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THE SCALAR FIELD IN CURVED SPACE

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PREFACE

These notes are based on a series of lectures we gave at the Institute of Mathematical Sciences in Madras. The lectures were intended to be pedagogical in nature, and no prior familiarity with the material was assumed. Although we have tried to provide some references where the reader can obtain more details than are presented here we have not attempted to be systematic about it. Many of the results presented here are due to the work of others, both published and unpublished. More complete references can be found in the references we have cited.

These notes would not have been possible without the high degree of audience participation. We owe a special debt to Biswajit Chakraborty and Sumitra Ranganathan who took the notes and asked many insightful questions which helped us to polish the presentation. We also want to thank Mrs.E.Gayathri for her careful typing of the manuscript. We hope eventually to give a more detailed presentation of this material. We would therefore appreciate the reader's comments.

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Lecture 1 : Introduction

In this introduction to quantum field theory in curved spacetime we shall treat the curved spacetime as a given, classical background. We shall not attempt to quantize the gravitational field. Instead, we shall deal with the quantization of the scalar field on this classical gravitational background ignoring any back reaction.

It should be emphasized that only "elementary" properties of general relativity and quantum field theory will be used. Specifically, from GR the concepts used are covariant differentiation and the curvature tensor, whereas from QFT we need only the standard Fock space quantization of the Klein-Gordon scalar field and the notion of Bogoliubov transformations.

The remarkable thing is that equipped only with these simple tools physically interesting results can be calculated which are both new and profound. The principle example is Hawking radiation, which will be discussed in lecture 3.

At the same time, it should also be emphasized that the rigorous mathematical foundations of this approach are not at all satisfactory. During these lectures we will attempt to point out some of the places where this occurs. A good (in factthe only) general reference is Birrell & Davies , chapters 2,3,4.

General Relativity

A spacetime is a manifold M equipped with a metric tensor g_{ab}. In general, n will denote the dimension of M, but we will usually consider only n = 2 or n = 4. The signature of g_{ab} is taken to be (- + + + ...), and we set $\hbar = c = G = k = 1$ (geometrical units). Covariant derivatives with respect to the (metric compatible, torsion free) Levi-Civita connection will be denoted by ∇_a and the Ricci scalar will be denoted by R.

The d'Alembertian operator $\Box$ is defined by

$$\Box \Phi = g^{ab} \nabla_a \nabla_b \Phi = \frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g}\, g^{ab} \partial_b \Phi \right) \tag{1}$$

where $g = \det (g_{ab})$.

The topology of M will not be explicitly specified, but in most cases it may be taken to be $\mathbb{R}^n$. We will also usually assume the existance of a timelike Killing vector field X^a, i.e.

$$\mathcal{L}_X g_{ab} \equiv \nabla_a X_b + \nabla_b X_a = 0 \tag{2}$$

where $\mathcal{L}$ denotes Lie derivative.

Klein-Gordon equation

We now show how the Klein-Gordon equation for the scalar field can be generalized to curved space. In flat space we have

$$(\Box - m^2)\Phi = 0 \tag{3}$$

In curved space the Klein-Gordon equation is

$$(\Box - \xi R - m^2)\Phi = 0 \tag{4}$$

where ξ is an arbitrary coupling constant and m is the usual mass parameter. Notice that the coupling of the scalar field to gravity occurs in two ways: due to the covariant derivatives in $\Box$ and due to the presence of R. The ξR term is included as the only possible mass independent scalar constructed from the metric and curvature tensors which has the right dimension. A more intuitive argument for its presence is due to the fact that one expects the massless scalar field to propagate along null directions (i.e. at the speed of light) and thus be conformally invariant. As an example consider n = 4. Then if

$$g_{ab} = \Omega^2 \hat{g}_{ab} \tag{5}$$

we also have

$$\Box \Phi = \Omega^{-2} \hat{\Box} \Phi + 2\Omega^{-3} \hat{g}^{ab} \hat{\nabla}_a \Omega \hat{\nabla}_b \Phi \tag{6}$$

and

$$R = \Omega^{-2} \hat{R} - 6\Omega^{-3} \hat{\Box} \Omega \tag{7}$$

(the numerical coefficients are dimension-dependent). Thus, setting

$$\Phi = \Omega^{-1} \hat{\Phi} \tag{8}$$

we obtain

$$(\Box - \tfrac{1}{6} R)\Phi \equiv \Omega^{-3}(\hat{\Box} - \tfrac{1}{6}\hat{R})\hat{\Phi} \tag{9}$$

so that the (massless) Klein-Gordon equation is conformally invariant in 4 dimensions if and only if ξ = 1/6, which is called conformal coupling. The choice $\xi = 0$ is referred to as minimal coupling.

In 2 dimensions it turns out that conformal coupling and minimal coupling coincide, i.e. that $\xi = 0$ gives a conformally invariant equation. Furthermore, here $\Phi = \hat{\Phi}$. The general relatio are

$$\xi = \frac{1}{4}\,\frac{n-2}{n-1} \tag{10}$$

$$\Phi = \Omega^{(2-n)/2}\,\hat{\Phi} \tag{11}$$

The equation of motion (4) can be derived from the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}\sqrt{-g}\left[g^{ab}\partial_a\Phi\,\partial_b\Phi + (\xi R + m^2)\,\Phi^2\right] \tag{12}$$

Let us first consider flat space in n dimensions. The metric is just $g_{ab} = \eta_{ab}$ i.e.

$$ds^2 = -dt^2 + d\vec{x}^{\,2} \tag{13}$$

and the Klein-Gordon equation is (Δ denotes the n-1 dimensional spatial Laplacian)

$$\left(-\partial_t^2 + \Delta - m^2\right)\Phi = 0 \tag{14}$$

There is a natural (i.e. conserved) scalar product, called the Klein-Gordon product, defined by

$$(\Phi, \tilde{\Phi}) = i\int\left(\Phi^*\dot{\tilde{\Phi}} - \dot{\Phi}^*\tilde{\Phi}\right)d^{n-1}x \tag{15}$$

where the dot refers to time derivatives and the surface of integration is taken to be $\{t = \text{constant}\}$.

<u>Assuming</u> that the surface term resulting from integration by parts vanishes one can see that

$$(\Phi, \tilde{\Phi})^{\dot{}} \equiv i\int\left(\Phi^*\Delta\tilde{\Phi} - \tilde{\Phi}\Delta\Phi^*\right)d^{n-1}x = 0 \tag{16}$$

We can generalize this to <u>any</u> spacelike hypersurface Σ with unit (timelike, future pointing) normal n^a as

$$(\Phi,\tilde{\Phi}) = i\int_{\Sigma} n^a(\Phi^*\partial_a\tilde{\Phi} - \tilde{\Phi}\partial_a\Phi^*)\,d\Sigma \tag{17}$$

and one can show (again, up to boundary terms which are assumed to be zero) that $(\Phi,\tilde{\Phi})$ is independent of Σ for any two solutions of (4).

Some properties of this scalar product are

1) $(\Phi,\Phi) = 0$ for Φ real
2) $(\tilde{\Phi},\Phi) = (\Phi,\tilde{\Phi})^*$
3) $(\Phi^*,\tilde{\Phi}^*) = -(\Phi,\tilde{\Phi})^*$ (18)

From 2) we see that the Klein-Gordon product is Hermitian, so that $(\Phi,\Phi)\in\mathbb{R}$ but from 3) with $\tilde{\Phi}=\Phi$ we see that it is <u>not</u> positive definite.

A Hilbert space is by definition a complex vector space equipped with a Hermitian, positive definite inner product and which is Cauchy complete. We will only consider <u>separable</u> Hilbert spaces, i.e. we will assume the existence of a countable basis. (Almost all Hilbert spaces in physics are separable.)

We therefore proceed as follows. Choose a countable set of solutions $\{u_k\}$, $k\in\mathbb{Z}$, to (4) which are orthonormal, i.e.

$$(u_k,u_{k'}) = \delta_{kk'} = -(u_k^*,u_{k'}^*)\ ;\ (u_k,u_{k'}^*) = 0 \tag{19}$$

and complete in the sense that "any" <u>real</u> solution can be written

$$\Phi = \sum_k (a_k u_k + a_k^* u_k^*) \tag{20}$$

<u>Classical Solutions:</u> We now construct the Hilbert space of classical, positive frequency solutions $\mathcal{H}$ by taking the Cauchy completion of the span of $\{u_k\}$ (but <u>not</u> including $\{u_k^*\}$) with respect to the

Klein-Gordon product. $\{u_k\}$ is an orthonormal basis of $\mathcal{H}$ and the Klein-Gordon product is clearly positive definite there.

We can also consider the space $\mathcal{H}^*$ of negative frequency solutions. In general, the space of complex solutions $V^{\mathbb{C}}$ is given by

$$V^{\mathbb{C}} = \mathcal{H} \oplus \mathcal{H}^* \tag{21}$$

One thing to be noted here is that specifying $V^{\mathbb{C}}$ does <u>not</u> uniquely determine the decomposition (21); the choice of positive/negative frequency depends on the choice of u_k. As a simple example of this, consider a new set $\{v_k\}$ given by

$$v_k = A u_k + B u_k^* \tag{22}$$

with $A^2 - B^2 = 1$. Then $\{v_k\}$ satisfies (19) and (20) and, denoting the corresponding Hilbert spaces by $\mathcal{H}_u$ and $\mathcal{H}_v$, we have

$$\mathcal{H}_u \oplus \mathcal{H}_u^* = V^{\mathbb{C}} = \mathcal{H}_v \oplus \mathcal{H}_v^* \tag{23}$$

although $\mathcal{H}_v$ is clearly different from $\mathcal{H}_u$.

<u>Second quantization</u>

So far we have treated the coefficients a_k and a_k^* in (20) as being C-numbers. Here their status is uplifted to that of operators, and a_k^* is replaced by $a_k^\dagger$.

So Φ itself becomes an operator, acting on a Fock space which we now construct. We impose the canonical commutation relations

$$[a_k, a_{k'}^\dagger] = \delta_{kk'} \quad ; \quad [a_k, a_{k'}] = 0 = [a_k^\dagger, a_{k'}^\dagger] \tag{24}$$

and define the vacuum state $|0\rangle$ by

$$a_k|0\rangle = 0 \quad \forall k \quad ; \quad \langle 0|0\rangle = 1 \tag{25}$$

The Hilbert space of one particle states is defined as the span of the orthonormal basis $\{a_k^{\dagger}|0\rangle\}$, and this space is isomorphic to $\mathcal{H}$. Multiparticle states are obtained as usual by constructing the Fock space F over $\mathcal{H}$ (cf.Wald)

$$F = \mathbb{1} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes_s \mathcal{H}) \oplus \ldots \tag{26}$$

where $\otimes_s$ denotes the symmetrized tensor product.

Bogoliubov transformations

As above, if we consider some other definition of positive frequency $\{V_\ell\}$ satisfying (19) and (20) then we get a different decomposition of F, i.e. usually $F_u = F_v$ although $\mathcal{H}_u \neq \mathcal{H}_v$. We now show in more detail how this works.

In addition to (20) we now assume that we can expand Φ as

$$\Phi = \sum_\ell (b_\ell V_\ell + b_\ell^* V_\ell^*) \tag{27}$$

or at the level of second quantization

$$\sum_k (a_k u_k + a_k^{\dagger} u_k^*) = \Phi = \sum_\ell (b_\ell V_\ell + b_\ell^{\dagger} V_\ell^*) \tag{28}$$

But since the bases are different, the notion of particle is different! In particular, there is no reason for b_ℓ to annihilate the same vacuum as a_k!

Since we are assuming completeness, we can write

$$V_\ell = \sum_K \left(\alpha_{\ell K} u_K + \beta_{\ell K} u_K^* \right) \tag{29}$$

where $\alpha_{\ell K} = (V_\ell, u_K)$, $\beta_{\ell K} = -(V_\ell, u_K^*)$.

Similarly,

$$u_K = \sum_\ell \left(\alpha_{\ell K}^* V_\ell - \beta_{\ell K} V_\ell^* \right) \tag{30}$$

Equivalently, using (28) and (19) we can write

$$\begin{aligned} b_\ell &= (V_\ell, \Phi) \\ &= \sum_K \left(\alpha_{\ell K}^* a_K - \beta_{\ell K}^* a_K^\dagger \right) \end{aligned} \tag{31}$$

$$\begin{aligned} a_K &= (u_K, \Phi) \\ &= \sum_\ell \left(\alpha_{\ell K} b_\ell + \beta_{\ell K}^* b_\ell^\dagger \right) \end{aligned} \tag{32}$$

Completeness forces these transformations to be invertible, i.e.

$$\begin{aligned} \sum_K \left(\alpha_{\ell K} \alpha_{\ell' K}^* - \beta_{\ell K} \beta_{\ell' K}^* \right) &= \delta_{\ell \ell'} \\ \sum_K \left(\alpha_{\ell K} \beta_{\ell' K} - \beta_{\ell K} \alpha_{\ell' K} \right) &= 0 \end{aligned} \tag{33}$$

and

$$\begin{aligned} \sum_\ell \left(\alpha_{\ell K}^* \alpha_{\ell K'} - \beta_{\ell K} \beta_{\ell K'}^* \right) &= \delta_{K K'} \\ \sum_\ell \left(\alpha_{\ell K}^* \beta_{\ell K'} - \beta_{\ell K} \alpha_{\ell K'}^* \right) &= 0 \end{aligned} \tag{34}$$

We can now ask how many "v" particles there are in the "u" vacuum $|0_u\rangle$. The number operator associated with "v" particles is $\sum_\ell b_\ell^\dagger b_\ell$, so the answer is (using (31)).

$$\sum_{\ell} \langle 0_u | b_\ell^\dagger b_\ell | 0_u \rangle$$
$$= \sum_{\ell,k,k'} \beta_{\ell k} \beta^*_{\ell k'} \langle 0_u | a_k a^\dagger_{k'} | 0_u \rangle$$
$$= \sum_{\ell,k} \beta_{\ell k} \beta^*_{\ell k} \tag{35}$$
$$= tr(\beta\beta^\dagger)$$

so that $tr(\beta\beta^\dagger)$ gives the total number of "v" type particles "created" in the "u" vacuum. Note that β is the coefficient which mixes positive and negative frequencies.

Plane Waves

In practice, however, the standard procedure is to choose a "basis" of plane waves. For instance, in flat space one usually considers

$$u_k = \frac{1}{\sqrt{(2\pi)^{n-1} 2\omega}} e^{ik\cdot x} e^{-i\omega t} \qquad k, x \in \mathbb{R}^{n-1} \tag{36}$$

which satisfy

$$(u_k, u_{k'}) = \delta^{(n-1)}(k - k') \tag{37}$$

It can not be emphasized too strongly that these u_k are not a basis because they are not even in the space V^C of "suitable" solutions because they are not normalizable! This difficulty is usually hidden by statements like "only suitable linear combinations of u_k are allowed".

The above discussion of Bogoliubov transformations can nevertheless be carried over to the case of a continuous parameter k by replacing sums over k with $\int d^{n-1}k$. However, if

$\{u_K\}$ and $\{v_\ell\}$ are not bases then it becomes very hard to decide if the two resulting expansions of "suitable" functions are equivalent, i.e. if the two sets of "suitable" functions define the same space.

One can easily see that even though $\alpha_{\ell K}$ and $\beta_{\ell K}$ now acquire the dimensions of (length)$^{n-1}$,

$$tr\,(\beta\beta^{+}) = \iint \beta_{K\ell}\,\beta^{+}_{K\ell}\; d^{n-1}K\, d^{n-1}\ell \tag{38}$$

remains dimensionless and still represents the total number of "v" particles in the "u" vacuum.

Finally, it should be pointed out that there is another use of the term Bogoliubov transformation in the special case where

$$\begin{aligned} \alpha_{KK'} &= \hat{\alpha}_K\, \delta^{(n-1)}(K-K') \\ \beta_{KK'} &= \hat{\beta}_K\, \delta^{(n-1)}(K-K') \end{aligned} \tag{39}$$

in which case one often refers to $\hat{\alpha}_K$, $\hat{\beta}_K$ instead of $\alpha_{KK'}$, $\beta_{KK'}$ as being the Bogoliubov coefficients. Note that in this case we have

$$\begin{aligned} tr\,(\beta\beta^{+}) &= tr\,(\hat{\beta}\hat{\beta}^{+})\; \delta^{(n-1)}(0) \\ &= tr\,(\hat{\beta}\hat{\beta}^{+})\,\frac{1}{(2\pi)^{n-1}}\int d^{n-1}x \end{aligned} \tag{40}$$

so that $\frac{1}{(2\pi)^{n-1}}\, tr\,(\hat{\beta}\hat{\beta}^{+})$ is the number density of created particles.

Lecture 2: Rindler Space

In this lecture we shall consider 2-dimensional Minkowski space with metric

$$ds^2 = -dt^2 + dx^2 \tag{2.1}$$

and consider the scalar field as seen by an observer undergoing constant acceleration. We again consider the Klein-Gordon equation

$$\left(-\partial_t^2 + \partial_x^2 - m^2\right)\Phi = 0 \tag{2.2}$$

and for ease of calculation we will work with the standard, Dirac delta function normalized set of modes

$$u_k = \frac{1}{\sqrt{4\pi\omega}}\, e^{i(kx-\omega t)} \tag{2.3}$$

with $\omega = +\sqrt{k^2+m^2}$ and $k \in \mathbb{R}$.

However, before proceeding to the actual calculation let us recall where the u_k come from and why other solutions are not considered.

Separable Solutions

The u_k are clearly separable, i.e. satisfy

$$\Phi(x,t) = X(x)\,T(t) \tag{2.4}$$

What if we consider all separable solutions? For the massless case (m=0) we would get the following possibilities for the functions X, T:

$$\begin{pmatrix} X \\ T \end{pmatrix} = \begin{pmatrix} Ae^{ikx}+Be^{-ikx} \\ Ce^{ikt}+De^{-ikt} \end{pmatrix}, \begin{pmatrix} Ae^{kx}+Be^{-kx} \\ Ce^{kt}+De^{-kt} \end{pmatrix}, \begin{pmatrix} A+Bx \\ C+Dt \end{pmatrix} \tag{2.5}$$

where A, B, C, D are arbitrary constants. For $m \neq 0$ the solutions are similar. Note that the u_k are a special case of the first possibility. The second possibility ("imaginary frequency") consists of solutions which diverge as x(or t) goes to $\pm\infty$. Thus, they can not even be normalized to Dirac delta functions. We will say more below about the last possibility ("zero frequency").

