^\dagger - B], \quad (3.40)$$

$$G = \frac{1}{2}(c_{00}\mathcal{I} + B^\dagger + B). \quad (3.41)$$

Finally, the expression for  $\mathcal{L}$  is

$$\mathcal{L}\rho_s(t) = -i[H, \rho_s(t)] + \{G, \rho_s(t)\} + \sum_{ij} c_{ij} A_i \rho_s(t) A_j^\dagger. \quad (3.42)$$

Since  $d\rho_s(t)/dt$  should be traceless, the restriction

$$\text{tr} \left\{ G\rho_s(t) + \rho_s(t)G + \sum_{ij} c_{ij} A_i \rho_s(t) A_j^\dagger \right\} = 0, \quad (3.43)$$

$$\text{i.e.,} \quad \text{tr} \left\{ \left( 2G + \sum_{ij} c_{ij} A_j^\dagger A_i \right) \rho_s(t) \right\} = 0 \quad (3.44)$$

applies for all  $\rho_s(t)$ , which leads to

$$G = -\frac{1}{2} \sum_{ij} c_{ij} A_j^\dagger A_i \quad (3.45)$$

The matrix  $c = [c_{ij}]$  is a positive operator and hence is diagonalizable through conjugation with a unitary operator. We can always find a unitary matrix  $u$  such

that

$$ucu^\dagger = \gamma \quad (3.46)$$

where  $\gamma$  is  $\text{diag}(\gamma_0, \gamma_1, \dots, \gamma_{d-1})$  with non-negative entries. We define new operators  $\{F_i\}$  such that

$$A_i = \sum_{k=1}^{d^2-1} u_{ki} F_k \quad (3.47)$$

and Eq.(3.42) takes the form

$$\mathcal{L}\rho_S(t) = -i[H, \rho_S(t)] + \left\{ -\frac{1}{2} \sum_i \gamma_i F_i^\dagger F_i, \rho_S(t) \right\} + \sum_k \gamma_k F_k \rho_S(t) F_k^\dagger. \quad (3.48)$$

This is the Lindblad form of the Markovian master equation [115], or to put it more succinctly the Lindblad master equation. The physical meaning of all the terms on the right hand side of the equation will be clear in the following section where we will derive the above equation in the microscopic description. But before going to that let us have a look at different pictures in quantum mechanics. We will need them in the microscopic derivation of Master equation.

### 3.3 Heisenberg and Interaction Picture

The time evolution of a state vector  $|\psi(t)\rangle$  is given by the Schrödinger equation:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle \quad (3.49)$$

where  $H$  is the Hamiltonian. If  $H$  is time independent, then the solution of the Schrödinger equation is:

$$|\psi(t)\rangle = \exp\left(-i \frac{H(t-t_0)}{\hbar}\right) |\psi(t_0)\rangle \quad (3.50)$$

### Chapter 3. Quantum operations and Master equation

This allows us to calculate the expectation value of an observable  $O$  when the system is in state  $|\psi(t)\rangle$ :

$$\langle O \rangle = \langle \psi(t) | O | \psi(t) \rangle. \quad (3.51)$$

However, it is sometimes useful and necessary to use time-dependent bases in quantum mechanics, different choices of which give rise to the Heisenberg, Schrödinger and Interaction picture. When we choose to make the state vector time-dependent we get Schrödinger picture. Therefore, the solution the Eq.(3.49) results in the Schrödinger picture.

The *Heisenberg picture* is obtained by making every vector in the Hilbert space move according to the Schrödinger Eq. (3.49) so that none of the vector evolve in time, i.e,

$$|\psi_{HP}\rangle = U^\dagger(t, t_0)|\psi(t)\rangle \quad (3.52)$$

where subscript  $HP$  represents the Heisenberg picture and  $U(t, t_0) = \exp\left(-i\frac{H(t-t_0)}{\hbar}\right)$ .

The expectation value of an observable is a physical quantity and therefore, should be independent of representation. This implies that

$$\langle O_{HP} \rangle = \langle \psi_{HP} | O_{HP} | \psi_{HP} \rangle = \langle \psi(t) | O | \psi(t) \rangle. \quad (3.53)$$

This equation gives us the relation between an observable in the Heisenberg picture  $O_{HP}$  and an observable in the Schrödinger picture  $O$ :

$$O_{HP} = U^\dagger(t, t_0) O U(t, t_0) \quad (3.54)$$

where the observables in the Heisenberg picture satisfies the equation:

$$i\hbar \frac{d}{dt} O_{HP} = [O_{HP}, H]. \quad (3.55)$$

In systems where  $H$  can be written as  $H_0 + H_1$ , where  $H_0$  has no explicit time dependence and  $H_1$  is a small time dependent perturbation, the *interaction picture* is quite useful. Like in Heisenberg picture, in interaction picture as well we make

state vector to rotate with a unitary matrix  $U_0$  given by:

$$U_0 = \exp\left(-i\frac{H_0(t-t_0)}{\hbar}\right). \quad (3.56)$$

The state vector in interaction picture is related to the one in Schrödinger picture by the relation

$$|\psi_I\rangle = U_0^\dagger(t, t_0)|\psi(t)\rangle \quad (3.57)$$

and satisfies the equation

$$i\hbar\frac{d}{dt}|\psi_I\rangle = H_I|\psi_I\rangle \quad (3.58)$$

where  $H_I = U_0^\dagger H_1 U_0$ . The expectation value of an observable in interaction picture is:

$$\langle O_I \rangle = \langle \psi_I | O_I | \psi_I \rangle = \langle \psi(t) | O | \psi(t) \rangle \quad (3.59)$$

and hence  $O_I = U_0^\dagger O U_0$  and the observable  $O_I$  satisfies the equation:

$$i\hbar\frac{d}{dt}O_I = [O_I, H]. \quad (3.60)$$

### 3.3.1 Microscopic derivation of the Markovian master equation

Dynamical semigroup approach for the derivation of the Lindblad master equation [Eq. (3.48)] gives us the formal expression for the equation. But the form of the operators  $F_k$ 's and the expression for  $\gamma_k$ 's in this equation depends on the system Hamiltonian, bath Hamiltonian and the way system and bath are interacting. Therefore, it is important to derive the generator of the quantum dynamical semigroup from the Hamiltonian dynamics of the system. In this section we will start with the Hamiltonian of the system and the bath and the interaction between them and derive the Lindblad master equation for the system.

Let the Hamiltonian  $H_S$  and  $H_B$  describes the dynamics of the system  $S$  – which can be any finite or infinite dimensional quantum system – and the infinite

### Chapter 3. Quantum operations and Master equation

dimensional bath  $B$  respectively, whereas the interaction between them is governed by  $H_I$ . A two-level quantum system interacting with a bath of an infinite number of non-interacting harmonic oscillators or a quantum dipole in the presence of an electromagnetic field are typical examples of such a setup. We assume that the system-bath combine is in the Schrödinger picture, which means that there is no explicit time dependence in the system Hamiltonian. In this picture, the density matrix of the composite system  $\rho_{tot}$  evolves according to the Liouville-von-Neumann equation,

$$\frac{d}{dt}\rho_{tot} = -i[H, \rho_{tot}] \quad (3.61)$$

where  $H = H_S + H_B + H_I$  is the Hamiltonian for the entire system. We have taken  $\hbar = 1$ .

However, it is more convenient to work in the *interaction picture* for two reasons. One is that the interaction picture Hamiltonian is simpler since it has lesser number of terms. The other reason is that in the interaction picture, we treat states and observables on an equal footing. This means that both the states and the observables are time dependent, unlike that in the Schrödinger or the Heisenberg picture. The state  $\rho$  (of the entire system) in the interaction picture can be written as

$$\rho_I(t) = \exp\{i(H_S + H_B)t\} \rho_{tot}(t) \exp\{-i(H_S + H_B)t\} \quad (3.62)$$

and the equation of motion for the density operator as,

$$\frac{d}{dt}\rho_I = -i[H_I^I(t), \rho_I(t)] \quad (3.63)$$

where,

$$H_I^I(t) = \exp\{i(H_S + H_B)t\} H_I(t) \exp\{-i(H_S + H_B)t\} \quad (3.64)$$

The density operator  $\rho_S(t)$  is used to characterize the system and is given by,

$$\rho_S(t) = \text{tr}_B(\rho_{tot}(t)), \quad (3.65)$$

or,

$$\rho_S(t) = \text{tr}_B(\exp\{-i(H_S + H_B)t\} \rho_I(t) \exp\{i(H_S + H_B)t\}). \quad (3.66)$$

Since  $[H_S, H_B] = 0$ ,

$$\rho_S(t) = \text{tr}_B (\exp(-iH_S t) \exp(-iH_B t) \rho_I(t) \exp(iH_S t) \exp(iH_B t)). \quad (3.67)$$

This can further be reduced to

$$\begin{aligned} \rho_S(t) &= \exp(-iH_S t) \text{tr}_B \{ \exp(iH_B t) \exp(-iH_B t) \rho_I(t) \} \exp(iH_S t), \\ &= \exp(-iH_S t) \text{tr}_B (\rho_I(t)) \exp(iH_S t), \end{aligned} \quad (3.68)$$

We need to know  $\rho_I(t)$  to get  $\rho_S(t)$ . For this, we need to solve Eq.(3.63) whose formal solution can be written as a recursive relation:

$$\rho_I(t) = \rho_I(0) - i \int_0^t dt' [H^I(t'), \rho_I(t')]. \quad (3.69)$$

$$\begin{aligned} \rho_I(t) &= \rho_I(0) - i \int_0^t dt' [H^I(t'), \rho_I(0)] \\ &\quad - \int_0^t dt' \int_0^{t'} dt'' [H^I(t'), [H^I(t''), \rho_I(t'')]]. \end{aligned} \quad (3.70)$$

We terminate this series beyond the second order by assuming that the system-bath interaction is weak. Using this, we rewrite Eq. (3.63) as

$$\frac{d}{dt} \rho_I(t) = -i [H^I(t), \rho_I(0)] - \int_0^t dt' [H^I(t), [H^I(t'), \rho_I(t')]]. \quad (3.71)$$

We assume as before, that the system and bath were initially uncorrelated. Thus, the initial state of the system-bath combine can be written as:

$$\rho_{tot}(0) = \rho_S(0) \otimes \rho_B. \quad (3.72)$$

We also assume that the state of the bath is *sufficiently random*. If we choose  $\{A_i\}$  and  $\{B_i\}$  some set of operators acting on the Hilbert spaces  $\mathcal{H}_S$  and  $\mathcal{H}_B$  of the system and the bath respectively such that the interaction Hamiltonian takes the form  $H_I = \sum A_i \otimes B_i$ , then sufficient randomness of the state of the bath implies that the expectation value  $\text{tr}(\rho_B B_i)$  of the operators  $B_i$  vanish and hence

$$\text{tr}_B [H^I, \rho_{tot}(0)] = \text{tr} [H^I, \rho_I(0)] = 0. \quad (3.73)$$

### Chapter 3. Quantum operations and Master equation

This assumption is motivated from the fact that the expectation value of all the field operators in the case of bath of electromagnetic field (EMF) in the thermal state, is zero, i.e,  $\langle a_i \rangle = \langle a_i^\dagger \rangle = 0$ .

Tracing over both sides of Eq. (3.71), and using Eq. (3.73), we get

$$\begin{aligned} \frac{d}{dt}\rho_s(t) &= -\text{tr}_B \int_0^t dt' [H^I(t), [H^I(t'), \rho(t')]] \\ &= - \int_0^t dt' \text{tr}_B [H^I(t), [H^I(t'), \rho(t')]]. \end{aligned} \quad (3.74)$$

Consider now a quantum system consisting of two subsystems, where the inertia of one subsystem is much higher than the other. An atom is one such example – electrons zip around in their orbitals at high frequency around the much more massive nucleus. In such cases, the *Born approximation* is applicable and we can write the total state of the system in product form. Thus Born approximation can be applied to our system-bath combine because a bath, by definition, has much more resistance to perturbation, while the system has a small number of degrees of freedom. This allows us to write the system-bath combine state as

$$\rho_{tot}(t) \approx \rho_s(t) \otimes \rho_B. \quad (3.75)$$

This simplifies Eq.(3.74) and gives,

$$\frac{d}{dt}\rho_s(t) = - \int_0^t dt' \text{tr}_B [H^I(t), [H^I(t'), \rho_s(t') \otimes \rho^B]]. \quad (3.76)$$

This can be further simplified using the Markov approximation. This has two parts – we first replace  $\rho_s(t')$  on the RHS of Eq. (3.76) with  $\rho_s(t)$  to make it local in time and hence obtain the *Redfield equation* [144, 145, 140]. We note that the Redfield equation is not Markovian as yet – there is still the dependency on the initial time. To get rid of that, we transfer the initial time back in time to  $t \rightarrow \infty$  and thus remove all information pertaining to the initial state of the system. This entire procedure is the Markov approximation which, when carried out on Eq. (3.76), ultimately yields

$$\frac{d}{dt}\rho(t) = - \int_0^\infty d\tau \text{tr}_B [H^I(t), [H^I(t - \tau), \rho(t) \otimes \rho^B]]. \quad (3.77)$$

This equation is called *Quantum Markovian Master Equation*.

### 3.3.2 Lindblad form of the Master Equation

In general in the Schrödinger picture interaction Hamiltonian can always be written as

$$H_I = \sum_i \mathcal{A}_i \otimes \mathcal{B}_i \quad (3.78)$$

where the  $\mathcal{A}_i$ 's act on the Hilbert space  $\mathcal{H}_S$  corresponding to system  $S$  and  $\mathcal{B}_i$ 's act on Hilbert space  $\mathcal{H}_B$  corresponding to the bath  $B$ . This is analogous to the form of the interaction Hamiltonian  $H_I$  in Sec. 3.3.1. We now project  $\mathcal{A}_i$ 's into the eigenspace of  $H_S$  using the projection operators  $\Pi(\omega)$  and write them as the eigenoperators of  $H_S$ :

$$\mathcal{A}_i(\omega) = \sum_{\epsilon' - \epsilon = \omega} \Pi(\epsilon) \mathcal{A}_i \Pi(\epsilon'). \quad (3.79)$$

Here we have assumed that the eigenvalues  $\epsilon$  of  $H_S$  form a discrete spectrum and thus have a fixed energy gap between the energy levels of  $H_S$  which we have denoted by  $\omega = \epsilon' - \epsilon$ .

Eq. (3.79) leads to the commutation relations

$$[H_S, \mathcal{A}_i(\omega)] = -\omega \mathcal{A}_i(\omega) \quad (3.80)$$

$$[H_S, \mathcal{A}_i^\dagger(\omega)] = \omega \mathcal{A}_i^\dagger(\omega). \quad (3.81)$$

Converting  $\mathcal{A}_i(\omega)$  into the interaction picture Hamiltonian, we have

$$\mathcal{A}_i^I(\omega) = \exp(iH_S t) \mathcal{A}_i(\omega) \exp(-iH_S t) \quad (3.82)$$

Using the Hadamard lemma,

$$\exp(iP)Q \exp(-iP) = Q + [P, Q] + \frac{1}{2!}[P, [P, Q]] + \dots, \quad (3.83)$$



### Chapter 3. Quantum operations and Master equation

we get,

$$\begin{aligned}\mathcal{A}_i^I(\omega) &= \mathcal{A}_i(\omega) - \omega t \mathcal{A}_i(\omega) + \frac{1}{2} \omega^2 t^2 \mathcal{A}_i(\omega) - \dots \\ &= \exp(-i\omega t) \mathcal{A}_i(\omega)\end{aligned}\tag{3.84}$$

$$\begin{aligned}\text{and } \mathcal{A}_i^{I\dagger}(\omega) &= \mathcal{A}_i^\dagger(\omega) + \omega t \mathcal{A}_i^\dagger(\omega) - \frac{1}{2} \omega^2 t^2 \mathcal{A}_i^\dagger(\omega) + \dots \\ &= \exp(i\omega t) \mathcal{A}_i^\dagger(\omega).\end{aligned}\tag{3.85}$$

Now, we note that,

$$\begin{aligned}[H_S, \mathcal{A}_i^\dagger(\omega) \mathcal{A}_i(\omega)] &= 0, \\ \mathcal{A}_i^\dagger &= \mathcal{A}_i(-\omega), \\ \text{and } \sum_{\omega} \mathcal{A}_i(\omega) &= \sum_{\omega} \mathcal{A}_i^\dagger(\omega) = \mathcal{A}_i.\end{aligned}\tag{3.86}$$

Hence we can write the  $H_I$  as,

$$H_I = \sum_{i,\omega} \mathcal{A}_i(\omega) \otimes \mathcal{B}_i = \sum_{i,\omega} \mathcal{A}_i^\dagger(\omega) \otimes \mathcal{B}_i^\dagger\tag{3.87}$$

which in the interaction picture becomes,

$$H^I(t) = \sum_{i,\omega} \exp(-i\omega t) \mathcal{A}_i(\omega) \otimes \mathcal{B}_i(t) = \sum_{i,\omega} \exp(i\omega t) \mathcal{A}_i^\dagger(\omega) \otimes \mathcal{B}_i(t)^\dagger\tag{3.88}$$

where,

$$\mathcal{B}_i(t) = \exp(iH^B t) \mathcal{B}_i \exp(-iH^B t)\tag{3.89}$$

We also note that the condition,

$$\begin{aligned}\text{tr}_B [H^I(t), \rho(0)] &= 0 \\ \text{becomes } \langle \mathcal{B}_i(t) \rangle &= \text{tr}_B \{ \mathcal{B}_i(t) \rho^B \} = 0.\end{aligned}\tag{3.90}$$

Now, we can write the commutator of Eq.(3.77) as,

$$\begin{aligned}
 [H^I(t), [H^I(t - \tau), \rho(t) \otimes \rho^B]] &= \{H^I(t)H^I(t - \tau) (\rho(t) \otimes \rho^B)\} \\
 &\quad - \{H^I(t - \tau) (\rho(t) \otimes \rho^B) H^I(t)\} \\
 &\quad + \{(\rho(t) \otimes \rho^B) H^I(t - \tau)H^I(t)\} \\
 &\quad - \{H^I(t) (\rho(t) \otimes \rho^B) H^I(t - \tau)\} \quad (3.91)
 \end{aligned}$$

We note that since the LHS of the the master equation Eq.(3.77) is Hermitian, the RHS also needs to be Hermitian. This implies that

$$\begin{aligned}
 \frac{d}{dt}\rho^S(t) &= - \int_0^\infty d\tau \text{tr}_B \{H^I(t)H^I(t - \tau) (\rho^S(t) \otimes \rho^B) \\
 &\quad - H^I(t - \tau) (\rho^S(t) \otimes \rho^B) H^I(t) + h.c.\} \quad (3.92)
 \end{aligned}$$

where *h.c.* is the Hermitian conjugate. In terms of the  $\mathcal{A}_i(t)$ s and  $\mathcal{B}_i(t)$ s, we then get,

$$\begin{aligned}
 \frac{d}{dt}\rho^S(t) &= - \int_0^\infty d\tau \left[ \sum_{\omega, \omega'} e^{i(\omega - \omega')t} e^{-i\omega\tau} \text{tr}_B \left\{ \mathcal{A}_j(\omega) \rho^S(t) \mathcal{A}_I^\dagger(\omega') \otimes \mathcal{B}_j(t - \tau) \rho^B \mathcal{B}_i^\dagger(t) \right\} \right. \\
 &\quad \left. - \text{tr}_B \left\{ \mathcal{A}_I^\dagger(\omega') \mathcal{A}_j(\omega) \rho^S(t) \otimes \mathcal{B}_i^\dagger(t) \mathcal{B}_j(t - \tau) \rho^B + h.c. \right\} \right]. \quad (3.93)
 \end{aligned}$$

Since the trace is over the bath, we can write

$$\begin{aligned}
 \frac{d}{dt}\rho^S(t) &= - \sum_{\omega, \omega'} \left\{ \mathcal{A}_j(\omega) \rho^S(t) \mathcal{A}_i^\dagger(\omega') - \mathcal{A}_i^\dagger(\omega') \mathcal{A}_j(\omega) \rho^S(t) \right\} \\
 &\quad \times \int_0^\infty d\tau e^{-i\omega\tau} \text{tr}_B \left\{ \mathcal{B}_i^\dagger(t) \mathcal{B}_j(t - \tau) \rho^B \right\} + h.c. \quad (3.94)
 \end{aligned}$$

where we have used the cyclic properties of trace.

Now,  $\text{tr}_B(\mathcal{B}_i^\dagger(t) \mathcal{B}_j(\tau) \rho^B) = \langle \mathcal{B}_i^\dagger(t) \mathcal{B}_j(\tau) \rangle$ . We define,

$$\Gamma_{ij}(\omega) = \int_0^\infty d\tau e^{-i\omega\tau} \langle \mathcal{B}_i^\dagger(t) \mathcal{B}_j(t - \tau) \rangle \quad (3.95)$$

At this point we apply the *rotating wave approximation* and drop all terms in Eq.(3.94) where  $\omega \neq \omega'$ . This is because they cause fast oscillations and will average

### Chapter 3. Quantum operations and Master equation

out to zero. We get:

$$\frac{d}{dt}\rho^S(t) = \sum_{i,j} \sum_{\omega} \Gamma_{ij}(\omega) \left[ \mathcal{A}_j(\omega)\rho^S(t)\mathcal{A}_i^\dagger(\omega') - \mathcal{A}_i^\dagger(\omega')\mathcal{A}_j(\omega)\rho^S(t) \right] + h.c. \quad (3.96)$$

Since  $\Gamma_{ij}$  is a complex number, we can write,

$$\Gamma_{ij}(\omega) = \frac{1}{2}\gamma_{ij}(\omega) + iS_{ij}(\omega) \quad (3.97)$$

where,  $S_{ij}(\omega)$  is an anti-Hermitian matrix and  $\gamma_{ij}(\omega)$  is positive, defined as,

$$S_{ij}(\omega) = \frac{1}{2i} (\Gamma_{ij}(\omega) - \Gamma_{ji}^*(\omega)) \quad (3.98)$$

$$\begin{aligned} \gamma_{ij}(\omega) &= (\Gamma_{ij}(\omega) + \Gamma_{ji}^*(\omega)) \\ &= \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \langle \mathcal{B}_i^\dagger(\tau)\mathcal{B}_j(0) \rangle \end{aligned} \quad (3.99)$$

Using these substitutions, we can write the Lindblad form of the master equation as,

$$\frac{d}{dt}\rho^S(t) = -i [H_{LS}, \rho^S(t)] + \mathcal{D}[\rho^S(t)] \quad (3.100)$$

where

$$H_{LS} = \sum_{\omega} \sum_{ij} S_{ij}(\omega)\mathcal{A}_i^\dagger(\omega)\mathcal{A}_j(\omega), \text{ so that } [H_S, H_{LS}] = 0$$

$$\text{and } \mathcal{D}[\rho^S(t)] = \sum_{\omega} \sum_{ij} \gamma_{ij}(\omega) \left[ \mathcal{A}_j(\omega)\rho^S(t)\mathcal{A}_i^\dagger(\omega) - \frac{1}{2} \left\{ \mathcal{A}_i^\dagger(\omega)\mathcal{A}_j(\omega), \rho^S(t) \right\} \right]$$

$H_{LS}$  is called *Lamb shift* Hamiltonian. It has the same eigenstates as the free Hamiltonian of the system but the energy levels are shifted. The second term  $\mathcal{D}[\rho^S(t)]$ , is the dissipator in Lindblad form. The second term in the dissipator is responsible for damping in the system. The first term is physically equivalent to a jump (or collapse) process. For the case of a two-level atom interacting with electro magnetic field (EMF),  $\mathcal{A}_i$  is  $\sigma_-$  which causes the system to jump from an excited state to the ground state while  $\mathcal{A}_i^\dagger$  is  $\sigma_+$  which does the opposite.

Since the matrix  $[\gamma_{ij}]$  is a hermitian matrix, it is possible to diagonalize it by

unitary operator  $U$ . Thus

$$D_\gamma = U\gamma U^\dagger, \quad (3.101)$$

where  $D_\gamma$  is a  $\text{diag}(\gamma_1, \gamma_2, \dots, \gamma_d)$  with non-negative entries and the  $\gamma$  is the matrix  $[\gamma_{ij}]$ .

The same unitary transformation will cause the mixing of the  $\mathcal{A}_i$  operators to give rise to new operators  $A_j$ , i.e.,

$$A_j = \sum_i U_{ji} \mathcal{A}_i. \quad (3.102)$$

Hence we can write the final form of the Lindblad dissipator as

$$\mathcal{D}[\rho^S(t)] = \sum_{j,\omega} \gamma_j(\omega) \left[ A_j(\omega) \rho^S(t) A_j^\dagger(\omega) - \frac{1}{2} \left\{ A_j^\dagger(\omega) A_j(\omega), \rho^S(t) \right\} \right]. \quad (3.103)$$

Eq. (3.100) together with the dissipation term  $\mathcal{D}[\rho^S(t)]$ , given in Eq. (3.103), is the Lindblad form of Markovian Master equation, Derives independently by Gorini, Kossakowski, Sudarshan [82] and by Lindblad [115]. State of quantum optical systems, in contact with the environments, often satisfy such type of master equation. As mentioned earlier, evolution under this equation guarantees the complete positivity property.



# 4

## Quantum trajectories

In the previous chapter we discussed various methods – operator sum representation and master equation – to study the dynamics of an open quantum system. In this chapter we discuss a numerical approach to solve the master equation namely, quantum trajectory approach. This technique was first introduced in [43, 163, 44, 45, 42].

In this chapter we will discuss the basics of quantum trajectory approach. We start with the perturbative expansion for the density operator  $\rho$  for the system  $S$  interacting with the bath  $B$ . This is followed by the section on unravelling the Lindblad master equation and the Monte Carlo simulations.

### 4.1 Perturbative expansion for the density operator

Consider the master equation

$$\dot{\rho} = \mathcal{L}\rho \tag{4.1}$$

for the density operator  $\rho$ .  $\mathcal{L}$  is the Liouvillian. In general, the Liouvillian  $\mathcal{L}$  can be written as the sum of an unperturbed part  $\mathcal{L}_0$  and a small perturbation  $S$ , such that

$$\mathcal{L} = \mathcal{L}_0 + S. \tag{4.2}$$

## Chapter 4. Quantum trajectories

The formal solution of Eq. (4.1) is

$$\rho_S(t) = e^{(\mathcal{L}_0+S)t} \rho_S(0). \quad (4.3)$$

For the perturbation expansion of  $\rho(t)$ , consider the operator:

$$\mathcal{A} = \exp((\mathcal{L}_0 + S)t) \quad (4.4)$$

and let

$$\mathcal{B} = e^{-\mathcal{L}_0 t} \mathcal{A}. \quad (4.5)$$

From here we get

$$\begin{aligned} \dot{\mathcal{B}} &= -\mathcal{L}_0 e^{-\mathcal{L}_0 t} \mathcal{A} + e^{-\mathcal{L}_0 t} (\mathcal{L}_0 + S) \mathcal{A} \\ &= e^{-\mathcal{L}_0 t} (S) \mathcal{A} \\ &= e^{-\mathcal{L}_0 t} S e^{\mathcal{L}_0 t} \mathcal{B}. \end{aligned} \quad (4.6)$$

This give rise to a formal solution

$$\mathcal{B}(t) = \mathcal{I} + \int_0^t \dot{\mathcal{B}}(s) ds \quad (4.7)$$

$$\Rightarrow \mathcal{A} = e^{\mathcal{L}_0 t} \left( \mathcal{I} + \int_0^t e^{-\mathcal{L}_0 s} S e^{\mathcal{L}_0 s} \mathcal{B} ds \right). \quad (4.8)$$

Repeating this process give rise to the identity

$$\begin{aligned} \mathcal{A} &= \sum_{m=0}^{\infty} \int_0^t dt_m \int_0^{t_m} dt_{m-1} \cdots \int_0^{t_2} dt_1 \\ &\times e^{\mathcal{L}_0(t-t_m)} S e^{\mathcal{L}_0(t_m-t_{m-1})} S \dots S e^{\mathcal{L}_0 t_1} \end{aligned} \quad (4.9)$$

and hence we get

$$\begin{aligned} e^{(\mathcal{L}_0+S)t} &= \sum_{m=0}^{\infty} \int_0^t dt_m \int_0^{t_m} dt_{m-1} \cdots \int_0^{t_2} dt_1 \\ &\times e^{\mathcal{L}_0(t-t_m)} S e^{\mathcal{L}_0(t_m-t_{m-1})} S \dots S e^{\mathcal{L}_0 t_1} \end{aligned} \quad (4.10)$$

with  $\{t_m\}$  a monotonically increasing sequence.

The integrand in Eq. (4.10) describes a single quantum trajectory for the initial state  $\rho(0)$ . The terms  $\exp[\mathcal{L}_0(t_m - t_{m-1})]$  represents continuous time-evolution in the intervals  $[t_{m-1}, t_m)$ , while  $S$  represents discontinuous quantum jumps at time  $\{t_m\}$ . Eq (4.10) can be interpreted as a generalized sum over all the possible “jump” pathways that the system might follow during its evolution from time  $t = 0$  to time  $t$ .

One can choose different  $\mathcal{L}_0$  and  $S$  depending on the experimental setup or the requirement of the problem. Different choices of  $\mathcal{L}_0$  and  $S$  give rise to different trajectories or unravellings. In the next section we will discuss the quantum trajectory for a specific  $\mathcal{L}_0$  and  $S$  for Lindblad master equation.

## 4.2 Unravelling the Lindblad master equation

Consider the Lindblad form of master equation (see Eq. (3.48)):

$$\begin{aligned}\dot{\rho}(t) &= -i[H, \rho] + \mathcal{D}\rho \\ &= -i[H, \rho] + \sum_j 2\gamma_j F_j \rho F_j^\dagger - \gamma_j [F_j^\dagger F_j \rho + \rho F_j^\dagger F_j].\end{aligned}\quad (4.11)$$

In this equation we choose  $F_i$  to be jump operators. The terms  $F_i^\dagger F_i \rho / 2$  and  $\rho F_i^\dagger F_i / 2$  describes the loss of population from the current states. The term  $F_i \rho F_i^\dagger$  can be understood as the density matrix after the transition described by  $F_i$ ; such transition can be interpreted as “quantum trajectory”.

Now we can write:

$$\mathcal{L}_0 = -i[H, \rho] + \sum_j 2\gamma_j - \gamma_j [F_j^\dagger F_j \rho + \rho F_j^\dagger F_j] \quad (4.12)$$

and

$$S = \sum_i S_i, \quad (4.13)$$



where

$$S_i \rho = F_i \rho F_i^\dagger. \quad (4.14)$$

The unperturbed Liouvillian  $\mathcal{L}_0$  can be written as

$$\mathcal{L}_0 = -i[H_{eff}\rho - \rho H_{eff}], \quad (4.15)$$

where

$$H_{eff} = H - i \sum_j \gamma_j [F_j^\dagger F_j]. \quad (4.16)$$

This non-hermitian Hamiltonian generates non-unitary time-evolution and we can write the non-unitary Schrödinger equation

$$i \frac{d}{dt} |\psi(t)\rangle = H_{eff} |\psi(t)\rangle. \quad (4.17)$$

This confirms that the unperturbed part of the Liouvillian  $\mathcal{L}_0$  give rise to non-unitary coherent time-evolution for the initial state  $\rho(0)$ . This coherent evolution is interrupted by non-deterministic jumps  $S_i$ . The probability of the jump in a particular time interval  $\Delta t$  can be calculated as:

$$\begin{aligned} p_{c,i}(t) &= \text{tr}(S_i \rho_c(t)) \Delta t, \\ &= 2\gamma_i \text{tr}(F_i \rho_c(t) F_i^\dagger) \Delta t, \\ &= 2\gamma_i \langle F_i^\dagger F_i \rangle \Delta t, \end{aligned} \quad (4.18)$$

where  $\rho_c$  is the density matrix in a particular trajectory. It is often called conditioned density operator. The probability to find at least one jump in the time interval  $[t, t + \Delta t)$  due to any of the  $S_i$  is given by:

$$p_c(t) = \sum_i p_{c,i}(t). \quad (4.19)$$

### 4.3 Monte Carlo simulation

Monte Carlo simulation provides the most useful implementation of the quantum trajectory. This section outlines a Monte Carlo algorithm for the generation of stochastic quantum trajectories based on the unravelling of the Lindblad master equation developed thus far.

In the simulation, time is discrete with a time-step  $\Delta t$ . We start with an arbitrary state  $|\Psi(0)\rangle$ . Let  $|\Psi(t)\rangle$  be the state at time  $t$ . Then we

1. evolve the state  $|\Psi(t_{n+1})\rangle$  as:

$$|\Psi(t_{n+1})\rangle = \exp(-iH_{eff}\Delta t)|\Psi(t_n)\rangle. \quad (4.20)$$

2. Normalise the state:

$$|\Psi(t_{n+1})\rangle = \frac{|\Psi(t_{n+1})\rangle}{\sqrt{\langle\Psi(t_{n+1})|\Psi(t_{n+1})\rangle}}. \quad (4.21)$$

3. Calculate the probability for the jump:

$$\begin{aligned} p_{c,i}(t_n) &= \langle\Psi(t_n)|F_i^\dagger F_i|\Psi(t_n)\rangle\Delta t, \\ P_c(t_n) &= \sum_i p_{c,i}(t_n). \end{aligned} \quad (4.22)$$

4. Draw a random number  $r_n$  from a uniform distribution in the interval  $[0, 1)$  and compare it with  $P_c$ .

- (a) If  $P_c(t_n) \geq r_n$ , a jump occurs. The new instantaneous state of the system is  $|\Psi(t_{n+1})\rangle = F_i|\Psi(t_n)\rangle$ .
- (b) If  $P_c(t_n) < r_n$ , go to step 1



# 5

## Channels and states

This chapter is devoted to some special relations between states of a bipartite system and any channel (or bath) acting on one subsystem of it. In the first section we will discuss channel-state duality which is known as Choi-Jamiolkowski [50, 101] isomorphism. This is, as suggested by the name, an isomorphism between the set of completely positive maps and the set of positive operators. This section is followed by a section discussing *factorization law for entanglement decay* [107]. After this we will discuss some special type of channels called *entanglement breaking channels*. We conclude the chapter with some examples of single-qubit channels.

