

PROCEEDINGS OF THE CONFERENCE
ON
STOCHASTIC PROCESSES AND APPLICATIONS

OOTACAMUND

11-14, December 1980

MATSCIENCE
THE INSTITUTE OF MATHEMATICAL SCIENCES
MADRAS-600 020 (India)

FEBRUARY, 1982.

Proceedings of the Conference

on

STOCHASTIC PROCESSES AND APPLICATIONS

Ootacamund
December, 11-14, 1980

Convener:

Professor Alladi Ramakrishnan
Director, MATSCIENCE

Editor:

Dr. R. Jagannathan
MATSCIENCE, Madras.

THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS-600 020

(India)

February 1982

C O N T E N T S

| | | | |
|--|-----|-----|-----|
| PREFACE | ... | ... | ... |
| ALLADI RAMAKRISHNAN | | | |
| New Concepts in Probability Theory | ... | | i |
| AGARWAL, G.S. | | | |
| Non-linear Optical Process in the field of Random pump-application of the theory of multiplicative stochastic processes | ... | ... | 1 |
| SUBRAMANYAM, P.V., BALAKRISHNA REDDY, K., and RANGACHARI, M.S. | | | |
| Matkowski's Fixed Point Theorem and Random Operators | | | 19 |
| SUBRAHMANYAM, K.H.S., and GOPALAKRISHNA, J. | | | |
| Study of Multidimensional distributions and processes associated with Multiple Dirichlet series and appli- cations. | ... | ... | 31 |
| JAIN, N.K. | | | |
| An optimum control criterion for a limited capacity queueing system | ... | ... | 36 |
| JAYANNAVAR, A.M. and KUMAR, N. | | | |
| Stochastic interpretation of van Leeuwen Theorem on the absence of diamagnetism in classical electron gas | | | 53 |
| NUTHAKI, P. | | | |
| On the concepts of probability in classical and quan- tum theories, and an alternative interpretation of quantum theory. | ... | ... | 61 |
| VITTAL, P.R. | | | |
| Collective risk theories and ruin probabilities | | | 86 |
| JAGANNATHAN, R. | | | |
| On the concept of product density in the theory of stochastic point processes | ... | | 107 |
| RADHAKRISHNAN, V. | | | |
| Role of probability in non-equilibrium processes | | | 120 |
| VENUGOPALA RAO, D. | | | |
| Coordinatizing Baer* -semigroups and event - state - Operation structure | ... | ... | 123 |

PREFACE

This report contains a collection of papers presented at the MATSCIENCE Conference on 'Stochastic Processes and Applications' held at the H.P.F. Club House, Ootacamund from 11th to 14th December 1980.

This Conference, the 37th of a series of Conferences conducted by MATSCIENCE on various topics in mathematical sciences, was inaugurated by Professor Alladi Ramakrishnan. In the inaugural address Professor Alladi Ramakrishnan recalled his association with Professor H.J. Bhabha three decades ago when they were engaged in research work in the theory of cosmic ray cascades. It was Bhabha who first realised that this stochastic problem involved a ^{new} feature, the random distribution of discrete number of particles in continuous energy space. After two years of effort Ramakrishnan was able to formulate the theory of product densities and publish his results at Manchester through famous English statisticians Bartlett and Kendall. This theory has now become a common currency under the title of Point Processes with wide applications in Physics, Chemistry, Biology, Medicine, Operations Research, Engineering etc.

We have to thank all the participants for the lively discussions and the enthusiastic cooperation in making the conference a success. We are grateful to Mr. P.R.S. Rao, the Managing Director, H.P.F., Ootacamund and his colleagues for providing the conference hall, accommodation for all the participants and all other necessary facilities and the kind hospitality shown to us. We also wish to thank Mr. N.S. Sampath, Mr. K. Ramesh and other supporting staff of our Institute for help rendered in organizing the conference and bringing out this report.

NEW CONCEPTS IN PROBABILITY THEORY

Alladi Ramakrishnan
MATSCIENCE, The Institute of Mathematical Sciences, Madras-20
INDIA.

+*+*

The author gave a series of four lectures on his recent contributions¹⁻⁴⁾ to the theory of probability and stochastic processes in which new concepts called 'Activity', 'Disparity' and 'Duality' have been introduced explaining clearly the meaning of the evolution of a stochastic process and the characteristics of 'earlier' (cause) and 'later' (effect) distributions.

References:-

1. Alladi Ramakrishnan, 'Approach to stationarity in stochastic processes', Proc. Tamil Nadu Acad. Sci., 2, (1979) 189.
2. Alladi Ramakrishnan, 'A mathematical feature of evolutionary stochastic processes', Sonderdruck aus Methods of Operations Research 36 (Oelgeschlager, Gunn and Hain, USA) 1980, the paper presented at Obervolfach Conference in November 1978.
3. Alladi Ramakrishnan, 'A New concept in Probability Theory' J. Math. Anal. Appl. 83, 408, (1981).
4. Alladi Ramakrishnan, 'Duality in stochastic Processes', J. Math. Anal. Appl. (to appear in 84 (1981))

NON LINEAR OPTICAL PROCESS IN THE FIELD OF RANDOM PUMP-APPLI-
CATION OF THE THEORY OF MULTIPLICATIVE STOCHASTIC PROCESSES*

G.S. Agarwal
School of Physics, University of Hyderabad P.O. Hyderabad 500134

This review lecture will be concerned with the applications of the theory of stochastic processes to various systems in the area of quantum optics.

It is well known that the generation of light by a source is basically random in nature and therefore in a proper description, the light field is to be treated as a stochastic process. (1) For example the light produced by a thermal source is described mathematically by a Gaussian stochastic process. On the other hand, the light produced by laser is characterized by much more complicated statistics. (2) In my lecture I would primarily be concerned with the interaction of a random light field with the material medium. In particular I would discuss the effect of the linewidth of the external radiation, which is used to excite the material medium, on the various types of line shapes obtained in a number of processes such as resonance fluorescence, Raman scattering, second harmonic generation etc. Such a question is of great importance in connection with high resolution spectroscopy particularly when one is devising techniques by which resolution beyond natural line width could be achieved.

We first indicate the nature of the stochastic equations that one comes across. In optical problems, usually the dipole approximation is adequate and hence the interaction Hamiltonian between the atomic/molecular system and the external fields $\vec{E}(t) = \sum_{\ell} (b_{\ell}(t) e^{-i\omega_{\ell}t} \vec{\epsilon}_{\ell} + c.c.)$ may be written as $-\vec{d} \cdot \vec{E}(t)$, where $\vec{d}(t)$ is the dipole moment operator of the system. Since the field is stochastic in nature, one way of characterizing the dynamics of $\vec{E}(t)$ is to prescribe the Langevin equations⁽²⁾ for $\vec{b}_{\ell}(t)$:

$$\dot{b}_{\ell}(t) = f_{\ell}(\{b\}) + F_{\ell}(t) \quad (1)$$

where $F_{\ell}(t)$ are Gaussian delta correlated random forces.

The nature of f_{ℓ} will depend on the external field. For example if the external field were Gaussian with Lorentzian spectrum then $f_{\ell}(\{b\}) = -\Gamma_{\ell} b_{\ell}$; $\langle b_{\ell}^*(t) b_{\ell}(0) \rangle = \langle |b_{\ell}|^2 \rangle e^{-\Gamma_{\ell} t}$.

For a laser oscillating in a single mode and far above threshold, we have⁽³⁾ in place of (1)

$$b = |b| e^{-i\phi}, \quad |b| = \text{deterministic}, \quad \dot{\phi} = \mu(t), \quad \langle \mu(t_1) \rangle = 0, \quad \langle \mu(t_1) \mu(t_2) \rangle = 2\gamma_c \delta(t_1 - t_2) \quad (2)$$

and $\mu(t)$ has Gaussian statistics. Note that again for the model (2), $\langle b^*(t) b \rangle = \langle |b|^2 \rangle e^{-\gamma_c t}$; so that the spectrum is again Lorentzian. Note also that a broad band source may be represented by white noise i.e. $\vec{E}(t)$ is Gaussian and delta correlated. The density matrix of the

atomic/molecular system satisfies the Liouville equation

$$\frac{\partial \rho_{ij}}{\partial t} = i \sum_{\ell} \left[\vec{d} \cdot (\vec{E}_{\ell}(t) e^{-i\omega_{\ell}t} b_{\ell}(t) + c.c., \rho \right]_{ij} + (R\rho)_{ij} \quad (3)$$

where $R\rho$ denotes the contributions arising from various types of radiative and non-radiative relaxations. The spontaneous emission is contained in $R\rho$. Various forms of $R\rho$ depending on collisions, the structure of the energy levels of the atomic/molecular system are known in literature^{2,4}. In view of the stochastic nature of b_{ℓ} , the density matrix elements $\rho_{ij}(t)$ become stochastic; and hence we have a set of coupled non-linear Langevin equations (1) and (3), the exact solution of which will enable us to obtain all the characteristics of the system under irradiation by a stochastic field. However analytical solution to such a set is not known even if the field $b_{\ell}(t)$ were Gaussian. In what follows we will be give examples where it is possible to obtain exact solutions when $b_{\ell}(t)$ happens to be a laser field oscillating well above threshold. For this purpose the results from the theory of multiplicative stochastic processes will be used. Before we discuss the general theory, we present some simple examples, which illustrate the importance of the band width of the exciting light in the context of optical resonance.

(A) Consider the spectrum of resonance fluorescence produced by a two level atom under the condition that the exciting radiation is weak. In such a case it can be shown⁽⁵⁾ that the spectrum has the structure $\gamma_c [\gamma^2 + \omega_0^2]^{-1} [\gamma_c^2 + \omega^2]^{-1}$, which in the following special cases leads to the celebrated result of Heitler⁶ namely

- (i) if $\gamma_c \gg \gamma$ (γ = natural linewidth), then the spectrum has a width γ and is centered at atomic frequency.
- (ii) if $\gamma_c \ll \gamma$, then the spectrum has a width γ_c and is centered at the laser frequency. The evidence for such line widths less than natural linewidths has been obtained in the experiments of Gibbs and Venkatesan⁷.

(B) Consider the intensity of fluorescence in light fields that are modulated at some frequency Ω . In such a case the phase shift of the modulated radiation has a remarkable dependence on γ_c . The general result⁸ for a two level system, under the resonant condition

(ω = atomic frequency) has the form

$$\tan \frac{\phi}{\Phi} = -\Omega \left\{ \frac{\Omega^2 + 4\gamma^2 + 2\gamma_c^2 + 6\gamma\gamma_c}{\gamma(\Omega^2 + 4\gamma^2) + \gamma_c(8\gamma^2 + 4\gamma\gamma_c - \Omega^2)} \right\}$$

$$\rightarrow -\Omega\tau, \quad \gamma_c \rightarrow \infty$$

$$\rightarrow -2\Omega\tau, \quad \gamma_c \rightarrow 0$$

demonstrates the crucial role of the laser linewidth in modulation studies ; which have been used extensively in the measurements of lifetimes.

(C) We next give an example which will illustrate the differences in the spectrum of the scattered light arising from the internal stochastic modulation and the external modulation. Take an harmonic oscillator of frequency ω_0 interacting with an external radiation field of bandwidth γ_c . In such a case one can show that the correlation function involving the complex amplitude σ of the oscillator has the structure

$$\lim_{t \rightarrow \infty} \langle \sigma^*(t+\tau) \sigma(t) \rangle = \langle \sigma^* \sigma \rangle e^{(i\omega_0 - \gamma)\tau} + \left\{ \frac{E^* \langle \sigma \rangle}{[\gamma - \gamma_c - i(\omega_0 - \omega_l)]} \right\} \left\{ \begin{array}{l} e^{-\gamma_c \tau + i\omega_l \tau} \\ - e^{(i\omega_0 - \gamma)\tau} \end{array} \right\} \quad (5)$$

On the other hand if the external radiation has zero bandwidth, but the frequency is randomly modulated and for the random modulation, if we take white noise model, then

$$\lim_{t \rightarrow \infty} \langle \sigma^*(t+\tau) \sigma(t) \rangle = \exp\{i(\omega_0 - \gamma - \gamma_c)\tau\} \langle \sigma^* \sigma \rangle + \left\{ \frac{E^* \{ \exp[(i\omega_0 - \gamma - \gamma_c)\tau] - \exp(i\omega_l \tau) \}}{(i(\omega_0 - \omega_l) - \gamma - \gamma_c)} \right\} \quad (6)$$

The spectrum of the scattered radiation is proportional to the Fourier transforms of the above correlations of the complex amplitudes. The results (5) and (6) demonstrate how the different type of stochastic fluctuation could lead to very different spectra of the scattered radiation. For example in (5) the line centered at the oscillator frequency has width γ whereas in (6) it has width $(\gamma + \gamma_c)$. Another important distinction is the absence of the coherent component in (5).

It is evident from the foregoing that many systems in quantum optics involve dynamical equations of the form

$$\dot{x}_i = \sum_j M_{ij} x_j + \sum_j F_{ij}(t) \quad (7)$$

where $F_{ij}(t)$ are Gaussian delta correlated processes with zero mean

$$\langle F_{ij}(t) \rangle = 0, \quad \langle F_{ij}(t) F_{kl}(t') \rangle = 2Q_{ijkl} \delta(t-t') \quad (8)$$

The random processes satisfying (7) are known as multiplicative stochastic processes for which following results⁽⁹⁾ could be proved -

(i) $x_i(t)$ are Markov variables,

$$(ii) \langle \dot{x}_i \rangle = \sum_j M_{ij} \langle x_j \rangle + \sum_{kj} Q_{ikkj} \langle x_j \rangle = \sum_j m_{ij} \langle x_j \rangle \quad (9)$$

$$\begin{aligned}
 (111) \quad \overline{\langle x_k x_l \rangle} &= \sum_{ij} Q_{kij} \langle x_i x_j \rangle \\
 &+ \sum_j (m_{kj} \langle x_j x_l \rangle + m_{lj} \langle x_j x_k \rangle)
 \end{aligned} \tag{10}$$

The markov property implies that if

$$\langle x_i(t+\tau) \rangle = \sum_j U_{ij}(\tau) \langle x_j(t) \rangle, \quad U(t) = \exp(mt)$$

then the correlation function of x_i 's is given by

$$\langle x_i(t+\tau) x_l(t) \rangle = \sum_j U_{ij}(\tau) \langle x_j(t) x_l(t) \rangle \tag{11}$$

The results (9) - (11) are very basic and have been used in studying the following systems:-

- (1) Absorption and emission spectra of two level systems under the influence of laser fields of arbitrary bandwidth and intensity ^{5,10}.
- (2) Optical double resonance ¹¹, various types of Hanle effect ¹² and Raman scattering in fluctuating beams.
- (3) Modulated Fluorescence studies ^{8,13} in fields that are fluctuating. The incident fields may be either amplitude or frequency modulated ¹⁴.
- (4) Coherent antistokes Raman scattering ⁽¹⁵⁾.

(5) Multiphoton absorption processes - this problem also has been solved numerically¹⁶ for the Gaussian Statistics of pump field. In all the above cases exact results have been obtained. Since such results are discussed extensively in recent literature, we do not pursue these any more.

Let us now examine some nonlinear optical systems briefly. We will be making use of the parametric approximation i.e. we will ignore the depletion of the pump field assuming that the pump field is strong enough.

(A) Parametric Amplification :

In the parametric approximation the interaction Hamiltonian describing parametric amplification in the field of random pump could be written as

$$H_1 = g (a^\dagger b^\dagger e^{-i\Phi(t)} + a b e^{i\Phi(t)}) \quad , \quad (12)$$

where we have assumed that the pump frequency is equal to the sum of the signal and idler frequencies and

(12) is written in interaction picture. The Hamiltonian (12) for the case of a nonfluctuating pump has been discussed at length by Mollow and Glauber⁽¹⁷⁾.

On making a canonical transform

$$\rho' = V(t) \rho V^\dagger(t), \quad V(t) = \exp \left\{ i \int \frac{u(t) dt}{2} (a^\dagger a + b^\dagger b) \right\}, \quad (13)$$

we find the following Liouville equation for ρ'

$$\dot{\rho}' = -ig [a^\dagger b^\dagger + ab, \rho'] + \frac{i\mu(t)}{2} [a^\dagger a + b^\dagger b, \rho'] \quad (14)$$

We now use the result (9) from the theory of multiplicative stochastic process to find that the ensemble average of ρ' over the distribution of $\mu(t)$ satisfies the equation

$$\begin{aligned} \langle \dot{\rho}' \rangle &= \frac{-\gamma_c}{4} [a^\dagger a + b^\dagger b, [a^\dagger a + b^\dagger b, \langle \rho' \rangle]] \\ &\quad - ig [a^\dagger b^\dagger + ab, \langle \rho' \rangle] \end{aligned} \quad (15)$$

It is evident from the definition (13) that all expectation values of the form

$$\langle (a^\dagger)^\alpha (b^\dagger)^\beta (a)^\gamma (b)^\delta \rangle \text{ with } \alpha + \beta + \gamma + \delta = 0$$

could be as well computed using the density matrix ρ' since

$$\begin{aligned} \text{Tr} \left\{ \rho (a^\dagger)^\alpha (b^\dagger)^\beta (a)^\gamma (b)^\delta \right\} &= \text{Tr} \left\{ \rho' (a^\dagger)^\alpha (b^\dagger)^\beta (a)^\gamma (b)^\delta \right\} \\ \text{if } \alpha + \beta + \gamma + \delta &= 0 \end{aligned} \quad (16)$$

Thus a solution of (15) will yield moments of the form

$\langle (a^\dagger)^\alpha (b^\dagger)^\beta (a)^\gamma (b)^\delta \rangle$, $\alpha + \beta + \gamma + \delta = 0$ for arbitrary values of g and the pump band-width γ_c . In particular

from (15) we find that

$$\langle \langle a^\dagger a + b^\dagger b + 1 \rangle \rangle = -2ig \langle \langle a^\dagger b^\dagger e^{i\Phi} - \text{c.c.} \rangle \rangle$$

$$\begin{aligned}
\langle\langle \overline{a^\dagger b^\dagger e^{i\Phi} - cc} \rangle\rangle &= -\gamma_c \langle\langle a^\dagger b^\dagger e^{i\Phi} - cc \rangle\rangle \\
&+ 2ig \langle\langle a^\dagger a + b^\dagger b + 1 \rangle\rangle \\
\langle\langle \overline{a^\dagger b^\dagger e^{i\Phi} + cc} \rangle\rangle &= -\gamma_c \langle\langle a^\dagger b^\dagger e^{i\Phi} + cc \rangle\rangle \\
\langle\langle \overline{a^\dagger a - b^\dagger b} \rangle\rangle &= 0
\end{aligned}$$

(17)

The double brackets have now been used to indicate the averaging with respect to the fluctuations of the pump, in addition to the averaging over the quantum states of the system. The time dependence of the number of photons in idler or

signal mode is now governed by

$$\exp\left(-\frac{\gamma_c}{2} \pm \frac{1}{2}\sqrt{\gamma_c^2 + 16g^2}\right)t$$

showing that the growth rate of photons is reduced due to the pump fluctuations. The fluctuations in the photon numbers can also be obtained from (15). To obtain the spectrum of the amplitude fluctuations, we note the following equations that follow from Eqs. (9), (11) and (12) and the markov property of the system

$$\begin{aligned}
\frac{\partial}{\partial \tau} \langle\langle a^\dagger(t+\tau)a(t) \rangle\rangle &= ig \langle\langle b(t+\tau) e^{i\Phi(t+\tau)} a(t) \rangle\rangle, \\
\left(\frac{\partial}{\partial \tau} + \gamma_c\right) \langle\langle b(t+\tau) e^{i\Phi(t+\tau)} a(t) \rangle\rangle &= -ig \langle\langle a^\dagger(t+\tau)a(t) \rangle\rangle
\end{aligned}$$

(18)

The eigenvalues associated with the amplitude fluctuations

are now $\left(\frac{-\gamma_c}{2} \pm \frac{1}{2} \sqrt{\gamma_c^2 + 4g^2} \right)$

It may be noted that a similar approach may be used for parametric frequency conversion described by the Hamiltonian

$$H_1 = g (a^\dagger b e^{-i\Phi}(t) + \text{c.c.}) \quad (19)$$

(B) Second Harmonic Generation :

The second harmonic generation (SHG) in the slowly varying amplitude approximation and in the situation when the group velocities of the waves at the pump and the second harmonic are different, is known to be described by the equations (18)

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_1} \frac{\partial}{\partial t} \right) A_1 = 0, \quad \left(\frac{\partial}{\partial z} + \frac{1}{v_2} \frac{\partial}{\partial t} \right) A_2 = -2igA_1^2 \quad (20)$$

where A_1 and A_2 are the complex field amplitudes of the waves at the fundamental and second harmonic frequency. The pump field A_1 is a function of $(z - v_1 t)$ only and its phase is assumed to satisfy Eq. (2). It can be shown from (20) either by using the results (9) - (11) or by direct integration (19) that

$$\langle |A_2|^2 \rangle = \frac{8g^2 |A_1|^4}{(4\gamma_c v)^2} [X - 1 + e^{-X}], \quad X = 4\gamma_c v z, \quad v = \frac{1}{v_2} - \frac{1}{v_1} \quad (21)$$

The linewidth of the second harmonic radiation is found to be

$$\Delta\omega = \frac{8 \gamma_c}{x} \left[1 - \frac{1}{x} + \frac{e^{-x}}{x} \right], \quad (22)$$

which depends in a cumbersome way on the pump linewidth.

(C) Stimulated Raman Scattering :

As a next illustration of the application of the theory of multiplicative stochastic process, we consider stimulated Raman scattering under the situation that both pump and stokes beams are fluctuating and each is characterized by the model (2). For simplicity we treat only the resonant situation i.e. we assume a three-level model for the atom with the ground level $|3\rangle$ [final level $|2\rangle$] and the excited level $|1\rangle$ nearly resonant with the pump frequency ω_p (stokes frequency ω_s). The fields can be of arbitrary intensity. Details of scattering by such a model can be found in ref. 20, where all the fields have been taken to be monochromatic. The stimulated Raman scattering can be shown to be related to $\langle\langle \rho_{32} e^{i\Phi_s(t)} \rangle\rangle$ using the theory of multiplicative stochastic processes one can show that the results for stimulating Raman scattering in fluctuating beams can be obtained from those for the monochromatic fields, if we use the following rule⁽²¹⁾, which is valid for fields of arbitrary intensity

$$\Gamma_{13} \rightarrow \Gamma_{13} + \gamma_{\ell}, \quad \Gamma_{12} \rightarrow \Gamma_{12} + \gamma_s, \quad \Gamma_{23} \rightarrow \Gamma_{23} + \gamma_{\ell} + \gamma_s \quad (23)$$

where γ_{ℓ} and γ_s are the linewidths of the incident waves at ω_{ℓ} and ω_s respectively. Γ_{ij} denotes various relaxation rates $(\dot{\rho}_{ij})_R = -\Gamma_{ij} \rho_{ij} (i \neq j)$. The situation however is different for spontaneous scattering in fields of arbitrary intensity. Using again the techniques of the multiplicative stochastic processes it is possible to account for the propagation effects in Raman scattering (22).

We conclude this talk by mentioning how the theory of multiplicative stochastic processes can be used to study the wave propagation in a random medium (23). It is known that the

wave propagation in a medium is approximately described by

$$2ik \frac{\partial \psi}{\partial z} + \nabla_{\vec{r}}^2 \psi + k^2 \epsilon_1(\vec{r}, z) \psi = 0, \quad k = \omega/c$$

$$\nabla_{\vec{r}}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}; \quad \vec{r} = (x, y, 0) \quad (24)$$

The dielectric function ϵ_1 is random in nature and one often assumes that ϵ_1 is a Gaussian, delta correlated random process with zero mean i.e.

$$\langle \epsilon_1(\vec{r}, z) \epsilon_1(\vec{r}', z') \rangle = 2\delta(z-z') f(\vec{r}, \vec{r}') \quad (25)$$

$$\langle \epsilon_1(\vec{r}, z) \rangle = 0.$$

The structure of (24) is similar to (7) and the general results from the theory of multiplicative processes could be directly

applied to (24) to obtain the equations for the ensemble averaged fields and their correlation functions. The linearity of the operator $\nabla_{\vec{r}}$ is also to be used. A straightforward application of (9) to (24) leads to $[z^-]$ is now playing the role of the variable t

$$2ik \frac{\partial}{\partial z} \langle \psi \rangle + \nabla_T^2 \langle \psi \rangle + \frac{ik^3}{2} f(\vec{p}, \vec{p}') \langle \psi \rangle = 0 \quad (26)$$

Similarly starting from (24) we have the equation for the bilinear product $\psi^*(\vec{p}', z) \psi(\vec{p}, z)$:

$$\begin{aligned} \frac{\partial}{\partial z} (\psi^*(\vec{p}', z) \psi(\vec{p}, z)) &= -\frac{1}{2ik} (\nabla_T^2 - \nabla_T'^2) (\psi^*(\vec{p}', z) \psi(\vec{p}, z)) \\ &\quad - \frac{k}{2i} [\epsilon_1(\vec{p}, z) - \epsilon_1(\vec{p}', z)] (\psi^*(\vec{p}', z) \psi(\vec{p}, z)) \end{aligned} \quad (27)$$

Equation (27) again has the structure of multiplicative stochastic process with the new Gaussian delta correlated random process $\epsilon_1(\vec{p}, z) - \epsilon_1(\vec{p}', z)$. Again a straight-forward application of (9) and the linearity of the operators ∇_T^2 lead to the equation for the correlation function $\langle \psi^*(\vec{p}', z) \psi(\vec{p}, z) \rangle$:

$$\begin{aligned} 2ik \frac{\partial}{\partial z} \langle \psi^*(\vec{p}', z) \psi(\vec{p}, z) \rangle &+ (\nabla_T^2 - \nabla_T'^2) \langle \psi^*(\vec{p}', z) \psi(\vec{p}, z) \rangle \\ &+ \frac{ik^3}{2} [f(\vec{p}, \vec{p}) + f(\vec{p}', \vec{p}') - 2f(\vec{p}, \vec{p}')] \langle \psi^*(\vec{p}', z) \psi(\vec{p}, z) \rangle \\ &= 0. \end{aligned} \quad (28)$$

Equations for higher order correlations are similarly obtained. Correlations at two different points z and z' can be obtained from the markov property (11). We have thus rederived well known equations ⁽²³⁾ describing the wave propagation in random media by using the theory of multiplicative stochastic processes. For various other derivations of (26) and (28), see ref. (23). Using equation (28) one can, for example, study how the amplitude correlations over a plane change, as the wave propagates through the random medium.

