DENSITY MATRIX METHODS IN NUCLEAR REACTIONS

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PREFACE

This Report consists of the lectures given at the Institute of Mathematical Sciences, Matscience, Madras-600020, India during November 1977, March 1978, on the applications of density matrix methods to muon capture processes. The reason for the choice of muon capture process is the author's familiarity with it and the techniques given here can be applied to other reactions as well. A brief discussion of radiative pion capture and gamma decay is also given.

These lectures are mainly based upon the following books and research papers.

(1) Polarization Phenomeno in Beta and gamma Emission.
M.E.Rose, Lectures in Theoretical Physics, Brandeis Summer Institute, Vol.2, 1961, W.A.Benjamin, Inc.

(2) Angular Correlations
H.Frauenfelder, R.M.Steffen in
α, β, γ ray spectroscopy, Vol.2, Ed.K.Siegbahn, North-Holland Pub. Co. 1965.

(3) Muon Capture and β-decay, M. Morita (1973), W.A.Benjamin Inc.

(4) V.Devanathan, R.Parthasarathy and P.R.Subramanian
(Recoil nuclear polarization in muon capture)
Ann. Phys. 73 (1972) 291.

(5) V.Devanathan, R.Parthasarathy and G.Ramachandran
(Polarization of emitted neutron in muon capture)
Ann. Phys. 72 (1972) 498.

(6) P.R.Subramanian, R.Parthasarathy and V.Devanathan
(Nuclear Orientation following muon capture by Spin Zero nucleus)
Nucl. Phys. A262 (1976) 433.

(7) R.Parthasarathy
Lectures given at Louvain University - 1975 (unpublished)

(8) J.Bernabeu
(Restrictions for asymmetry and polarizations of recoil in muon capture)
Phys. Lett. 55B (1975) 313.

(9) V.Devanathan and P.R.Subramanian
Ann. Phys. 93 (1975) 25.

(10) R.Parthasarathy and V.N.Sridhar
(Gamma-Neutrino angular correlations in muon capture by ^{28}Si)
Phys. Rev. C. July issue 1978.

I take this opportunity to thank Professor A.Ramakrishnan for the constant encouragement throughout the course of lectures, without which, the lectures could not have been put in the form of a Matscience Report. I enjoyed the discussions with Professor J.Bernabeu at CERN, Professor N.Vinh Mau and Dr.A.Boussy at Orsay, Professor J.P.Deutsch, Professor L.Grenacs and Professor N.C.Mukhopadhyay at Louvain and Professor V.Devanathan and Professor K.Srinivasa Rao at Madras, which benefitted me a great amount in understanding the various aspects of the problem.

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Lecture 1:

We shall begin with an introduction to pure and mixed states in quantum mechanics. Quantum Mechanics deals with phenomena in which a maximum of information is available about the system under consideration. States of maximum information are called 'pure states'. A pure state is characterised by the existence of an experiment that gives a result predictable with certainty when performed on a system in that state and in that state only. How to represent a pure state? In fact, one can construct a variety of Hermitian operators which the given pure state as an eigen state. When it is not convenient to identify a pure state by specifying the relevant experiment or by the corresponding operator, the state may be identified as a linear superposition of eigenstates of any suitable complete set of operators. The representation of a pure state either as an eigenstate of a particular operator or as a superposition of eigenstates of another arbitrary operator, is usually called a state vector .

However, quantum-mechanical systems also occur for which no complete experiment gives a unique result predictable with certainty. In such a case we say that the information on the system is less than the maximum, with reference to the lack of a complete experiment. Such states are called mixed states since they can be described by the incoherent superposition of pure states. What do we mean by incoherent superposition? By incoherent superposition, we mean that to calculate the probability of finding a certain experimental result with a system in the mixed state, one must first calculate the probability for each of the pure state and take an average, attributing to each of the pure state an assigned weight. It should be remembered that the description of a mixed state as the incoherent superposition of pure states is not unique. For example, unpolarized light (a mixed state) can be described by two linear polarizations (pure states) or by two circular polarizations (pure states). Although given unpolarized light, one can tell by suitable experiment, whether it is made up of two linearly polarized or circularly polarized beams, the non-uniqueness means that, two linear or circular polarization can produce unpolarized beam.

Definition of the Density Matrix:-

A pure state can in general be written as

$$\psi = \sum_n a_n u_n \tag{1.1}$$

where u_n's are eigenvectors of some complete set of operators. The expectation value of an operator Q with respect to this pure state is

$$\langle Q \rangle_i = \sum_n{}' \sum_m{}' a_n^{*} a_m Q_{nm} \tag{1.2}$$

When a non-pure state is expressed by a linear combination of pure states ψ_i (given in 1.1) with corresponding statistical weights p_i, the expectation value of Q with respect to the non-pure state is the sum of the expectation values in each pure state ψ_i multiplied by the weight factor p_i. Let us call this expectation value by 'average expectation value' and it is given by

$$\langle \widetilde{Q} \rangle = \sum_i{}' p_i \langle Q \rangle_i \tag{1.3}$$

Substituting for $\langle Q \rangle_i$ from (1.2) and defining the matrix element

$$\rho_{nm} = \sum_i{}' p_i \, a_m^{(i)*} a_n^{(i)} \tag{1.4}$$

we get

$$\langle \widetilde{Q} \rangle = \sum_{n,m}{}' \rho_{mn} Q_{nm} = \mathrm{Tr}\,[Q\rho] \tag{1.5}$$

Thus the average expectation value is given by the trace of the product of Q and a matrix ρ whose matrix elements are defined through (1.4). This matrix ρ is called the 'density matrix' for the non-pure state and eqn.(1.5) itself is considered to be the definition of the density matrix ρ .

Among the properties of ρ , we shall mention a few. For details, please refer to U.Fano, Rev. Mod. Phys.

Properties of Density Matrix:-

(1) Since for every Hermitian operator Q, $\langle \tilde{Q} \rangle$ has to be real, it immediately follows from eqn.(1.5)

$$\rho^{*}_{mn} = \rho_{nm} \qquad (1.6)$$

i.e. ρ is Hermitian.

(2) The unit operator has the average expectation value 1.

$$Tr[\rho] = \sum_{n}' \rho_{nn} = 1 \qquad (1.7)$$

(3) The number of independent parameters that specify the density matrix depends upon the number of rows and columns. This number N is then the number of orthogonal pure states over which the impure state is built in. This number is usually finite (especially when we consider the spin orientation of the nucleus) but it can as well be infinite. Since $Tr\,\rho = 1$, there are (N^2-1) independent real parameters. Thus, we require (N^2-1) separate measurements to identify the state of a system

which possesses N independent pure states.

(4) Consider the transition from a level A to B. Let us denote the set of quantum numbers describing the eigenstates of level A by a, a' ..... and of level B by b, b'...... . Let H be the transition from A to B.

Suppose, the system is initially in an eigenstate $|a\rangle$ of A, then the transition will result in a state $H|a\rangle$. The probability of finding the state after transition, in a certain eigenstate $|b\rangle$ is then given by $|\langle b|H|a\rangle|^2 = P_B(b)$.

Suppose, the system initially is not in a pure eigenstate like $|a\rangle$ but in a mixed state which can be described by the density matrix ρ_A, and makes a transition by H, then the density matrix ρ_B for the final state is given by

$$\rho_B = H \rho_A H^\dagger \tag{1.8}$$

Therefore

$$\langle b|\rho_B|b'\rangle = \sum_{a\,a'}' \langle b|H|a\rangle\langle a|\rho_A|a'\rangle\langle a'|H^\dagger|b'\rangle$$

$$= \sum_{a\,a'}' \langle b|H|a\rangle\langle a|\rho_A|a'\rangle\langle b'|H|a'\rangle^* \tag{1.9}$$

This expression which gives the matrix element of final state density matrix in terms of initial state density matrix and transition operator, will be frequently referred to.

(5) For an randomly oriented state with sharp angular momentum J, the density matrix is proportional to the unit matrix.

$$\rho_{mm'} = \frac{1}{2J+1}\,\delta_{mm'}$$

Nuclear Spin Orientation:-

We now derive a formula for the average expectation value of a set of statistical tensors which specify the nuclear spin orientations. Consider a nuclear transition from an angular momentum state $|J_i M_i\rangle$ to $|J_f M_f\rangle$ and let the transition operator be t which can be thrown in spherical tensor form. Let ρ_i and ρ_f denote the density matrices for the initial and final states respectively. As mentioned earlier, ρ_i will be a Hermitian matrix of dimension $(2J_i+1)\times(2J_i+1)$ and ρ_f will be a Hermitian matrix of dimension $(2J_f+1)\times(2J_f+1)$. From eqn.(1.9), the typical matrix element of in the spin space of the final nucleus is given by,

$$(\rho_f)_{M_f M_f'} = \sum_{M_i, M_i'}{}' \langle J_f M_f | t | J_i M_i\rangle (\rho_i)_{M_i M_i'} \langle J_f M_f' | t | J_i M_i'\rangle^{*} \qquad (1.10)$$

The spin orientation of the final nucleus can be represented by a set of statistical tensors T_K^{μ} whose average expectation value, in accordance with eqn.(1.5) is given by

$$\langle T_K^{\mu} \rangle = Tr\,[\,T_K^{\mu}\,\rho_f\,] \Big/ Tr\,\rho_f \tag{2.1}$$