### 5.1 Channel-State Duality

A quantum channel is a channel for the transmission of quantum as well as classical information, and is essentially a completely positive map between spaces of operators. Any such physical channel, acting on a  $d$ -dimensional quantum state, can be mapped to a positive operator in  $d^2$  dimensions. Similarly, a positive operator acting on a  $d^2$ -dimensional Hilbert space can be mapped to a physical channel acting on  $d$  dimensions. Such a two-way mapping between a quantum channel acting on  $d$ -dimensional system and a quantum state in  $d^2$ -dimensions is called channel-state duality [50, 101].

Let us consider two  $d$ -dimensional Hilbert space  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . The set of all bounded linear operators acting on  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are  $\mathcal{B}(\mathcal{H}_1)$  and  $\mathcal{B}(\mathcal{H}_2)$ . Let  $\phi$  be

the map such that:

$$\phi : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2). \quad (5.1)$$

The matrix representation of  $\phi$  can be given by looking at the action of  $\phi$  on a basis in  $\mathcal{B}(\mathcal{H}_1)$ . Choose the orthonormal basis  $\{E_{ij}\}$  of  $\mathcal{B}(\mathcal{H}_1)$ . The  $ij$ -th element of the matrix  $E_{ij}$  is 1 and rest all are zero. The action of  $\phi$  on  $E_{ij}$  can be written as:

$$\phi(E_{ij}) = F_{ij} \quad (5.2)$$

where  $F_{ij} \in \mathcal{B}(\mathcal{H}_2)$ . Thus, the matrix element  $\rho_{\phi ij,kl}$  corresponding to  $\phi$  are:

$$\rho_{\phi ij,kl} = (F_{ij})_{kl} \quad (5.3)$$

$$\Rightarrow \rho_{\phi} = \sum_{ij,kl} E_{ij} \otimes F_{ij} \quad (5.4)$$

$$\Rightarrow \rho_{\phi} = \sum_{ij} E_{ij} \otimes \phi(E_{ij}) \quad (5.5)$$

$$\Rightarrow \rho_{\phi} = (\mathcal{I} \otimes \phi) (|\phi^+\rangle\langle\phi^+|), \quad (5.6)$$

where  $|\phi^+\rangle = \sum_{i=1}^d |ii\rangle/\sqrt{d}$  and  $\mathcal{I}$  is the identity operator. Note that  $|\phi^+\rangle\langle\phi^+| = (\sum_{ij} E_{ij} \otimes E_{ij})/d$ .

If  $\phi$  is a completely positive map then Choi theorem on completely positive map [50] says that the operator  $\rho_{\phi}$  is a positive operator. This is quite straightforward to see that  $\rho_{\phi}$  is a positive operator for completely positive map  $\phi$  from Eq. (5.6). The operator  $|\phi^+\rangle\langle\phi^+| \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_1)$  is a positive operator and  $\phi$  is acting on one side of this operator. Therefore,  $\rho_{\phi}$  is positive. Thus we have established the duality between the channel (the map  $\phi$ ) and the state  $\rho_{\phi}$ .

Making use of channel-state duality, we will derive the factorization law of entanglement decay in the next section.

## 5.2 Factorization law of entanglement decay

One of the focus of this thesis is the dynamics of entanglement. We will be studying a number of systems and the evolution of entanglement in different environments.

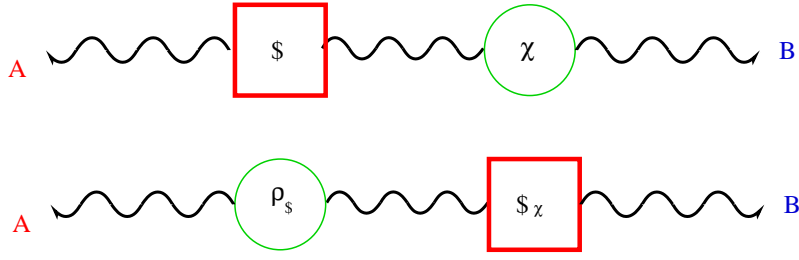


Figure 5.1: Schematic diagram for channel state duality.

It is interesting to note that the dynamics of entanglement in a two-qubit system under the influence of local bath can be fully characterized by its action on a maximally entangled state.

Suppose a bipartite system  $S = A + B$  is subjected to a local bath  $\$$  acting on subsystem  $A$ . The entanglement in state  $\rho_{AB}(t) = (\$ \otimes \mathcal{I})(\rho_{AB}(0))$  at time  $t$  can be written as:

$$\mathcal{C}(\rho_{AB}(t)) = \mathcal{C}(\rho_{AB}(0))\mathcal{C}((\$ \otimes \mathcal{I})(|\phi^+\rangle\langle\phi^+|)) \quad (5.7)$$

where  $\$$  is a single-qubit channel,  $\rho_{AB}(0)$  is a pure two-qubit state and  $\mathcal{C}(\rho_{AB})$  is the concurrence in  $\rho_{AB}$  [25, 91, 179, 181, 180]. This is *factorization law of entanglement decay* [107].

To prove the above equation consider an entangled pair of qubits in state  $|\chi\rangle_{AB}$

$$|\chi\rangle_{AB} = \sqrt{\omega}|00\rangle + \sqrt{1-\omega}|11\rangle \quad (5.8)$$

where  $0 \leq \omega \leq 1$ . The concurrence in the state  $|\chi\rangle_{AB}$  is  $\sqrt{\omega(1-\omega)}$ . The qubit  $A$  is exposed to an arbitrary bath  $\$$ , as shown in Fig. (5.1).

The action of the bath on the state  $|\chi\rangle_{AB}\langle\chi|$  can be written as

$$\rho_{AB} = (\$ \otimes \mathcal{I})(|\chi\rangle_{AB}\langle\chi|). \quad (5.9)$$

The motive is to calculate entanglement in the state  $\rho_{AB}$ . To do so let us turn to the dual picture where two-qubit state  $|\chi\rangle_{AB}\langle\chi|$  acts as a single-qubit channel and

## Chapter 5. Channels and states

a single-qubit bath  $\mathcal{S}$  acts as two-qubit state, i.e,

$$\begin{aligned}
 |\chi\rangle_{AB}\langle\chi| &= (\mathcal{I} \otimes \mathcal{S}_\chi)(|\phi^+\rangle\langle\phi^+|) \\
 \rho_{AB} &= (\mathcal{S} \otimes \mathcal{I})(|\chi\rangle_{AB}\langle\chi|) \\
 \Rightarrow \rho_{AB} &= (\mathcal{S} \otimes \mathcal{S}_\chi)(|\phi^+\rangle\langle\phi^+|) \\
 &= (\mathcal{I} \otimes \mathcal{S}_\chi)(\mathcal{S} \otimes \mathcal{I})(|\phi^+\rangle\langle\phi^+|) \\
 &= (\mathcal{I} \otimes \mathcal{S}_\chi)(\rho_{\mathcal{S}}).
 \end{aligned}$$

Here  $\mathcal{S}_\chi$  is the single-qubit channel corresponding to the two-qubit state  $|\chi\rangle$  and  $\rho_{\mathcal{S}}$  is two-qubit state corresponding to single-qubit channel  $\mathcal{S}$ .

The matrix representation  $\rho_{\mathcal{S}_\chi}$  for  $\mathcal{S}_\chi$  is:

$$\rho_{\mathcal{S}_\chi} = \begin{pmatrix} \omega & 0 & 0 & \sqrt{\omega(1-\omega)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{\omega(1-\omega)} & 0 & 0 & 1-\omega \end{pmatrix}. \quad (5.10)$$

The action of this channel on state  $\rho_{\mathcal{S}}$  can be seen as:

$$\rho_{AB} = (\mathcal{I} \otimes K) \rho_{\mathcal{S}} (\mathcal{I} \otimes K) \quad (5.11)$$

where

$$K = \begin{pmatrix} \sqrt{\omega} & 0 \\ 0 & \sqrt{1-\omega} \end{pmatrix}. \quad (5.12)$$

Setting  $\rho_{AB}$  in this form simplifies the proof of Eq. (5.7). It is now enough to show that the entanglement in  $\rho_{AB}$  is proportional to that in  $\rho_{\mathcal{S}}$ . Consider the matrix  $R = \rho_{AB}\tilde{\rho}_{AB}$ , where  $\tilde{\rho} = (\sigma_2 \otimes \sigma_2)\rho^*(\sigma_2 \otimes \sigma_2)$  and  $\sigma_2$  is the Pauli matrix  $\sigma_y$ . Let the eigenvalues of  $R$  be  $\{\xi_i\}$ . The eigenvalue equation for  $R$  can be written as

$$\det(R - \xi\mathcal{I}) = 0, \quad (5.13)$$

while that for the matrix  $R_{\S} = \rho_{\S} \tilde{\rho}_{\S}$ , with eigenvalues  $\{\mu_i\}$  is

$$\det(R_{\S} - \mu \mathcal{I}) = 0. \quad (5.14)$$

It is interesting to note that

$$\begin{aligned} R &= \rho_{AB}(\sigma_2 \otimes \sigma_2) \rho_{AB}^*(\sigma_2 \otimes \sigma_2) \\ &= (\mathcal{I} \otimes K) \rho_{\S}(\sigma_2 \otimes K \sigma_2 K) \rho_{\S}^*(\sigma_2 \otimes K \sigma_2 K). \end{aligned} \quad (5.15)$$

The matrix  $K$  is invertible as long as  $|\chi\rangle$  is entangled, therefore,

$$(\mathcal{I} \otimes K^{-1}) R (\mathcal{I} \otimes K) = \rho_{\S}(\sigma_2 \otimes K \sigma_2 K) \rho_{\S}^*(\sigma_2 \otimes K \sigma_2 K). \quad (5.16)$$

We have  $K \sigma_2 K = \sqrt{\omega(1-\omega)} \sigma_2$ . Substituting this in the previous equation gives rise to

$$(\mathcal{I} \otimes K^{-1}) R (\mathcal{I} \otimes K) = \omega(1-\omega) R_{\S}. \quad (5.17)$$

Since a similarity transformation does not change the eigenvalues of a matrix, the eigenvalues of  $(\mathcal{I} \otimes K^{-1}) R (\mathcal{I} \otimes K)$  are the same as  $R$  and are equal to the eigenvalues of  $R_{\S}$  times  $\omega(1-\omega)$ . The entanglement in  $\rho_{AB}$  can be calculated as

$$\begin{aligned} \mathcal{C}(\rho_{AB}) &= \max\{0, \sqrt{\xi_1} - \sqrt{\xi_2} - \sqrt{\xi_3} - \sqrt{\xi_4}\} \\ &= \sqrt{\omega(1-\omega)} \max\{0, \sqrt{\mu_1} - \sqrt{\mu_2} - \sqrt{\mu_3} - \sqrt{\mu_4}\} \\ &= \sqrt{\omega(1-\omega)} \mathcal{C}(\rho_{\S}) \end{aligned} \quad (5.18)$$

This completes the proof for a pure state  $|\chi\rangle$ .

### 5.3 Entanglement breaking channels

For some systems a study of the dynamics of entanglement requires knowledge of the separability of the output state. This can be estimated using a family of channels called *entanglement breaking channels* [97, 124, 108, 94]. These are:

**Definition 5** *A stochastic map  $\phi$  is called an entanglement breaking map if  $(\mathcal{I} \otimes$*



$\phi)(\rho)$  is always separable, i.e., any bipartite entangled density matrix  $\rho$  is mapped to a separable state.

The following is a simple criterion to check whether a given map is of the entanglement breaking type.

**Theorem 8** *A map is entanglement breaking if and only if it can be written as [97, 94, 108, 124]:*

$$\phi(\sigma) = \sum_k |\psi_k\rangle\langle\psi_k| \langle\phi_k, \sigma\phi_k\rangle, \quad (5.19)$$

where  $\sigma$  is a  $d$  dimensional density operator and  $|\psi_k\rangle$  and  $|\phi_k\rangle$  are normalized states.

The Kraus operators for these kind of maps can be written as:

$$A_k = |\psi_k\rangle\langle\phi_k| \quad (5.20)$$

and  $\rho_\phi$  is

$$\rho_\phi = \sum_k |\psi_k\phi_k\rangle\langle\psi_k\phi_k|, \quad (5.21)$$

where  $|\psi_k\phi_k\rangle = |\psi_k\rangle \otimes |\phi_k\rangle$ . Therefore, the  $\rho_\phi$  matrix corresponding to an entanglement breaking map is always separable and vice versa, i.e, if  $\rho_\phi$  matrix corresponding to a channel is separable, the corresponding channel is of the entanglement breaking type. This result is useful for the study of entanglement sudden death in open quantum systems.

In the next section we discuss some examples of single-qubit channels and the Kraus operators corresponding to those channels.

## 5.4 Example of qubit channels

### 5.4.1 Depolarizing channels

A depolarizing channel  $\Lambda_{dp}$  is a quantum channel in which a density matrix  $\rho$  can either be mapped to a maximally mixed state  $\mathcal{I}/2$  with a probability  $p$ , or can

maintain status quo with probability  $(1 - p)$  [130, 63]. Therefore, for a qubit,

$$\Lambda_{dp} : \rho \rightarrow \tilde{\rho} = p \frac{\mathcal{I}}{2} + (1 - p)\rho, \quad (5.22)$$

for  $0 \leq p \leq 1$ . This channel can be written in the operator sum representation as:

$$\tilde{\rho} = \left(1 - \frac{3p}{4}\right) \rho + \frac{p}{3}(\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z), \quad (5.23)$$

where  $\sigma_{x,y,z}$  are the Pauli matrices and  $\left\{\sqrt{1 - 3p/4}\mathcal{I}, \sqrt{p/3}\sigma_{x,y,z}\right\}$  are the Kraus operators corresponding to this channel. And the  $\rho_{dp}$  for this channel can be written as:

$$\rho_{dp} = \begin{pmatrix} 1 - \frac{p}{2} & 0 & 0 & 1 - p \\ 0 & \frac{p}{2} & 0 & 0 \\ 0 & 0 & \frac{p}{2} & 0 \\ 1 - p & 0 & 0 & 1 - \frac{p}{2} \end{pmatrix}. \quad (5.24)$$

Note that the depolarizing channel becomes entanglement breaking channel when  $\rho_{dp}$  is separable, i.e,  $p \geq 2/3$ .

### 5.4.2 Amplitude damping channel

Processes such as spontaneous emission or dissipation are characterized by a QO known as an *amplitude damping channel*. The operator sum representation for this qubit channel can be written as:

$$\tilde{\rho} = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger, \quad (5.25)$$

where

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \gamma} \end{pmatrix},$$

$$E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}.$$

The corresponding  $\rho_{ad}$  for this channel is:

$$\rho_{ad} = \begin{pmatrix} 1 & 0 & 0 & \sqrt{1-\gamma} \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1-\gamma} & 0 & 0 & 1-\gamma \end{pmatrix}. \quad (5.26)$$

The amplitude damping channel becomes entanglement breaking only at  $\gamma = 1$ .

### 5.4.3 Phase damping channel

A phase damping channel destroys quantum information without affecting the energy of the system. This is possible because, in this channel the system hamiltonian commutes with the interaction hamiltonian  $\mathcal{H}_I$ . This channel is special in that it has no classical analog. For the case of a qubit, a phase damping channel corresponds to the decay of off-diagonal terms. Also, the Kraus operators for this channels are diagonal operators, and can be written as:

$$F_0 = \sqrt{\beta}\mathcal{I}, \\ F_{1,2} = \sqrt{1-\beta}(\mathcal{I} \pm \sigma_z)$$

The  $\rho_{pd}$  for this channel is:

$$\rho_{pd} = \begin{pmatrix} 1 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \beta & 0 & 0 & 1 \end{pmatrix}. \quad (5.27)$$

This channel becomes entanglement breaking only at  $\beta = 0$ .

In this chapter we have discussed the relation between channels and quantum states. Using that relation we have discussed the factorization law of entanglement decay in two-qubit systems and we have discussed some special channels for example entanglement breaking channels, depolarizing channels, amplitude damping channels and phase damping channels.

# 6

## Non-Markovian Dynamics

We have until now restricted ourselves to a description of Markovian dynamics, which is characterized by a memoryless bath. The master equation approach discussed in Chapter (3), and the quantum trajectory approach discussed in chapter (4), are both examples of Markovian dynamics. However, a Markovian bath is an approximation rather than a norm in real-world scenarios, and as such, it becomes necessary to consider non-Markovian dynamics as well.

This chapter will be devoted to giving the readers a flavor of some of the techniques used when dealing with non-Markovian evolution. We will review two such techniques : the Nakajima-Zwanzig technique [126, 192] and the time convolutionless technique [152, 48, 151], both of which use projection operators and lead to exact dynamical equations for the system density operator. We will conclude the chapter with a brief discussion on the post-Markovian dynamical equation [150, 120], which describes a dynamical process that falls, in character, between Markovian and exact dynamics.

### 6.1 The Nakajima-Zwanzig technique

Define two super-operators  $\mathcal{P}$  and  $\mathcal{Q}$  as:

$$\mathcal{P}\rho = \text{tr}_B\{\rho\} \otimes \rho_B \equiv \rho_S \otimes \rho_B, \quad (6.1)$$

$$\mathcal{Q}\rho = \rho - \mathcal{P}\rho. \quad (6.2)$$

## Chapter 6. Non-Markovian Dynamics

where  $\rho_S$  is the system density operator,  $\rho_B$  is the bath density operator and  $\rho$  is the density operator for system-bath combine.

In the literature,  $\mathcal{P}\rho$  is called the *relevant* part of the density operator and  $\mathcal{Q}\rho$  is considered to be the *irrelevant* part of the density operator. We note that  $\mathcal{P} + \mathcal{Q} = \mathbb{I}$ ,  $\mathcal{P}^2 = \mathcal{P}$  and  $\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = 0$ .

The Hamiltonian  $H$  of the system-bath combine is written as:

$$H = H_0 + \alpha H_I, \quad (6.3)$$

where  $H_0$  is the free Hamiltonian of the system and the bath,  $H_I$  is the interaction Hamiltonian between the two and  $\alpha$  is the strength of the interaction. The dynamical equation for the density operator  $\rho(t)$  for the system-bath combine can be written in the interaction picture as:

$$\frac{d\rho(t)}{dt} = -i\alpha[H_I(t), \rho(t)] \equiv \alpha\mathcal{L}(t)\rho(t), \quad (6.4)$$

where  $H_I(t) = \exp(iH_0t)H_I \exp(-iH_0t)$  is the interaction Hamiltonian in the interaction picture. Our aim is to find the equation of motion of the relevant part  $\mathcal{P}\rho(t)$ . The following relations are true for both the relevant and irrelevant parts:

$$\mathcal{P}\frac{d\rho(t)}{dt} = \frac{d\mathcal{P}\rho(t)}{dt} = \alpha\mathcal{P}\mathcal{L}(t)\rho(t), \quad (6.5)$$

$$\mathcal{Q}\frac{d\rho(t)}{dt} = \frac{d\mathcal{Q}\rho(t)}{dt} = \alpha\mathcal{Q}\mathcal{L}(t)\rho(t). \quad (6.6)$$

Substituting the identity  $\mathcal{P} + \mathcal{Q} = \mathbb{I}$  in these equations results in:

$$\frac{d\mathcal{P}\rho(t)}{dt} = \alpha\mathcal{P}\mathcal{L}(t)\mathcal{P}\rho(t) + \alpha\mathcal{P}\mathcal{L}(t)\mathcal{Q}\rho(t), \quad (6.7)$$

$$\frac{d\mathcal{Q}\rho(t)}{dt} = \alpha\mathcal{Q}\mathcal{L}(t)\mathcal{Q}\rho(t) + \alpha\mathcal{Q}\mathcal{L}(t)\mathcal{P}\rho(t). \quad (6.8)$$

We need to eliminate the irrelevant part to get the equation of motion for the relevant part in decoupled form. We note that the Eq.(6.8) is a non-homogeneous differential equation of the form:

$$\left( \frac{d}{dt} - M(t) \right) X(t) = Z(t), \quad (6.9)$$

where  $M(t) = \alpha \mathcal{Q}\mathcal{L}(t)$ ,  $X(t) = \mathcal{Q}\rho(t)$  and  $Z(t) = \mathcal{P}\rho(t)$ . We solve this equation using Green's function

$$\mathcal{Q}\rho(t) = \mathcal{G}(t, t_0)\mathcal{Q}\rho(t_0) + \alpha \int_s^t \mathcal{G}(t, s)\mathcal{Q}\mathcal{L}(s)\mathcal{P}\rho(s), \quad (6.10)$$

where

$$\mathcal{G}(t, s) = T_{\leftarrow} \exp \left\{ \alpha \int_s^t ds' \mathcal{Q}\mathcal{L}(s') \right\}. \quad (6.11)$$

Here  $T_{\leftarrow}$  represents the time ordering. Substituting Eq.(6.10) into Eq.(6.7) results in

$$\frac{d\mathcal{P}\rho(t)}{dt} = \alpha \mathcal{P}\mathcal{L}(t)\mathcal{P}\rho(t) + \alpha \mathcal{P}\mathcal{L}(t)\mathcal{G}(t, t_0)\mathcal{Q}\rho(t_0) + \alpha^2 \int_s^t \mathcal{P}\mathcal{L}(t)\mathcal{G}(t, s)\mathcal{Q}\mathcal{L}(s)\mathcal{P}\rho(s). \quad (6.12)$$

This is the Nakajima-Zwanzig (NZ) equation and is an exact equation of motion for the dynamics of the relevant part. The term  $\alpha^2 \int_s^t \mathcal{P}\mathcal{L}(t)\mathcal{G}(t, s)\mathcal{Q}\mathcal{L}(s)\mathcal{P}$  is called the convolution or memory kernel and is represented by  $\mathcal{K}(t, s)$ . This equation consists of a term containing information about the initial state as well as a time-integral that describes the history of the evolution and as such describes the complete non-Markovian dynamics. However, NZ equations are often quite difficult to solve analytically as well as numerically. In the next section we will discuss a different approach that simplifies the solution of the NZ equation.

## 6.2 Time convolutionless technique

The time-convolutionless (TCL) technique simplifies the NZ equation (Eq. 6.12) by removing its dependency on history and thus making it local in time. Let us define a backward propagator  $G(t, s)$  which maps the state  $\rho(t)$  to the state  $\rho(s)$  where  $s < t$ . Therefore, we can write

$$\rho(s) = G(t, s)\rho(t) = G(t, s)\mathcal{P} + \mathcal{Q}\rho(t), \quad (6.13)$$

$$\text{where } G(t, s) = T_{\rightarrow} \exp \left\{ -\alpha \int_s^t ds' \mathcal{L}(s') \right\}. \quad (6.14)$$

Substituting this in Eq.(6.10) results in

$$\begin{aligned} \mathcal{Q}\rho(t) &= \mathcal{G}(t, t_0)\mathcal{Q}\rho(t_0) \\ &\quad + \alpha \int_s^t \mathcal{G}(t, s)\mathcal{Q}\mathcal{L}(s)\mathcal{P}\mathcal{G}(t, s)(\mathcal{P} + \mathcal{Q})\rho(t), \end{aligned} \quad (6.15)$$

$$= \mathcal{G}(t, t_0)\mathcal{Q}\rho(t_0) + \Sigma(t)(\mathcal{P} + \mathcal{Q})\rho(t), \quad (6.16)$$

$$\Rightarrow [1 - \Sigma(t)]\mathcal{Q}\rho(t) = \mathcal{G}(t, t_0)\mathcal{Q}\rho(t_0) + \Sigma(t)\mathcal{P}\rho(t). \quad (6.17)$$

Note that, for small enough  $\alpha$ , the operator  $1 + \Sigma(t)$  is invertible. Thus we can write:

$$\mathcal{Q}\rho(t) = [1 - \Sigma(t)]^{-1}\mathcal{G}(t, t_0)\mathcal{Q}\rho(t_0) + [1 - \Sigma(t)]^{-1}\Sigma(t)\mathcal{P}\rho(t). \quad (6.18)$$

Substituting this in Eq.(6.7) gives:

$$\begin{aligned} \frac{d\mathcal{P}\rho(t)}{dt} &= \alpha\mathcal{P}\mathcal{L}(t)\mathcal{P}\rho(t) + \alpha\mathcal{P}\mathcal{L}(t)[1 - \Sigma(t)]^{-1}\mathcal{G}(t, t_0)\mathcal{Q}\rho(t_0) \\ &\quad + \alpha\mathcal{P}\mathcal{L}(t)[1 - \Sigma(t)]^{-1}\Sigma(t)\mathcal{P}\rho(t), \end{aligned} \quad (6.19)$$

$$= \alpha\mathcal{P}\mathcal{L}(t)\mathcal{P}\rho(t) + \mathcal{I}(t)\mathcal{Q}\rho(t_0) + \mathcal{K}(t)\mathcal{P}\rho(t), \quad (6.20)$$

where  $\mathcal{K}(t) = \alpha\mathcal{P}\mathcal{L}(t)[1 - \Sigma(t)]^{-1}\Sigma(t)\mathcal{P}$  is the TCL generator and  $\mathcal{I} = \alpha\mathcal{P}\mathcal{L}(t)[1 - \Sigma(t)]^{-1}\mathcal{G}(t, t_0)\mathcal{Q}$  is the inhomogeneity. If the initial state is uncorrelated, that is, if  $\rho(t_0) = \rho_S(t_0) \otimes \rho_B$ , then  $\mathcal{Q}\rho(t_0) = 0$ , and thus the inhomogeneous part of the NZ equation vanishes.

As an application of this technique we will discuss an exact solution for spontaneous decay of two-level atom in the next section.

### 6.3 Spontaneous decay of two-level atom: Exact master equation

The spontaneous decay of a two-level atom is one of the rare examples where it is possible to write down an exact master equation. This was first derived by Garraway [79]. In this we consider a two-level atom interacting with a bath of

harmonic oscillators. The Hamiltonian for such a setup is:

$$H = H_S + H_B + H_I, \quad (6.21)$$

$$H_S = \omega \sigma_z, \quad (6.22)$$

$$H_B = \sum_k \omega_k a_k^\dagger a_k, \quad (6.23)$$

$$H_I = \sigma_+ \otimes B + \sigma_- \otimes B^\dagger, \quad (6.24)$$

$$B = \sum_k g_k a_k. \quad (6.25)$$

In the Garraway solution, one considers three type of states: a) when both the system and the bath are in their respective ground states, b) when the system is in its first excited state and the bath is in its ground state and c) when the system is in its ground state and the bath is in its first excited state with one photon in it. Mathematically we can write

$$|\psi_0\rangle = |0\rangle_S \otimes |0\rangle_B, \quad (6.26)$$

$$|\psi_1\rangle = |1\rangle_S \otimes |0\rangle_B, \quad (6.27)$$

$$|\psi_k\rangle = |0\rangle_S \otimes |k\rangle_B, \quad (6.28)$$

where  $|0\rangle_S$  ( $|1\rangle_S$ ) is the ground state (excited state) of the system and  $|k\rangle_B$  denotes the state with one photon in  $k$ -th mode. The dynamical equation can be written in interaction picture as:

$$\frac{d}{dt} |\phi(t)\rangle = -i H_I(t) |\phi(t)\rangle, \quad (6.29)$$

where  $H_I(t) = \exp(iH_0 t) H_I \exp(-iH_0 t)$ .

Since the number operator  $N = \sigma_+ \sigma_- + \sum_k a_k^\dagger a_k$  commutes with the total Hamiltonian, the subspaces with different number of particles in it will remain invariant. Thus a state of the form

$$|\phi(0)\rangle = c_0 |\psi_0\rangle + c_1(0) |\psi_1\rangle + \sum_k c_k(0) |\psi_k\rangle \quad (6.30)$$



## Chapter 6. Non-Markovian Dynamics

will evolve into

$$|\phi(t)\rangle = c_0|\psi_0\rangle + c_1(t)|\psi_1\rangle + \sum_k c_k(t)|\psi_k\rangle \quad (6.31)$$

From Eq.(6.29) and Eq.(6.31) we can write

$$\dot{c}_1(t) = -i \sum_k g_k \exp\{i(\omega - \omega_k)t\} c_k(t), \quad (6.32)$$

$$\dot{c}_k(t) = -i g_k^* \exp\{-i(\omega - \omega_k)t\} c_1(t). \quad (6.33)$$

If we assume that initially the bath was in ground state, i.e,  $c_K(0) = 0$ , then the solution of Eq.(6.33) can be written as:

$$c_k(t) = -i g_k^* \int_0^t ds \exp\{-i(\omega - \omega_k)s\} c_1(s). \quad (6.34)$$

Substituting this in Eq.(6.32) results in

$$\dot{c}_1(t) = - \int_0^t dt \sum_k g_k^* g_k \exp\{i(\omega - \omega_k)(t - s)\} c_1(s), \quad (6.35)$$

$$= - \int_0^t ds f(t - s) c_1(s), \quad (6.36)$$

where

$$f(t - s) = \sum_k g_k^* g_k \exp\{i(\omega - \omega_k)(t - s)\} = \text{tr}_B\{B(t)B^\dagger(s)\rho_B\} \exp\{i\omega(t - s)\}, \quad (6.37)$$

$$B(t) = \sum_k g_k a_k \exp\{-i\omega_k t\}, \quad (6.38)$$

$$\rho_B = |0\rangle\langle 0|. \quad (6.39)$$

Using Eq.(6.31), we can write the reduced density operator for the system:

$$\rho_S(t) = \text{tr}_B\{|\phi(t)\rangle\langle\phi(t)|\} = \begin{pmatrix} |c_1(t)|^2 & c_0^* c_1(t) \\ c_0 c_1^*(t) & 1 - |c_1(t)|^2 \end{pmatrix}. \quad (6.40)$$

From here we get the dynamical equation

$$\dot{\rho}_S(t) = \begin{pmatrix} \frac{d}{dt}|c_1(t)|^2 & c_0^* \dot{c}_1(t) \\ c_0 \dot{c}_1^*(t) & -\frac{d}{dt}|c_1(t)|^2 \end{pmatrix}, \quad (6.41)$$

$$= -\frac{i}{2}S(t)[\sigma_+\sigma_-, \rho_S(t)] \\ + \gamma(t) \left\{ \sigma_-\rho_S(t)\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho_S(t) - \frac{1}{2}\rho_S(t)\sigma_+\sigma_- \right\}, \quad (6.42)$$

where  $S(t)$  is the time-dependent Lamb shift and  $\gamma(t)$  is the decay rate, and are given by:

$$S(t) = -2\mathcal{I} \left( \frac{\dot{c}_1(t)}{c_1(t)} \right), \quad (6.43)$$

$$\gamma(t) = -2\mathcal{R} \left( \frac{\dot{c}_1(t)}{c_1(t)} \right). \quad (6.44)$$

Eq.(6.42) is in the Lindblad form and describes Markovian dynamics as long as the decay rate is positive. However, it is not altogether impossible in a physical scenario for the decay rate to become negative, in which case the process deviates from Markovianity.

We note that Eq.(6.42) is local in time and hence can be written as:

$$\dot{\rho}_S(t) = \mathcal{K}_S(t)\rho_S(t), \quad (6.45)$$

where  $\mathcal{K}_S(t)$  is the TCL generator. This form of the dynamical equation will be used in the following chapters.

## 6.4 Post-Markovian processes

Any Markovian master equation, being of Lindblad type, always gives rise to a completely positive dynamics for the density matrix of the system. Unfortunately most of the non-Markovian master equations which are potentially solvable, do not satisfy this criterion, as these equations are generated mostly from some phenomenological aspects. Shabani and Lidar [150] recently derived, from measurement perspective, a non-Markovian master equation which satisfies complete pos-

itivity as well as it is solvable.

Dynamical evolution in the measurement picture can be described by two contrasting viewpoints. In Markovian dynamics, evolution is followed by continuous measurements on the bath, and this interpretation leads to the quantum jump approach to the solution of the dynamics (see Chapter (4)). On the other hand, exact dynamics (given by operator sum representation) involves a single measurement on the bath in the end of the evolution. Post-Markovian dynamics (as described in [150]) falls somewhere between these two extremes, in that here the measurement is carried out at some intermediate time  $t'$  with the probability determined by the memory kernel  $k(t', t)$ . Let us consider the process where the system and the bath evolve jointly for some time  $t'$  so that the state of the system at this time is  $\Lambda(t')\rho(0)$ , where  $\Lambda$  is the dynamical map. Let, at this moment, a non-selective measurement be performed on the bath degree of freedom, after which the system and the bath resume their evolution till time  $t$ , when the final measurement is applied. The dynamical equation for such a process (under the assumption that  $\Lambda(t')$  comes from Lindblad generator  $\mathcal{L}$  and  $k(t', t) \equiv k(t')$ ) can be written as:

$$\frac{d\rho}{dt} = \mathcal{L} \int_0^t dt' k(t') \exp(\mathcal{L}t') \rho(t - t'), \quad (6.46)$$

where  $\Lambda(t) = \exp(\mathcal{L}t)$  and  $k(t') = k(t', t)$  is the memory kernel.

Using Laplace transform of both sides of Eq.(6.46) and using thereby the notion of damping basis [37], one can show that this equation gives rise to a completely positive trace preserving map  $\Phi(t)(\rho_s(0)) = \rho_s(t)$ . Moreover, this equation can also be cast in the form of NZ equation as well as in the form of time convolutionless master equation.