It would be worthwhile to examine the non-linear optical resonance phenomena in fields for which the model (2) is not adequate and to see if there are some other models for which exact solutions to the non-linear optical problems in the field of random pumps could be obtained. Some exact results are already available ^{24, 25}.

Acknowledgement

The author would like to thank Prof. A. Ramakrishnan and Prof. V. Radhakrishnan for their kind invitation to the Conference and for their hospitality at Ooty.

REFERENCES

1. L. Mandel and E. Wolf, Rev. Mod. Phys. 37, 231 (1965)
2. H. Haken, Laser Theory, Handbuch der Physik ed. S. Flugge (Springer, New York (1970)) Vol. XXV/2C ; M. Lax in Statistical Physics, Phase Transitions and Superfluidity editors M. Chretien, E.P. Gross and S. Deser (Gordon and Breach, New York 1968), Vol. II, p. 269.
3. cf. J.R. Klauder and E.C.G. Sudarshan, "Fundamentals of Quantum Optics" (W.A. Benjamin, New York 1968) p.228.
4. G.S. Agarwal in Progress in Optics, Vol. XI, Ed. E. Wolf (North Holland, Amsterdam 1973), p.1 ;
G.S. Agarwal, Quantum Optics, Springer Tracts in Modern Physics, Vol. 70, Ed. G. Hohler (Springer, Berlin 1974).
5. G.S. Agarwal, Phys. Rev. A18, 1490 (1978) ; appendix A.
6. W. Heitler, Quantum Theory of Radiation (Oxford, New York 1954) Sec. 20.
7. H.M. Gibbs and T.N.C. Venkatesan, Opt. Commun. 17, 87 (1976).
8. R. Saxena and G.S. Agarwal, J. Phys. B12, 1939 (1979) and the references therein.
9. Reference 5, appendix B.
10. J.H. Eberly, Phys. Rev. Letters 37, 1387 (1976) ; P. Avan and C. Cohen Tannoudji, J. Phys. B10, 155 (1977) ;
H.J. Kimble and L.Mandel, Phys. Rev. A15, 689 (1977) ;
P. Zoller, J. Phys. B10, L 321 (1977) ; K. Wodkiewicz, Phys. Rev. A19, 1686 (1979).

11. A.T. Georges and P. Lambropoulos, Phys. Rev. A18, 587 (1978) ; G.S. Agarwal and P.A. Narayana, Opt. Commun. 30, 364 (1979).
12. P. Avan and C. Cohen Tannoudji, J. Phys. B10, 171 (1977); P.A. Lakshmi and G.S. Agarwal, Phys. Rev. A23, 2553 (1981).
13. R. Saxena and G.S. Agarwal, Phys. Rev. A in press.
14. G.S. Agarwal, Phys. Rev. A23, 1375 (1981).
15. G.S. Agarwal and S. Singh, submitted to Phys. Rev. A ; N.K. Dutta, J. Phys. B13, 411 (1980).
16. P. Zoller, Phys. Rev. A19, 1151 (1979) ; A.T. Georges, P. Lambropoulos and P. Zoller, Phys. Rev. Lett. 42, 1609 (1979).
17. B.R. Mollow and R.J. Glauber, Phys. Rev. 160, 1076 (1967); *ibid* 160, 1097 (1967).
18. S.A. Akhmanov, A.S. Chirkin and V.G. Tunkin, Opto - electronics 1, 196 (1969).
19. cf. N.K. Dutta, Optical and Quantum Electronics 11, 217 (1979).
20. G.S. Agarwal and S.S. Jha, J. Phys. B12, 2655 (1979) ; C. Cohen Tannoudji and S. Reynand, J. Phys. B10, 365 (1977).

21. Similar rules in the context of scattering from two level systems are given in ref. 5 and in J.H. Eberly (ref. 10) and in Laser Spectroscopy IV, eds. H. Walther and K.W. Rothe (Springer, Berlin 1979), p.80.
22. M.G. Raymer, J. Mostowski and J.L. Carlsten, Phys. Rev. A19, 2304 (1979).
23. The literature on this subject is very extensive. We refer to a recent review article J.W. Strohbehn, in Laser Beam Propagation in the Atmosphere; Ed. J.W. Strohbehn (Springer, Berlin (1978) p. 45.
24. S.N. Dixit, P. Zoller and P. Lambropoulos, Phys. Rev. A21, 1289 (1980) ; J.J. Yeh and J.H. Eberly, to be published. The phase diffusion model is generalised to include the finite correlation time for the time derivative of the phase of the field.
25. G.S. Agarwal, Z. Physik B33, 111 (1979)
In this reference the amplitude of the stochastic field has been taken to be represented by a two state markov process.

MATKOWSKI'S FIXED POINT THEOREM AND RANDOM OPERATORS*

K.Balakrishna Reddy, M.S.Rangachari
The Ramanujan Institute of Advanced Study in Mathematics, The -
University of Madras, Madras 600 005.

and

P.V.Subrahmanyam
The School of Mathematics, The University of Hyderabad,
Hyderabad 500 134
and (now at)
Department of Mathematics, Indian Institute of Technology,
Madras 600 036, India.

+++++

Introduction:

The mathematical models studying the nondeterministic nature of phenomena in the realm of biological, engineering, Oceanographic and physical sciences often lead to random or stochastic equations. Although several concrete examples of random operators and random operator equations were known for a long time, a systematic study of probabilistic operator theory and its applications was initiated by the Prague school of probabilists under the direction of Antonin Spacek in the 1950's. For a history of probabilistic operator theory Bharucha-Reid ((3) (4)) and Tsokos and Padgett (10) may be referred. Many of these random operator equations can be solved using certain fixed point theorems, called random fixed point theorems.

* Paper presented by P.V. Subrahmanyam.

Matkowski ((8), (9)) proved a fixed point theorem generalizing Banach's contraction principle. Altman (1) gave an elaborate theory of contractors and contractor directions and one of his basic theorems generalized the contraction principle. In (2) Matkowski's fixed point theorem and Altman's contractor theory were unified. Hans (5) obtained the random analogue of the contraction principle, while Lee and Padgett ((6), (7)) developed the random version of Altman's contractor theory.

In what follows we prove a fundamental lemma for a system of random inequalities and this lemma could be used to obtain the random analogue of Matkowski's fixed point theorem. The random analogues of the existence theorems proved in (2) could also be obtained and these, generalizing the random contractor theory will be published elsewhere.

1. Preliminaries

Let $(\Omega, \mathcal{A}, \mu)$ be a complete probability measure space and $(a_{ik}(\omega))$ be an $n \times n$ positive random matrix, i.e. $a_{ik} : \Omega \rightarrow \mathbb{R}^+$ is a real-valued random variable, $i, k = 1, \dots, n$. Define the sequence of random matrices $\{ a_{ik}^1(\omega) \}$ as follows:

$$(1) a_{ik}^I(\omega) = \begin{cases} a_{ik}(\omega) & i \neq k \\ 1 - a_{ik}(\omega), & i = k \end{cases}$$

$i, k = 1, 2, \dots, n$

$$(2) a_{ik}^{Q+1}(\omega) = \begin{cases} a_{i+1k+1}^Q(\omega) a_{i+1k+1}^Q(\omega) + a_{i+1i}^Q(\omega) \\ a_{ik+1}^Q(\omega), & i \neq k \\ a_{i+1i}^Q(\omega) a_{i+1k+1}^Q(\omega) - a_{i+1i}^Q(\omega) a_{ik+1}^Q(\omega), & i \neq k, \end{cases}$$

$i, k = 1, \dots, n-1; Q = 1, \dots, n-1$

Let (X, β) and (Y, \mathcal{C}) be two measurable spaces, where X and Y are Banach spaces of the same scalar type and β and \mathcal{C} are σ -algebras of Borel subsets of X and Y respectively.

Definition 1.1 A mapping $x : \Omega \rightarrow X$ is said to be an X -valued random variable if the inverse image under the mapping x for each $B \in \beta$ belongs to \mathcal{A} ; i.e. $x^{-1}(B) \in \mathcal{A}$ for all $B \in \beta$.

Definition 1.2 Let T be an operator from X to Y with domain $D(T)$. A mapping $T : \Omega \times D(T) \rightarrow Y$ is said to be a random operator if $y(\omega) = T(\omega)$ is a Y -valued random variable for every $x \in D(T)$.

Definition 1.3 Any x -valued random variable $x(\omega)$ which satisfies the condition $\mu(\{\omega \in T(\omega) x(\omega) = y(\omega)\}) = 1$ is said to be a random solution of the random operator equation $T(\omega)(x(\omega)) = y(\omega)$.

Definition 1.4 An X -Valued random variable $\xi(\omega)$ is said to be a fixed point of the random operator $T(\omega)$ if $\xi(\omega)$ is a random solution of the equation $T(\omega)\xi(\omega) = \xi(\omega)$.

2. A lemma on the solution of a system of random inequalities

We establish the analogue of Matkowski's lemma for random variables from which a random version of Matkowski's fixed point theorem could be obtained. In the following we assume that the inequality and equalities hold almost surely in Ω and so we do not state this explicitly. Further, the Random matrices $(a_{ik}^1(\omega))$ are as defined in (1) and (2).

Lemma 2.1 Let $a_{ik}^1(\omega)$ be an $n \times n$ positive random matrix. The system of random inequalities

$$(3) \quad \sum_{k=1}^n a_{ik}^1(\omega) r_k(\omega) < r_i(\omega), \quad \omega \in \Omega$$

$i = 1, 2, \dots, n$ has a random solution

$r_i(\omega) > 0$, $i = 1, \dots, n, \omega \in \Omega$, if and only if

$$(4) \quad a_{ii}^1(\omega) > 0, \quad i = 1, \dots, n, \quad i = 1, \dots, n+1-l$$

Proof Necessity Suppose $a_{ik}^1(\omega) > 0, i, k = 1, \dots, n;$
 $n \geq 2$ and $\sum_{k=1}^n a_{ik}(\omega) r_k(\omega) < r_i(\omega)$ a.s.,

$i = 1, \dots, n, \omega \in \Omega$ has a positive random solution $r_i(\omega)$.

We shall show that $a_{ii}^l(\omega) > 0, l = 1, \dots, n; i = 1, \dots,$

$n + 1 - l$. First we dispose of the case when $n = 2$. In

this case we shall prove that $a_{ii}^l(\omega) > 0, l = 1, 2; i = 1,$
 $\dots, 2+1-l, \omega \in \Omega$. That is, we shall prove that $a_{11}^1(\omega)$

$> 0, a_{22}^1(\omega) > 0$ and $a_{11}^2(\omega) > 0, \omega \in \Omega$. In view of

the hypothesis it suffices to show that $a_{11}^2(\omega) > 0, \text{ a.s.}$

When $n = 2$, (3) can be written as

$$a_{11}(\omega) r_1(\omega) + a_{12}(\omega) r_2(\omega) < r_1(\omega)$$

$$a_{21}(\omega) r_1(\omega) + a_{22}(\omega) r_2(\omega) < r_2(\omega)$$

Using (1) these inequalities can

be written as

$$(5) \begin{cases} a_{12}^1(\omega) r_2(\omega) < a_{11}^1(\omega) r_1(\omega) \\ a_{21}^1(\omega) r_1(\omega) < a_{22}^1(\omega) r_2(\omega) \end{cases}$$

From (5) we have

$$\frac{a_{21}^1(\omega)}{a_{22}^1(\omega)} < \frac{r_2(\omega)}{r_1(\omega)} < \frac{a_{11}^1(\omega)}{a_{12}^1(\omega)}$$

This implies that

$$a_{11}^2(\omega) = a_{11}^1(\omega) a_{22}^1(\omega) - a_{21}^1(\omega) a_{12}^1(\omega) > 0.$$

To treat the general case we apply mathematical induction. Consider the system of inequalities (3). Assume that this system has a random solution $r_1(\omega) > 0$, $i = 1, \dots, n$.

We shall show that $a_{i1}^{\ell}(\omega) > 0$, $\ell = 1, 2, \dots, n$, $i = 1, \dots, n+1-\ell$. From (3) we have, when $i = 1$,

$$a_{11}(\omega)r_1(\omega) + a_{12}(\omega)r_2(\omega) + \dots + a_{1n}(\omega)r_n(\omega) < r_1(\omega)$$

$$\text{i.e. } a_{12}^1(\omega)r_2(\omega) + \dots + a_{1n}^1(\omega)r_n(\omega) < a_{11}^1(\omega)r_1(\omega)$$

By hypothesis $a_{11}^1(\omega) > 0$, a.s. and hence the above inequality can be written as

$$(6) \quad \frac{1}{a_{11}^1(\omega)} \left[\sum_{k=2}^n a_{1k}^1(\omega)r_k(\omega) \right] < r_1(\omega). \text{ For } i > 1, (3)$$

$$\text{can be written as (7) } a_{i1}^1(\omega)r_1(\omega) + \sum_{k=2}^n a_{ik}^1(\omega)r_k(\omega) < a_{ii}^1(\omega)r_i(\omega)$$

From (6) and (7) we have

$$\frac{a_{i1}^1(\omega)}{a_{11}^1(\omega)} \sum_{k=2}^n a_{1k}^1(\omega)r_k(\omega) + \sum_{\substack{k=2 \\ k \neq i}}^n a_{ik}^1(\omega)r_k(\omega) < a_{ii}^1(\omega)r_i(\omega) \\ i = 1, 2, \dots, n.$$

$$\sum_{\substack{k=2 \\ k \neq i}}^n \left[a_{11}^1(\omega)a_{ik}^1(\omega) + a_{i1}^1(\omega)a_{1k}^1(\omega) \right] r_k(\omega) < (a_{11}^1(\omega)a_{ii}^1(\omega) - a_{i1}^1(\omega)a_{1i}^1(\omega))r_i(\omega) \\ i = 1, 2, \dots, n$$

Using the definitions

$$a_{i-1, i-1}^2(\omega) = a_{11}^1(\omega)a_{ii}^1(\omega) - a_{i1}^1(\omega)a_{1i}^1(\omega)$$

and

$$a_{i-1, k-1}^2(\omega) = a_{i1}^1(\omega) a_{ii}^1(\omega) + a_{i1}^1(\omega) a_{1k}^1(\omega)$$

we have,

$$\sum_{k=2, \neq i}^n a_{i-1, k-1}^2(\omega) \gamma_k(\omega) < a_{i-1, i-1}^2(\omega) \gamma_i(\omega), \quad i=1, 2, \dots, n$$

$$(8) \quad \sum_{k=1, \neq i}^n a_{ik}^2(\omega) \gamma_{k+1}(\omega) < a_{ii}^2(\omega) \gamma_{i+1}(\omega), \quad i=1, 2, \dots, n-1$$

We observe that $a_{ik}^2(\omega) > 0$, $i, k = 1, 2, \dots, n-1$, For $i \neq k$

we have, by definition,

$$a_{ik}^2(\omega) = a_{i1}^1(\omega) a_{1+k+1}^1(\omega) + a_{i1}^1(\omega) a_{1k+1}^1(\omega).$$

Since

, it follows that

$$a_{ik}^2(\omega) > 0, \quad i \neq k, \quad i, k = 1, \dots, n-1$$

$$a_{ii}^1(\omega) > 0$$

If $a_{ii}^2 < 0$, then (8) shows that $\sum_{k=1, \neq i}^n a_{ik}^2(\omega) \gamma_{k+1}(\omega) \leq 0$,

which is a contradiction in view of $a_{ik}^2(\omega) > 0$, $i, k = 1, 2, \dots, n-1$, $i \neq k$ and $\gamma_2(\omega), \dots, \gamma_n(\omega) > 0$. Thus, we have shown that

$$a_{ik}^2(\omega) > 0, \quad i, k = 1, \dots, n-1. \quad \text{So, the inductive hypothesis}$$

applies to the system (8) and therefore $(a_{ii}^2)^l(\omega) > 0$, $l=1, \dots, n-2$

$i=1, \dots, n+1-l$. That is $a_{ii}^l(\omega) > 0$, $l=3, \dots, n$; $i=1, \dots,$

$n+1-l$. As we have already shown $a_{ii}^2(\omega) > 0$, $i=1,$

$2, \dots, n+1-l$, $a_{ii}^l(\omega) > 0$, for $l=1, 2, \dots, n$; $i=1, \dots, n+1-l$

Sufficiency Suppose $a_{ik}^1(\omega) > 0$ and $a_{ii}^l(\omega) > 0$, $i=1, \dots, n+1-l$,

$l=1, 2, \dots, n$. Then the system of random inequalities (3) has a solution

$\gamma_1(\omega), \dots, \gamma_n(\omega) > 0$. Consider the case when $n=2$. By

hypothesis $a_{11}^1(\omega), a_{12}^1(\omega), a_{21}^1(\omega), a_{22}^2(\omega) > 0$. By

definition $a_{11}^2(\omega) = a_{11}^1(\omega) a_{12}^1(\omega) - a_{21}^1(\omega) a_{12}^1(\omega) > 0$ i.e.

$\frac{a_{21}^1(\omega)}{a_{22}^1(\omega)} < \frac{a_{11}^2(\omega)}{a_{12}^1(\omega)}$. We can find random variables $r_1(\omega)$
 > 0 and $r_2(\omega) > 0$ such that $\frac{a_{21}^1(\omega)}{a_{22}^1(\omega)} < \frac{r_2(\omega)}{r_1(\omega)} < \frac{a_{11}^1(\omega)}{a_{22}^1(\omega)}$

i.e. $a_{12}^1(\omega)r_2(\omega) < a_{11}^1(\omega)r_1(\omega)$
 $a_{21}^1(\omega)r_1(\omega) < a_{22}^1(\omega)r_2(\omega)$

Thus in the case when $n = 2$, the system (3) has a positive solution. Suppose $a_{i1}^l(\omega) > 0$, $i = 1, \dots, n+1-l$; $l = 1, \dots, n$. Assume the validity of the sufficiency part of Lemma 1.1 for $n-1$. We shall prove its validity for n by induction, again consider the system of inequalities (3).

Rewriting,

$$(9) \quad \sum_{k=1, k \neq i}^n a_{ik}^1(\omega)r_k(\omega) < a_{ii}^1(\omega)r_i(\omega), \quad i = 1, 2, \dots, n.$$

For $i = 1$, (9) gives

$$a_{12}^1(\omega)r_2(\omega) + \dots + a_{1n}^1(\omega)r_n(\omega) < a_{11}^1(\omega)r_1(\omega)$$

Since $a_{11}^1(\omega) > 0$, we deduce (6) and therefrom (8). By definition, for $i \neq k$, $a_{ik}^2(\omega) = a_{11}^1(\omega) a_{i+1k+1}^1(\omega) + a_{i+11}^1(\omega) a_{1k+1}^1(\omega) > 0$. By hypothesis $a_{ik}^l(\omega) > 0$, $l = 1, \dots, n, i = 1, \dots, n+1-l$. Since by induction hypothesis the system (8) has a random solution $r_2(\omega)$, $\dots, r_n(\omega) > 0$. Now to find a positive random variable

$r_p(\omega)$ so that (3) is true, for each $i = 2, \dots, n$, we find random variables $\epsilon_i(\omega)$ with $\epsilon_i(\omega) > 0$ (in view of

$$(6)) \text{ such that } (10) \quad a_{ii}^i(\omega) \left[\frac{1}{a_{ii}^i(\omega)} \sum_{k=2}^n a_{ik}^i(\omega) r_k(\omega) + \epsilon_i(\omega) \right] \\ + \sum_{k=2, k \neq i}^n a_{ik}^i(\omega) r_k(\omega) < a_{ii}^i(\omega) r_i(\omega)$$

Define $\epsilon(\omega) = \min(\epsilon_1(\omega), \dots, \epsilon_n(\omega))$ and

$$r_1(\omega) = \frac{1}{a_{11}^1(\omega)} \sum_{k=2}^n a_{1k}^1(\omega) r_k(\omega) + \epsilon(\omega)$$

Clearly $r_1(\omega) > 0$ and it follows from (10) that

$$a_{ii}^i(\omega) r_i(\omega) + \sum_{k=2, k \neq i}^n a_{ik}^i(\omega) r_k(\omega) < a_{ii}^i(\omega) r_i(\omega), \quad i = 1, 2, \dots, n.$$

Thus,

$$\sum_{k=1, k \neq i}^n a_{ik}^i(\omega) r_k(\omega) < a_{ii}^i(\omega) r_i(\omega), \quad i = 1, 2, \dots, n.$$

i.e.,

$$\sum_{k=1}^n a_{ik}^i(\omega) r_k(\omega) < r_i(\omega), \quad i = 1, 2, \dots, n.$$

Then, the sufficiency part of Lemma 1.1 is valid for n .

The following theorem is a random version of Matkowski's fixed point theorem in the set-up of separable Banach spaces.

Theorem 2.1 Let X_i , $i=1, \dots, n$ be separable Banach Spaces and $(\Omega, \mathcal{A}, \mu)$ be a complete probability measure space and $T_i(\omega): \Omega \times X_1 \times \dots \times X_n \rightarrow X_i$, $i=1, \dots, n$ be mappings such that

$$(1) \quad \|T_i(\omega)(x_1, \dots, x_n) - T_i(\omega)(y_1, \dots, y_n)\| \leq \sum_{k=1}^n a_{ik}(\omega) \|x_k - y_k\|$$

$$i = 1, \dots, n, \quad x_k, y_k \in X_k, \quad k = 1, \dots, n,$$

where the random variables $a_{ik}(\omega)$ are defined by (1) and (2) fulfill (4). Then the system of random operator equations

$$T_i(\omega)(x_1(\omega), \dots, x_n(\omega)) = x_i(\omega), \quad i = 1, \dots, n$$

has exactly one random solution $(x_1(\omega), \dots, x_n(\omega))$ such that $x_i: \Omega \rightarrow X_i$ is an X_i -valued random variable, $i = 1, \dots, n$. For any arbitrary random variable $x_i^0: \Omega \rightarrow X_i$, the sequence of successive approximations

$$(12) \quad x_i^{m+1}(\omega) = T_i(\omega)(x_1^m(\omega), \dots, x_n^m(\omega)), \quad m=0, 1, \dots, \quad i=1, \dots, n$$

converges a.s and $x_i(\omega) = \lim_{m \rightarrow \infty} x_i^m(\omega)$, $i = 1, \dots, n$

Moreover, there exist positive real-valued random variables

$$\gamma_i(\omega) = \gamma_i^0(\omega)(x_i^0(\omega), \dots, x_n^0(\omega)) > 0 \quad \text{and an } \delta: \Omega \rightarrow \mathbb{R}^+$$

such that $\delta(\omega) < 1$ a.s. and

$$\|x_i^{m+1}(\omega) - x_i^m(\omega)\| \leq [\delta(\omega)]^m \gamma_i(\omega), \quad i=1, \dots, n.$$

Remarks 1 The proof of this random fixed point theorem and an application to an infinite system of random equations will be published elsewhere.

2. When $n = 1$, the above theorem reduces to the contraction principle for random operators.
3. It is possible to obtain the random analogues of the existence theorems in (2), thereby generalizing the results of Lee and Padgett on random contractors. These will be presented elsewhere.

References

1. Altman, M., 'Contractors and Contractor directions, Theory and Applications', Marcel Dekker, New York (1977).
2. Balakrishna Reddy, K., and Subrahmanyam, P.V., Altman's contractors and Matkowski's fixed point theorem (To appear in Nonlinear Analysis: Theory, methods and Applications, 1981).
3. Bharucha-Reid, A.T., 'Random Integral Equations', Academic Press, New York, (1972).
4. _____, Fixed point theorems in probabilistic analysis, Bull. Amer Math. Soc 62 (1976), 641 - 657.
5. Hans, O., Random fixed point theorems, Trans. 1st Prague Conf. on Information Theory, Statist. Division Functions and Random Processes (Liblice, 1956), Czechoslovak Acad. Sci. Prague, 1957, pp. 61-103.

6. Lee, A.C.H. and Padgett, W.J., Random Contractors and the solution of random nonlinear equations, Non-linear Analysis, Theory, Methods and Applications, 1 (1977), 175-185.
7. _____, Random contractors with random nonlinear majorant functions, Ibid, 3 (1979), 707-715.
8. Matkowski, J., Some inequalities and a generalization of Banach's principle, Bull. Acad. Polon Sci. Set Math. Astronom. Phys. 21 (1973), 323-324.
9. _____, Integrable solutions of functional equations, Dissertationes Math. (Rozprawy Mat.) (XXVII) (1975), 1-68.
10. Tsokos, C.P. and Padgett, W.J., 'Random integral equations with applications to life sciences and engineering', Acad. Press. New York.