These quantities T_K^{μ} of rank K and projection μ are defined in the spin-space of the final nucleus and satisfy the condition

$$Tr\,[\,(T_K^{\mu})^{\dagger}\;T_{K'}^{\mu'}\,] = (2J_f+1)\,\delta_{KK'}\,\delta_{\mu\mu'} \tag{2.2}$$

These statistical tensors T_K^{μ} serve the purpose of expansion for ρ_f by the following equation,

$$\rho_f = \sum_{K,\mu} C_{K\mu}\;T_K^{\mu} \tag{2.3}$$

Now from eqn.(2.1),

$$\langle T_K^{\mu} \rangle^{*} = Tr\,[\,\rho_f\,T_K^{\mu\,*}\,] \Big/ Tr\,\rho_f$$

and substituting for ρ_f from eqn.(2.3),

$$\langle T_K^{\mu} \rangle^{*} = Tr\,[\,\sum_{K',\mu'} C_{K'\mu'}\;T_{K'}^{\mu'}\;T_K^{\mu\,*}\,] \Big/ Tr\,\rho_f$$

and now using (2.2)

$$\langle T_K^{\mu} \rangle^{*} = (2J_f+1)\,C_{K\mu} \Big/ Tr\,\rho_f$$

or

$$C_{K\mu} = (Tr\,\rho_f)\,\langle T_K^{\mu}\rangle^*/(2J_f+1) \tag{2.4}$$

Thus

$$\rho_f = \frac{(Tr\,\rho_f)}{2J_f+1} \sum_{K,\mu}{}' \langle T_K^{\mu}\rangle^* \, T_K^{\mu}$$

It is always possible to choose one of the spherical tensor operators as unit operator and so

$$\rho_f = \frac{(Tr\,\rho_f)}{2J_f+1} \left[\mathbb{1} + \sum_{K,\mu} \langle T_K^{\mu}\rangle^* \, T_K^{\mu} \right] \tag{2.5}$$

Now, we construct a method of evaluating $\langle T_K^{\mu}\rangle$. The transition operator t is given the spherical tensor form

$$t = \sum_{\lambda, m_\lambda}{}' O_\lambda^{m_\lambda} \tag{2.6}$$

The initial nucleus will usually be randomly oriented and so ρ_i is given by eqn. $(\rho_i)_{M_i M_i'} = 1/(2J_i+1)\,\delta_{M_i M_i'}$. So, substituting for t from (2.6) in (1.10) and using (2.1), we obtain

$$Tr\,[T_K^{\mu}\,\rho_f] = \frac{1}{2J_i+1} \sum_{M_i}{}' \langle J_f M_f |\sum_{\lambda, m_\lambda} O_\lambda^{m_\lambda}| J_i M_i\rangle$$

$$\langle J_f M_f | T_K^{\mu} | J_f M_f'\rangle$$

$$\langle J_f M_f' | \sum_{\lambda' m_\lambda'} O_{\lambda'}^{m_\lambda'} | J_i M_i\rangle^{*}$$

Applying Wigner-Eckart theorem,

$$Tr\,[T_K^{\mu}\,\rho_f] = \frac{1}{2J_i+1} \sum_{M_i, m_\lambda}{}' \sum_{m_\lambda'} \sum_{\lambda, \lambda'} C(J_i \lambda J_f ; M_i m_\lambda)$$

$$C(J_f K J_f ; M_f' \mu M_f)\, C(J_i \lambda' J_f ; M_i m_\lambda' M_f')$$

$$\langle J_f \| O_\lambda \| J_i\rangle \langle J_f \| T_K \| J_f\rangle \langle J_f \| O_\lambda' \| J_i\rangle^{*}$$

The three Clebsch-Gordon coefficients can be rearranged so that when sum over M_i is carried out, one obtains

$$Tr\,[T_K^{\mu}\,\rho_f] = \frac{1}{2J_i+1} \sum_{\lambda, \lambda', m_\lambda}{}' (-1)^{\lambda - m_\lambda} W(\lambda J_i K J_f ; J_f \lambda')$$

$$\frac{[J_f]^3}{[K]} C(\lambda \lambda' K ; m_\lambda - m_\lambda' - \mu) \langle J_f \| O_\lambda \| J_i\rangle$$

$$\langle J_f \| O_\lambda' \| J_i\rangle^{*} \quad \langle J_f \| T_K \| J_f\rangle \qquad (2.7)$$

If K=0 is put in eqn.(2.7), one immediately obtains $Tr\,\rho_f$ and hence eqn.(2.7) can be used to calculate $\langle T_K^{\mu}\rangle$. The above equation forms the basis for our further analysis.

Lecture 2:

Density Matrix, Population and Fano's Statistical Tensor:-

If we consider a nucleus in a sharp angular momentum state J, it can exist in any one of the $(2J + 1)$ sublevels with (in general) unequal populations. If the populations in these magnetic sublevels are same, then we say that the nucleus is randomly oriented. If not, let the population pertaining to a magnetic level M be denoted by P_M. In discussing the angular distributions of the emitted radiation, it is convenient to use Fano's statistical tensors. They determine the effect of the initial emitting state on the angular distribution and polarization of the emitted radiation. These Fano's tensors show that not the individual population P_M but certain moments of P_M are important. These are defined through

$$G_\nu (J) = \sum_M (-1)^{J-M} P_M \, C(J J \nu ; M -M 0) \qquad (2.8)$$

In the case of unoriented nuclei, if the populations are normalised so that $\sum_M P_M = 1$ and $P_M = 1/(2J+1)$

$$G_\nu (J) = \frac{1}{2J+1} \sum_M (-1)^{J-M} C(J J \nu ; M -M 0)$$

Multiplying by $C(J 0 J ; M 0 M) \, (=1)$ we can show that

$$G_\nu (J) = \frac{1}{\sqrt{2J+1}} \delta_{\nu 0} \qquad (2.9)$$

In the case of oriented nuclei, we can evaluate G_0, G_1 etc. and see what they represent

$$G_0(J) = \frac{1}{\sqrt{2J+1}} \sum_M P_M \tag{2.9a}$$

and so $G_0(J)$ represents total population.

$$G_1(J) = \sum_M{}' (-)^{J-M} P_M \, C(J\,J\,1; M\,-M\,0)$$

By actually putting the value for C.G. coefficient,

$$G_1(J) = \sqrt{\frac{3}{2J+1}} \frac{1}{\sqrt{J(J+1)}} \sum_M M P_M \tag{2.10}$$

Suppose the orientation of the nucleus is such that the 'first moment' of P_M is not zero, then $G_1(J) \neq 0$ and we say that the nucleus is 'polarized'. In a similar way

$$G_2(J) = \left[\sqrt{5}\Big/\sqrt{J(J+1)(2J-1)(2J+1)(2J+3)}\right] \sum_M P_M \left\{3M^2 - J(J+1)\right\} \tag{3.1}$$

If the orientation of the nucleus is such that the 'second moment' of P_M is not zero, then $G_2(J) \neq 0$ and we say that nucleus is aligned.

It will be sometimes useful to introduce another statistical tensor α_ν by expanding P_M in a polynomial of degree 2J in m and since $C(J\lambda J; M0M)$ is also a polynomial of degree λ in M,

$$P_M = \sum_{\lambda=0}^{2J} \alpha_\lambda C(J\lambda J; M0M) \tag{3.2}$$

Then substituting for P_M in eqn.(2.8) and using the orthogonal properties,

$$G_\nu(J) = \sqrt{\frac{2J+1}{2\nu+1}}\, \alpha_\nu \tag{3.3}$$

From (2.10) and (2.9a), we can define polarization P_N in terms of α_ν.

$$P_N(J) = \frac{1}{J} \sum_M M P_M \Big/ \sum_M P_M$$

$$= \frac{1}{J} \sqrt{\frac{2J+1}{3}} \frac{\sqrt{J(J+1)}}{\sqrt{(2J+1)}}\, G_1(J) \Big/ G_0(J)$$

$$= \frac{1}{J} \left[\frac{(2J+1)J(J+1)(2J+1)}{3\times 3(2J+1)(2J+1)} \right]^{1/2} \frac{\alpha_1}{\alpha_0}$$

$$P_N(J) = \frac{1}{3} \sqrt{\frac{J+1}{J}}\, \alpha_1 / \alpha_0 \tag{3.4}$$

We shall now show that P_M and G_ν are transforms of one another. We had from (2.8)

$$G_\nu(J) = \sum_M (-1)^{J-M} P_M \, C(JJ\nu; M-M0) \qquad (2.8)$$

Multiplying by $C(JJ\nu; M'-M'0)(-1)^{J-M'}$ and summing over ν

$$\sum_\nu (-1)^{J-M'} C(JJ\nu; M'-M'0) \, G_\nu(J)$$
$$= P_M$$

by the orthogonal property of Clebsch-Gordon coefficients. Hence

$$P_M = \sum_\nu (-1)^{J-M} G_\nu(J) \, C(JJ\nu; M-M0) \qquad (3.5)$$

From (2.8) and (3.5), P_M and G_ν are transforms of one another.

<u>Theorem:-</u>

If the nuclear system is initially in a state of orientation given by a statistical tensor of rank λ and if it makes a transition to a final state whose orientation is given by a statistical tensor of rank λ' then $\lambda' = \lambda$ if the transition is a parity conserving one, $\lambda' = \lambda \pm 1$ if the transition is a parity-violating one.