Comparing the solution of the post-Markovian master equation Eq.(6.46) for a two-level atom interacting resonantly with a quantized mode of an empty high-Q cavity with that of the exact solution given by Garraway [79] and that of a memory kernel non-Markovian master equation (not completely positive). Maniscalco and Petruccione [120] have shown that in the weak coupling limit, the solution of the post-Markovian master equation approximates the exact solution in much better way compared to the case of strong coupling limit. So, although the post-Markovian master equation is completely positive, it does not provide, in general, a good approximation to the exact dynamics.

Part II

Applications



# 7

## Entanglement dynamics: quantum trajectory approach

This chapter is devoted to the entanglement dynamics from quantum trajectory point of view. As we have already discussed (see Sec. 4), quantum trajectory technique is a numerical approach to the dynamics of an open quantum system. In this approach a state of a system evolves in a non-unitary coherent way caused by non-hermitian Hamiltonian  $H_{eff}$ . The coherent evolution is interrupted by non-deterministic jumps  $S_i$ . The non-deterministic nature of jumps causes a fixed initial state  $|\psi\rangle$  to evolve to different final states. This approach mimics the real physical systems subject to constant supervision. Therefore, the different final states are the states which we observe in experiments.

Before applying the quantum trajectory approach to study the evolution of the entanglement, we need to settle down the measure of entanglement. A mixed state  $\rho$  can be decomposed into an ensemble of pure states in an infinitely many ways:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad (7.1)$$

where  $p_i \geq 0$  and  $\sum_i p_i = 1$ . For a given ensemble, the entanglement may be quantified as:

$$E(\rho) = \sum_i p_i E(|\psi_i\rangle\langle\psi_i|). \quad (7.2)$$

For bipartite pure states the von-Neumann entropy of the reduced density operator

of one of the subsystem is a good measure of entanglement. Thus we can choose  $E(|\psi_i\rangle\langle\psi_i|)$  to be that measure.

The entanglement of formation (EoF) is defined as:

$$EoF(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle\langle\psi_i|), \quad (7.3)$$

i.e, the ensemble average entanglement minimized over all decomposition. In [129] the authors took a different approach. Instead of taking the minimal decomposition, i.e, the decomposition that gives rise to the minimum entanglement, they fixed the decomposition and calculated the ensemble averaged entanglement in that fixed decomposition. The decomposition is fixed by the particular unravelling we choose in the quantum trajectory.

In this chapter we will follow the work in [129]. The system under consideration [129] consists of a harmonic oscillator  $A$  coupled to a two-level atom  $B$ . The atom is driven by a resonant external field. Both the systems  $A$  and  $B$  are exposed to thermal baths. The total Hamiltonian of the system bath combine is:

$$H = H_{AB} + H_{ext}^B + H_{res} \quad (7.4)$$

where

$$H_{AB} = H_A + H_B + H_I, \quad (7.5)$$

$$\text{with } H_A = \hbar\omega a^\dagger a, \quad (7.6)$$

$$H_B = \hbar\omega\sigma_z \quad (7.7)$$

$$\text{and } H_I = i\hbar g [a^\dagger\sigma^- - a\sigma^+], \quad (7.8)$$

is the free Hamiltonian the system  $S = A + B$  and

$$H_{ext}^B = i\hbar\Omega [\sigma^+ e^{-i\omega t} - \sigma^- e^{i\omega t}] \quad (7.9)$$

is the interaction Hamiltonian of the atom with the external field and

$$H_{res} = H_{R_A} + H_{R_B} + H_{A R_A} + H_{B R_B} \quad (7.10)$$

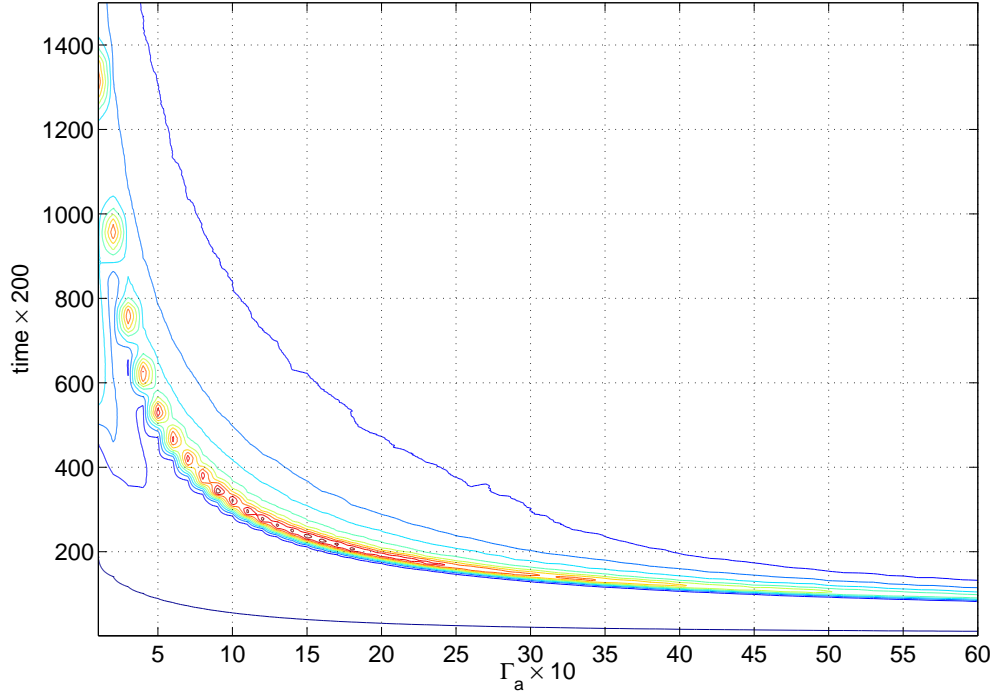


Figure 7.1: The contour plot for entanglement dynamics. On  $x$ -axis we have  $\Gamma_A$  and time is on  $y$ -axis. The amount of entanglement depends on the color of the contour. It increases from blue to red.

is the Hamiltonian of the two reservoirs attached to the harmonic oscillator and the atom and their interactions with reservoir. The Lindblad master equation for the system can be written as:

$$\frac{d\rho}{dt} = -i \left[ \tilde{H}_{AB} + \tilde{H}_{ext}^B, \rho \right] + (\mathcal{L}_A + \mathcal{L}_B)\rho, \quad (7.11)$$

where  $\tilde{H}_{AB} = i(a^\dagger \sigma^- - a \sigma^+)$ ,  $\tilde{H}_{ext}^B = i\hbar\Omega(\sigma^+ - \sigma^-)/g$  and  $\mathcal{L}_{A,B} = \Gamma_{A,B}(2o.o^\dagger - o^\dagger o - .o^\dagger o)$  where ( $o = a, \sigma^-$  for  $A$  and  $B$  respectively) and  $\Gamma_{A,B}$  determines the reservoir interaction strength.

For the purpose of studying the entanglement dynamics in this system we choose the initial state of the system to be  $|\Omega\rangle|g\rangle = |\Omega\rangle \otimes |g\rangle$ . Here  $|\Omega\rangle$  is the coherent state of harmonic oscillator with eigenvalue  $\Omega$  and  $|g\rangle$  is the ground state of the atom. This is a product state and happens to be an eigenstate of the



$\tilde{H}_{AB} + \tilde{H}_{ext}^B$ . Since the atom is in ground state, in the case where harmonic oscillator does not interact with the bath, i.e,  $\Gamma_A = 0$ , the state  $|\Omega\rangle|g\rangle$  will remain unchanged. Thus, we are interested in the case when  $\Gamma_A \neq 0$ .

The dynamics of entanglement for a range of  $\Gamma_A$  can be seen in the contour plot (7.1). This particular system shows an interesting entanglement dynamics in which first the amount of entanglement increases with time for fixed parameters ( $\Gamma_A$  and  $\Gamma_B$ ) and then the entanglement goes to zero asymptotically since the steady state of the system is a separable one.

# 8

## Entanglement dynamics in finite dimensional systems

Entanglement is considered to be one of the most useful resources in Quantum Information Theory [130] and is essential for various quantum information tasks such as quantum teleportation, superdense coding, communication complexity problems and one-way computation. Creating an entangled state is a non-trivial task; however, storing or transmitting entangled states are much more difficult, if not altogether impossible. This is due to the very fragile nature of quantum systems – every quantum system has a high probability of interacting with its surrounding and thereby getting entangled with it. This gives rise to the phenomenon of decoherence [191].

In general, the purity of any initial state of the quantum system decreases with time in the presence of decoherence. The time for the complete loss of coherence (i.e, decoherence time) depends on the system as well as on the character of the interaction of the system with its environment. By the monogamy property of entanglement [52, 131], the initial entanglement (if any) of a bipartite or multipartite quantum system will, in general, decay (to zero) when each individual system undergoes a decoherence procedure. What can be said about the associated rate of the above-mentioned decay in entanglement? How does one compare the rate of decoherence of the individual subsystems and the rate of decay in initial entanglement among the subsystems? In this connection, Yu and Eberly [165] described a phenomenon called *entanglement sudden death (ESD)* in which the entanglement decay rate is shown to be exponentially larger than the rate of decoherence. This

happens when the individual qubits of a two-qubit system undergo evolution under local Markovian heat bath action at zero temperature.

The monotonic decay of entanglement is absent in the non-Markovian dynamics. In such dynamics we can see the ESD as well as rebirth of entanglement [18, 55, 116]. This happens because of the back action of the system on the bath which is suppressed in the Markovian evolutions.

In this chapter we will study the evolution of finite dimensional quantum systems which are initially entangled and are under the influence of thermal, squeezed thermal or quantum non-demolition baths. We shall be focusing on the time at which the entanglement in the system vanishes and we will show explicit calculations for the time to ESD [86]. Further we will discuss the entanglement dynamics in the non-Markovian evolution and study the phenomenon of rebirth of entanglement.

## 8.1 Master equation for single-qubit system

Let us consider a single-qubit system interacting with a bath of infinitely many non-interacting harmonic oscillators at temperature  $T$ . The Hamiltonian for this setting can be written as:

$$H = H_0 + H_I, \quad (8.1)$$

where

$$H_0 = \omega_0 \sigma_z + \sum_k \omega_k a_k^\dagger a_k \quad (8.2)$$

is the free Hamiltonian for the combined system of qubit and harmonic oscillators and  $H_I$  represents the interaction between the qubit and the bath. In dipole approximation, this interaction Hamiltonian can be taken as:

$$H_I = \sum_k \left( g \sigma^- a_k^\dagger + g^* \sigma^+ a_k \right), \quad (8.3)$$

which automatically satisfies the rotating wave approximation.

The irreversible time evolution of the state  $\rho(t)$  of a single-qubit under the

Hamiltonian  $H$  is described by the Lindblad master equation [115]:

$$\begin{aligned} \frac{d\rho}{dt} = & \frac{(N+1)\gamma}{2} [2\sigma_- \rho \sigma_+ - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_-] \\ & + \frac{N\gamma}{2} [2\sigma_+ \rho \sigma_- - \sigma_- \sigma_+ \rho - \rho \sigma_- \sigma_+], \end{aligned} \quad (8.4)$$

where  $N = 1/(e^{\beta\omega_0/k_B T} - 1)$  is the mean occupation number of quanta in the reservoir,  $\gamma$  is the spontaneous decay rate of the qubits,  $\sigma_+ = |1\rangle\langle 0|$  and  $\sigma_- = |0\rangle\langle 1|$  where  $\sigma_z|0\rangle = -|0\rangle$  and  $\sigma_z|1\rangle = |1\rangle$ . Here we have ignored the unitary part of the evolution which is irrelevant for our purpose [7, 33]. We can rewrite Eq. (8.4) as:

$$\dot{\rho} = \mathcal{L}\rho \Rightarrow \dot{\rho}_{ij} = \sum_{kl} L_{ij,kl} \rho_{kl} \quad (8.5)$$

where  $L$  is the matrix representation for  $\mathcal{L}$  (called Lindblad operator [115]).  $L$  can be calculated from the Eq. (8.4) as:

$$\begin{aligned} L = & \frac{(N+1)\gamma}{2} [2\sigma_- \otimes \sigma_- - \sigma_+ \sigma_- \otimes \mathcal{I} - \mathcal{I} \otimes \sigma_+ \sigma_-] \\ & + \frac{N\gamma}{2} [2\sigma_+ \otimes \sigma_+ - \sigma_- \sigma_+ \otimes \mathcal{I} - \mathcal{I} \otimes \sigma_- \sigma_+], \end{aligned} \quad (8.6)$$

$$= \begin{pmatrix} -(N+1)\gamma & 0 & 0 & N\gamma \\ 0 & -\frac{\gamma}{2}(2N+1) & 0 & 0 \\ 0 & 0 & -\frac{\gamma}{2}(2N+1) & 0 \\ (N+1)\gamma & 0 & 0 & -N\gamma \end{pmatrix}. \quad (8.7)$$

The solution for Eq. (8.5) is:

$$\rho_{ij}(t) = \sum_{kl} V_{ij,kl} \rho_{kl}(0) = \sum_{kl} M_{ik,jl} \rho_{kl}(0). \quad (8.8)$$

Here  $V = \exp(Lt)$  [8] represents a completely positive map and  $M$  is the positive operator corresponding to the map  $V$ . The matrix representation of  $V$  for a one-

qubit under the action of the thermal bath can be written as:

$$V = \begin{bmatrix} b & 0 & 0 & y_2 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ y_1 & 0 & 0 & c \end{bmatrix} \quad (8.9)$$

where

$$\begin{aligned} x &= \exp \left[ -\frac{1}{2} \gamma (1 + 2N) t \right], \\ b &= \frac{N(1 + x^2) + x^2}{2N + 1}, \\ c &= \frac{N(1 + x^2) + 1}{2N + 1}, \\ y_1 &= \frac{(N + 1)(1 - x^2)}{2N + 1}, \\ y_2 &= \frac{N(1 - x^2)}{2N + 1}. \end{aligned}$$

The matrix  $M$  which corresponds to thermal evolution is:

$$M(t) = \frac{1}{2} \begin{bmatrix} b & 0 & 0 & x \\ 0 & y_2 & 0 & 0 \\ 0 & 0 & y_1 & 0 \\ x & 0 & 0 & c \end{bmatrix}. \quad (8.10)$$

If a system consists only of qubits, then it is enough to study the evolution of entanglement in  $M(t)$  by use of the factorization law for entanglement decay.

## 8.2 Evolution of entanglement in two-qubit system

Consider a two-qubit system  $S = A + B$ , consisting of the qubits  $A$  and  $B$ . Let us assume that one of the qubits (say  $B$ ) interacts with a thermal bath (represented by  $\$$ ) and also that the initial state  $|\chi\rangle$  of  $S$  is pure and entangled. We want to study the evolution of entanglement in this system.

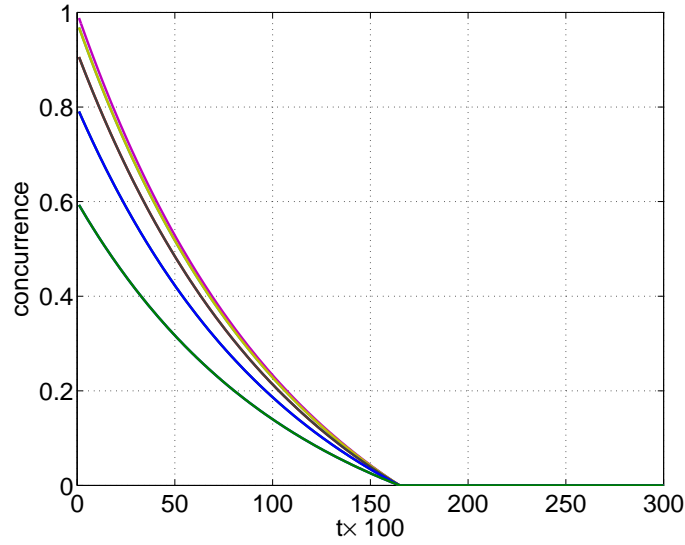


Figure 8.1: In this plot we have plotted the evolution of entanglement for different initial states.

If  $|\Phi^+\rangle$  is the maximally entangled state  $(|00\rangle + |11\rangle)/\sqrt{2}$ , then the factorization law for entanglement decay gives

$$E(\rho_{AB}(t)) = E(|\chi\rangle\langle\chi|)E((\mathcal{I} \otimes V)(|\phi^+\rangle\langle\phi^+|)) \quad (8.11)$$

where  $\rho_{AB}(t) = (\mathcal{I} \otimes V)(|\chi\rangle\langle\chi|)$ ,  $M = (\mathcal{I} \otimes V)(|\phi^+\rangle\langle\phi^+|)$ ,  $V$  is the CP map corresponding to the action of  $\$$  and  $E(\cdot)$  denotes the concurrence. Since the entanglement in the state  $\rho_{AB}$  at time  $t$  is the entanglement in  $M(t)$  times the entanglement in the initial state  $|\chi\rangle$ , which is constant, it is enough to study the evolution of entanglement in  $M$  in order to study the entanglement evolution in  $|\chi\rangle$ .

Notice that  $M(t)$  represents an entangled state iff the partial transpose of  $M(t)$  is non-positive. This requirement of positivity under partial transpose for

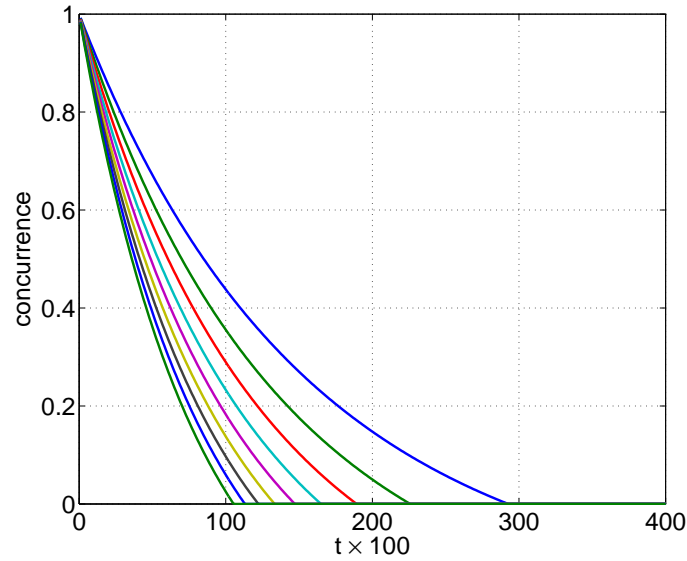


Figure 8.2: Evolution of entanglement for different bath temperatures. This plot shows that the time to ESD decreases as we increase the temperature.

separability can be written as

$$M^{T_B}(t) \geq 0 \Leftrightarrow \frac{1}{2} \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & y_2 & x & 0 \\ 0 & x & y_1 & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \geq 0 \quad (8.12)$$

where  $T_B$  represents the transposition on subsystem  $B$ . The eigenvalues of matrix  $M^{T_B}(t)$  are  $b/2$ ,  $c/2$ ,  $(y_1 + y_2 \pm \sqrt{(y_1 - y_2)^2 + 4x^2})/4$ . Therefore, the only eigenvalue which can be negative is  $(y_1 + y_2 - \sqrt{(y_1 - y_2)^2 + 4x^2})/4$ . The time  $t$  at which this eigenvalue becomes zero (if initially it starts from a positive value) is the time at which the entanglement in  $M(t)$  goes to zero. Thus the equation for separability is:

$$\frac{y_1 + y_2 - \sqrt{(y_1 - y_2)^2 + 4x^2}}{4} = 0 \quad (8.13)$$

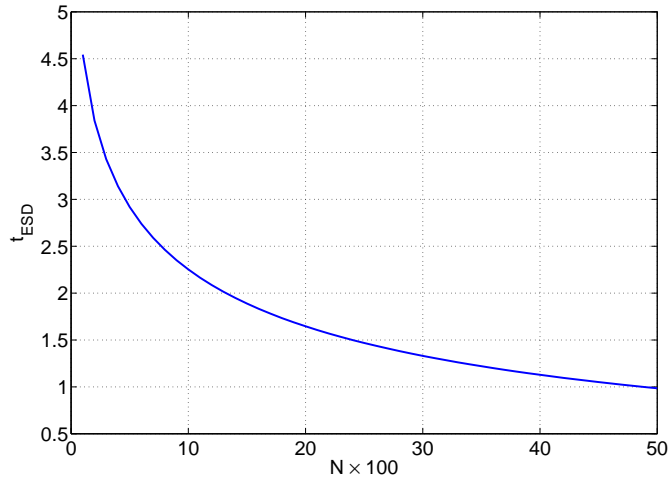


Figure 8.3: Time for ESD for different values of bath temperature

which gives

$$\sinh\left(\frac{\gamma(1+2N)t}{2}\right) = \frac{1}{2} \frac{(1+2N)}{\sqrt{N(N+1)}} \quad (8.14)$$

this implies that ESD occurs at time  $t_c = \frac{2}{\gamma(2N+1)} \sinh^{-1}\left(\frac{1}{2} \frac{(1+2N)}{\sqrt{N(N+1)}}\right)$ .

At non-zero temperature,  $t_c$  is finite. In fact,  $t_c \rightarrow 0$  as the temperature  $T \rightarrow \infty$ . However, when  $T = 0$ ,  $t_c \rightarrow \infty$ . This implies that no pure two-qubit state, when one of the qubits being exposed to a thermal bath at zero temperature, will show ESD. However, for  $T > 0$ , all such states will show ESD.

Mixed states are convex combination of pure states. At  $T > 0$ , they too show ESD under the same setup. Surprisingly however, it has been found [165] that certain mixed states do show ESD even at  $T = 0$ .

In section 8.4, we will extend this result to the  $n$ -qubit case. Before that we will derive the generalized factorization law for entanglement decay, which is simply the factorization law for  $d \otimes d$  systems.



### 8.3 Factorization law for entanglement decay for $d \otimes d$ systems

The purpose of this section is to extend the factorization law from  $2 \otimes 2$  systems to  $d \otimes d$  systems. The law for  $d \otimes d$  systems can further be generalized to  $d_1 \otimes d_2$  systems. We can use this to show ESD in  $n$ -qubit systems by choosing  $d_1 = 2$  and  $d_2 = 2^{n-1}$ . This will be shown in Sec. 8.4. Further discussion on this topic are available in [164, 185]. Let us consider the same setup as in Sec. 8.2, except that the systems  $A$  and  $B$  are now each  $d$ -dimensional. Local action of the bath will cause the entanglement to decay. Our aim is to calculate the entanglement in this  $d \otimes d$  system at time  $t$ .

This can be done by using G-Concurrence ([84], Sec 2.4.5) as the measure of entanglement. For a bipartite pure state  $|\chi\rangle$  in  $d \otimes d$ , G-concurrence is the geometric mean of its  $d$  Schmidt coefficients. For a most general  $|\chi\rangle = \sum_{i,j=1}^d A_{ij}|i\rangle|j\rangle$ , the G-concurrence can be written as

$$G_d(|\chi\rangle\langle\chi|) = d[\det(A^\dagger A)]^{1/d}. \quad (8.15)$$

For a mixed state  $\rho$  the G-concurrence can be defined as

$$G_d(\rho) = \inf \left\{ \sum_i p_i G_d(|\psi_i\rangle\langle\psi_i|) \mid \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \right\}. \quad (8.16)$$

The action of the bath  $\$$  on the state  $|\chi\rangle$  can be seen as:

$$\begin{aligned} \rho(t) &= (\mathcal{I} \otimes \$)(|\chi\rangle\langle\chi|) \\ &= (\mathcal{I} \otimes \$)(\$_\chi \otimes \mathcal{I})(|\phi^+\rangle\langle\phi^+|) \\ &= (\$_\chi \otimes \mathcal{I})(\mathcal{I} \otimes \$)(|\phi^+\rangle\langle\phi^+|) \\ &= (\$_\chi \otimes \mathcal{I})(\rho_\$) \end{aligned} \quad (8.17)$$

where  $\rho_\$$  is the result of the action of  $\$$  on one side of the maximally entangled state  $|\phi^+\rangle = 1/\sqrt{d} \sum_i |ii\rangle$  and  $(\$_\chi \otimes \mathcal{I})(\rho_\$) = (K \otimes \mathcal{I})(\rho_\$)(K^\dagger \otimes \mathcal{I})$  with  $K = \sum_{i,j}^d A_{ij}|i\rangle\langle j|$ . Note that the map  $\$_\chi$  is not trace preserving.

It is to be noted that the G-concurrence for the states  $|\psi\rangle$  and  $(K \otimes \mathcal{I})|\psi\rangle$  are

related as

$$G_d(|\psi\rangle\langle\psi|) = \frac{1}{|\det K|^{2/d}} G_d((K \otimes \mathcal{I})(|\psi\rangle\langle\psi|)(K^\dagger \otimes \mathcal{I})). \quad (8.18)$$

Let us assume that the minimal decomposition for  $\rho_{\mathfrak{S}}$ , i.e, the decomposition for which the  $G_d(\rho_{\mathfrak{S}})$  is minimum is:

$$\rho_{\mathfrak{S}} = \sum_i q_i |\theta_i\rangle\langle\theta_i| \quad (8.19)$$

which gives

$$G_d(\rho_{\mathfrak{S}}) = \sum_i q_i G_d(|\theta_i\rangle\langle\theta_i|) \quad (8.20)$$

From here the G-concurrence in  $(K \otimes \mathcal{I})(\rho_{\mathfrak{S}})(K^\dagger \otimes \mathcal{I})$  can be calculated as:

$$\begin{aligned} G_d((K \otimes \mathcal{I})(\rho_{\mathfrak{S}})(K^\dagger \otimes \mathcal{I})) &= \sum_i q_i G_d((K \otimes \mathcal{I})(|\theta_i\rangle\langle\theta_i|)(K^\dagger \otimes \mathcal{I})) \\ &= |\det K|^{2/d} \sum_i q_i G_d(|\theta_i\rangle\langle\theta_i|). \end{aligned} \quad (8.21)$$

Here  $|\det K|^{2/d}$  is the entanglement in the initial state  $|\chi\rangle$ . Finally we get

$$G_d(\rho(t)) = G_d(|\chi\rangle\langle\chi|) G_d((\mathcal{I} \otimes \mathfrak{S})(|\phi^+\rangle\langle\phi^+|)). \quad (8.22)$$

This is the general  $d \otimes d$  form of Eq. (8.11). The same framework also allows us to consider any  $d_1 \otimes d_2$  system (for different  $d_1$  and  $d_2$ ), since the pure states of such a system can exhibit entanglement in atmost  $\min\{d_1, d_2\}$  levels due to their Schmidt representation.

## 8.4 ESD in $n$ -qubit system

We now extend the results of the two-qubit systems to  $n$ -qubit systems. We prove that, for a finite non-zero temperature, every  $n$ -qubit state with all qubits connected to a local thermal bath shows ESD. We provide the proof of this statement for a three-qubit system. The proof for a general  $n$ -qubit system is extremely

cumbersome, but can essentially be carried out using the same method.

Let us consider a three-qubit system with all three qubits being individually connected to a local thermal bath at  $T \neq 0$ . Mathematically the action of a local thermal bath is analogous to the action of the map  $V$  on the state of the system. Hence, if the initial state of this system be  $|\psi\rangle_{ABC}$  — where  $A$ ,  $B$  and  $C$  are the single qubit subsystems — then the state of the system at time  $t$  can be written as the action of a map  $V \otimes V \otimes V$  on  $|\psi\rangle_{ABC}\langle\psi|$ :

$$\begin{aligned}
 \rho(t) &= (V \otimes V \otimes V) (|\psi\rangle_{ABC}\langle\psi|) \\
 &= (\mathcal{I} \otimes \mathcal{I} \otimes V)(\mathcal{I} \otimes V \otimes \mathcal{I})(V \otimes \mathcal{I} \otimes \mathcal{I}) (|\psi\rangle_{ABC}\langle\psi|) \\
 &= (\mathcal{I} \otimes \mathcal{I} \otimes V)(\mathcal{I} \otimes V \otimes \mathcal{I})(\rho_{A:BC}) \\
 &= (\mathcal{I} \otimes \mathcal{I} \otimes V)(\rho_{A:B:C})
 \end{aligned} \tag{8.23}$$

where  $\rho_{A:BC} = (V \otimes \mathcal{I} \otimes \mathcal{I}) (|\psi\rangle_{ABC}\langle\psi|)$  and  $\rho_{A:B:C} = (\mathcal{I} \otimes V \otimes \mathcal{I})(\rho_{A:BC})$ . We have already seen in Sec, 8.2, that under the action of a map  $V$ , the maximally entangled state  $|\Phi^+\rangle$  ultimately evolves to a separable state. Using the factorization law for entanglement decay for a  $2 \otimes d$  system, we can now say that the state  $\rho_{A:BC}$  — which is  $2 \otimes d$  state — will be separable in the partition  $A : BC$  after some finite time  $t_c$ .

This also means that, the time  $t_c$  is the maximum time any state can take to lose all its entanglement under the action of a channel  $V$  on one of its subsystems. This implies that the reduced state  $\rho_{BC} = \text{tr}_A \rho_{A:BC}$  can take *at most* time  $t_c$  to become separable in the partition  $B : C$ . Thus we have shown that the entire three-qubit system shows ESD at  $T \neq 0$ . This result can be generalized to mixed states in the same fashion as before — pure states showing ESD imply that mixed states of the same dimensionality and in the same setup also show ESD. This result can be generalized to any  $n$ -qubit case.

## 8.5 Squeezed thermal bath

So far we have analysed the evolution of entanglement in  $n$ -qubit system under the action of local thermal bath. We have found that at non-zero finite temperature all pure states of the system show ESD but none of them do so at zero temperature.

Some mixed states are exception to this rule as discussed earlier. In the remaining chapter we change the nature of the bath and study its effects on the evolution of entanglement. We start with squeezed thermal bath. In the next section we will consider bath with quantum non-demolition (QND) interactions.

Let  $S = S_1 + S_2 + \dots + S_n$  be an  $n$ -qubit system where each subsystem interacts with a squeezed thermal bath locally. The evolution of the reduced density matrix of the subsystem  $S_1$  in the interaction picture is given by the Master equation [14, 125, 33]:

$$\begin{aligned}
 \frac{d}{dt}\rho^s(t) &= \mathcal{L}_{sq}\rho^s(t) \\
 &= \gamma_0(N+1) \left( \sigma_- \rho^s(t) \sigma_+ - \frac{1}{2} \sigma_+ \sigma_- \rho^s(t) - \frac{1}{2} \rho^s(t) \sigma_+ \sigma_- \right) \\
 &\quad + \gamma_0 N \left( \sigma_+ \rho^s(t) \sigma_- - \frac{1}{2} \sigma_- \sigma_+ \rho^s(t) - \frac{1}{2} \rho^s(t) \sigma_- \sigma_+ \right) \\
 &\quad - \gamma_0 M \sigma_+ \rho^s(t) \sigma_+ - \gamma_0 M^* \sigma_- \rho^s(t) \sigma_-.
 \end{aligned} \tag{8.24}$$

Here  $\gamma_0$  is the spontaneous decay rate,  $\sigma_+ = |1\rangle\langle 0|$  and  $\sigma_- = |0\rangle\langle 1|$  are the standard raising and lowering operators respectively. The quantities  $N$  and  $M$  are given by  $2N+1 = \cosh(2r)(2N_{th}+1)$  and  $M = -\frac{1}{2} \sinh(2r)e^{i\phi}(2N_{th}+1)$  where  $N_{th} = (e^{\hbar\omega/k_B T} - 1)^{-1}$  is the mean occupation number of thermal photons at frequency  $\omega$  with  $T$  being the temperature of the bath and  $k_B$  is the Boltzman constant. Also,  $r$  and  $\phi$  are the squeezing parameters. We have neglected the unitary evolution part in Eq. (8.24) similar to what we did in Eq. (8.4). Moreover, Eq. (8.24) is in a Lindblad form and hence corresponds to a completely positive map  $V$  [33, 125]. The matrix representation  $L$  of the superoperator  $\mathcal{L}_{sq}$  is:

$$\begin{aligned}
 L_{sq} &= \frac{(N+1)\gamma}{2} [2\sigma_- \otimes \sigma_- - \sigma_+ \sigma_- \otimes \mathcal{I} - \mathcal{I} \otimes \sigma_+ \sigma_-] \\
 &\quad + \frac{N\gamma}{2} [2\sigma_+ \otimes \sigma_+ - \sigma_- \sigma_+ \otimes \mathcal{I} - \mathcal{I} \otimes \sigma_- \sigma_+], \\
 &\quad - \gamma_0 M \sigma_+ \sigma_- - \gamma_0 M^* \sigma_- \sigma_+
 \end{aligned} \tag{8.25}$$

$$= \begin{pmatrix} -(N+1)\gamma_0 & 0 & 0 & N\gamma_0 \\ 0 & -\frac{\gamma_0}{2}(2N+1) & -\gamma_0 M & 0 \\ 0 & -\gamma_0 M^* & -\frac{\gamma_0}{2}(2N+1) & 0 \\ (N+1)\gamma_0 & 0 & 0 & -N\gamma_0 \end{pmatrix}. \tag{8.26}$$

$V_{sq} = \exp(L_{sq}t)$  is:

$$V_{sq} = \begin{bmatrix} \alpha & 0 & 0 & \beta \\ 0 & y & z & 0 \\ 0 & z^* & y & 0 \\ \mu & 0 & 0 & \nu \end{bmatrix}, \quad (8.27)$$

where

$$\begin{aligned} \alpha &= \frac{N(1+x^2)+x^2}{2N+1}, \quad \beta = \frac{N(1-x^2)}{2N+1}, \\ \mu &= \frac{(N+1)(1-x^2)}{2N+1}, \quad \nu = \frac{N(1+x^2)+1}{2N+1}, \\ x^2 &= \exp[-\gamma_0(2N+1)t], \quad y = \cosh\left(\frac{\gamma_0 at}{2}\right)x, \\ z &= \sinh\left(\frac{\gamma_0 at}{2}\right)x \exp[i\Phi], \quad a = \sinh(2r)(2N_{th}+1). \end{aligned}$$

The two-qubit state corresponding to the channel  $V_{sq}$  is given by the Choi-Jamiolkowski isomorphism [101, 50] which is:

$$\begin{aligned} (\mathcal{I} \otimes V_{sq})(|\phi^+\rangle\langle\phi^+|) &= M_{sq} \\ &= \begin{bmatrix} \alpha & 0 & 0 & ye^{-i\omega t} \\ 0 & \beta & ze^{-i\omega t} & 0 \\ 0 & z^*e^{i\omega t} & \mu & 0 \\ ye^{i\omega t} & 0 & 0 & \nu \end{bmatrix}. \end{aligned} \quad (8.28)$$

This matrix  $M_{sq}$  is a positive semi-definite matrix and is separable iff the following conditions hold good simultaneously:

$$\alpha\nu - |z|^2 \geq 0, \quad \beta\mu - y^2 \geq 0. \quad (8.29)$$

These are equivalent to the following:

$$\frac{N(N+1)}{2N+1} 4 \cosh^2\left(\frac{\gamma_0(2N+1)t}{2}\right) - \sinh^2\left(\frac{\gamma_0 at}{2}\right) \geq 0, \quad (8.30)$$

$$\frac{N(N+1)}{2N+1} 4 \sinh^2\left(\frac{\gamma_0(2N+1)t}{2}\right) - \cosh^2\left(\frac{\gamma_0 at}{2}\right) \geq 0. \quad (8.31)$$

Since  $\cosh^2\left(\frac{\gamma_0(2N+1)t}{2}\right) \geq \sinh^2\left(\frac{\gamma_0(2N+1)t}{2}\right)$  and  $\sinh^2\left(\frac{\gamma_0 at}{2}\right) \leq \cosh^2\left(\frac{\gamma_0 at}{2}\right)$  we can write

$$\begin{aligned} \frac{N(N+1)}{2N+1} 4 \sinh^2\left(\frac{\gamma_0(2N+1)t}{2}\right) - \cosh^2\left(\frac{\gamma_0 at}{2}\right) \leq \\ \frac{N(N+1)}{2N+1} 4 \cosh^2\left(\frac{\gamma_0(2N+1)t}{2}\right) - \sinh^2\left(\frac{\gamma_0 at}{2}\right). \end{aligned} \quad (8.32)$$

This implies that the relation given by Eq. (8.31) is enough for determining the separability of  $M_{sq}$ . At time  $t = 0$ , Eq. (8.31) does not hold. However, at very large  $t$  it does hold if  $2N + 1 > a$ , i.e,  $2N + 1 > \sinh(2r)(2N_{th} + 1)$ . However, we know that  $2N + 1 = \cosh(2N_{th} + 1)$  and since  $\cosh(2r) \geq \sinh(2r)$  is always true, therefore, Eq. (8.31) is always true at large  $t$ . This shows that although  $M_{sq}$  starts off entangled, it loses its entanglement at some finite time  $t_c$  – this time being given by the equality condition corresponding to Eq. (8.31). Since  $M_{sq} = (\mathcal{I} \otimes V_{sq})(|\Phi^+\rangle\langle\Phi^+|)$  by definition, we can use the factorization law for entanglement decay for entangled state in the  $d \otimes 2$  case as before and show that the existence of ESD for any single subsystem can be extended to the entire  $n$ -qubit system under the action of a squeezed thermal bath.