Fourier expansion in t

We can also obtain the u_k if we Fourier decompose solutions of (2.2) as

$$\Phi = \int e^{-i\omega t}\tilde{\Phi}_\omega(x)\,d\omega \tag{2.6}$$

This approach is especially attractive when, as here, t corresponds to an isometry of spacetime, i.e. ∂_t is a Killing vector. But if we now consider the massive case ($m \neq 0$) and use (2.2) we find that the $\tilde{\Phi}_\omega$ must satisfy

$$(\omega^2 - m^2)\tilde{\Phi}_\omega + \tilde{\Phi}_\omega'' = 0 \tag{2.7}$$

For $|\omega| > m$ this indeed yields the u_k, but for $|\omega| < m$ one is again led to divergent, non-normalizable solutions. Also for $|\omega| = m$ one possible solution is $\tilde{\Phi}_\omega = x$, which leads to the same problem.

Fourier decomposition in x

If we instead try

$$\Phi = \int e^{-ikx} \hat{\Phi}_k(t)\, dk \tag{2.8}$$

we obtain the following equation for $\hat{\Phi}_k$

$$(k^2 + m^2)\hat{\Phi}_k + \ddot{\hat{\Phi}}_k = 0 \tag{2.9}$$

which does yield the U_k except for the case k=0=m, for which $\hat{\Phi}_0 = t$ would be a solution.

The motivation for this discussion is to emphasize that it is not enough just to give a set of modes U_k, especially when they are not normalizable. In addition one must also specify in some physical manner, e.g. by giving boundary conditions, which combinations of the modes are acceptable.

Zero frequency

For m ≠ 0 there are two perfectly well-behaved $(k=0)$ modes namely $U_0 \sim e^{\pm imt}$. But for m = 0, not only is there a degeneracy in that $U_0 = U_0^*$ but the normalization factor $1/\sqrt{\omega}$ diverges. In practice this is usually ignored, but in fact this is one of the reasons that the m = 0 case (especially in 2 dimensions) is the hardest to justify rigorously.

However, since "zero modes" may be important if the space is compact, we point out here that there does exist a nonstandard choice of modes which does have a zero frequency limit, namely

$$\Psi_k = \frac{1}{\sqrt{4\pi}} e^{ikx}\left(\cos\omega t - i\,\frac{\sin\omega t}{\omega}\right) \tag{2.10}$$

In particular, for m = 0, the zero mode

$$\psi_0 = \frac{1}{\sqrt{4\pi}}(1 - it) \qquad (2.11)$$

is normalizable to a Dirac delta function.

Accelerating Observer: Consider now an observer moving along the trajectory

$$x^2 - t^2 = \frac{1}{a^2} \qquad (2.12)$$

We claim that this observer moves with uniform acceleration a as measured in his instantaneous rest frame. To see this, let τ denote the proper time of the observer, i.e. the time measured by his watch. Then we can view x, t as being functions of τ. The 2-velocity of the observer is defined by

$$U^a = \begin{pmatrix} \partial t/\partial\tau \\ \partial x/\partial\tau \end{pmatrix} \quad ; \quad U_a U^a = -1 \qquad (2.13)$$

which can easily be solved to give

$$t = \frac{1}{a}\sinh a\tau \quad , \quad x = \frac{1}{a}\cosh a\tau \qquad (2.14)$$

where we have assumed t(0) = 0 and $X > 0$. The 2-acceleration is now given by

$$A^a = U^b \nabla_b U^a = \frac{\partial U^a}{\partial\tau} = \begin{pmatrix} a\sinh a\tau \\ a\cosh a\tau \end{pmatrix} \qquad (2.15)$$

Note that A^a is orthogonal to U^a, i.e

$$U_a A^a = 0 \qquad (2.16)$$

But u^a is precisely the instantaneous time axis, which means that the acceleration is proportional to the instantaneous space axis. Therefore, the acceleration measured by the observer (the "spatial" component of A^a) is just

$$(A_a A^a)^{1/2} \equiv a \tag{2.17}$$

as claimed.

We now introduce Rindler coordinates, which are the coordinates appropriate to uniformly accelerating observers. Define coordinates ρ, τ implicitly by

$$\begin{aligned} t &= \frac{1}{a} e^{a\rho} \sinh a\tau \\ x &= \frac{1}{a} e^{a\rho} \cosh a\tau \end{aligned} \tag{2.18}$$

so that

$$x^2 - t^2 = \frac{e^{2a\rho}}{a^2} \tag{2.19}$$

From the above discussion, we see that the trajectories $\{\rho = \text{constant}\}$ correspond to observers undergoing constant acceleration $ae^{-a\rho}$. Note furthermore that coordinate time τ is in general only proportional to the proper time of such an observer, and is only equal to the proper time for the observer $\rho = 0$, corresponding to acceleration a.

These relations are depicted in Figure 1.

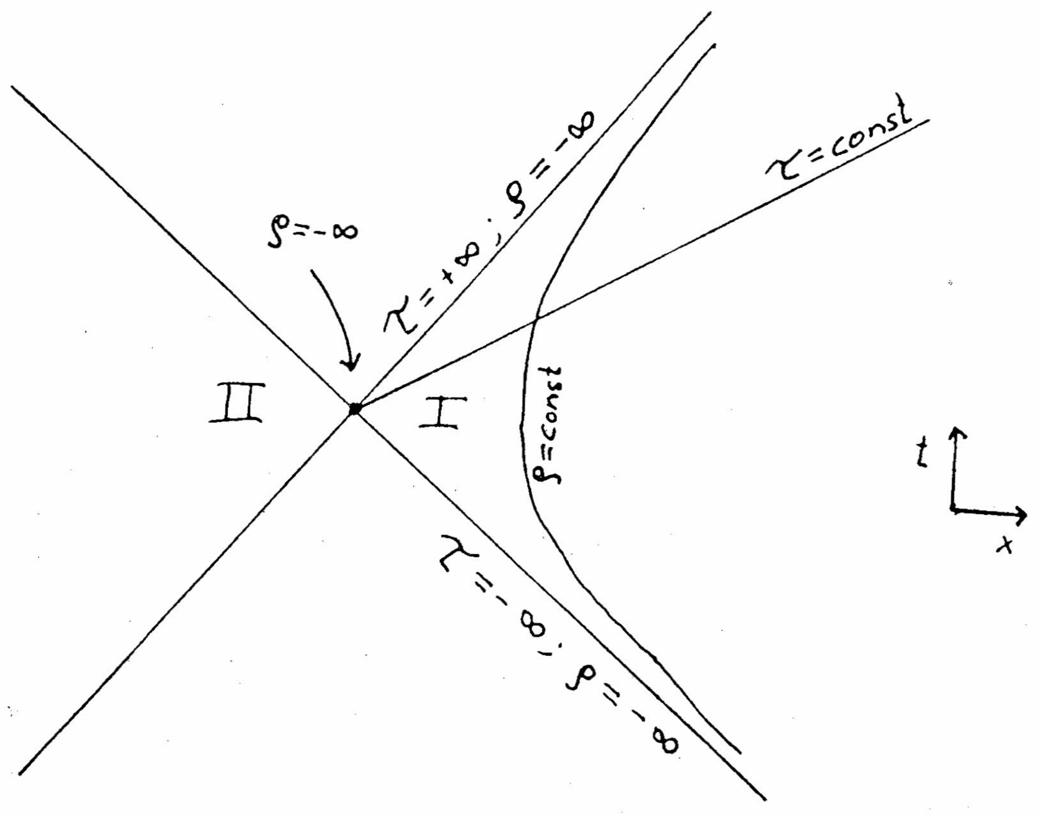


Figure 1: The Rindler coordinates (ρ, τ) of (2.18) cover only one wedge (I) of Minkowski space.

The metric in these coordinates becomes

$$ds^2 = e^{2a\rho}(-d\tau^2 + d\rho^2) \tag{2.20}$$

Note that this metric is conformally flat. (It is of course also flat since it equals (2.1).) Furthermore, τ corresponds to an isometry, i.e. ∂_τ is a (timelike) Killing vector.

For the other wedge (II) one replaces $e^{a\rho}$ in (2.18) by $-e^{a\rho}$. To cover both wedges at once, introduce

$$r = \begin{cases} \frac{1}{a} e^{a\rho} & \text{for the right wedge (I)} \\ -\frac{1}{a} e^{a\rho} & \text{for the left wedge (II)} \end{cases} \tag{2.21}$$

so that $r > 0$ corresponds to the right and $r < 0$ to the left.

In Rindler coordinates the massless Klein-Gordon equation becomes

$$(-\partial_\tau^2 + \partial_\rho^2)\,\Phi = 0 \tag{2.22}$$

and the Klein-Gordon product (in one wedge) is

$$(\Phi, \tilde{\Phi}) = i \int_{-\infty}^{+\infty} (\Phi^* \partial_\tau \tilde{\Phi} - \tilde{\Phi}\, \partial_\tau \Phi^*)\, d\rho \tag{2.23}$$

where the integral is over a τ = constant surface. But (2.22) and (2.23) are formally identical to their Minkowski analogues! This leads one to write down immediately the Dirac delta function normalized modes

$$V_\omega = \frac{1}{\sqrt{4\pi|\omega|}}\, e^{i(\omega\rho - |\omega|\tau)}$$

$$= \frac{1}{\sqrt{4\pi|\omega|}}\, e^{i\left(\frac{\omega}{a}\ln|ar| - |\omega|\tau\right)} \tag{2.24}$$

for $\omega \in \mathbb{R}$. (This is not the same ω as in (2.3).)

In Minkowski coordinates we have the second quantized expansion

$$\Phi_M = \int_{-\infty}^{\infty} \left(a(k)u_k + a^\dagger(k)u_k^*\right) dk \tag{2.25}$$

with

$$\left[a(k), a^\dagger(k')\right] = \delta(k-k') \tag{2.26}$$

Similarly, in Rindler coordinates we have in the right wedge

$$\Phi_R = \int_{-\infty}^{\infty} \left(b(\omega)V_\omega + b^\dagger(\omega)V_\omega^*\right) d\omega \tag{2.27}$$

and in the left wedge

$$\Phi_R = \int_{-\infty}^{\infty} \left(d(\omega)V_\omega + d^\dagger(\omega)V_\omega^*\right) d\omega \tag{2.28}$$

(Technically we should distinguish between the ${}_{I}V_\omega$ of (2.27) and the ${}_{II}V_\omega$ of (2.28) which have support in the right/left wedge, respectively). Furthermore, due to causality, the expressions (2.27) and (2.28) should commute, i.e. symbolically

$$[b, d] = 0 \tag{2.29}$$

We can now set these expansions equal, i.e.

$$\Phi_M \equiv \Phi_R \tag{2.30}$$

and calculate the Bogoliubov coefficients as explained in the last lecture. However, before doing this let us ask whether (2.30) holds <u>classically</u>, i.e. whether the expansions (2.25) and (2.27), (2.28) are equivalent. To do so requires us (finally!) to discuss boundary conditions in more detail.

<u>Boundary conditions</u>

One criterion which is often suggested is that Φ should be square integrable, i.e.

$$\int_{t=const} \Phi_M^* \Phi_M \, dx < \infty \tag{2.31}$$

(one writes $\Phi_M \in L^2(dx)$ if this is satisfied.) But this is too <u>weak</u>, because the time derivatives of Φ_M are unrestricted. Thus, this is <u>not</u> sufficient to guarantee that the Klein-Gordon norm (Φ_M, Φ_M) of Φ_M will be finite, let alone that the necessary boundary terms are zero in the proof of the time independence of the Klein-Gordon product.

So assume that <u>both</u> Φ_M and $\partial_t \Phi_M$ are square integrable. This is sufficient to satisfy the criteria of the preceding paragraph, but it turns out to be too strong. In this context, requiring $\Phi_M \in L^2(dx)$ and $\partial_t \Phi_M \in L^2(dx)$ is equivalent to demanding that $\Phi_M \in H^1(dx)$, where H^1 is a Sobolev space. (The "energy norm" of Φ_M is finite.) But because of the formal symmetry between Minkowski space and Rindler space it is clear that if we require $\Phi_M \in H^1(dx)$ we had better require $\Phi_R \in H^1(d\rho)$.

The immediate question is this: Are $H^1(dx)$ and $H^1(d\varrho) \oplus H^1(d\varrho)$ (one copy for each wedge) equal in an appropriate sense? (i.e. is the intersection dense in each of them?). The answer turns out to be no (cf. Dray & Manogue).

<u>Theorem:</u> There exists a function $f \in H^1(dx)$ which is orthogonal to every function in $H^1(d\varrho) \oplus H^1(d\varrho)$.

To avoid this problem, one must reexamine the physical conditions one wishes to impose. What one really wants is for the Klein-Gordon norm of positive frequency solutions to be finite, and this turns out to be equivalent to demanding

$$\Phi_M^+ \in H^{1/2}(dx) \tag{2.32}$$

(where the "+" means that the integral (2.25) has no u_k^* terms). This avoids all of the problems discussed above.

We now return to the comparison of Φ_M with Φ_R and the calculation of Bogoliubov coefficients. (The following calculation was done by Pierre van Baal for a course taught by Gerard 't Hooft).

We introduce the following notation:

$$\begin{aligned} \epsilon &= \operatorname{sgn}(r) \\ \delta &= \operatorname{sgn}(k) \end{aligned} \tag{2.33}$$

so that $\epsilon = +1$ in the right wedge(I) and $\epsilon = -1$ in the left wedge (II) etc. We will also write $b(\omega)$ for <u>both</u> expansions (2.27) and (2.28). We shall use the following useful relation

$$\int_{-\infty}^{\infty} e^{i\omega\tau} e^{\pm\left[ikre^{-\delta a\tau}\right]} d\tau$$

$$= \frac{1}{a} e^{i\frac{\omega}{a}\delta \ln|kr|} \Gamma\left(\frac{-i\omega}{a}\right) e^{\pm\epsilon\frac{\pi\omega}{2a}} \qquad (2.34)$$

Define new operators

$$\alpha_{\pm}(\omega) = \frac{1}{a\sqrt{2\pi}} \Gamma\left(\frac{\mp i\omega}{a}\right) \int_{0}^{\infty} \frac{1}{\sqrt{4\pi K}} a(\pm K) e^{i\frac{\omega}{a}\ln\frac{K}{a}} dK \qquad (2.35)$$

But now multiplying Φ_M by $e^{i\omega\tau}$, replacing (x,t) by (r,τ), and integrating over τ yields

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi_M e^{i\omega\tau} d\tau$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi|k|}} \begin{bmatrix} a(k) e^{ikre^{-\delta a\tau}} e^{i\omega\tau} \\ + a^{\dagger}(k) e^{-ikre^{-\delta a\tau}} e^{i\omega\tau} \end{bmatrix} dk\, d\tau \qquad (2.36)$$

$$= e^{i\frac{\omega}{a}\ln|ar|}\left(\alpha_{+}(\omega) e^{\epsilon\frac{\pi\omega}{2a}} + \alpha_{+}^{\dagger}(-\omega) e^{-\epsilon\frac{\pi\omega}{2a}}\right) + e^{-i\frac{\omega}{a}\ln|ar|}\left(\alpha_{-}(\omega) e^{\epsilon\frac{\pi\omega}{2a}} + \alpha_{-}^{\dagger}(-\omega) e^{-\epsilon\frac{\pi\omega}{2a}}\right) \qquad (2.37)$$

It is to be noted that the operators $\alpha_{\pm}(\omega)$ are just linear combinations of the a(k). Thus, the vacuum annihilated by the $\alpha_{\pm}(\omega)$ is the same as that annihilated by the a(k), namely the usual Minkowski vacuum $|0_M\rangle$.

On the other hand, replacing Φ_M by Φ_R yields directly

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi_R\, e^{i\omega\tau}\,d\tau \tag{2.38}$$

$$=\begin{cases}\frac{1}{\sqrt{2|\omega|}}\left[b(\omega)\,e^{i\frac{\omega}{a}\ln|ar|}+b(-\omega)\,e^{-i\frac{\omega}{a}\ln|ar|}\right] & (\omega>0)\\ \frac{1}{\sqrt{2|\omega|}}\left[b^{\dagger}(\omega)\,e^{-i\frac{\omega}{a}\ln|ar|}+b^{\dagger}(-\omega)\,e^{i\frac{\omega}{a}\ln|ar|}\right] & (\omega<0)\end{cases}$$

In what follows we will always assume $\omega>0$. Comparing (2.38) with (2.37) using (2.30) we can express b and $b^{\dagger}$ in terms of $\alpha_{\pm}$ as follows

$$b(\omega)=\sqrt{2\omega}\left[\alpha_+(\omega)\,e^{\epsilon\frac{\pi\omega}{2a}}+\alpha_+^{\dagger}(-\omega)\,e^{-\epsilon\frac{\pi\omega}{2a}}\right]$$

$$b(-\omega)=\sqrt{2\omega}\left[\alpha_-(\omega)\,e^{\epsilon\frac{\pi\omega}{2a}}+\alpha_-^{\dagger}(-\omega)\,e^{-\epsilon\frac{\pi\omega}{2a}}\right]$$

$$b^{\dagger}(\omega)=\sqrt{2\omega}\left[\alpha_+^{\dagger}(\omega)\,e^{\epsilon\frac{\pi\omega}{2a}}+\alpha_+(-\omega)\,e^{-\epsilon\frac{\pi\omega}{2a}}\right]$$

$$b^{\dagger}(-\omega)=\sqrt{2\omega}\left[\alpha_-^{\dagger}(\omega)\,e^{\epsilon\frac{\pi\omega}{2a}}+\alpha_-(-\omega)\,e^{-\epsilon\frac{\pi\omega}{2a}}\right] \tag{2.39}$$

One can now check the commutators of the $b(\omega)$ directly, obtaining

$$\left[b(\omega),b^{\dagger}(\omega')\right]=2\omega\left(\left[\alpha(\omega),\alpha^{\dagger}(\omega')\right]e^{\frac{\epsilon\pi\omega}{a}}+\left[\alpha^{\dagger}(-\omega),\alpha(-\omega')\right]e^{-\frac{\epsilon\pi\omega}{a}}\right)$$

$$=\epsilon\,\delta(\omega-\omega') \tag{2.40}$$

where we have used

$$\left[\alpha(\omega), \alpha^{\dagger}(\omega')\right] = \frac{\delta(\omega-\omega')}{2\omega\left(e^{\frac{\pi\omega}{a}} - e^{-\frac{\pi\omega}{a}}\right)} \tag{2.41}$$

(Note that each of (2.40) and (2.41) corresponds to two equations, one for $\omega > 0$ (and $\alpha \equiv \alpha_+$), the other for $\omega < 0$ (and $\alpha \equiv \alpha_-$).)

But while (2.40) is the expected result in the right wedge, it appears to be wrong in the left wedge! The reason for this is that in the left wedge τ runs <u>backwards</u> in time and should really have been replaced by $-\tau$. This would of course interchange the roles of positive and negative frequency solutions and corresponds to replacing the "annihilation" operator $d(\omega)$ by the new creation operator $c^{\dagger}(\omega)$. Making this change, (2.40) now corresponds to

$$\left[b(\omega), b^{\dagger}(\omega')\right] = \delta(\omega-\omega') = \left[c(\omega), c^{\dagger}(\omega')\right] \tag{2.42}$$

as desired.