We shall illustrate the proof by using general expressions for parity conserving and parity-violating interactions. However, the above theorem is strictly valid for restricted interactions.

Part A:-

Consider the transition from $|\mathring{j}\, m\rangle$ to $|\mathring{j}_1\, m_1\rangle$ and let the nucleus in the initial state $|\mathring{j}\, m\rangle$ be oriented such that it can be described by α_ν. We want to investigate the orientation of $|\mathring{j}_1\, m_1\rangle$. Let the parity-conserving interaction be chosen as $H = \vec{M}\cdot\vec{M}^*$ where M can be represented by a spherical tensor. If we denote the populations of the initial nucleus in its various magnetic sublevels by P_m, then the diagonal element of the density matrix of the final state is

$$\rho_{m_1 m_1} = \sum_m{}' P_m\, \vec{M}\cdot\vec{M}^*$$

writing $P_m = \sum_{\lambda} \alpha_\lambda(\mathring{j})\, C(\mathring{j}\lambda\mathring{j};\, m0m)$ and using spherical tensor decomposition for $\vec{M}$,

$$\rho_{m_1 m_1} = \sum_{\lambda, m}{}' \alpha_\lambda(\mathring{j})\, C(\mathring{j}\lambda\mathring{j};\, m0m)\, C(\mathring{j}\nu\mathring{j}_1;\, m\, m_\nu\, m_1)^2$$
$$|\langle \mathring{j}\, \| M_\nu \| \mathring{j}_1\rangle|^2$$

where Wigner-Eckart theorem has been applied.

The three Clebsch-Gordon coefficients can be arranged so that when summed over m gives

$$\rho_{m_1 m_1} = |\langle j_1 \| M_\nu \| j \rangle|^2 \sum_\lambda \alpha_\lambda(j)(-1)^{j_1 - j - \nu - \lambda} W(j j j_1 j_1; \lambda \nu)[j][j_1] C(j_1 \lambda j_1; m_1 0 m_1) \tag{3.6}$$

Now comparing the sum in (3.6) with the equation P_m (3.2), it is interesting to define

$$\alpha_\lambda(j)(-1)^{j_1 - j - \nu - \lambda} W(j j j_1 j_1; \lambda \nu)[j][j_1] = \alpha_\lambda(j_1) \tag{3.7}$$

so that

$$\rho_{m_1 m_1} = |\langle j_1 \| M_\nu \| j \rangle|^2 \sum_\lambda \alpha_\lambda(j_1) C(j_1 \lambda j_1; m_1 0 m_1) \tag{3.8}$$

$$= |\langle j_1 \| M_\nu \| j \rangle|^2 P_{m_1} \tag{3.9}$$

Thus $\alpha_\lambda(j_1)$ plays the same role of $\alpha_\lambda(j)$ for the initial state. Since α_λ define the orientation, because of (3.7), the parity conserving interaction carries every tensor orientation $\alpha_\lambda(j)$ of rank λ to a tensor orientation $\alpha_\lambda(j_1)$ of same rank λ. If the initial nucleus was unpolarized, it remains unpolarized and so on.

The exact relationship between the initial and final polarization can now be derived. From eqn.(3.4),

$$\frac{P_N(\mathring{j}_1)}{P_N(\mathring{j})} = \sqrt{\frac{(\mathring{j}_1+1)\,\mathring{j}}{\mathring{j}_1(\mathring{j}+1)}}\;\frac{\alpha_1(\mathring{j}_1)}{\alpha_1(\mathring{j})}\;\frac{\alpha_0(\mathring{j})}{\alpha_0(\mathring{j}_1)},$$

From eqn.(3.7) it is easy to show

$$\alpha_0(\mathring{j}_1) = \alpha_0(\mathring{j})$$

$$\frac{P_N(\mathring{j}_1)}{P_N(\mathring{j})} = \sqrt{\frac{(\mathring{j}_1+1)\,\mathring{j}}{\mathring{j}_1(\mathring{j}+1)}}\,(-1)^{\mathring{j}_1-\mathring{j}-\nu-1}\,W(\mathring{j}\mathring{j}\mathring{j}_1\mathring{j}_1;1\nu)\,[\mathring{j}\,J\,[\mathring{j}_1]^{(3.10)}$$

If we consider for example a dipole transition, then $\nu = 1$ and

$$\frac{P_N(\mathring{j}_1)}{P_N(\mathring{j})} = [\mathring{j}(\mathring{j}+1) - \mathring{j}_1(\mathring{j}_1+1) - 2] \Big/ 2\mathring{j}_1(\mathring{j}+1) \qquad (4.1)$$

The above formula is true for any parity conserving dipole transition like dipole emission of x-rays or Auger electrons following muon capture.

Part-B:-

A typical parity non-conserving transition which can be represented by

$$H = \vec{\nu}\cdot(\vec{M} \times \vec{M}^*)$$

will have the diagonal element for the final state density matrix

$$\rho_{m_1 m_1} = \vec{\nu}\cdot \sum_m P_m (\vec{M} \times \vec{M}^*)$$

For the sake of illustration let

$$\vec{M} = \sum_{\mu} \langle j_1 m_1 | \sigma_1^{\mu} | j m \rangle \, \hat{\xi}_1^{-\mu} (-1)^{\mu}$$

so that ($\hat{\xi}_1$ spherical basis vectors)

$$\vec{M} \times \vec{M}^* = |\langle j_1 \| \sigma_1 \| j \rangle|^2 \sum_{\mu} c(j 1 j_1 ; m \mu m_1)^2 \, (-1)^{\mu} \, \hat{\xi}_1^{-\mu} \times \hat{\xi}_1^{\mu}$$

Using

$$\hat{\xi}_1^{\mu} \times \hat{\xi}_1^{\mu'} = i\sqrt{2}\, c(111; \mu \mu' \mu+\mu') \, \hat{\xi}_1^{(\mu+\mu')}$$

$$\vec{M} \times \vec{M}^* = |\langle j_1 \| \sigma_1 \| j \rangle|^2 \sum_{\mu} c(j 1 j_1 ; m \mu m_1)^2 \, i\sqrt{2} \; c(111; -m_1 + m, -m + m_1) \, (-1)^{m_1 - m} \, \hat{\xi}_1^{0} \qquad (4.2)$$

so that

$$\rho_{m_1 m_1} = v_3 \sqrt{2} \, |\langle j_1 \| \sigma_1 \| j \rangle|^2 \sum_{m} (-1)^{m_1 - m} P_m \; c(j 1 j_1 ; m \; m_1 - m)^2 \, c(111; m - m_1, m_1 - m) \qquad (4.3)$$

Expanding P_m as given in eqn.(3.2),

$$\rho_{m_1 m_1} = v_3 \sqrt{2} \, |\langle j_1 \| \sigma_1 \| j \rangle|^2 \, S(m_1) \qquad (4.4)$$

with

$$S(m_1) = \sum_{m,\lambda} \alpha_\lambda(j)(-1)^{m_1-m} c(j\lambda j; m0m)$$
$$c(j1j_1; m\, m_1-m)^2 c(111; m-m_1, m_1-m) \qquad (4.5)$$

Comparing eqn.(4.4) with (3.9) it is natural to interpret $S(m_1)$ as having the same status as P_{m_1}. Then $S(m_1)$ can be given a polynomial form as

$$S(m_1) = \sum_{\nu=0}^{2j_1} \beta_\nu\, c(j_1\nu j_1; m_1 0 m_1) \qquad (4.6)$$

Once the above interpretation is accepted, then β_ν has the same role as α_λ, in determining the orientation of the final nucleus. We want to investigate β_ν. The inverse of eqn.(4.6) is immediate and so

$$\beta_\nu = \frac{[\nu]}{[j_1]} \sum_{m_1} S(m_1) c(j_1\nu j_1; m_1 0 m_1) \qquad (4.7)$$

(β_ν & S are transforms of one another)

Now substituting for $S(m_1)$ from (4.5)

$$\beta_\nu = \frac{[\nu]}{[j_1]} \sum_{m_1} \sum_{m,\lambda} \alpha_\lambda(j)(-1)^{m_1-m} c(j\lambda j; m0m)$$
$$c(j1j_1; m, m_1-m)^2 c(111; m-m_1, m_1-m)$$
$$c(j_1\nu j_1; m_1 0 m_1)$$

The sum over m and m_1 can be carried out. Although the procedure is tedious it is straightforward. The result is

$$\beta_\nu = [1][\nu][j][j_1] \sum_\lambda \alpha_\lambda(j)\, C(\lambda 1 \nu;\, 000)$$
$$\sum_x W(\nu j_1, j 1;\, j, x)\, W(1 1 j_1, j;\, 1 x)$$
$$W(j \lambda x 1;\, j \nu)\, (-1)^{j-x-1} [x]^2 \tag{4.8}$$

From the above equation, it is seen that the rank of the orientation of the final nucleus ν is related to λ, the rank of the orientation of the initial nucleus through the parity Clebsch-Gordon coefficient $C(\lambda 1 \nu;\, 000)$ and so

$$\begin{aligned} \nu &\neq \lambda \\ \nu &= \lambda \pm 1 \end{aligned} \tag{4.9}$$

Thus, the rank increases or decreases by unity. So, if an unpolarized nucleus emits β-particles, it will get polarized, if a polarized nucleus emits β-particle, it may be aligned or depolarized. For further details of polarization of recoil nucleus after β-emission, see M.E.Rose, Lectures on Theoretical Physics, Vol.2, Brandeis Summer Institute, 1962. We shall apply these ideas later in attempting a test for weak neutral currents in inelastic electron scattering by nuclei.