The results given above are valid for a finite non-zero  $T$ . We have seen earlier at  $T = 0$ , there is no ESD for a pure 2-qubit entangled state where the initial state of the thermal bath is vacuum and the evolution is governed by Eq. (8.4). However, this is not true for a squeezed thermal bath, since at  $T = 0$ , the mean occupation number  $N$  is not zero and hence, all pure states connected to a squeezed thermal bath show ESD even at  $T = 0$ .

The effect of squeezing on the evolution of entanglement is thus twofold: at  $T = 0$  it kills off the entanglement faster and is thus a nuisance. However, at finite temperature, squeezing manages to extend the lifetime of entanglement and is thus a valuable tool in quantum information tasks.

## 8.6 Quantum non demolition (QND) interaction

We now study baths that interacts with a  $n$ -qubit system under QND interaction. In open quantum systems under QND type interactions, the system Hamiltonian commutes with the interaction Hamiltonian. Consider the system Hamil-

tonian  $H_S = \hbar\omega/2\sigma_3$ , where  $\sigma_3$  is the Pauli spin matrix for the  $z$ -axis, the bath Hamiltonian  $H_B = \sum_k \omega_k A_k^\dagger a_k$  and the system bath interaction to be of the form  $H_I = \sigma_3 \otimes \sum_k (g_k a_k^\dagger + g_k^* a_k)$ . We can see that the interaction Hamiltonian commutes with the system Hamiltonian. The reduced density operator  $\rho^S(t)$  for the system for such dynamics can be written as [14, 170, 166, 186]:

$$\rho_{11}^S(t) = \rho_{11}^S(0), \quad (8.33)$$

$$\rho_{12}^S(t) = e^{-i\omega t} e^{-(\hbar\omega)^2 \gamma(t)} \rho_{12}^S(0), \quad (8.34)$$

$$\rho_{21}^S(t) = e^{i\omega t} e^{-(\hbar\omega)^2 \gamma(t)} \rho_{21}^S(0), \quad (8.35)$$

$$\rho_{22}^S(t) = \rho_{22}^S(0). \quad (8.36)$$

The completely positive map  $V_{QND}$  and  $M_{QND}$  can be written as:

$$V_{QND} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-i\omega t} e^{-(\hbar\omega)^2 \gamma(t)} & 0 & 0 \\ 0 & 0 & e^{i\omega t} e^{-(\hbar\omega)^2 \gamma(t)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (8.37)$$

$$M_{QND} = \begin{pmatrix} 1 & 0 & 0 & e^{-i\omega t} e^{-(\hbar\omega)^2 \gamma(t)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{i\omega t} e^{-(\hbar\omega)^2 \gamma(t)} & 0 & 0 & 1 \end{pmatrix}, \quad (8.38)$$

where  $\omega$  is the natural frequency of the system and  $\gamma(t)$  is the time dependent spontaneous decay parameter (see [14]). We see that the matrix  $M_{QND}$  is not positive under partial transposition and hence, under the one-sided action of  $V_{QND}$ ,  $|\phi^+\rangle\langle\phi^+|$  will never become separable and hence ESD will never be observed under this map.

## 8.7 ESD and rebirth of entanglement: non-Markovian evolution

By definition the entanglement in a bipartite systems can not be increased by local operations on the subsystems. Therefore, we see only the decay of entanglement

in the Markovian dynamics as we have seen in earlier sections. However, in the non-Markovian dynamics we can see some counter intuitive results regarding entanglement dynamics [18, 55, 116]. In non-Markovian processes we do not discard the back action of the bath on the system. Therefore, as the system loses its entanglement the bath becomes entangled and it can impart some of this entanglement back to the system and hence one can observe the phenomenon called *rebirth of entanglement*.

To study the entanglement dynamics in a non-Markovian evolution, let us consider a two-qubit system interacting locally with a bath of Harmonic oscillators. The Hamiltonian for individual qubits interacting with corresponding (separate) baths can be written as:

$$H = H_S + H_B + H_I, \quad (8.39)$$

$$H_S = \omega \sigma_z, \quad (8.40)$$

$$H_B = \sum_k \omega_k a_k^\dagger a_k, \quad (8.41)$$

$$H_I = \sigma_+ \otimes B + \sigma_- \otimes B^\dagger, \quad (8.42)$$

$$B = \sum_k g_k a_k. \quad (8.43)$$

We consider the exact dynamics of each qubit [79] and we can quote the results from Sec. (6.3) and write the state  $\rho(t)$  of the system at time  $t$  as:

$$\rho(t) = \begin{pmatrix} \rho_{00}(0)c(t) & \rho_{01}(0)\sqrt{c(t)} \\ \rho_{10}(0)\sqrt{c(t)} & \rho_{11}(0) + \rho_{00}(0)(1 - c(t)) \end{pmatrix}, \quad (8.44)$$

where

$$c(t) = e^{-\lambda t} \left\{ \cos\left(\frac{dt}{2}\right) + \frac{\lambda}{d} \sin\left(\frac{dt}{2}\right) \right\}^2, \quad (8.45)$$

$$d = \sqrt{2\gamma_0\lambda - \lambda^2}. \quad (8.46)$$

Here  $\gamma_0$  is the decay rate and the  $\lambda$  is the spectral width of the spectral density



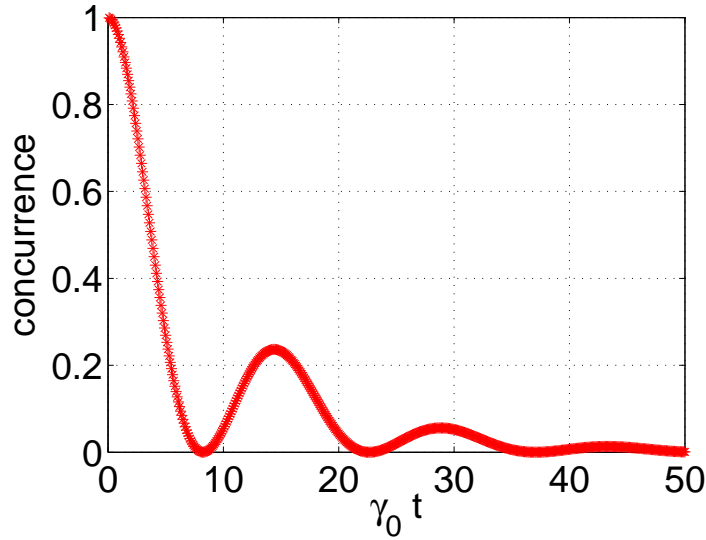


Figure 8.4: In this plot we have plotted the entanglement evolution for the non-Markovian dynamics. Here we have concurrence on the y-axis and  $\gamma_0 t$  on the x-axis. The parameter  $\lambda$  is set to be  $0.1\gamma_0$  and the initial state considered is  $(|01\rangle + |10\rangle)/\sqrt{2}$ . One can clearly see the phenomenon of ESD and rebirth of entanglement.

$J(\omega)$  which is defined as:

$$J(\omega) = \frac{1}{2\pi} \frac{\gamma_0 \lambda^2}{(\omega_0 - \omega)^2 + \lambda^2}. \quad (8.47)$$

In Fig.(8.4) we have shown the evolution of entanglement of two-qubit system (prepared in the state  $(|01\rangle + |10\rangle)/\sqrt{2}$ ) when the above said non-Markovian dynamics is applied separately on both the qubits. If we increase the strength of the memory kernel (i.e, if we make  $\lambda/\gamma_0$  smaller and smaller), there will be more and more time gap between sudden death of entanglement and the revival (see. for example, Fig 4 in Bellomo [18]).

In this chapter, we have studied the evolution of entanglement for a  $n$ -qubit states under the action of different thermal baths. We have found that entanglement decays at different rates for different baths at different temperatures. Whereas thermal baths are more useful at  $T = 0$  in that there is no ESD, squeezed thermal baths should be preferred at  $T > 0$ . However, the third type, bath undergoing QND interactions with the systems, appears to be the best overall because

the system in this setup never show any sudden death of entanglement.



# 9

## Evolution of entanglement and quantum correlations in the system of harmonic oscillators

In this chapter we will study the evolution of entanglement in an infinite-dimensional system interacting with local baths through various types of interactions. Unlike previous chapter the Hilbert space of the present system is infinite dimensional, for example the Hilbert space of a quantum harmonic oscillator. The complexity of the system increases with dimensions which makes the study of entanglement dynamics in harmonic oscillator systems difficult. Fortunately, when the state of the harmonic oscillator is Gaussian, it can be fully described by its first two moments. This eases the problem of studying evolution of entanglement in such systems up to some extent.

In this chapter we consider the system to be in two-mode Gaussian state (for example, the ground state of a system of two independent one-dimensional quantum harmonic oscillators) and the bath to consist of infinitely many non-interacting harmonic oscillators in thermal equilibrium [87].

The study of the evolution of entanglement as well as that of its sudden death in continuous variable systems has received increased attention over the past few years. Dodd *et al.* [58] dealt with ESD from the point of view of a separable representation of the joint Wigner function of two-mode Gaussian states. They stressed that entanglement is destroyed by the same mechanism which destroys interference. In [56], Diósi has given a bound on the time of ESD by using a

theorem on entanglement breaking quantum channels (see [155, 93, 149] and also Sec. 5.3). [121] Marek *et al.* have addressed a different problem; in this paper they obtained a class of states which is tolerant against the decoherence at zero temperature. There have been a number of studies on the decoherence in infinite dimensional systems interacting with different kinds of bath and with different system-environment models [177, 143, 146, 167]. Asymptotic nature of the entanglement evolution in a bipartite harmonic oscillator system has been addressed in [76, 100]. In these papers the authors have tried to find the system and bath setups where the asymptotic entanglement at high temperature is non-zero. For baths acting on both the systems together, the dynamics of entanglement is not monotonic. In some cases the entanglement can increase from zero and this phenomena has been called *sudden death and rebirth of entanglement* [135, 109, 136].

Since classicality subsumes disentanglement, the ESD problem can be embedded into the corresponding quantum-to-classical transition problem. Thus, the time taken for the system to attain classicality will be an upper bound on the time for ESD. We use the criterion due to Simon [158, 9] to check the classicality of Gaussian states in terms of their covariance matrices  $G$  and hence characterize the quantum-to-classical transition. If  $\mathcal{I}$  be the covariance matrix corresponding to the vacuum state, then Simon's criterion implies that Gaussian state with covariance matrix  $G$  attain classicality if and only if the difference  $(G - \mathcal{I})$  becomes positive semi-definite. The relation  $G - \mathcal{I} \geq 0$ , is analogous to the relation  $P(\alpha) \geq 0$ , where  $P(\alpha)$  is Sudarshan-Glauber  $P$  function [39, 38]. We use this condition to find out the transition time to classicality ( $t_c$ ) and hence the time to ESD. Diósi *et al.* [57] discuss the positivity of Wigner function and  $P$  function under Markovian evolution. We find that the behavior of the Gaussian system, as regards to ESD, depends on the state and the temperature of the bath it is coupled to. In this chapter, we have shown that every two-mode Gaussian state coupled to a local thermal bath shows ESD at finite non-zero temperature, whereas some such states do not do so when the temperature is lowered to zero. Squeezing the thermal bath causes every Gaussian state to show ESD even at  $T = 0$  [87].

We begin our analysis by writing down the master equation for a single harmonic oscillator [Sec. 9.1] in contact with a thermal bath of infinitely many non-interacting oscillators at some finite non-zero temperature. We express the states of this system in the Sudarshan-Glauber  $P$ -representation and derive the corre-

sponding quantum analog of the classical linear Fokker-Planck equation [Sec. 9.2]. This enables us to use covariance matrices to characterize the classicality of the single-mode system in the presence of local thermal baths [Sec. 9.3]. Once we have this mechanism in place, we use it to determine the action of the thermal bath [Sec. 9.4] and the squeezed thermal bath [Sec. 9.5] on a two-mode Gaussian state. Using Simon's criterion, we calculate the time to classicality and consequently determine the time to ESD. Using the results of the two-mode analysis, and making the important observation that Simon's criterion  $G - \mathcal{I} \geq 0$  is independent of the number of modes of the system, we extend our analysis to any  $n$ -mode Gaussian state and then generalize our earlier statements to  $n$ -mode systems [Sec. 9.6].

## 9.1 Master equation for a single mode system

The irreversible time evolution of the state  $\rho_s(t)$  of a single harmonic oscillator coupled to a bath is described by the Lindblad master equation (ME) [114],

$$\begin{aligned} \frac{d}{dt}\rho_s(t) = & -i\omega_0[a^\dagger a, \rho_s(t)] \\ & + \gamma_0(N+1) \left\{ a\rho_s(t)a^\dagger - \frac{1}{2}a^\dagger a\rho_s(t) - \frac{1}{2}\rho_s(t)a^\dagger a \right\} \\ & + \gamma_0(N) \left\{ a^\dagger\rho_s(t)a - \frac{1}{2}aa^\dagger\rho_s(t) - \frac{1}{2}\rho_s(t)aa^\dagger \right\}. \end{aligned} \quad (9.1)$$

The initial state of the system-bath combine is of the form  $\rho_s(0) \otimes \rho_{th}$ , where  $\rho_s(0)$  is the initial state of the system and  $\rho_{th}$  is the thermal state of bath. The first term of the ME describes free evolution generated by the system Hamiltonian  $H = \omega_0 a^\dagger a$  while the rest are interaction terms with the bath. The dissipative coupling is provided through the damping rate  $\gamma_0$ . Here  $N = (e^{\beta\omega_0} - 1)^{-1}$  is the mean number of quanta in a mode with frequency  $\omega_0$ . This ME can be used to describe, for example, the damping of an electromagnetic field inside a cavity where  $a$  and  $a^\dagger$  denote the creation and annihilation operators of the cavity mode and the mode outside the cavity plays the role of the environment with a dissipative coupling rate  $\gamma_0$ .

## 9.2 Coherent state representation

Coherent states are minimum uncertainty Gaussian states of the quantum harmonic oscillator. Mathematically, a coherent state  $|\alpha\rangle$  is defined to be the eigenstate of the annihilation operator  $a$ , i.e.,

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad (9.2)$$

where the eigenvalue  $\alpha$  is a complex number. In the Fock basis, the coherent state  $|\alpha\rangle$  can be written as:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (9.3)$$

We transform the master equation described by Eq. (9.1) into a continuous diffusion process by representing  $\rho_s$  in terms of Sudarshan-Glauber  $P$  representation:

$$\rho_s(t) = \int d^2\alpha P(\alpha, \alpha^*, t) |\alpha\rangle\langle\alpha|. \quad (9.4)$$

Here  $P(\alpha, \alpha^*, t)$  is the Sudarshan-Glauber  $P$ -function [81, 160] and the integration in Eq. (9.4) is over the entire complex plane. The quasiprobability distribution, called so because  $P(\alpha, \alpha^*, t)$  can take negative values for some  $\alpha$ , satisfies the normalization

$$\text{tr}_s \rho_s(t) = \int d^2\alpha P(\alpha, \alpha^*, t) = 1. \quad (9.5)$$

Substituting Eq. (9.4) into Eq. (9.1) and using the properties

$$\begin{aligned} a|\alpha\rangle\langle\alpha| &= \alpha|\alpha\rangle\langle\alpha|, \\ a^\dagger|\alpha\rangle\langle\alpha| &= \left( \frac{\partial}{\partial\alpha} + \alpha^* \right) |\alpha\rangle\langle\alpha|, \end{aligned} \quad (9.6)$$

we get the following equation for the evolution of  $P(\alpha, \alpha^*, t)$ :

$$\begin{aligned} \frac{\partial}{\partial t} P(\alpha, \alpha^*, t) = & - \left[ \left( -i\omega_0 - \frac{\gamma_0}{2} \right) \frac{\partial}{\partial \alpha} \alpha + \left( i\omega_0 - \frac{\gamma_0}{2} \right) \frac{\partial}{\partial \alpha^*} \alpha^* \right] P(\alpha, \alpha^*, t) \\ & + \gamma_0 N \frac{\partial^2}{\partial \alpha \partial \alpha^*} P(\alpha, \alpha^*, t). \end{aligned} \quad (9.7)$$

This is structurally similar to the classical linear Fokker-Planck equation [148] and can be solved by using the Gaussian ansatz [33]

$$P(\alpha, \alpha^*, t) = \frac{1}{\pi \sigma^2(t)} \exp \left[ -\frac{|\alpha - \beta(t)|^2}{\sigma^2(t)} \right], \quad (9.8)$$

given the initial condition  $P(\alpha, \alpha^*, 0) = \delta^2(\alpha - \alpha_0)$ , where  $|\alpha_0\rangle$  is the initial coherent state. Here the mean amplitude is  $\beta(t) = \int d^2\alpha \alpha P(\alpha, \alpha^*, t) = \alpha_0 e^{(-i\omega_0 - \gamma_0/2)t}$  and the variance is  $\sigma^2(t) = N(1 - e^{-\gamma_0 t})$ . Since we are interested in dissipation, we choose to ignore the  $\omega_0$  term which only contributes to free evolution. The linearity of the evolution map ensures that any density matrix  $\rho(0) = \int P(\lambda, \lambda^*, 0) |\lambda\rangle\langle\lambda| d^2\lambda$  will evolve to  $\rho(t) = \int P(\alpha, \alpha^*, t) |\alpha\rangle\langle\alpha| d^2\alpha$ , where

$$P(\alpha, \alpha^*, t) = \int P(\lambda, \lambda^*, 0) \exp \left( -\frac{|\alpha - \beta(t)|^2}{\sigma^2(t)} \right) d^2\beta$$

with  $\beta = \lambda e^{-\gamma_0 t/2}$ . Therefore, thermal evolution of the  $P$ -distribution manifests as a convolution of the  $P$  and the thermal distribution  $P_{th} = \exp(-|\alpha|^2/\sigma^2(t))$ .

### 9.3 Covariance matrix

In classical probability theory, the covariance matrix of a vector of random variables  $X = [X_1 \ X_2 \ \dots \ X_n]^T$  is defined as  $C = \langle (X - \langle X \rangle)(X^T - \langle X^T \rangle) \rangle$ . Elementwise, it can be written as:

$$C_{ij} = \text{cov}(X_i, X_j) = \langle (X_i - \langle X_i \rangle)(X_j - \langle X_j \rangle) \rangle, \quad (9.9)$$

where angular brackets represent average over the sample space. For the case of a quantum harmonic oscillator, the covariance matrix  $G$  is defined as the matrix of



quadratures, i.e,

$$G_{ij} = \langle \{R_i - \langle R_i \rangle, R_j - \langle R_j \rangle\} \rangle \quad (9.10)$$

where  $R$  is a vector of position and momentum operators, i.e,

$$R = [r_1 \ p_1 \ r_2 \ p_2 \ \cdots \ r_n \ p_n]^T. \quad (9.11)$$

The matrix  $G$  is by construction real and symmetric and it satisfies the uncertainty relation

$$G + i\hbar\sigma \geq 0 \quad (9.12)$$

where

$$\sigma = \bigoplus_{j=1}^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (9.13)$$

Any arbitrary real symmetric matrix  $\Sigma$  represents a covariance matrix of a quantum state if and only if it satisfies the uncertainty relation.

It turns out [39, 38] that the symmetric characteristic function

$$\chi(\alpha, t) = e^{-\frac{|\alpha|^2}{2}} \mathcal{F}[P(\alpha, \alpha^*, t)] = \exp\left(-\frac{X^T G X}{4}\right), \quad (9.14)$$

is a function of the elements of the covariance matrix  $G$ . Here  $\mathcal{F}$  denotes the Fourier transform and the vector  $X^T$  is  $(q, p)$ , where  $q = \frac{1}{\sqrt{2}}(\alpha + \alpha^*)$  and  $p = \frac{i}{\sqrt{2}}(\alpha^* - \alpha)$  are the position and momentum variables. The evolution of  $\chi(\alpha, t)$  follows from that of  $P(\alpha, \alpha^*, t)$  (Eq. (9.7)) and thus we have, from Eq. (9.14)

$$\chi(\alpha, t)e^{\frac{|\alpha|^2}{2}} = \chi(\beta e^{-\frac{\gamma_0}{2}t}, 0)e^{-\frac{|\alpha|^2 e^{-\gamma_0 t}}{2}} e^{-\frac{|\alpha|^2 \sigma^2(t)}{4}}. \quad (9.15)$$

The covariance matrix  $G$  can be written as:

$$G_{ij} = \int (X X^T)_{ij} \mathcal{F}^{-1} \chi(\alpha, t) d^2 q d^2 p, \quad (9.16)$$

and hence

$$G(t) = e^{-\gamma_0 t} G(0) + \left(\frac{N}{2} + 1\right) (1 - e^{-\gamma_0 t}) \mathcal{I}. \quad (9.17)$$

A *Gaussian channel* is a completely positive map that takes Gaussian states to Gaussian states, an example of which is the evolution, given by Eq. (9.1), for a harmonic oscillator coupled with a thermal bath. The evolution of the covariance matrix  $V$  of the system, under the action of a general Gaussian channel, can be characterized by two matrices  $A$  and  $B$ :

$$G_f = A G A^T + B, \quad (9.18)$$

where  $B$  is a positive operator [90]. For a thermal bath, we have from Eq. (9.17),  $A = e^{-\gamma_0 t/2} \mathcal{I}$  and  $B = \left(\frac{N}{2} + 1\right) (1 - e^{-\gamma_0 t}) \mathcal{I}$ . We can thus characterize the action of a thermal bath completely using these two matrices. The characterization, given by Eq. (9.18), guarantees that  $G(t)$  of Eq. (9.17) is a bona fide covariance matrix for all finite time  $t$ .

## 9.4 ESD of two-mode Gaussian state

Consider a two-mode system coupled to two identical local thermal baths. Let us assume that the initial two-mode ( $4 \times 4$ ) covariance matrix  $G_0$  represents an entangled state. Its subsequent evolution is given by

$$\begin{aligned} G(t) &= (A \oplus A) G(0) (A \oplus A)^T + (B \oplus B) \\ &= e^{-\gamma_0 t} G(0) + \left(\frac{N}{2} + 1\right) (1 - e^{-\gamma_0 t}) \mathcal{I}. \end{aligned} \quad (9.19)$$

where  $A = e^{-\gamma_0 t/2} \mathcal{I}_4$  and  $B = \left(\frac{N}{2} + 1\right) (1 - e^{-\gamma_0 t}) \mathcal{I}_4$ . From quantum optics point of view, we know [158] that the state  $\rho = \int d^2\alpha d^2\beta P(\alpha, \beta) |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|$ , written in the  $P$  representation, is classical if and only if  $P(\alpha, \beta)$  is non-negative for all  $\alpha$  and  $\beta$ . This interpretation of classicality can be translated into the language of the covariance matrix  $G(t)$ . Thus, a two-mode Gaussian state will be classical at some time  $t$  if and only if  $G(t) \geq \mathcal{I}$  [158]. Clearly, being entangled, the initial covariance matrix satisfies the relation  $G(0) < \mathcal{I}$ . However, since the evolution

given by Eq. (9.19) is dissipative, the system will attain classicality after a time  $t_c$  so that  $G(t_c) \geq \mathcal{I}$ . This condition is equivalent to  $n_{min}(t_c) \geq 1$ , where  $n_{min}(t_c)$  is the smallest eigenvalue of  $G(t_c)$ . Using Eq. (9.19), it can be written as the evolution of  $n_{min}(0)$  which is the smallest eigenvalue of  $G_0$ . Since  $G(0)$  is not classical,  $n_{min}(0) < 1$ .

$$n_{min}(t_c) = e^{-\gamma_0 t_c} n_{min}(0) + \left(\frac{N}{2} + 1\right) (1 - e^{-\gamma_0 t_c}). \quad (9.20)$$

Here we find that  $n_{min}(t_c) \geq 1$  always for  $t_c \geq -\frac{1}{\gamma_0} \ln\left(\frac{N}{N+2-2n_{min}(0)}\right)$ . An appropriate choice of  $G(0)$  allows us to make  $n_{min}(0)$  arbitrarily small (which is the case in EPR states which are maximally entangled initially) and thus we get an upper bound on the transition time  $t_c$ . Thus the upper bound is given by  $t_{max} = -\frac{1}{\gamma_0} \ln\left(\frac{N}{N+2}\right)$ . For  $T > 0$ , we have  $N > 0$  and hence  $t_{max}$  is finite and non-zero. This proves that there is always ESD at finite non-zero temperatures. However, when  $T = 0$ , *i.e.*  $N = 0$ , we have  $t_{max} \rightarrow \infty$  and hence no quantum-to-classical transition is seen at finite times. This does not, however, rule out ESD since non-classicality does not necessarily imply that there is entanglement.

For  $T = 0$ , let us consider the following particular form of the initial covariance matrix representing a symmetric two-mode Gaussian state:

$$G(0) = \begin{pmatrix} n & 0 & k_x & 0 \\ 0 & n & 0 & -k_y \\ k_x & 0 & n & 0 \\ 0 & -k_y & 0 & n \end{pmatrix}. \quad (9.21)$$

Here  $k_x$  and  $k_y$  are positive and satisfy  $n^2 - (\max\{k_x, k_y\})^2 \geq 1$  (in order that  $G(0)$  be a bona fide covariance matrix). This covariance matrix represents an entangled state when  $(n-k_x)(n-k_y) < 1$  [80]. Under evolution given by Eq. (9.19) and taking  $k_x = k_y$ , we have  $n(t) = ne^{-\gamma_0 t} + (1 - e^{-\gamma_0 t})$  and  $k_x(t) = k_x e^{-\gamma_0 t}$ . At the quantum-to-classical transition time  $t_c$ , the state becomes separable:  $(n(t_c) - k_x(t_c))^2 \geq 1$ . This condition is, in this case, equivalent to  $n(t_c) - k_x(t_c) \geq 1$ . However, we have verified that there is no positive finite value of  $t_c$  that satisfies this condition. Hence we see that, for the zero-temperature case, there are states given by Eq. (9.21) that do not show ESD. This can be easily generalized to the case when  $k_x \neq k_y$ .

This result is similar to the two-qubit case where we see that ESD occurs for all states at non-zero temperature. For  $T = 0$ , there exist states which do not show ESD [86].

## 9.5 Squeezed thermal bath

If the initial state of the bath is a squeezed thermal state, then the master equation (9.1) is replaced by [1]:

$$\begin{aligned}
 \frac{d}{dt}\rho_s(t) = & -i\omega_0[a^\dagger a, \rho_s(t)] \\
 & + \gamma_0(N+1) \left\{ a\rho_s(t)a^\dagger - \frac{1}{2}a^\dagger a\rho_s(t) - \frac{1}{2}\rho_s(t)a^\dagger a \right\} \\
 & + \gamma_0 N \left\{ a^\dagger \rho_s(t)a - \frac{1}{2}aa^\dagger \rho_s(t) - \frac{1}{2}\rho_s(t)aa^\dagger \right\} \\
 & - \frac{\gamma_0}{2}M^* \{2a\rho a - a^2\rho - \rho a^2\} \\
 & - \frac{\gamma_0}{2}M \{2a^\dagger \rho a^\dagger - (a^\dagger)^2 \rho - \rho (a^\dagger)^2\}, \tag{9.22}
 \end{aligned}$$

where  $M = -\frac{1}{2}\sinh(2r)e^{i\phi}(2N_{th} + 1)$  and  $2N + 1 = \cosh(2r)(2N_{th} + 1)$ . The quantity  $N_{th}$  is the average number of photons in the thermal state while  $r$  and  $\phi$  are the squeezing parameters. Repeating the same procedure as earlier, we finally write down the evolution in terms of covariance matrices:

$$G(t) = e^{-\gamma_0 t}G(0) + (1 - e^{-\gamma_0 t})G_\infty, \tag{9.23}$$

where  $G_\infty$  is given by

$$G_\infty = \begin{pmatrix} \frac{N}{2} + 1 + \text{Re}\{M\} & \text{Im}\{M\} \\ \text{Im}\{M\} & \frac{N}{2} + 1 - \text{Re}\{M\} \end{pmatrix}. \tag{9.24}$$

In this case, for a squeezed thermal bath,  $A = e^{-\gamma_0 t}\mathcal{I}$  and  $B = (1 - e^{-\gamma_0 t})G_\infty$ . If there is no squeezing (*i.e.* if we set  $M = 0$  and  $N = N_{th}$ ), we get  $G_\infty = (N/2 + 1)\mathcal{I}$  and thus recover the unsqueezed result namely, Eq. (9.19).

The evolution of the two-mode Gaussian state can now be written as

$$G(t) = e^{-\gamma_0 t} G(0) + (1 - e^{-\gamma_0 t})(G_\infty \oplus G_\infty). \quad (9.25)$$

If the smallest eigenvalue of  $G(0)$  is  $n(0)$  then the classicality condition  $G(t) \geq \mathcal{I}$  can be written as [95]:

$$n(t) \geq e^{-\gamma_0 t} n(0) + (1 - e^{-\gamma_0 t}) \left( \frac{N}{2} + 1 - |M| \right) \geq 1, \quad (9.26)$$

$$\text{i.e.,} \quad t \geq -\frac{1}{\gamma_0} \ln \left( \frac{N - 2|M|}{N + 2 - 2|M| - 2n(0)} \right), \quad (9.27)$$

where in Eq. (9.26),  $n(t)$  is the smallest eigenvalue of  $G(t)$  and  $\left(\frac{N}{2} + 1 - |M|\right)$  is the smallest eigenvalue of  $G_\infty$ . Since  $n(0) < 1$ ,  $t_c$  is finite and positive. Therefore, the transition to classicality and thus ESD is ensured for the squeezed thermal bath for non-zero temperatures. However, unlike the thermal bath case, we see here that  $N$  does not become zero with temperature  $T$  and hence a quantum-to-classical transition always happens at zero temperature for the squeezed thermal bath. This ensures ESD.

## 9.6 ESD for $n$ -mode Gaussian states

In this section we generalize the result mentioned previously to  $n$ -mode Gaussian states in the presence of local thermal and squeezed thermal baths. Consider an initial covariance matrix  $G_n(0)$  corresponding to an  $n$ -mode Gaussian state. Let the matrices  $A$  and  $B$  characterize the single mode Gaussian channel for a given bath. The evolution of the covariance matrix can be written as [as a generalization of Eq. (9.19)],

$$G_n(t) = (A \oplus A \oplus \dots \oplus A) G_n(0) (A \oplus A \oplus \dots \oplus A)^T + (B \oplus B \oplus \dots \oplus B). \quad (9.28)$$

If  $n(0)$  is the smallest eigenvalue of  $G_n(0)$ , then the classicality condition  $G_n(t) \geq \mathcal{I}$  gives rise to an equation identical to Eq.(9.20) (Eq. (9.26)) representing  $n_{min}(t)(n(t))$  as the smallest eigenvalue of  $V_n(t)$  ( a covariance matrix for  $n$ -mode Gaussian state)

in the case of the thermal (squeezed thermal) bath. Therefore, all  $n$ -mode Gaussian states show quantum-to-classical transitions at finite temperatures.