STUDY OF MULTIDIMENSIONAL DISTRIBUTIONS AND PROCESSES
ASSOCIATED WITH MULTIPLE DIRICHLET SERIES AND APPLICATIONS. *

J. Gopala Krishna and K.H.S. Subrahmanyam,
 Department of Mathematics, Andhra University,
 WALTAIR - 530 003.

BRIEF SUMMARY :

In the first place this work studies the probability distributions and processes associated with entire Dirichlet series in one or several complex variables (which essentially includes the entire power series as a particular case) so that certain classical distributions and processes like that of Poisson are included in the study. The results lead to some basic growth theorems in one and several complex variables (which are to be detailed in a paper scheduled for presentation at the Annual Conference of the Ind. Math. Soc., Dec., '80), and generalize a number of results known in the special case of power series.

DETAILED SUMMARY :

Let f be an entire Dirichlet series in C^k , the cartesian product of k -copies of the complex plane C (so that it converges over the entire C^k), specified by

$$f(z) = \sum_{n \in I} a_n e^{\lambda_n z}, \quad \forall z = (z_1, \dots, z_k) \in C^k,$$

where $I = I^k$ is the set of all ordered k -tuples of non-negative

* Presented by : K.H.S. Subrahmanyam.

integers, the net of coefficients $\{a_n; n \in I\}$ is in \mathbb{C} , the net of exponents $\{\lambda_n = (\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,k}); n \in I\}$ is in \mathbb{R}^k (satisfying other conditions detailed below) and " $\lambda_n z$ " denotes the formal Hadamard product $\prod_{j=1}^k \lambda_{n,j} z_j$.

We assume that the net of exponents $\{\lambda_n; n \in I\}$ is strictly increasing and $\rightarrow +\infty = (+\infty, \dots, +\infty)$ in \mathbb{R}^k i.e. the net is in \mathbb{R}^k and is such that $\lambda_m < \lambda_n$ whenever $m, n \in I$ and $m < n$ and $\lambda_{n,j} \rightarrow +\infty$ as $n \rightarrow +\infty$ for $j = 1, 2, \dots, k$, where $x = (x_1, \dots, x_k) < y = (y_1, \dots, y_k)$ for $x, y \in \mathbb{R}^k$ means that $x \neq y$ but $x_j \leq y_j$ for $j = 1, 2, \dots, k$ (and this relation restricted to I is understood to be directing the same during our references to nets indexed by I - confirming to General Topology by Kelley, J.L., Van Nostrand Co., Princeton (1955)).

We also assume that

$$\limsup_{\|n\| \rightarrow +\infty} (\log \|n\|) / \|\lambda_n\| < +\infty$$

where $\|z\|$ denotes $\sum_{j=1}^k |z_j|$, $\forall z \in \mathbb{C}^k$

Let the supremum modulus M , the maximum term μ and the central index $\nu = (\nu_1, \dots, \nu_k)$ of f be defined by

$$M(x) = \sup [|f(z)| : z \in \mathbb{C}^k \text{ and } \operatorname{Re} z = (\operatorname{Re} z_1, \dots, \operatorname{Re} z_k) = x],$$

$$\mu(x) = \max [|a_n| e^{\lambda_n x} : n \in I]$$

and

$$\nu_j(x) = \begin{cases} \max [n_j \in I^1: |a_n| e^{\lambda_n x} = \mu(x)], & \text{if } \mu(x) > 0 \\ 0 & \text{if } \mu(x) = 0 \end{cases}$$

for $1 \leq j \leq k$ and $\forall x \in \mathbb{R}^k$.

It is clear that the theory of such Dirichlet series includes that of power series in one and several complex variables, taking $\lambda_n = n$, $\forall n \in I$. And the extensive literature on Valiron-Wiman theory of entire power series in \mathbb{C}^1 and \mathbb{C}^k and that of entire Dirichlet series in \mathbb{C}^1 and their applications motivate the present work.

We establish the absolute convergence of the series f and consider the distributions and processes associated with the series defined by

$$F(x) = \sum_{n \in I} |a_n| e^{\lambda_n x}, \quad \forall x \in \mathbb{R}^k$$

Let \mathcal{F} denote the mapping over \mathbb{R}^k with values as probability distributions such that the distribution \mathcal{F}_x , associated with $x \in \mathbb{R}^k$ is the discrete distribution over \mathbb{R}^k having mass $|a_n| e^{\lambda_n x} / F(x)$ at each $n \in I$.

It is some time convenient to look upon \mathcal{F} as describing a multidimensional stochastic process over \mathbb{R}^k , the inter-relations between whose variates at different parametric points are left un-restricted. We write λ_n , for $n \in I$, to denote the n -th cumulant of \mathcal{F} (so that $\lambda_n(x)$ is the n -th cumulant of \mathcal{F}_x).

In the special case when $\lambda_n = n$, $\forall n \in I$, f is extensively studied by Gopala Krishna (see, Probabilistic techniques leading to a Valiron-type theorem in several complex variables, The Annals of Math. Stat. Vol.41 (1970) pp.2126-2129) and Gopala Krishna and Nagaraja Rao (see, Generalised Inverse and Probability Techniques and some fundamental growth theorems in C^k , J. Ind. Math. Soc. Vol.41 (1977), pp.203 - 219). This work extends all these results to the general context described above. As in the case of the references cited above we develop the relevant versions of Chebyshev's inequality, in k -dimensions, which involve the centres of mass, the dispersion matrices and their generalised inverses of the distributions of \mathcal{F} and establish a number of relations among the cumulants and the modal mass of the same. We finally arrive at the following asymptotic results, in particular :

THEOREM 1. The following three statements are equivalent :

- a. $\log F$ is of finite Ritt order, equivalently

$$\log F(x) \leq e^{\alpha x} \quad \text{as } x \rightarrow +\infty \quad \text{for some } \alpha \in \mathbb{R}^k$$
 where each $\alpha_j > 0$;
- b. each cumulant of \mathcal{F} is of finite Ritt order;
- c. there exists a positive integer p such each cumulant of degree p (i.e., X_n with $n_1 + \dots + n_k = p$ - usually referred to as '... of order p ') is of finite Ritt order.

REMARK 1. It is observed that in case F itself is of finite Ritt order (instead of $\log F$), the Dirichlet series under consideration reduces to a finite summation and all the distributions of \mathfrak{F} have only finitely many positive mass points.

THEOREM 2. Let $\log F$ be of finite Ritt order. Then the reciprocal of the modal mass of \mathfrak{F} , viz F/μ is of finite Ritt order.

THEOREM 3. $\log F(x) \sim \log \mu(x)$, as $x \rightarrow +\infty$ when $\log F$ is of finite Ritt order.

REMARK 2. Actually $\log F$ would be of finite Ritt order if the entire Dirichlet series f is of finite Ritt order i.e. $\log M$ is of finite Ritt order (consistent with the usage in the theory of a complex variable). But a justification of this would need the authors extension of the results of Gopala Krishna and Nagaraja Rao (mentioned above). If such extensions are accepted it further follows that

$$\log F(x) \sim \log M(x) \sim \log \mu(x)$$

as $x \rightarrow +\infty$, when f is of finite Ritt order.

AN OPTIMUM CONTROL CRITERION FOR A LIMITED CAPACITY QUEUEING SYSTEM

N.K. JAIN

Department of Mathematics,
Kurukshetra University, Kurukshetra-132119ABSTRACT:

The end of this paper is to develop an optimum control criterion for a single server limited capacity queueing system with Poisson arrivals and exponential service time distribution. To this purpose we obtain the joint distribution of the number of customers in the system at any time t and the number of customers lost in time interval $(0, t]$, therefrom the distribution of the number of customers lost in time interval $(0, t]$ is derived explicitly. The expectation and variance of the distribution are also found explicitly. Some relations between $G_N(t)$ and $E_N(t)$ are established which are considered to be of extra importance for further critical studies of the various queueing models. The existence and uniqueness of the optimum number, expressing the capacity size, is established.

INTRODUCTION:

The single server limited capacity queueing system with Poisson arrivals and exponential service distribution has widely been investigated for various important characteristics, viz; the number of customers in the system at any time t , [2]; the number of transitions made in time interval $(0, t]$, [5] and the

number of times the system reached a preassigned capacity in time interval $(0, t]$, [3]. This paper deals with the number of customers lost in time interval $(0, t]$; a customer arriving when the capacity of the system is fully occupied is considered lost for the system. The study of this characteristic of the system becomes more important and essential, especially, when it is desired to control the loss of such customers most economically. To this end we obtain the joint distribution of the number of customers in the system at time t and the number of customers lost in time interval $(0, t]$; therefrom the distribution of the number of customers lost in time interval $(0, t]$ is found explicitly. The expectation and variance for the distribution are also obtained explicitly. Eventually, we develop an optimum control criterion for the capacity size of the system keeping in view the loss of customers for the system. The existence and uniqueness of the optimum size of the capacity is established. However, in the way we have established some relations among $G_N(t)$, $E_N(t)$ and $Q_N(t)$ etc. which are of additional importance for further studies of queueing models.

MODEL

Consider a single server limited capacity queueing system with customers arriving 'at random' at instants $t_0 < t_1 < t_2 \dots$; the interarrival times $u_k = t_{k+1} - t_k$ ($k \geq 0$) are independent random variables having the common distribution function

$$dA(u) = \lambda e^{-\lambda u} du \quad (0 \leq u < \infty)$$

and the service times

being independent of the interarrival time. The capacity of the system is N . A customer arriving when there are already N customers in the system is considered lost for the system. To this purpose we assume that every $(N+1)$ th arrival to the system is exponentially served at the end of the queue with service rate $v \rightarrow \infty$, i.e. the customer is lost for the system with probability one. The server is busy if there is at least one customer in the system.

Let $X(t)$ and $Y(t)$ be the random variables denoting the number of customers in the system at time t and the number of customers lost in time interval $(0, t]$. The time t is reckoned from the instant when $X(0) = Y(0) = 0$.

DEFINITIONS AND NOTATIONS

Define

$$Q_{m,n}(t) = \text{Prob} [X(t) = m, Y(t) = n], 0 \leq m \leq N+1, 0 < n \quad (1)$$

Obviously, $Q_{0,n}(0) = \delta_{0,n}$ (Kronechor's delta),

and

$$Q_{m,\cdot}(t) = \sum_{n=0}^{\infty} Q_{m,n}(t) = \left\{ \begin{array}{l} \text{the probability that there} \\ \text{are } m \text{ customers in the system} \\ \text{at time } t, \end{array} \right.$$

$$Q_{\cdot,n}(t) = \sum_{m=0}^{N+1} Q_{m,n}(t) = \left\{ \begin{array}{l} \text{the probability that the} \\ \text{number of the customers lost} \\ \text{in time interval } (0, t] \text{ is } n. \end{array} \right.$$

The probability generating functions:

$$G_m(x, t) = \sum_{n=0}^{\infty} Q_{m,n}(t) x^n, \quad |x| \leq 1 \quad \dots (2)$$

$$H(x, y; t) = \sum_{m=0}^{N+1} G_m(x, t) y^m, \quad \dots (3)$$

$L_{2(a)}^x \equiv$ is an operator defined as

$$L_{2(a)}^x f(x) = f(x+a) - f(x-a)$$

and

$$L_{2(a,b,c,\dots)}^x \equiv L_{2(a)}^x L_{2(b)}^x L_{2(c)}^x \dots$$

Let $f^*(s)$ be the Laplace transform (L.T) of $f(t)$: given as

$$f^*(s) = \int_0^{\infty} e^{-st} f(t) dt, \quad \text{Re}(s) > 0.$$

JOINT DISTRIBUTION

$$\frac{d}{dt} Q_{0,n}(t) = -\lambda Q_{0,n}(t) + u Q_{1,n}(t), \quad 0 \leq n$$

$$\frac{d}{dt} Q_{m,n}(t) = -\alpha Q_{m,n}(t) + \lambda Q_{m-1,n}(t) + u Q_{m+1,n}(t), \\ 0 < m < N, \quad 0 \leq n$$

$$\frac{d}{dt} Q_{N,0}(t) = -\alpha Q_{N,0}(t) + \lambda Q_{N-1,0}(t)$$

$$\frac{d}{dt} Q_{N,n}(t) = -\alpha Q_{N,n}(t) + \lambda Q_{N-1,n}(t) + v Q_{N+1,n-1}(t) + u Q_{N+1,n}(t), \\ 1 \leq n$$

$$\frac{d}{dt} Q_{N+1,n}(t) = -(v+u) Q_{N+1,n}(t) + \lambda Q_{N,n}(t), \quad 0 \leq n \\ \text{--- (4)}$$

where $\alpha = \lambda + u$.

Taking the L.T. of (4), we have

$$\delta Q_{0,n}^*(s) = -\lambda Q_{0,n}^*(s) + u Q_{1,n}^*(s) + \delta_{0,n}, \quad 0 \leq n$$

$$\delta Q_{m,n}^*(s) = -\alpha Q_{m,n}^*(s) + \lambda Q_{m-1,n}^*(s) + u Q_{m+1,n}^*(s), \quad \begin{matrix} 0 < m < N, \\ 0 \leq n \end{matrix}$$

$$\delta Q_{N,0}^*(s) = -\alpha Q_{N,0}^*(s) + \lambda Q_{N-1,0}^*(s),$$

$$\delta Q_{N,m}^*(s) = -\alpha Q_{N,m}^*(s) + \lambda Q_{N-1,m}^*(s) + v Q_{N+1,m-1}^*(s) + u Q_{N+1,m}^*(s), \quad \begin{matrix} 1 \leq n \end{matrix}$$

$$\delta Q_{N+1,m}^*(s) = -(v+u) Q_{N+1,m}^*(s) + \lambda Q_{N,m}^*(s), \quad 0 \leq n$$

—(5)

Multiplying (5) by the appropriate powers of x and summing over n , we have

$$\delta G_0^*(x, s) = -\lambda G_0^*(x, s) + u G_1^*(x, s) + 1,$$

$$\delta G_m^*(x, s) = -\alpha G_m^*(x, s) + \lambda G_{m-1}^*(x, s) + u G_{m+1}^*(x, s),$$

$$\delta G_N^*(x, s) = -\alpha G_N^*(x, s) + (\lambda + vx) G_{N-1}^*(x, s) + u G_{N+1}^*(x, s), \quad 0 < m < N$$

$$\delta G_{N+1}^*(x, s) = -(v+u) G_{N+1}^*(x, s) + \lambda G_N^*(x, s).$$

—(6)

Again multiplying (6) by the appropriate powers of y and summing over m , we have

$$H^*(x, y, s) = \frac{y + u(y-1) - \lambda [v(y-x) + \lambda y(y-1)] y^{N+1} G_N^*(x, s) / (s+v+u)}{sy + (y-1)(u - \lambda y)} \quad (7)$$

Since

$$G_{N+1}^*(x, s) = \lambda G_N^*(x, s) / (s+v+u).$$

Putting the denominator of (7) equal to zero, we have the roots

$$\alpha_i(s) = \frac{(s+\alpha) \pm \sqrt{[(s+\alpha)^2 - 4\lambda u]}}{2\lambda}, \quad i=1,2. \quad (8)$$

Here the value of the square root with positive real part is taken, i.e., $\alpha_1(s)$ has $\text{Re}(s) > 0$. $\alpha_1(s)$ has the positive sign before the radical.

By Rouches' theorem it is trivial to show that the denominator of (7) has exactly one zero with absolute value < 1 . This zero must be $\alpha_2(s)$ since $|\alpha_2(s)| < |\alpha_1(s)|$ and there is no zero on $|y| = 1$ (see [6], p 89). Letting $\alpha_i(s) \equiv \alpha_i$, we have

$$\alpha_1 \alpha_2 = \beta^{-2}, \quad s = -\lambda (1 - \alpha_2)(1 - \alpha_1)$$

where

$$\beta = \sqrt{\lambda/u}$$

Since $H^*(x, y; s)$ is a polynomial in y , therefore, for $H^*(x, y; s)$

to remain finite, the numerator of (7) must vanish for $\alpha_i, i=1,2$, i.e. $u(1-\alpha_i) G_0^*(x, s) + \lambda [v(\alpha_i - x) + \lambda \alpha_i(\alpha_i - 1)] \alpha_i^{N+1} G_N^*(x, s) / (s+v+u) = 0$ (9)

Equations (9) give :

$$G_0^*(x, s) = \frac{\sum_{i=1}^2 (-1)^i [v(\alpha_{2-i} - x) + \lambda \alpha_{2-i} (\alpha_{2-i} - 1)] \alpha_{2-i}^N}{\sum_{i=1}^2 (-1)^i \lambda (1 - \alpha_{1+i}) [v(\alpha_{2-i} - x) + \lambda \alpha_{2-i} (\alpha_{2-i} - 1)] \alpha_{2-i}^{N+1}} \quad (10)$$

$$G_N^*(x, s) = \frac{(s+v+u)(\alpha_1 - \alpha_2)}{\sum_{i=1}^2 (-1)^{i+1} \lambda (1 - \alpha_{1+i}) [v(\alpha_{2-i} - x) + \lambda \alpha_{2-i} (\alpha_{2-i} - 1)] \alpha_{2-i}^{N+1}} \quad (11)$$

And we get

$$G_{N+1}^*(x, s) = \frac{\alpha_1 - \alpha_2}{\sum_{i=1}^2 (-1)^i (1 - \alpha_{1+i}) [v(\alpha_{2-i} - x) + \lambda \alpha_{2-i} (\alpha_{2-i} - 1)] \alpha_{2-i}^{N+1}} \quad (12)$$

Taking limit $v \rightarrow \infty$, we see that $G_{N+1}^*(x, s) = 0$, and

$$\begin{aligned} H^*(x, y; s) &= \sum_{m=0}^N G_m^*(x, s) y^m \\ &= \frac{y + u(1-y)G_0^*(x, s) - \lambda(y-x)y^{N+1}G_N^*(x, s)}{sy + (y-1)(u - \lambda y)} \end{aligned} \quad (13)$$

$$G_0^*(x, s) = \frac{\sum_{i=1}^2 (-1)^i (\alpha_{2-i} - x) \alpha_{2-i}^N}{\sum_{i=1}^2 (-1)^i \lambda (1 - \alpha_{1+i}) (\alpha_{2-i} - x) \alpha_{2-i}^{N+1}} \quad (14)$$

$$G_N^*(x, s) = \frac{\alpha_1 - \alpha_2}{\sum_{i=1}^2 (-1)^{i+1} \lambda (1 - \alpha_{1+i}) (\alpha_{2-i} - x) \alpha_{2-i}^{N+1}} \quad (15)$$

Now, $\frac{1}{n!} \frac{d^m}{dy^m} (H^*(x, y; s)) = G_m^*(x, s) = \frac{\beta^{2m}}{\lambda V(1)} [\lambda T(m) G_0^*(x, s) - V(m)]$, $0 \leq m \leq N$

-(16)

Where

$$V(m) = \alpha_1^m - \alpha_2^m,$$

$$T(m) = V(m+1) - \beta^{-2} V(m),$$

for $x = 0$, (16) gives

$$Q_{0,0}^*(s) = \frac{\sum_{i=1}^2 (-1)^i \alpha_{2-i}^{N+1}}{\sum_{i=1}^2 (-1)^i \lambda (1 - \alpha_{1+i}) \alpha_{2-i}^{N+1}} \quad (17)$$

Differentiating (14), n times with respect to x , setting $x = 0$ and simplifying, we have

$$Q_{0,n}^*(s) = [\beta^{-N} V(1)] \frac{\left[\sum_{i=1}^2 (-1)^i (1 - \alpha_{1+i}) \alpha_{2-i}^{N+1} \right]^{n-1}}{\left[\sum_{i=1}^2 (-1)^i \lambda (1 - \alpha_{1+i}) \alpha_{2-i}^{N+2} \right]^{n+1}} \quad (18)$$

Similarly from (16), we get

$$Q_{m,0}^*(s) = \frac{\beta^{2m}}{\lambda V(1)} [\lambda T(m) Q_{0,0}^*(s) - V(m)], \quad (19)$$

$$Q_{m,n}^*(s) = \frac{\beta^{2m}}{V(1)} T(m) Q_{0,n}^*(s), \quad 1 \leq n$$

Since $|((1 - \alpha_2) \alpha_1^{N+1})| > |(1 - \alpha_2) \alpha_2^{N+1}|$,

therefore, (19) and (20)

can be expanded as :

$$Q_{m,0}^*(s) = \left(\frac{1}{\lambda} \right) \sum_{\substack{(l, \omega, m-r, \omega, k) \\ (r, i, j, k, l)}} (-1)^{R+r+1} \binom{R}{l} \binom{R+i}{i} \beta^{(m-r-i-l+1)} \times \\ \sum_{l=0}^p \binom{p}{2(N+1)} R^{-(p+1)} - \beta^{m+1} \sum_{i=0}^{m-1} R^{-(\alpha_1^0 - m + 1)} \quad (21)$$

$$Q_{m,n}^*(z) = \left(\frac{1}{\lambda}\right) \sum_{(r,i,j,k,l)} \frac{(1,\omega,n-1,\omega,j+k)}{(-1)^{j+l+r+1} \binom{n-1}{j} \binom{n+k}{k} \binom{j+k}{l}} \times \binom{j+k+l+1}{i} \beta^{(m+n-r-i-l+1)q} L_{2(r-m-1,1)} R^{-q}$$

Where

$$p = 2[(j+1)+k(N+2)] - [m-r-i+l],$$

$$q = 2[(j+1)(N+1)+k(N+2)] + [n+i-l+1],$$

$$R = \beta \alpha_1,$$

and

| | | | |
|-------------|-----|-----|-----|
| (a,b,c,...) | a | b | c |
| Σ | ≡ | Σ | Σ |
| (i,j,k,...) | i=0 | j=0 | k=0 |

Using the Erdélyi tables [1], we have the inverse L.T's of (21) and (22)

$$Q_{m,0}(t) = \sum_{(r,i,j,k,l)} \frac{(1,\omega,m-r,\omega,k)}{(-1)^{R+r} \binom{k}{l} \binom{k+i}{i} \beta^{(m-r-i-l)X}} \times L_{2(N+1,1)}^p I_{p+N} (2\sqrt{\lambda u} t) e^{-\alpha t} - \frac{\beta^{m+1}}{\lambda} \sum_{i=0}^{m-1} (2i-m+1) X \quad (23)$$

$$Q_{m,n}(t) = \sum_{(r,i,j,k,l)} \frac{(1,\omega,n-1,\omega,j+k)}{(-1)^{j+l+r} \binom{n-1}{j} \binom{n+k}{k} \binom{j+k}{l} \binom{j+k+i+1}{i} X} \beta^{(m+n-r-i-l)q} L_{2(r-m-1,1)}^q I_q (2\sqrt{\lambda u} t) e^{-\alpha t} \quad (24)$$

Note that

$$(2v/z)I_{\gamma}(z) = I_{v-1}(z) - I_{v+1}(z).$$

NUMBER OF CUSTOMERS LOST

Putting $y = 1$, in (13), we obtain

$$H^*(x, 1; s) = (1/s) \left[1 + \frac{(x-1)(\alpha_1 - \alpha_2)}{\sum_{i=1}^2 (-1)^{i+1} (1 - \alpha_{1+i}) (\alpha_{2-i}^{-x}) \alpha_{2-i}^{N+1}} \right] \quad (26)$$

For $x = 0$, (26) gives

$$Q_{\cdot, 0}^*(s) = (1/s) \left[1 - \frac{\alpha_1 - \alpha_2}{\sum_{i=1}^2 (-1)^{i+1} (1 - \alpha_{1+i}) \alpha_{2-i}^{N+1}} \right] \quad (27)$$

Differentiating (26), n times with respect to x , setting $x = 0$ and simplifying, we get

$$Q_{\cdot, n}^*(s) = V(1) V(N+1) \frac{\left[\sum_{i=1}^2 (-1)^i (1 - \alpha_{1+i}) \alpha_{2-i}^{N+1} \right]^{n-1}}{\left[\sum_{i=1}^2 (-1)^i (1 - \alpha_{1+i}) \alpha_{2-i}^{N+2} \right]^{n+2}} \left(\frac{1}{\lambda} \right), \quad 1 \leq n \quad (28)$$

Expanding (27) and (28) into series, we get

$$Q_{\cdot, 0}^*(s) = (1/s) \left[1 + \sum_{(i, j, k)}^{(\infty, \infty, j)} (-1)^R \binom{j}{k} \binom{j+i}{i} \beta^{(N-i+k+1)} L_{2(1)}^{\gamma} R^{-\gamma} \right] \quad (29)$$

$$Q_{\cdot, n}^*(s) = (1/\lambda) \sum_{(i, j, k, l)}^{(\infty, n-1, \infty, j+k)} (-1)^{j+l} \binom{n-1}{j} \binom{n+k}{k} \binom{j+k}{l} \binom{j+k+i+1}{i} \beta^{(N+n-i-l+1)} L_{2(1, N+1)}^{\gamma} R^{-\gamma} \quad (30)$$

here $r = (2j+1)(N+2) + (i-k)$.