Lecture 3:

In this lecture, I shall indicate the use of the density matrix in evaluating the polarization of the outgoing nucleon in a reaction

$$a + A \rightarrow B + b + n \tag{4.10}$$

where a and b are elementary particles and we shall restrict to spin-zero target. The extension to non-zero spin targets is straightforward by defining an initial state density matrix. The aim will be to apply the formalism to two typical reactions

$$\mu^- + A \rightarrow B + \nu_\mu + n$$
$$\gamma + A \rightarrow B + \pi + N$$

Polarization of Emitted neutron in muon capture:-

First of all, the Hamiltonian for muon capture process need be written. This has been derived from the universal V-A interaction for leptons and using Lorentz invariance the general form for the hadrons. This has been given a non-relativistic reduction by using the two-component theory for neutrino (it is to be noted that the Dirac equation for mass-less particle violate parity conservation) and a two-component for the nucleon spinors. The use of non-relativistic reduction for nucleons is justified as the motion of nucleons in the nucleus is believed to be non-relativistic. We shall give the final expression for the muon-capture Hamiltonian as,

$$H = \frac{1}{2}\, \tau_L^{+} (1 - \vec{\sigma}_L \cdot \hat{\nu}) \sum_{i=1}^{A} \tau_i^{-} \Big[G_V \mathbb{1}_i \mathbb{1}_L + G_A\, \vec{\sigma}_L \cdot \vec{\sigma}_i - G_P (\vec{\sigma}_L \cdot \hat{\nu})(\vec{\sigma}_i \cdot \hat{\nu}) - \frac{g_V}{M} (\vec{\sigma}_L \cdot \hat{\nu})(\vec{\sigma}_L \cdot \vec{p}_i) - \frac{g_A}{M} (\vec{\sigma}_L \cdot \hat{\nu})(\vec{\sigma}_i \cdot \vec{p}_i) \Big] \delta(\vec{r} - \vec{r}_i) \qquad (5.1)$$

where τ_L^{+}, $\vec{\sigma}_L$, $\mathbb{1}_L$ are the lepton Iso-spin, spin and unit operators, τ_i^{-}, $\vec{\sigma}_i$, $\vec{p}_i$, $\mathbb{1}_i$ are the nucleon Iso-spin, spin, momentum and unit operators respectively, $\hat{\nu}$ is the unit vector along the direction of neutrino motion, M is the nucleon mass and G_V, G_A, G_P, g_V and g_A are muon capture coupling constants whose explicit expressions are given in standard literature on muon capture.

The matrix element of H has to be taken in the lepton and nucleon space. Denoting the matrix element in nucleon space by Ω, the complete matrix element of H is

$$Q = \langle u_\nu | \Omega | u_\mu \rangle \qquad (5.2)$$

where $|u_\mu\rangle$ and $|u_\nu\rangle$ are muon and neutrino spinors. Comparing (5.2) with (5.1), it is straightforward to see

$$\Omega = \frac{1}{2}\, \tau_L^{+} (1 - \vec{\sigma}_L \cdot \hat{\nu}) (\mathcal{M}_1 + \vec{\sigma}_L \cdot \vec{\mathcal{M}}_2) \qquad (5.3)$$

where

$$\mathcal{M}_1 = G_V M_1 - \frac{g_V}{M}(\hat{\nu}\cdot\vec{M}_3)$$

$$\vec{\mathcal{M}}_2 = G_A \vec{M}_2 - G_P(\hat{\nu}\cdot\vec{M}_2)\hat{\nu} - \frac{i g_V}{M}(\hat{\nu}\times\vec{M}_3) - \frac{g_A}{M} M_4 \hat{\nu} \tag{5.4}$$

with

$$M_I = \langle f | \sum_{i=1}^{A} \tau_i^- \exp(-i\hat{\nu}\cdot\vec{r}_i)\,\phi_\mu(r_i)\,O_I | i\rangle \tag{5.4a}$$

$$O_1 = 1\!\!1_i\,;\ O_2 = \sigma_i\,,\ O_3 = p_i,\ O_4 = \sigma_i\cdot p_i$$

Following eqn.(1.10), the density matrix for the final state is given by

$$\rho_f = T_r^L\,[\Omega(1+\vec{\sigma}_L\cdot\vec{P}_\mu)\Omega^+]/2 \tag{5.5}$$

where T_r^L means trace over Lepton operators and the factor $\frac{1}{2}(1+\vec{\sigma}_L\cdot\vec{P}_\mu)$ is the density matrix for muon of polarization $\vec{P}_\mu$. If we are dealing with unpolarized muons, this factor will not be included in (5.5). Substituting for Ω from (5.3), the lepton traces could be evaluated and since this is a straight forward procedure, we give the final result only as

$$\rho_f = \frac{1}{2}(A+B) \tag{5.6}$$

where

$$A = G_V^2\, M_1 M_1^* + G_A^2\, \vec{M}_2 \cdot \vec{M}_2^* + (G_P^2 - 2G_P G_A)\,|\hat{\nu} \cdot \vec{M}_2|^2$$

$$- 2\,\frac{G_V g_V}{M}\, M_1\,(\hat{\nu} \cdot \vec{M}_3^*) + 2\,(G_P - G_A)\,\frac{g_A}{M}\,(\hat{\nu} \cdot \vec{M}_2)\,M_4^*$$

$$- \vec{P}_\mu \cdot \Big[G_V^2\, M_1 M_1^*\, \hat{\nu} - G_A^2\, \vec{M}_2 \cdot \vec{M}_2^*\, \hat{\nu} + G_P^2\, |\hat{\nu} \cdot \vec{M}_2|^2\, \hat{\nu}$$

$$- 2\,\frac{G_A g_A}{M}\, M_4 M_2^* + 2\,(G_A - G_P)\,G_A\,(\hat{\nu} \cdot \vec{M}_2)\,\vec{M}_2^*$$

$$+ 2\,\frac{G_P g_A}{M}\,(\hat{\nu} \cdot \vec{M}_2)\,M_4^*\,\hat{\nu} - 2\,\frac{G_V g_V}{M}\, M_1(\hat{\nu} \cdot \vec{M}_3^*)\,\hat{\nu} \Big] \tag{5.7}$$

$$B = 2\,(G_P - G_A)\,G_V\,(\hat{\nu} \cdot \vec{M}_2)\,M_1^* - i\,G_A^2\, \hat{\nu} \cdot (\vec{M}_2 \times \vec{M}_2^*)$$

$$- 2\,\frac{G_P g_V}{M}\,(\hat{\nu} \cdot \vec{M}_2)(\hat{\nu} \cdot \vec{M}_3^*) + 2\,\frac{G_V g_A}{M}\, M_4 M_1^*\,(1 - \vec{P}_\mu \cdot \hat{\nu})$$

$$+ 2\,\frac{G_A g_V}{M}\, \vec{M}_2 \cdot \vec{M}_3^*\,(1 + \vec{P}_\mu \cdot \nu) + 2\,G_V G_A\, M_1\,(\vec{P}_\mu \cdot \vec{M}_2^*)$$

$$- 2\,G_V G_P\,(\vec{P}_\mu \cdot \hat{\nu})\, M_1\,(\hat{\nu} \cdot \vec{M}_2^*) - i\,G_A^2\, \vec{P}_\mu \cdot (\vec{M}_2 \times \vec{M}_2^*)$$

$$- 2\,\frac{G_A g_V}{M}\,(\hat{\nu} \cdot \vec{M}_3)(\vec{P}_\mu \cdot \vec{M}_2) - 2\,(G_A - G_P)\,\frac{g_V}{M} \left\{ (\hat{\nu} \cdot \vec{M}_2)(\vec{P}_\mu \cdot \vec{M}_3^*) \right\} \tag{5.8}$$

By writing $\wp_{f}$ as A+B in eqn.(5.6) we understand (by actual calculation or by inspection) that terms in A contribute to the capture-rate (scalar quantity) and terms in B contribute to the polarization (pseudoscalar quantity). The terms in A and B are mutually exclusive and further the muon capture coupling constants in A and B appear in different combinations, different from one another. If the nuclear matrix elements which occur in pairwise in A and B (have the form given in eqn.(5.4a))are evaluated, then we have calculated $\wp_{f}$. Depending upon our interest, the evaluation of the nuclear matrix element proceeds.