In this chapter we have shown that at finite temperature all the two-mode Gaussian states achieve classicality in finite time and hence become separable when the system is coupled to a local thermal or squeezed thermal bath, though there are some states which do not achieve classicality when the bath is at zero temperature. Squeezing can substitute for the temperature and cause the system to achieve classicality in finite time even at zero temperature.



# 10

## Effect of control procedures on entanglement dynamics

Up until now we have discussed ESD in various systems. This chapter is devoted to the study of control procedures to suppress the entanglement decay and delay ESD. Given the obvious importance of ESD regarding the success of quantum tasks, it is thus a worthwhile exercise to investigate ways and means of controlling the rate of loss of entanglement. Error-correcting codes [40, 41, 154, 106] and error-avoiding codes [186, 59, 113] ( which are also known as decoherence-free subspaces) are such attempts. Open loop decoherence control strategies [170, 10, 60, 171, 172, 169, 3, 5, 6] are another class of widely used strategies used to this effect, where the system of interest is subjected to external, suitably designed, time-dependent drivings that are independent of the system dynamics. The aim is to cause an effective dynamic decoupling of the system from the ambient environment. A comparative analysis of some of these methods has been made in [67]. Another mechanism known to slow down the process of decoherence is through manipulation of the density of states. This has been put to use in photonic band-gap materials, which is used to address questions related to the phenomenon of localization of light [102, 182, 103, 183, 104]. In [88] we have studied the effect of control procedures on the evolution of entanglement. This chapter is based on that study.

In this chapter, we analyze the evolution of entanglement in two-qubit systems connected to local baths (or reservoirs). A number of studies of entanglement in open quantum systems have been made [12, 13, 47]. Here we address the need to have a control on the resulting nonunitary evolution, as motivated by the above



discussion, and study several methods of doing so. These include manipulation of the density of states in photonic crystals, modulation of the frequency of the system-bath coupling and modulation of external driving on two-qubit systems. A significant part of the chapter is devoted the study of control methods in two-qubit systems undergoing non-Markovian evolution. The first of these is dynamic decoupling — which is an open-loop strategy — on a two-qubit system that is in contact with a harmonic oscillator bath. This system undergoes a quantum non-demolition interaction, where dephasing occurs without the system getting damped. The second is a Josephson-junction charge qubit subject to random telegraph ( $1/f$ ) noise due to charge impurities.

The surprising aspect of this study is that suppression of decoherence due to a control procedure need not necessarily mean preservation of entanglement. In fact, application of resonance fluorescence on a two-level atom exposed to a thermal bath or dynamical decoupling on the Josephson junction charge qubit, undergoing non-Markovian evolution, results in faster ESD even though decoherence gets suppressed.

## 10.1 Evolution of entanglement in the presence of photonic crystals

Let us consider a system of two level atoms interacting with a periodic dielectric crystal, this particular structure of which gives rise to the photonic band gap [102, 92, 183]. The effect of this on electromagnetic waves is analogous to the effect semiconductor crystals have on the propagation of electrons, and leads to interesting phenomena like strong localization of light [103], inhibition of spontaneous emission [183] and atom-photon bound states [104, 112, 184]. The origin of such phenomena can ultimately be traced to the photon density of states changing at a rate comparable to the spontaneous emission rates. The photon density of states are of course estimated from the local photon mode density which constitutes the reservoir. It is this photonic band that suppress decoherence [174].

Let us consider a two-qubit system, one qubit of which is locally coupled to a photonic crystal reservoir kept at zero temperature . In this case, entanglement dynamics can be obtained by studying any one of the qubits individually. We start

with the following Hamiltonian:

$$H = \frac{\omega_0}{2}\sigma_z + \sum_k \omega_k a_k^\dagger a_k + \sum_k (g_k a_k^\dagger \sigma_- + g_k^* a_k \sigma_+), \quad (10.1)$$

where  $\omega_0$  is the natural frequency of the two level atom,  $\omega_k$  is the energy of the  $k$ -th mode and  $g_k$  is the frequency dependent coupling between the qubit and the photonic crystal, the latter acting as the reservoir here. Also,  $\sigma_z$  and  $\sigma_\pm = \sigma_x \pm i\sigma_y$  are the Pauli matrices, with  $a_k$  and  $a_k^\dagger$  being the annihilation and creation operators for  $k$ -th mode. If we restrict the total atom-reservoir system to the case of a single excitation [79], the evolution of a given state of the qubit is then given by [174]:

$$\rho(t) = \begin{pmatrix} \rho_{11}(0)|c(t)|^2 & \rho_{01}(0)c(t) \\ \rho_{10}(0)c^*(t) & \rho_{00}(0) + \rho_{11}(0)(1 - |c(t)|^2) \end{pmatrix}, \quad (10.2)$$

where

$$c(t) = \varepsilon \left( \lambda_+ e^{i\lambda_+^2 t} [1 + \Phi(\lambda_+ e^{i\pi/4} \sqrt{t})] - \lambda_- e^{i\lambda_-^2 t} [1 + \Phi(\lambda_- e^{i\pi/4} \sqrt{t})] \right),$$

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{(2k+1)!!} \text{ is the error function,}$$

$$\varepsilon = \frac{e^{i\delta t}}{\sqrt{\alpha^2 - 4\delta}},$$

$$\lambda_{\pm} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\delta}}{2},$$

$$\alpha \approx \frac{\omega_0^2 d^2}{8\omega_c \epsilon_0 (\pi A)^{3/2}}.$$

Here  $\delta = \omega_0 - \omega_c$  is the detuning of the atomic frequency and  $\omega_c$  is the upper band-edge frequency. We have made use of the following photon-dispersion relation near the band edge:  $\omega_k \approx \omega_c + A(k - k_0)^2$ , where  $A \approx \omega_c/k_0^2$ ,  $d$  is the atomic dipole moment and  $\epsilon_0$  is the vacuum dielectric constant.

The density matrix  $\rho(t)$  in Eq. (10.2) is related to the initial density matrix

$\rho(0)$  by a map  $\Lambda$ , given by  $\rho(t) = \Lambda_{pc}[\rho(0)]$ , whose matrix representation is

$$V_{pc} = \begin{pmatrix} |c(t)|^2 & 0 & 0 & 0 \\ 0 & c(t) & 0 & 0 \\ 0 & 0 & c^*(t) & 0 \\ 1 - |c(t)|^2 & 0 & 0 & 1 \end{pmatrix}. \quad (10.3)$$

Channel-state duality, explained earlier in Chapter 5, ensures that there exists a two-qubit density matrix  $M_{pc}$  for every single-qubit channel  $V_{pc}$ . This matrix  $M_{pc}$  can be written as

$$\begin{aligned} M_{pc} &= (\mathcal{I} \otimes \Lambda_{pc})(|\Phi^+\rangle\langle\Phi^+|) \\ &= \frac{1}{2} \begin{pmatrix} |c(t)|^2 & 0 & 0 & c(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 - |c(t)|^2 & 0 \\ c^*(t) & 0 & 0 & 1 \end{pmatrix}, \end{aligned} \quad (10.4)$$

where  $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$  is a two-qubit maximally entangled state. The concurrence of  $M_{pc}$  is  $|c(t)|^2$ , where  $c(t)$  is a complex-valued function of the detuning parameter  $\delta$  and time  $t$ . Therefore, we need to see the effect of  $\alpha$  on entanglement in  $M_{pc}$ . If we assume that  $\delta = \Delta\alpha^2$ ,  $c(t)$  can then be written in the following simplified form:

$$c(t) = \frac{e^{i\Delta\tau}}{\sqrt{1-4\Delta}} \frac{1}{2} \left( d_+ e^{id_+^2\tau} [1 + \Phi(d_+ e^{i\pi/4} \sqrt{\tau})] - d_- e^{id_-^2\tau} [1 + \Phi(d_- e^{i\pi/4} \sqrt{\tau})] \right),$$

where  $d_{\pm} = -1 \pm \sqrt{1-4\Delta}$  and  $\tau = \alpha^2 t$ . Since the entanglement in  $M_{pc}$  is  $|c(t)|^2$ , it is now a function of  $\delta$  and  $\tau$ . Invoking the factorization law of entanglement decay, it is sufficient to study entanglement in  $M_{pc}$  in order to understand the nature of evolution of entanglement in the two-qubit system.

We show the evolution of entanglement in  $M_{pc}$  for different values of  $\Delta$  in Figs. (10.1(a)), (10.1(b)). The insets of the figures depict the evolution of entanglement – computed using concurrence (see appendix) – for the usual case of zero band gap, while the main panels show the evolution of entanglement for increasing influence

of the band gap. In FIG. (10.1(a)), the system is within a gap in the photonic spectrum, indicated by the negative value of  $\Delta$  and hence also  $\delta$ , as a result of which coherence is preserved and the decay of entanglement is arrested. This feature is further highlighted in FIG. (10.1(b)), which is also for the case of negative  $\Delta$  of higher order of magnitude than that in FIG. (10.1(a)), and as a result there is a greater persistence of entanglement. Thus we find that with the increase of the influence of the photonic band gap on the evolution, entanglement is preserved longer. From Eq. (10.4), it can be seen that, following the arguments of the previous section, there is no ESD in this case, a feature corroborated by the FIGS. (10.1).

Apart from these, Fig. (10.1) shows another interesting phenomenon – the temporally damped oscillations in the entanglement. This phenomenon is a signature of the emergence of non-Markovian characteristics in the evolution and implies that the action of detuning changes the character of the dynamics itself, turning it non-Markovian.

## 10.2 Frequency modulation

Agarwal and coworkers [3, 5, 6] introduced an open-loop control strategy which involved modulation of the system-bath coupling, with the proviso that the frequency modulation (to be introduced below) should be carried out at a time scale which is faster than the correlation time scale of the heat bath. The technique of frequency modulation has been used earlier to demonstrate the existence of population trapping states in a two-level system [4]. Raghavan *et al.* [142] showed the connection between trapping in a two-level system under the action of frequency-modulated fields in quantum optics and dynamic localization of charges moving in a crystal under the action of a time-periodic electric field.

Consider the Hamiltonian given in Eq. (10.1). Frequency modulation essentially involves a modification of the coupling  $g_k$  — the modulated coupling is  $g_k \exp\{-im \sin \nu t\}$ , where  $m$  is the amplitude and  $\nu$  is the frequency of the modulation. The decay of the excited state population can be significantly arrested by choosing  $m$  such that  $J_0(m) = 0$ , where  $J_0$  are the Bessel functions of order zero. The resulting master equation in the interaction picture, when applied to

the evolution of a two-level system, is [3, 5, 6]:

$$\begin{aligned} \frac{\partial \rho}{\partial t} = & -\frac{2(\kappa - i\Delta)J_1^2(m)}{(\kappa - i\Delta)^2 + \nu^2} \{C_0^{-+}(\sigma^+ \sigma^- \rho - \sigma^- \rho \sigma^+) \\ & + C_0^{+-}(\rho \sigma^- \sigma^+ - \sigma^+ \rho \sigma^-)\} + h.c., \quad \Delta = (\omega_0 - \omega). \end{aligned} \quad (10.5)$$

Here  $\sigma^\pm$  are the Pauli matrices. We have used the Bessel function expansion  $e^{-im \sin(\nu t)} = \sum_{l=-\infty}^{l=\infty} J_l(m) e^{-il\nu t}$  where  $J_1(m)$  is the Bessel function of order one. Additionally, the modified bath correlation functions are assumed to have the forms  $C^{-+}(t) = C_0^{-+} e^{-\kappa t} e^{i\omega t}$  and  $C^{+-}(-t) = C_0^{-+} e^{-\kappa t} e^{i\omega t}$ , where  $\kappa$  is the bath correlation frequency. Now, we have

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \mathcal{L}_{fm}[\rho], \\ \Rightarrow \rho(t) &= \exp(\mathcal{L}_{fm} t) \rho(0), \\ \Rightarrow \rho(t)_{ij} &= \sum_{kl} \{\exp(\mathcal{L}_{fm} t)\}_{ij,kl} \rho(0)_{kl} = \sum_{kl} \{V_{fm}(t)\}_{ij,kl} \rho(0)_{kl}, \end{aligned} \quad (10.6)$$

where  $V_{fm}(t) = \exp(L_{fm} t)$  and  $L_{fm}$  is the matrix representation of  $\mathcal{L}_{fm}$ . We obtain the matrices  $L_{fm}$  and  $V_{fm}$  using Eq. (10.5).

$$L_{fm} = \left( \begin{array}{cc|cc} -2\text{Re}(\alpha)C_0^{-+} & 0 & 0 & 2\text{Re}(\alpha)C_0^{+-} \\ 0 & -\alpha(C_0^{-+} + C_0^{+-}) & 0 & 0 \\ \hline 0 & 0 & -\alpha^*(C_0^{-+} + C_0^{+-}) & 0 \\ 2\text{Re}(\alpha)C_0^{-+} & 0 & 0 & -2\text{Re}(\alpha)C_0^{+-} \end{array} \right), \quad (10.7)$$

$$V_{fm} = \exp(L_{fm} t) \quad (10.8)$$

$$= \left( \begin{array}{cccc} \frac{1}{T} (C_0^{-+} e^{-2\text{Re}(\alpha)Tt} + C_0^{+-}) & 0 & 0 & \frac{1}{T} (C_0^{+-} (1 - e^{-2\text{Re}(\alpha)Tt})) \\ 0 & e^{-\alpha Tt} & 0 & 0 \\ 0 & 0 & e^{-\alpha^* Tt} & 0 \\ \frac{1}{T} (C_0^{-+} (1 - e^{-2\text{Re}(\alpha)Tt})) & 0 & 0 & \frac{1}{T} (C_0^{+-} e^{-2\text{Re}(\alpha)Tt} + C_0^{-+}) \end{array} \right), \quad (10.9)$$

where  $\alpha = \frac{2(\kappa - i\Delta)J_1^2(m)}{(\kappa - i\Delta)^2 + \nu^2}$  and  $T = C_0^{-+} + C_0^{+-}$ . If  $M_{fm} = (\mathcal{I} \otimes V_{fm})(|\phi^+\rangle\langle\phi^+|)$ ,

then we have

$$M_{fm} = \begin{pmatrix} M_{11} & 0 & 0 & e^{-\alpha Tt} \\ 0 & M_{22} & 0 & 0 \\ 0 & 0 & M_{33} & 0 \\ e^{-\alpha^* Tt} & 0 & 0 & M_{44} \end{pmatrix}, \quad (10.10)$$

where

$$\begin{aligned} M_{11} &= \frac{1}{T} (C_0^{-+} e^{-2\text{Re}(\alpha)Tt} + C_0^{+-}), \\ M_{22} &= \frac{1}{T} (C_0^{+-} (1 - e^{-2\text{Re}(\alpha)Tt})), \\ M_{33} &= \frac{1}{T} (C_0^{-+} (1 - e^{-2\text{Re}(\alpha)Tt})), \\ M_{44} &= \frac{1}{T} (C_0^{+-} e^{-2\text{Re}(\alpha)Tt} + C_0^{-+}). \end{aligned}$$

If  $M_{fm}$  is separable at some time  $t$ , the factorization law for entanglement decay [107] allows us to assert that all states will show ESD. The state  $M_{fm}$  is separable if only if it is positive under partial transposition, i.e,

$$1 + X^2 - 2X - \frac{T^2}{C_0^{-+}C_0^{-+}}X \geq 0, \quad (10.11)$$

where  $X = \exp(-2\text{Re}(\alpha)Tt)$ . Therefore,  $M_{fm}$  is separable when LHS of Eq. (10.11) is zero. The roots of the above equation are

$$X_{\pm} = \frac{1}{2} \left[ \left( 2 + \frac{T^2}{C_0^{-+}C_0^{-+}} \right) \pm \sqrt{\left( 2 + \frac{T^2}{C_0^{-+}C_0^{-+}} \right)^2 - 4} \right]. \quad (10.12)$$

The root  $X_-$  is less than unity, implying that there exists, always, a finite and positive time  $t_{ESD}$  at which the system loses all its entanglement. This is given by

$$t_{ESD} = -\frac{1}{2\text{Re}(\alpha)T} \log(X_-). \quad (10.13)$$

The modulation factor  $\nu$  appears in the numerator of Eq. (10.13) and, therefore, it can be expected that a higher frequency of modulation should sustain entanglement longer. This is confirmed in the plot of  $t_{ESD}$  against  $\nu$  (FIG. (10.2)). This result is

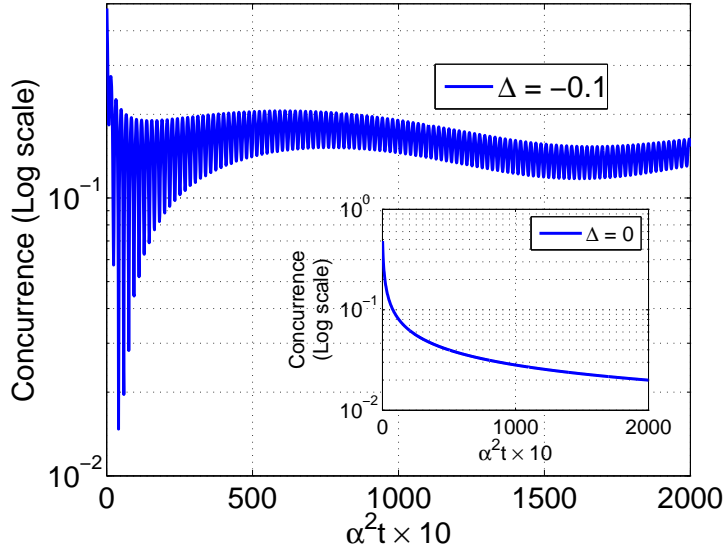
not altogether surprising, for a higher degree of modulation is naturally expected to increase the coherence by filtering out the influence of the bath, which ultimately results in entanglement sustaining for a longer period of time.

### 10.3 Resonance fluorescence

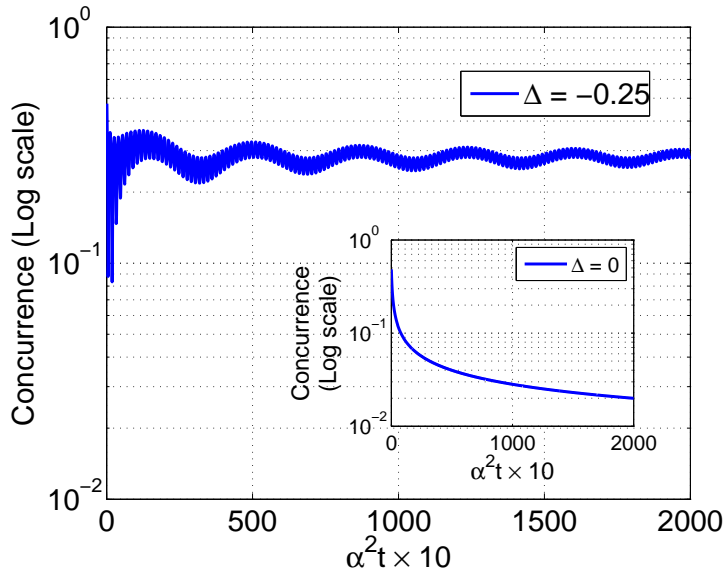
In the previous section, we focused on the decrease in the time to ESD by increasing the degree of frequency modulation of the system-bath coupling. In this section, we study a system where a two-level atomic transition is driven by an external coherent single-mode field which is in resonance with the transition itself. We shall show that, in this situation, an increase in the Rabi frequency — which plays the role of the modulator — produces the opposite effect by speeding up ESD. The behavior of such driven systems has been well studied in the literature and has found many applications. In contrast to the situation here, Lam and Savage [110] have investigated a two-level atom driven by polychromatic light. The phenomenon of tunneling in a symmetric double-well potential perturbed by a monochromatic driving force was analyzed by Grossmann *et al.*, [89], while photon-assisted tunneling in a strongly driven double-barrier tunneling diode has been studied by Wagner [173].

The analysis of the said driven system begins with its Hamiltonian which, when written in the interaction picture, is  $H_{SR} = -E(t) \cdot D(t)$ . Here  $E(t) = \varepsilon e^{-i\omega_0 t} + \varepsilon^* e^{i\omega_0 t}$  is the electric field strength of the driving mode (treated classically),  $\omega_0$  is the atomic transition frequency and  $D(t)$  is the dipole moment operator in the interaction picture. The driven two-level system is coupled to a thermal reservoir of radiation modes. If  $\gamma_0$  be the spontaneous rate due to coupling with the thermal reservoir and  $N = N(\omega_0)$  be the Planck distribution at the atomic transition frequency  $\omega_0$ , the evolution of this composite system is given by the following master equation [33]:

$$\begin{aligned} \frac{d}{dt}\rho(t) = & \frac{i\Omega}{2}[\sigma_+ + \sigma_-, \rho(t)] \\ & + \frac{\gamma_0(N+1)}{2} [2\sigma_- \rho(t) \sigma_+ - \sigma_+ \sigma_- \rho(t) - \rho(t) \sigma_+ \sigma_-] \\ & + \frac{\gamma_0(N)}{2} [2\sigma_+ \rho(t) \sigma_- - \sigma_- \sigma_+ \rho(t) - \rho(t) \sigma_- \sigma_+], \end{aligned} \quad (10.14)$$



(a) Entanglement (concurrency) of  $M$  as a function of  $\tau = \alpha^2 t$  for detuning parameter  $\Delta = -0.1$ . In this plot the behaviour of entanglement is different from the one in the case for  $\Delta = 0$  (see inset). Here entanglement is seen to converge to a non-zero value at large  $\tau$ .



(b) Entanglement (concurrency) of  $M$  as a function of  $\tau = \alpha^2 t$  for detuning parameter  $\Delta = -0.25$ . This plot shows that higher the magnitude of detuning, larger will be the asymptotic value of entanglement.

Figure 10.1: Evolution of entanglement in photonic band gap crystals at zero temperature.



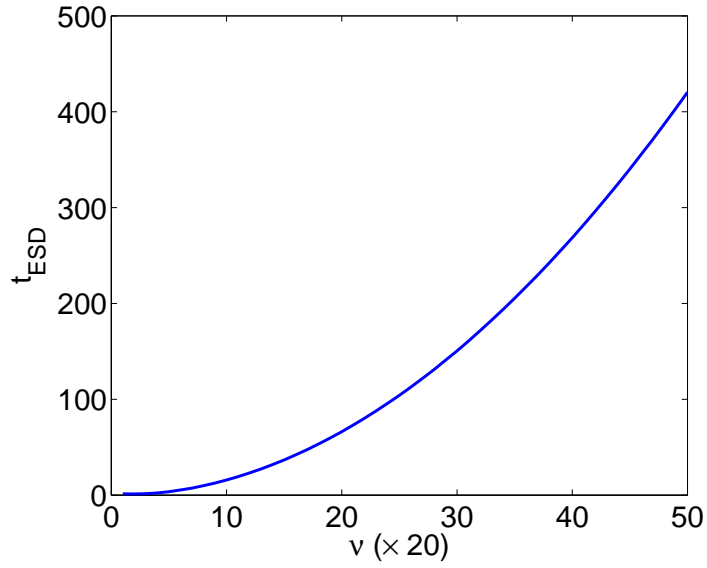


Figure 10.2: In this plot we have time of ESD, i.e, the time at which a maximally entangled initial state loses all its entanglement and become separable when exposed to a bath, against the frequency of modulation  $\nu$ . In this case we have kept the value of  $m$  to be the first zero of the Bessel function  $J_0$ , i.e,  $m = 2.4048$ . Also, at  $\nu = 0$  the value of  $t_{ESD}$  is 1.4. The value of the other parameters are:  $\kappa = 0.1$ ,  $\Delta = 0.1$ ,  $C_0^{+-} = C_0^{-+} = 0.1$ .

where  $\Omega = 2\varepsilon \cdot d^*$  is the Rabi frequency and  $d$  is the transition matrix element of the dipole operator. The term  $-(\Omega/2)[\sigma_+ + \sigma_-]$  characterizes the interaction between the atom and the external driving field in the rotating wave approximation. As usual,  $\sigma_{\pm}$  are the atomic raising and lowering operators, respectively.

Let us consider two identical qubits and, as before, assume that one of them interacts locally with a thermal bath and is subject to monochromatic driving by an external coherent field. The master equation (Eq. 10.14) yields the corresponding matrices  $V_{rf}$  and  $M_{rf}$  (where the subscript  $rf$  stands for resonance fluorescence):

$$V_{rf} = \begin{pmatrix} a_1 & a_2 & a_2^* & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ b_1^* & b_3^* & b_2^* & b_4^* \\ d_1 & -a_2 & -a_2^* & d_4 \end{pmatrix}, \quad (10.15)$$

$$M_{rf} = \frac{1}{2} \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_2^* & a_4 & b_3 & b_4 \\ b_1^* & b_3^* & d_1 & -a_2 \\ b_2^* & b_4^* & -a_2^* & d_4 \end{pmatrix}, \quad (10.16)$$

where

$$a_1 + a_4 = 1 + \left( 1 - X^3 \left( \cos(\mu t) - \frac{\gamma}{4\mu} \sin(\mu t) \right) \right) S_3 + \frac{i\Omega}{\mu} X^3 \sin(\mu t) (S_- + S_+),$$

$$a_1 - a_4 = X^3 \left[ \cos(\mu t) - \frac{\gamma}{4\mu} \sin(\mu t) \right],$$

$$a_2 = \frac{i\Omega}{\mu} X^3 \sin(\mu t),$$

$$b_1 + b_4 = -X^2(S_+ + S_-) - \frac{i\Omega}{\mu} X^3 \sin(\mu t) S_3 + X^3 \left( \cos(\mu t) + \frac{\gamma}{4\mu} \sin(\mu t) \right) (S_- - S_+),$$

$$b_1 - b_4 = \frac{i\Omega}{\mu} X^3 \sin(\mu t),$$

$$b_{2,3} = \frac{1}{2} X^2 \pm X^3 \left( \cos(\mu t) + \frac{\gamma}{4\mu} \sin(\mu t) \right),$$

$$d_1 + d_4 = 2 - (a_1 + a_4),$$

$$\begin{aligned}
 d_1 - d_4 &= -(a_1 - a_4), \\
 X &= e^{-\frac{\gamma t}{4}}, \\
 S_+ &= -\frac{i\Omega\gamma_0}{\gamma^2 + 2\Omega^2}, \\
 S_- &= S_+^*, \\
 S_3 &= -\frac{\gamma_0\gamma}{\gamma^2 + 2\Omega^2}, \\
 \gamma &= \gamma_0(2N + 1), \\
 \mu &= \sqrt{\Omega^2 - (\gamma/4)^2}.
 \end{aligned}$$

Using these, we plot, in FIG. (10.3), concurrence vs the time to ESD for different values of the Rabi frequency  $\Omega$  and observe that  $t_{ESD}$  decreases for an increase in  $\Omega$ . This is contrary to the result derived in the previous section, where an increase in the modulation frequency  $\nu$  delayed the loss of entanglement. The decrease in  $t_{ESD}$  does not however continue indefinitely, but rather saturates to a certain value for large values of the Rabi frequency. Figure (10.4) depicts an increase in the single-qubit coherence with an increase in the Rabi frequency  $\Omega$ , bringing out the fact that here coherence and entanglement behave in a different fashion. This puts into perspective the fact that coherence, a local property, need not be monotonic with entanglement, a non-local property of quantum correlations.

Let us now consider the situation where the system, consisting of the excited two-level atom, is at zero temperature. Let us also consider the evolution of entanglement for two cases demarcated by the relation between the Rabi frequency and the spontaneous rate of coupling with the thermal reservoir. For the underdamped case when  $\Omega > \gamma_0/4$ , the quantity  $\mu$  is real (since  $N = 0$  at  $T = 0$ ) and hence both the upper level occupation and coherence exhibit exponentially damped oscillations. Conversely, in the overdamped case,  $\Omega < \frac{\gamma_0}{4} \Rightarrow \mu$  is purely imaginary and both these quantities decay monotonically to their stationary values. The evolution of entanglement, however, works in an opposite way. Entanglement decays faster for the underdamped case than for overdamping, where the  $t_{ESD}$  is higher. One possible reason for this could be the relative positions of the three Lorentzian peaks of the inelastic part of the resonance fluorescence spectrum. The central peak is at  $\omega = \omega_0$  and the rest are at  $\omega = \omega_0 \pm \mu$  [33] for the underdamped case,

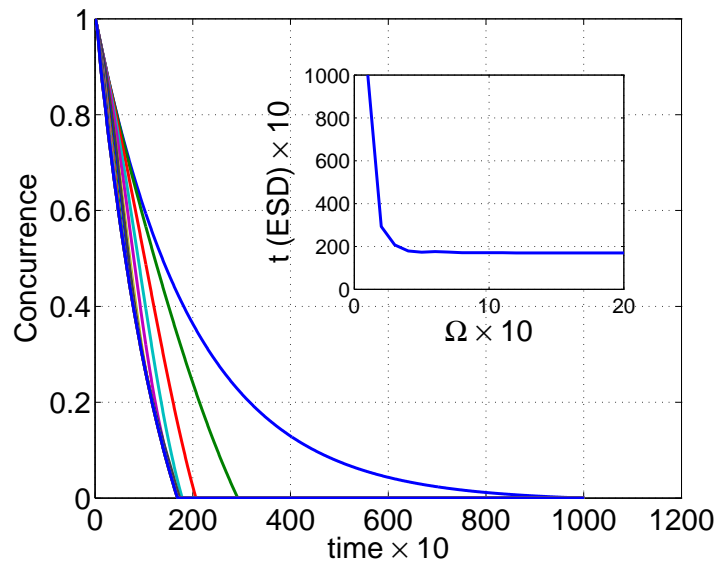


Figure 10.3: Entanglement (concurrence) of  $M_{r,f}$  as a function of time for different values of Rabi frequency  $\Omega$ , varying from 0 to 0.5: 0 (the last curve on the right hand side) corresponding to pure damping and 0.5 (first curve on the left hand side) corresponding to the underdamped case, i.e., it covers both the overdamped as well as the underdamped cases. In the inset one can see that as we increase the  $\Omega$  the  $t_{ESD}$  seem to converge at  $t = 17.0$ . Here  $\gamma = 0.1$ .

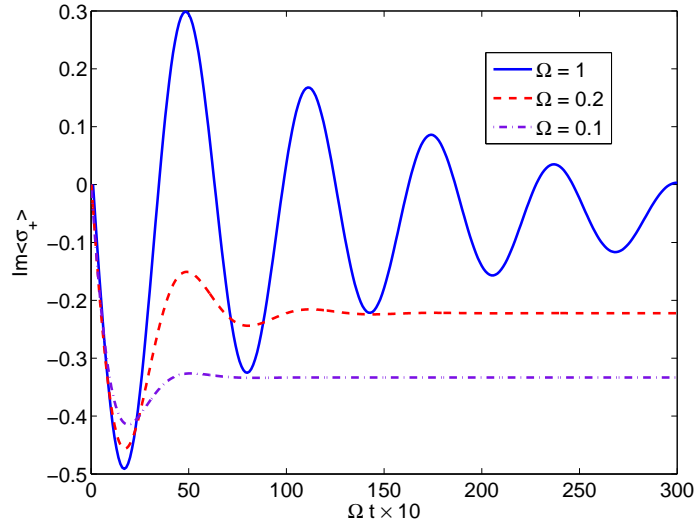


Figure 10.4: The plot for the evolution of coherence for a single qubit when the initial state of the qubit is  $|\psi\rangle = |0\rangle$  in the presence of thermal bath. We can see that the coherence increases as the Rabi frequency  $\Omega$  is increased.

whereas all three peaks are at  $\omega = \omega_0$  for the overdamped case. This indicates that the decay of entanglement in the underdamped should be closely dependent on the quantity  $\mu$ . This in turn depends on both the dissipation parameter  $\gamma$  and the Rabi frequency, the latter in itself a function of the driving strength of the external field and the dipole transition matrix elements. Thus, in the underdamped case, there exists greater avenues for the decay of quantum coherences as well as entanglement than the overdamped case. Phenomenologically, for the underdamped case ( $\Omega > \gamma_0/4$ ), the two-level atom interacts with the external monochromatic field multiple times before spontaneously radiating a photon (see, for example, chapter 10 of ref. [125]). Such numerous interactions allows quantum correlations to develop between the two atomic levels and the quantized levels of the field. The phenomenon of monogamy of entanglement [131] thus ensures that the amount of quantum correlation between the two qubits will decrease. Additionally, it can be seen that at a higher Rabi frequency,  $\Omega$  dominates the dissipation and thus causes a saturation of the time to ESD, as shown in FIG. (10.3).

## 10.4 Dynamic decoupling and the effect on ESD

As discussed earlier, open-loop control strategies involve the application of suitably tailored control fields on the system of interest, with the aim of achieving dynamic decoupling of the system from the environment [170, 171, 169, 3, 5, 6]. Bang-Bang control is a particular form of such decoupling where the decoupling interactions are switched on and off at a rate faster than the rate of interaction set by the environment. The application of suitable radio frequency (RF) pulses, applied fast enough, averages out unwanted effects of the environment and suppresses decoherence. In this section, we compare the effect of Bang-Bang decoupling on the evolution of entanglement, using channel-state duality and factorization law of entanglement decay, in systems connected to two different types of baths. One bath type is composed of infinitely many harmonic oscillators at a finite non-zero temperature  $T$  and couples locally to a two-level atom acting as the qubit, while the other adds random telegraph noise to a Josephson-junction charge qubit. It should be kept in mind that our result in [86] that every two-qubit state shows ESD at all non-zero temperature, was based on a particular type of Markovian master equation. So for the case of telegraphic noise, it may not be wise to extend our result bluntly.