Taking the inverse L.T's of (29) and (30), we have

$$q_{.,0}(t) = 1 - \lambda \sum_{(i,j,k)}^{(\infty, \infty, j)} (-1)^k \binom{j}{k} \binom{j+i}{i} \beta^{(N-i+k)} \int_0^t dW L_{2(1,1)}^r I_r(2\sqrt{\lambda u} W) e^{-\alpha W} \quad (31)$$

$$q_{.,n}(t) = \sum_{(i,j,k,l)}^{(\infty, n-1, \infty, j+k)} (-1)^{j+l} \binom{n-1}{j} \binom{n+k}{k} \binom{j+k}{l} \binom{j+k+i+1}{i} \beta^{(N+n-i-l)} \\ \times L_{2(1, N+1, -1)}^\alpha I_\alpha(2\sqrt{\lambda u} t) e^{-\alpha t} \quad (32)$$

Expectation:

Let $G_N(t)$ be the expected number of customers lost in time interval $(0, t]$ when the capacity of the system is N . Then

$$G_N(t) = \lambda^2 \sum_{k=0}^{\infty} \beta^{N-1} \int_0^t (t-w) L_{2(1,1)}^k I_k(2\sqrt{\lambda u} W) e^{-\alpha W} dW \quad (33)$$

Properties:

$$(1) \quad G_N'(t) = \lambda^2 \sum_{k=0}^{\infty} \beta^{N-1} \int_0^t L_{2(1,1)}^k I_k(2\sqrt{\lambda u} W) e^{-\alpha W} dW \quad (34)$$

$$= \lambda q_{N, \cdot}(t).$$

since

$$q_{N, \cdot}(t) = \lambda \sum_{k=0}^{\infty} \beta^{N-1} \int_0^t L_{2(1,1)}^k I_k(2\sqrt{\lambda u} W) e^{-\alpha W} dW \quad (35)$$

$$G_N''(t) = \lambda^2 \sum_{k=0}^{\infty} \beta^{N-1} L_{2(1,1)}^k I_k(2\sqrt{\lambda u} t) e^{-\alpha t} \quad (36)$$

(2)

Let $E_N(t)$ be the expected number of times the system reaches its capacity N in time interval $(0, t]$. Then

$$E_N(t) = \beta^{-2} G_N(t) + Q_{N, \cdot}(t), \quad (37)$$

where

$$E_N(t) = \lambda \sum_{k=0}^{\infty} \beta^{N-1} \int_0^t [1 + u(t-w)] L_{2(l,l)}^h \Gamma_h(2\sqrt{\lambda u} w) e^{-\alpha w} dW \quad (38)$$

(i) Differentiating (27), w.r.t, t, we have

$$G_N''(t) + \beta^{-2} G_N'(t) = \lambda Q_{N-1, \cdot}(t), \quad (39)$$

Using (34) and (36) in (39), we can get $Q_{N-1, \cdot}(t)$ explicitly.

(ii) From (37), clearly

$$G_N(t) < \beta^2 E_N(t), \quad 0 < t < \infty \quad (40)$$

hence $\beta^2 E_N(t)$ is an upper bound of $G_N(t)$.

(iii) Evidently,

$$\begin{aligned} |E_N(t) - \beta^{-2} G_N(t)| &< 1, \\ \text{or } |\lambda E_N(t) - \mu G_N(t)| &< \lambda, \end{aligned} \quad \dots(41)$$

$$\text{(iv) } \lim_{t \rightarrow \infty} [E_N(t) - \beta^{-2} G_N(t)] = \lambda Q_{N, \cdot} \quad \dots(42)$$

where

$Q_{N, \cdot} \equiv$ The steady state probability that there are N customers in the system.

(v) If $Q_{N, \cdot}$ exists, then in any interval (t_1, t_2) , we have

$$G_N(t_2 - t_1) = (t_2 - t_1) \lambda Q_{N, \cdot}, \quad \dots(43)$$

Similarly, we have

$$E_N(t_2 - t_1) = (t_2 - t_1) \lambda Q_{N-1, \cdot}, \quad \dots(44)$$

Therefore, (43) and (44) give

$$G_N(t_2 - t_1) = \beta^2 E_N(t_2 - t_1) \quad \dots(45)$$

(3) Replacing N by $N+k$, $k = 1, 2, 3, \dots$, in $G_N(t)$, we see that $G_{N+k}(t)$ is a monotonically decreasing function of k , i.e.

$$G_{N+k}(t) > G_{N+k+1}(t) \quad \dots (46)$$

$$\text{and } \lim_{k \rightarrow \infty} G_{N+k}(t) = 0 \quad \dots (47)$$

Variance

Let $V_N(t)$ be the variance of the distribution of the number of customers lost in time interval $(0, t]$, when the capacity of the system is N .

We know that

$$V_N(t) = \sum_{n=2}^{\infty} n(n-1) Q_{\cdot, n}(t) + G_N(t)[1-G_N(t)] \quad \dots (48)$$

Now, to obtain $V_N(t)$ explicitly it suffices to calculate the first term on the right hand side of (48), since the second term is already known. Therefore

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1) Q_{\cdot, n}(t) &= (1/\lambda) G_N(t) - \lambda \sum_{k=0}^{\infty} \beta^{N-2(k+1)} \\ &\quad \times \int_0^t (t-w)^L {}_2^h(-N, -1, -1) I_h(2\sqrt{\lambda} w) e^{-\alpha w} dw \quad \dots (49) \end{aligned}$$

OPTIMUM CONTROL CRITERION

Let C_1 be the average cost associated with each customer and C_2 be the average cost associated with each additional capacity unit provided to the system for a unit time. Then

$$C_1 G_{N+k}(t) = \text{The expected cost incurred when the capacity of the system is } N+k,$$

$$C_2 kt = \text{The expected cost incurred on account of the } k \text{ additional capacity units provided to the system for time } t.$$

Now, it is profitable to provide k additional capacity units to the system only if

$$\frac{C_2}{C_1} = \bar{C} < \frac{G_N(t) - G_{N+k}(t)}{kt}, \quad (50)$$

Define a cost function

$$Z(k) = C_1 G_{N+k}(t) + C_2 kt \quad (51)$$

with the constraint that the capacity of the system cannot be less than N (a pre-assigned capacity).

Now we are interested in finding a number $k \in \mathbb{N} = [1, 2, \dots]$ such that $Z(k)$ is minimum i.e.

$$Z(k+1) \geq Z(k) \leq Z(k-1)$$

$$\text{or } G_{N+k-1}(t) - G_{N+k+1}(t) \geq \bar{C}t \quad \dots (52)$$

$$G_{N+k}(t) - G_{N+k+1}(t) \leq \bar{C}t$$

The $k \in \mathbb{N}$ and satisfying (52) is said to be the optimum k and is denoted by k_0 . This k_0 provides the optimum control of the customers.

The k_0 is unique and exists if any only if

$$G_N(t) - G_{N+1}(t) > \bar{C}t > 0, \quad \dots (53)$$

Proof:- See 4

Numerical Illustration:

The purpose of this section of the paper is to illustrate, numerically, the use of our theoretical study by taking various parameters arbitrarily. The computed values of $E_{N+k}(t)$, $G_{N+k}(t)$ and $Q_{N+k}(t)$ are given in the table below. Here $N = 4$, $k = 1, 2, \dots, 5$. $t = 10$ units, $\mu = 3.2$

TABLE

| $k \backslash \lambda$ | $E_{N+k}(t)$ | | | | $\beta^{-2} G_{N+k}(t)$ | | | | $G_{N+k, \cdot}(t)$ | | | |
|------------------------|--------------|--------|--------|--------|-------------------------|--------|--------|--------|---------------------|--|--|--|
| | 3.4 | 3.2 | 3.0 | 3.4 | 3.2 | 3.0 | 3.4 | 3.2 | 3.0 | | | |
| 1 | 1.3140 | 1.1247 | 0.9428 | 1.2655 | 1.0831 | 0.9047 | 0.0485 | 0.0416 | 0.0381 | | | |
| 2 | 1.0551 | 0.8769 | 0.7097 | 1.0128 | 0.8416 | 0.6787 | 0.0423 | 0.0353 | 0.0310 | | | |
| 3 | 0.8626 | 0.6943 | 0.5411 | 0.8249 | 0.6637 | 0.5153 | 0.0377 | 0.0306 | 0.0258 | | | |
| 4 | 0.7030 | 0.5471 | 0.4099 | 0.6692 | 0.5205 | 0.3885 | 0.0338 | 0.0266 | 0.0214 | | | |
| 5 | 0.5668 | 0.4262 | 0.3070 | 0.5369 | 0.4034 | 0.2894 | 0.0299 | 0.0228 | 0.0176 | | | |
| 6 | 0.4520 | 0.3283 | 0.2272 | 0.4258 | 0.3090 | 0.2130 | 0.0262 | 0.0193 | 0.0142 | | | |

8

It is quite evident from the table that the values of $E_{N+k}(t)$, $G_{N+k}(t)$ and $Q_{N+k}(t)$ are decreasing (increasing) as the k is increasing (decreasing). Moreover, these values behave in accordance with λ . Also, clearly, $G_{N+k}(t)$ is a monotonically decreasing function of k .

Remarks:

It may specifically be mentioned that with the use of the equation (37), $G_N(t)$, $E_N(t)$ and $Q_N(t)$ may easily be computed for given any two of the three and λ, μ . Also the criterion provides us a tool to find the optimum capacity size for the system such that the loss of the customers is most economical.

ACKNOWLEDGEMENTS

My indebtedness is due to Professor C.Mohan, Head of the Department of Mathematics, Kurukshetra University, Kurukshetra for the able guidance in the investigation of the problem.

REFERENCES:

- 1 Erdelyi, A.et. al. (1954) Tables of integral Transforms! Vol.I, McGraw-Hill Book Company, Inc., New York.
- 2 Heathcote, C.R. and Moyal, J.E.(1959) The Random walk in continuous time and its applications to the Theory of Queues! Biometrika, Vol.16 pp.400-411.
- 3 Jain,N.K. (1979) Study of some Queues with special Reference to the development of Optimality Criteria! Ph.D. thesis, Department of Mathematics, Kurukshetra University, Kurukshetra. (India).

- 4 _____ An Optimality Criterion for a Limited Capacity Queueing System; Math. Operations for sch. Statist. Ser. Optimization, Vol.10 (1979) No.4, 597-611.
- 5 Neuts, M.F.
(1964) The Distribution of the Maximum length of Poisson Queue during a Bussy period Opns. Res. Vol.12,pp.281-285.
- 6 Saaty, T.L.(1961) Elements of Queueing Theory with Application; McGraw Hill Book Company, New York. '

STOCHASTIC INTERPRETATION OF van LEEUWEN THEOREM ON THE ABSENCE
OF DIAMAGNETISM IN CLASSICAL ELECTRON GAS. *

A.M. JAYANNAVAR and N. KUMAR
Department of Physics, Indian Institute of Science,
Bangalore 560 012, India.

ABSTRACT:

Following a real space-time approach based on the stochastic Langevin equation, we have examined explicitly the role of confinement in causing the orbital diamagnetism of a classical system of charged particles in equilibrium. Our results reaffirm the absence of diamagnetism in such a system even for arbitrarily weak confinement, and thus complement the classic theorem due to van Leeuwen. The present treatment also predicts an experimentally verifiable transient effect namely flux expulsion associated with the abrupt insulator - to - semimetal transition, induced by pressure, in the presence of static magnetic field.

The equilibrium statistical mechanics of a classical system of charged particles in an external magnetic field was treated by van Leeuwen ¹⁾ in a now classic paper in which she had demonstrated the absence of orbital diamagnetism in such a system in general. The essential point of the proof of the theorem is that

* Presented by : A.M. JAYANNAVAR.

the partition function becomes independent of the magnetic field implying vanishing susceptibility, and hence the absence of diamagnetism. This can readily be seen by recalling that the magnetic field \vec{B} , or equivalently the vector potential \vec{A} , can be incorporated in the Hamiltonian by the canonical replacement $\vec{p} \rightarrow (\vec{p} - \frac{e}{c} \vec{A})$. Thus the vector potential disappears when the momenta are integrated over in the evaluation of the partition function. (This partial tracing over the momenta is, of course, not possible in a quantum mechanical treatment owing to the commutation problem). The demonstrated absence of the classical diamagnetism is, however, somewhat paradoxical in as much as each particle must trace a cyclotron orbit and, therefore, contribute the diamagnetic moment of the type $(-|e|/2c) (\vec{v} \times \vec{r})$. This was resolved by Lorentz²⁾ by noting that the cuspidal orbits of the electron skipping the boundary generate a paramagnetic moment equal and opposite to that due to the carriers in the bulk. The statistical mechanical treatment of van Leeuwen, however, makes no appeal to such a boundary effect. This may at once be taken as the strength and the weakness of the equilibrium statistical mechanical approach. The purpose of this paper is thus two-fold: First, to make explicit the role of the confinement (boundary) via a real space-time approach, and secondly to study the effect of a finite life-time of the carriers. Our main results are that (i) the equilibrium orbital magnetic moment of a classical but unbounded system of charged

particles is non-zero, (ii) on the introduction of a confining potential, however weak, the equilibrium diamagnetism vanishes, and χ stays zero for arbitrarily weak confinement, that is even in the limit as the strength of the confining potential tends to zero. This should imply that a confinement, or equivalently a boundary, is required only virtually.. This is presumably implicit but obscured in any equilibrium statistical mechanical treatment, such as the one due to van Leeuwen. In this real space-time approach we shall follow the 'time-dependent statistical mechanics' a la Langevin³⁾. We shall first consider the case of an infinitely extended system without any confining potential, i.e., no boundary. Consider the stochastic evolution of a classical system of identical charged particles e.g. a non-degenerate electron gas, each having mass m and charge $-|e|$ in the presence of a uniform external magnetic field of magnitude B_0 directed along the positive z -axis. The motion of an arbitrarily chosen 'test' particle in the plane perpendicular to the field is described by the Langevin equation³

$$m\ddot{x} = -\Gamma\dot{x} - \frac{|e|}{c} B_0 \dot{y} + f_x(t) \quad \dots (1)$$

$$m\ddot{y} = -\Gamma\dot{y} + \frac{|e|}{c} B_0 \dot{x} + f_y(t) \quad \dots (2)$$

where the fluctuating random force-field $f_\alpha(t)$ is a gaussian white-noise process, i.e.,

$$\langle f_\alpha(t) f_\beta(t') \rangle = A \delta_{\alpha\beta} \delta(t-t') \quad \dots(3a)$$

with $\alpha, \beta = x, y$. Here

$$A = 2K_B T \Gamma / m^2 \quad \dots(3b)$$

to ensure asymptotic thermalization, i.e.,

$$\frac{1}{2} m \langle \dot{x}^2 \rangle = \frac{1}{2} m \langle \dot{y}^2 \rangle = \frac{1}{2} K_B T$$

as $t \rightarrow \infty$. Here the angular bracket denotes subensemble average for an arbitrary chosen initial condition, namely

$\dot{x} = u_0, \dot{y} = v_0$ at $t = 0$. Γ is the frictional coefficient.

Multiplying Eq. (2) by $i (= \sqrt{-1})$ and adding to Eq. (1), we get

$$\ddot{z} = -\beta \dot{z} + F(t), \quad \dots(4)$$

where $\beta = (\omega_r - i \omega_c)$ with $\omega_r = \Gamma/m$ and $\omega_c = |e| B_0/mc$ the cyclotron frequency, and $F(t) = (1/m) [f_x(t) + i f_y(t)]$

We note that $z = x + iy$ occurs holomorphically in Eq. (4).

This simplifies the problem considerably. The quantity of interest is the magnetic moment

$$\langle m(t) \rangle = \frac{|e|}{2c} \langle \vec{v} \times \vec{r} \rangle = -\frac{|e|}{2c} \text{Im} \langle z^* \dot{z} \rangle \quad \dots(5)$$

where Im denotes imaginary part of the expression that follows.

Formal solution of Eq.(4) can be written as

$$Z = \frac{\dot{z}_0}{\beta} (1 - e^{-\beta t}) + \int_0^t d\eta e^{-\beta \eta} \int_0^\eta d\xi e^{\beta \xi} F(\xi) \quad \dots(6)$$

where we have, without loss of generality, assumed the

particle to be initially at the origin. Here $\dot{z}_0 = u_0 + i v_0$.

From Eqs. (5) and (6) and recalling that

$$\langle F(\xi)F^*(\eta) \rangle = \frac{4k_B T \Gamma}{m^2} \delta(\xi - \eta)$$

we get

$$\begin{aligned} \langle m(t) \rangle = & - \left\{ |\dot{z}_0|^2 \frac{|e|}{2c} \frac{1}{(\omega_r^2 + \omega_c^2)} e^{-\omega_r t} \left[\omega_r \sin \omega_c t \right. \right. \\ & \left. \left. - \omega_c (\cos \omega_c t - e^{-\omega_r t}) \right] + \frac{|e|}{mc} \frac{k_B T \omega_c}{(\omega_r^2 + \omega_c^2)} (1 - e^{-2\omega_r t}) \right. \\ & \left. - \frac{2|e|}{mc} \frac{k_B T \omega_r}{(\omega_r^2 + \omega_c^2)} e^{-\omega_r t} \sin \omega_c t \right\} \dots (7) \end{aligned}$$

It is readily seen now that $\langle m(t) \rangle$ tends to a non-zero value as $t \rightarrow \infty$. More explicitly, the magnetic moment per particle in equilibrium,

$$\langle m(t) \rangle_{t \rightarrow \infty} = \frac{-|e|}{mc} \frac{k_B T \omega_c}{(\omega_r^2 + \omega_c^2)} \dots (8)$$

This clearly shows that for an unbounded classical system of charged particles, one indeed has a non-zero, in fact very large, diamagnetic susceptibility. This result brings into sharp focus the question of boundedness of the system.

We will now examine explicitly the effect of confinement by introducing a confining potential. For simplicity we shall take it to be a harmonic potential $\sum_i \frac{1}{2} \omega^2 (x_i^2 + y_i^2)$ where

summation runs over all the particles. Now Eq.(4) for the arbitrarily chosen particle gets modified to

$$\ddot{z} = -\beta \dot{z} - \omega^2 z + F(t) \quad \dots(9)$$

where the meaning of the various terms remains unchanged.

Proceeding as before, the expression for the magnetic moment for non-zero ω now turns out to be

$$\langle m(t) \rangle = \frac{-|e|}{2c} \operatorname{Im} \left\{ \frac{|\dot{z}_0|^2}{(\mu_1 - \mu_2)(\mu_1^* - \mu_2^*)} \left[\mu_1 \exp(\mu_1 + \mu_1^*)t + \mu_2 \exp(\mu_2 + \mu_2^*)t - \mu_2 \exp(\mu_1^* + \mu_2)t - \mu_1 \exp(\mu_2^* + \mu_1)t \right] + \frac{f_0}{(\mu_1 - \mu_2)(\mu_1^* - \mu_2^*)} \left[\frac{\mu_1}{(\mu_1^* + \mu_1)} \exp(\mu_1^* + \mu_1)t + \frac{\mu_2}{(\mu_2^* + \mu_2)} \exp(\mu_2^* + \mu_2)t - \frac{\mu_2}{(\mu_1^* + \mu_2)} \exp(\mu_1^* + \mu_2)t - \frac{\mu_1}{(\mu_2^* + \mu_1)} \exp(\mu_2^* + \mu_1)t \right] \right\}$$

Where $\mu_1 = -\frac{1}{2}\beta + (\beta^2/4 - \omega^2)^{1/2}$, $\mu_2 = -\frac{1}{2}\beta - (\beta^2/4 - \omega^2)^{1/2}$, $f_0 = 4K_B T \omega_1 / m v$ --- (10)

It can readily be seen that the expression in Eq.(10) does verify that $\langle m(t) \rangle$ goes to zero as t tends to infinity in conformity with the van Leeuwen theorem. What is remarkable is that this remains true for arbitrarily small, but non-zero, value of the confining potential, i.e., arbitrary small ω . The order of limits is, however, important. Indeed, one verifies

$$\lim_{\omega \rightarrow 0} (\lim_{t \rightarrow \infty} \langle m(t) \rangle) \rightarrow 0 \quad \dots(11a)$$

but $\lim_{t \rightarrow \infty} (\lim_{\omega \rightarrow 0} \langle m(t) \rangle) \not\rightarrow 0 \quad \dots(11b)$

Since the limit $t \rightarrow \infty$ corresponds to equilibrium the limits in Eq.(11a) only are physically meaningful. This order of limits is physically quite understandable in that only for $t \gg \Gamma/m\omega$ does the particle sense the confinement. This is an exact result. The conceptually important point to note here is that $\langle m(t) \rangle \rightarrow 0$ as $t \rightarrow \infty$ for arbitrarily small ω . This implies that confinement is necessary only virtually. This is presumably subsumed in the statistical mechanical treatment of van Leeuwen. It must be noted that the arbitrarily chosen initial value $|\dot{z}_0|^2 = u_0^2 + v_0^2$ in Eq.(10) should be replaced by the thermal average, i.e.

$\langle \dot{z}_0 \rangle^2 = 2 K_B T/m$. This is so because the stochastic evolution is regarded as an ongoing process having started in the infinite past.

The explicit time dependence of $\langle m(t) \rangle$ given by Eq.(10) has an observable significance for the following rather interesting experimental situation. We consider an insulator undergoing a pressure induced discontinuous insulator-to-semimetal (or semiconductor) transition in the presence of a static external magnetic field. Here the system of charged carriers is effectively created at time $t = 0$ and the induced orbital magnetic moment should evolve essentially according to Eq.(10) starting with $\langle m(t) \rangle = 0$ at $t = 0$ and approaching equilibrium value $\langle m(t) \rangle = 0$ asymptotically as $t \rightarrow \infty$. We suggest such a transient experiment in which diamagnetism will

manifest as expulsion of static magnetic flux threading the sample.

ACKNOWLEDGEMENTS

One of us (AMJ) would like to thank Dr.H.R.Krishnamurthy for useful discussions and UGC for financial assistance. The other (NK) would like to thank Dr. S.D.Mahanti for helpful comments.

REFERENCES:

- 1) J.H. Van Leeuwen, J.de Physique 2, (1921), 361
- 2) See J.H. van Vleck, The theory of Electric and Magnetic susceptibilities, Oxford University Press (1932) p 102.
- 3) S.Chandrasekhar, Rev. Mod.Phys. 15 (1943), 1.

ON THE CONCEPTS OF PROBABILITY IN CLASSICAL AND QUANTUM THEORIES,
AND AN ALTERNATIVE INTERPRETATION OF QUANTUM THEORY.

P. NUTHAKKI
Department of Physics,
Andhra University, Waltair, (AP) 530 003, India.

ABSTRACT :

As a fundamental principle, we assume that (I) similar phenomena in similar circumstances proceed in similar manner. A special case of (I) is that same phenomena in same circumstances proceed in same manner. Hence, any physical phenomenon in its associated circumstances proceeds only in one manner, i.e., it takes only one course of action. Thus, (i) physical phenomena are governed by the laws of certainties, and not of chance (probabilities). Hence, the concept that the experimental apparatus selects out of a large number of possibilities one of actuality is incorrect; it may only change the course of action. The probability description (such as any toss or all tosses of a perfect coin have the equal probability to get the head and tail, etc.) is incorrect. On the otherhand, the statistical description (such as the tosses and the circumstances favour the outcome of approximately the same number of heads and tails in a large number of tosses, etc.) is correct; it is an incomplete description. Uncertainties consist of physical and nonphysical parts; only the former plays part in physical phenomena. The later is actually zero, but takes in general

finite values. The mathematical quantities in the so-called configuration space, etc., correspond, either directly or indirectly to real physical quantities in the 3-dimensional physical world. Hence, the concept of such spaces, etc., are incorrect. It is a secondary point to know whether one can construct a complete theory in a given case. The primary point is to recognise the correctness of the general principles such as (I) and their consequences such as (i). The principle (I) also plays a fundamental role in the proposed alternative interpretation of special and general relativity theories.

1. INTRODUCTION :

In previous considerations⁽¹⁾, an alternative interpretation of special relativity, general relativity and quantum theories is proposed, basing on common principles. This is further developed here as regards the quantum theory. The presently accepted interpretation of the quantum theory and some proposed alternative interpretations along with their references can be found, for example, in refs. (2), (3) and so on. It is gratifying to know that some of the ideas expressed here are similar to the ones given in these references.

Some of the implicit or explicit statements (abbreviated hereafter as Sts.) of the presently accepted interpretation of the quantum theory are as follows :

(A) The same thing need not happen under the same conditions.

(B) The probability is a new kind of "objective" physical reality and it is closely related to the concept of possibility, probability wave is related to an individual process and can be conceived as representing a statistical assembly only in so far as the experiment concerned can be repeated as often as desired.

(C) The individual and large number of physical phenomena are governed by the laws of chance (probabilities).

(D) The description of physical phenomena in terms of probabilities is a complete description.

(E) Uncertainties are inherent in Nature.

(F) Matter, say an electron, exhibits some times particles aspects and at other times wave aspects, and so on.

We shall state later some more points of the usual interpretation and discuss them as we go along. It may be noted here that the usual Sts. (A) - (F) are inter-dependent. For example, (C) is valid if (A) is valid, and vice versa, and so on.

2. THE PRINCIPLE OF PHYSICAL PHENOMENA.

As a fundamental law of Nature, we assume that (I) similar phenomena in similar circumstances proceed in similar manner. Some special cases of (I) are that (1) equivalent phenomena in equivalent circumstances proceed in equivalent manner, (2) same phenomena in same circumstances proceed in some manner, etc. The St. (I) can also be stated in another way :

The laws of similar phenomena in similar circumstances take the same mathematical form. It may be noted that (I) includes the principle of causality, etc. The St. (I), which may be called the Principle of Physical Phenomena, is true for every physical property. It is true even for the so-called random motions, fluctuations, etc.