Since, we are interested in the polarization of the outgoing nucleon, we have to evaluate them in the spin space of the outgoing nucleon. This, we shall illustrate for one typical term and the evaluation of other terms, proceed in a similar way. In the initial state, we have a proton in the bound state represented by the wave function $|u_{nL}(r), L\tfrac{1}{2} J M\rangle$ where $u_{nL}(r)$ is the radial part of the wave function which can be taken to be Harmonic-Oscillator type or Saxon-Woods type. In the final state, we have the neutron in the continuam state which can, in the absence of final state interactions, be represented by exp $(i\vec{n}\cdot\vec{r})$. Representing the outgoing neutrino as a plane wave exp$(i\hat{\nu}\cdot\vec{r})$ we can write

$$M_4 = \langle \vec{n}, \tfrac{1}{2} m_s | e^{-i\hat{\nu}\cdot r} \phi_{\mu}(r)\, \vec{\sigma}\cdot\vec{P} | u_{nL}(r) ; L\tfrac{1}{2} J M\rangle \quad (5.9)$$

where $|\frac{1}{2}\, m_s\rangle$ is the spin-part of the outgoing neutron and $\phi_\mu(r)$ the muon wave function which is averaged over the nuclear volume. Decoupling the initial proton angular momentum part,

$$M_4 = \langle \tfrac{1}{2}\, m_s | \sum_{m_L} e^{-i\vec{P}_R\cdot\vec{r}}\, \phi_\mu(r)\, \vec{\sigma}\cdot\vec{P}\, C(L\tfrac{1}{2}J;\, m_L\,\mu\, M)$$

$$|u_{nL}(r);\, Y_L^{m_L}(\theta\phi);\, \tfrac{1}{2}\,\mu\rangle \tag{5.10}$$

where $\vec{P}_R = \vec{n} + \vec{\nu}$ the momentum of the recoil nucleus. Expanding $\exp(-i\,\vec{P}_R\cdot\vec{r})$ in partial waves, it is easy to see that only one partial wave $l = L$ contributes.

$$M_4 = \sum_{m_L} 4\pi\, (i)^L\, Y_L^{m_L}(\hat{P}_R)^*\, C(L\tfrac{1}{2}J;\, m_L\,\mu\, M)$$

$$\langle \tfrac{1}{2}\, m_s | \vec{\sigma}\cdot\vec{P} | \tfrac{1}{2}\,\mu\rangle \int_0^\infty u_{nL}(r)\, j_L(P_R r)\, \phi_\mu(r)\, r^2\, dr. \tag{6.1}$$

Denoting the radial integral by $F(n, L, P_R)$ and proceeding in a similar way

$$(\hat{\nu}\cdot\vec{M}_2) = \langle \tfrac{1}{2}\, m_s | \vec{\sigma}\cdot\hat{\nu} | \tfrac{1}{2}\,\mu\rangle \sum_{m_L} 4\pi (i)^L\, Y_L^{m_L}(\hat{P}_R)^*$$

$$C(L\tfrac{1}{2}J;\, m_L\,\mu\, M)\; F(n, L, P_R) \tag{6.2}$$

Thus

$$M_4(\hat{\nu}\cdot\vec{M}_2^*) = \sum_\mu \langle \tfrac{1}{2}\, m_s | \vec{\sigma}\cdot\vec{P} | \tfrac{1}{2}\,\mu\rangle\langle \tfrac{1}{2}\,\mu | \vec{\sigma}\cdot\hat{\nu} | \tfrac{1}{2}\, m_s'\rangle$$

$$\frac{(2J+1)}{8\pi}\, 16\pi^2\, |F(n, L, P_R)|^2 \tag{6.3}$$

where we have made use of (See V. Devanathan, Ann. Phys 43(1967)74)

$$\sum_{m_L, m_L'} C(L\tfrac{1}{2}J; m_L \mu M)\, C(L\tfrac{1}{2}J; m_L' \mu' M)\, Y_L^{m_L *}(\hat{P}_R)\, Y_L^{m_L'}(\hat{P}_R)$$

$$= \frac{(2J+1)}{8\pi}\, \delta_{\mu\mu'} \qquad (6.3a)$$

writing $\vec{p} = \vec{n} + \vec{\nu}$

$$M_4(\hat{\nu}\cdot\vec{M}_2^*) = 2\pi(2J+1)\,|F(n,L,P_R)|^2 \langle \tfrac{1}{2} m_s | \vec{\sigma}\cdot(\vec{n}+\hat{\nu})\,(\vec{\sigma}\cdot\hat{\nu})\,| \tfrac{1}{2} m_s' \rangle \qquad (6.4)$$

Thus the term $M_4(\hat{\nu}\cdot\vec{M}_2^*)$ of ρ_f is expressed in the spin space of the outgoing neutron. In a similar way, the remaining terms can be expressed and so

$$\begin{aligned}
\rho_A = 2\pi(2J+1)\beta\,|F(n,L,P_R)|^2 \Big[& G_V^2 + G_P^2 - 2G_P G_A + G_A^2(\vec{\sigma}\cdot\vec{\sigma}) + \frac{2}{M}(G_P g_A - G_A g_A - G_V g_V)\,\hat{\nu}\cdot(\vec{n}+\vec{\nu}) \\
& - (\vec{P}_\mu\cdot\hat{\nu})\Big\{ G_V^2 + 2G_A^2 + G_P^2 - 2G_P G_A - G_A^2(\vec{\sigma}\cdot\vec{\sigma}) \\
& + \frac{2}{M}(G_P g_A - G_V g_V)\,\hat{\nu}\cdot(\vec{n}+\vec{\nu})\Big\} + 2\frac{G_A g_A}{M}\,\vec{P}_\mu\cdot(\vec{n}+\hat{\nu})\Big]
\end{aligned} \qquad (6.5)$$

and

$$\begin{aligned}
\rho_B = 2\pi(2J+1)\beta\,|F(n,L,P_R)|^2 \Big[& \Big\{ 2G_V(G_P - G_A) + 2G_A^2 \\
& - 2G_V G_P(\vec{P}_\mu\cdot\hat{\nu}) - 2\frac{G_P g_V}{M}\,\hat{\nu}\cdot(\vec{n}+\vec{\nu}) - 2(G_A - G_P)\frac{g_V}{M} \\
& \mp \vec{P}_\mu\cdot(\vec{n}+\vec{\nu})\Big\}(\vec{\sigma}\cdot\hat{\nu}) + \Big\{(2G_A^2) + 2G_V G_A - 2\frac{G_A g_V}{M}
\end{aligned}$$

$$\hat{\nu}\cdot(\vec{n}+\vec{\nu})\}(\vec{\sigma}\cdot\vec{P}_\mu) + \frac{2}{M}(G_V g_A + G_A g_V)\,\vec{\sigma}\cdot(\vec{n}+\vec{\nu})$$

$$+ \frac{2}{M}(G_A g_V - G_V g_A)(\vec{P}_\mu\cdot\hat{\nu})(\vec{\sigma}\cdot(\vec{n}+\vec{\nu}))] \qquad (6.6)$$

where β is the final state phase-space factor

$$\beta = (2\pi)^{-5} M n \nu^2$$

If we are interested in the polarization of the outgoing neutron, the neutrino being not observed, its angles have to be integrated out. This is not simple since the radial integrals involve P_R which involves ν. So defining

$$D_0 = \int |F(n,L,P_R)|^2 \, d\Omega_\nu$$

$$D_1 = \int |F(n,L,P_R)|^2 (\vec{n}\cdot\vec{\nu})\, d\Omega_\nu$$

$$D_2 = \int |F(n,L,P_R)|^2 (\vec{n}\cdot\hat{\nu})^2\, d\Omega_\nu$$

we can evaluate the longitudinal polarization of the neutron as

$$\langle\vec{\sigma}\cdot\vec{n}\rangle = \mathrm{Tr}[\vec{\sigma}\cdot\vec{n}\cdot\rho_f] / \mathrm{Tr}\,\rho_f \qquad (6.7)$$

This turns out to be

$$\langle\vec{\sigma}\cdot\vec{n}\rangle = (a + b\,\vec{P}_\mu\cdot\hat{n}) / (c + d\,(\vec{P}_\mu\cdot\hat{n})) \qquad (6.8)$$

with

$$a = (2\pi)(2J+1)\beta\Big[\Big(G_V g_A + G_A g_V\Big)\frac{2n}{M} D_0 + \Big\{2G_V(G_P - G_P) + 2G_A^2 + \frac{2\nu}{M}(G_V g_A + G_A g_V - G_P g_V)\Big\} D_1 - \frac{2n}{M} G_P g_V D_2\Big]$$

$$b = 2\pi(2J+1)\beta\Big[\Big(2G_A^2 + 2G_V G_A - \frac{2\nu}{M} G_A g_V\Big) D_0 + \frac{2n}{M}(G_P g_V - G_V g_A - G_A g_V) D_1 - \Big\{2G_V G_P + \frac{2\nu}{M}(G_V g_A - G_P g_V)\Big\} D_2\Big]$$

$$c = 2\pi(2J+1)\beta\Big[\Big\{G_V^2 + 3G_A^2 + G_P^2 - 2G_P G_A + \frac{2\nu}{M}(G_P g_A - G_A g_A - G_V g_V)\Big\} D_0 + \frac{2n}{M}\Big\{(G_P - G_A) g_A - G_V g_V\Big\} D_1\Big]$$

$$d = 2\pi(2J+1)\beta\Big[\frac{2n}{M} G_A g_A D_0 - \Big\{G_V^2 - G_A^2 + G_P^2 - 2G_P G_A + \frac{2\nu}{M}(G_P g_A - G_V g_V - G_A g_A)\Big\} D_1 - \frac{2n}{M}(G_P g_A - G_V g_V) D_2\Big]$$

Since the denominator of (6.8) is $Tr\,\rho_f$, the asymmetry coefficient in the angular distribution of the emitted neutron with respect to the muon polarization direction is given by

$$\alpha = d/c \tag{6.9}$$

Further, if we neglect nucleon momentum - dependent terms, then

$$\langle \vec{\sigma} \cdot \hat{n} \rangle = \frac{[2G_A^2 + 2G_V(G_P - G_A)\} D_1}{[G_V^2 + 3G_A^2 + G_P^2 - 2G_P G_A] D_0} \tag{6.10}$$

and for muon capture by Hydrogen $D_1 = - D_0$,

$$\langle \vec{\sigma} \cdot \hat{n} \rangle = \frac{2G_V(G_A - G_P) - 2G_A^2}{G_V^2 + 3G_A^2 - 2G_P G_A + G_P^2} \tag{7.1}$$

I shall conclude this section by giving the numerical values for $\langle \vec{\sigma} \cdot \hat{n} \rangle$ for Hydrogen and for some nuclei

$$\langle \vec{\sigma} \cdot \hat{n} \rangle_{Hydrogen} = -0.975$$

$\langle \vec{\sigma} \cdot \hat{n} \rangle$ for other nuclei after integrating over neutron energy will depend upon neutron angle of emission. For comparison, we give the values for $\theta = 90^\circ$ and 180°.