### 10.4.1 Bang-Bang decoupling when the bath consists of harmonic oscillators

#### Quantum Non-Demolition Interaction

Let us consider the interaction of a qubit with a bath of harmonic oscillators where the system Hamiltonian commutes with the interaction Hamiltonian so that there is no exchange of energy between the system and the bath — this is *quantum non-demolition dynamics* [11, 14]. The only effect of the bath will be on the coherence elements of the qubit density operator, which will decay in time at the rate  $\gamma$ . The

total Hamiltonian for the system-bath combine is:

$$\begin{aligned}
 H_0 &= H_q + H_B + H_I; & (10.17) \\
 H_q &= \omega_0 \sigma_z, \\
 H_B &= \sum_k \omega_k b_k^\dagger b_k, \\
 H_I &= \sum_k \sigma_z (g_k b_k^\dagger + g_k^* b_k).
 \end{aligned}$$

Here the system Hamiltonian  $H_q$  commutes with the interaction Hamiltonian  $H_I$  and the evolution of such a system is called *pure dephasing*. For simplicity we will work in the interaction picture where the density matrix of the system-bath combine and the interaction Hamiltonian transform as:

$$\tilde{\rho}(t) = e^{i(H_q+H_B)t} \rho(t) e^{-i(H_q+H_B)t}, \quad (10.18)$$

$$\tilde{H}(t) = \sigma_z \sum_k (g_k b_k^\dagger e^{i\omega_k t} + g_k^* b_k e^{-i\omega_k t}). \quad (10.19)$$

From here we can write the total time evolution operator for the system plus bath as

$$\begin{aligned}
 \tilde{U}(t_0, t) &= T \exp \left\{ -i \int_{t_0}^t ds \tilde{H}(s) \right\} \\
 &= \exp \left\{ \frac{\sigma_z}{2} \sum_k [b_k^\dagger e^{i\omega_k t_0} \xi_k(t-t_0) - b_k e^{-i\omega_k t_0} \xi_k^*(t-t_0)] \right\}, & (10.20)
 \end{aligned}$$

where  $\xi_k(t) = \frac{2g_k}{\omega_k} (1 - \exp(i\omega_k t))$ . We are interested in calculating

$$\tilde{\rho}_{01}(t) = \langle 0 | \text{Tr}_B \left\{ \tilde{U}(t_0, t) \tilde{\rho}(t_0) \tilde{U}^\dagger(t_0, t) \right\} | 1 \rangle. \quad (10.21)$$

Assuming that the bath and the qubit were uncorrelated in the beginning and that the bath is in a thermal state, we have [170]:

$$\tilde{\rho}_{01}(t) = \tilde{\rho}_{01}(t_0) e^{-\gamma(t_0, t)}, \quad (10.22)$$

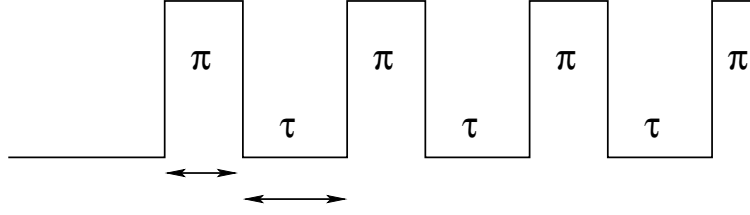


Figure 10.5: Sketch of the pulse sequence used in bang-bang decoupling procedure.

where

$$\gamma(t_0, t) = \sum_k \frac{|\xi_k(t - t_0)|^2}{2} \coth\left(\frac{\omega_k}{2T}\right). \quad (10.23)$$

The matrix representation of the evolution operator  $V_{QND}$  can be written from here as:

$$V_{QND} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-\gamma(t_0, t)} & 0 & 0 \\ 0 & 0 & e^{-\gamma(t_0, t)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (10.24)$$

The evolution of the maximally entangled state  $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$  provided sufficient information considering the evolution of entanglement. The evolution of one subsystem in state  $|\phi^+\rangle$  gives rise to the density matrix:

$$M_{QND} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & e^{-\gamma(t_0, t)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{-\gamma(t_0, t)} & 0 & 0 & 1 \end{pmatrix}. \quad (10.25)$$

The concurrence in the state  $M$  is directly proportional to  $e^{-\gamma(t_0, t)}$ .

### Dephasing under Bang-Bang dynamics

The function of Bang-Bang decoupling is to hit the system of interest with a sequence of fast radio-frequency pulses with the aim of slowing down decoherence (see FIG. 10.5). Adding the radio frequency term to the system-plus-bath Hamiltonian



$H_0$  (Eq. (10.17)), we get

$$H(t) = H_0 + H_{RF}(\omega_0, t), \quad (10.26)$$

$$H_{RF}(t) = \sum_{n=1}^{n_p} U^{(n)}(t) \{ \cos[\omega_0(t - t_p^{(n)})] \sigma_x + \sin[\omega_0(t - t_p^{(n)})] \sigma_y \}, \quad (10.27)$$

where  $t_p^{(n)} = t_0 + n\Delta t$ ,  $n = 1, 2, \dots, n_p$ , and

$$U^{(n)}(t) = \begin{cases} U & t_p^{(n)} \leq t \leq t_p^{(n)} + \tau_p \\ 0 & \text{elsewhere.} \end{cases} \quad (10.28)$$

The term  $H_{RF}$  acts only on the system of interest which here is the qubit. It represents a sequence of  $n_p$  identical pulses, each of duration  $\tau_p$ , applied at instants  $t = t_p^{(n)}$ . The separation between the pulses is  $\tau = \Delta t$ . The decay rate for this pulsed sequence evolution is [170]:

$$\gamma_p(N, \Delta t) = \sum_k \frac{|\eta_k(N, \Delta t)|^2}{2} \coth\left(\frac{\omega_k}{2T}\right), \quad (10.29)$$

where

$$|\eta_k(N, \omega_k \Delta t)|^2 = 4(1 - \cos(\omega_k \Delta t))^2 \times \left( N + \sum_{n=0}^{N-1} 2n \cos[2(N-n)\omega_k \Delta t] \right). \quad (10.30)$$

In [170] it has also been shown that  $|\eta_k|^2 \leq |\xi_k|^2$  which implies that decoherence is suppressed. Also, it is evident that a lower value of  $\eta$  implies a lower value of  $\gamma$ . Consequently, we conclude that Bang-Bang decoupling slows down entanglement decay.

### 10.4.2 Josephson Junction qubit

Although solid state nanodevices satisfy the requirements of large scale integrability and flexibility in design, they are subject to various kinds of low-energy excitations in the environment and suffer from decoherence problems. There have been a number of proposals in this context about the implementation of quantum

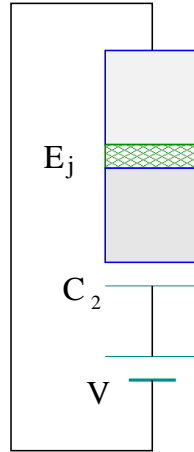


Figure 10.6: Schematic diagram for Josephson-junction charge qubit

computers using superconducting nanocircuits [118, 68]. Experiments highlighting the quantum properties of such devices have already been performed [127, 74]. Here the concept of a Josephson-junction qubit comes into prominence. A charge-Josephson qubit is a superconducting island connected to a circuit via a Josephson junction and a capacitor. The computational states are associated with charge  $Q$  in the island and are mixed by Josephson tunneling. For temperatures much lower than the Josephson energy,  $k_B T \ll E_j$  [153, 119, 133, 132], we have the Hamiltonian

$$H_Q = \frac{\epsilon}{2} \sigma_z - \frac{E_j}{2} \sigma_x, \quad (10.31)$$

with the charging energy  $E_C$  dominating the Josephson energy. Here,  $\epsilon \equiv \epsilon(V) = 4E_C(1 - C_2 V/e)$ ,  $C_2$  is the capacitance of the capacitor connected to the island and  $V$  is the external gate voltage (see Fig. (10.6)).

Fluctuating background charges (BCs) (charge impurities) are an important source of decoherence in the operation of Josephson charge qubits. These are believed to originate in random traps for single electrons in dielectric materials surrounding the superconducting island. These fluctuations cause at low frequencies, the  $1/f$  noise which is also known as random telegraph noise, and is directly observed in single electron tunneling devices [190, 128]. This has also been studied in the context of fractional statistics in the Quantum Hall Effect [105]. This noise,

arising out of decoherence, is modeled [133, 132] by considering each of the BCs as a localized impurity level connected to a fermionic band, i.e., the quantum impurity is described by the Fano Anderson model. This is the quantum analogue of the classical model of  $N$  independent, randomly activated bistable processes. For a single impurity, the total Hamiltonian is:

$$H = H_Q - \frac{v}{2}b^\dagger b \sigma_z + H^I, \quad (10.32)$$

where

$$H^I = \epsilon_c b^\dagger b + \sum_k \left[ T_k c_k^\dagger b + h.c. \right] + \sum_k \epsilon_k c_k^\dagger c_k. \quad (10.33)$$

Here  $H^I$  describes the BC Hamiltonian,  $b$  represents the impurity charge in the localized level  $\epsilon_c$ ,  $c_k$  the electron in the band with energy  $\epsilon_k$ , and  $H_Q$  is as in Eq. (10.31). The impurity electron may tunnel to the band with amplitude  $T_k$ . The BC produces an extra bias  $v$  for the qubit via the coupling term  $(v/2)b^\dagger b \sigma_z$ . An important scale is the switching rate  $\gamma = 2\pi\rho(\epsilon_c)|T|^2$ , where  $\rho(\epsilon_c)$  is the density of states of the band. It is assumed that we are working in the relaxation regime of the BC where the tunneling rate to all fermionic bands are approximately same. The fraction  $v/\gamma$  determines whether the operational regime of the qubit is weak ( $v/\gamma \ll 1$ ) or strong ( $v/\gamma > 1$ ). Studying the single BC case is important, since it has been shown [133] that the effect of multiple BCs can be trivially extended from that of a single BC. For multiple strongly coupled BCs producing  $1/f$  noise, the effect of a large number of slow fluctuators is minimal and pronounced features of discrete dynamics such as saturation and transient behavior are seen. There are two special operational points for the qubit related to Eq. (10.31): (a)  $\epsilon = 0$ , corresponding to charge degeneracy and (b)  $E_j = 0$ , for the case of pure dephasing [26, 2], where tunneling can be neglected. We will consider this case later in detail and make a comparison of ESD, for the case of pure dephasing, between the harmonic oscillator and  $1/f$  baths.

The general procedure for studying the effect of the BC on the dynamics of the qubit is to calculate the unitary evolution of the entire system-bath combine and then trace out the bath degree of freedom. Thus, i.e,  $\rho_Q(t) = \text{tr}_E\{W(t)\}$ ,  $W(t)$  being the the full density matrix. In the weak coupling limit a master equation for

$\rho_Q(t)$  can be written [53]. The results in the standard weak coupling approach are obtained at lowest order in the coupling  $v$ , but it has been pointed out that higher orders are important for a  $1/f$  noise [153, 119, 133].

The failure of the standard weak coupling approach is due to the fact that the  $1/f$  environment includes fluctuators which are very slow on the time scale of the reduced dynamics. To circumvent this problem one considers another approach in which a part of the bath is treated on the same footing as the system [132]. We study the evolution of this new system and later trace out the extra part which belongs to the bath, i.e.,  $\rho(t) = \text{Tr}_{\text{fb}}\{W(t)\}$ . We then obtain  $\rho_Q(t)$  from  $\rho(t)$  as  $\rho_Q(t) = \text{Tr}_{\text{b}}\{\rho(t)\}$ , where the subscript  $fb$  stands for fermionic band. In that context we split the Hamiltonian (10.32) into a system Hamiltonian  $H_0 = H_Q - \frac{v}{2}b^\dagger b\sigma_z + \epsilon_c b^\dagger b$  and environment Hamiltonian  $H_E = \sum_k \epsilon_k c_k^\dagger c_k$  coupled by  $\mathcal{V} = \sum_k [T_k c_k^\dagger b + h.c.]$ . The eigenstates of  $H_0$  are product states of the form  $|\theta\rangle|n\rangle$ , e.g,

$$\begin{aligned} |a\rangle &= |\theta_+\rangle|0\rangle, \\ |b\rangle &= |\theta_-\rangle|0\rangle, \\ |c\rangle &= |\theta'_+\rangle|1\rangle, \\ |d\rangle &= |\theta'_-\rangle|1\rangle, \end{aligned}$$

with corresponding energies

$$-\frac{\Omega}{2}, \frac{\Omega}{2}, -\frac{\Omega'}{2} + \epsilon_c, \frac{\Omega'}{2} + \epsilon_c.$$

Here  $|\theta_\pm\rangle$  are the two eigenstates of  $\sigma_{\hat{n}}$ , the direction being specified by the polar angle  $\theta$  and  $\phi = 0$ . The two level splittings are  $\Omega = \sqrt{\epsilon^2 + E_j^2}$  and  $\Omega' = \sqrt{(\epsilon + v)^2 + E_j^2}$ , and  $\cos(\theta) = \epsilon/\Omega$ ,  $\sin(\theta) = E_j/\Omega$ ,  $\cos(\theta') = (\epsilon + v)/\Omega'$ ,  $\sin(\theta') = E_j/\Omega'$ .

The master equation for the reduced density matrix  $\rho(t)$ , in the Schrödinger representation and in the basis of the eigenstates of  $H_0$  reads:

$$\frac{d\rho_{ij}(t)}{dt} = -i\omega_i\rho_{ij}(t) + \sum_{mn} R_{ij,mn}\rho_{mn}(t), \quad (10.34)$$

where  $\omega_{ij}$  is the difference of the energies and  $R_{ij,mn}$  are the elements of the Redfield tensor [53]. These are given by :

$$R_{ij,mm} = \int_0^\infty d\tau \left\{ c_{njmi}^>(\tau) e^{i\omega_{mi}\tau} + c_{njmi}^<(\tau) e^{i\omega_{jn}\tau} - \delta_{nj} \sum_k c_{ikmk}^>(\tau) e^{i\omega_{mk}\tau} - \delta_{im} \sum_k c_{nkjk}^<(\tau) e^{i\omega_{kn}\tau} \right\}, \quad (10.35)$$

where

$$c_{ijkl}^{\gtrless}(t) = [\langle i|b|j\rangle\langle l|b^\dagger|k\rangle + \langle i|b^\dagger|j\rangle\langle l|b|k\rangle] iG^{\gtrless}(t). \quad (10.36)$$

Here  $iG^>(\omega) = \gamma/(1 - e^{-\beta\omega})$  is the Fourier transform of  $G^>(t)$  and  $G^<(\omega) = G^>(-\omega)$ , therefore,  $G^<(t) = G^>(-t)$ . This problem has a very interesting symmetry: the diagonal and off diagonal elements do not mix if the initial state of the charge particle is a diagonal density matrix in the BC. Therefore, we can divide the Redfield tensor elements in two parts, one corresponding to population (diagonal elements) and other corresponding to coherence (off diagonal elements).

The  $R_{ii,nn}$  elements which affect the population are:

$$R_{ii,nn} = \int_0^\infty \left\{ \chi_{in} iG^>(\tau) e^{i\omega_{ni}\tau} + \chi_{in} iG^<(\tau) e^{-i\omega_{ni}\tau} \right\} = \chi_{in} [iG^>(\omega_{ni})]. \quad (10.37)$$

Here  $n \neq i$  and  $\chi_{in} = (|\langle n|b|i\rangle|^2 + |\langle n|b^\dagger|i\rangle|^2)$ , and

$$R_{ii,ii} = - \sum_k \chi_{ik} [iG^>(\omega_{ik})]. \quad (10.38)$$

Now we calculate the elements which are responsible for the coherence part. In the adiabatic regime we have  $\gamma \sim \Omega - \Omega' \ll \Omega \sim \Omega'$ , i.e., where the BCs are not static and the mixing of  $\rho_{ab}$  and  $\rho_{cd}$  (10.34), as well as their conjugates, cannot be neglected. Hence the non-zero elements of  $R$  tensor – which affect the coherence

– are the following:

$$\begin{aligned} R_{ab,ab} &= -\frac{\gamma}{2} [1 - c^2\delta - s^2\delta' + i(c^2w + s^2w')], \\ R_{cd,cd} &= -\frac{\gamma}{2} [1 + c^2\delta + s^2\delta' + i(c^2w - s^2w')], \\ R_{ab,cd} &= \frac{c^2\gamma}{2} [1 + \delta - iw], \\ R_{cd,ab} &= \frac{c^2\gamma}{2} [1 - \delta - iw]. \end{aligned}$$

Here

$$\begin{aligned} c &= \cos[(\theta - \theta')/2], \\ s &= \sin[(\theta - \theta')/2], \\ \delta &= t_{ca} + t_{db}, \\ \delta' &= t_{da} + t_{cb}, \\ w &= w_{ca} - w_{cb}, \\ w' &= w_{da} - w_{cb}, \\ t_{ij} &= \frac{1}{2} \tanh\left(\frac{\beta\omega_{ij}}{2}\right), \\ w_{ij} &= -\frac{1}{\pi} \mathcal{R} \left\{ \psi\left(\frac{\pi + i\beta\omega_{ij}}{2\pi}\right) \right\}, \end{aligned}$$

and  $\psi(z)$  is the digamma function.

Now we can construct the explicit form of the matrix  $R$ :

$$R = \left( \begin{array}{cccc|cccc|cccc|cccc} R_{1,1} & 0 & 0 & 0 & 0 & R_{1,2} & 0 & 0 & 0 & 0 & R_{1,3} & 0 & 0 & 0 & 0 & R_{1,4} \\ 0 & z_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & z_-^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_+^* & 0 \\ R_{2,1} & 0 & 0 & 0 & 0 & R_{2,2} & 0 & 0 & 0 & 0 & R_{2,3} & 0 & 0 & 0 & 0 & R_{2,4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ R_{3,1} & 0 & 0 & 0 & 0 & R_{3,2} & 0 & 0 & 0 & 0 & R_{3,3} & 0 & 0 & 0 & 0 & R_{3,4} \\ 0 & y_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_+ & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_-^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_+^* & 0 \\ R_{4,1} & 0 & 0 & 0 & 0 & R_{4,2} & 0 & 0 & 0 & 0 & R_{4,3} & 0 & 0 & 0 & 0 & R_{4,4} \end{array} \right), \quad (10.39)$$

where

$$z_- = -\frac{\gamma}{2} [1 - c^2\delta - s^2\delta' + i(c^2w + s^2w')],$$

$$z_+ = -\frac{\gamma}{2} [1 + c^2\delta + s^2\delta' + i(c^2w - s^2w')],$$

$$y_+ = \frac{c^2\gamma}{2} [1 + \delta - iw],$$

$$y_- = \frac{c^2\gamma}{2} [1 - \delta - iw],$$

$$R_{i,j} = R_{ii,jj}.$$

Channel-state duality implies that the exponential of the matrix  $R$  is the matrix representation of the evolution channel. Therefore, we have  $V = \exp(Rt)$  which gives us the evolution for the qubit plus the charge impurity. This map is in the basis  $\{|\theta_{\pm}\rangle|i\rangle\} \otimes \{|\theta_{\pm}\rangle|j\rangle\}$ . To make it computationally easier we need to write it

in the basis  $\{|\theta_{\pm}\rangle|\theta_{\pm}\rangle\} \otimes \{|i\rangle|j\rangle\}$ , since the matrix representation of the map acting on the qubit is in the basis  $\{|\theta_{\pm}\rangle|\theta_{\pm}\rangle\}$ . To change the basis we need the assistance of a unitary matrix (in this case permutation matrix)  $P$  which is defined as:

$$P(|a\rangle|b\rangle|c\rangle|d\rangle) = |a\rangle|c\rangle|b\rangle|d\rangle; \quad (10.40)$$

$$|a\rangle|b\rangle|c\rangle|d\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \otimes \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \quad (10.41)$$

$$|a\rangle|c\rangle|b\rangle|d\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \otimes \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}. \quad (10.42)$$

$$\Rightarrow P = \mathcal{I} \otimes p \otimes \mathcal{I}, \quad (10.43)$$

where

$$p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

After conjugating the matrix  $V$  by  $P$  we get:

$$\tilde{V} = PV P^T. \quad (10.44)$$

We can write this  $16 \times 16$  matrix  $\tilde{V}$  as a  $4 \times 4$  matrix, where each of the element itself is a  $4 \times 4$  matrix  $Z_{ij}$  where  $i, j \in \{1, 2, 3, 4\}$ . Then the map  $V_s$  acting on qubit is simply  $V_{sij} = \text{Tr}(Z_{ij})$ . From here we can get the corresponding  $M$  matrix. It is not easy to solve it analytically in the present case. Therefore, we use numerical methods to calculate the evolution operator and entanglement evolution for a system of qubits.

The two parameters  $\epsilon$  and  $E_j$  in the Hamiltonian for the charge Josephson qubit  $H_Q$  play a crucial role in the decoherence properties of the system. For example, if  $E_j = 0$ , the system Hamiltonian  $H_Q$  commutes with the interaction Hamiltonian. This situation, as mentioned earlier, is called non-demolition evolution or pure dephasing. In this case there is no energy exchange between system and the bath.



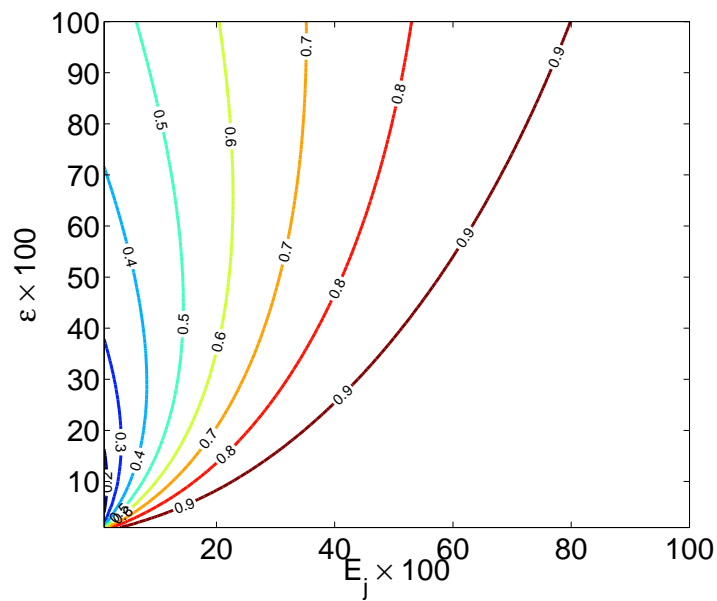


Figure 10.7: Contour of the entanglement after time  $t = 5$  for all values of  $E_j$  and  $\epsilon$  (Josephson junction Hamiltonian parameters) The temperature in this case is equal to 0,  $\gamma = 1$ ,  $\kappa = v/\gamma = 0.45$ .

On the other hand when we have  $\epsilon = 0$ , the system Hamiltonian does not commute with the interaction Hamiltonian. Therefore, the two situations are qualitatively different. We present, in Fig. (10.7), a plot of entanglement in the phase space of  $\epsilon$  and  $E_j$  by evolving a maximally entangled state of two qubits with the bath (of charge impurities) acting only on one qubit. The qubit is evolved for a fixed time  $t$  and the entanglement is calculated for different values of  $\epsilon$  and  $E_j$ . Fig. (10.7) shows that the entanglement in the system increases with an increase in  $E_j$  when  $\epsilon$  is held fixed, but decreases with an increase in  $\epsilon$  when  $E_j$  is held fixed. This is counterintuitive because dissipation increases with the increase in the value of  $E_j$ . Fig. (10.8) compares the time-evolution of entanglement for the harmonic oscillator bath with the charge-impurities bath, both under pure dephasing. While entanglement decay is exponential for the case of  $1/f$  noise, it is slower for a bath of harmonic oscillators. We compare, in FIG. (10.9), the time-evolution of entanglement for various values of Josephson energy ( $E_j$ ) starting with the pure dephasing case given by  $E_j = 0$ . We see that the entanglement remaining in the system increases with an increase in the value of Josephson energy. This is consistent with FIG. (10.7).

Decoherence produced by background charges depends qualitatively on the ratio  $\kappa = v/\gamma$ , where  $\kappa \ll 1$  denotes the weak-coupling regime and  $\kappa > 1$  is the strong coupling regime. The latter gives rise to qualitatively new properties. We find that (see FIG. (10.10(b))), for  $\kappa > 1$ , the time-evolution of entanglement does not depend on  $\kappa$ . This is in contrast to the weak coupling regime, where the time-evolution of entanglement does depend on  $\kappa$ , as seen in FIG. (10.10(a)), where an increase in  $\kappa$  leads to a decrease in entanglement. Naturally, decoherence due to the bath forces entanglement to decay with time for both cases.

### Evolution operator with Bang-Bang interaction

The Josephson charge qubit in contact with an  $1/f$  bath is now subject to fast pulses, under the Bang-Bang dynamical decoupling scheme. The Hamiltonian for this radio frequency pulse is the same as in Eq. (10.27). If the time for which a pulse is active is  $\pi$ , then the evolution operator for the pulse may be written as

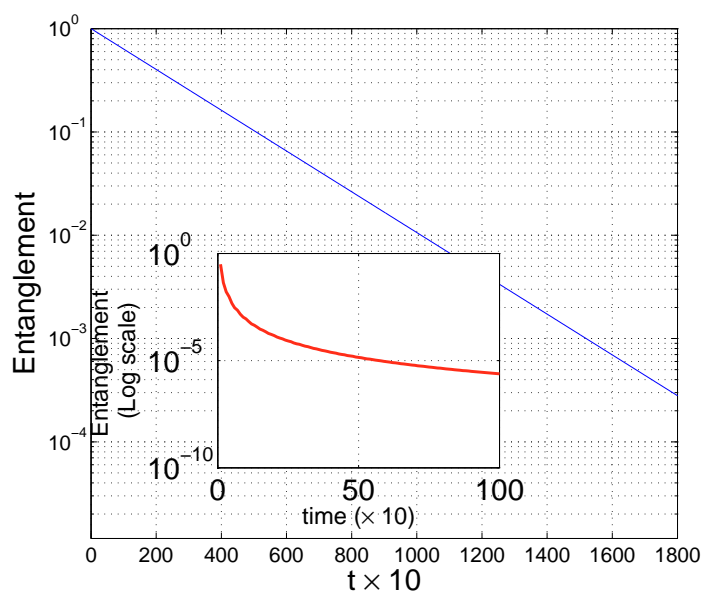


Figure 10.8: Evolution of entanglement for the pure dephasing case, i.e.,  $E_j = 0$ . We can see that whereas entanglement decay is exponential for the  $1/f$  noise, it slows down for a bath of harmonic oscillator (inset) at zero temperature.

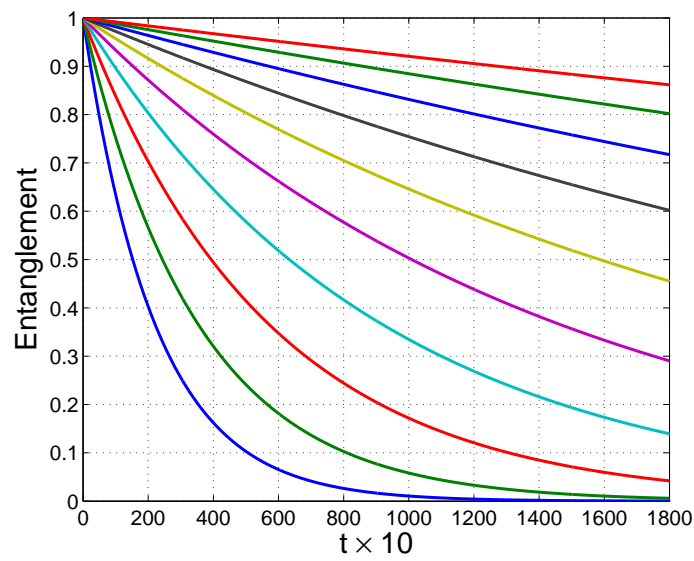
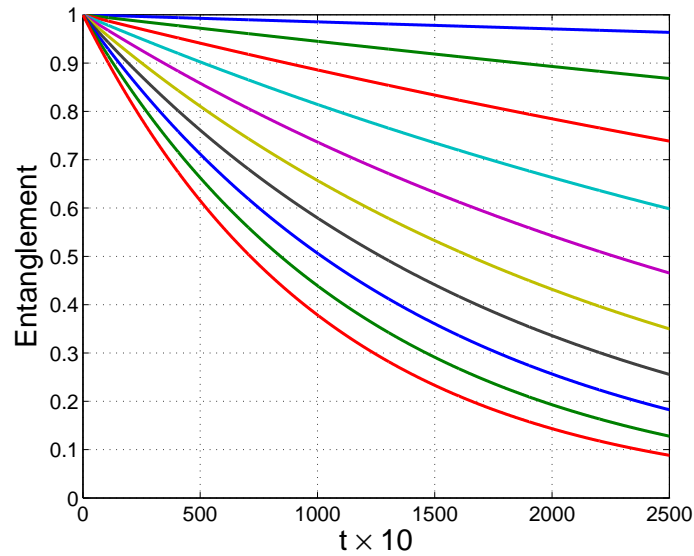
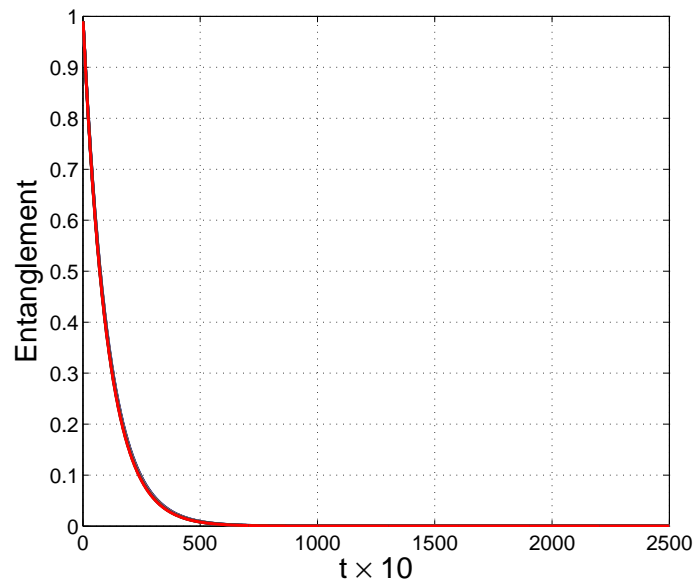


Figure 10.9: Evolution of entanglement for  $1/f$  (telegraph) noise at zero temperature. Here different curves represent the evolution of entanglement for different values of  $E_j$  (from 0 to 1), with the curve at the bottom corresponding to  $E_j = 0$  and that at the top to  $E_j = 1$ , while  $\epsilon$  is fixed and equal to 1.



(a) Plot of the entanglement (concurrency) as a function of time ( $t$ ) for different values of coupling strength  $\kappa = v/\gamma$ , in the weak coupling regime, i.e.  $\kappa \ll 1$ . Here the range of  $\kappa$  is from 0.05 to 0.5, with 0.05 corresponding to the uppermost curve, and 0.5 to the lowest (bottom) one. We can see that as we increase  $\kappa$ , entanglement decreases.



(b) Plot of the entanglement (concurrency) as a function of time ( $t$ ) for different values of coupling strength  $\kappa = v/\gamma$ , in the strong coupling regime, i.e.  $\kappa > 1$ . Here the range of  $\kappa$  is from 5.05 to 5.5. We can see that in the strong coupling region all the curves converge.

Figure 10.10: Evolution of entanglement with respect to time, for different coupling strengths and temperature  $T = 0$ ,  $E_j = 1 = \epsilon$ . 140

$V_p = \mathcal{I} \otimes i\sigma_x$  where  $2U\tau_p = \pm\pi$ . The total evolution can therefore be written as

$$V_{total} = (V_p V_S(\tau))^{2N} \quad (10.45)$$

where  $2N\tau = t$ . Since the RF pulses act on the system for very short amounts of time, the evolution of the system can safely be assumed to be governed only by the dynamical map  $V_p$  for the time period during which the pulse is operating. As can be seen from Figs. (10.11) to (10.13) and FIG. (10.16), the system exhibits the phenomenon of ESD on the application of bang-bang pulses.

Let us consider the case where  $E_j = \epsilon = 1$ . Let us also fix the pulse strength to be  $U = 50\pi$  and ensure that the pulses act for very short times. As defined earlier, the ratio of the BC bias  $v$  and the switching rate  $\gamma$  defines the weak and strong coupling regimes, the former designated by  $\frac{v}{\gamma} \ll 1$  and the latter by  $\frac{v}{\gamma} > 1$ . In FIG. (10.11), we plot the time-evolution of entanglement, with the coupling strength as parameter. For weak coupling, we find that  $t_{ESD}$  initially increases with coupling strength. This continues till a turning point is reached at  $\frac{v}{\gamma} = 0.38$  when  $t_{ESD} \simeq 880$ . After this, with increase in coupling strength,  $t_{ESD}$  starts to decrease. As a result, a kink appears in the corresponding entanglement vs time plot, FIGS. (10.11), (10.12). The receding of  $t_{ESD}$  with increase in coupling strength continues well into the strong coupling regime, i.e. for  $5.05 < \frac{v}{\gamma} < 5.5$ . It, however, does not go to zero, but rather chooses to saturate at the threshold value of  $t_{ESD} \simeq 10$ , see FIG. (10.13). The ‘‘turning’’ and the ‘‘saturation’’ features are well captured in FIG. (10.16), where we plot  $t_{ESD}$  against  $\frac{v}{\gamma}$  and keep the pulse strength and durations fixed. We observe a crossover phenomenon around  $\frac{v}{\gamma} \simeq 0.38$ , where the value of  $t_{ESD}$  rises sharply, only to fall back again even quicker.

The evolution of coherence with respect to time, for the Josephson charge qubit subjected to  $1/f$  noise, is shown in Figs. (10.14), (10.15), for the weak and strong coupling regimes, respectively. Both show an improvement in the coherence with the application of the bang-bang decoupling pulses, in contrast to the corresponding behavior of entanglement, thereby reiterating that coherence is not synonymous with entanglement.