We can now solve (2.39) for $\alpha_{\pm}(\omega)$ in terms of $b(\omega)$ and $c(\omega)$ obtaining

$$\frac{1}{\sqrt{2\omega}}\alpha_{\pm}(\omega) = b(\pm\omega)e^{\frac{\pi\omega}{2a}} - c^{\dagger}(\pm\omega)e^{-\frac{\pi\omega}{2a}}$$
$$\frac{1}{\sqrt{2\omega}}\alpha_{\pm}(-\omega) = c(\pm\omega)e^{\frac{\pi\omega}{2a}} - b^{\dagger}(\pm\omega)e^{-\frac{\pi\omega}{2a}} \tag{2.43}$$

But since $\alpha_{\pm}(\omega)|0_M\rangle = 0$ we are led to

$$b(\omega)|0_M\rangle = c^{\dagger}(\omega)e^{-\frac{\pi|\omega|}{a}}|0_M\rangle$$
$$c(\omega)|0_M\rangle = b^{\dagger}(\omega)e^{-\frac{\pi|\omega|}{a}}|0_M\rangle \tag{2.44}$$

This enables us to express the Minkowski vacuum in the Fock space appropriate to Rindler space, i.e. with respect to a basis $|m,n\rangle$ corresponding to m "b -type" particles in the right wedge and n "c-type" particles in the left wedge. The result is

$$|0_M\rangle = \prod_{\omega}\sum_{n} e^{-\frac{\pi n|\omega|}{a}} N(\omega)\,|n,n\rangle \tag{2.45}$$

where $N(\omega)^2 = 1 - e^{-\frac{2\pi|\omega|}{a}}$. The Minkowski vacuum contains Rindler particles! An accelerated observer in the usual Minkowski vacuum observes particles!

To obtain the observed spectrum, one computes the expectation value of the number operator

$$\begin{aligned}\langle 0_M|\,b^{\dagger}(\omega)b(\omega)\,|0_M\rangle &\equiv \langle 0_M|\,c^{\dagger}(\omega)c(\omega)\,|0_M\rangle \\ &= \left(1 - e^{-\frac{2\pi|\omega|}{a}}\right)\sum n\, e^{-\frac{2\pi n|\omega|}{a}} \\ &= 1\Big/\left(e^{\frac{2\pi|\omega|}{a}} - 1\right)\end{aligned} \tag{2.46}$$

which is just a Planckian spectrum with temperature

$$T = \frac{a}{2\pi} \equiv \frac{\hbar a}{2\pi k c} \tag{2.47}$$

We have shown that an accelerated observer sees a thermal spectrum of particles. But we have only considered the 2-dimensional, massless scalar field. It turns out that this doesn't matter, e.g. the massive scalar field in 4 dimensions can also be solved exactly and yields the same temperature. Also, other fields, e.g. fermions, again lead to (2.47).

Lecture 3: Hawking Radiation

In this lecture we will show how to use the results of the Rindler calculation to derive Hawking's famous prediction that a quantum mechanical black hole radiates with a thermal spectrum. We will first consider the mathematically simpler case of an "eternal" black hole, and then briefly discuss the procedure used in the more physically interesting case of stellar collapse to form a black hole.

The Schwarzschild metric is given by

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{3.1}$$

with $f = 1 - \frac{2m}{r}$. This is the unique, stationary, spherically symmetric, vacuum solution of Einstein's equations corresponding to mass m.

The Klein-Gordon equation cannot be solved for this metric! Therefore, for the time being we shall not bother about the angular coordinates (θ, φ) and shall instead work in 2 dimensions with metric

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} \tag{3.2}$$

We can rewrite this in a suggestive form by introducing the "tortoise coordinate" r^* for $r > 2m$ via

$$r^* = \int \frac{dr}{f} = r + 2m \ln\left|\frac{r}{2m} - 1\right| \tag{3.3}$$

so that the metric becomes

$$ds^2 = f(-dt^2 + dr^{*2}) \qquad (3.4)$$

The massless scalar field equation can now easily be solved using standard plane wave modes. Note that both (3.2) and (3.4) are badly behaved at $r = 2m$, and that $\lim_{r \to 2m} r^* = -\infty$.

Introduce Kruskal coordinates (T, R) via

$$\begin{aligned} T &= 4m\, e^{r^*/4m} \sinh(t/4m) \\ R &= 4m\, e^{r^*/4m} \cosh(t/4m) \end{aligned} \qquad (3.5)$$

leading to the metric

$$ds^2 = \frac{2m}{r} e^{-r/2m} (-dT^2 + dR^2) \qquad (3.6)$$

Note that since r in (3.6) is a function of T (and R), ∂_T is not a Killing vector. Thus, the justification for working in Kruskal coordinates is not obvious. We will defer a discussion of this point until later when we discuss stellar collapse. (∂_T is, however, a conformal Killing vector)

But (3.6) is perfectly regular at $r = 2m$! We can illustrate this by drawing the (R, T) plane; see Figure 2. The corresponding Penrose conformal diagram is shown in Figure 3.

Comparing Figures 1 & 2 we see that the Kruskal coordinates (T, R) correspond to Minkowski coordinates (t, x), whereas the Schwarzschild coordinates (t, r^*) correspond to Rindler coordinates (τ, ρ) .

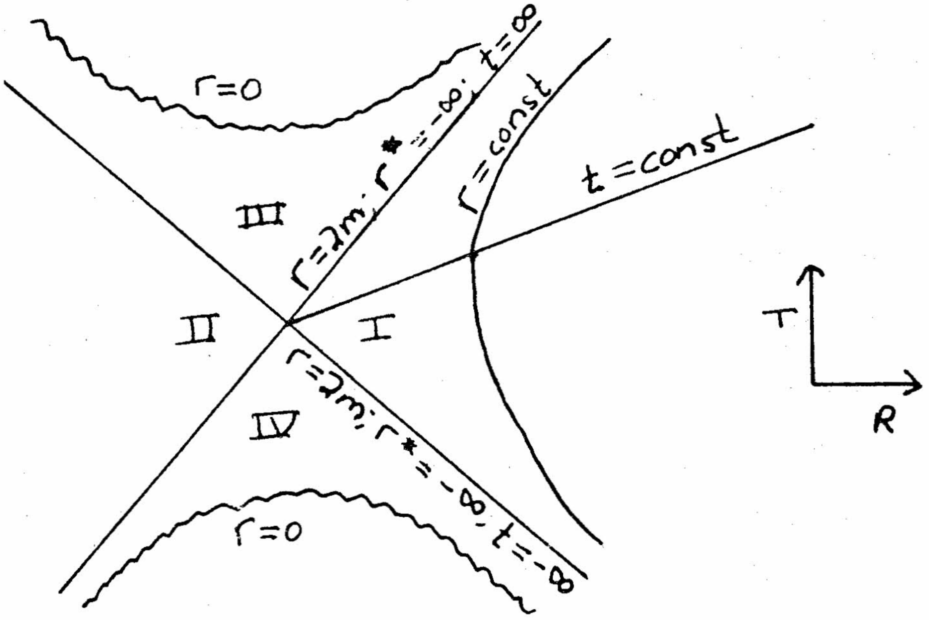


Figure 2: The Schwarzschild black hole is shown in Kruskal coordinates (T,R). The Schwarzschild coordinates (t, r^*) cover only one wedge (I) as shown. Compare Figure 1.

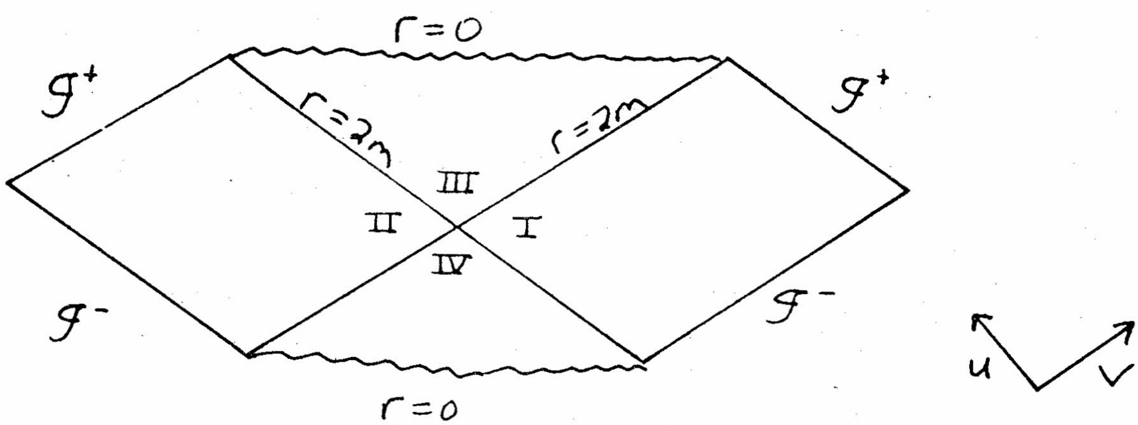


Figure 3: The Penrose conformal diagram for the (maximally extended) Schwarzschild black hole. Each point represents a 2-sphere, and light moves at 45°. Region III is the black hole, Region IV is a white hole, and Regions I & II represent two different external universes.

The above discussion shows that the Schwarzschild trajectories $\{r = \text{constant}\}$ correspond to the Rindler trajectories $\{\rho = \text{constant}\}$ which are uniformly accelerated. But this makes physical sense: The stationary observer at fixed distance from the black hole experiences a constant gravitational force due to the black hole. To remain stationary, such an observer must accelerate uniformly away from the black hole (e.g. with a space ship).

The correspondence between Minkowski/Rindler coordinates and Kruskal/Schwarzschild coordinates can also be seen directly by comparing (3.5) with (2.18) if one replaces a by 1/4m. But the Bogoliubov transformations calculated in Lecture 2 depended only on the coordinate transformations! Another way of saying this is that since (3.6) is conformally the same as the flat metric (2.1), if one considers the conformally coupled ($\xi = 0$ in 2-dimensions!), massless scalar field in each case the results should be the same.

The conclusion is that in order to determine the particle spectrum seen by a Schwarzschild observer in the Kruskal vacuum $|0_K\rangle$, it is sufficient to replace a by 1/4m in (2.47). Thus such an observer sees a Planckian spectrum of particles with temperature

$$T = \frac{1}{8\pi m} = \frac{\hbar c^3}{8\pi k G m} \tag{3.7}$$

This is Hawking's result, and is called the Hawking temperature.

Hawking's original calculation was for a 4-dimensional collapsing star scenario. While it turns out that the result (3.7) does not depend on the fact that we considered a very special case, namely the 2-dimensional, massless, conformally coupled scalar field, this fact is not at all obvious. We will therefore briefly discuss a (2-dimensional) collapse scenario. Our treatment follows chapter 8 of Birrell & Davies and is very similar to the case of a moving mirror in Minkowski space.

We first introduce null coordinates. For Rindler, define

$$u = \tau - \rho$$
$$v = \tau + \rho \tag{3.8}$$

and for Minkowski define

$$\bar{u} = t - x \equiv -\frac{1}{a} e^{-au}$$
$$\bar{v} = t + x \equiv \frac{1}{a} e^{av} \tag{3.9}$$

so that the Minkowski metric becomes

$$ds^2 = -d\bar{u}\,d\bar{v} \equiv -e^{2a\rho}\,du\,dv \tag{3.10}$$

Analogously, for Schwarzschild define

$$u = t - r^*$$
$$v = t + r^* \tag{3.11}$$

and for Kruskal define

$$\bar{u} = T - R \equiv -4m e^{-u/4m}$$
$$\bar{v} = T + R \equiv 4m e^{v/4m} \tag{3.12}$$

so that the Schwarzschild metric becomes

$$ds^2 = -\frac{2m}{r} e^{-2m/r} d\bar{u} d\bar{v} \equiv -f du dv \tag{3.13}$$

Comparing (3.9) and (3.12) again shows the exact analogy between the two sets of coordinate transformations which is obtained by replacing a by 1/4m.

<u>Collapse</u>

Let us now consider a 2-dimensional model of a collapsing spherical star. The surface of the star will describe some timelike trajectory, as shown in Figure 4. Outside the star, the metric will be the Schwarzschild metric i.e.

$$ds^2 = -f du dv \tag{3.14}$$

whereas inside the star the metric will be assumed to be arbitrary, i.e.

$$ds^2 = -g dU dV \tag{3.15}$$

Since there are only two discrete null directions we can assume that U depends only on u, and V depends only on v, i.e.

$$U = A(u)$$
$$v = B(V) \tag{3.16}$$

We could of course choose A and B to be the identity and put all the information about the interior of the star into the function g. But for reasons which will become clear, we prefer to require that

$$V - U = 2r \tag{3.17}$$

which, as can be seen from (3.11), is not satisfied by u,v.

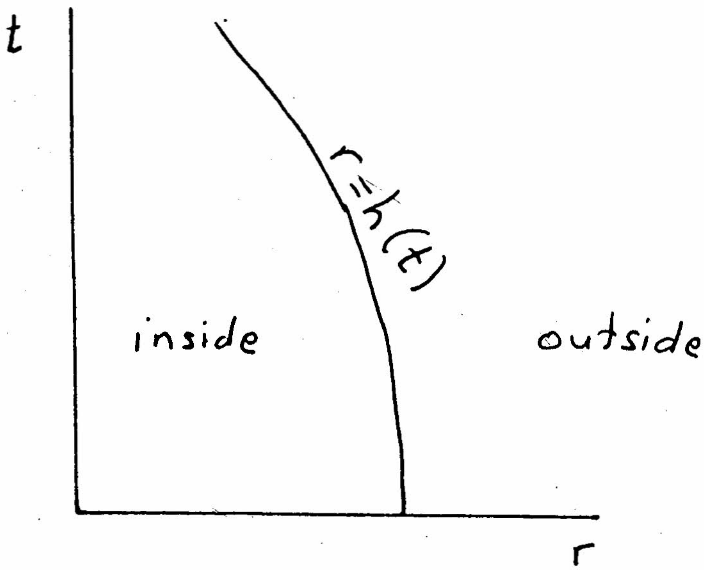


Figure 4: A schematic, 2-dimensional representation of stellar collapse. The curve $r = h(t)$ represents the surface of the star.

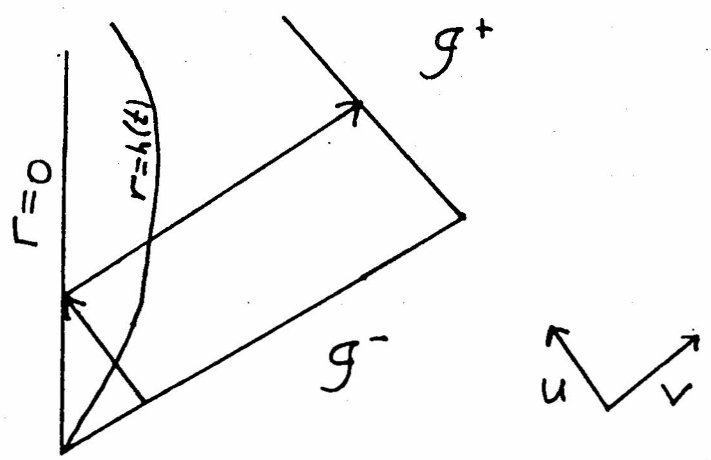


Figure 5: The (partial) Penrose diagram for a collapsing star, as in Figure 4. The arrow shows an incoming wave at early times which passes through the star and is "reflected" into an outgoing wave at late times.

In order to model 4-dimensional spherical symmetry in 2 dimensions, impose perfectly reflecting boundary conditions, i.e.

$$\Phi\big|_{r=0} = 0 \tag{3.18}$$

The reason for this is that a wave approaching the origin from one direction (θ, φ) in spherical coordinates leaves in the antipodal direction $(\pi - \theta, \pi + \varphi)$, corresponding to a reflection in the 2-dimensional (r,t) plane.

So consider the situation depicted in Figure 5, namely an incoming wave which, at early times $(\mathscr{I}^-)$ is just the usual positive frequency mode (up to normalization)

$$\Phi_{in} = e^{-i\omega v} \tag{3.19}$$

(One can easily check, using (3.11), that this corresponds to an ingoing wave.) But (again using the conformal invariance of the massless equation in two dimensions), (3.19) is clearly an <u>exact</u> solution of the Klein-Gordon equation, which is now

$$\partial_u \partial_v \Phi = 0 \tag{3.20}$$

It therefore only remains to impose the boundary condition (3.18) so that

$$\Phi = e^{-i\omega v} - e^{-i\omega v}\big|_{r=0} \tag{3.21}$$

But

$$\begin{aligned} v\big|_{r=0} &= B(V)\big|_{r=0} \\ &\equiv B(U)\big|_{r=0} \equiv B(A(u))\big|_{r=0} \end{aligned} \tag{3.22}$$

Thus, the simple ingoing wave (3.19) is turned into the complicated outgoing wave

$$\Phi_{out} = e^{-i\omega B(A(u))} \tag{3.23}$$

A more detailed examination (highly nontrivial!) shows that, <u>independent</u> of the details of the collapse (i.e. the choice of g and h),

$$A(u) \sim e^{-u/4m}$$
$$B(V) \sim V \qquad (3.24)$$

i.e. U corresponds to the <u>Kruskal</u> coordinate $\bar{u}$, and V corresponds to the <u>Schwarzschild</u> coordinate v. Despite the fact that V does not correspond to the Kruskal $\bar{v}$, the Bogoliubov coefficients are essentially the same, and in particular the same temperature (3.7) results.

<u>Different vacua</u>

In fact, there are <u>three</u> different vacua floating around. The first is the Hartle-Hawking vacuum which is just the Kruskal vacuum $|0_K\rangle$ obtained using ∂_T to define positive frequency. This corresponds to the Minkowski vacuum $|0_M\rangle$ in flat space and to the modes $e^{-i\omega\bar{u}}, e^{-i\omega\bar{v}}$, i.e.

$$|0_K\rangle \longleftrightarrow |0_M\rangle \sim e^{-i\omega\bar{u}}, e^{-i\omega\bar{v}} \qquad (3.25)$$

The Hartle-Hawkingvacuum describes thermal <u>equilibrium</u> at the Hawking temperature, i.e. it corresponds to a black hole in a perfectly reflecting cavity or in a heat bath at the Hawking temperature.

The vacuum which is invariant under the isometry ∂_t is called the <u>Boulware</u> vacuum and is just the Schwarzschild vacuum $|0_S\rangle$, corresponding to the Rindler vacuum $|0_R\rangle$ in flat space, i.e.

$$|0_S\rangle \leftrightarrow |0_R\rangle \sim e^{-i\omega u}, e^{-i\omega v} \qquad (3.26)$$

Since the Schwarzschild spacetime is asymptotically flat, Schwarzschild observers are "almost Minkowskian" far from the black hole. This means that the Boulware vacuum $|0_S\rangle$ "looks like" the Minkowski vacuum $|0_M\rangle$ far from the black hole.