	$P_\mu = 10\%$	$P_\mu = 15\%$	$P_\mu = 20\%$
^{28}Si	- 0.8175	- 0.8175	- 0.8175
	- 0.8243	- 0.8285	- 0.8341
^{32}S	- 0.8275	- 0.8275	- 0.8275
	- 0.8352	- 0.8389	- 0.8431
^{40}Ca	- 0.8849	- 0.8849	- 0.8849
	- 0.8901	- 0.8932	- 0.8947

It can be seen that for nuclei, the values of $\langle \vec{\sigma} \cdot \hat{n} \rangle$ are not very different from single nucleon value (Hydrogen) and hence the nuclear structure effects in this planewave approximation, do not affect the polarization of the emitted neutron.

However, when the final state interaction of the outgoing neutron with the residual nucleus is taken into account by means of an optical potential, the above conclusion is changed. The reason is: when a plane wave is used (or eikonal approximation), the neutron momentum operator operating on planewave gives the free momentum or eigenvalue while when the distorted wave is used it will not be an eigenfunction of momentum operator. The entire formalism has to be modified. Such a difficulty is not expected to arise in $\gamma + A \rightarrow B + N + \pi$ as will be explained now.

<u>Polarization of emitted nucleon in Pion Photoproduction:-</u>

$$\gamma + A \rightarrow B + N + \pi$$

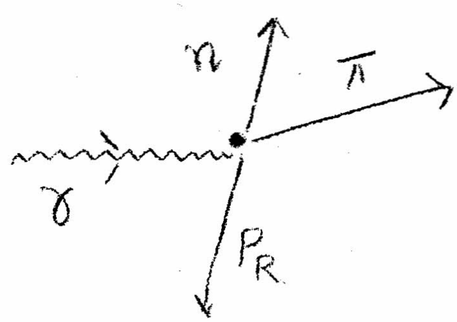
The amplitude for photoproduction of pions from nuclei has been taken from the amplitude for the analogous process $\gamma + N \rightarrow \pi + N$ by impulse approximation. The often used Chew-Low-Goldberger-Nambu amplitude has the structure

$$(\vec{\sigma} \cdot \vec{K} + L) \qquad (7.2)$$

where $\vec{K}$ and L depend upon, photon polarization, Pion momentum, Photon momentum and not on nucleon momentum. In this way it is interesting when compared to muon capture process. We have initial a proton in the bound state represented by $| u_{nL}(r); L\frac{1}{2} JM\rangle$ which interacts with photon through $(\vec{\sigma}\cdot\vec{K}+L)$ and becomes a pion and nucleon, both in the continuam state. For the sake of illustration, the two are represented by plane waves besides a plane wave description for the incoming photon. The appropriate matrix element is then,

$$\langle \vec{p}, \tfrac{1}{2} m_s | (\vec{\sigma}\cdot\vec{K}+L)\, e^{i\vec{q}\cdot\vec{r}} | u_{nL}(r); L\tfrac{1}{2} JM\rangle \qquad (7.3)$$

By conservation of momentum



we have

$$\vec{P}_R = \vec{\gamma} - \vec{n} - \vec{\pi}$$

$$\langle \tfrac{1}{2} m_s | (\vec{\sigma}\cdot\vec{K}+L)\, e^{i\vec{P}_R\cdot\vec{r}} | u_{nL}(r); L\tfrac{1}{2} JM\rangle \qquad (7.4)$$

Expanding $\exp(i\vec{P}_R\cdot\vec{r})$ in partial waves and decoupling $|L\frac{1}{2} JM\rangle$ into $Y_L^{m_L}(\theta\phi)\,|\frac{1}{2}\mu\rangle$, we obtain for the matrix element,

$$\langle \tfrac{1}{2} m_s | \vec{\sigma} \cdot \vec{K} + L | \tfrac{1}{2} \mu \rangle \sum_{m_L} 4\pi (i)^L Y_L^{m_L} (\hat{P}_R)^* C(L \tfrac{1}{2} J; m_L \mu M)$$

$$\int_0^\infty u_{n_L}(r) \, j_L(P_R r) \, r^2 dr \qquad (7.5)$$

Denoting the radial integral by $F(n, L, P_R)$ and making use of (6.3a), we obtain,

$$(\rho_f)_{m_s m_s'} = 2\pi(2J+1) \langle \tfrac{1}{2} m_s | (\vec{\sigma} \cdot K + L)(\vec{\sigma} \cdot K^* + L^*) | \tfrac{1}{2} m_s' \rangle \, |F(n, L, P_R)|^2 \qquad (7.6)$$

$$\rho_f = 2\pi(2J+1)\beta \, |F(n, L, P_R)|^2 \{ \vec{K} \cdot \vec{K}^* + LL^* + i \vec{\sigma} \cdot (\vec{K} \times \vec{K}^*) + (\vec{\sigma} \cdot \vec{K}) L^* + (\vec{\sigma} \cdot \vec{K}^*) L \}$$

The differential cross-section for the process is

$$\frac{d\sigma}{dE_n \, dE_\pi \, d\Omega_n \, d\Omega_\pi} = 2\pi(2J+1)\beta \, |F(n, L, P_R)|^2 (\vec{K} \cdot \vec{K}^* + LL^*) \qquad (7.7)$$

The longitudinal polarization of the emitted nucleon is

$$\langle \vec{\sigma} \cdot \hat{p} \rangle = \frac{|F(n, L, P_R)|^2 [i \hat{p} \cdot (\vec{K} \times \vec{K}^*) + (\hat{p} \cdot \vec{K}) L^* + (\hat{p} \cdot \vec{K}^*) L]}{|F(n, L, P_R)|^2 [\vec{K} \cdot \vec{K}^* + LL^*]} \qquad (7.8)$$

Now, if the outgoing nucleon and pion are detected in coincidence and then if the nucleon polarization is measured, the $|F(n, L, P_R)|^2$ factor cancels out in (7.8) and the nucleon polarization in nuclear pion photoproduction is the same as that for the single nucleon case. In fact, this has been experimentally measured by Ananin et. al. JETP. Letters. 23 (1976) 269. It is usually claimed that this will be the case in the absence of final state interaction. This needs some explanation. If the final state interaction for the pion is neglected and for the nucleon is taken into account, then the conclusion remains the same. This is because of the fact that neither $\vec{K}$ nor L has any nucleon-momentum dependence and so use of a distorted wave for nucleon, although changes $F(n, L, P_R)$ in (7.8) they get cancelled out. On the other hand, if the final state interaction for the outgoing pion is taken into account, since $\vec{K}$ and L involve pion momentum, the expression for $\langle \vec{\sigma} \cdot \hat{p} \rangle$ will become more complicated.

Now, suppose, the outgoing pion is not observed, then one has to integrate over pion angles and since $F(n, L, P_R)$ involves pion angles and since the numerator and denominator in (7.8) have different combinations of $\vec{K}$ and L, $F(n, L$ will not get cancelled out. Further, the experimental value for $\langle \vec{\sigma} \cdot \hat{p} \rangle$ is rather large -0.5 to +0.4, a detailed study of this is in progress.