To start with, let us note that we use the term process in a broad sense. For example, an electron (or a clock) may be considered as a process or a bundle of processes, depending upon the case we consider. Moreover, when we say that a process proceeds (developes), we mean that it is actual, i.e., something that happens. Whatever that happens is covered by the term proceeds.

It is well known that there are many physical processes in the universe interacting with each other. Any physical process is in interaction with some other physical processes. The processes which influence a given process (in the time interval under consideration) may be considered as circumstances associated with that process. Thus, any physical process (between any two instants) is associated with a set of circumstances. It is incorrect to consider a physical process without its associated circumstances. For, it is in interaction with the circumstances. There is no such thing as a physical process that is free from or independent of its associated circumstances. The development of a physical process in its associated circumstances is always to be considered, to be

specific, between two instants. The time interval is an important factor. In what follows, we always assume some circumstances and some time interval irrespective of whether they are mentioned explicitly or not.

It may be noted that we generally make relative comparison of physical processes. In this case, it may be sufficient to take into account only those circumstances that influence the relative comparison of physical processes under investigation. Of course, the circumstances are not static, but change with time. The interaction of the process with the circumstances changes not only the process, but also the circumstances. It is well known that the experimental apparatus influence the process under investigation. Thus, whenever measurements are made, one has to include the apparatus also in the circumstances. One has to take into account all such factors.

Let us now consider two processes p_1, p_2 in their associated circumstances C_1, C_2 respectively, for convenience, between the same two (events) instants T_i and T_f . The idea is to see whether the processes P_1, p_2 in their associated circumstances C_1, C_2 proceed in the same manner between T_i and T_f or not, when the processes p_1, p_2 are (a) the same in every respect at T_i , and the circumstances C_1, C_2 are (b) the same in every respect at T_i and (c) change by the same (external) factors at any instant between T_i and T_f . One may combine (b) and (c), and say that the circumstances C_1, C_2 are the same at any instant between

T_i and T_f . If one or more of (a)-(c) are not satisfied, then the processes p_1 , p_2 cannot obviously proceed in the same manner; there is nothing special about it. Since it is incorrect to consider a physical process without its associated circumstances, it is necessary to satisfy (b) for the validity of (a), and vice versa. Hence, (a)-(c) are satisfied only if, at any instant between T_i and T_f , the interaction of p_1 with C_1 is the same in every respect as the interaction of p_2 with C_2 . Thus, unless the above conditions are satisfied, it is incorrect to say that p_1 , p_2 are the same and C_1 , C_2 are the same. If the above conditions are satisfied, then p_1 , p_2 are the same and C_1 , C_2 are the same, and they proceed in the same manner. If they do not proceed in the same manner, then one or more of the above conditions are not satisfied and they are not the same. In any case, the usual St. (A) is incorrect.

3. THE LAWS OF CERTAINITIES

From the above discussion we see that the same physical process in the same circumstances proceeds in the same manner. Thus, (3) any physical process in its associated circumstances proceeds only in one manner, i.e., it takes only one course of action. The course of action followed is, for example, the motion of a particle in a straight line or in a zig-zag path. The point is that the same path is followed whenever everything is exactly the same.

Let us now consider the probability statement that (a) in any single toss of a perfect coin, the probability to get the

head is $P_h = 0.5$, and the probability to get the tail is $P_t = 0.5$, i.e., there is equal probability to get the head and tail; the head and tail are equally likely. If the toss and its associated circumstances between the two instants T_i and T_f of the beginning and the end of the toss favour equally the head and tail, then such a probability statement is correct. In this case, we would get neither a head nor a tail. This is not what happens. The correct statement is that (a') the toss and its associated circumstances between the two instants T_i and T_f favour the outcome of one and not the other; as a result, it has a certainty to happen in one way and not in the other way. Thus, in any given toss in its associated circumstances between T_i and T_f , the coin takes only one course of action. The course of action followed is the one that happens between T_i and T_f . A motion picture of the coin as it moves between T_i and T_f gives approximately the course of action followed by the coin. The important point is to consider the net effect of the entire course of action of the toss and its associated circumstances between T_i and T_f . They favour, as mentioned before, the outcome of the one and not the other. Hence, in any case, the probability St. (a) is incorrect. Similarly, the probability statement that (b) all tosses of a perfect coin have the same probability to get the head and tail is incorrect. The St. (b) becomes correct if all the tosses and their associated circumstances are the same, and they favour equally the head and

tail. The fact is that the tosses and the circumstances are not the same, but generally different (similar). The correct statement is that (b') the tosses and the circumstances favour the outcome of approximately the same number of heads and tails in a large number of tosses. The different tosses in their different associated circumstances are such that there is certainty to get head in one toss, tail in the other, and so on. The final result is that the number of heads and tails are approximately the same in a large number of tosses. Thus, the correct meaning one can attach to the quantities $P_h = 0.5$ and $P_t = 0.5$ is that (b'') in a large number of tosses n , the number of heads, n_h , is approximately nP_h and the number of tails, n_t , is approximately nP_t . Hence, the quantities P_h and P_t refer to the happening of a large number of cases, not in the sense of the incorrect probability concept (b), but in the sense of the correct statistical concept (b''). Thus, the above quantities cannot refer to the happening of an individual case. Contrary to the usual interpretation, this is so irrespective of whether one tries to verify it experimentally or not. The only difference is that when measurements are made in a large number of cases as required, one gets approximately the same number of heads and tails, confirming the St. (b''), otherwise it remains as an unverified statement.

The generalization of the above discussion to the case where the processes happen in m ways is obtained from P_i where i goes

from 1 to m . Instead of calling P_i the probabilities, one may simply call them the statistical factors.

Let us now consider the probability statement that at any instant, all the atoms in a radioactive substance have the same probability to decay. As before, such a probability statement is correct if the state of all the atoms and their associated circumstances are identical in all respects. In this case, all the atoms decay at the same time. But this is not the case. The different atoms have certainties to decay at different times. The fact that the different atoms decay at different times means that either the atoms are not in the same stage of development in the process of decay at the given instant or the circumstances they encounter during the time interval under consideration are different or both. Hence, in any case, the above probability statement is incorrect.

One may say that the toss and the circumstances favour, say, the head slightly more than the tail in a toss. In order to see the incorrectness of this argument, let us consider a (hypothetical) case of a heavily loaded coin (or of a process) for which P_t is, say, 0.7 and $P_h = 0.3$. Now the usual statement is that in any toss of this coin, the probability to get the head is 0.7 and the probability to get the tail is 0.3. This is correct if the toss and the circumstances in any toss favour the head more than the tail by the factor 0.7/0.3. In this case we should get the head all the time. This is not

what happens. Favouring the head or tail slightly more or slightly less is not going to alter these conclusions. Hence the above statements are incorrect.

One may say that possibilities or probabilities do not exist in Nature, but they exist in our knowledge (predictions) of Nature. Our knowledge should, as far as possible, be in accordance, but not in contradiction, with Nature. Instead of saying incorrectly that in any toss of a perfect coin there is equal (or, more or less equal) probability to get the head and tail, etc, one should say that in any toss, there is certainty to happen in one way and not in the other way, and in similar circumstances involving large number of tosses, one gets approximately equal number of heads and tails, etc.

One may say that the expression "probability of an event" simply means the number of cases where this event happens divided by the total number of cases. Now, any event can only happen once. Hence, the expression "the number of cases where this event happens" is incorrect. One may replace the term "this event" by "this type of events" in the above expression. In this case, the result is not the probability of an event, but the probability of one type of events. Hence, one may say that the probability of a type of events means the number of cases where this type of events happen divided by the total number of cases. For example, one may say that the probability of getting heads, P_h , in a large number of tosses on an average is the number of

heads, n_h , divided by the total number of tosses, n . Nothing much will be gained by the use of the word probability here. At any rate, the above probability concept refers not to an individual process, but to a large number of processes of different types. Moreover, one should at least say that the probability concept is an approximation to reality. In this case, such an approximation should be incorporated in the probability statements. On the otherhand, the statistical Sts. (b'), (b''), etc. contain explicitly the concept of approximation, as they should. At any rate, we consider that the probability concept is incorrect, irrespective of whether it refers to an individual case or to a large number of cases. If the same physical process in the same circumstances proceeds in different ways at different times, then one may say that the concept of probability is correct and the physical phenomena are governed by the laws of chance (probabilities). Since the same physical process in the same circumstances proceeds in the same manner at all times, the concept of probability is incorrect, and the physical phenomena are governed by the laws of certainties and not of chance (probabilities).

We make a distinction between probability and statistical concepts. While the probability description (such as (a), (b), etc.) is incorrect, the statistical description (such as (b'), (b''), etc) is correct. The other point is that the statistical description is an approximate and incomplete description.

Any theory based on it is an approximate and incomplete theory. A complete theory takes into account every factor of the physical processes and of their associated circumstances, and gives the information of the ways in which the processes proceed, etc. Thus a complete theory of our previous examples tells us exactly (and not approximately) how many heads and tails one gets in a given number of tosses in their associated circumstances, and at what instants the atoms in the radioactive substance decay, etc. Whether it is possible to construct a complete theory or not is a different matter. The point here is that the usual Sts. (B)-(D) are, in any case, incorrect.

4. PARTICLES AND WAVES.

Before taking up the usual St. (E), let us first consider the usual St. (F), viz. an electron, for example, behaves some times like a particle and at other times like a wave (wave-particle dualism).

(i) As an example, let us consider an idealized electron diffraction experiment involving a source of electrons, a diaphragm with two slits and a screen. The interference pattern appears when a large number of electrons are accumulated at the screen. The intensity of the electron beam at any time during the experiment is not important for the final appearance of the interference pattern. Thus, this is true even when only one electron at a time is in transit between the source and the screen. Now, according to the usual interpretation, we have the following

situation: When measurements are made soon after the slits, one finds that an electron comes either through the slit 1 or slit 2. In this case of measurements, the interference pattern disappears. When no measurements are made soon after the slits, the interference pattern appears. In this case, the wave packet representing the electron goes through both the slits at once. The interference pattern comes about by linear superposition of the wave packets. Thus, the interference pattern appears if the electron goes through both the slits at once. Hence, when no measurements are made soon after the slits, it is incorrect to say that the electron goes either through the slit 1 or slit 2, and if one says that and starts to make deduction from the statement, then he will make errors in the analysis. Thus, according to the usual interpretation, (a) when measured soon after the slits, the electron comes either through the slit 1 or slit 2, and when not measured, the electron goes through both the slits at once. It is, of course, understandable that the measurements can influence the objects under measurement. But it seems that this cannot help the usual interpretation in explaining the St. (a). Hence, the St. (a) is still rightly to be considered as a paradox.

(ii) As another example, let us consider the incidence of a 'photon' on a semi-transparent mirror. The usual interpretation gives the following description : On reaching the semi-transparent mirror, the wave packet representing the

photon decomposes into two wave packets, a reflected packet and a transmitted packet. It is incorrect to say that on reaching the mirror, the photon makes a choice between the reflected packet and the transmitted packet. After a sufficient time the two packets will be separated by any distance desired. By placing a reflecting mirror at a suitable position, one gets interference effects between the two packets. If, instead of the reflecting mirror, two photographic plates are placed at the reflected and transmitted packets, then the photon appears in one of the two packets. Thus, one is obliged to say in the usual interpretation, that (b) in one case, the photon is present (in a potential state) in both the packets, and in the other case, it shows its presence only in one of the two packets. It is very difficult to explain the St. (b). Hence, this is to be considered as another paradox. We consider that the situation is much more serious than the Sts. (a) and (b).

(iii) According to the usual St. (F), an electron behaves sometimes like a particle and at other times like a wave packet. The usual interpretation assumes that the wave packet, in general, moves, changes its size and shape, i.e., it spreads. It decomposes, depending upon the case, into two or more parts which can be separated by any distance desired, etc. Now, when the electron behaves like a particle, there is no particle other than the electron. Similarly, when the electron behaves like a wave packet, there is no wave packet other than the electron except for

errors, about which we talk later. There are no two physical things, one electron and one wave packet (or particle). There is only one physical thing in the usual interpretation, viz, the electron. Hence, if one assumes that when the wave packet representing the electron goes through both the slits at once, spreads and decomposes into two or more parts, etc, which play part in the development of physical phenomena, then one should say that the electron goes through both the slits at once, spreads and decomposes into two or more parts, etc.

The electron that has spread out, etc, should somehow show its presence in a smaller volume when measured. These considerations will naturally lead to many difficulties. Hence, it is incorrect to say that the electron behaves sometimes like a particle and at other times like a wave packet with the physical properties that the wave packet spreads, etc.

(iv) The usual interpretation assumes that the electron exists in a potential state throughout the volume of the wave packet. There is a definite probability of finding the electron in every region of the wave packet. If an experiment yields the result that the electron is in one region of the wave packet, then the probability of finding the electron in every other region of the wave packet immediately becomes zero. The experiment at the region of the wave packet where the electron is found thus exerts a kind of action (reduction of wave packet) at every other region of the wave packet. This action is propagated

instantly with a velocity greater than the velocity of light. The experiment singles out of a large number of possibilities one of which is the one which has happened. If the wave packet spreads and decomposes into two or more packets, then the above considerations are applicable for all the packets.

Thus, in the usual interpretation, something happens or becomes actual only when a measurement is made. One is, in general, not permitted to say what happens between measurements and when no measurements are made at all. Between measurements, only possibilities, probabilities, potential states, etc, exist. The experiment singles out of a large number of possibilities one of actuality, etc.

Now, as mentioned before, the experimental apparatus is a part of the circumstances. The electron at any instant in its associated circumstances proceeds only in one way. This, at any instant, is actual, i.e. something that happens irrespective of whether we make any measurements or not. There are no possibilities or probabilities or potential states at any time, either before or after the introduction of the apparatus in the circumstances. The electron proceeds in one way when the apparatus is not included in the circumstances; now, it is possible that the electron proceeds in another way when the apparatus is included in the circumstances. The apparatus can only change from one course of action to another course of action. The experiment tells us the course of action followed when the apparatus is

included in the circumstances. Hence, the considerations that the electron exists in a potential state throughout the volume of the wave packet, there is a definite probability of finding the electron in every region of the wave packet, the experiment singles out of a large number of possibilities one of actuality, the experiment exerts a kind of action at distant points instantly (reduction of wave packet), etc, do not arise at all. For, as mentioned before, there are no possibilities or probabilities or potential states at any time. The physical phenomena happen irrespective of whether we make measurements or not. The happening of phenomena when not measured is as actual as the phenomena when measured. The difference is that when not measured, we do not know the ways in which the phenomena happen. One should make a distinction between the happening of phenomena and our realization of the same. It is understandable that the measurement of a physical quantity changes its value as well as the values of some other physical quantities. At least to this extent, what we measure is different from what we do not. There is no subjective character involved in the interaction of the apparatus with the object under measurement. We should say atleast in general principles what happens not only when measured, but also between measurements and when no measurements are made at all. In all such cases, the general principle (I) and its special cases (1), (2), etc, are valid. Thus the above concepts of the usual interpretation are incorrect.

(V) The usual interpretation assumes that the extension of the wave packet corresponds to the uncertainty about the position of the electron. Thus, the statement that the position of the electron is known to within a certain accuracy Δx at the time t (in the x -direction) can be visualized by the pictures of the wave packet in the proper position with an approximate extension Δx . In time, the wave packet spreads, i.e., Δx increases. If a measurement is made and the electron is found in a smaller region, then the electron is represented by the wave packet of this smaller region. Thus, the wave packet which has spread out is replaced by a smaller one which represent the result of this measurement. Now, one may say that the electron somehow extends throughout the volume of the wave packet and spreads, and shows its presence in a small volume when measured, etc, or the electron(or at least most of the electron) exists in a small volume throughout its life time. The former case leads to the considerations (iii). In the later case, the electron goes through one of the two slits in the diffraction experiment and there is no physical object in the usual interpretation to go through the other slit that can interfere with the electron. This point will become more clear later. Thus, the latter case cannot explain the observed results. It seems that certain extended physical wave-like quantities are necessary to explain the diffraction results.

(vi) The usual interpretation assumes that the waves in a $3n$ -dimensional configuration space are probability waves (where n is the number of electrons involved). The probability waves represent not a reality in the classical sense, but rather the possibility of such a reality. The usual interpretation tries to argue that the configuration space is abstract and not real, the mathematical quantities in the configuration space do not correspond to real physical quantities in the classical sense. For one electron, the configuration space reduces to the 3-dimensional physical space. This makes no difference in the usual interpretation as regards the spreading of the wave packet, selecting out of a large number of possibilities one of actuality, etc, as discussed before.

Now, as in many cases, there are two important points here. The first point is that the concepts such as points, velocities, probability waves, etc, in $3n$ -dimensional configuration space are incorrect; the concept of configuration space with $3n$ -dimension is itself incorrect. For, there are no such things at all. There is only one space with 3-dimensions. The second point is that the mathematical quantities correspond, either directly or indirectly, to real physical quantities in the 3-dimensional physical world, with or without certain mathematical operations. The mathematical operations, if any are of secondary importance here. The important point is that there is nothing abstract or unreal about the mathematical

quantities. For example, all the quantities $dx_1, dx_2, \dots, dx_{3n}$ are related to the positions of the n particles in the 3-dimensional physical world. Thus, we object here to the interpretations, concepts and names, and not to the mathematical quantities. This is similar to the case where we object to the probability concept and not to the mathematical (statistical) quantities P_i . Another example involves the theory of relativity.

Because of the above reasons, there must be some other explanation other than the usual one about particles and waves. In order to resolve the situation, one may proceed in the following way: The electron is a particle of finite size in a given circumstances at an instant. It generates waves as it moves in a medium. The medium may be aether.

It is assumed in the usual interpretation of the Special Theory of Relativity that the concept of aether is incorrect and unnecessary. The fact is that one of the old concepts, viz, the aether is at rest relative to space, is found to be incorrect (at least inside the galaxies). It seems that the aether in one form or another is a fundamental part of Nature (4-8). For example, the usual interpretation of the Quantum Field Theory ascribes without any explanation many physical properties to the so-called vacuum such as the vacuum state (from which particles can be created), vacuum polarization, zero-point oscillations, zero-point fluctuations, etc. These physical properties make sense if one ascribes them to the aether.

At any rate, a part of the waves generated by the electron goes through the slit 1 and another through the slit 2 in the electron diffraction experiment. They along with the freshly created waves interfere with each other and with the direction of motion of the electron, thus giving rise to the observed interference pattern. On the otherhand, the electron (or atleast the main body of the electron) goes through only one of the two slits irrespective of whether any measurements are made soon after the slits or not. The difference is that when measured, such measurements interfere with the waves and the electron, and hence no interference is observed at the screen. Thus, we consider that the Sts. (F), (a), (b), etc, are incorrect.

It may be that the wave function is directly related to the waves generated by a particle or particles depending upon the case. The square of the absolute value of the wave function may correspond to the intensity of the waves. The waves will naturally spread out, but there are always waves that are just created. The freshly created waves of, say, an electron, give many of the properties of the electron. The conservation laws, etc, arise from the description of the individual cases. When one considers a large number of cases, the square of the absolute value of the wave function times the volume element approximately gives also the number of the electrons contained (or the number of times an electron is present) in the volume element. This gives rise to statistical laws, etc.

5. THE CONCEPT OF UNCERTAINTIES

We are now ready to take up the topic of uncertainties (St. E). As mentioned before, any physical process in its associated circumstances proceeds only in one manner. This is true for every physical quantity. Hence, contrary to the usual St. (E), no uncertainties are inherent in Nature. However, it may not be possible to determine physical quantities without any uncertainties.

Let us say that the position and momentum of an electron are determined within Δx and Δp respectively at a certain time t in a certain circumstances C . The electron has a certain size (spread) in position, Δx_p , at the time t in the circumstances C . It cannot be perfectly rigid. Its non-rigidity, etc, gives rise to certain amount of spread in momentum, Δp_p . The spreads Δx_p , Δp_p are physical quantities. Now, Δx and Δp are physical to the extent that they contain Δx_p and Δp_p of the electron; they play fundamental part in the physical phenomena. The rest of Δx and Δp are non-physical and play no part in the physical phenomena. The non-physical parts, Δx_n and Δp_n , are actually zero, but take finite unknown values because Nature may not permit us to know all of its details.

As mentioned before, in the usual interpretation, the extension in position and the range in wave number of the wave packet correspond to the uncertainties about the position and momentum of the electron. Hence, in the usual interpretation,

the wave packet is physical only to the extent that it contains the electron. The rest of it is non-physical and plays no part in the physical phenomena. The physical part remains essentially the same in a given circumstances. But the uncertainties about the position and momentum of the electron increase with time and hence the non-physical part increases with time, i.e., the wave packet spreads. This is, of course, the reason in the usual interpretation why there is no physical quantity to go through the other slit if the electron goes through one of the two slits and why it is necessary for the electron to go through both the slits at once to produce the interference pattern, etc. These considerations will naturally lead to many difficulties as mentioned before.

Contrary to the usual interpretation, the wave packet is, as mentioned before, assumed to be generated by the electron. Here, besides the electron, the entire spreads of wave packet (except for experimental errors when measured) are physical and play fundamental part in the physical phenomena. Moreover, it may be that the quantities Δx and Δp that occur in the uncertainty relations corresponds to the spreads of the wave packet. In this case, Δx and Δp are physical quantities and not uncertainties. Obviously Δx and Δp have certain relations with the extension in position and spread in momentum of the electron, especially when one considers the spreads of the freshly created wave packet. Many of the properties which are in accordance with the uncertainty

relations are due to the wave spreads. The sizes of the orbits of the electrons in an atom, for example, are determined by the waves generated by the electrons.

6. CONCLUSION

From our measurements, experiences, intuitions, reasoning, etc, it is possible for us to know correctly many general principles of nature. However, it may only be possible to determine their course of action within certain approximations and assumptions. It is a secondary point to know whether one can construct a complete theory in a given case. The primary point is to recognise the correctness of the general principles such as similar phenomena in similar circumstances proceed in similar manner, etc, and their consequences such as same phenomena in same circumstances proceed in same manner, and hence, physical phenomena are governed by the laws of certainties and not of chance (probabilities), etc.

ACKNOWLEDGEMENT

The present work was done in the Nuclear Physics Department of the Andhra University, Waltair.

REFERENCES:

- (1) P.Nuthakki, Bull.Am.Phys. Soc. 18, 768, 769 (1973);
Int. J. Theo. Phys. 13, 27 (1975); 8th Int. Conf. GRG,
Waterloo (Canada), 275 (1977); 8th Ind. Conf. GRG,
13 (1977); Proc. Ind. Sci. Conf. 64th, III, 48 (1977);
66th, III, 50 (1979); An Alternative Interpretation of
Special Relativity (Andhra University Physics Report
AUPH R01/1979); An Alternative Interpretation of General
Relativity (AUPH R02/1979).
- (2) W.Heisenberg, The Physical Principles of Quantum
Theory (Chicago, 1930).
- (3) W.Heisenberg, in Neils Bohr and the Development of
Physics, W.Pauli (Ed) (Pergamon, London, 1955).
- (4) H.A. Lorentz, Proc. Amst. Acad. Sci. 6, 809 (1904).
- (5) A. Einstein, Uber den Ather. Verh. d. Schaeiger,
Nat. Ges. 105, Teil II, 85 (1924).
- (6) P.A.M. Dirac, Nature 168, 906 (1951).
- (7) H.E.Ives, J.Opt. Soc. Am. 43, 217 (1953).
- (8) L.Janossy, Theory of Relativity Based on Physical
Reality (Akademiaia Kiade, Budapest, 1971).

COLLECTIVE RISK THEORIES AND RUIN PROBABILITIES

R. Vasudevan, MATSCIENCE, Madras and
 P.R. Vittal, Dept. of Mathematics, Vivekananda College, Madras.

+*+*

1. Introduction.

The collective theory of risk has been the object of series of investigations during the last few decades. A significant contribution to this has been made by Harald Cramer (1954), Filip Lundberg (1903), Sparre Anderson (1957), Segedahl (1939, 1959, 1970), Olof Thorin (1970, 1971, 1974, 1975), Michael Harison (1977) and many others. The main problems of interest in this field are (1) the probability distribution of the random variable $X(t)$, representing the net reserve of an insurance company at time t and (2) the probability of ruin of the company in a given time t .

The classical theory of collective risk using the notion of probability was initiated by Filip Lundberg in the beginning of this century (1908). The primary model devoted to the study of this stochastic process is

$$S(t) = U + ct - \sum_{i=1}^{N(t)} Z_i \quad (1)$$

In this model u represents the initial reserve at time $t = 0$ and c is the constant income to the company by means of interest and premiums. The liabilities to the company are the claims and these claims are assumed to occur in a Poissonian way

with intensity λ . $N(t)$ represents the number of Poisson claims in the time interval $(0, t)$ and $(Nt) > 0$. Z_i , representing the amounts of claims are assumed to be independent and identically distributed random variables with a common density function $a(z)$. Obviously $x(t)$ represents the net reserve at time t .