Finally, the vacuum appropriate to collapse is the <u>Unruh</u> vacuum $|0_U\rangle$, and corresponds to the case of an accelerating mirror in flat space, i.e.

$$|0_u\rangle \leftrightarrow \text{accelerating mirror} \sim e^{-i\omega\bar{u}}, e^{-i\omega v} \qquad (3.27)$$

This vacuum is the physically interesting one corresponding to a collapse scenario, and represents a thermal flux (black body spectrum) of particles leaving the black hole at the Hawking temperature.

<u>Penrose diagrams</u>

We have given the Penrose diagram of an eternal black hole in Figure 3. One can also draw the Penrose diagrams appropriate to a collapsing star and to Hawking radiation. These are shown in Figures 6 & 7 respectively. Note that since the Hawking temperature is <u>inversely</u> proportional to the mass it gets hotter and hotter as it decreases in size. This has led Hawking to predict the eventual explosion of a black hole, at the point labelled P. It should be emphasized that all calculations presented here assume a fixed, background spacetime, whereas the evaporating black hole scenario depicted in **Figure** 7 assumes some sort of back reaction on the spacetime due to Hawking radiation, e.g. the mass of the black hole decreases.

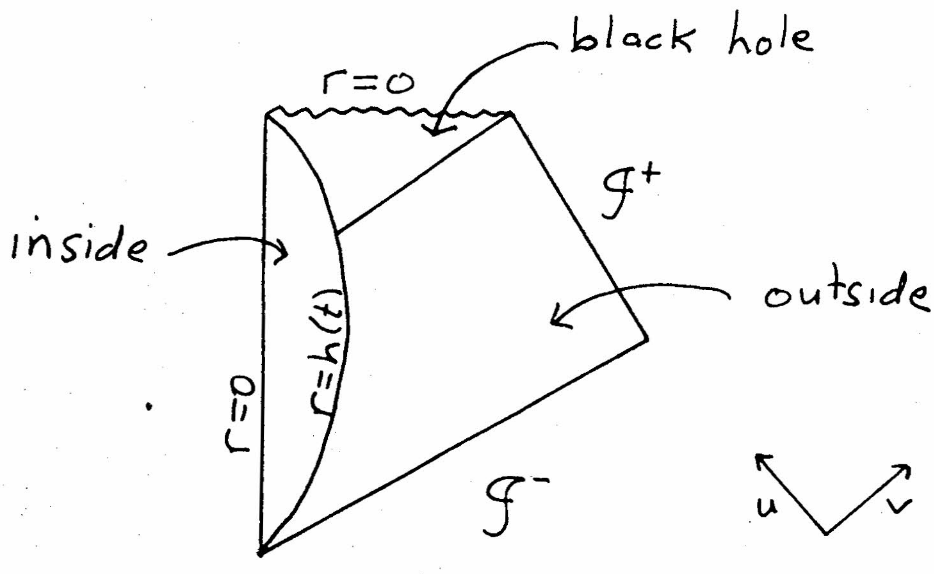


Figure 6: The Penrose diagram of a collapsing star which forms a black hole. Compare Figure 5.

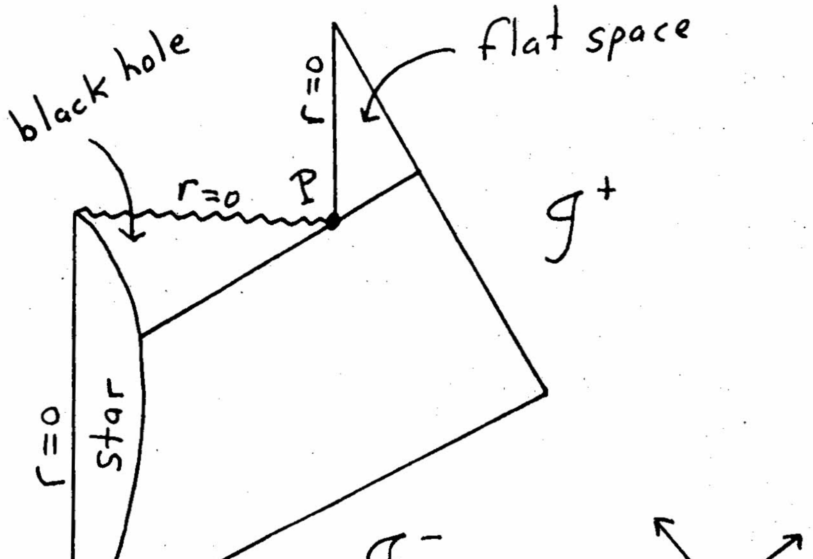


Figure 7: The Penrose diagram appropriate to a black hole which is first formed by stellar collapse as in Figure 6, then loses mass due to Hawking radiation, and finally evaporates or explodes at the point P.

Hawking radiation made easy

There is a trick related to the requirement that the Euclidean Green function be analytic which enables one to calculate the Hawking temperature much more quickly.

Consider polar coordinates in two dimensions

$$ds^2 = dr^2 + r^2 d\theta^2 \qquad (3.28)$$

This metric is badly behaved at $r = 0$ but we know that the underlying spacetime $\mathbb{R}^2$ is well-behaved there so long as the periodicity of θ is chosen to be 2π. If, however, we choose some other periodicity for θ, e.g. identifying $\theta = \frac{3\pi}{2}$ with $\theta = 0$, then the resulting manifold, although flat, will be a cone and will not be differentiable at $r = 0$. (Try carrying out the above construction using paper and a pair of scissors and tape to make the identification)

This can be made precise as follows. Consider the metric

$$ds^2 = G(r)\, dr^2 + F(r)\, d\theta^2 \qquad (3.29)$$

where the periodicity of θ is assumed to be A, i.e. $\theta \in [0, A)$ and where $F(r_0) = 0$. This metric is badly behaved at $r = r_0$ so the question is whether or not there is a conical singularity there.

To avoid a conical singularity we must have

$$\lim_{r \to r_0} \frac{\text{geodesic circumference}}{\text{geodesic radius}} = 2\pi \qquad (3.30)$$

or in other words

$$2\pi = \lim_{r\to r_0} \frac{\int_0^A \sqrt{F}\, d\theta}{\int_{r_0}^{r} \sqrt{G}\, dr}$$

$$= \lim_{r\to r_0} \frac{\frac{AF'}{2\sqrt{F}}}{\sqrt{G}} \tag{3.31}$$

where we have used l'Hôpital's rule in the last step. Thus, the spacetime is regular at $r = r_0$ if and only if the periodicity is chosen to satisfy

$$A = \left.\frac{4\pi\sqrt{FG}}{F'}\right|_{r_0} \tag{3.32}$$

For the Rindler metric (2.20), make the substitutions

$$\theta = i\tau\,, \quad r = \rho\,, \quad r_0 = -\infty \tag{3.33}$$
$$F = G = e^{2ar}$$

This brings (2.20) to the form (3.29). Then (3.32) yields

$$A_{\text{Rindler}} = \frac{4\pi}{2a} = \frac{2\pi}{a} \tag{3.34}$$

For the Schwarzschild metric (3.2), make the substitutions

$$\theta = it\,, \quad r_0 = 2m \tag{3.35}$$
$$F = f\,, \quad G = 1/f$$

which result in

$$A_{\text{Schwarzschild}} = \left.\frac{4\pi}{f'}\right|_{r=2m} = \left.\frac{4\pi}{\frac{2m}{r^2}}\right|_{2m} = 8\pi m \tag{3.36}$$

In both cases the temperature is just the reciprocal of the periodicity!

Lecture 4: Robertson-Walker Spacetimes

In this lecture we will consider particle definitions in Robertson-Walker spacetimes. The presentation is based on a series of 3 papers by Dray et al. (Dray, Renn & Salisbury and 2 papers by Dray & Renn). But first we present some new notation (cf. Ashtekar & Magnon).

Complex Structure

Let V be the space of (suitable) real solutions of the Klein-Gordon equation, and let V_+ denote a choice of positive frequency solutions. Then the space $V^{\mathbb{C}}$ of complex solutions can be decomposed as

$$V^{\mathbb{C}} = V \oplus iV = V_+ \oplus V_- \qquad (4.1)$$

where $V_- = V_+^*$. We have already considered the Hilbert space of positive frequency solutions as (the Cauchy completion of)

$$\mathcal{H}_+ = \{V_+ \, , \, (\; , \;)\} \qquad (4.2)$$

where (,) denotes the Klein-Gordon product. But this corresponds to the quantization of the real scalar field, so we expect there to be a representation in terms of V. But there is clearly a map

$$V_+ \longrightarrow V$$

$$\Phi_+ \longmapsto \Phi_+ + \Phi_+^* \equiv 2 \operatorname{Re} \Phi_+ \qquad (4.3)$$

What about the other direction? From (4.1), given a choice of positive frequency there is a unique decomposition of $\Phi \in V$ as

$$\Phi = \Phi_+ + \Phi_- = \Phi_+ + \Phi_+^* \tag{4.4}$$

with $\Phi_+ \in V_+$; this can be used to define a map from $V \rightarrow V_+$ as follows.

First define a map

$$J : V \longrightarrow V$$

$$\begin{aligned} \Phi_+ + \Phi_- \longmapsto & \; i(\Phi_+ - \Phi_-) \\ & \equiv -2\,\mathrm{Im}\,\Phi_+ \end{aligned} \tag{4.5}$$

Then clearly

$$\Phi_+ = \frac{\Phi - iJ\Phi}{2} \tag{4.6}$$

This is the desired map. Furthermore,

$$\begin{aligned} J(J\Phi) &= i(J\Phi)_+ - i(J\Phi)_- \\ &= i(i\Phi_+) - i(-i\Phi_-) \\ &\equiv -\Phi \end{aligned} \tag{4.7}$$

so that

$$J^2 = -\mathbb{1} \tag{4.8}$$

Thus, one can use J to turn the real space V into a vector space over $\mathbb{C}$. For this reason, J is called a <u>complex structure</u>.

We can now use (4.6) to express $(\Phi_+, \tilde{\Phi}_+)$ in terms of $\Phi, \tilde{\Phi}$:

$$\begin{aligned} (\Phi_+, \tilde{\Phi}_+) &= \tfrac{1}{4}(\Phi - iJ\Phi, \tilde{\Phi} - iJ\tilde{\Phi}) \\ &= \tfrac{1}{4}\left[\begin{aligned} &(\Phi, \tilde{\Phi}) + (J\Phi, J\tilde{\Phi}) \\ &+ i(J\Phi, \tilde{\Phi}) - i(\Phi, J\tilde{\Phi}) \end{aligned} \right] \end{aligned} \tag{4.9}$$

However, it follows directly from

$$(\Phi_+, \tilde{\Phi}_-) \equiv 0 \tag{4.10}$$

that

$$(\Phi, J\tilde{\Phi}) \equiv (-J\Phi, \tilde{\Phi}) \tag{4.11}$$

We can now introduce the Hilbert space of real solutions as (the Cauchy completion of)

$$\mathcal{H} = \{V, J, <\, ,\, >\} \tag{4.12}$$

where

$$\langle \Phi, \tilde{\Phi} \rangle = \frac{1}{2}\Omega(\Phi, J\tilde{\Phi}) + \frac{i}{2}\Omega(\Phi, \tilde{\Phi}) \tag{4.13}$$

where we have written

$$(\Phi, \tilde{\Phi}) \equiv i\,\Omega(\Phi, \tilde{\Phi}) \tag{4.14}$$

(Ω is called the symplectic structure.) In general, we can start from (4.12), (4.13), (4.14) for any J satisfying (4.8) and

$$\Omega(J\Phi, J\tilde{\Phi}) = \Omega(\Phi, \tilde{\Phi}) \tag{4.15}$$

$$\Omega(\Phi, J\Phi) \geqslant 0 \tag{4.16}$$

(4.15) is equivalent to (4.11), while (4.16) insures that the scalar product is positive definite. A choice of J amounts to a choice of positive and negative frequency.

We will also work with initial data on a hypersurface instead of the actual solution of the Klein-Gordon equation. Let Σ be a Cauchy surface (spacelike hypersurface; usually $\{t=\text{constant}\}$) for the spacetime and define data

$$\begin{pmatrix} \varphi \\ \pi \end{pmatrix} = \begin{pmatrix} \Phi \\ \dot{\Phi} \end{pmatrix}_{\Sigma} \tag{4.17}$$

Then there is a 1-1 correspondence between solutions $\Phi(x,t)$ and data $(\varphi(x), \pi(x))$. In this representation, J becomes an operator-valued matrix. As an example, the standard definition of positive frequency in Minkowski space corresponds to

$$J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \tag{4.18}$$

where $\omega = \sqrt{-\Delta + m^2}$. (This can easily be checked by considering the data corresponding to the real solution $\Phi = e^{i(kx-\omega t)} + e^{-i(kx-\omega t)}$.)

We have considered J and hence $\mathcal{H}$ to be defined with respect to <u>one</u> Cauchy surface Σ, but we can also define a complex structure J_t and hence $\mathcal{H}_t$ on each Cauchy surface Σ_t. This gives us a definition of particles at each time. The natural question is, how are the particle definitions J_{t_1} and J_{t_2} related, i.e. how many particles are created between $t = t_1$ and $t = t_2$?

<u>Robertson-Walker spacetimes</u>

We consider the (4-dimensional) metric

$$ds^2 = -dt^2 + a(t)^2 \, d\vec{x}^{\,2} \tag{4.19}$$

for which the Klein-Gordon equation becomes

$$\ddot{\Phi} + 3\frac{\dot{a}}{a}\dot{\Phi} + \omega^2 \Phi = 0 \tag{4.20}$$

where

$$\omega^2 = -\frac{\Delta}{a^2} + m^2 + \xi R \tag{4.21}$$

and Δ is the standard Laplacian in $\mathbb{R}^3$. This metric describes a universe with flat spatial cross-sections $\Sigma_t = \{t = \text{constant}\}$ undergoing an expansion (or contraction) described by a(t). We Fourier decompose Φ to obtain the modes

$$\Phi = \Phi_k(t)\, e^{i\vec{k}\cdot\vec{x}} \tag{4.22}$$

where Φ_k satisfies (4.20) with ω^2 replaced by

$$\omega_k^2 = \frac{k^2}{a^2} + m^2 + \xi R \tag{4.23}$$

Note the several differences between (4.20), (4.23) and the corresponding statements in flat space. First of all, due to the presence of the functions a(t) and R(t), ω_k is not constant. Furthermore, (4.20) contains a term in $\dot{\Phi}$. However, we can remove this term by changing variables.

Introduce new variables

$$S = \int^t \frac{dt}{g(t)} \tag{4.24}$$

$$\Psi = h(t)\,\Phi \tag{4.25}$$

Then if

$$\frac{gh^2}{a^3} = 1 \tag{4.26}$$

(4.19) with (4.22) becomes

$$\Psi_k'' + \Omega_k^2 \Psi_k = 0 \tag{4.27}$$

with

$$\Omega_k^2 = g^2 \omega_k^2 - h''/h \tag{4.28}$$

and where we have denoted s derivatives by prime. The choice of g(t) (and hence also h(t)) will be referred to as a choice of <u>normal-form</u>. (The most common choice is g=a=h, which exploits the fact that (4.19) is conformally flat.)

But <u>any</u> (normalized) solution to (4.27) can be written (with the suffix k dropped)

$$\psi = \frac{1}{\sqrt{2W}} e^{-i\int W ds} \tag{4.29}$$

(this is not obvious!) where W satisfies

$$W^2 = \Omega^2 - \frac{1}{2}\frac{W''}{W} + \frac{3}{4}\frac{W'^2}{W^2} \tag{4.30}$$

We will treat J as a multiplication operator on the Fourier decomposed modes, i.e.

$$J = \begin{pmatrix} e & -f \\ \frac{e^2+1}{f} & -e \end{pmatrix} \tag{4.31}$$

where e and f > 0 are functions of k and t. This corresponds to making the choice of positive frequency at time t given by

$$\begin{pmatrix} \varphi_k^+ \\ \pi_k^+ \end{pmatrix} = \frac{1}{\sqrt{2fa^3}} \begin{pmatrix} f \\ e-i \end{pmatrix} \Bigg|_{\Sigma_t} \tag{4.32}$$

((4.32) is just the (normalized) eigenvector of (4.31) with eigenvalue +i; cf.(4.5).)

By expanding e, f, w in powers of 1/k (and setting g=1), and using approximate solutions of (4.30) to compare data at different times, one can calculate the Bogoliubov coefficients α_K, β_K relating the definitions (4.32) of positive frequency at two different times. (For details see Dray, Renn & Salisbury.) But recall from Lecture 1 that the number density of particles "created" between these two times is just

$$D = \int |\beta_K|^2 \, d^3K \tag{4.33}$$

One can also consider the energy density

$$E = \int \omega_k |\beta_k|^2 \, d^3k \tag{4.34}$$

If one requires that J reduce to (4.18) if $\dot{a}=0$ and that D, E be finite one can show that the expansion for J in powers of 1/k is

$$J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} + \begin{pmatrix} \frac{c}{3\omega} & 0 \\ 0 & -\frac{c}{3\omega} \end{pmatrix} + \begin{pmatrix} 0 & -\frac{R}{12}\omega^{-3} \\ \left(\frac{c^2}{9} - \frac{R}{12}\right)\omega^{-1} & 0 \end{pmatrix} + \cdots \tag{4.35}$$

where $c = -\frac{3\dot{a}}{a}$ is the trace of the extrinsic curvature of Σ_t. The first term is just the flat space result, the second guarantees $D < \infty$, and the third guarantees $E < \infty$.

Ashtekar & Magnon give a prescription for finding J which here yields only the first term of (4.35). Note that this leads to an infinite density of created particles between any two instants of time!

Adiabatic particle definitions

Approximate solutions to (4.30) and hence to (4.20) can be constructed as follows

$$W_0^2 = \Omega^2$$
$$\left(W_{2N+2}\right)^2 = \Omega^2 - \frac{1}{2}\frac{W_{2N}''}{W_{2N}} + \frac{3}{4}\left(\frac{W_{2N}'}{W_{2N}}\right)^2 \qquad (4.36)$$

leading to Ψ_{2N} and Φ_{2N}.

These solutions are approximate in the sense that for each N there is an exact solution Φ such that

$$|\Phi_{2N} - \Phi| = \frac{1}{\sqrt{K}}\, O\left(\frac{1}{K^{2N+1}}\right) \qquad (4.37)$$

The $2N^{th}$ order adiabatic particle definition at time t is obtained by inserting the approximate solution w_{2N} into (4.29) and (4.25) and defining positive frequency to be the exact solution whose data at time t agrees with the resulting Φ_{2N}.

Again comparing the particle definitions at different times one can easily show that $N \geq 0$ implies $D < \infty$; $N \geq 1$ implies $E < \infty$. For the full $(N \to \infty)$ adiabatic particle definition β_K falls off faster than any power of 1/k.