Lecture 4:

In this lecture, I shall discuss the spin orientations of final nucleus in a nuclear reaction, with particular reference to muon capture. In general the spin orientation of a nucleus can be well described by a set of tensor parameters whose average expectation value can be calculated by methods developed in earlier lectures. For the sake of illustration, let us consider

$$\mu^- + {}^{12}C\,(0^+) \rightarrow {}^{12}B\,(1^+;\, g.s) + \nu_\mu \tag{7.9}$$

The muon capture Hamiltonian (5.1) describes the above process and can be used to construct the density matrix of ${}^{12}B\,(1^+;\, g.s)$ in its spin-space. The density matrix ρ_f will now be a 3 x 3 Hermitian matrix with the nine parameters as $\langle \tilde{T} \rangle$, $\langle \tilde{T}_1^{\pm 1, 0} \rangle$, $\langle \tilde{T}_2^{\pm 2, \pm 1, 0} \rangle$. Writing the nuclear matrix element for (7.9) as Ω and using density matrix for polarized muons, the density matrix can be evaluated. The procedure is the same as outlined in Lecture 3. The density matrix can be conveniently split up into two parts

$$\rho_f = \frac{1}{2}(A + B) \tag{7.10}$$

similar to (5.6). However because of the difference in the final states, A will have the following additional terms

$$\frac{2G_A g_V}{M} i \, \vec{M}_2 \cdot (\hat{\upsilon} \times \vec{M}_3^*)$$

and B will have

$$2G_P G_A \vec{P}_\mu \cdot (\hat{\upsilon} \times \vec{M}_2)(\hat{\upsilon} \cdot \vec{M}_2^*) - \frac{2G_A g_A}{M} \vec{P}_\mu \cdot (\hat{\upsilon} \times \vec{M}_2) M_4^*$$

Further, the evaluation of the nuclear matrix element will be different now as we are dealing with the bound to bound state transition. The A-part of ρ_f will contribute to the partial transition rate while the B-part will contribute to the pseudo-scalar observable. The recoil nuclear polarization can be given by

$$\langle T_K^\mu \rangle = \mathrm{Tr}\,[T_K^\mu \rho_f] / \mathrm{Tr}\,\rho_f \tag{8.1}$$

and when no restrictions are imposed on the Kinematics, all the possible tensor moments can be evaluated. However when the out-going neutrino is not observed, an integration over $d\Omega_\upsilon$ is to be carried out and this gives the condition δ_{K1}. This would mean, that the recoil nucleus can have only vector polarization. This is an expected result since when neutrino directions are averaged we have only one vector namely $\vec{P}_\mu$ and as the final state is two body state, consisting of neutrino and recoil nucleus moving back to back (muon is at rest) integration over $d\Omega_\upsilon$ is equivalent to integration over recoil nuclear directions. So, the recoil nuclear polarization is an average quantity and will be specified by $\vec{P}_\mu$. In order to evaluate nuclear matrix elements, one has to assume a model for the nucleus. The detailed expressions in the particle-hole model are given in Devanathan, Parthasarathy and Subramanian (Ann. Phys. (N.Y) 73 (1972) 291). For the sake of illustration, a typical term would be

$$Tr\{i\, T_K^{\mu}\, \vec{P}_{\mu}\cdot \vec{M}_2 \times \vec{M}_2^{*}\} = 16\pi^2\sqrt{2}\sqrt{\frac{4\pi}{3}}\,|\phi|^2_{av.}\sum_{\ell}{}' (-1)^{J_f-\ell}$$

$$\frac{[J_f]^3}{[K]} W(J_f 1 J_f 1; \ell 1)\langle J_f \| T_K \| J_f\rangle \,|\langle J_f \| \sum_{i=1}^{A} j_\ell(\nu r_i) \tag{8.2}$$

$$(Y_\ell(\hat{r}_i)\times\sigma_1)_{J_f} \| 0\rangle|^2\, P_\mu\, Y_1^{\mu}(\hat{P}_\mu)\,\delta_{K1}$$

The nuclear reduced matrix can be evaluated using standard 'angular momentum algebra'. The recoil nuclear polarization is

$$\vec{P}_N = \sqrt{\frac{2}{3}}\,\frac{\mathcal{A}}{\mathcal{B}}\,\vec{P}_\mu \tag{8.3}$$

where $\mathcal{A}$ and $\mathcal{B}$ will contain nuclear matrix elements and muon capture coupling constants. $\mathcal{B}$ will give the partial transition rate. For the process (7.9), neglecting nucleon momentum dependent terms and restricting to S-wave neutrinos,

$$\vec{P}_N = \frac{2G_A^2 - \frac{4}{3}G_A G_P}{3G_A^2 - 2G_A G_P + G_P^2}\,\vec{P}_\mu \tag{8.4}$$

This is same as eqn.(6) of Wolfenstein (Nuovo. Cimento 13 (1959) 319), and for a pure G.T. transition (neglecting strong interaction induced effects)

$$\vec{P}_N = \frac{2}{3}\vec{P}_\mu \tag{8.5}$$

which is same as that of Jackson, Wyld and Treiman (Phys. Rev. 107 (1957) 327). In this way, the use of density matrix methods give a general expression for the recoil nuclear polarization. I will close this discussion by giving some feeling for the numerical values.

<u>Partial Capture Rate</u> (Process 7.a)

Pure Shell-model	$40.17 \times 10^{3}\ sec^{-1}$
Cohen - Kurath	$6.13 \times 10^{3}\ sec^{-1}$

for = 0.983 G, = -1.23 , =3.7 , =7.5

Experiment (E.J.Maier et. al. Phys. Rev.B133 (1964) 663)

$$6.75^{+0.3}_{-0.75} \times 10^{3}\ sec^{-1}$$

<u>Recoil Polarization</u>

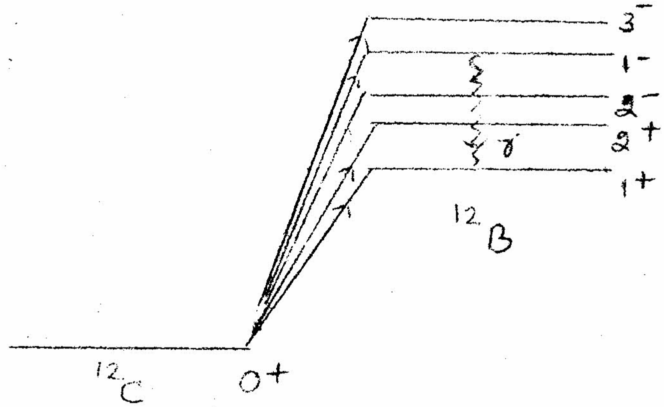
Pure Shell Model	0.5322
Cohen - Kurath	0.4810

Experiment (A.Possoz et. al. Phys. Lett. 70B (1977) 265)

$$0.452 \pm 0.042.$$

The above results indicate that while <u>capture rate is strongly nuclear model dependent, the recoil nuclear polarization is not</u> and hence can be used to examine the Induced pseudoscalar coupling.

In discussing the $^{12}B(1^{+}:g.s)$ recoil polarization, it has been assumed that the muon capture by $^{12}C(0^{+})$ leads to $^{12}B(1^{+}: g.s)$ only. However, the muon capture process excites other low-lying levels, as shown in the following figure.



with the following capture rates (in 10^3 sec^{-1})

$$\lambda(1^+) = 6.2 \pm 0.3$$
$$\lambda(2^+) = 0$$
$$\lambda(2^-) \quad 0.12$$
$$\lambda(1^-) = 0.84 \pm 0.09$$

Miller et. al. Phys. Lett. 41B (1972) 50.

So about 15% population ~~goes~~ to 1^- state which feed $^{12}B(1^+)$ by gamma de-excitation. This will change the $^{12}B(1^+)$ recoil polarization. As gamma decay is an e.m. interaction, the life time of $^{12}B(1^-)$ will be small. The effective polarization of $^{12}B(1^+)$ will be now the statistical sum of (1) directly from $^{12}C(0^+)$ by muon cap. (2) indirectly: by gamma decay of $^{12}B(1^-)$.

The density matrix method developed here can be applied to take into account the correction.

Consider the gamma decay from $|j m\rangle$ to $|j_1 m_1\rangle$ and let the initial level $|j m\rangle$ be polarized, so that the magnetic sublevel populations can be denoted by p_m. The question, how much $|j_1 m_1\rangle$ will be polarized due to gamma feed? Denoting the gamma decay interaction by H_γ the diagonal density matrix element pertaining to $|j_1 m_1\rangle$ can be written as

$$\rho_{m_1 m_1} = \sum_{m}' p_m \, |\langle j_1 m_1 | H_\gamma | j m\rangle|^2 \tag{8.6}$$

Following Rose (loc. cit), $H_\gamma = \vec{j}_N \cdot \vec{A}_p$ where $\vec{j}_N$ is the e.m. current and A_p is the vector potential of the photon of polarization p. Expanding $\vec{A}_p$ in terms of multipoles,

$$A_p = \sum_{\tau = \varepsilon, m}' \sum_{LM} a_\tau A_L(\tau) \, D^{L}_{Mp}(\phi, \theta, 0) \tag{8.7}$$

where $a(\tau)$ is a constant, ε, m stand for electric or magnetic transition. Substituting (8.7) in (8.6), applying Wigner - Eckart theorem and integrating over photon directions,

$$\rho_{m_1 m_1} = |a(\tau)|^2 \, |\langle j_1 \| L(\tau) \| j\rangle|^2 \sum_{mM}' \frac{4\pi \, p_m}{(2L+1)} \, c(jLj_1; m M m_1)^2 \tag{8.8}$$

where the reduced matrix element contains the dynamics. Writing p_m as a polynomial of degree $2j$ in m (3.2) and summing over m,

$$\rho_{m_1 m_1} = \frac{4\pi}{(2L+1)} |a(\tau)|^2 \, |\langle j_1 \| L(\tau) \| j\rangle|^2 \sum_{\lambda=0}^{2j} \alpha_\lambda(j)$$

$$(-1)^{j + j_1 - L - \lambda} \, W(j_1 j j_1 j; L\lambda) \, [j][j_1] \, c(j_1 \lambda j_1; m_1 0 m_1)$$

(8.9)

If we define

$$\alpha_\lambda(j_1) = (-)^{j+j_1-L-\lambda}\, W(j_1 j j_1 j; L\lambda)\,[j][j_1]\,\alpha_\lambda(j) \tag{8.10}$$

then

$$\rho_{m_1 m_1} = \frac{4\pi}{(2L+1)} |a(\tau)|^2 |\langle j_1 \| L(\tau) \| j\rangle|^2 \sum_{\lambda=0}^{2j}{}' \alpha_\lambda(j_1)\, C(j_1 \lambda j_1; m_1 0 m_1) \tag{9.1}$$

which when compared with (3.2) and the fact $\rho_{m_1 m_1} \propto P_{m_1}$ gives, $\alpha_\lambda(j_1)$ same role as $\alpha_\lambda(j)$. Thus, every tensor orientation of $|jm\rangle$ is carried over to $|j_1 m_1\rangle$. This is an example of eqn.(3.7). The polarization of $|j_1 m_1\rangle$ can be obtained from (4.1).