Further complications of problems (1) and (2) are due to the fact that the random claims may follow different distributions - general and special. But closed form solutions are available only in a few special cases. The study of ruin probability in general corresponds to the study of the motion of a particle with random jumps and linear upward drift with an absorbing barrier at $X = 0$. This single barrier problem has been studied by Keilson (1963) using the technique of compensation function. The classical problem of ruin probability when there is an upper limit K to the risk reserve is completely studied by Segerblom (1970). This problem corresponds to the motion of a particle executing random walk along the real line with barrier at $X = 0$ and $X = K$. In this two barrier problem the ruin may occur after a number of crossings of the upper barrier K . Treating $X = K$ as a reflecting barrier excess of inputs cut off by the barrier $X = K$ before ruin takes place is also studied by Segedahl (1970). In this model Segedahl has taken the income rate to the company as a constant and the random claims to be governed by exponential density functions. One valid criticism of this classical model is the assumed independence

of the income rate c and the level of the risk reserve.

Davidson suggested that the instantaneous income rate can be taken as $c(y)$ where y is taken as the risk reserve to study the realistic process

$$X(t) = u + \int_0^t c[X(t')] dt' - \sum_{i=1}^{N(t)} Z_i \quad (1.2)$$

The stationary transition probabilities are to be obtained starting with $X(0) = u$. The jumps are Poisson with intensity ν and the jump amount are independent and identically distributed random variables. In between the jumps $X'(t) = c X(t)$.

Segerdahl also gave an explicit expression for the ultimate ruin probability function $\psi(u)$ in terms of incomplete gamma function by taking the income process as $c(y) = \beta + \alpha y$. Here he treats the problem as a single barrier problem.

In this Chapter we consider the process (1.2) with

$$X'(t) = \beta + \alpha X \quad (1.3)$$

This means that the income to the company has two components.

(1) the income through premiums coming at a constant rate β and (2) the interest earned continuously at a rate αX on the current capital. We study the process $x(t)$ as a two barrier problem with the barrier $X = K$ as the upper limit for the risk reserve. The following probability distributions are arrived at.

(1) the probability distribution of ruin at time t given that the risk reserve has not reached its upper limit in time t . This means that the barrier $X=0$ is to be crossed at time t before the risk reserve reaches the value $X=K$.

(2) the probability distribution for ruin when $X=K$ is a reflecting barrier.

(3) the probability distribution for the risk reserve to reach the value $X=K$ before ruin takes place.

We obtain analytical solutions in closed form for above problems by assuming that the claims occur with a Poisson frequency λ and the density of the random claims as exponential.

In section 2 we obtain the Laplace transform for the ruin probability function before the risk reserve reaches the value K . We also deduce the probability function for ultimate ruin. Section 3 is devoted to the study of the ruin probability function by taking the barrier $X=K$ as a reflecting barrier. In section 4 we obtain the probability function for the risk reserve to reach the value K before ruin takes place.

2. First passage time for ruin before the risk reserve reaching the limit K .

The first passage time problems for a jump process with linear drift have been studied by Vasudevan et. al. (1978,1979) by the imbedding approach. The imbedding method is also seen to be an effective tool for studying the first passage problems for process with exponential decay (Vasudevan et. al. 1980).

In Chapters II, III and IV we used this technique to study problems relating to neuron discharge activity and storage models. In this section we exploit this approach to study the problem of determining the probability density function for the risk reserve to reach the value $X < 0$ for the first time before reaching the value $X = K$. Let $X(t)$ be the stochastic variable representing the net capital reserve in time t . The equations defining $X(t)$ is given by (1.2) and (1.3). (1.3) means that there is a regular income at a constant rate α in addition to the income which is proportional to risk reserve at that time. Define $P(u, K, t)$ as the probability that the insurance company starting with an initial reserve $X = u$ is ruined between time t and $t+dt$ given that the net reserve has not reached its upper limit K . Considering the different possibilities of the happenings in the initial interval $(0, dt)$ and proceeding to the limit as dt tends to zero we arrive at the integro-differential equation

$$\begin{aligned} & \frac{\partial P}{\partial t} - (\beta + \alpha u) \frac{\partial P}{\partial u} + \nu P \\ = & \nu \int_{-u}^0 P(u+z, K, t) dz + \delta(t) \nu \int_{-\infty}^{-u} a(z) dz \end{aligned} \quad (2.1)$$

Defining $\bar{P}(u, K, \ell)$ as the Laplace transform of $P(u, K, t)$ we have from equation (2.1)

$$\begin{aligned} & -(\beta + \alpha u) \frac{\partial \bar{P}}{\partial u} + (\ell + \nu) \bar{P} \\ = & \nu \int_{-u}^0 a(z) \bar{P}(u+z, K, \ell) dz + \nu \int_{-\infty}^{-u} a(z) dz \end{aligned} \quad (2.2)$$

In order to arrive at a closed form solution we consider the special case

$$a(z) = -\eta e^{\eta z} H(-z) \quad (2.3)$$

and obtain after transformation the equation

$$\begin{aligned} e^{\eta u} \left[-(\beta + \alpha u) \frac{\partial \bar{P}}{\partial u} + (\ell + \nu) \bar{P} \right] \\ = \nu \eta \int_0^u e^{\eta y} \bar{P}(y, \kappa, \ell) dy + \nu \end{aligned} \quad (2.4)$$

Differentiating (2.4) with respect to u and simplifying we get the second order differentiation equation

$$-(\beta + \alpha u) \frac{\partial^2 \bar{P}}{\partial u^2} + [\ell + \nu - \alpha - \eta(\beta + \alpha u)] \frac{\partial \bar{P}}{\partial u} + \eta \ell \bar{P} = 0 \quad (2.5)$$

Letting $\beta + \alpha u = -\alpha w / \eta$ we have from (2.5)

$$\eta w \frac{\partial^2 \bar{P}}{\partial u^2} + \left[1 - \frac{\ell + \nu}{\alpha} - w \right] \frac{\partial \bar{P}}{\partial u} + \frac{\ell}{\alpha} \bar{P} = 0 \quad (2.6)$$

This equation being of Kummer's type (Slater 1960) the solution in terms of the original variable u is

$$\begin{aligned} \bar{P}(u, \kappa, \ell) \\ = A_1 {}_1F_1 \left[-\frac{\ell}{\alpha}, 1 - \frac{\ell + \nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha u) \right] \\ + B_1 \left[-\frac{\eta}{\alpha} (\beta + \alpha u) \right]^{\frac{\ell + \nu}{\alpha}} {}_1F_1 \left[\frac{\nu}{\alpha}, 1 + \frac{\ell + \nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha u) \right] \end{aligned} \quad (2.7)$$

where $A_1 = A_1(\kappa, \ell)$ and $B_1 = B_1(\kappa, \ell)$ are functions of κ

and l independent of u . These functions are to be determined using the boundary conditions. From the definition of

$P(u, k, t)$ we have the boundary condition

$$\bar{P}(k, k, l) = 0 \quad (2.8)$$

Using the boundary condition (2.8) in (2.7) we get

$$\begin{aligned} & A_1 {}_1F_1 \left[-\frac{l}{\alpha}, 1 - \frac{l+\nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha k) \right] \\ & + B_1 \left[-\frac{\eta}{\alpha} (\beta + \alpha k) \right]^{\frac{l+\nu}{\alpha}} {}_1F_1 \left[\frac{\nu}{\alpha}, 1 + \frac{l+\nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha k) \right] \quad (2.9) \\ & = 0 \end{aligned}$$

The second boundary condition is the equation (2.4)

which takes care of the crossing of the boundary at $X=0$ at

time t . Hence substituting the solution (2.8) in (2.4) we get

$$\begin{aligned} & A_1 e^{\eta u} \left\{ -(\beta + \alpha u) \frac{\partial}{\partial u} {}_1F_1 \left[-\frac{l}{\alpha}, 1 - \frac{l+\nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha u) \right] \right. \\ & \quad + (l+\nu) {}_1F_1 \left[-\frac{l}{\alpha}, 1 - \frac{l+\nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha u) \right] \\ & \quad \left. - (l+\nu) {}_1F_1 \left[-\frac{l}{\alpha}, -\frac{l+\nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha u) \right] \right\} \\ & + B_1 e^{\eta u} \left\{ -(\beta + \alpha u) \frac{\partial}{\partial u} \left(-\frac{\eta}{\alpha} (\beta + \alpha u) \right)^{\frac{l+\nu}{\alpha}} {}_1F_1 \left[\frac{\nu}{\alpha}, 1 + \frac{l+\nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha u) \right] \right. \\ & \quad + (l+\nu) \left(-\frac{\eta}{\alpha} (\beta + \alpha u) \right)^{\frac{l+\nu}{\alpha}} {}_1F_1 \left[\frac{\nu}{\alpha}, 1 + \frac{l+\nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha u) \right] \\ & \quad \left. + \left[-\frac{\eta}{\alpha} (\beta + \alpha u) \right]^{\frac{l+\nu}{\alpha} + 1} \left(\frac{\nu \alpha}{l+\nu+\alpha} \right) {}_1F_1 \left[1 + \frac{\nu}{\alpha}, 2 + \frac{l+\nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha u) \right] \right\} \end{aligned}$$

$$\begin{aligned}
&= -(l+\nu) A_1 {}_1F_1 \left[-\frac{l}{\alpha}, -\frac{l+\nu}{\alpha}, -\eta\beta/\alpha \right] \\
&+ B_1 \left(-\frac{\eta\beta}{\alpha} \right)^{\frac{l+\nu}{\alpha} + 1} \left(\frac{\nu\alpha}{l+\nu+\alpha} \right) {}_1F_1 \left[1+\frac{\nu}{\alpha}, 2+\frac{l+\nu}{\alpha}, -\frac{\eta\beta}{\alpha} \right] \\
&+ \nu
\end{aligned}
\tag{2.10}$$

Using the standard properties of ${}_1F_1$ (Slater 1960) functions it can be easily shown, as seen in earlier chapters, that the coefficients of $A_1 e^{\eta u}$ and $B_1 e^{\eta u}$ in LHS of equation (2.11) vanish. Hence we have the second condition for determining A_1 and B_1 as:-

$$\begin{aligned}
&A_1 {}_1F_1 \left[-\frac{l}{\alpha}, -\frac{l+\nu}{\alpha}, -\eta\beta/\alpha \right] \\
&- B_1 \left(-\eta\beta/\alpha \right)^{\frac{l+\nu}{\alpha} + 1} \frac{\nu\alpha}{(l+\nu)(l+\nu+\alpha)} {}_1F_1 \left[1+\frac{\nu}{\alpha}, 2+\frac{l+\nu}{\alpha}, -\frac{\eta\beta}{\alpha} \right] \\
&= \nu/(l+\nu)
\end{aligned}
\tag{2.11}$$

Letting

$$L = {}_1F_1 \left[-\frac{l}{\alpha}, 1 - \frac{l+\nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha k) \right]
\tag{2.12}$$

$$M = \left[-\frac{\eta}{\alpha} (\beta + \alpha k) \right]^{\frac{l+\nu}{\alpha}} {}_1F_1 \left[\frac{\nu}{\alpha}, 1 + \frac{l+\nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha k) \right]
\tag{2.13}$$

$$N = {}_1F_1 \left[-\frac{l}{\alpha}, -\frac{l+\nu}{\alpha}, -\eta\beta/\alpha \right] \quad (2.14)$$

$$Q = \left(-\frac{\eta\beta}{\alpha} \right)^{\frac{l+\nu}{\alpha} + 1} {}_1F_1 \left[1 + \frac{\nu}{\alpha}, 2 + \frac{l+\nu}{\alpha}, -\eta\beta/\alpha \right] \quad (2.15)$$

We get from (2.9) and (2.11)

$$A_1 L + B_1 M = 0 \quad (2.16)$$

$$A_1 N - B_1 \frac{\nu\alpha Q}{(l+\nu)(l+\nu+\alpha)} = \frac{\nu}{(l+\nu)} \quad (2.17)$$

Solving (2.16) and (2.17)

$$A_1 = \frac{\nu(l+\nu+\alpha)M}{(l+\nu)(l+\nu+\alpha)MN + \nu\alpha QL} \quad (2.18)$$

$$B_1 = \frac{-\nu(l+\nu+\alpha)L}{(l+\nu)(l+\nu+\alpha)MN + \nu\alpha QL} \quad (2.19)$$

In the expression (2.8) since the term multiplying the coefficient of B happens to contain the term $(-1)^{\frac{l+\nu}{\alpha}}$ and since $\frac{l+\nu}{\alpha}$ can be a fraction doubt may arise whether the solutions contains an imaginary part. This cannot be the case since l can be taken as real and $\bar{P}(u, k, l)$ is the L.T. of a probability

density which is real. However it is to be noted that M and Q contains similar terms as multiplicative factors. In view of the expressions (2.18) and (2.19) for A_1 and B_1 it is easily seen that for real l , whatever be the values of ν and α , the total solution can never become complex. (2.7) together with (2.8) and (2.19) determines the complete solution for

$\bar{P}(u, k, l)$, the L.T. of the ruin probability function $P(u, k, t)$.

Deductions. (1) The case when the income is only proportional to the level of the risk reserve: In this case $\beta = 0$. Hence we note

$$Q = 0, \quad N = 1$$

$$A_1 = \nu / (l + \nu), \quad B_1 = -\nu L / (l + \nu) M \Big|_{\beta=0} \quad (2.20)$$

$$\bar{P}(u, k, l) \Big|_{\beta=0} = \left(\frac{\nu}{l + \nu} \right) {}_1F_1 \left[-\frac{l}{\alpha}, 1 - \frac{l + \nu}{\alpha}, -\eta u \right]$$

$$- \left(\frac{\nu}{l + \nu} \right) \frac{{}_1F_1 \left[-\frac{l}{\alpha}, 1 - \frac{l + \nu}{\alpha}, -\eta k \right] {}_1F_1 \left[\frac{\nu}{\alpha}, 1 + \frac{l + \nu}{\alpha}, -\eta u \right]}{(-\eta k)^{\frac{l + \nu}{\alpha}} {}_1F_1 \left[\frac{\nu}{\alpha}, 1 + \frac{l + \nu}{\alpha}, -\eta k \right]} \quad (2.21)$$

(2) The probability for ultimate ruin:

The probability for ultimate ruin is got by putting $l = 0$ in the solution for $\bar{P}(u, k, l)$

$$\bar{P}(u, k, l) = \int_0^{\infty} P(u, k, t) dt \quad (2.22)$$

when $l = 0$ we note that

$$A_1(0) = \frac{(\alpha + \nu) M(0)}{(\nu + \alpha) M(0) + \alpha Q(0)} \quad (2.23)$$

$$B_1(0) = \frac{-(\nu + \alpha)}{(\nu + \alpha) M(0) + \alpha Q(0)} \quad (2.24)$$

and

$$\bar{P}(u, k, l) \Big|_{l=0} = A_1(0) + B_1(0) \left[-\frac{\eta}{\alpha} (\beta + \alpha u) \right]^{\nu/\alpha} {}_1F_1 \left[\frac{\nu}{\alpha}, \frac{1+\nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha u) \right] \quad (2.25)$$

Using the property of the confluent hypergeometric function (page 262, Abramowitz and Stegun 1965)

$${}_1F_1(a, a+1, -x) = a x^{-a} \gamma(a, x) \quad (2.26)$$

where $\gamma(a, x)$ is the incomplete gamma function defined by

$$\gamma(a, x) = \int_0^x e^{-z} z^{a-1} dz \quad (2.27)$$

we arrive at the expression for the probability of ultimate ruin,

$$\bar{P}(u, k, 0) = \frac{\frac{\nu}{\alpha} \gamma \left[\frac{\nu}{\alpha}, \frac{\eta}{\alpha} (\beta + \alpha k) \right] - \frac{\nu}{\alpha} \gamma \left[\frac{\nu}{\alpha}, \frac{\eta}{\alpha} (\beta + \alpha u) \right]}{\frac{\nu}{\alpha} \gamma \left[\frac{\nu}{\alpha}, \frac{\eta}{\alpha} (\beta + \alpha k) \right] - \gamma \left[\frac{\nu}{\alpha} + 1, \eta \beta / \alpha \right]} \quad (2.28)$$

When there is no upper limit to the risk reserve the probability of ultimate ruin, allowing an unlimited growth of the risk reserve, is got on proceeding to the limit as k tends to infinity. When k tends to infinity we observe that $\gamma(a, k)$ tends to $\Gamma(a)$ where

$$\Gamma(a) = \int_0^{\infty} e^{-x} x^{a-1} dx \quad (2.29)$$

Hence

$$\bar{P}(u, \infty, 0) = P(u) = \frac{\frac{\nu}{\alpha} \Gamma\left(\frac{\nu}{\alpha}\right) - \frac{\nu}{\alpha} \gamma\left(\frac{\nu}{\alpha}, \frac{\eta}{\alpha} (\beta + \alpha u)\right)}{\frac{\nu}{\alpha} \Gamma\left(\frac{\nu}{\alpha}\right) - \gamma\left(\frac{\nu}{\alpha} + 1, \eta\beta/\alpha\right)} \quad (2.30)$$

$P(u)$ being the probability of ultimate ruin. Further using the properties

$$\gamma(a+1, x) = a \gamma(a, x) - x^a e^{-x} \quad (2.31)$$

$$\Gamma(a, x) = \Gamma(a) - \gamma(a, x) \quad (2.32)$$

where $\Gamma(a, x) = \int_x^{\infty} e^{-z} z^{a-1} dz \quad (2.33)$

from
we get/(2.30)

$$P(u) = \frac{\nu \Gamma\left[\frac{\nu}{\alpha}, \frac{\eta}{\alpha} (\beta + \alpha u)\right]}{\left\{ \nu \Gamma\left(\frac{\nu}{\alpha}, \frac{\eta\beta}{\alpha}\right) + \alpha \left(\frac{\eta\beta}{\alpha}\right)^{\nu/\alpha} e^{-\eta\beta/\alpha} \right\}} \quad (2.34)$$

This agrees with the result obtained by Segerdahl (1959). When $\beta=0$ the result (2.28) becomes

$$\bar{P}(u, K) \Big|_{\beta=0} = \frac{\Gamma(\frac{\nu}{\alpha}, \eta K) - \frac{\nu}{\alpha} \gamma(\frac{\nu}{\alpha}, \eta u)}{\frac{\nu}{\alpha} \gamma(\frac{\nu}{\alpha}, \eta K) - \Gamma(\frac{\nu}{\alpha} + 1)} \quad (2.35)$$

and the probability of ultimate ruin is

$$P(u) = \Gamma(\frac{\nu}{\alpha}, \eta u) / \Gamma(\nu/\alpha) \quad (2.36)$$

3. Ruin probability when K is a reflecting barrier.

In the previous section we determined the ruin probability density function for an insurance business with the condition that the ruin is attained at time t and the risk reserve has not reached the value K in this time. In this section we shall evaluate the probability function for ruin allowing the risk reserve to reach the value K and treating $X=K$ as a reflecting barrier. This means the excess of the reserve over K is cut off as in the case of a dam of finite capacity when overflow takes place. In the language of insurance the excess over K can be considered as taken out of the risk reserve for bonus purposes, for policy holders. In such a situation how is the probability density function is different from the one discussed in the previous model is described in this section. For the stochastic process defined in (1.2) and (1.3) we shall define

$\psi(u, k, t)$ as the probability that for an insurance company with the initial reserve $X=u$ ruin takes place between time t and $t+dt$ given that $X=k$ is a reflecting barrier.

The imbedding equation for $\psi(u, k, t)$ is given by the integro differential equation

$$\frac{\partial \psi}{\partial t} - (\beta + \alpha u) \frac{\partial \psi}{\partial u} + \nu \psi = \nu \int_0^{-u} a(z) \psi(u+z, k, t) dz + \delta(t) \nu \int_{-\infty}^{-u} a(z) dz \quad (3.1)$$

Denoting $\bar{\psi}(u, k, \ell)$ as the L.T. of $\psi(u, k, t)$ and taking the density of claims as in (2.3) we get

$$e^{\eta u} \left[-(\beta + \alpha u) \frac{\partial \bar{\psi}}{\partial u} + (\ell + \nu) \bar{\psi} \right] = \nu \eta \int_0^u e^{\eta y} \bar{\psi}(y, k, \ell) dy + \nu \quad (3.2)$$

As done in the previous section, differentiating (3.2) with respect to u and cancelling $e^{\eta u}$ throughout we get the second order differential equation

$$-(\beta + \alpha u) \frac{\partial^2 \bar{\psi}}{\partial u^2} + \left[\ell + \nu - \alpha - \frac{\eta}{\alpha} (\beta + \alpha u) \right] \frac{\partial \bar{\psi}}{\partial u} + \eta \ell \bar{\psi} = 0 \quad (3.3)$$

The above equation can be reduced to the Kummer's type confluent hypergeometric equation and hence the solution of equation (3.3) is

$$\begin{aligned} \bar{\psi}(u, k, \ell) = & A_2 {}_1F_1 \left[-\frac{\ell}{\alpha}, 1 - \frac{\ell + \nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha u) \right] \\ & + B_2 \left[-\frac{\eta}{\alpha} (\beta + \alpha u) \right]^{\frac{\ell + \nu}{\alpha}} {}_1F_1 \left[\frac{\nu}{\alpha}, 1 + \frac{\ell + \nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha u) \right] \end{aligned} \quad (3.4)$$

where A_2 and B_2 are functions of K and ℓ and they can be determined by suitable boundary conditions. As K is a reflecting barrier the risk reserve once reaches the upper value K stops at K without moving up in the absence of claims and when there is a claim it goes down the value K . As $\bar{\psi}$ is continuous at $X=K$ one expects the boundary condition for K to be a reflecting barrier as

$$\left[\frac{\partial \bar{\psi}}{\partial u} \right]_{u=K} = 0 \quad (3.5)$$

This condition can also be visibly seen by writing down the integral equation of the process starting from $X=K$ and comparing it with (3.2). Differentiating (3.4) partially with respect to u and using the condition (3.5) we get

$$A_2 L_1 \frac{\ell}{\ell + \nu - \alpha} + B_2 \left[\frac{\ell + \nu}{\alpha} M_1 + \frac{\nu}{\ell + \nu + \alpha} R \right] = 0 \quad (3.6)$$

$$\text{where } L_1 = {}_1F_1 \left[1 - \frac{\ell}{\alpha}, 2 - \frac{\ell + \nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha K) \right] \quad (3.7)$$

$$M_1 = \left[-\frac{\eta}{\alpha} (\beta + \alpha K) \right]^{\frac{\ell + \nu}{\alpha} - 1} {}_1F_1 \left[\frac{\nu}{\alpha}, 1 + \frac{\ell + \nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha K) \right] \quad (3.8)$$

$$N_1 = \left[-\frac{\eta}{\alpha} (\beta + \alpha k) \right]^{\frac{l+\nu}{\alpha}} {}_1F_1 \left[1 + \frac{\nu}{\alpha}, 2 + \frac{l+\nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha k) \right] \quad (3.9)$$

The second boundary condition which will serve in getting a relation between A and B is the integral equation (3.2). As done in the earlier model substituting the solution (3.4) in (3.2) and simplifying we get the relation

$$A_2 N - \frac{B_2 \nu \alpha Q}{(l+\nu)(l+\nu+\alpha)} = \frac{\nu}{l+\nu} \quad (3.10)$$

where N and Q are given by (2.14) and (2.15). Solving (3.6) and (3.10) we arrive at

$$A_2 = \frac{\nu(l+\nu+\alpha)}{\left\{ (l+\nu)(l+\nu+\alpha) N + \frac{\nu \alpha^2 l (l+\nu+\alpha) Q L}{(l+\nu+\alpha) [(l+\nu+\alpha)(l+\nu) M_1 + \nu \alpha R]} \right\}} \quad (3.11)$$

and

$$B_2 = \frac{-\nu(l+\nu+\alpha)}{\left\{ (l+\nu) \frac{N}{\alpha} \left[(l+\nu+\alpha)(l+\nu) M_1 + \nu \alpha R \right] \frac{(l+\nu-\alpha)}{l\nu_1} + \nu \alpha Q \right\}} \quad (3.12)$$

Here also by an argument similar to the one given in previous section we can show that the solution $\bar{\Psi}$ is real for real l . The complete analytical solution for $\bar{\Psi}(u, k, l)$ is got by substituting the values of A_2 and B_2 given by (3.11) and (3.12)

in (3.4).

Deductions. (1): The probability of ultimate ruin in this case is got by putting $\ell=0$ in the solution for $\bar{\psi}(u, k, \ell)$.

When $\ell=0$ we easily see from (3.11) and (3.12) that

$$A_2 = A_2(k, 0) = 1 \quad (3.15)$$

$$B_2 = B_2(k, 0) = 0 \quad (3.16)$$

where $A_2(k, 0)$ and $B_2(k, 0)$ are the values of A_2 and B_2 at $\ell=0$. Thus from the solution for $\bar{\psi}(u, k, \ell)$ we get

$$\begin{aligned} \psi(u) = \bar{\psi}(u, k, 0) &= \int_0^{\infty} \psi(u, k, t) dt \\ &= 1, \text{ as could be expected.} \end{aligned} \quad (3.17)$$

4. First passage density for risk reserve to reach the limit K before ruin.

In the previous sections we have analysed the probability of ruin when there is an upper limit to the risk reserve. The probability distributions for the risk reserve reaching the level K for the first time before ruin is also worth investigating in order to study the possibilities of giving bonus for the policy holders. We shall obtain this probability distributions in this section.

Define

$\chi(u, k, t)$ = the probability that the risk reserve reaches the level K for the first time between time t and $t+dt$ given that the ruin has not taken place before time t .