Furthermore, the full adiabatic particle definition is independent of the choice of variables (choice of normal-form) used in its construction. This is not generally true of other quantization prescriptions (e.g. Hamiltonian diagonalization).

Stable adiabatic vacua

Finally, one can ask for spacetimes in which the adiabatic vacuum leads to zero particle production. The simplest case of this is when $W_0 = \Omega$ is an <u>exact</u> solution of (4.30) i.e.

$$\Phi_K = \frac{1}{\sqrt{2\Omega_K a^3 g^{-1}}} e^{-i\int \Omega_K g^{-1} dt} \tag{4.38}$$

is an <u>exact</u> solution of (4.20) for all k and defines the adiabatic vacuum at <u>all</u> instants of time.

One can show that this leads to two conditions, one on the normal-form and one on the spacetime. The condition on the normal form is

$$g = a \tag{4.39}$$

(or one other, very messy choice) and the condition on the spacetime is

$$\left(\xi - \tfrac{1}{6}\right)\dot{R} + \frac{2\dot{a}}{a}\left[m^2 + \left(\xi - \tfrac{1}{6}\right)R\right] = 0 \tag{4.40}$$

The solutions of (4.40) are

$$\begin{aligned}
\text{I} &: \xi = \tfrac{1}{6} && \Rightarrow m = 0 \text{ or } \dot{a} = 0 \\
\text{II} &: \xi \neq \tfrac{1}{6}, m = 0 && \Rightarrow a^2 = At^2 + Bt + C \\
\text{III} &: \xi \neq \tfrac{1}{6}, m \neq 0 && \Rightarrow a^2 = Ae^{\lambda t} + Be^{-\lambda t} + C
\end{aligned} \tag{4.41}$$

where $\lambda^2 = -\frac{2m^2}{6\xi - 1}$. Case I implies that the conformally coupled, massless scalar field produces no particles. This is just what one expects since (4.19) is conformally flat. Note that Case II with B=C=0 is the Milne universe and with A=C=0 is the radiation dominated Friedmann universe, while Case III with C=0 and $\xi < \frac{1}{6}$ includes part of the de Sitter universe as a special case.

Further details of the above results involving the adiabatic particle definition can be found in the two papers by Dray & Renn.

Lecture 5: The Klein Paradox

Here we will consider particle creation due to an electromagnetic background instead of a gravitational background, using the techniques developed in Lecture 1.* The action for a massive scalar field Φ minimally coupled to an electromagnetic potential is

$$S = -\int \left(\Phi^*_{\cdot\mu}\Phi_{\cdot}^{\ \mu} + m^2\Phi^*\Phi\right) d^4x \tag{5.1}$$

where the gauge covariant derivative $\cdot$ is defined such that

$$\begin{aligned}\Phi_{\cdot\mu} &= (\partial_\mu - ieA_\mu)\,\Phi \\ \Phi^*_{\cdot\mu} &= (\partial_\mu + ieA_\mu)\,\Phi^*\end{aligned} \tag{5.2}$$

Variation of the action (5.1) with respect to Φ^* leads to the dynamical equation

$$\Phi_{\cdot\mu}^{\ \ \mu} - m^2\Phi = 0 \tag{5.3}$$

We will specialize to electric potentials of the form $A^\mu = (\Phi(x),0,0,0)$ where $\Phi(x)$ is a barrier which goes to a constant value $-\frac{V}{2}$ as $x \longrightarrow -\infty$ (on the "left") and to $\frac{V}{2}$ as $x \longrightarrow +\infty$ (on the "right"). See figure 8.

We can Fourier decompose in the $\underline{y} = (y,z)$ and t directions giving

$$\Phi = N e^{i(\underline{k}\cdot\underline{y} - \omega t)} f(x) \tag{5.4}$$

where f(x) must satisfy the separated dynamical equation

$$\left[\frac{d^2}{dx^2} + (\omega - e\Phi(x))^2 - (\underline{k}^2 + m^2)\right] f(x) = 0 \tag{5.5}$$

* This presentation is based on a paper by C.A.Manogue, "The Klein Paradox and Superradiance", to appear in Annals of Physics.

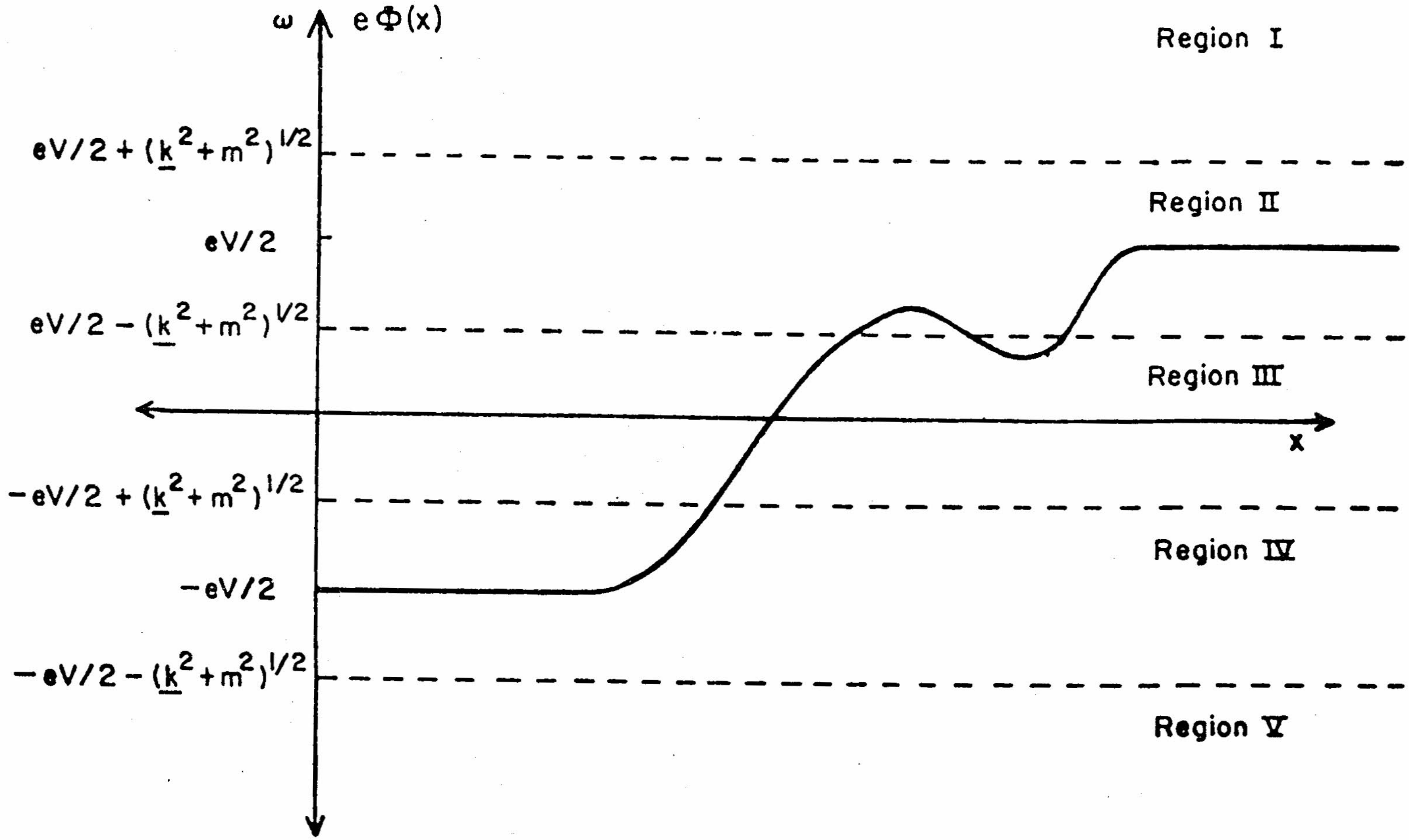


Figure 8: An example of the type of potential $\Phi(x)$ considered, showing the different frequency regions which determine the behaviour of the modes.

Far from the barrier the solutions of (5.5) look like linear combinations of plane waves $e^{\pm iqx}$ on the left, and like combinations of $e^{\pm irx}$ on the right, where

$$q^2 = \left(\omega + \frac{eV}{2}\right)^2 - \left(\underline{k}^2 + m^2\right)$$
$$r^2 = \left(\omega - \frac{eV}{2}\right)^2 - \left(\underline{k}^2 + m^2\right) \qquad (5.6)$$

Notice that q is imaginary for ω such that

$$-\frac{eV}{2} - \left(\underline{k}^2 + m^2\right)^{1/2} < \omega < -\frac{eV}{2} + \left(\underline{k}^2 + m^2\right)^{1/2} \qquad (5.7)$$

(see figure 8) leading to exponentially damped solutions on the left so that waves cannot be "sent in" from the left for frequencies in this range. Similarly r is imaginary for ω such that

$$\frac{eV}{2} - \left(\underline{k}^2 + m^2\right)^{1/2} < \omega < \frac{eV}{2} + \left(\underline{k}^2 + m^2\right)^{1/2} \qquad (5.8)$$

(see figure 8) so that waves cannot be "sent in" from the right for frequencies in this range.

The dispersion relations (5.6) do not determine the signs of q and r for q and r real. These signs are determined by using the group velocity to specify the direction of travel. For example, a wave on the left of the form $e^{i(qx-\omega t)}$ has group velocity

$$\frac{\partial \omega}{\partial q} = \frac{q}{\omega + \frac{eV}{2}} \qquad (5.9)$$

This wave will travel to the right (left) if its group velocity is positive (negative), i.e. if the sign of q is the same as (opposite) to) that of $\omega + \frac{eV}{2}$. Similarly, a wave on the right of the form

$e^{i(rx-\omega t)}$ has group velocity

$$\frac{\partial\omega}{\partial r} = \frac{r}{\omega - \frac{eV}{2}} \tag{5.10}$$

i.e. the sign of r is compared to the sign of $\omega - \frac{eV}{2}$.

A complete set of solutions of the dynamical equation (5.3) looks asymptotically like

$$\vec{u}_{in} = \vec{N} e^{i(\underline{k}\cdot\underline{y}-\omega t)} \begin{cases} e^{iqx} + \vec{R} e^{-iqx} & (x \to -\infty) \\ \vec{T} e^{irx} & (x \to +\infty) \end{cases} \tag{5.11}$$

$$\overleftarrow{u}_{in} = \overleftarrow{N} e^{i(\underline{k}\cdot\underline{y}-\omega t)} \begin{cases} \overleftarrow{T} e^{iqx} & (x \to -\infty) \\ e^{-irx} + \overleftarrow{R} e^{irx} & (x \to +\infty) \end{cases} \tag{5.12}$$

If the signs of q and r are chosen so that

$$\begin{aligned} q, r > 0 \quad & \text{for} \quad \frac{eV}{2} + (\underline{k}^2 + m^2)^{1/2} < \omega \\ q > 0,\ r < 0 \quad & \text{for} \quad -\frac{eV}{2} + (\underline{k}^2+m^2)^{1/2} < \omega < \frac{eV}{2} - (\underline{k}^2+m^2)^{1/2} \\ q, r < 0 \quad & \text{for} \quad \omega < -\frac{eV}{2} - (\underline{k}^2+m^2)^{1/2} \end{aligned} \tag{5.13}$$

(see Figure 8) then the modes (5.11) correspond to the infinite plane wave limit of a wave packet which comes in from the left at early times with both a reflected piece back to the left and a transmitted piece to the right at late times. (See Figure 9a) The modes (5.12) are the same with the words left and right interchanged.

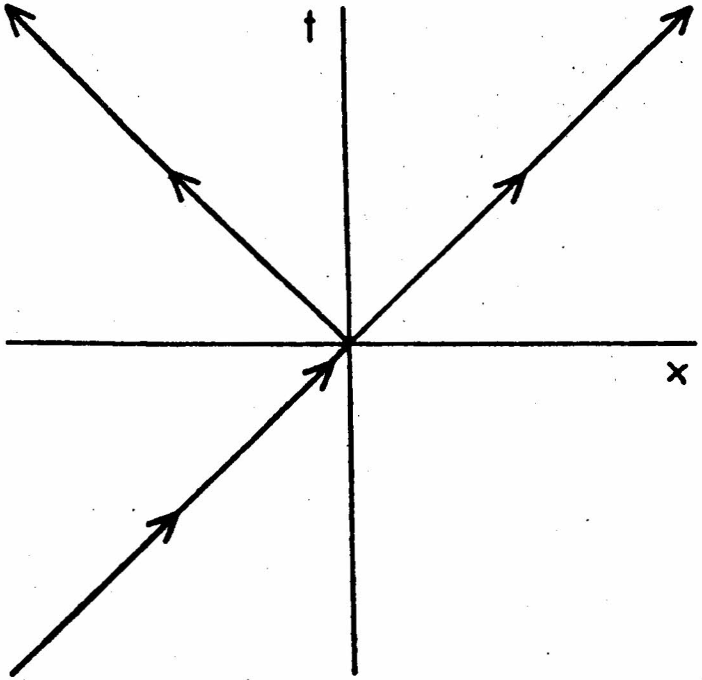


Figure 9a: Spacetime diagram for modes of type (5.11)

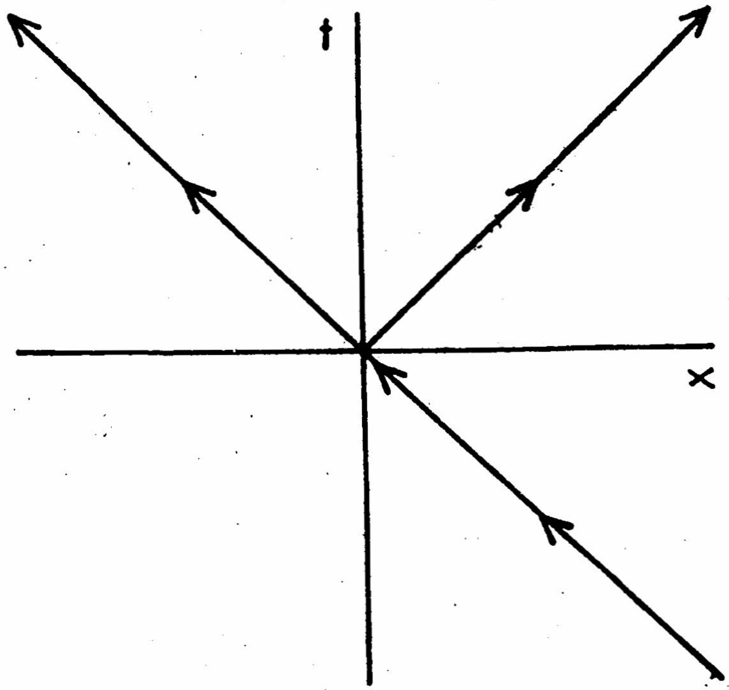


Figure 9b: Spacetime diagram for modes of type (5.12)

(See figure 9b) This set of modes will be called "in" modes because they come <u>in</u> from infinity at early times.

Another complete set of solutions ("out"-modes) is given asymptotically by

$$\overleftarrow{u}_{out} = \overleftarrow{M} e^{i(\underline{k}\cdot\underline{y}-\omega t)} \begin{cases} e^{-iqx} + \overleftarrow{Q} e^{iqx} & (x \to -\infty) \\ \overleftarrow{S} e^{-irx} & (x \to +\infty) \end{cases} \tag{5.14}$$

$$\overrightarrow{u}_{out} = \overrightarrow{M} e^{i(\underline{k}\cdot\underline{y}-\omega t)} \begin{cases} \overrightarrow{S} e^{iqx} & (x \to -\infty) \\ e^{irx} + \overrightarrow{Q} e^{-irx} & (x \to +\infty) \end{cases} \tag{5.15}$$

With the same choices of sign for q and r , these correspond to waves which come in from infinity from both sides at early times in such a way that they conspire to go <u>out</u> to infinity at late times in only one direction; to the left (5.14) or to the right (5.15). (See figure 10.)

The Klein Gordon product appropriate to the action (5.1) is

$$(u, \tilde{u}) = i \int (u^* \tilde{u}_{.t} - u^*_{.t} \tilde{u})\, dx\, d^2\underline{y} \tag{5.16}$$

cf. (1.15). In particular notice the appearance of the gauge covariant derivative.

To normalize the in-(out-) modes it is sufficient to normalize the incoming (outgoing) wave in the asymptotic potential which it sees. This corresponds to taking the plane wave limit of the normalization of a broad packet. Delta function normalization can be imposed. The modes (5.11) and (5.14) are positive (negative) normed if q is positive (negative). The sign of the norm of the modes (5.12) and (5.15) follows the sign of r .