In FIG. (10.17), we plot the behavior of  $t_{ESD}$  with  $E_j$  and find that, as we

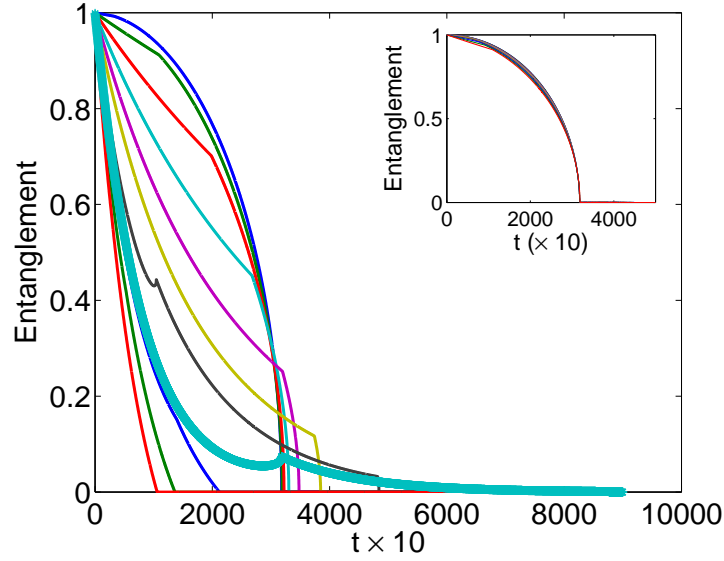


Figure 10.11: Effect of bang-bang decoupling on entanglement, in the weak coupling regime. If we compare this plot with FIG. (10.10(a)), with the curves corresponding to the same values of  $\kappa$ , we see that the bang-bang decoupling causes entanglement to disappear faster in time for a fixed value of the coupling strength. Here the parameters are same as in FIG. (10.10(a)) and the pulse strength is  $U = 50\pi$  with time for which the pulse was activated is  $\tau_p = 0.01$ . In the inset we have the evolution of entanglement for very small range (0.01 to 0.1) of coupling  $\kappa$ . The thickest curve is the one corresponding to  $\kappa = 0.38$ . This curve is important in the sense that it has the largest  $t_{ESD}$ .

increase  $E_j$  and thus move away from the pure dephasing situation, the time to ESD keeps increasing. As discussed earlier, this is a counterintuitive result because dissipation increases with  $E_j$ .

## 10.5 Summing up

As stated earlier, the aim of most control procedures is to suppress decoherence. For the case of photonic crystals, the design allows the system to conserve coherence when it is within the photonic band gap. Modulating the frequency of the system-bath coupling aims to suppress decoherence by shifting the system out of the spectral influence of the bath. In both these cases it is found that the suppression

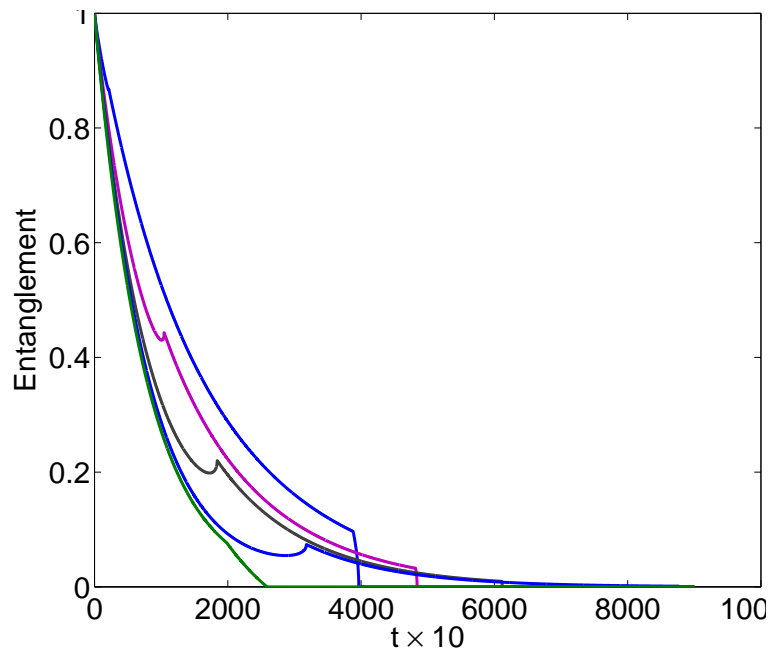


Figure 10.12: Entanglement evolution for the coupling parameter range  $0.3 < \kappa < 0.4$ , with the uppermost curve corresponding to  $\kappa = 0.3$  and the lowest (bottom) curve corresponding to  $0.4$ . One can see from this plot the formation and disappearance of the kink.



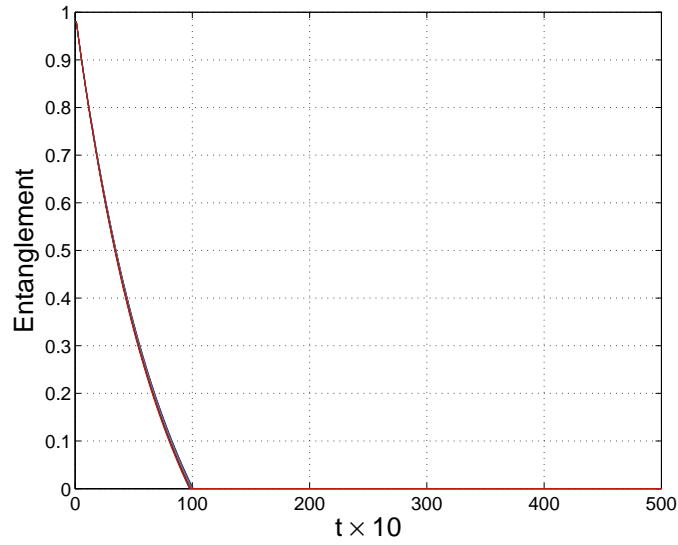


Figure 10.13: Effect of bang-bang decoupling on entanglement in the strong coupling region. Here again we can see the effect of bang-bang decoupling on the entanglement if we compare this plot with FIG.(10.10(b)). Here the parameters are same as in FIG. (10.10(b)) and the pulse strength is  $U = 50\pi$  with the pulse duration  $\tau = 0.01$ .

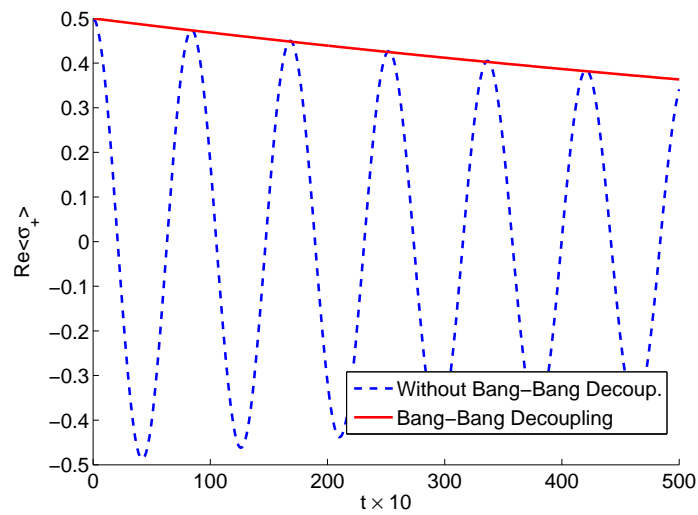


Figure 10.14: Plot for the evolution of coherence in the case of Telegraph noise in weak coupling region, i.e,  $\kappa < 1$ . Here all the parameters has the value same as in Fig. (10.12) and (10.11) and  $\kappa = 0.38$ .

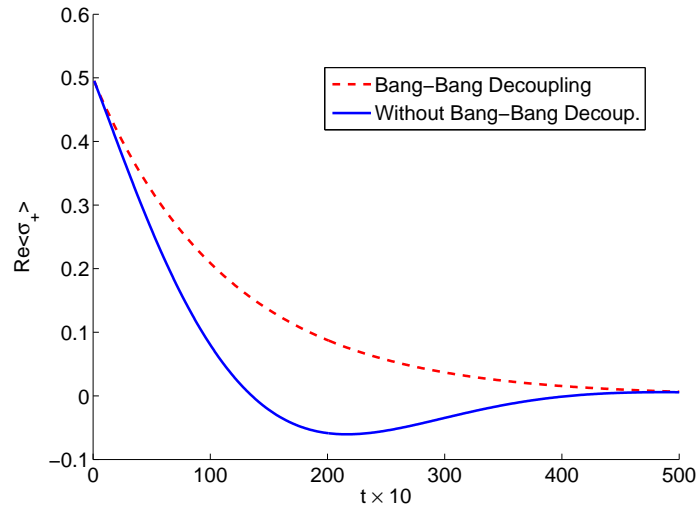


Figure 10.15: Plot for the evolution of coherence in the case of Telegraph noise in strong coupling region, i.e,  $\kappa \geq 1$ . Here the value of all the parameters are same as in Fig. (10.13) and  $\kappa = 5.38$ .

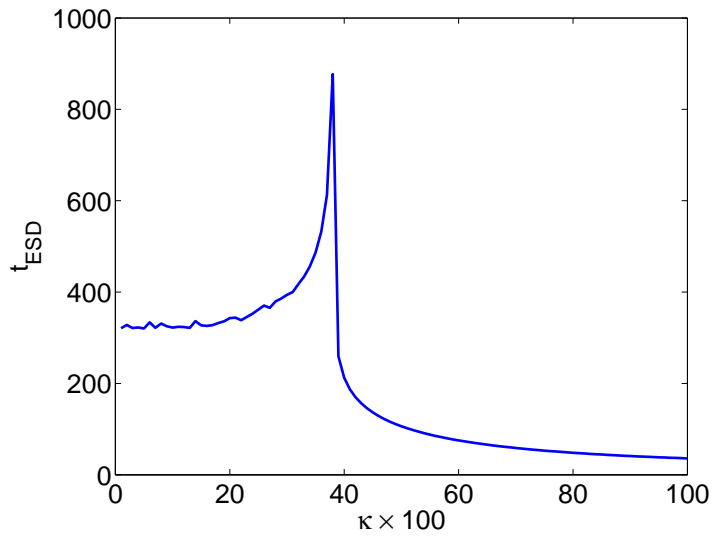


Figure 10.16:  $t_{ESD}$  is plotted as a function of coupling strength  $\kappa$ . Here we can see that there is a clear distinction between the strong and the weak coupling region. As we increase  $\kappa$  the  $t_{ESD}$  tends to freeze and asymptotic value of  $t_{ESD}$  is around 10. The parameters used are as in the previous plots.

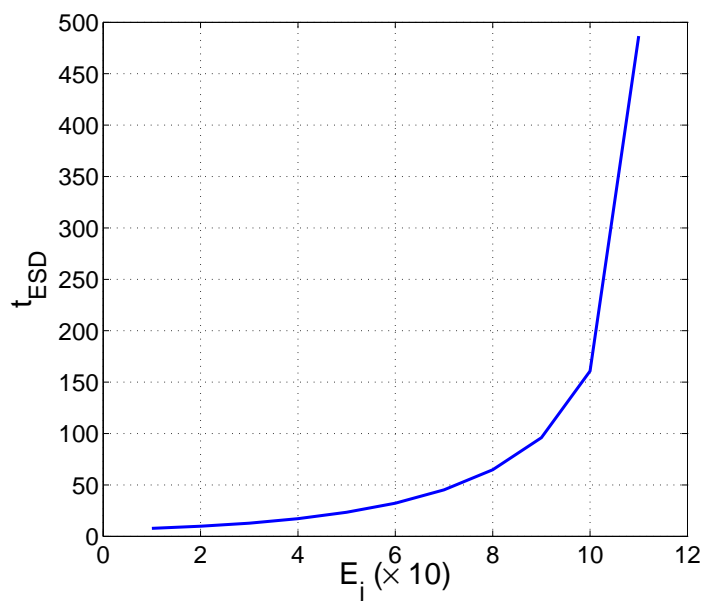


Figure 10.17:  $t_{ESD}$  is plotted as a function of  $E_j$ . This shows that as we go away from pure dephasing the  $t_{ESD}$  increases. The parameters used are same as in the previous plots.

of decoherence is accompanied by a corresponding increase in  $t_{ESD}$ .

However, it will be erroneous to naïvely suppose that this is the norm. Exactly the opposite phenomenon is observed for the case of resonance fluorescence, where the coupling between the bath and a two-level atomic system forced by an external resonant field, is modulated. It is seen that an increase in the external field frequency  $\Omega$ , the Rabi frequency, results in a faster decay of entanglement (FIG. (10.3)). A further non-trivial effect observed is the saturation in the time to ESD:  $t_{ESD}$  does not go below a threshold value no matter what the Rabi frequency. In what could point towards a possible explanation of this phenomenon, we observe that the sudden death time stops being dependent on the Rabi frequency at  $\Omega = \frac{\gamma_0}{4}$ ; strikingly, this happens to be the boundary between the overdamped  $\Omega < \frac{\gamma_0}{4}$  and underdamped  $\Omega > \frac{\gamma_0}{4}$  regimes.

In dynamic decoupling schemes RF pulses, applied at short time-intervals, smooth out unwanted effects due to environmental interactions. We discuss two qualitatively different system-bath models: the first being the usual qubit and harmonic oscillator bath pair with pure dephasing or QND interaction; and the second being a bath of charge impurities, simulating  $1/f$  (telegraph) noise, acting on a Josephson-junction charge qubit. Entanglement decays to zero asymptotically in both these models. The application of fast RF pulses to the former manages to speedup the rate of the still-asymptotic loss of entanglement, whereas the same RF pulse applied to the latter kills off entanglement in finite time and thus shows ESD. A very interesting phenomenon, observed in the strong coupling regime, is the decrease in the time to ESD with increasing pulse strengths. This is extremely counterintuitive, and brings into perspective the fact that in the non-Markovian strong coupling regime, the dynamics of entanglement can be different than that of decoherence. This feature gets further highlighted by the behavior of coherence with time, both for the case of resonance fluorescence and Josephson-junction charge qubit subjected to  $1/f$  noise. Here coherence –which is a local property – is seen to vary in a non-monotonic fashion with entanglement which happens to be a non-local property of the system.

Chapter 10. Effect of control procedures on entanglement dynamics

Control Procedure	Decoherence suppression	Entanglement decay suppression
Photonic Crystals	✓	✓
Frequency Modulation	✓	✓
Resonance Fluorescence	✓	×
DD in EMF bath	✓	✓
DD in Telegraph noise	✓	×

Table 10.1: Summary of the results.



## Assisted suppression of decoherence

In this chapter we will discuss a scheme to suppress the decoherence which is qualitatively different from the schemes we discussed in the previous chapters. The scheme presented here involves introduction of an ancilla in the bath close to the system of interest. The presence of the ancilla causes a shielding effect on the system and hence suppresses the decoherence. The ancilla and the system do not interact directly but by *exchange interactions* through the bath they interact [85].

### 11.1 Dynamics of the system plus ancilla in the presence of thermal bath

Consider a bipartite system  $S_1 = A_1 + B_1$  initially in an entangled state. One of the subsystem (say  $A_1$ ) is exposed to a thermal bath. We have already studied the evolution of entanglement in such systems in chapter 8. We have seen that in the case of thermal bath with non-zero temperature the entanglement in the system  $S_1$  dies out in a short span of time. Let us introduce another bipartite system, namely ancilla,  $S_2 = A_2 + B_2$  where  $A_1$  and  $A_2$  both interacts with the same bath (see Fig. (11.1)), i.e, the bath is acting on  $A_1$  and  $A_2$  together to generate a kind of bath induced interaction between  $A_1$  and  $A_2$ . The Lindblad master equation for

the system  $A_1$  before we introduce  $S_2$  (neglecting the unitary evolution part) is:

$$\frac{d\rho_{A_1}}{dt} = \mathcal{L}_1\rho_{A_1} \quad (11.1)$$

$$= \frac{(N+1)\gamma}{2} [2\sigma_{-}\rho_{A_1}\sigma_{+} - \sigma_{+}\sigma_{-}\rho_{A_1} - \rho_{A_1}\sigma_{+}\sigma_{-}] \\ + \frac{N\gamma}{2} [2\sigma_{+}\rho_{A_1}\sigma_{-} - \sigma_{-}\sigma_{+}\rho_{A_1} - \rho_{A_1}\sigma_{-}\sigma_{+}], \quad (11.2)$$

where  $N$  is the mean occupation number of quanta in the reservoir,  $\gamma$  is the spontaneous decay rate of the qubits,  $\sigma_{+} = |1\rangle\langle 0|$  and  $\sigma_{-} = |0\rangle\langle 1|$ .

Once we introduce  $S_2$  the Lindblad master equation for  $A_1$  will change. To see that consider the master equation for  $A_1A_2$ :

$$\frac{d\rho}{dt} = \frac{(N+1)\gamma}{2} [2J_{-}\rho J_{+} - J_{+}J_{-}\rho - \rho J_{+}J_{-}] \\ + \frac{N\gamma}{2} [2J_{+}\rho J_{-} - J_{-}J_{+}\rho - \rho J_{-}J_{+}], \quad (11.3)$$

where  $J_{\pm} = \sigma_{\pm}^1 + \sigma_{\pm}^2 = \sigma_{\pm} \otimes \mathcal{I} + \mathcal{I} \otimes \sigma_{\pm}$ .

To get the master equation for  $A_1$  we need to trace out  $A_2$  which will result in

$$\dot{\rho}_{A_1} = \mathcal{L}_1\rho_{A_1} + \mathcal{L}_{eff}(t)\rho_{A_1}, \quad (11.4)$$

where

$$\mathcal{L}_{eff}\rho_{A_1} = \text{Tr}_{A_2} \left\{ \frac{(N+1)\gamma}{2} [2\sigma_{-}^1\rho\sigma_{+}^2 + 2\sigma_{-}^2\rho\sigma_{+}^1 - \sigma_{+}^1\sigma_{-}^2\rho - \sigma_{+}^2\sigma_{-}^1\rho - \rho\sigma_{+}^1\sigma_{-}^2 - \rho\sigma_{+}^2\sigma_{-}^1] \right. \\ \left. + \frac{N\gamma}{2} [2\sigma_{+}^1\rho\sigma_{-}^2 + 2\sigma_{+}^2\rho\sigma_{-}^1 - \sigma_{-}^1\sigma_{+}^2\rho - \sigma_{-}^2\sigma_{+}^1\rho - \rho\sigma_{-}^1\sigma_{+}^2 - \rho\sigma_{-}^2\sigma_{+}^1] \right\}. \quad (11.5)$$

The presence of time dependent  $\mathcal{L}_{eff}$  may change the dynamics of the system  $S_1$ . Note that even if the joint dynamics of  $S_1 + S_2$  is of Markovian type, there is, in general, no guarantee that the dynamics of  $S_1$  (or  $S_2$ ) will be Markovian type (see for example, J. Piilo *et al.* [138], H.-P. Breuer *et al.* [36], H.-P. Breuer *et al.* [35], D. Chruściński *et al.* [51]).

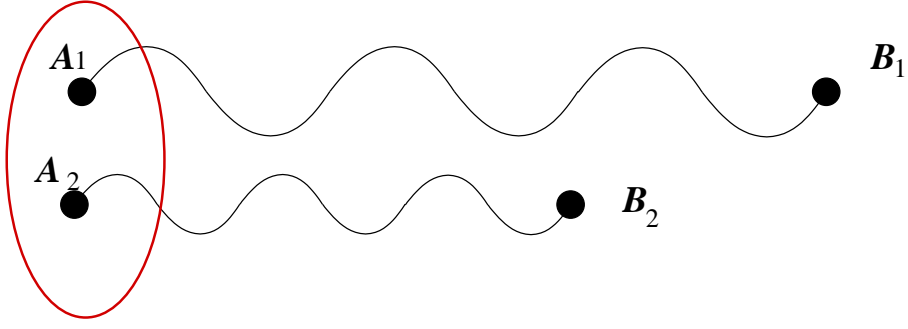


Figure 11.1: Schematic representation of system plus bath plus ancilla.

## 11.2 Dynamics in terms of Kraus operators

To get a better understanding of the evolution of the system  $S_1$  in the presence of ancilla  $S_2$  and the bath, let us consider the four-partite initial density matrix  $\rho_{A_1 A_2 B_1 B_2}$ . Suppose that  $\{A_k\}$  are the two-qubit Kraus operators, the evolution of the state  $\rho_{A_1 A_2 B_1 B_2}$  when the bath is acting on the qubits  $A_1$  and  $A_2$  can be written as

$$\rho(t)_{A_1 A_2 B_1 B_2} = \sum_k (A_k \otimes \mathcal{I}_{B_1 B_2}) \rho_{A_1 A_2 B_1 B_2} (A_k^\dagger \otimes \mathcal{I}_{B_1 B_2}) \quad (11.6)$$

Let us assume that the single qubit bath be of full rank, i.e., one needs at least four Kraus operators for this bath. If  $\{a_m\}_{m=1}^4$  are the single qubit Kraus operators then the operators  $\{a_m \otimes a_n\}_{m,n=1}^4$  will form a basis for operators acting on two-qubit states. Thus, the two-qubit Kraus operators  $\{A_k\}_{k=1}^{16}$  can be written as:

$$A_k = \sum_{mn} \alpha_{mn}^k a_m \otimes a_n^*, \quad (11.7)$$

where  $\{\alpha_{mn}^k\}$  are the expansion coefficients. We can now write

$$\rho_{A_1 A_2 B_1 B_2}(t) = \sum_k \sum_{mn, m'n'} \alpha_{mn}^k \alpha_{m'n'}^{k*} (a_m \otimes a_n^* \otimes \mathcal{I}_{B_1 B_2}) \rho_{A_1 A_2 B_1 B_2} (a_{m'}^\dagger \otimes a_{n'}^T \otimes \mathcal{I}_{B_1 B_2}). \quad (11.8)$$



Let us take the initial density operator  $\rho_{A_1 A_2 B_1 B_2}$  of the entire system to be of the form

$$\rho_{A_1 A_2 B_1 B_2} = \rho_{A_1 B_1} \otimes |\chi\rangle_{A_2 B_2} \langle \chi|, \quad (11.9)$$

where  $\rho_{A_1 B_1}$  is an arbitrary state of the two-qubit system  $A_1 + B_1$  and  $|\chi\rangle_{A_2 B_2}$  is a fixed state of ancilla. Therefore, we can write the Eq.(11.8) as

$$\text{tr}_{A_2 B_2} (\rho_{A_1 A_2 B_1 B_2}(t)) = \sum_{mm'} Y_{mm'} (a_m \otimes \mathcal{I}_{B_1}) \rho_{A_1 B_1} (a_{m'}^\dagger \otimes \mathcal{I}_{B_1}), \quad (11.10)$$

where

$$Y_{mm'} = \text{tr}_{A_2 B_2} \sum_k \sum_{mn, m'n'} \alpha_{mn}^k \alpha_{m'n'}^{k*} (a_n^* \otimes \mathcal{I}) (|\chi\rangle_{A_2 B_2} \langle \chi|) (a_{n'}^T \otimes \mathcal{I}). \quad (11.11)$$

Interesting thing to notice is that the matrix  $Y$  is positive semi-definite and hence we can write it as

$$Y_{mm'} = \sum_{kl} U_{mk} d_{kl} (U^\dagger)_{lm'} \quad (11.12)$$

where  $d_{kl} = d_k \delta_{kl}$  are the eigenvalues of  $Y$  and are non-negative real numbers. Substituting this in Eq.(11.10) results in

$$\tilde{\rho}_{A_1 B_1}(t) = \sum_k d_k (b_k \otimes \mathcal{I}_{B_1}) (\rho_{A_1 B_1}) (b_k^\dagger \otimes \mathcal{I}_{B_1}), \quad (11.13)$$

where  $b_k = \sum_m U_{mk} a_m$  are new Kraus operators. Absorbing the  $d_k$  into the new Kraus operator give rise to  $\tilde{b}_k = \sqrt{d_k} b_k$ . Our main task is to calculate the entanglement in the  $M$  matrix corresponds to Kraus operators  $a_i$  and  $\tilde{b}_k$  and compare the entanglement between these two states.

## 11.3 Results

Numerically it is straight forward to calculate the evolution of entanglement and coherence in the system  $S_1$  (where the initial state of  $S_1$  is chosen to be  $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ ) after introducing the ancilla  $S_2$  with different initial states  $|\chi\rangle$

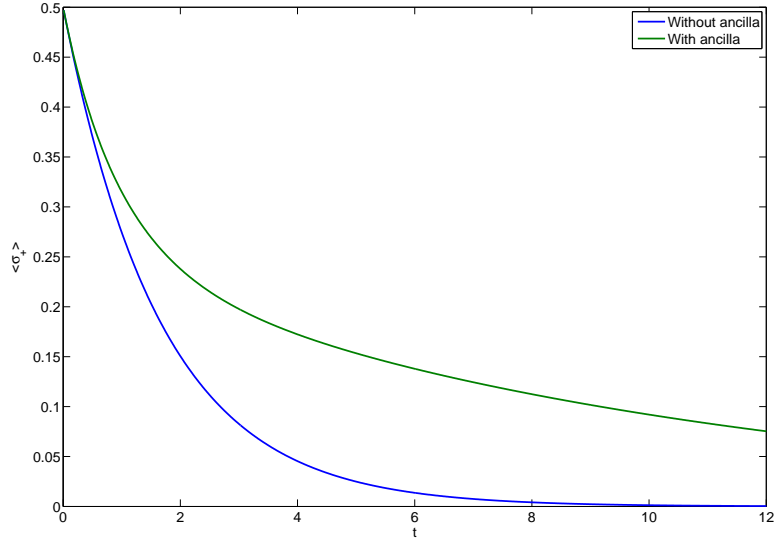


Figure 11.2: Here we are plotting the coherence (real part of  $\langle \sigma_+ \rangle$ ) for a single qubit. The blue curve shows the evolution of coherence in the absence of ancilla whereas the green curve is in the presence of ancilla. One can see that the green curve is always above the blue one and hence the presence of ancilla arrest the decay of coherence. Here the initial state of the system qubit is  $(|0\rangle + |1\rangle)/\sqrt{2}$  and ancilla is in ground state  $(|0\rangle)$ .

of  $S_2$ . In Fig. (11.2) and Fig. (11.3) we have shown the effect of ancilla on the evolution of the coherence  $\langle \sigma_+ \rangle$  and  $\langle \sigma_z \rangle$  of the system  $S_1$ . In Fig. (11.4) we can see the entanglement evolution for  $S_1$  for different  $|\chi\rangle$ . We see that for  $|\chi\rangle = |0\rangle \otimes |\psi\rangle$  where  $|\psi\rangle$  can be arbitrary single-qubit state, the entanglement last for much longer time than for other  $|\chi\rangle$ 's and for  $|\chi\rangle = |1\rangle \otimes |\psi\rangle$  the entanglement decays fastest. This shows the 'shielding effect' of ancilla on  $S_1$ . This effect is more prominent when the number of qubits in ancilla are more. In Fig. (11.5) we show the evolution of entanglement in  $S_1$  for two-qubit, four-qubit and six-qubit ancilla. The improvement in the ESD time for  $S_1$  is eminent.

Because of the exchange interactions, the presence of ancilla in the ground state may result in reducing the effective temperature of the bath. This might be a reason for the observed shielding effect.

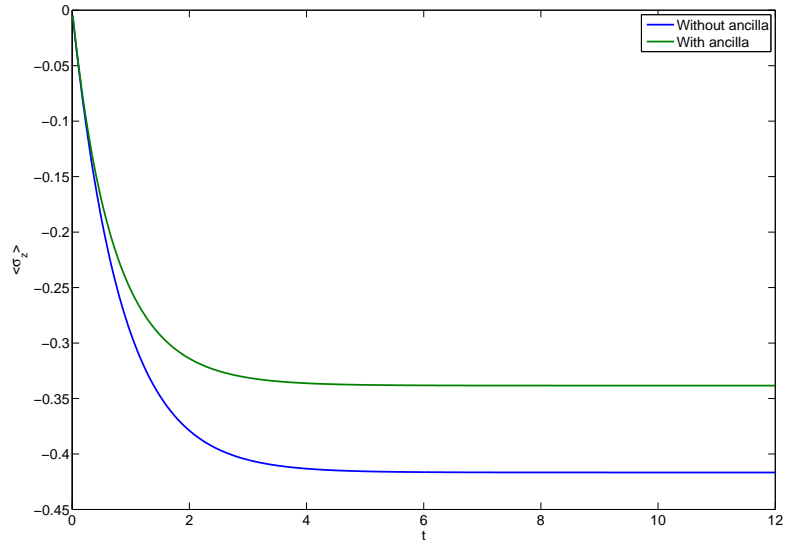


Figure 11.3: Here we are plotting the coefficient of  $\sigma_z$  for a single qubit for the same setup as in Fig. (11.2). Again the improvement due to the presence of ancilla is obvious from the plot.

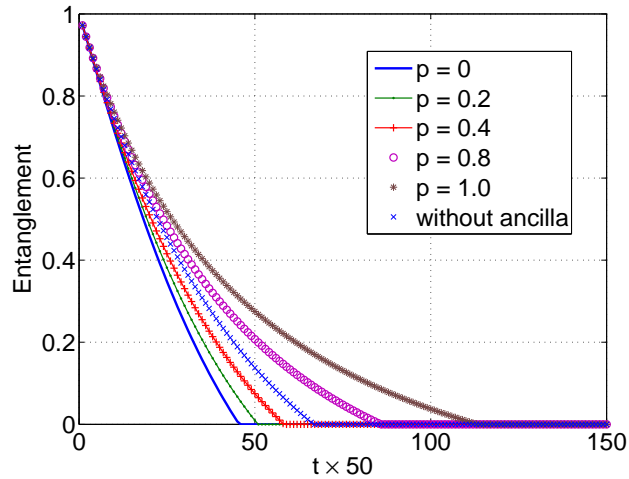


Figure 11.4: The effect of two-qubit ancilla on the evolution of entanglement when the initial state of the system  $A_1 + B_1$  is maximally entangled state  $(|00\rangle + |11\rangle)/\sqrt{2}$ . The state of ancilla is  $|\chi\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle$ . The average number of photons in the thermal cavity is  $N = 0.3$ .

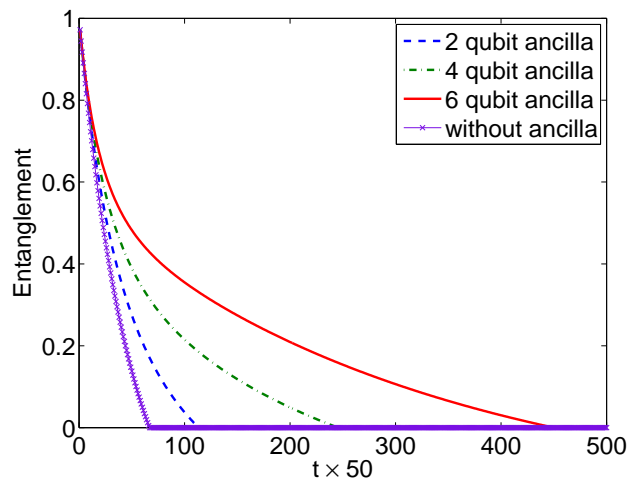


Figure 11.5: The effect of the increasing number of qubits in ancilla on the evolution of entanglement when the initial state of the system  $A_1 + B_1$  is maximally entangled state  $(|00\rangle + |11\rangle)/\sqrt{2}$ . All the qubits in ancilla are in the ground state  $|0\rangle$ .



# 12

## Conclusion

The study of entanglement in quantum systems has assumed great importance in recent times. With the advent of quantum information theory (QIT), the creation, analysis and manipulation of quantum systems has now entered the realm of everyday possibility. One of the primary resources of this powerful theory is entanglement – indispensable for essential quantum information tasks such as quantum teleportation, super-dense coding, communication complexity problems and one-way computation, among others.

Most, if not all, quantum systems are fragile by nature. This is because they are “open” systems and can not be shielded from dissipative interactions with the surrounding environment which creates entanglement between the system and its surrounding environment. This entanglement with the surrounding environment causes a loss of the quantum nature of the system – a phenomenon known as decoherence – and leads to an inevitable loss in the entanglement in the system.

In this thesis, we have studied the evolution of entanglement in open quantum systems by varying the dimensionality of the system as well as the nature of the dissipative interaction between the bath and the system. Our focus has been on the time of the onset of Entanglement Sudden Death — a phenomenon characterized by the complete loss of entanglement in a finite amount of time. For finite dimensional systems connected to a local thermal bath, we have found that ESD does not take place for any pure state at zero temperature, but does so for all states at non-zero temperatures. Squeezing the bath delays the onset of ESD for finite non-zero temperatures, but allows ESD to take place even at zero temperatures. However, best results are obtained by setting the system-bath inter-

action to be of the quantum non-demolition type, in which case ESD is never seen at any temperature whatsoever. As regards the evolution of two-mode Gaussian states of an infinite system of coupled harmonic oscillators, we have looked for the transition of the system to classicality which ensures that the system has reached separability. We have found that, when the system interacts with a local thermal bath at non-zero temperature, every two-mode Gaussian state makes the transition to classicality within a finite amount of time. This does not always happen at zero temperatures. For squeezed thermal baths, we have found that classicality is achieved in finite times even at zero temperatures.