The barrier $X=K$ will be reached exactly in this case as there is a continuous change in the positive direction. The imbedding equation for $\bar{\chi}(u, k, l)$, the L.T. of $\chi(u, k, t)$ is

$$e^{\eta u} \left[-(\beta + \alpha u) \frac{\partial \bar{\chi}}{\partial u} + (l + \nu) \bar{\chi} \right] = \nu \eta \int_0^u e^{\eta y} \bar{\chi}(y, k, l) dy \quad (4.1)$$

This on differentiation with respect to u reduces to Kummer's type of confluent hypergeometric equation (Slater 1960) and hence the solution is

$$\begin{aligned} \bar{\chi}(u, k, l) = & A_3 {}_1F_1 \left[-\frac{l}{\alpha}, 1 - \frac{l + \nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha u) \right] \\ & + B_3 \left[-\frac{\eta}{\alpha} (\beta + \alpha u) \right]^{\frac{l + \nu}{\alpha}} {}_1F_1 \left[\frac{\nu}{\alpha}, 1 + \frac{l + \nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha u) \right] \end{aligned} \quad (4.2)$$

where A_3 and B_3 are to be determined by suitable boundary conditions. From the definition of $\chi(u, k, t)$ we have as a boundary condition

$$\bar{\chi}(K, K, l) = 1 \quad (4.3)$$

Hence one relation connecting A_3 and B_3 is

$$A_3 L + B_3 M = 1 \quad (4.4)$$

where L and M are as defined in (2.12) and (2.13). Another relation connecting A_3 and B_3 is got by feeding the solution (4.2) into the first order differential equation (4.1) for

$\bar{\chi}(u, k, l)$. This equation serves as the second boundary condition. Hence we get the relation

$$A_3 N - B_3 \frac{\nu \alpha Q}{(l+\nu)(l+\nu+\alpha)} = 0 \quad (4.5)$$

where N and Q are given by (2.14) and (2.15) respectively. Solving equations (4.4) and (4.5) we get

$$A_3 = \frac{\nu \alpha Q}{(l+\nu)(l+\nu+\alpha)MN + \nu \alpha Q L} \quad (4.6)$$

$$B_3 = \frac{(l+\nu)(l+\nu+\alpha)N}{(l+\nu)(l+\nu+\alpha)MN + \nu \alpha Q L} \quad (4.7)$$

Then the complete analytical solution for $\bar{\chi}(u, k, l)$ is given by (4.3), (4.6) and (4.7).

Deductions:

(1) When $l=0$ we have the probability of ever reaching the upper limit K as

$$\begin{aligned}
 \chi(u, k) &= \int_0^{\infty} \chi(u, k, t) dt \\
 &= A_3(0) + B_3(0) \left[-\frac{\eta}{\alpha} (\beta + \alpha u) \right]^{\nu/\alpha} \times \\
 &\quad \times {}_1F_1 \left[\frac{\nu}{\alpha}, \frac{1+\nu}{\alpha}, -\frac{\eta}{\alpha} (\beta + \alpha u) \right] \quad (4.8)
 \end{aligned}$$

where

$$A_3(0) = \frac{\alpha Q(0)}{(\nu + \alpha) M(0) + \alpha Q(0)} \quad (4.9)$$

$$B_3(0) = \frac{\nu + \alpha}{(\nu + \alpha) M(0) + \alpha Q(0)} \quad (4.10)$$

Here $M(0)$ and $N(0)$ stand for the expressions M and Q at $l=0$ (2.15 and 2.12).

(2) Further if we assume that the income to the company is only proportional to the risk reserve at time t we have $\beta=0$. In this case the probability of the risk reserve ever reaching the value K is

$$\chi(u, k) \Big|_{\beta=0} = \frac{{}_1F_1 \left[\frac{\nu}{\alpha}, 1 + \frac{\nu}{\alpha}, -\eta u \right]}{{}_1F_1 \left[\frac{\nu}{\alpha}, 1 + \frac{\nu}{\alpha}, -\eta k \right]} \quad (4.11)$$

(3) Lastly we verify that well known fact in infinite time the sum of the probabilities $P(u, k)$ and $\chi(u, k)$ is unity. When $l=0$ one gets from (2.23), (2.24), (4.9) and (4.10)

$$P(u, k) + \chi(u, k) = 1 \quad (4.12)$$

as could be expected.

5. Conclusion.

In our analysis of collective risk theory to study the ruin probability we have taken the claims as a Poisson while the income to the company at any time t with reserve $X(t)$ is taken as $\beta + \alpha X$. We have obtained the L.T. for the probability ruin density assuming that there is an upper limit K to the risk reserve. We have taken the barrier $X=K$ as an absorbing barrier in section 2 and as a reflecting barrier in section 3. We have also arrived at the results for the probability of ultimate ruin in these cases. Lastly we have studied the probability for the risk reserve reaching the value $X=K$ before ruin takes place. We have also verified the result that

$$P(u, K) + \chi(u, K)$$

tends to unity as ℓ tends to zero.

ON THE CONCEPT OF PRODUCT DENSITY IN THE THEORY OF STOCHASTIC
POINT PROCESSES

R. Jagannathan
MATSCIENCE, THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS-600 113
(India)

The purpose of this note is to point out how it is incorrect to refer to the literature on stochastic processes earlier to Ramakrishnan's work in 1950 as the origin of the "product density technique" in the theory of stochastic point processes ^{in general.} To this end let us proceed as follows.

First let us recall briefly the theory of product densities as given by Ramakrishnan (1950, 51, 59, 72 and references therein for details). If $M(E)$ is the number of particles with energy less than E , $dM(E)$ is the number of particles in the range dE . Let it be assumed that

$$\text{Prob} \{dM(E) = n\} = p(n) = O(dE^n), n = 1, 2, \dots \quad (1)$$

If $\mathcal{E}\{dM(E)\}$ is the average number of particles in dE then let

$$\mathcal{E}\{dM(E)\} = f_1(E)dE \quad (2)$$

Thus

$$\begin{aligned} p(1) &= f_1(E)dE + O(dE^2) = \mathcal{E}\{dM(E)\} + O(dE^2) \\ p(0) &= 1 - f_1(E)dE - O(dE^2) \\ p(n) &= O(dE^n), \forall n > 1 \end{aligned} \quad (3)$$

Hence

$$\mathbb{E}\{m^2\} = \mathbb{E}\{[dM(E)]^2\} = \mathbb{E}\{m\} = \mathbb{E}\{dM(E)\} \quad (4)$$

so that all the moments of the stochastic variable $dM(E)$ are equal to the probability that the stochastic variable assumes value 1. It should be noted that while $f_1(E)dE$ is a probability its integral over E gives only the mean number of particles in the range of integration as the integration does not correspond to the addition of infinitesimal probabilities relating to mutually exclusive events. So

$$\mathbb{E}\{M(E_f) - M(E_i)\} = \int_{E_i}^{E_f} dE \mathbb{E}\{dM(E)\} = \int_{E_i}^{E_f} f_1(E)dE \quad (5)$$

Now the concept of product density of order 2 is introduced through

$$\mathbb{E}\{dM(E_1)dM(E_2)\} = f_2(E_1, E_2)dE_1dE_2 \quad (6)$$

which gives also the joint probability for one particle to lie in each of the nonoverlapping intervals (E_1, E_1+dE_1) and (E_2, E_2+dE_2) .

When the intervals overlap a degeneracy occurs and

$$\mathbb{E}\{[dM(E_1)]^2\} = \mathbb{E}\{dM(E_1)\} = f_1(E_1)dE_1 \quad (7)$$

In view of this degeneracy

$$\begin{aligned} \mathcal{E}\{[M(E_f) - M(E_i)]^2\} &= \int_{E_i}^{E_f} \int_{E_i}^{E_f} \mathcal{E}\{dM(E_1) dM(E_2)\} dE_1 dE_2 \\ &= \int_{E_i}^{E_f} f_1(E) dE + \int_{E_i}^{E_f} \int_{E_i}^{E_f} f_2(E_1, E_2) dE_1 dE_2 \end{aligned} \quad (8)$$

Similarly the product densities of higher order are introduced by

$$f_n(E_1, E_2, \dots, E_n) dE_1 dE_2 \dots dE_n = \mathcal{E}\{dM(E_1) dM(E_2) \dots dM(E_n)\} \quad (9)$$

whereagain the intervals should not overlap. If the (n-1)th and the n-th intervals overlap

$$\mathcal{E}\{dM(E_1) \dots dM(E_{n-1})\} = f_{n-1}(E_1, E_2, \dots, E_{n-1}) dE_1 dE_2 \dots dE_{n-1} \quad (10)$$

Taking into account such degeneracies r-th moment can be shown to be

$$\mathcal{E}\{[M(E_f) - M(E_i)]^r\} = \sum_{s=1}^r C_s^r \int_{E_i}^{E_f} \dots \int_{E_i}^{E_f} f_s(E_1, E_2, \dots, E_s) dE_1 dE_2 \dots dE_s$$

(11)

where the coefficients $\{C_s^r\}$ are given by

$$N^r = \sum_{s=1}^r C_s^r N(N-1)\dots(N-s+1) \quad (12)$$

independent of the functions $\{f_s\}$. Thus in summary for the situation when particles are distributed in the continuous \mathbf{E} -space Ramakrishnan defines a function $f_n(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_n)$ called the product density of order n such that $f_n(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_n) d\mathbf{E}_1 d\mathbf{E}_2 \dots d\mathbf{E}_n$ represents the probability that there is a particle/the range $(\mathbf{E}_1, \mathbf{E}_1+d\mathbf{E}_1)$, a particle in the range $(\mathbf{E}_2, \mathbf{E}_2+d\mathbf{E}_2), \dots$, and a particle in the range $(\mathbf{E}_n, \mathbf{E}_n+d\mathbf{E}_n)$ and while $f_n(\mathbf{E}_1, \mathbf{E}_2, \dots, d\mathbf{E}_2) d\mathbf{E}_1, d\mathbf{E}_2 \dots d\mathbf{E}_n$ is a probability magnitude, $f_n(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_n)$ is not a probability density. Then it is clear from (11)-(12) that for the calculation of the r -th moment of $[M(\mathbf{E}_r) - M(\mathbf{E}_1)]$ we should know the product densities of order less than or equal to r .

In his earliest pioneering works (1950, 1951) Ramakrishnan introduced this general theory/product density technique as above and applied it to the cosmic ray problem. In the cosmic ray problem there are two types of particles, electrons and photons. So here he had to define $f_n(\mathbf{E}_1, \dots, \mathbf{E}_n; t)$ and $g_n(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_n; t)$ as the product densities of order n of electrons and photons respectively and $fg_{n,m}(\mathbf{E}_1, \dots, \mathbf{E}_n, \mathbf{E}_{n+1}, \dots, \mathbf{E}_{n+m}; t)$ as the mixed product density of order (n,m) of electrons and photons i.e.

$f_{n,m}(E_1, \dots, E_n, E_{n+1}, \dots, E_{n+m}; t) dE_1 \dots dE_{n+m}$ is the probability that there is an electron in the interval $(E_1, E_1 + dE_1)$, an electron in the interval $(E_2, E_2 + dE_2), \dots$, an electron in the interval $(E_n, E_n + dE_n)$ a photon in the interval $(E_{n+1}, E_{n+1} + dE_{n+1})$, a photon in the interval $(E_{n+2}, E_{n+2} + dE_{n+2}), \dots$, and a photon in the interval $(E_{n+m}, E_{n+m} + dE_{n+m})$. Then it was shown how to express the variation of $f_n(E_1, \dots, E_n; t)$ with respect to the time 't' in terms of the f_i, g_i and fg_{ij} functions and detailed theory of calculation of $\mathbb{E}\{M(E, t)\}$ and $\mathbb{E}\{M(E, t)^2\}$ were presented including numerical results.

The fundamental significance of the introduction of product densities by Ramakrishnan in the above manner in a general fashion to the theory of stochastic point processes in general is very clear from the observation of Bartlett (1955) as follows:

"Many stochastic processes ----- deal with particles or individuals distributed in a continuous infinity of states ----- the behaviour of individuals depends on their age x and of particles on their energy ϵ the immediate difficulty is that only probability densities can be attached to particular values of the age x or energy ϵ and not non-zero probabilities (stochastic processes of this type will be called point processes). Integration of the probability density can only yield

the first order expectation of the number of particles or individuals in prescribed ranges of ϵ or x whereas we require ^{also to study fluctuations in these numbers} ' .

To overcome this obstacle in the cosmic ray problem, Bhabha and Ramakrishnan introduced higher order density functions called by Ramakrishnan product densities. A somewhat similar procedure was introduced independently in the population problem by D.G.Kendall, who noted its connection with the use of characteristic functional.

The differences between the contributions of Bhabha (Cf. Bhabha, 1950, Bhabha and Ramakrishnan, 1950), ~~and~~ Kendall (1949) and Ramakrishnan (1950) are clearly stated by Ramakrishnan (1972) himself as follows " The analytical solution to the problem came with the independent introduction of product density function by Bhabha (1950) and Ramakrishnan for cosmic ray cascades and cumulant densities for the population process by Kendall (1949). While Bhabha and Kendall confined themselves to second order densities, Ramakrishnan defined product densities of general order n . Bhabha's treatment followed closely that of Scott and Uhlebeck (1942) and involved a transition from a discrete E -space to continuous E while that of the author dealt with continuous E -space abinitio".

While writing about the Molecular theory of fluids (Yvon, 1935, Born and Green, 1949) Bartlett (1955) observes in p.170 of his famous book " in statistical models of dense gases and liquids the motion of the entire system isonly

determined from the simultaneous detailed positions and velocities of all the particles. Logically the system is classifiable as a point process ----- in particular, the simultaneous probability of a particle in an element dr of position - velocity phase-space and another in ds will be of the form $f_2 \frac{dr}{ds}$ where f_2 is a 'product density' of the second order". In fact Ramakrishnan himself in his Ph.D. thesis (1951) which gives the detailed account of his original development of the general theory of product density functions and their applications to physical problems has given examples of the instances in physics literature where functions with the meaning of product densities were used earlier without realising their exact significance in the context of general probability theory. The examples are multiform distribution functions used by Born and Green (1949) in their formulation of a general kinetic theory of liquids (based on Yvon's work, 1935) and many - electron distribution functions used by Mott and Sneddon (1950). As noted by Ramakrishnan (1951) these functions are not distribution functions in the correct sense of probability theory but are really product density functions. Rice (1945) also employes briefly in the course of the work on the mathematical analysis of random noise, the simultaneous probability functions of the same nature as the multiform distribution functions of the kinetic theory of liquids. Though Bartlett (1955) discusses the work of Rice (1945) explicitly in p.205 of his book

the use of product densities therein is not mentioned. The situation is very clear from what Leadbetter (1978) says, "The statistical properties of the intervals between zero crossings or between crossings by some level by a stationary stochastic process have important engineering applications the interest in this problem again dates back to the pioneering work of S.O. Rice (1945) who gives a series expression and a very simple approximation for the case of successive zero crossings by a stationary normal process. More generally one may consider the distributions of times between an axis crossing and the n-th subsequent axis crossing or between an upcrossing of zero and the n-th subsequent downcrossing and so on. Problems of this latter type have been discussed by Longuet-Higgins (1962) with particular reference to the normal case. In particular Longuet-Higgins calculates by somewhat heuristic methods a series for the probability density of the time between an arbitrary upcrossing of zero and the (r+1)th subsequent upcrossing The series just referred to may be written as

$$f(\tau) = \sum_{i=0}^{\infty} (-1)^i \binom{r+1}{i} \int \dots \int W(0, t_2, \dots, t_{n-1}, \tau) / W(0) \times \\ 0 < t_2 < \dots < t_{n-1} < \tau \quad dt_2 \dots dt_{n-1}$$

(13)

where $W(t_1, \dots, t_n) dt_1, \dots, dt_n$ represents the probability of an upcrossing of the axis in each of the intervals $(t_i, t_i + dt_i)$. These W-functions were used originally by Ramakrishnan (Cf. Bartlett, 1955) and termed 'product densities'."

The relevant portion of Rice's paper of 1945 reads 'The problem of determining the distribution function for the distance between two successive zeros seems to be quite difficult and apparently nobody has as yet given a satisfactory solution. Here we shall give some results which are related to the general problem

..... . Consider the class of curves having a zero at $x = 0$. Then in theory our methods will allow us to compute the functions

$p_0(\tau)$, $p_1(r, \tau)$, $p_2(r, s, \tau)$, associated with this class where

- $p_0(\tau) d\tau$ is the probability of curve having zero in $d\tau$
- $p_1(r, \tau) dr d\tau$ " " " zero in dr and $d\tau$
- $p_2(r, s, \tau) dr ds d\tau$ " " " zeros in dr, ds and $d\tau$

The method of inclusion and exclusion then leads to an expression for $P_0(\tau) d\tau$, the probability of having a zero at 0 and a zero in τ and $\tau + d\tau$ but none between 0 and τ . It is

$$P_0(\tau) = p_0(\tau) - \int_0^\tau \int_0^\tau p_1(r, \tau) dr d\tau + \frac{1}{2!} \int_0^\tau \int_0^\tau \int_0^\tau p_2(r, s, \tau) dr ds d\tau - \dots \tag{14}$$

Here again we run into difficult integrals^{*}. Now it is clear that here too simultaneous probability functions, to be identified only later, after Ramakrishnan's work as having the meaning of product densities have been employed without realising at all their full significance with respect to the general theory of point processes as remarked by Ramakrishnan (1951) in the case of the works of Born and Green (1949) and Mott and Sneddon (1950). In fact in all these works prior to Ramakrishnan (1950) the relation between product densities used ^{as} simultaneous probability functions and the moments of the stochastic variable embodied in (11)-(12) has not been realised at all.

In recent literature some times works (such as Yvon (1935), and Rice (1945)) earlier to Ramakrishnan (1950) are referred to as the the origin of the concept of product density functions (Fischer (1972), Snyder (1975) Srinivasan and Mehata (1976), Brillinger (1978)). In the review article of Brillinger (1978) the basic properties of product densities are defined and discussed in equations (2.2.4) and (2.5.1 - 2.5.6) * (change of notation is obvious)

$$* \quad p_k(t_1, \dots, t_k) dt_1 \dots dt_k = \text{Prob} \left\{ dN(t_1)=1, \dots, dN(t_k)=1 \right\}_{t_1, \dots, t_k \text{ distinct}} \quad (2.24)$$

$$p_k(t_1, \dots, t_k) dt_1 \dots dt_k = E \left\{ dN(t_1) \dots dN(t_k) \right\} \quad (2.2.1)$$

$$\text{Prob} \{ N(I) = m \} < L_g |I|^m \quad (2.5.2)$$

$$E \{ dN(t) dN(t) \} = E \{ dN(t) \} + O(dt^2) = p_1(t) dt + O(dt^2) \quad (2.5.3)$$

$$\int_0^t p_1(u) du = E \{ N(t) \} \quad (2.5.4)$$

$$\int_0^t \int_0^t p_2(u_1, u_2) du_1 du_2 = E \left\{ \int_{u_1 \neq u_2} dN(u_1) dN(u_2) \right\} = E \left\{ \left(\int_0^t dN(u) \right)^2 - \int_0^t (dN(u))^2 \right\} \quad (2.5.5)$$

$$\int_I \dots \int_I p_k(t_1, \dots, t_k) dt_1 dt_2 \dots dt_k = E \{ N(I)^{(k)} \} \quad (2.5.6)$$

$$N^{(k)} = N(N-1) \dots (N-k+1), \quad k=1, 2, \dots$$

exactly following the original treatment of Ramakrishnan (1950, 1951) as given above while the references given following these definitions are: Macchi (1945) and Brillinger (1972, 1975). The above considerations show clearly how such accounts of the development of the product density technique in the general theory of point processes do not give a proper historical perspective. In fact the very name 'product density' for $f_n(E_1, \dots, E_n)$ was coined by Ramakrishnan as clearly mentioned by Bartlett (1955).

Thanks are due to Professor R. Vasudevan for discussions.

REFERENCES

- Ramakrishnan, Alladi (1950), 'Stochastic Processes relating to particles distributed in continuous infinity of states', Proc. Camb. Phil. Soc. 46, 595.
- Ramakrishnan, Alladi (1951) 'Stochastic Processes and their applications to physical problems', Ph.D. Thesis, Univ. of Manchester.
- Ramakrishnan, Alladi (1959), 'Probability and Stochastic Processes', Handbuch der Physik, Springer-Verlag III/2, 524.
- Ramakrishnan, Alladi (1972) in 'Stochastic Point Processes, Statistical Analysis, Theory and Applications', Ed. P.A.W. Lewis, Wiley-Interscience, p. 533.
- Bartlett, M.S. (1955), 'An Introduction to Stochastic Processes, with special reference to methods and applications' (1966 Edition).

- Bhabha, H.J. (1950), 'On the Stochastic Theory of Continuous parametric systems and its Applications to electron cascades', Proc. Roy. Soc. A202, 301.
- Bhabha, H.J. and Alladi Ramakrishnan (1950), Proc. Indian Acad. Sci. 32, 141.
- Kendall, D.G. (1949), 'Stochastic Processes and Population Growth J.R. Statist. Soc. B11, 230.
- Scott, W.T. and Uhlenbeck, G.E. (1942), 'On the theory of Cosmic Ray Showers', Phys. Rev. 62, 497.
- Yvon, (1935), 'La Theories Statistique des Fluids et l'Eqn d'Etat', Hermann, Paris.
- Born, M. and Green, H.S. (1949), 'A general Kinetic Theory of liquids', Cambridge Univ. Press.
- Mott, N.F. and Sneddon, I. (1950), 'Wave Mechanics and its Applications' OXFORD.
- Rice, S.O. (1945) 'Mathematical Analysis of Random Noise', Bell Syst. Tech. J. 24, 46.
- Leadbetter, M.R. (1972) in 'Stochastic Point Processes, statistical Analysis, Theory and Applications', Ed. P.A.W. Lewis, Wiley-Interscience.
- Longuet-Higgins, M.S. (1962), 'The distribution of Intervals between Zeros of a stationary Random function', Phil. Trans. Roy. Soc. 254, 557.
- Fischer, L. (1972) in 'Stochastic Point Processes: Statistical Analysis, Theory and Applications', Ed. P.A.W. Lewis, Wiley-Interscience.
- Snyder, D.L. (1975), 'Random Processes', John-Wiley.

- Srinivasan, S.K. and Mehata, K.M. (1976), 'Stochastic Processes', Tata-Mc-Graw Hill, New Delhi.
- Brillinger, D.R. (1978), in 'Developments in Statistics', Vol.1, Ed. P.R.Krishnaiah, Academic Press, p.34.
- Macchi, O., (1975) 'The coincidence approach to stochastic point processes', Advan. Appl. Prob. 7, 83.
- Brillinger, D.R. (1972), 'The spectral Analysis of Stationary interval functions', p.483 in Proc. Berkeley Sym- 6th (Eds. L.M.LeCam, J.Neyman, and E.L.Scott) Vol.1, Univ. of California, Berkeley.

ROLE OF PROBABILITY IN NON-EQUILIBRIUM PROCESSES

V. Radhakrishnan

MATSCIENCE, The Institute of Mathematical Sciences, MADRAS-600 113

+*+*+

The concept of 'equilibrium' is very familiar through mechanics. In very important circumstances one would like to know the equilibrium state of the system, and the stability of that state against perturbations. The concept of 'equilibrium' in thermodynamics is conveniently introduced through the concept of Entropy, while in mechanics the parameter energy was the one usually made to be minimum to find the equilibrium state. In thermodynamics, Entropy is maximised or negative of entropy minimised. The connection between the laws of thermodynamics and mechanics, as is well known, is through statistical mechanics. The element of probability is introduced therefore at this stage. Therefore probability theory plays an important role in the development of thermodynamics from the laws of mechanics. In this study usually one separates the two classes of problems, first one called 'equilibrium processes' and the second one 'Non-equilibrium processes'. The second topic is obviously more difficult to deal with. Here we shall qualitatively mention a few important examples of non-equilibrium processes and their resemblance and difference from equilibrium processes. For quantitative details reference is given to a few articles at the end.

The employment of quantum field theory in statistical physics, has enabled great progress in recent years, and the branch of many body problems emerged as an important topic. Great successes achieved

during recent years include the understanding of super conductivity, superfluidity, coherent systems such as lasers and masers, etc. The recognition of long range order existing in such coherent states, and their description in terms of a single 'order parameter', has not only clarified the concepts but also simplified the mathematical treatments. Apart from these during recent years it has become evident that many biological processes have great resemblances to these systems and many models have been constructed to parallel the mathematical treatment with reasonable successes. While discussing such non-equilibrium processes in these contexts, generally one is interested in 'stationary Non-equilibrium processes'.

In the superfluid system, which is 'equilibrium' case, the probability element enters through the partition function, which once specified, determines all the physical properties of the system in that equilibrium state. The appropriate theory for this, the famous Ginzburg-Landau Theory, gives this as

$$P = \text{Const. } e^{-a |\psi|^2 - b |\psi|^4} \quad \text{where } \psi \text{ is the}$$

complex order parameter. Due to the non-linearity the equilibrium state is achieved as one with $|\psi| \neq 0$, which is the true, stable minimum for the superfluid state, in the plot of the Free-Energy versus the order parameter. The minimum at $|\psi| = 0$ which was stable in the non-superfluid state becomes unstable equilibrium point and the new equilibrium is one at which free energy obtains a new minimum corresponding to $|\psi| \neq 0$ in the plot $F = -a |\psi|^2 + b |\psi|^4$.