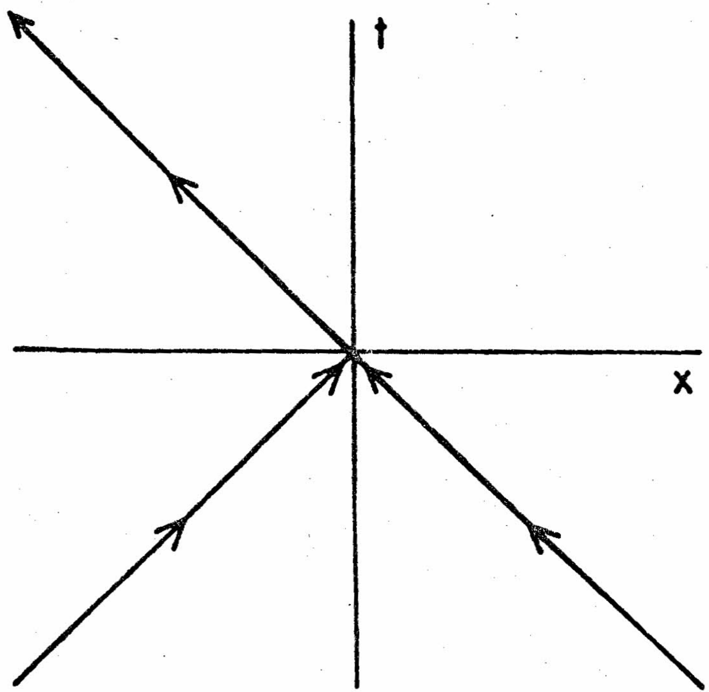


Figure 10a: Spacetime diagram for modes of type (5.14)

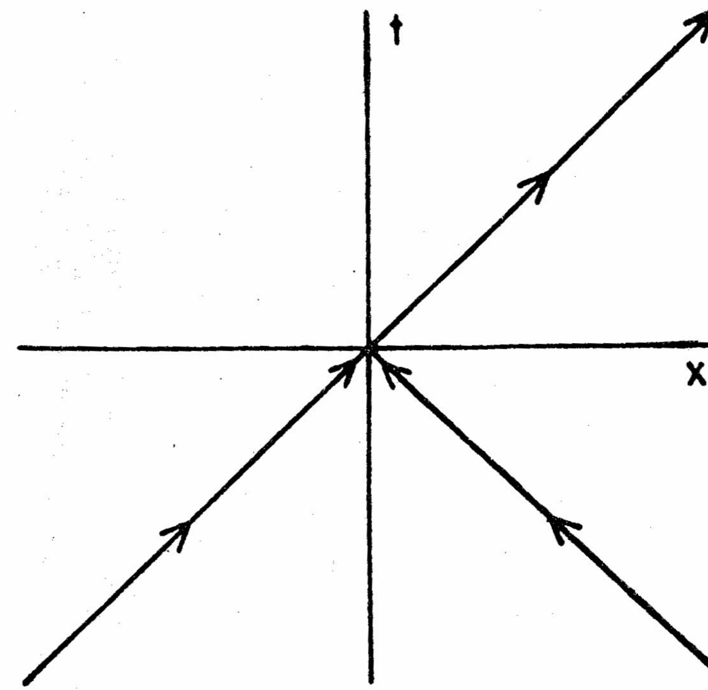


Figure 10b: Spacetime diagram for modes of type (5.15)

Notice that for a given basis of solutions it is the sign of the norm and not the sign of ω which determines the definition of "positive and negative frequency", i.e. which determines which modes are to be associated with creation operators and which with annihilation operators. Suppose u is a negative normed solution of the dynamical equation and suppose you mistakenly associate u with the annihilation operator a (instead of the creation operator $a^\dagger$). Then, since $a = -(u,\Phi)$ (since u is negative normed) and $a^\dagger = (u^*,\Phi^\dagger)$ we have

$$
\begin{aligned}
[a,a^\dagger] &= [-(u,\Phi),(u^*,\Phi^\dagger)] \\
&= \left[-i\int(u^*\Phi_{,t}-u^*_{,t}\Phi)\,dx\,d^2\underline{y}\,,\ i\int(u\Phi^\dagger_{,t}-u_{,t}\Phi^\dagger)\,d\tilde{x}\,d^2\tilde{\underline{y}}\right] \\
&= +i\iint(u^*u_{,t}-u^*_{,t}u)\,\delta(x-\tilde{x})\,\delta^2(\underline{y}-\tilde{\underline{y}})\,dx\,d\tilde{x}\,d^2y\,d^2\tilde{y} \\
&= +(u,u) \\
&= -1
\end{aligned}
\tag{5.17}
$$

where the canonical commutation relation $[\Phi(x,\underline{y}),\Phi^\dagger_{,t}(\tilde{x},\tilde{\underline{y}})] = i\,\delta(x-\tilde{x})\,\delta^2(\underline{y}-\tilde{\underline{y}})$ has been used to obtain the third line. We see that we do <u>not</u> have the canonical commutation relation for a and $a^\dagger$. The creation and annihilation operators have been misidentified. Nowhere does the sign of ω enter the discussion!

Although the reflection and transmission coefficients in (5.11, 5.12, 5.14, 5.15) depend upon the detailed shape of the barrier, general relations among the coefficients (known as Wronskian relations) can be obtained from current conservation. If f_1 and f_2 are any two solutions of (5.5) for the same values of ω and k, then

$$
f_1\left(\frac{d}{dx}f_2\right)-\left(\frac{d}{dx}f_1\right)f_2 \tag{5.18}
$$

s a conserved current, i.e. is independent of x. This is easy to rove by taking the x derivative and using (5.5). This is analogous o the proof of time invariance of the Klein-Gordon product on pacelike hypersurfaces. In particular, for a solution of the type 5.11) and its complex conjugate, evaluating (5.18) on the left nd on the right gives

$$1 - |\vec{R}|^2 = \frac{r}{q} |\vec{T}|^2 \tag{5.19}$$

In the Klein region (i.e. in the region where q and r have opposite signs, cf. (5.13)) then $|\vec{R}|^2 > 1$. The reflected current is greater than the incident current!— a phenomenon known as super-radiance or stimulated emission.

It is important to mention that most textbook discussions of the fermionic Klein paradox incorrectly state that fermions are also superradiant. The analysis of the Dirac equation in an electro-magnetic potential is completely analogous to this discussion of the scalar wave equation. However q and r in the analogue of (5.19), for the case of fermions, are multiplied by ϵ and ϵ' respectively, where ϵ is +1 when q is positive and -1 when q is negative (similarly for ϵ' and r). Thus $|\vec{R}|^2 < 1$ always. Fermions do not superradiate. (Physically, this is due to the Pauli exclusion principle.) The usual treatment of the fermionic Klein paradox gets the signs of ϵ and ϵ' correct but assumes q and r positive by failing to use the argument about the group velocities. Thus the reflected current appears to be greater than the incoming current $(|\vec{R}|^2 > 1)$.

What has really happened is that a mode which was thought to be of the type in figure 9a is actually of the type in figure 10a. $|\vec{T}|^2$ is then the current from an additional incoming piece and the single outgoing current $|\vec{R}|^2$ is the sum of the 2 incoming currents as expected.

The reason for the superradiance is that the potential barrier is dissolving to make particle-antiparticle pairs. (You must continually pump energy into the barrier to maintain its original l.ape.) Even in the vacuum, there is a constant flow of particles from the barrier. Several different calculations give a measure of the number of particles produced.

<u>Current</u>

Varying the action (5.1) with respect to A^μ gives an expression for the current in terms of Φ and Φ^*. Expanding these in terms of modes and creation and annihilation operators, and taking the vacuum expectation value, gives asymptotic values for the current in the x-direction namely

$$\langle 0_{in} | J^x | 0_{in} \rangle = \frac{|e|}{(2\pi)^3} \int \frac{r}{q} |\vec{T}|^2 d\omega \, d^2\underline{K} \qquad (x \to \pm\infty) \tag{5.20}$$

where the integrals extend only over modes in the Klein region. Since q and r have opposite signs in this region, we see that there is a current flowing from the right to the left. Notice that the spontaneous emission in the vacuum (5.20) is related to the stimulated emission rate given by $\frac{r}{q}|\vec{T}|^2$.

Momentum

Varying the action (5.1) with respect to $g_{\mu\nu}$ gives an expression for the stress-energy tensor $T^{\mu\nu}$ in terms of Φ and Φ^*. Analogously to the computation of the current, we obtain an asymptotic expression for the momentum in the x-direction

$$\langle O_{in}|T^{tx}|O_{in}\rangle = \frac{1}{(2\pi)^3}\int\left(\omega+\frac{eV}{2}\right)\frac{r}{q}|\vec{T}|^2\, d\omega\, d^2\underline{k} \qquad (x\to \tag{5.21}$$

$$= \frac{1}{(2\pi)^3}\int\left(\omega-\frac{eV}{2}\right)\frac{r}{q}|\vec{T}|^2\, d\omega\, d^2\underline{k} \qquad (x\to$$

where again the integrals extend only over modes in the Klein region. We see that momentum flows away from the barrier in both directions (cf. (5.20)).

Number Density of Created Particles

For a complex field such as we are studying here, unlike a real field, the negative normed antiparticle modes ("negative frequency") are not the complex conjugates of the positive normed particle modes ("positive frequency"). Therefore, when the field is expanded in terms of a complete set of states, it looks like

$$\Phi = \sum_i a_i u_i + \sum_p b_p^\dagger V_p \tag{5.22}$$

where the u_i are the positive normed modes and the V_p are the negative normed modes. The a_i and b_p operators are now independent and the vacuum must be defined by

$$a_i|0\rangle = 0 = b_p|0\rangle \tag{5.23}$$

Define Bogoliubov transformations via

$$u_{i\,out} = \sum_j \alpha_{ij} u_{j\,in} + \sum_q \beta_{iq} V_{q\,in}$$

$$V_{p\,out} = \sum_j \gamma_{pj} u_{j\,in} + \sum_q \epsilon_{pq} V_{q\,in} \tag{5.24}$$

Then, as in lecture 1 (cf. (1.35)), the number of particles, labelled by i, at late time in the vacuum of early time is

$$\langle 0_{in} | a^{\dagger}_{i\,out}\, a_{i\,out} | 0_{in} \rangle = \sum_q \beta_{iq} \beta^{*}_{qi} \tag{5.25}$$

Similarly the number of antiparticles, labelled by p, is

$$\langle 0_{in} | b^{\dagger}_{p\,out}\, b_{p\,out} | 0_{in} \rangle = \sum_j \gamma^{*}_{pj} \gamma_{jp} \tag{5.26}$$

The relevant Bogoliubov coefficients are obtained by a simple calculation comparing modes in the asymptotic regions:

$$\beta_{ip} \equiv \beta(\omega, \underline{k}, \leftarrow | \omega', \underline{k}', \leftarrow) = \left|\frac{r}{q}\right|^{1/2} \vec{T}\, \delta(\omega - \omega')\, \delta^2(\underline{k} - \underline{k}')$$

$$\gamma_{pi} \equiv \gamma(\omega, \underline{k}, \rightarrow | \omega', \underline{k}', \rightarrow) = -\left|\frac{r}{q}\right|^{1/2} \vec{T}^{*} \delta(\omega - \omega')\, \delta^2(\underline{k} - \underline{k}') \tag{5.27}$$

showing that the total number <u>current</u> of out particles <u>or</u> antiparticles in the in-vacuum is

$$\frac{1}{(2\pi)^3} \int \left|\frac{r}{q}\right| \left|\vec{T}\right|^2 d\omega\, d^2\underline{k} \tag{5.28}$$

where again the integrals extend only over modes in the Klein region (cf. (5.20), (5.21)).

Even though fermions do not superradiate, electromagnetic potentials in the vacuum do create fermions, according to equations analogous to (5.20), (5.21) and (5.28). (The Pauli exclusion principle still holds: $|\vec{T}|^2$ is much smaller for fermions.)

Lecture 6: The Rotating Vacuum

In 1948 Casimir made the astounding prediction that two :lean flat conducting parallel plates in the vacuum would experience . mutual attractive force inversely proportional to the fourth power ›f the distance between them. This force can be understood in terms ›f the Van der Waals forces between the atoms of the plates, but this is ɪot how Casimir made his prediction. His prediction is based on the $\sum \frac{1}{2}\hbar\omega$ per mode vacuum energy of electromagnetic field theory. 'his infinite vacuum energy is ordinarily thrown away as being ›hysically irrelevant. However, in the presence of the two conductors, :he normal mode expansion of the electromagnetic field will be altered .n order to satisfy the new boundary conditions. The new sum $\sum \frac{1}{2}\hbar\omega_{new}$ ɪill also be infinite but different. For high enough frequencies, ›hysical plates will fail to be perfect conductors and become transparent ɪnd the terms in the sums become identical. The physically relevant ɪuantity is the difference between the two sums, i.e. the difference in ɪacuum energy due to the presence of the plates. The failure of the ›erfect conductor boundary conditions introduces a natural cut-off .nto this difference, leaving a finite answer which turns out to be .ndependent of the cut-off. The finite energy differance gives :ise to the force between the plates. When Casimir's prediction was ɪerified by Sparnaay in 1958, it was the first physical evidence that :he infinite energy of the vacuum has to be taken ›eriously and should not simply be thrown away unthinkingly in all :ircumstances. Since that time, the "Casimir effect" has been :alculated for many other kinds of fields, and shapes and types of ›oundaries. The Rindler problem, lecture 2, is essentially a generalization of the Casimir effect to moving (i.e. accelerating) boundaries. In this, lecture we will discuss the calculation of the Casimir effect for static boundaries in more detail and then examine the results of a calculation for rotating boundaries. The presentation is based on (Manogue).

Casimir's method of summing over modes gives as a result the total energy between the plates. A more modern approach is to calculate the vacuum expectation value of the stress tensor $\langle 0|T^{\mu\nu}|0\rangle$ which contains much more information. First it gives the energy density between the plates as a function of position. Second it gives the other components of the stress tensor such as momentum density and pressure density.

An expression for the stress energy tensor $T^{\mu\nu}$ is obtained by varying the action with respect to the metric $g_{\mu\nu}$. (In this talk we will, for simplicity, consider the massless scalar field in four dimensions instead of the electromagnetic field. In all known cases the results are qualitatively the same.) The expression for $T^{\mu\nu}$ is then quadratic in derivatives of Φ. As in the other lectures, Φ can be expanded in terms of a complete set of modes and the vacuum expectation value of $T^{\mu\nu}$ can be evaluated. However, in this picture it is difficult to implement the renormalization procedure which subtracts off the infinite vacuum expectation value of the stress tensor which corresponds to the absence of boundaries. Alternatively one can use the method of Green's functions. A Green's function is a function of two spacetime points x and x' which satisfies the equation

$$\Box G(x,x') = \delta^4(x-x') \qquad (6.1)$$

where $\Box$ is the field operator defined in (1.1). It can be shown that (where T denotes time ordering)

$$G(x,x') = \langle 0|T\Phi(x)\Phi(x')|0\rangle \tag{6.2}$$

so that the vacuum expectation value of $T^{\mu\nu}$ for a given problem can be written as the limit as x' approaches x of a differential operator acting on the Green's function appropriate to that problem (in particular the Green's function as a function of either x or x' should satisfy the same boundary conditions as the field Φ), i.e.

$$\begin{aligned}\langle 0|T^{\mu\nu}|0\rangle &\\ = \lim_{x'\to x}\Big[&-(\tfrac{1}{2}-2\xi)(g^{\mu\nu}g^{\rho\sigma'}D_{\rho}D_{\sigma'})\\ &+\tfrac{1}{2}(1-2\xi)(g^{\mu\rho}g^{\nu\sigma'}+g^{\mu\sigma'}g^{\nu\rho})D_{\rho}D_{\sigma'}\\ &-\xi(g^{\mu\rho'}g^{\nu\sigma'}D_{\rho'}D_{\sigma'}+g^{\mu\rho}g^{\nu\sigma}D_{\rho}D_{\sigma})\Big](-i)G(x,x')\end{aligned} \tag{6.3}$$

where $g^{\mu\nu'}(x,x')$ is the bivector of parallel transport which transports vectors from x' to x, and D_μ is the covariant derivative operator. It can be seen that once the proper Green's function is found, the calculation of $\langle 0|T^{\mu\nu}|0\rangle$ is the purely mechanical one of taking derivatives and limits.

The Method of Images

For boundary conditions with a high degree of symmetry, it is often easiest to use the method of images to find the appropriate Green's function. For example, consider Casimir's original case of 2 parallel plates separated by a distance a (see figure 11),

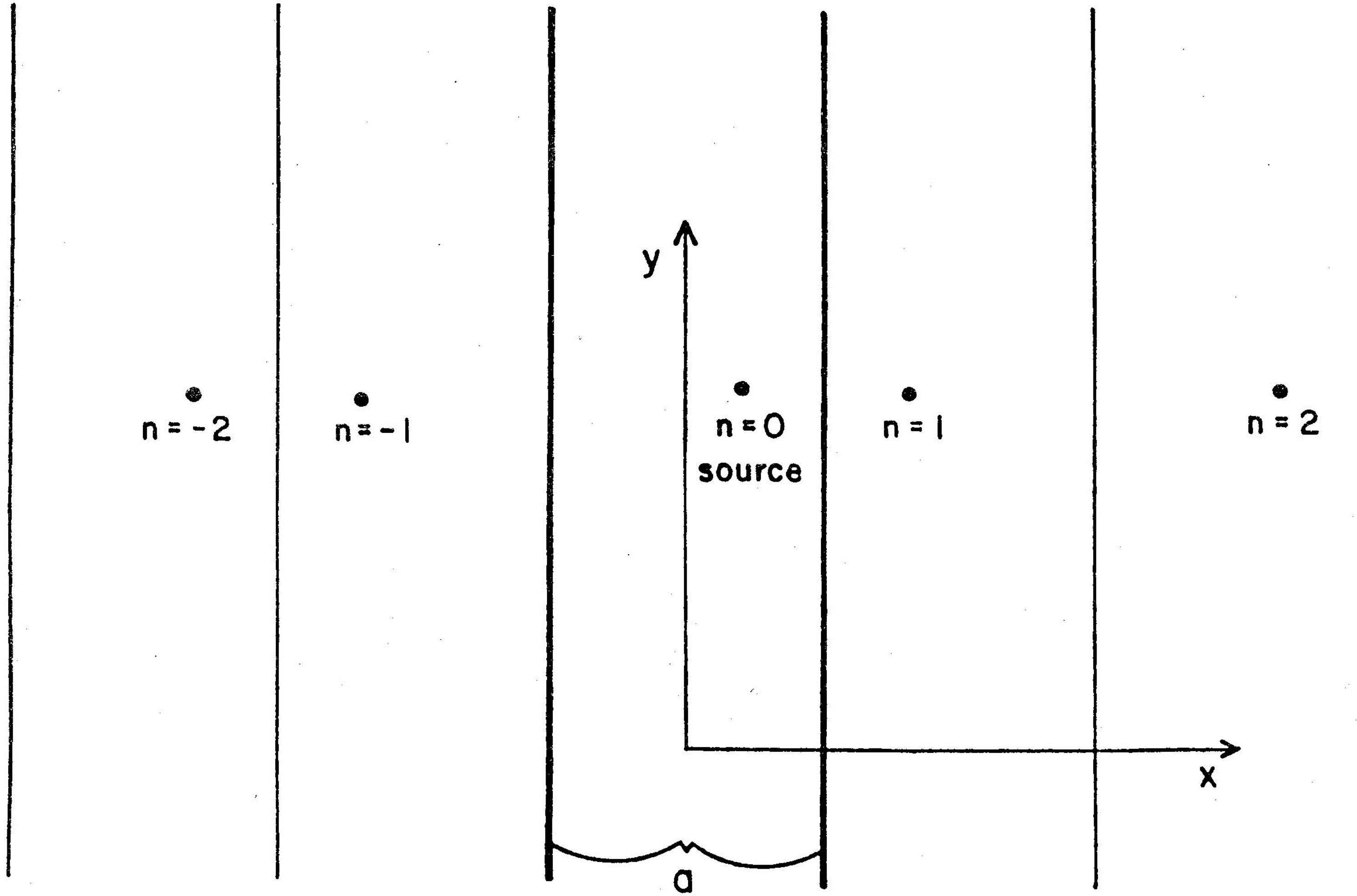


Figure 11: Casimir's parallel plates, separated by the distance a. The first few images for the Green's function are shown.

where the Green's function should satisfy Dirichlet boundary conditions on the plates, i.e.

$$G(x,x')\Big|_{x=\pm\frac{a}{2}} = 0 \tag{6.4}$$

The Green's function for unbounded flat Minkowski space is

$$G(x,x') = \frac{i}{4\pi^2} \frac{1}{-(t-t')^2+(x-x')^2+(y-y')^2+(z-z')^2} \tag{6.5}$$

If the point x is on the left-hand plate $\left(x=-\frac{a}{2}\right)$ and x' is between the plates then the Green's function has a particular value. Write down a similar Green's function with x' shifted to the "mirror image" position using the left-hand plate as the mirror. Since the Green's function depends only on the distance between x and x' and since x' and its mirror image are at the same distance from x, the values of these two Green's functions are identical. Therefore the difference of these two Green's functions is a new Green's function which is zero for x on the left-hand plate. One of the two boundary conditions is satisfied. To satisfy the boundary condition on the right-hand plate one must subtract off two more Green's functions, reflecting both the original x' and its first image, now using the right-hand plate as a mirror. But this destroys the left-hand boundary condition which must be resurrected by yet more mirror images etc. The final Green's function is an infinite series

$$G(x,x') = \frac{i}{4\pi^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{-(t-t')^2+(x-(-1)^n x'+na)^2+(y-y')^2+(z-z')^2} \tag{6.6}$$

(See figure 11.)