In addition to the study of entanglement dynamics in various settings, we have also studied and modified existing ways of controlling the loss of entanglement in open quantum systems, and have also suggested and executed new methods of doing so. Our analysis of the effectiveness of decoherence control procedures on the evolution of entanglement has led to counterintuitive and surprising results. We have found that, while these control procedures are proficient in delaying the loss of coherence in the system, some of them are counter-productive when it comes to preservation of entanglement in the system. For the case of a thermal bath of photonic crystals interacting locally with a two-qubit system, we have found that the amount of entanglement at the asymptotic limit increases with an increase in the detuning parameter  $\Delta$ . Modulation of the system-bath coupling at a frequency much higher than the correlation time scale of the local thermal bath is another method of controlling decoherence. We have found that for a two-qubit system in contact with a local thermal bath, the time to ESD increased with the modulation frequency of the system-bath coupling. A third control procedure is the technique of resonance fluorescence, where the atomic transition is driven using an external coherent single-mode field in resonance with the transition itself, and the only control parameter is the transition frequency  $\Omega$ , also called the Rabi frequency. We have found that, for the same two-qubit system in contact with a local thermal bath and exposed to such an external field, an increase in  $\Omega$  decreases decoherence but at the same time also decreases the time to ESD. Finally, the technique of dynamical decoupling involves switching the decoupling interactions on and off at a rate much faster than the one set by the environment. Bang-bang decoupling is one such technique where fast radio frequency pulses are applied in order to average out unwanted effects of the environment and thus control decoherence.

However, as in the previous case, dynamical decoupling also causes a faster loss of entanglement.

In the final part of the thesis, we have proposed a new method of shielding the entanglement in the system from dissipative environmental effects. In this scheme, we install an ancillary system in a fixed initial state and let it interact with the main system through exchange interactions. The presence of the ancilla modifies the rate at which the system evolves — mathematically, it changes the form of the superoperator  $\mathcal{L}$  which governs the dynamics of the system density operator with its own time derivative. We have, through numerical computations, shown that the ancilla, in its ground state, is able to enhance the lifespan of the entanglement in the system. For the case of a two-qubit system interacting with a local thermal bath as well as with an  $n$ -qubit ancilla, we have further shown that increasing the size of the ancilla, that is, increasing  $n$  improves its performance.

In future work, we hope to find out the exact time to ESD for mixed entangled states via a generalization of the factorization law of entanglement decay, both for finite as well as infinite dimensional systems. Following the work of Shabani and Lidar [150], we plan to study the non-Markovian master equation from the measurement perspective, putting particular emphasis on understanding the role of memory in such non-Markovian entanglement dynamics. As to the control of entanglement, we plan to characterize, in greater detail, the decoherence control mechanisms currently available according to their effect of ESD. Finally, we would like to extend our shielding control procedure to non-Markovian dynamics and compare its effects to Markovian dynamics.



## List of publication

- **Quantum to classical transition and entanglement sudden death in Gaussian states under local heat bath dynamics**

Sandeep K Goyal and Sibasish Ghosh,  
*Phys. Rev. A.* **82** 042337 (2010)  
(<http://arxiv.org/abs/1005.4224>)

- **Effect of control procedures on the evolution of entanglement in open quantum systems**

Sandeep K Goyal, Subhashish Banerjee and Sibasish Ghosh,  
*Phys. Rev. A.* **85** 012327 (2012)  
(<http://arxiv.org/abs/1102.4403>)

- **Entanglement dynamics under local Lindblad evolution**

Sandeep K. Goyal and Sibasish Ghosh  
(<http://arxiv.org/abs/1003.1248>)

- **Spatial entanglement using a quantum walk on a many-body system**

Sandeep K. Goyal and C. M. Chandrashekar,  
*J. Phys. A: Math. Theor.* **43** (2010) 235303  
(<http://arxiv.org/abs/0901.0671>)

- **Entanglement generation in spatially separated systems using quantum walk**

C. M. Chandrashekar, Sandeep K. Goyal and Subhashish Banerjee,  
( Submitted to Euro. J. Phys. D)  
(<http://arxiv.org/abs/1005.3785>)

- **Geometry of the generalized Bloch sphere for qutrit**

Sandeep K. Goyal, B. Neethi Simon, Rajeev Singh and Sudhavathani Simon  
(<http://arxiv.org/abs/1111.4427>)

- **Anomalous quantum Hall states in an optical lattice**

S. R. Hassan, Sandeep Goyal, R. Shankar, David Sénéchal (<http://arxiv.org/abs/1201.5874>)

# Bibliography

- [1] A. Ferraro, S. Olivares, and M. G. A. Paris. Gaussian states in continuous variable quantum information. *ArXiv:quant-ph/0503237*, 2005.
- [2] B. Abel and F. Marquardt. *Phys. Rev. B*, 78:201302(R), 2008.
- [3] G. S. Agarwal. *Phys. Rev. A*, 61:013809, 1999.
- [4] G. S. Agarwal and W. Harshawardhan. *Phys. Rev. A*, 50:R4465, 1994.
- [5] G.S. Agarwal, M. O. Scully, and H. Walther. *Phys. Rev. Lett.*, 86:4271, 2001.
- [6] G.S. Agarwal, M. O. Scully, and H. Walther. *Phys. Rev. A*, 63:044101, 2001.
- [7] A. Al-Qasimi and D. F. V. James. *Phys. Rev. A*, 77:012117, 2008.
- [8] E. Andersson, J. D. Cresser, and M. J. W. Hall. *J. Mod. Opt.*, 54:15, 2007.
- [9] Arvind, B Dutta, N Mukunda, and R Simon. The real symplectic groups in quantum mechanics and optics. *Pramana*, 45:471–497, 1995.
- [10] M. Ban. *J. Mod. Opt.*, 45:2513, 1998.
- [11] S. Banerjee and R. Ghosh. *J. Phys. A: Math. Theo.*, 40:13735, 2007.
- [12] S. Banerjee, V. Ravishankar, and R. Srikanth. *Eur. Phys. Jr. D*, 121:587, 2010.
- [13] S. Banerjee, V. Ravishankar, and R. Srikanth. *Ann. of Phys. (N. Y.)*, 128:588, 2010.
- [14] S. Banerjee and R. Srikant. *Eur. Phys. J. D*, 46:335–344, 2008.
- [15] Adriano Barenco, Charles H. Bennett, Richard Cleve, David P. DiVincenzo, Norman Margolus, Peter Shor, Tycho Sleator, John A. Smolin, and Harald Weinfurter. Elementary gates for quantum computation. *Phys. Rev. A*, 52(5):3457–3467, Nov 1995.
- [16] Howard Barnum, Christopher A. Fuchs, Richard Jozsa, and Benjamin Schumacher. General fidelity limit for quantum channels. *Phys. Rev. A*, 54(6):4707–4711, Dec 1996.

## Bibliography

- [17] J. S. Bell. On the einstein podolsky rosen paradox. *Physics*, 1:195, 1964.
- [18] B. Bellomo, R. Lo Franco, and G. Compagno. Non-markovian effects on the dynamics of entanglement. *Phys. Rev. Lett.*, 99:160502, Oct 2007.
- [19] Rafael Benguria and Mark Kac. Quantum langevin equation. *Phys. Rev. Lett.*, 46(1):1–4, Jan 1981.
- [20] C. H. Bennett and G. Brassard. Quantum cryptography: Public key distribution and coin tossing. *Proc. IEEE Int. Conf. Computers*, pages 175 – 179, 1984.
- [21] C.H. Bennett and P.W. Shor. Quantum information theory. *Information Theory, IEEE Transactions on*, 44(6):2724 –2742, oct 1998.
- [22] Charles H. Bennett. Quantum cryptography using any two nonorthogonal states. *Phys. Rev. Lett.*, 68(21):3121–3124, May 1992.
- [23] Charles H. Bennett, François Bessette, Gilles Brassard, Louis Salvail, and John A. Smolin. Experimental quantum cryptography. *J. Cryptology*, pages 3–28, 1992.
- [24] Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters. Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels. *Phys. Rev. Lett.*, 70(13):1895–1899, Mar 1993.
- [25] Charles H. Bennett, David P. DiVincenzo, John A. Smolin, and William K. Wootters. Mixed-state entanglement and quantum error correction. *Phys. Rev. A*, 54(5):3824–3851, Nov 1996.
- [26] J. Bergli and L. Faoro. *Phys. Rev. B*, 75:054515, 2007.
- [27] Alexandre Blais, Jay Gambetta, A. Wallraff, D. I. Schuster, S. M. Girvin, M. H. Devoret, and R. J. Schoelkopf. Quantum-information processing with circuit quantum electrodynamics. *Phys. Rev. A*, 75(3):032329, Mar 2007.
- [28] David Bohm. A suggested interpretation of the quantum theory in terms of "hidden" variables. i. *Phys. Rev.*, 85(2):166–179, Jan 1952.

- [29] David Bohm. A suggested interpretation of the quantum theory in terms of "hidden" variables. ii. *Phys. Rev.*, 85(2):180–193, Jan 1952.
- [30] Luc Bouten, Madalin Guta, and Hans Maassen. Stochastic schrödinger equation. *J. Phys. A: Math. Gen.*, 37:3189, 2004.
- [31] Samuel L. Braunstein and H. J. Kimble. Teleportation of continuous quantum variables. *Phys. Rev. Lett.*, 80(4):869–872, Jan 1998.
- [32] Samuel L. Braunstein and A. Mann. Measurement of the bell operator and quantum teleportation. *Phys. Rev. A*, 51(3):R1727–R1730, Mar 1995.
- [33] H. P. Breuer and F. Petruccione. *The theory of open quantum systems*. Oxford University Press, 2002.
- [34] Heinz-Peter Breuer. Separability criteria and bounds for entanglement measures. *J. Phys. A: Math. Gen.*, 39:11847, 2006.
- [35] Heinz-Peter Breuer, Elsi-Mari Laine, and Jyrki Piilo. Measure for the degree of non-markovian behavior of quantum processes in open systems. *Phys. Rev. Lett.*, 103:210401, Nov 2009.
- [36] Heinz-Peter Breuer and Bassano Vacchini. Quantum semi-markov processes. *Phys. Rev. Lett.*, 101:140402, Oct 2008.
- [37] Hans-Jürgen Briegel and Berthold-Georg Englert. Quantum optical master equations: The use of damping bases. *Phys. Rev. A*, 47:3311–3329, Apr 1993.
- [38] K. E. Cahill and R. J. Glauber. Density operators and quasiprobability distributions. *Phys. Rev.*, 177(5):1882, 1969.
- [39] K. E. Cahill and R. J. Glauber. Ordered expansion in boson amplitude operators. *Phys. Rev.*, 177(5):1857, 1969.
- [40] S. Calderbank and P. Shor. *Phys. Rev.*, 54:1098, 1996.
- [41] S. Calderbank, P. Shor, and A. Steane. *Proc. Roy. Soc., London, Ser. A*, 452:2551, 1996.

## Bibliography

- [42] H. J. Carmichael. *Open systems approach to quantum optics: Lectures presented at the Universite Libre De Bruelles*. Springer-Verlag, 1993.
- [43] H. J. Carmichael. Quantum trajectory theory for cascaded open systems. *Phys. Rev. Lett.*, 70(15):2273–2276, Apr 1993.
- [44] H. J. Carmichael. *Statistical methods in quantum optics I: Master equations and Fokker-Plank equations*. Springer, 1999.
- [45] H. J. Carmichael. *Statistical methods in quantum optics II: Non-classical fields*. Springer-Verlag, 2008.
- [46] Yvan Castin and Klaus Mølmer. Monte carlo wave-function analysis of 3d optical molasses. *Phys. Rev. Lett.*, 74(19):3772–3775, May 1995.
- [47] I. Chakrabarty, S. Banerjee, and N. Siddharth. A study of quantum correlations in open quantum systems. *ArXiv:1006.1856*, 2010.
- [48] S. Chaturvedi and F. Shibata. Time-convolutionless projection operator formalism for elimination of fast variables. application to brownian motion. *Z. Phys.*, B35:297, 1979.
- [49] Kai Chen, Sergio Albeverio, and Shao-Ming Fei. Entanglement of formation of bipartite quantum states. *Phys. Rev. Lett.*, 95(21):210501, Nov 2005.
- [50] M. Choi. *Linear Algebra and Its Applications*, 10:285–290, 1975.
- [51] Dariusz Chruściński and Andrzej Kossakowski. Non-markovian quantum dynamics: Local versus nonlocal. *Phys. Rev. Lett.*, 104:070406, Feb 2010.
- [52] V. Coffman<sup>1</sup>, J. Kundu, and W. K. Wootters. Distributed entanglement. *Phys. Rev. A*, 61:052306, 2000.
- [53] C. Cohen-Tannoudji, B. Diu, and F. Laloe. Quantum mechanics, volume 1. *Wiley-VCH, June 1986.*, 1986.
- [54] Igor Devetak, Aram W. Harrow, and Andreas Winter. A family of quantum protocols. *Phys. Rev. Lett.*, 93(23):230504, Dec 2004.

- [55] Arend G. Dijkstra and Yoshitaka Tanimura. Non-markovian entanglement dynamics in the presence of system-bath coherence. *Phys. Rev. Lett.*, 104:250401, Jun 2010.
- [56] L. Diósi. Progressive decoherence and total environmental disentanglement. *Lect. Notes Phys.*, 622:157–163, 2003.
- [57] L. Diósi and C. Kiefer. Exact positivity of the wigner and  $p$ -function of a markovian open system. *J. Phys. A: Math. Gen.*, 35:2675–2683, 2002.
- [58] P. J. Dodd and J. J. Halliwell. Disentanglement and decoherence by open system dynamics. *Phys. Rev. A*, 69:052105, 2004.
- [59] L. M. Duan and G. C. Guo. *Phys. Rev. Lett.*, 79:1953, 1997.
- [60] L. M. Duan and G. C. Guo. *Phys. Lett. A*, 261:139, 1999.
- [61] R. Dum, A. S. Parkins, P. Zoller, and C. W. Gardiner. Monte carlo simulation of master equations in quantum optics for vacuum, thermal, and squeezed reservoirs. *Phys. Rev. A*, 46(7):4382–4396, Oct 1992.
- [62] R. Dum, P. Zoller, and H. Ritsch. Monte carlo simulation of the atomic master equation for spontaneous emission. *Phys. Rev. A*, 45(7):4879–4887, Apr 1992.
- [63] M. K. Durstberger. *Geometry of entanglement and decoherence in quantum systems*. PhD thesis, Universität Wien, 2005.
- [64] R. E. Edwards. *Functional analysis, theory and application*. Courier Dover Publications, 1995.
- [65] A. Einstein, B. Podolsky, and N. Rosen. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.*, 47(10):777–780, May 1935.
- [66] J Eisert, F G S L Brandão, and K M R Audenaer. Quantitative entanglement witnesses. *New J. Phys.*, 9:46, 2007.
- [67] P. Facchi, S. Tasaki, S. Pascazio, H. Nakazato, A. Tokuse, and D. A. Lidar. *Phys. Rev. A*, 71:022302, 2005.

## Bibliography

- [68] G. Falci, R. Fazio, G. M. Palma, J. Siewert, and V. Vedral. *Nature*, 407:355, 2000.
- [69] Heng Fan, Keiji Matsumoto, and Hiroshi Imai. Quantify entanglement by concurrence hierarchy. *J. Phys. A: Math. Gen.*, 36:4151, 2003.
- [70] G. W. Ford and M. Kac. On the quantum langevin equation. *Journal of Statistical Physics*, 46:803–810, 1987. 10.1007/BF01011142.
- [71] G. W. Ford, J. T. Lewis, and R. F. O’Connell. Quantum langevin equation. *Phys. Rev. A*, 37(11):4419–4428, Jun 1988.
- [72] F Franchini, A R Its, and V E Korepin. Renyi entropy of the xy spin chain. *J. Phys. A: Math. Theor.*, 41:025302, 2008.
- [73] V. A. Franke. On the general form of the dynamical transformation of density matrices. *Theoretical and Mathematical Physics*, 27:406–413, 1976. 10.1007/BF01051230.
- [74] J. R. Friedman, V. Patel, W. Chen, S. K. Tolpygo, and J. E. Lukens. *Nature*, 406:43, 2000.
- [75] A. Furusawa, J. L. SÃyrensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik. Unconditional quantum teleportation. *Science*, 282(5389):706–709, 1998.
- [76] F. Galve, L. A. PachÃn, and D. zueco. Bringing entanglement to the high temperature limit. *Phys. Rev. Lett.*, 105:180501, 2010.
- [77] C. W. Gardiner. *Quantum Noise*. Springer-Verlag, 1991.
- [78] C. W. Gardiner, A. S. Parkins, and P. Zoller. Wave-function quantum stochastic differential equations and quantum-jump simulation methods. *Phys. Rev. A*, 46(7):4363–4381, Oct 1992.
- [79] B. M. Garraway. *Phys. Rev. A*, 55:2290, 1997.
- [80] G. Giedke, M. M. Wolf, O. Kruger, R. F. Werner, and J. I. Cirac. *Phys. Rev. Lett.*, 91:107901, 2003.

- [81] R. J. Glauber. Coherent and incoherent states of the radiation field. *Phys. Rev.*, 131:2766, 1963.
- [82] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan. Completely positive dynamical semigroups for n-level systems. *J. Math. Phys.*, 17(5):821, 1976.
- [83] Vittorio Gorini, Alberto Frigerio, Maurizio Verri, Andrzej Kossakowski, and E. C. G. Sudarshan. Properties of quantum markovian master equations. *Reports on Mathematical Physics*, 13(2):149 – 173, 1978.
- [84] Gilad Gour. Family of concurrence monotones and its applications. *Phys. Rev. A*, 71(1):012318, Jan 2005.
- [85] S. K. Goyal. Controlling decoherence through shielding effect. (*To be Communicated*).
- [86] S. K. Goyal and S. Ghosh. Entanglement dynamics under local lindblad evolution. *arxiv:1003.1248*, 2010.
- [87] S. K. Goyal and S. Ghosh. Quantum to classical transition and entanglement sudden death in gaussian states under local heat bath dynamics. *Phys. Rev. A*, 82:042337, 2010.
- [88] S.K. Goyal, S. Banerjee, and S. Ghosh. Effect of control procedures on the evolution of entanglement in open quantum systems. *Phys. Rev. A*, 85, 2012.
- [89] F. Grossmann, T. Dittrich, P. Jung, and P. Hanggi. *Phys. Rev. Lett.*, 67:516, 1991.
- [90] T. Heinosaari, A. S. Holevo, and M. M. Wolf. The semigroup structure of gaussian chennals. *Arxiv:0909.0408*, 2009.
- [91] S. Hill and W. K. Wootters. *Phys. Rev. Lett.*, 78:5022, 1997.
- [92] K. M. Ho, C. T. Chan, and C. M. Soukoulis. *Phys. Rev. Lett.*, 65:3152, 1990.
- [93] A. S. Holevo. *Russian Math. Surveys*, 53:1295, 1999.
- [94] A. S. Holevo. Entanglement breaking channels in infinite dimensions. *Problems of Information Transmission*, 44:171, 2008.



## Bibliography

- [95] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.
- [96] M. Horodecki, P. Horodecki, and R. Horodecki. *Phys. Lett. A*, 223:1, 1996.
- [97] M. Horodecki, P. W. Shor, and M. B. Ruskai. Entanglement breaking channels. *Rev. Math. Phys.*, 15:629, 2003.
- [98] Michał Horodecki and Paweł Horodecki. Reduction criterion of separability and limits for a class of distillation protocols. *Phys. Rev. A*, 59(6):4206–4216, Jun 1999.
- [99] Michal Horodecki, Pawel Horodecki, and Ryszard Horodecki. Separability of mixed states: necessary and sufficient conditions. *Physics Letters A*, 223(1-2):1 – 8, 1996.
- [100] A. Isar. Decoherence and asymptotic entanglement in open quantum dynamics. *J. Russ. Laser Res.*, 28:439, 2007.
- [101] A. Jamiolkowski. *Rep. Math. Phys.*, 3:275, 1972.
- [102] S. John. *Phys. Rev. Lett.*, 53:2169, 1984.
- [103] S. John. *Phys. Rev. Lett.*, 58:2486, 1987.
- [104] S. John and T. Quang. *Phys. Rev. A*, 50:1764, 1994.
- [105] C. L. Kane. *Phys Rev. Lett*, 90:047901, 2003.
- [106] E. Knill and R. Laflamme. *Phys. Rev. A*, 55:1098, 1997.
- [107] T. Konrad, F. Melo, M. Tiersch, C. Kasztelan, A. Aragao, and A. Buchleitner. *Nature Physics*, 4:99, 2008.
- [108] J. K. Korbicz, M. L. Almeida, J. Bae, M. Lewenstein, and A. Acin. Structural approximation to positive maps and entanglement breaking channels. *Phys. Rev. A*, 78:062105, 2008.
- [109] A. Kowalewska-Kudlaszyk and W. Leoński. Sudden death of entanglement and its rebirth in a system of two nonlinear oscillators. *Phys. scr*, T140:014051, 2010.

- [110] P. K. Lam and C. M. Savage. *Phys. Rev. A*, 50:3500, 1994.
- [111] Jinhyoung Lee, M. S. Kim, and Hyunseok Jeong. Transfer of nonclassical features in quantum teleportation via a mixed quantum channel. *Phys. Rev. A*, 62(3):032305, Aug 2000.
- [112] M. Lewenstein, J. Zakrzewski, and T. W. Mossberg. *Phys. Rev. A*, 38:808, 1988.
- [113] D. A. Lidar, I. L. Chuang, and K. B. Whaley. *Phys. Rev. Lett.*, 81:2594, 1998.
- [114] G. Lindblad. *Commun. Math. Phys.*, 48:119, 1976.
- [115] G. Lindblad. On the generators of quantum dynamical semigroups. *Communications in Mathematical Physics*, 48(2):119–130, 1976.
- [116] C. E. López, G. Romero, F. Lastra, E. Solano, and J. C. Retamal. Sudden birth versus sudden death of entanglement in multipartite systems. *Phys. Rev. Lett.*, 101:080503, Aug 2008.
- [117] W. A. Majewski. Dynamical semigroups in the algebraic formulation of statistical mechanics. *Fortschritte der Physik*, 32(3):89–133, 1984.
- [118] Y. Makhlin, G. Schön, and A. Shnirman. *Nature*, 386:305, 1999.
- [119] Y. Makhlin, G. Schön, and A. Shnirman. *Rev. Mod. Phys.*, 73:357, 2001.
- [120] S. Maniscalco and F. Petruccione. Non-markovian dynamics of a system. *Phys. Rev. A*, 73:012111, 2006.
- [121] P. Marek, J. Lee, and M. S. Kim. vacuum as a less hostile environment to entanglement. *Phys Rev A*, 77:032302, 2008.
- [122] Florian Mintert, Andr   R.R. Carvalho, Marek Kus, and Andreas Buchleitner. Measures and dynamics of entangled states. *Physics Reports*, 415(4):207 – 259, 2005.
- [123] Klaus Mølmer, Yvan Castin, and Jean Dalibard. Monte carlo wave-function method in quantum optics. *J. Opt. Soc. Am. B*, 10(3):524–538, Mar 1993.

## Bibliography

- [124] L. Moravčiková and M. Ziman. Entanglement annihilation and entanglement breaking channels. *J. Phys. A: Math. Theor.*, 43:275306, 2010.
- [125] M.O.Scully and M.S. Zubairy. *Quantum optics*. Cambridge University Press, Cambridge, 1997.
- [126] S. Nakajima. On quantum theory of transport phenomena. *Progr. Theor. Phys.*, 20:948, 1958.
- [127] Y. Nakamura, Yu. A. Pashkin, and J. S. Tsai. *Nature*, 398:786, 1999.
- [128] Y. Nakamura, Yu. A. Pashkin, T. Yamamoto, and J. S. Tsai. *Phys. Rev. Lett.*, 88:047901, 2002.
- [129] Hyunchul Nha and H. J. Carmichael. Entanglement within the quantum trajectory description of open quantum systems. *Phys. Rev. Lett.*, 93(12):120408, Sep 2004.
- [130] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge, 2002.
- [131] T. J. Osborne and F. Verstraete. General monogamy inequality for bipartite qubit entanglement. *Phys. Rev. Lett.*, 96:220503, 2006.
- [132] E. Paladino, L. Faoro, and G. Falci. *Adv. in Solid State Phys.*, 43:747, 2003.
- [133] E. Paladino, L. Faoro, G. Falci, and R. Fazio. *Phys. Rev. Lett.*, 88:228304, 2002.
- [134] James Park and William Band. Generalized two-level quantum dynamics. i. representations of the kossakowski conditions. *Foundations of Physics*, 7:813–825, 1977. 10.1007/BF00708507.
- [135] J. P. Paz and A. J. Roncaglia. Dynamics of the entanglement between two oscillators in the same environment. *Phys. Rev. Lett.*, 100:220401, 2008.
- [136] J. P. Paz and A. J. Roncaglia. Entanglement dynamics during decoherence. *Quantum Information Processing*, 8:535, 2009.

- [137] Asher Peres. Separability criterion for density matrices. *Phys. Rev. Lett.*, 77(8):1413–1415, Aug 1996.
- [138] Jyrki Piilo, Sabrina Maniscalco, Kari Härkönen, and Kalle-Antti Suominen. Non-markovian quantum jumps. *Phys. Rev. Lett.*, 100:180402, May 2008.
- [139] M. B. Plenio. Logarithmic negativity: A full entanglement monotone that is not convex. *Phys. Rev. Lett.*, 95(9):090503, Aug 2005.
- [140] W.T. Pollard and R.A. Friesner. Solution of the redfield equation for the dissipative quantum dynamics of multilevel systems. *The Journal of chemical physics*, 100:5054, 1994.
- [141] Sandu Popescu and Daniel Rohrlich. Thermodynamics and the measure of entanglement. *Phys. Rev. A*, 56(5):R3319–R3321, Nov 1997.
- [142] S. Raghavan, V. M. Kenkre, D. H. Dunlap, A. R. Bishop, and M. I. Salkola. *Phys. Rev. A*, 54:1781, 1996.
- [143] A. K. Rajagopal and R. W. Rendell. Decoherence, correlation and entanglement in a pair of coupled quantum dissipative oscillators. *Phys. Rev. A*, 63:022116, 2001.
- [144] A. G. Redfield. On the theory of relaxation processes. *IBM JOURNAL OF RESEARCH AND DEVELOPMENT*, 1(1):19–31, 1957.
- [145] A. G. Redfield. The theory of relaxation processes. *Advan. Magn. Resonance*, 1:1–32, 1966.
- [146] R. W. Rendell and A. K. Rajagopal. Revivals and entanglement from initially entangled mixed states of a damped jaynes-cummings model. *Phys. Rev. A*, 67:062110, 2003.
- [147] A. Rényi. On measures of information and entropy. *Proceedings of the 4th Berkeley Symposium on Mathematics, Statistics and Probability 1960*, page 547, 1961.
- [148] H. Risken. *The Fokker-Planck equation – Methods of solution and applications*. Springer, 1989.

## Bibliography

- [149] M. B. Ruskai. *LA e-print arXiv-ph/0207100*, 2002.
- [150] A. Shabani and D. A. Lidar. *Phys. Rev. A*, 71:020101, 2005.
- [151] F. Shibata and T. Arimitsu. Expansion formulas in non-equilibrium statistical mechanics. *J. Phys. Soc. Jap.*, 49:891, 1980.
- [152] F. Shibata, Y. Takahashi, and N. Hashitsume. A generalized stochastic liouville equation. non-markovian versus memoryless master equations. *J. stat. phys.*, 17:171, 1977.
- [153] A. Shnirman, Y. Makhlin, and G. Schön. *Phys. Scr.*, T102:147, 2002.
- [154] P. W. Shor. *Phys. Rev. A*, 52:2493, 1995.
- [155] P. W. Shor. *J. Math. Phys.*, 43:4334, 2002.
- [156] Ralph Simmons and James Park. On completely positive maps in generalized quantum dynamics. *Foundations of Physics*, 11:47–55, 1981. 10.1007/BF00715195.
- [157] B. N. Simon, S. Simon, F. Gori, M. Santarsiero, R. Borghi, N. Mukunda, and R. Simon. Nonquantum entanglement resolves a basic issue in polarization optics. *Phys. Rev. Lett.*, 104(2):023901, Jan 2010.
- [158] R. Simon. *Phys. Rev. Lett.*, 84:2726, 2000.
- [159] Sudhavathani Simon, S. Rajagopalan, and R. Simon. The structure of states and maps in quantum theory. *Pramana*, 73:471–483, 2009. 10.1007/s12043-009-0100-1.
- [160] E. C. G. Sudarshan. Equivalence of semiclassical and quantum mechanical description of statistical light beam. *Phys. Rev. Lett.*, 10:277, 1963.
- [161] E. C. G. Sudarshan, P. M. Mathews, and J. Rau. Stochastic dynamics of quantum mechanical systems. *Phys. Rev.*, 121:920, 1961.
- [162] Barbara M. Terhal. Bell inequalities and the separability criterion. *Physics Letters A*, 271(5-6):319 – 326, 2000.

- [163] L. Tian and H. J. Carmichael. Quantum trajectory simulations of two-state behavior in an optical cavity containing one atom. *Phys. Rev. A*, 46(11):R6801–R6804, Dec 1992.
- [164] Markus Tiersch, Fernando de Melo, and Andreas Buchleitner. Entanglement evolution in finite dimensions. *Phys. Rev. Lett.*, 101(17):170502, Oct 2008.
- [165] T. Yu and J.H. Eberly. *Phys. Rev. Lett.*, 93:140404, 2004.
- [166] W. G. Unruh. Maintaining coherence in quantum computers. *Phys. Rev. A*, 51:992–997, Feb 1995.
- [167] R. Vasile, S. Olivares, M. G. A. Paris, and S. Maniscalco. Continuous variable entanglement dynamics in structured reservoirs. *Phys. Rev. A*, 80:062324, 2009.
- [168] G. Vidal and R. F. Werner. Computable measure of entanglement. *Phys. Rev. A*, 65(3):032314, Feb 2002.
- [169] L. Viola, E. Knill, and S. Lloyd. *Phys. Rev. Lett.*, 82:2417, 1999.
- [170] L. Viola and S. Lloyd. *Phys. Rev. A*, 58:2733, 1998.
- [171] D. Vitali and P. Tombesi. *Phys. Rev. A*, 59:4178, 1999.
- [172] D. Vitali and P. Tombesi. *Phys. Rev. A*, 65:012305, 2001.
- [173] M. Wagner. *Phys. Rev. A*, 51:798, 1995.
- [174] F. Wang, Z. Zhang, and R. Liang. *Phys. Rev. A*, 78:042320, 2008.
- [175] Reinhard F. Werner. Quantum states with einstein-podolsky-rosen correlations admitting a hidden-variable model. *Phys. Rev. A*, 40(8):4277–4281, Oct 1989.
- [176] Aleksander Weron, A. K. Rajagopal, and Karina Weron. Quantum theory of decaying systems from the canonical decomposition of dynamical semigroups. *Phys. Rev. A*, 31(3):1736–1744, Mar 1985.
- [177] D. Wilson, J. Lee, and M. S. Kim. Entanglement of a two-mode squeezed state in a phase-sensitive gaussian environment. *J. Mod. Opt.*, 50:1809, 2003.

## Bibliography

- [178] A. Wonderen and K. Lendi. Quantum theory of dissipative processes: The markov approximation revisited. *Journal of Statistical Physics*, 80:273–305, 1995. 10.1007/BF02178360.
- [179] W. K. Wootters. Quantum entanglement as a quantifiable resource. *Phil. Trans. R. Soc. Lond. A*, 356:1717, 1998.
- [180] W. K. Wootters. Entanglement of formation and concurrence. *Quantum information and computation*, 1:27, 2001.
- [181] William K. Wootters. Entanglement of formation of an arbitrary state of two qubits. *Phys. Rev. Lett.*, 80(10):2245–2248, Mar 1998.
- [182] E. Yablonovitch. *Phys. Rev. Lett.*, 59:2059, 1987.
- [183] E. Yablonovitch, T. J. Gmitter, and K. M. Leung. *Phys. Rev. Lett.*, 67:2295, 1991.
- [184] Y. Yang and S. Y. Zhu. *Phys. Rev. A*, 62:013805, 2000.
- [185] Chang-shui Yu, X. X. Yi, and He-shan Song. Evolution of entanglement for quantum mixed states. *Phys. Rev. A*, 78(6):062330, Dec 2008.
- [186] P. Zanardi and M. Rasetti. *Phys. Rev. Lett.*, 79:3306, 1997.
- [187] Qiang Zhang, Alexander Goebel, Claudia Wagenknecht, Yu-Ao Chen, Bo Zhao<sup>1</sup>, Tao Yang, Alois Mair, Jörg Schmiedmayer, and Jian-Wei Pan. Experimental quantum teleportation of a two-qubit composite system. *Nature Physics*, 2:678–682, 2006.
- [188] T. C. Zhang, K. W. Goh, C. W. Chou, P. Lodahl, and H. J. Kimble. Quantum teleportation of light beams. *Phys. Rev. A*, 67(3):033802, Mar 2003.
- [189] P. Zoller, Th. Beth, D. Binosi, R. Blatt, H. Briegel, D. Bruss, T. Calarco, J. I. Cirac, D. Deutsch, J. Eisert, A. Ekert, C. Fabre, N. Gisin, P. Grangiere, M. Grassl, S. Haroche, A. Imamoglu, A. Karlson, J. Kempe, L. Kouwenhoven, S. Kröll, G. Leuchs, M. Lewenstein, D. Loss, N. Lütkenhaus, S. Massar, J. E. Mooij, M. B. Plenio, E. Polzik, S. Popescu, G. Rempe, A. Sergienko, D. Suter, J. Twamley, G. Wendin, R. Werner, A. Winter, J. Wrachtrup, and

- A. Zeilinger. Quantum information processing and communication. *The European Physical Journal D - Atomic, Molecular, Optical and Plasma Physics*, 36:203–228, 2005. 10.1140/epjd/e2005-00251-1.
- [190] A. B. Zorin, F.-J. Ahlers, J. Niemeyer, T. Weimann, and H. Wolf. *Phys. Rev. B*, 53:13682, 1996.
- [191] W. H. Zurek. Decoherence and the transition from quantum to classical. *Physics today*, 44(10):36, 1991.
- [192] R. Zwanzig. Ensemble method in the theory of irreversibility. *J. Chem. Phys.*, 33:1338, 1960.