The laser operates above the threshold in a 'Non-Equilibrium steady state'. Several workers, in particular Haken and his collaborators have studied the quantum mechanical equations describing this state and have brought out the resemblances to the equilibrium superfluid state, both these states having the common property of 'Coherence'. They have shown, after drastic simplification justified in various stages, that the systems of coupled equations for the laser can be reduced to a single stochastic equation for the 'order parameter' which is the complex mode amplitude. This stochastic non-linear equation, after simplification is shown to yield the probability function for the mode amplitude to be essentially of the form,

$$P(q) \sim e^{-aq^2 - bq^4}$$

which is the form obtained earlier for the equilibrium superfluid case.

Frohlich has pointed out how in biological systems, energy can be absorbed at random and given out in an ordered coherent fashion. The analogy with laser operation is then evoked. It is therefore clear that the application of the theory of probability and stochastic processes, in the field of Non-equilibrium steady state processes is of great promise for the future research.

REFERENCES:

1. H.Haken, Handbuch der Physik Article, Vol XXV/2c, Springer-Verlag.
2. H.Haken, "Synergetics" A Workshop, Springer-Verlag, 1977, (1970)
3. H.Frohlich, ^(Ed) ⁱⁿ "Theoretical ^{Physics and} Biology", Ed. M. Marois, North Holland, 1969,

COORDINATIZING BAER*- SEMIGROUPS AND EVENT - STATE -OPERATION STRUCTURE

D. Venugopala Rao,

Institute of Mathematics, 38/656, T.D. Road, ERNAKULAM
Cochin

+*+*+

1. INTRODUCTION

An extremely important problem in the quantum logic approach to quantum mechanics is to characterize those orthomodular lattices that can be embedded in Hilbert space (1,7,10,11,15,17,22). Since it is well known (8,9) that an arbitrary orthomodular lattice L has no such embedding, one must add physically motivated conditions to L for an embedding to exist. In this paper we shall give a characterization in terms of a single physical condition. The idea behind this condition relies on concepts from the algebraic (5,13) and operational (3,4,19,20) approaches to quantum mechanics. In this way, the present paper obtains a unifying connection between these three approaches to axiomatic quantum mechanics.

In order to derive our result we obtain a characterization of Baer*- semigroups which are representable in Hilbert space. This characterization has an interest in its own right because Baer*-semigroups may be physically interpreted as sets of operations for quantum systems.

To illustrate our result, let H be a complex Hilbert space and let $B(H)$ be the set of bounded linear operators on H . For $A \in B(H)$, let A' denote the projection onto the null space of A . Let $S \subseteq B(H)$ be a semigroup containing 0 , I and A^* , A' whenever $A \in S$, and suppose $\|A\| \leq 1$ for every $A \in S$. Then S

is an example of a Baer*-Semigroup. For $x \in H$, define $\phi(A) = \langle Ax, x \rangle$, $A \in S$. Then $\phi(0) = 0$ and using Schwarz's inequality we see that ϕ is bounded. Moreover, if $\{\lambda_1, \lambda_2, \dots, \lambda_n\} \in \mathbb{C}$, $\{A_1, \dots, A_n\} \subseteq S$ we see that

$$\begin{aligned} \sum \bar{\lambda}_i \lambda_j \phi(A_i^* A_j) &= \sum \bar{\lambda}_i \lambda_j \langle A_j x, A_i x \rangle \\ &= \left\| \sum \lambda_j A_j x \right\|^2 \geq 0 \end{aligned}$$

We shall call such functions " *-definite." The *-definite function ϕ has an additional important property. If $\phi(A^* A) = 0$, then $Ax = 0$, which implies that $A'x = x$. Hence, $\phi(A') = \phi(I)$. We shall call *-definite functions with this property 'consistent'. We thus see that S possesses a separating set of consistent *-definite functions. Our main result shows that a slightly stronger condition characterizes Baer *-semigroups which are isomorphic to an $S \subseteq B(H)$.

2. Embedding and Representations.

Let L be an orthomodular lattice, H a complex Hilbert space, and $P(H)$ the lattice of orthogonal projections on H . A map $r: L \rightarrow P(H)$ is a morphism of L in H if r satisfies:

$$(M 1) \quad r(a \wedge b) = r(a) \wedge r(b) \quad \text{for all } a, b \in L$$

$$(M 2) \quad r(0) = 0, \quad r(1) = I$$

$$(M 3) \quad r(a') = r(a)' \quad \text{for all } a \in L.$$

It is easy to see that a morphism preserves order, all lattice operations, orthogonality, compatibility, etc. An injective morphism is called an embedding. If there exists an embedding of L , we call L embeddable.

Now let S be a Baer $*$ -semigroup (6,19) whose set of closed projections we denote by $p'(S)$, and let $B(H)$ denote the set of bounded linear operators on H . For $a \in B(H)$, a' denotes the projection onto the null space of a . A Baer $*$ -representation ρ of S in H is a map $\rho : S \rightarrow B(H)$ satisfying:

- (R 1) $\rho(0) = 0, \rho(1) = I,$
- (R 2) $\rho(st) = \rho(s)\rho(t)$ for all $s, t \in S$
- (R 3) $\rho(s^*) = \rho(s)^*$ for all $s \in S,$
- (R 4) $\rho(s') = \rho(s)'$ for all $s \in S,$
- (R 5) $\|\rho(s)\| \leq 1$ for all $s \in S.$

Since any orthomodular lattice can be coordinatized by a Baer $*$ -Semigroup (6,7), there is a close connection between morphisms of an orthomodular lattice and Baer $*$ -representations of Baer $*$ -semigroups.

Lemma 1. An orthomodular lattice L is embeddable in H if and only if some coordinatizing Baer $*$ -semigroup for L has an injective Baer $*$ -representation in H .

Because of Lemma 1, characterizations of embeddable orthomodular lattices can be found by investigating Baer $*$ -representations of Baer $*$ -semigroups.

3. Baer *-representations of Baer *-semigroups

A Baer *-representation $\rho : S \rightarrow B(H)$ is cyclic if there exists an $x \in H$ such that the linear hull of $\{\rho(s)x : s \in S\}$ is dense in H . It is not difficult to show that any Baer *-representation is a direct sum of cyclic ones. For this reason, we shall mainly restrict our attention to cyclic Baer *-representations.

A bounded complex-valued function ϕ on S is *-definite (2, 16) if $\phi(0) = 0$ and for any finite collections $\{\lambda_1, \dots, \lambda_n\} \subseteq \mathbb{C}$, $\{s_1, \dots, s_n\} \subseteq S$, we have

$$\sum_{i,j=1}^n \overline{\lambda_i} \lambda_j \phi(s_i^* s_j) \geq 0$$

Notice that if ϕ is *-definite and $t \in S$, then the function $\phi_t(s) = \phi(t^* s t)$ is also *-definite. We say that a *-definite function ϕ is consistent if $\phi(s^*s) = 0$ implies that $\phi(s') = \phi(1)$. In a Baer *-semigroup, $s^*s = 0$ implies that $s' = 1$. Hence, a consistent *-definite function is one that respects the above implication. For example, if ϕ is faithful ($\phi(s^*s) = 0$ implies that $s = 0$), then ϕ_t is consistent for all $t \in S$. Indeed,

$$\begin{aligned} \phi_t(s^*s) = 0 &\Rightarrow \phi[(s t)^* s t] = 0 \Rightarrow s t = 0 \Rightarrow s' t = t \\ &\Rightarrow \phi_t(s') = \phi(t^* s' t) = \phi(t^* t) = \phi_t(1). \end{aligned}$$

In the next paragraph we shall define ϕ to be strongly consistent if limits of certain *-definite functions derived from ϕ are consistent.

Let ϕ be a *-definite function on S and let $\{c_1, \dots, c_m\} \subseteq \mathbb{C}$, $\{t_1, \dots, t_m\} \subseteq S$. Define the function $\phi_1 \equiv \phi\{c_i, t_i\}$ by $\phi_1(s) = \sum_{k,l} \bar{c}_k c_l \phi(t_k^* s t_l)$.

It is not hard to show that ϕ_1 is again *-definite. Now

let $\{c_1^k, \dots, c_{n(k)}^k\} \subseteq \mathbb{C}$, $\{t_1^k, \dots, t_{n(k)}^k\} \subseteq S$, $k=1, 2, \dots$

be a sequence of finite sets and let $\phi_k \equiv \phi\{c_i^k, t_i^k\}$.

We say that the sequence ϕ_k converges if

$$\lim_{k \rightarrow \infty} \phi_k(s) = \operatorname{Re} \lim_{k, l \rightarrow \infty} \sum_{i, j} \bar{c}_i^l c_j^k \phi(t_i^{l*} t_j^k)$$

Lemma 2. If ϕ_k converges, then $\psi(s) \equiv \lim_{k \rightarrow \infty} \phi_k(s)$ exists

for all $s \in S$ and ψ is *-definite.

We call ψ in Lemma 2 a limit function for ϕ . We say that ϕ is strongly consistent if every limit function for ϕ is consistent.

Theorem 3. (a) let $\rho: S \rightarrow B(H)$ be a Baer *-representation, $x \in H$, and $\phi(s) = \langle \rho(s)x, x \rangle$, $s \in S$. Then ϕ is a strongly consistent *-definite function. Conversely, if ϕ

is a strongly consistent *-definite function on S , then there exists a Baer *-representation $\rho: S \rightarrow B(H)$ which is cyclic with cyclic $x \in H$ such that $\phi(s) = \langle \rho(s)x, x \rangle$ for all $s \in S$. [vector

(b) S admits an injective Baer *-representation if and only if S possesses a separating set of strongly consistent *-definite functions.

Corollary 4. An orthomodular lattice L is embeddable if and only if some coordinatizing Baer $*$ -semigroup for L possesses a separating set of strongly consistent $*$ -definite functions.

Let L be a σ -orthocomplete orthomodular lattice. A σ -embedding is an embedding $r: L \rightarrow P(H)$ satisfying $r(\bigvee a_i) = \bigvee r(a_i)$ for any sequence of mutually orthogonal elements a_1, a_2, \dots in L . Let S be a coordinatizing Baer $*$ -semigroup for L . A $*$ -definite function on S is σ -additive if $\phi_t(\bigvee e_i) = \sum \phi_t(e_i)$ for every $t \in S$ and every sequence of mutually orthogonal $e_i \in p'(S)$.

Corollary 5. A σ -orthocomplete orthomodular lattice L is σ -embeddable if and only if some coordinatizing Baer $*$ -semigroup for L possesses a separating set of strongly consistent σ -additive $*$ -definite functions.

It follows from Theorem 3 that if ϕ is a strongly consistent $*$ -definite function on S such that $\phi(1) = 1$, then $0 \leq \phi(e) \leq 1$ for every $e \in p'(S)$ and $\phi(\bigvee e_i) = \sum \phi(e_i)$ for every finite mutually orthogonal sequence $e_1, \dots, e_n \in p'(S)$.

Thus, $\phi|_{p'(S)}$ is a (finitely-additive) state on $p'(S)$. In this way, a strongly consistent $*$ -definite function on S is an extension of a state on $p'(S)$. Thus, we may interpret Corollary 4 as saying that L is embeddable if and only if L possesses a sufficiently large number of (finitely additive) states which have extensions to S . A similar remark holds in Corollary 5 for σ -embeddings and states on L .

The existence of Baer *-representations for finite Baer *-semigroups becomes much simpler, in this case one only needs *-definite function ϕ such that $\phi\{c_i, t_i\}$ is consistent

for all finite sets $\{c_1, \dots, c_n\} \subseteq \mathbb{C}$; $\{t_1, \dots, t_n\} \subseteq S$.

4. Event - State - Operation Structures.

We have characterized embeddable orthomodular lattices in terms of coordinatizing Baer *-semigroups. Following James Pool (19, 20) , we now show that a certain coordinatizing Baer *-semigroup has physical significance.

An event-state structure (19, 20) is a triple (E, S, P) where E, S are nonempty sets and $P : E \times S \rightarrow [0, 1]$ satisfies (ES 1) - (ES 7) below, where for $p \in E$, $S_1(p) = \{\alpha \in S \cdot P(p, \alpha) = 1\}$ and $S_0(p) = \{\alpha \in S : P(p, \alpha) = 0\}$.

(ES1) If $p, q \in E$, and $S_1(p) = S_1(q)$, then $p = q$.

(ES2) There exists an element $1 \in E$ such that $S_1(1) = S$.

(ES3) If $p, q \in E$ and $S_1(p) \subseteq S_1(q)$, then $S_0(q) \subseteq S_0(p)$.

(ES4) If $p \in E$, then there exists a $p' \in E$ such that

$$S_1(p') = S_0(p), S_0(p') = S_1(p).$$

(ES5) If $p_1, p_2, \dots \in E$ and $S_1(p_i) \subseteq S_0(p_j)$ for $i \neq j$, then there exists a $p \in E$ such that $S_1(p_i) \subseteq S_1(p)$ for all i , if $S_1(p_i) \subseteq S_1(q)$ for all i , then $S_1(p) \subseteq S_1(q)$ and $P(p, \alpha) = \sum P(p_i, \alpha)$ for all $\alpha \in S$.

- (ES6) If $\alpha, \beta \in S$ and $P(p, \alpha) = P(p, \beta)$ for all $p \in E$, then $\alpha = \beta$.
- (ES7) If $\alpha_1, \alpha_2, \dots \in S$, $t_1, t_2, \dots \in [0, 1]$ with $\sum t_i = 1$, then there exists an $\alpha \in S$ such that $P(p, \alpha) = \sum t_i P(p, \alpha_i)$ for all $p \in E$.

We call the elements of E events and the elements of S states. If $p, q \in E$ we define $p \leq q$ when $S_1(p) \subseteq S_1(q)$ and $p \perp q$ when $S_1(p) \subseteq S_1(q') = S_0(q)$. For $\alpha \in S$, define $\mu_\alpha : E \rightarrow [0, 1]$ by $\mu_\alpha(p) = P(p, \alpha)$ and let $\hat{S} = \{ \mu_\alpha : \alpha \in S \}$.

Theorem 6. (Pool) If (E, S, P) is an event-state structure, then (E, \leq, \perp) is a σ -orthocomplete orthomodular poset. If $\alpha \in S$, then $\mu_\alpha(1) = 1$ and $\mu_\alpha(\bigvee p_i) = \sum \mu_\alpha(p_i)$ for any sequence of mutually orthogonal $p_i \in E$.

Let (E, S, P) be an event-state structure, and let Σ denote the set of maps $x : D_x \subseteq S \rightarrow S$. We define $1 \in \Sigma$ to be the identity map on S and $0 \in \Sigma$ is defined by $D_0 = \phi$. If $x, y \in \Sigma$, then $x \circ y \in \Sigma$ is defined by $D_{x \circ y} = \{ \alpha \in D_y : y \alpha \in D_x \}$ and $(x \circ y) \alpha = x(y \alpha)$ for all $\alpha \in D_{x \circ y}$. An event-state-operation structure is a 4-tuple (E, S, P, Ω) where (E, S, P) is an event-state structure and $\Omega : E \rightarrow \Sigma$, $\Omega : p \mapsto \Omega_p$ satisfies:

(01) If $p \in E$, then $D_{\Omega_p} \equiv D_p = \{ \alpha \in S, P(p, \alpha) \neq 0 \}$.

(02) If $p \in E$, $\alpha \in D_p$ and $P(p, \alpha) = 1$, then $\Omega_p \alpha = \alpha$.

(03) If $p \in E$, and $\alpha \notin D_p$, then $P(p, \Omega_p \alpha) = 1$.

(04) If $p_1, \dots, p_n, q_1, \dots, q_m \in E$ and $\Omega_{p_1} \circ \dots \circ \Omega_{p_n}$
 $= \Omega_{q_1} \circ \dots \circ \Omega_{q_m}$, then $\Omega_{p_n} \circ \Omega_{p_{n-1}} \circ \dots \circ \Omega_{p_1}$
 $= \Omega_{q_m} \circ \Omega_{q_{m-1}} \circ \dots \circ \Omega_{q_1}$.

(05) If $x = \Omega_{p_1} \circ \dots \circ \Omega_{p_n}$, then there exists a $q_x \in E$
 such $S_1(q_x) = D_x^c$ (the complement of D_x).

(06) If $p, q \in E$, $q \leq p$ and $\alpha \in D_p$, then $P(q, \Omega_p \alpha) =$
 $= P(q, \alpha) / P(p, \alpha)$.

(07) If $p, q \in E$ are compatible and $\alpha \in D_p$, then
 $P(q, \Omega_p \alpha) = P(p \wedge q, \Omega_p \alpha)$.

For $p \in E$, $\alpha \in D_p$ we call $\Omega_p \alpha$ the state conditioned
on the event p and the state α . The elements of the set

$$S_\Omega = \{ \Omega_{p_1} \circ \dots \circ \Omega_{p_n} : p_1, \dots, p_n \in E \} \subseteq S$$

are called operations. We define the map $*$: $S_\Omega \rightarrow S_\Omega$ as
 follows:

if $x = \Omega_{p_1} \circ \dots \circ \Omega_{p_n} \in S_\Omega$ then $x^* = \Omega_{p_n} \circ \dots \circ \Omega_{p_1}$. It

is not hard to show that q_x in (05) is unique. For $x \in S_\Omega$

we define $x' = \Omega_{q_x}$, where q_x is this unique event.

Theorem 7. (Pool) Let (E, S, P, Ω) be an event-state-operation structure. Then $(E, \leq, ')$ is a σ -orthocomplete orthomodular lattice and $(S_\Omega, 0, *, ')$ is a coordinatizing Baer *-semigroup for $(E, \leq, ')$.

All the axioms for an event-state-operation structure have been physically motivated by Pool [19, 20]. Thus, we have physically motivated axioms for a σ -orthocomplete orthomodular lattice together with a coordinatizing Baer *-semigroup. It follows from Corollary 5 that an event-state-operation structure (E, S, P, Ω) is embeddable in a Hilbert space if and only if there exists a separating set of strongly consistent σ -additive *-definite functions on $(S_\Omega, 0, *, ')$.

5. Connection to the Algebraic Approach

In this section we give a physical significance for *-definite functions using a connection to the algebraic approach. Let S be an arbitrary *-semigroup with 0 and 1, and let \mathcal{A} be a Banach *-algebra with unit I. We say that \mathcal{A} is an enveloping Banach *-algebra for S if there is an injection $\rho: S \rightarrow \mathcal{A}$ satisfying (R1), (R2), (R3), (R5) and

$$(R6) \quad \mathcal{A} = \left\{ \sum_{i=1}^n c_i \rho(s_i) \mid c_i \in \mathbb{C}, s_i \in S, n \in \mathbb{N} \right\}^{\wedge}$$

where \wedge denotes the closure.

We now show that any $*$ -semigroup S with 0 and 1 possesses an enveloping Banach $*$ -algebra. Let $\mathcal{L}_1(S)$ be the set of functions $f: S \rightarrow \mathbb{C}$ satisfying $f(s) = 0$ except for countably many $s \in S$ and $\|f\| = \sum_{s \neq 0} |f(s)| < \infty$. Moreover, we identify any two functions f and g if $f(s) = g(s)$ for every $s \neq 0$. Under pointwise sums and scalar products it is clear that $\mathcal{L}_1(S)$ is a complex normed linear space. For $f, g \in \mathcal{L}_1(S)$ we define

$$(f \cdot g)(s) = \sum \{ f(r) g(t) : r, t \in S, r t = s \}$$

For $f \in \mathcal{L}_1(S)$ we define the involution f^* of f by $f^*(s) = \overline{f(s^*)}$. Let $\delta: S \rightarrow \mathcal{L}_1(S)$ be the map $[\delta(s)](t) = \delta_{s,t}$ (the Kronecker delta). The proof of the next lemma uses results from (14, 21).

Lemma 8. Under the map $\delta: S \rightarrow \mathcal{L}_1(S)$, $\mathcal{L}_1(S)$ is an enveloping Banach $*$ -algebra for S . Moreover, every element of $\mathcal{L}_1(S)$ has the form $f = \sum c_i \delta(s_i)$ where $\|f\| = \sum |c_i| < \infty$.

Recall that a linear functional ω on a Banach $*$ -algebra \mathcal{A} is positive if $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$. A state is a positive linear functional ω on \mathcal{A} satisfying $\omega(I) = 1$. Of course, this concept of state is different than our previous definition of a state on an orthomodular lattice.

Lemma 9. Let S be a $'$ -semigroup with $0,1$ and let \mathcal{A} be an enveloping Banach $*$ -algebra for S with enveloping map $\delta : S \rightarrow \mathcal{A}$. If ϕ is a $*$ -definite function on S , then there exists a unique positive linear functional ω on \mathcal{A} such that $\omega[\delta(s)] = \phi(s)$ for all $s \in S$. Conversely, if ω is a positive linear functional on \mathcal{A} , then $\phi(\cdot) = \omega[\delta(\cdot)]$ is a $*$ -definite function on S . If $\phi(1) = 1$, then ω is a state.

In the algebraic approach to quantum mechanics, the bounded observables of a quantum system are represented by self-adjoint elements of a Banach $*$ -algebra (usually a C $*$ -algebra) \mathcal{A} and the expectation functionals are represented by states on \mathcal{A} . The above lemma gives a physical interpretation for the $*$ -definite functions on S . The normalized $*$ -definite functions on S are the restrictions to S of a state on any enveloping Banach $*$ -algebra for S . The consistency condition has an important physical significance. To see this let S be a Baer $*$ -semigroup and let \mathcal{A} be an enveloping Banach $*$ -algebra with enveloping map $\delta : S \rightarrow \mathcal{A}$. Let ϕ be a normalized consistent $*$ -definite function on S and let ω be the corresponding unique state on \mathcal{A} . In some approaches to quantum mechanics (3,4,10) the positive elements of S (i.e. elements of the form $s*s$) are interpreted as quantum mechanical "effects". Then $\omega[\delta(s*s)] = \phi(s*s)$ is interpreted as the probability that the effect $s*s$ is observed when the system is in the state ω . The probabilistic interpretation is mathematically reasonable since a state ω is continuous with norm 1 [5, 18] and hence

$$0 \leq \omega [\delta(s*s)] \leq \|\delta(s*s)\| \leq 1.$$

Now if $s*s$ is an effect, $(s*s)'$ corresponds to the event which occurs if and only if the effect $s*s$ is not observed. It follows that if the effect $s*s$ has zero probability of being observed in the state ω , then $(s*s)'$ occurs with certainty in the state ω . Since $(s*s)' = s'$ [6], we should have $\omega [\delta(s*s)] = 0$ implies that $\omega \delta(s') = 1$. In terms of ϕ this becomes $\phi (s*s) = 0$ implies $\phi (s') = 1$. This is precisely the consistency condition.

The proofs of the theorems in Sections 2, 3 and 5 may be found in [11, 12].

REFERENCES

1. Birkhoff G., Lattice Theory, Amer.Math.Soc. Colloqui. Publ. Vol.25, Providence, R.I. (1967)
2. Birkhoff G. and J.von Neumann, The logic of quantum mechanics, Ann. Math. 37, 823-843 (1938)
3. Davies E.B., and J.Lewis, An operational approach to quantum probability, Communi. Math. Phys. 17, 239-269 (1970)
4. Davies E.B., Quantum Theory of Open Systems, Academic Press, N.Y.
5. Emch G., Algebraic methods in statistical mechanics and quantum field theory, Wiley-Interscience, N.Y. (1973)
6. Foulis, D., Baer *-semigroups, Proc.Amer.Math. Soc. 11, 648-654 (1960)
7. Foulis, D., A note on orthomodular lattices, Portugal,Math. 1, 65-72 (6).
8. Greechie R., An orthomodular lattice with full set of states not embeddable in Hilbert space, Carib. J.Sci.Math.2, 15-26, 1969.

9. Greechie R., Orthomodular lattices admitting no states,
J. Comb. Theory 10, 119-132, 1971.
10. R.Lindall and H.Maserick, Positive definite functions on
involution semi groups, Duke Math. J.39, 771-782, 1971.
11. Venugopala Rao, D. An algebraic approach to quantum field
theory, Math. Teacher, 71, 87-89, 1979.
12. Venugopala Rao, D., Hilbert space representation Baer *-semi-
groups, Mathematik der Univ., bonn, 38, 567-568, 78.
14. Hewitt E. and H.Zuckerman, The l_1 -algebra of commutative
semigroup, Trans. Amer. Soc. 83, 70-97, 1956.
15. Jauch, J. Foundations of quantum mechanics, Addison-Wesley,
r reading Mass.
16. Mackey, G., The mathematical foundations of quantum mechanics,
Benjamin, N.Y.
17. Zuckerman, Logical structure of quantum mechanics, Springer
Verlag, N.Y.
18. Naimark M. Normed Algebras, Noodhoff, Groningen, The Netherlands.
19. Piron C., Axiomatic Quantique, Helv. Phys. Acta., 37, 439-468
(1964).
20. Pool, J., Semi-modularity and the logic of quantum mechanics,
Comm. Math. Phys. 1, 173-182 (1972)
21. Pool, J., Baer *-Semigroups and the logic of quantum mechanics
Comm. Math. Phys. 9, 212-228 (1968).
22. Ruttiman G., On the logical structure of quantum mechanics,
Found. Phys. 1 , 173-182 (1972).
23. Varadarajan, V., Geometry of Quantum Theory, Vols. 1 and 2,
Van Nostrand, Princeton, N.J. (1968) and (1970).