The Stress Tensor

If this Green's function is used in (6.3) to calculate $\langle 0|T^{\mu\nu}|0\rangle$, the answer is of course infinite. This is just because $\langle 0|T^{\mu\nu}|0\rangle$ for unbounded Minkowski space is infinite and (6.6) contains the Minkowski term (n = 0). What we are really interested in is the difference between $\langle 0|T^{\mu\nu}|0\rangle$ in the presence of boundaries and in the absence of any boundaries. This is trivially accomplished by subtracting the ordinary Minkowski piece (n = 0) from (6.6).

In the conformally coupled choice $\left(\xi = \frac{1}{6}\right)$, the result is

$$\langle 0|T^{\mu\nu}|0\rangle = \frac{\pi^2}{1440a^4}\,\mathrm{diag}\,(-1,-3;1,1) \tag{6.7}$$

Notice that the energy density is constant between the plates. Except for a factor of ½ due to the scalar field having half the number of degrees of freedom of the electromagnetic field, this will integrate to give exactly Casimir's result for the total energy. For the minimally coupled choice $(\xi = 0)$, (6.7) will have an additional term

$$-\frac{\pi^2}{48a^4}\;\frac{3-2\sin^2\left(\frac{\pi(x+\frac{a}{2})}{a}\right)}{\sin^4\left(\frac{\pi(x+\frac{a}{2})}{a}\right)}\;\mathrm{diag}\,(-1,0,1,1) \tag{6.8}$$

Notice in particular that the energy density is not constant between the plates. In fact, it diverges as the plate is approached $(x = \pm\frac{a}{2})$. Originally, this divergence was considered to be a good reason for choosing the conformal stress tensor over the minimal one. However, it has been shown (Deutsch & Candelas) that such divergences near boundaries are generic even for the conformal stress tensor unless special symmetries (such as the flatness of the plates) are present. One can see how these divergences arise. The Green's function (6.5) diverges if x' approaches x. However, the contribution of this divergence to the stress tensor is subtracted off in the renormalization procedure. But when x' is on the boundary, it is also possible for the mirror image of x' to approach x, creating a divergence in the $n = \pm 1$ terms of (6.6). These divergences are not subtracted off in the renormalization procedure. It turns out that for the conformal stress tensor the Green's function terms from an odd number of reflections through a flat boundary do not contribute and so these divergences do not arise. In most other circumstances they do arise. The problem is really due to the imposition of perfect conductor boundary conditions. In physical situations boundaries will be transparent to waves whose wavelengths are small compared to the interatomic spacing in the plates. Thus a natural cut-off will be introduced. However, if the interatomic spacing is small compared to the spacing between the plates (or whatever other natural length scale is present in the problem under consideration), then, while the stress tensor will not actually become infinite near a boundary, it can become very large. This is a real physical effect which must not be thrown away.

As another example, consider two plates which come together to form a right angle. The Green's function appropriate to Dirichlet boundary conditions will have 4 terms: the original

Minkowski piece (6.5), the two mirror image terms each formed from a single reflection in one of the plates, and the mirror image term formed from a double reflection — first in one plate and then in the other. Again, renormalization throws away the original Minkowski piece. The conformal stress tensor does not have contributions from the two single reflection terms and therefore is finite on the plates. However, it does have contributions from the double reflection. This image can approach the corner where the plates meet and therefore the (perfect conductor, conformal) stress tensor does diverge there. The minimal stress tensor has the expected divergences on both the plates and in the corner.

Rotating Boundaries

The simplest example of rotating boundaries is an infinitely long circular cylinder rotating around its long central axis. However, in this case, both the boundary and the boundary conditions are invariant under rotation. There is no way to tell the vacuum that the boundary is rotating! Indeed, detailed calculation shows that there is no difference in the stress tensor between a rotating and nonrotating cylinder. (Both of these do, of course, contain a static Casmir piece.)

One way to make the vacuum sense the rotation of the boundary is to make the cross-section of the cylinder square instead of round. Then the corners will "push" the vacuum around. It is impossible to solve the rotating image problem exactly, but a perturbation expansion in powers of the angular velocity Ω can be made (perturbing around the Green's function for a nonrotating square cylinder). To first order in the angular velocity, only the momentum density terms in the stress tensor receive a correction over and above the static Casimir piece of the nonrotating square cylinder. In figure 12, this momentum density is plotted as a vector. (Details of this messy calculation can be found in (Manogue).)

The momentum density is diverging near the boundaries even for the conformal stress tensor. This is because the boundaries are curved in a space-<u>time</u> sense.

Since the energy density for a nonrotating square cylinder turns out to be negative, the velocity density describes rotation counter to the rotation of the box. But negative energy density is like negative mass; if you push on it, it moves back toward you. So the vacuum really does rotate the opposite way from the cylinder.

(An interesting historical aside: soon after Casimir's result, it was suggested that a stable electron could be formed as a spherical shell using the attractive Casimir force to balance the repulsive force due to concentration of the negative charge. The exact balancing of these two forces would predict a radius for the electron. Unfortunately, the Casimir energy inside a sphere is positive resulting in a repulsive Casimir force, so such an electron cannot be spherical. But the energy density inside a square cylinder is negative, maybe electrons are really square!)

Notice that the motion of the vacuum is not circular. In fact, for Neumann boundary conditions (i.e. the first derivative of ϕ is zero on the boundary), this effect is even more pronounced. (See figure 13.) There are actual whirlpools in this case. Can one ascribe a "viscosity" to the vacuum? Are we returning to an "ether"?

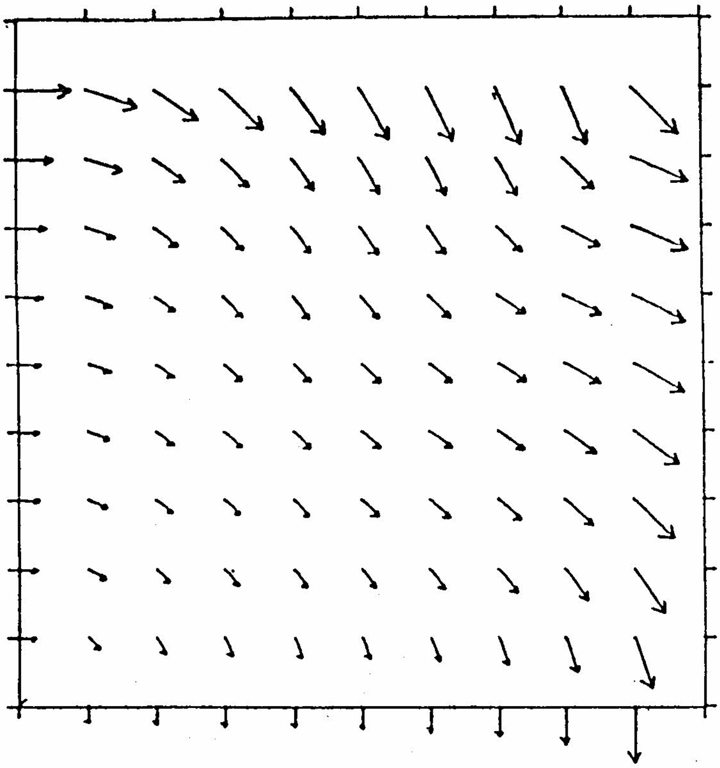


Figure 12: The vacuum expectation value of the momentum density inside a clockwise rotating square cylinder. Only the upper right-hand corner of the box is shown. The lengths of the vectors have been scaled by taking eighth roots in order to show motion near the center of the box. G(x,x') satisfies Dirichlet boundary conditions.

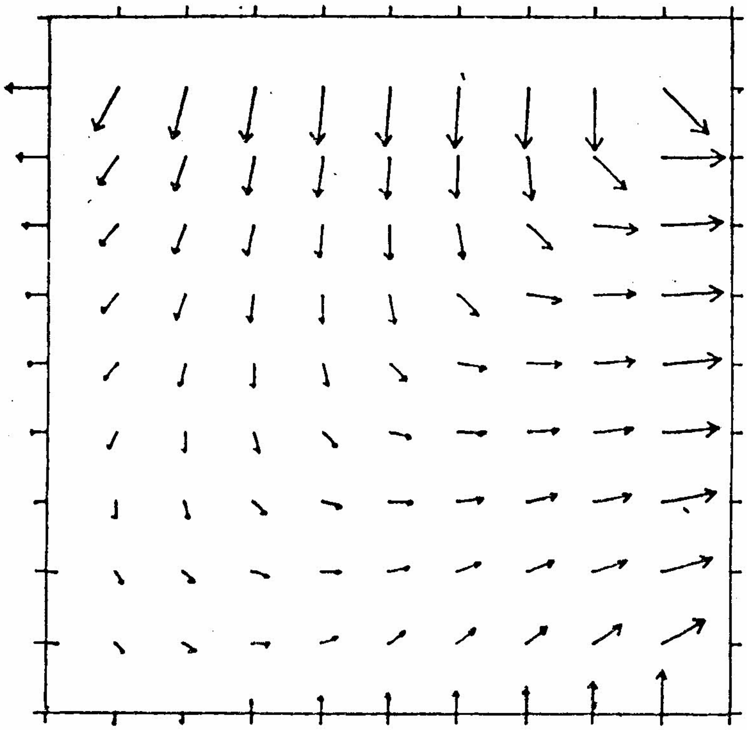


Figure 13: As in Figure 12 but G(x,x') satisfies Newmann

Lecture 7: The Trousers Problem

In this lecture we will consider the massless scalar wave equation propagating on a two dimensional spacetime whose spatial cross sections change topology, i.e. at early times the spacetime will be a flat, two dimensional cylinder with circumference 2λ. After some time, this cylinder splits into two disjoint cylinders, each with circumference λ. (See figure 14.) The spacetime looks like an inverted pair of trousers.

At early times and late times the cylinders can be chosen to be flat and the wave equation is just the usual one

$$\left(-\partial_t^2 + \partial_x^2\right)\phi = 0 \qquad (7.1)$$

However, if we take the region where the topology change occurs to be smooth, then there must be a coordinate patch at the crotch of the trousers which is locally Euclidean whereas the rest of the spacetime is locally Lorentzian. The metric changes signature. Changing from Lorentzian to Euclidean signature changes the sign of the determinant of the metric. If the metric is real, the determinant must pass through zero. Otherwise the determinant takes on complex values. In either case, the interpretation of the wave equation is unclear. To avoid these problems we will shrink the Euclidean patch to a single extremely singular point and then remove this point from the spacetime. The resulting manifold is Lorentzian everywhere and may be chosen to be flat everywhere so that (7.1) holds throughout. We are now left with the problem of specifying what happens to

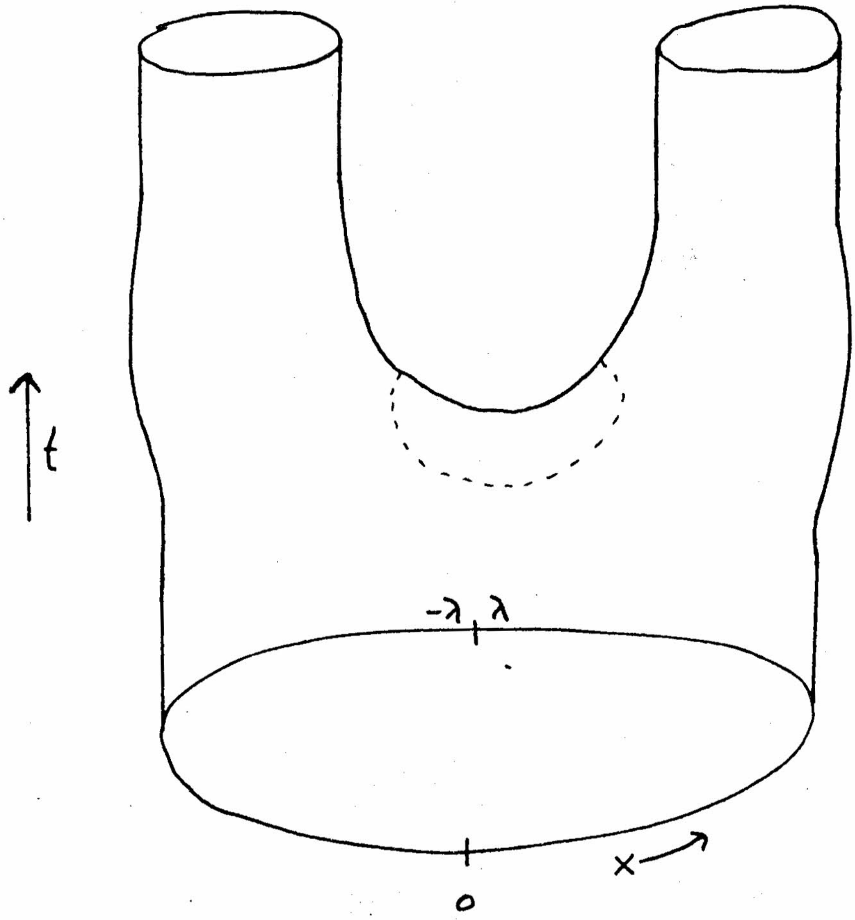


Figure 14: The trousers spacetime. The area enclosed by the dotted line is a Euclidean patch. Everywhere else the spacetime is Lorentzian.

solutions of (7.1) when they reach the singularity. It must be emphasized that the existing laws of physics can not handle this situation. We are inventing totally new physics. A warning to those interested in string theory.: The situation considered here is analogous to string interactions. The problems discussed here are avoided in string theory by pretending that the surface is Euclidean. This only avoids the problems by refusing to consider them.

However, at early and at late times the existing laws of physics do hold and we can write down a complete orthonormal set of solutions of (7.1) at each of these times. At early times, in the trunk, the solutions, called in or trunk modes, are

$$u_k = N_k e^{+ik(\pm x - t)}, \quad u_k^* = N_k e^{-ik(\pm x - t)}$$
$$\alpha \quad , \quad \beta t \qquad (7.2)$$

where N_k, α, β are normalization constants and $\pm$ refer to right and left moving modes respectively. For $k = \frac{n\pi}{\lambda}$, $n = 1, 2, \ldots$ all these modes are periodic. Unlike the case of unbounded Minkowski space, on the cylinder the constant mode and the mode proportional to t have finite Klein Gordon product with each other (and are orthogonal to the u_k and u_k^*), so they must be included in the complete set of modes. The field ϕ can be expanded as a sum over these solutions

$$\phi = \sum_k \left(a_k u_k + a_k^\dagger u_k^* \right) + q\alpha + p\beta t \qquad (7.3)$$

and the in vacuum is defined by

$$a_K |0_{in}\rangle = 0 = p|0_{in}\rangle \qquad (7.4)$$

(The correct quantization of the operators q and p will be discussed at the end of the lecture)

At late times the two legs are completely independent so that ϕ can be expanded using out or leg modes as

$$\phi = \sum_{\ell} \left(a_{\ell L} u_{\ell L} + a^{\dagger}_{\ell L} u^{*}_{\ell L} + a_{\ell R} u_{\ell R} + a^{\dagger}_{\ell R} u^{*}_{\ell R} \right) + q_L \alpha_L + p_L \beta_L t + q_R \alpha_R + p_R \beta_R t \qquad (7.5)$$

where the subscript L refers to modes which are analogous to (7.2) in the left leg (with k replaced by $\ell = \frac{2m\pi}{\lambda}$ to satisfy the periodicity condition) and which are zero in the right leg. The subscript R is similar, interchanging right and left. The out vacuum is defined by

$$a_{\ell L} |0_{out}\rangle = a_{\ell R} |0_{out}\rangle = 0 = p_L |0_{out}\rangle = p_R |0_{out}\rangle \qquad (7.6)$$

We are ultimately interested in calculating the vacuum expectation value

$$\langle 0_{in} | T^{\mu\nu}_{out} | 0_{in} \rangle \qquad (7.7)$$

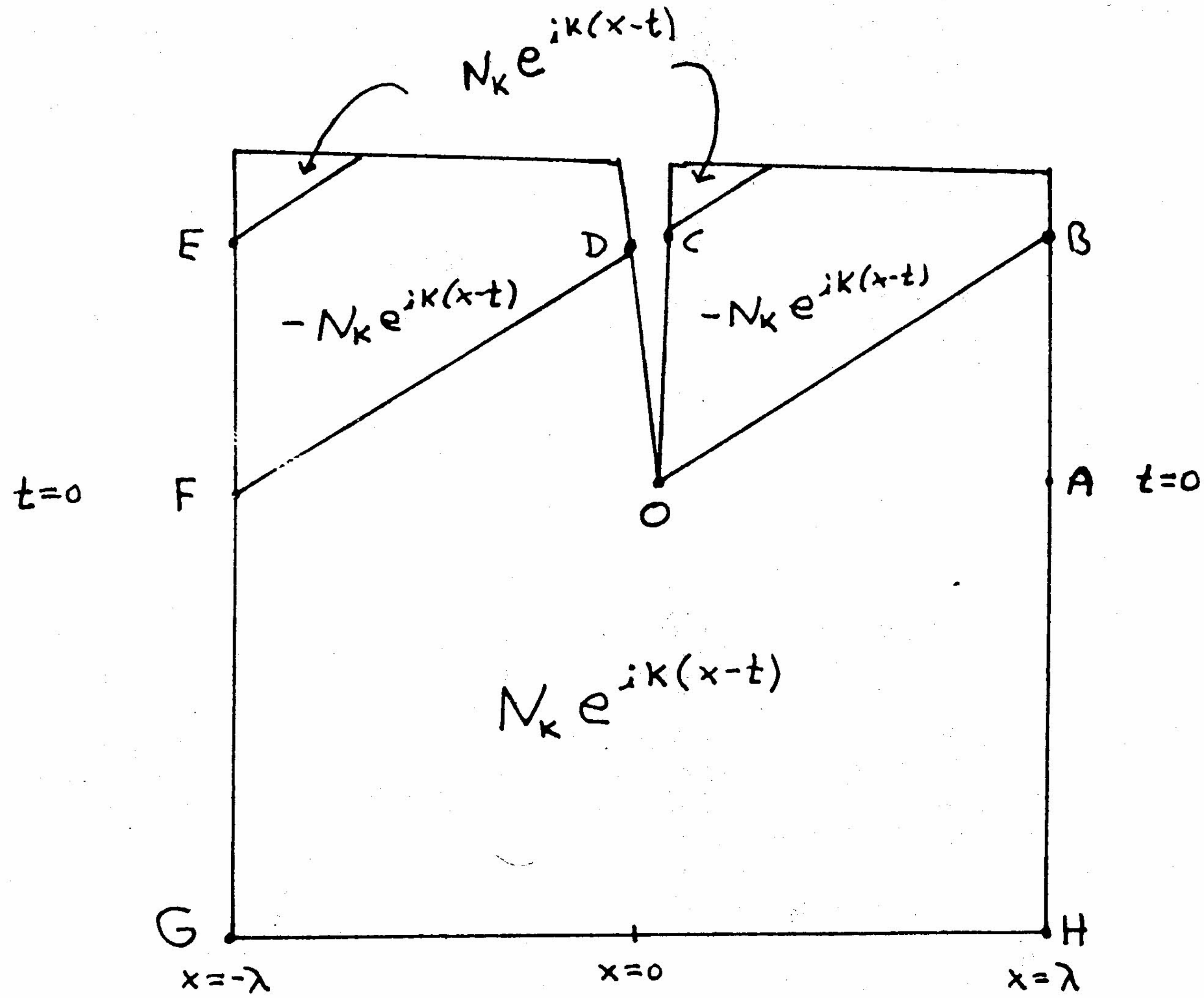
the stress tensor at late times in the early time vacuum. If this turns out to be finite everywhere, then only a finite number of particles are created during the topology change. If it turns out to be locally infinite, with finite integral over a spatial slice, then one might argue that the infinity is due to the artificial shrinking of the singularity to a single point. A suitably smoothed out version might then be finite everywhere. If, however, even the spatial integral of (7.7) is infinite, then an infinite number of particles are produced, requiring an infinite amount of energy, and the topological change will never take place.

In order to calculate (7.7), we first need to choose a propagation rule which propagates the trunk modes up into the legs and propagates the leg modes down into the trunk. We would certainly like such an propagation rule to preserve the orthonormality of the individual sets of modes. This problem was first considered in (Anderson & DeWitt). Let us examine their propagation rule, called the shadow rule. Two basic principles underlie the shadow rule. The first principle is inherent in all physically realistic propagation rules : Since we are considering a massless scalar field, information about the occurrence of the topology change can only propagate along the light cones leading out from the singularity. Until those light cones are reached, modes will propagate as usual, governed by (7.1). The second principle, which amounts to a choice of propagation rule, is that right moving solutions will continue to be only right moving and left moving solutions, left moving. No part of a solution is "reflected" off the singularity.

Let us consider some examples of the shadow rule. The trousers can be cut apart and unrolled. (See figure 15.) Since the trunk mode

$$u_k = N_k e^{ik(x-t)} \tag{7.8}$$

is right moving, it will continue to be right moving in the legs. The solution will continue to look like (7.8) throughout the triangular region OAB until it hits the light cone from the singularity



<u>Figure 15</u>: The unrolled trousers showing "barber pole" stripes appropriate to a right moving trunk mode with n odd. Identify the line segments $\overline{OC} = \overline{AB}$, $\overline{OD} = \overline{FE}$, $\overline{AH} = \overline{FG}$.

But we know that we must indentify the side $\overline{AB}$ $(x=\lambda)$ with
(x = 0), so the solution must be continuous across this join, i.e.

$$N_\kappa e^{i\kappa(\lambda-t)} = u_\kappa(0,t) \qquad (7.9)$$

:all that $\kappa = \frac{n\pi}{\lambda}$. If n is even, then we see that the mode
›ks like (7.8) everywhere. It does not see the singularity. However,
›n n is odd, then near $\overline{OC}$ the solution must look like

$$u_\kappa = -N_\kappa e^{i\kappa(x-t)} \qquad (7.10)$$

ıtinuing this analysis, keeping the mode right moving and matching
: solution across the cut in the trouser leg when necessary, we
: that the complete solution for n odd will form "barber pole"
:ipes up the trouser leg, alternating sign with every stripe.
›re will be identical stripes in the left leg. (See Figure 15.)
:t moving modes will form stripes which circle the legs in the
›osite direction.

Using the theta-function defined by

$$\Theta(x) = \begin{cases} 0 & (x<0) \\ \frac{1}{2} & (x=0) \\ 1 & (x>0) \end{cases} \qquad (7.11)$$

is possible to write down an expression for (7.8) in the legs.

$$u_\kappa = N_\kappa \sum_{n=0}^{\infty} (-1)^n e^{i\kappa(x-t)} \left[\Theta(t-x-n\lambda+\lambda) - \Theta(t-x-n\lambda)\right]$$

right leg

$$u_\kappa = N_\kappa \sum_{n=0}^{\infty} (-1)^n e^{-i\kappa(x-t)} \left[-\Theta(t-x-n\lambda-\lambda) + \Theta(t-x-n\lambda)\right] \qquad (7.12)$$

left leg

In a similar manner we can construct the propagation of the leg modes back into the trunk towards early times. The solutio

$$u_{\ell R} = \begin{cases} N_\ell e^{i\ell(x-t)} & \text{right leg} \\ 0 & \text{left leg} \end{cases} \tag{7.13}$$

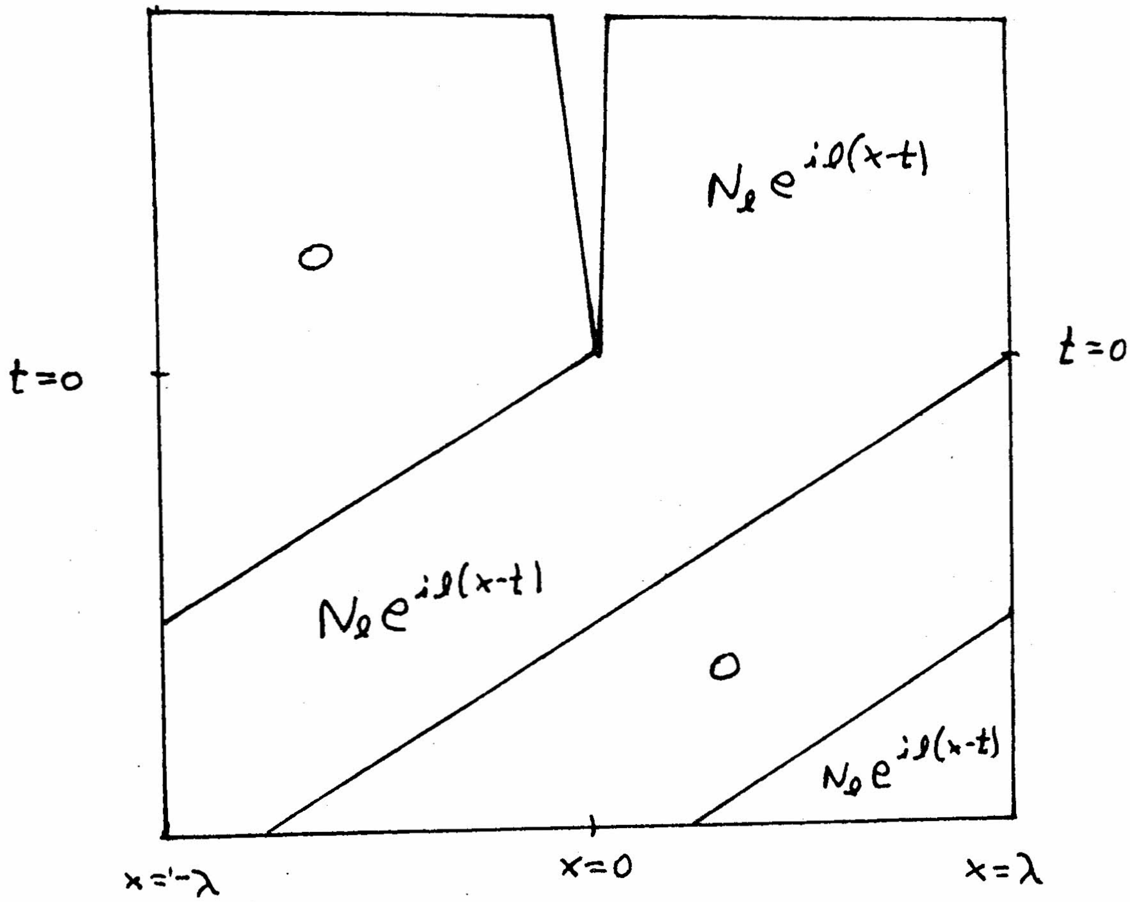
looks like

$$u_{\ell R} = N_\ell \sum_{n=0}^{\infty} e^{i\ell(x-t)} \left[\theta(t-x+2n\lambda+\lambda) - \theta(t-x+2n\lambda)\right] \tag{7.14}$$

in the trunk. (See Figure 16.) Because $\ell = \frac{2m\pi}{\lambda}$ and 2m is always even, so there is no sign flip in this case. However, (7.13) is zero in the left leg. The barber pole stripes here alternate between zero and $+N_\ell e^{i\ell(x-t)}$

Notice that since these solutions are purely right or left moving, they are trivially solutions of (7.1) everywhere except at the singularity.

As in the previous lectures, we obtain an expression for $T^{\mu\nu}$ by variation of the action with respect to $g_{\mu\nu}$. $T^{\mu\nu}$ is quadratic in derivatives of ϕ, Anderson and DeWitt argue that because of the theta functions in (7.12) and (7.14), the first derivatives of ϕ will have a term proportion to the delta function. Then, generically, $T^{\mu\nu}$ will have terms which are quadratic in delta functions, i.e. $T^{\mu\nu}$ is no only locally infinite, its spatial integral is infinite as well.



<u>Figure 16</u>: The unrolled trousers showing "barber pole" stripes appropriate to a right moving leg mode which is nonzero in the right leg.

What is not clear from their discussion is whether or not it is possible for the coefficient of this delta squared term to be zero. Let us examine this possibility in more detail.

To calculate (7.7), we first expand the ϕ 's in $T^{\mu\nu}$ as in (7.5). We then use Bogoliubov transformations to write the out operators in terms of in operators. Bogoliubov transformations are just Klein-Gordon products of in modes with out modes. These would be straightforward to calculate except that for the shadow rul the answer is different depending on whether we take the spacelike hypersurface to be before or after the singularity. What can be done about this undesirable situation?

It turns out that there are two extra solutions of the wave equation

$$\gamma_0 = \begin{cases} 0 & \text{in trunk} \\ \sum_{n=0}^{\infty} \left[\theta(t-x-n\lambda)+\theta(t+x-n\lambda-\lambda)\right] & \text{in right leg} \\ \sum_{n=0}^{\infty} \left[-\theta(t-x-n\lambda-\lambda)-\theta(t+x-n\lambda)\right] & \text{in left leg} \end{cases} \quad (7.15$$

and

$$\gamma = \begin{cases} \sum_{n=0}^{\infty} \begin{bmatrix} \theta(t-x+2n\lambda+\lambda)-\theta(t-x+2n\lambda) \\ -\theta(t+x+2n\lambda)+\theta(t+x+2n\lambda-\lambda) \end{bmatrix} & \text{in trunk} \\ 0 & \text{in right leg} \\ 0 & \text{in left leg} \end{cases} \quad (7.1$$

(See Figure 17.) γ_0 is orthogonal to all of the trunk modes and orthogonal to itself so the orthonormality of the trunk modes i

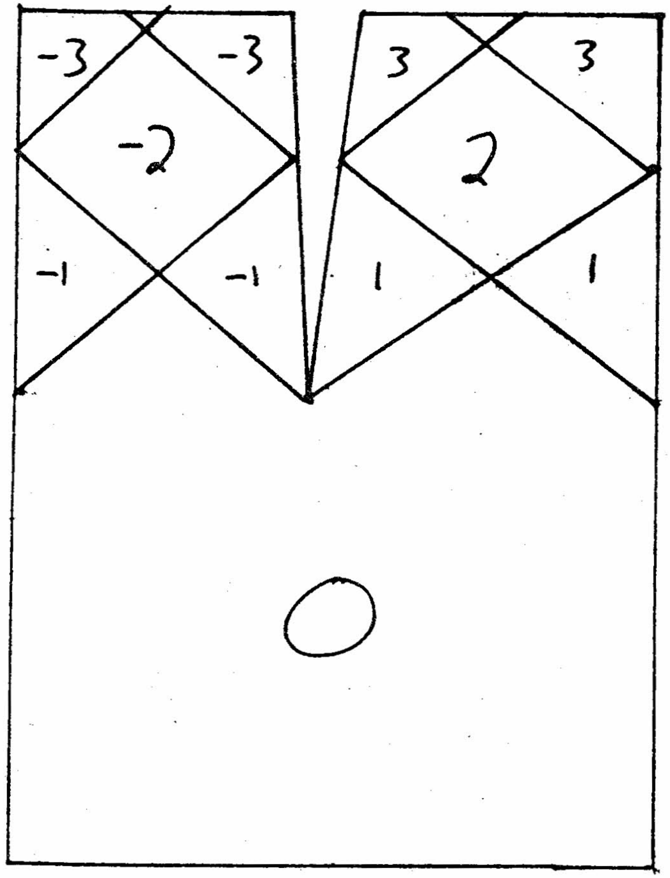


Figure 17a: The extra trunk mode γ_0.

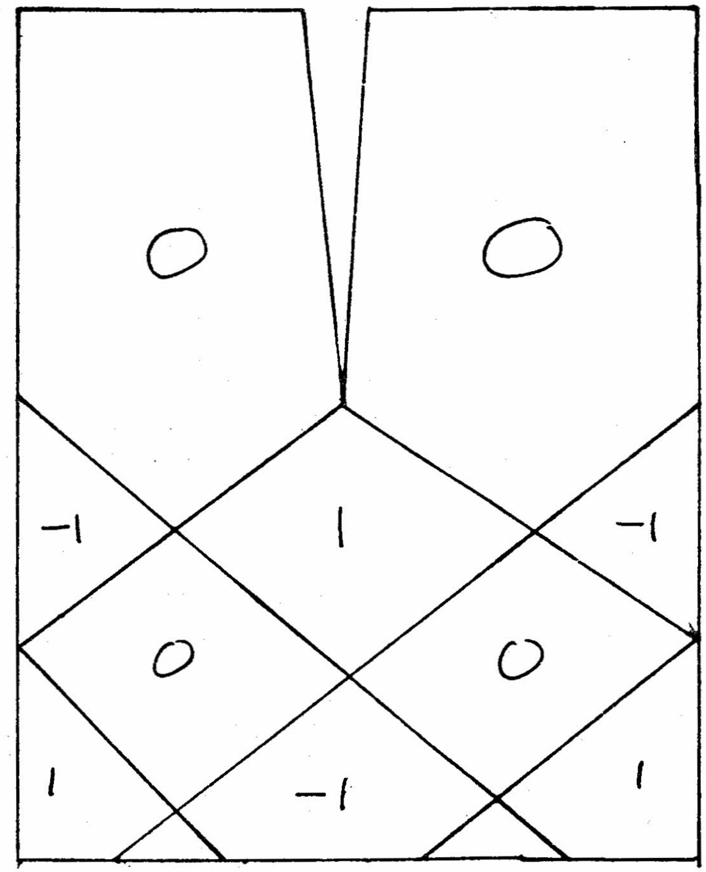


Figure 17b: The extra leg mode γ.

preserved if an arbitrary amount of γ_0 is added to each one, i.e

$$u_k \longmapsto u_k + A_k \gamma_0 \tag{7.17}$$

where A_k is an arbitrary parameter. Similarly γ is orthogon to all of the leg modes, so orthonormality of the leg modes is preserved under

$$u_\ell \longmapsto u_\ell + A_\ell \gamma \tag{7.18}$$

Since Bogoliubov transformations of trunk modes with γ and leg modes with γ_0 also depend on whether the hypersurface of integration occurs before or after the singularity, it might be possible to choose A_k and A_ℓ so that the total Bogoliubov transformation is independent of hypersurface. Surprisingly (there are more equations than parameters), it turns out that not only can this be done, but there is one leftover free parameter A.

Preliminary calculations nevertheless indicate that there is no choice of the free parameter A which eliminates the delta squared terms in $T^{\mu\nu}$. However, the preliminary calculations yield three terms, each infinite, which must be handled very carefully. One of these looks like the Minkowski infinity everywhe except on the light cone, one is the delta squared piece, and one looks like minus the Minkowski infinity everywhere. (This last piece arises from the renormalization procedure — normal ordering with respect to the out operators) Thus we see that a delta squared term is not only expected, it is desirable as long as it contributes just the expected Minkowski infinity which is missing from the first term. Unfortunately, the term proportional to A appears to be even more divergent than the Minkowski infinity.

There are other modes besides γ_0 and γ which might
p. These look like the delta function (or derivatives of the
ta function); flashes arising from the singularity and propagating
ng the light cone. These may affect the calculation of (7.7) in
ways. First they may represent extra freedom in the propagation
e (in a manner analogous to the above discussion of γ_0 and γ),
reby changing the Bogoliubov transformations. Second, they may,
ether with γ_0 and γ naturally form canonically
jugate pairs of modes which should be included in the expansions
Φ, (7.3) and (7.5). We are now pursuing these two possi-
ities.

Now a word about the zero frequency modes α and βt.
ce they are real, their norms are zero; but their Klein Gordon
duct with each other is nonzero. Demanding the commutation relation

$$\left[\Phi(x,t),\ \dot{\Phi}(x',t)\right] = i\,\delta(x-x') \tag{7.19}$$

oses canonical commutation relations on q and p, the operators
ociated with α and βt respectively, i.e.

$$[q, p] = i \tag{7.20}$$

iving the Hamiltonian H from the action in the usual way, we see
t H has a term $\frac{1}{2}p^2$. Also demanding $H|0\rangle = 0$, we can see
t we must have (since p and q are Hermitian)

$$p|0\rangle = 0 = \langle 0|p \tag{7.21}$$

The conditions (7.20) and (7.21) appear to be inconsistent. On the one hand, using (7.20)

$$\langle 0|[q,p]|0\rangle = \langle 0|i|0\rangle = i\langle 0|0\rangle \neq 0 \tag{7.22}$$

On the other hand, using (7.21)

$$\langle 0|[q,p]|0\rangle = \langle 0|qp-pq|0\rangle = 0 \tag{7.23}$$

This apparent inconsistency is related to the fact that q is not a well defined operator on the space of eigenfunctions of p. A correct resolution of this dilemna shows that quantities like

$$\langle 0|q^2|0\rangle \tag{7.24}$$

are divergent. It is interesting to note that while such terms do not appear in the calculation of $\langle O_{in}|T^{\mu\nu}_{out}|O_{in}\rangle$ which we have been considering, they do appear in the time reversed problem $\langle O_{out}|T^{\mu\nu}_{in}|O_{out}\rangle$. It is possible that these divergences could cancel the delta squared terms encountered above. If this turns out to be true, then topology change might be possible in only one direction. A single circle could not break up into two, but two circles could join into one. This is reminiscent of the classical black hole theorem which says that a black hole cannot break up into two black holes, but two can come together to form one.

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