Application to $^{12}B(1^-) \xrightarrow{\gamma} {}^{12}B(1^+)$ immediately gives

$$P_N^\gamma(1^+) = 0.5\, P_N^\mu(1^-) \tag{9.2}$$

where $P_N^\gamma(1^+)$ is the additional polarization of $^{12}B(1^+)$ due to gamma feeding and $P_N^\mu(1^-)$ is the polarization of $^{12}B(1^-)$ because of muon capture by $^{12}C(0^+)$. Denoting the partial capture rates to these states by $\lambda(1^+)$ and $\lambda(1^-)$, the effective polarization of $^{12}B(1^+)$ is

$$P_N^{eff}(1^+) = \frac{\lambda(1^+)}{\lambda(1^+)+\lambda(1^-)} P_N^\mu(1^+) + \frac{\lambda(1^-)}{\lambda(1^+)+\lambda(1^-)}\, 0.5\, P_N^\mu(1^-) \tag{9.3}$$

$P_N^{\mu}(1^+)$ and $P_{\mu}^{\mu}(1^-)$ can be obtained from (8.3). The numerical calculations can be carried out using explicit nuclear models. We give the results here for $g_P/g_A = 7.5$ and in IPM, Gillet-Vinh Mau and Donnelly and Walker models.

	$P_N^{\mu}(1^+)$	$P_N^{\mu}(1^-)$	$P_N^{eff}(1^+)$
IPM	0.5322	0.6285	0.5649
GV	0.5765	0.6664	0.5650
DW	0.5792	0.6523	0.5545

The above considerations have the neutrino angles integrated out. However, one can preserve the identity of neutrino directions which will then give <u>tensor polarizations</u> of the recoil nuclei. The experimental set up could require preservation at a given time the orientation of the recoil nucleus recoiling into one of the hemispheres. In such a case, the nuclear orientation in the other hemisphere is destroyed. Here, a direct calculation shows that the nucleus could be aligned. The direct calculation by Subramanian, Parthasarathy and Devanathan (Nucl. Phys. A262 (1976) 433) gives the alignment in forward and backward hemispheres. We shall discuss this later using helicity formalism.

Now, we shall proceed to apply the density matrix methods to calculate the recoil nuclear polarization in <u>radiative pion capture.</u>

$$\pi^- + {}^{12}C(0^+) \rightarrow {}^{12}B(1^+) + \gamma \qquad (9.4)$$

Confining to 1S-capture, the Hamiltonian can be given by

$$H = \sum_{i=1}^{A}{}' \, (\vec{\sigma}_i \cdot \hat{\varepsilon}) \, \tau_i^- \, e^{-i\vec{\gamma}\cdot\vec{r}_i} \tag{9.5}$$

where $\hat{\varepsilon}$, $\vec{\gamma}$ are polarization and momentum of photon. Expanding $exp(-i\vec{\gamma}\cdot\vec{r}_i)$ in partial waves and expressing $\vec{\sigma}_i \cdot \hat{\varepsilon}$ in spherical basis, the matrix element for (9.4) becomes

$$\begin{aligned} \langle J_f M_f | H | J_i M_i \rangle = \sum_{l m \lambda}{}' \, 4\pi (i)^{l} (-1)^{\lambda - l} \, Y_l^{m}(\hat{\gamma})^{*} \\ C(J_i \, l \, J_f; M_i \, m \, M_f) \langle J_f \| \sum_{i=1}^{A} \tau_i^- \, j_l(\gamma r_i) \\ [(Y_l(\hat{r}_i) \times \varepsilon_1)_\lambda \times \sigma_1]_l \| J_i \rangle \end{aligned} \tag{9.6}$$

where the reduced matrix element can be evaluated by using specific nuclear models. The density matrix of the recoil nucleus is given by

$$\rho_{M_f M_f'} = \sum_{M_i M_i'}{}' \langle J_f M_f | H | J_i M_i \rangle (\rho_i)_{M_i M_i'} \langle J_f M_f' | H | J_i M_i' \rangle^{\dagger} \tag{9.7}$$

Since ρ_i is given by $\frac{1}{2J_i + 1} \delta_{M_i M_i'}$, (9.7) can be evaluated using (9.6). Describing the orientations of the recoil nucleus by T_K^{μ} and using (2.7), $\langle T_K^{\mu} \rangle$ can be evaluated. The result is

$$Tr(T_K^{\mu} \rho) = \frac{1}{2J_i+1} \sum_{\ell\ell'}' \sum_{\lambda\lambda'}' \frac{16\pi^2}{(4\pi)^{1/2}} (i)^{\ell-\ell'} (-1)^{\lambda-\ell+\lambda'-\ell'}$$

$$(-1)^{L+\ell} \; Y_K^{\mu}(\hat{\gamma}) \; [\lambda][\lambda'][\ell][J_f]^4 \langle J_f \| T_K \| J_f \rangle$$

$$[K]^{-2} \; W(J_f J_i K J_f ; J_f J_f) \; W(J_f J_i K \ell' ; \ell J_f)$$

$$C(\ell\ell'K;000) \langle J_f \| \sum_{i=1}^{A} \tau_i^- \, j_\ell(\gamma r_i) \{ (Y_\ell(\hat{r}_i) \times \mathcal{E}_1)_\lambda$$

$$\times \sigma_1 \}_\ell \| J_i \rangle \langle J_f \| \sum_{i=1}^{A} \tau_i^- \, j_{\ell'}(\gamma r_i) \{ (Y_{\ell'}(\hat{r}_i) \times \mathcal{E}_1)_{\lambda'}$$

$$\times \sigma_1 \}_{\ell'} \| J_i \rangle^*$$

(9.8)

If photon direction is averaged over then K = 0 which means that will be unpolarized. This is quite obvious as we have initial spin zero target and a pion. However, when, photon directions are not integrated out, $\langle T_K^{\mu} \rangle$ can be obtained using (9.8) and the result is

$$\langle T_K^{\mu} \rangle = \sqrt{4\pi} \, (-1)^{K+J_f} \frac{[J_f]}{[K]^2} C(J_f J_f K;000) \langle J_f \| T_K \| J_f \rangle \, Y_K^{\mu}(\hat{\gamma}) \qquad (9.9)$$

For process (9.4), the parity C.G. coefficient in (9.9) gives K=2. Hence

$$\langle T_2^{\mu} \rangle = -\sqrt{\frac{8\pi}{25}} \langle 1 \| T_2 \| 1 \rangle \, Y_2^{\mu}(\hat{\gamma})$$

(9.10)

Using $\langle 1 \| T_2 \| 1 \rangle = \sqrt{5}$, the Fano's statistical tensor is

$$G_2(1) = -\sqrt{\frac{8\pi}{15}}\; Y_2^{\,0}(\hat{\gamma}) \tag{10.1}$$

we have already shown that G's and the magnetic sublevel popula-tions p_m are transforms of one another (eqn.3.5). Hence

$$p_m = \sum_{\nu}' (-1)^{J-M}\, G_\nu(J)\; C(J\,J\,\nu;\, M\, -M\, 0) \tag{10.2}$$

From (9.9), $\langle T_0^{\,0} \rangle$ and $\langle T_2^{\,\mu} \rangle$ can be calculated. $\langle T_1^{\,\mu} \rangle = 0$ because of parity C.G. coefficient. Then

$$P_M = \frac{1}{3}\left[1 - \sqrt{10}\; C(1\,2\,1;\, m\,0\,m)\, P_2(\cos\theta)\right] \tag{10.3}$$

Or

$$\begin{aligned} P_1 &= \frac{1}{3}\left[1 - P_2(\cos\theta)\right] \\ P_0 &= \frac{1}{3}\left[1 + 2P_2(\cos\theta)\right] \\ P_{-1} &= \frac{1}{3}\left[1 - P_2(\cos\theta)\right] \end{aligned} \tag{10.4}$$

The alignment of the nucleus is

$$A = \sum_m' P_m\left[m^2 - j(j+1)\right] \tag{10.5}$$

which is $-\frac{4}{3} P_2(\cos\theta)$.

APPENDIX:-

Here let us give some useful formulae.

$$G_1(j) = \sqrt{\frac{3}{2j+1}} \frac{1}{\sqrt{j(j+1)}} \sum_m' m P_m$$

$$G_2(j) = \left[\frac{5}{j(j+1)(2j-1)(2j+1)(2j+3)}\right]^{1/2} \sum_m' P_m [3m^2 - j(j+1)]$$

Since P_m are diagonal elements of ρ in $|jm\rangle$ representation

$$Tr[T_K^0 \rho] = \langle j \| T_K \| j \rangle \frac{[j]}{[K]} G_K(j)$$

$$T_0^0 = \frac{1}{\sqrt{2j+1}}$$

$$T_1^0 = 2\left[\frac{3(2j-1)!}{(2j+2)!}\right]^{1/2} J_z$$

$$T_2^0 = \left[\frac{180(2j-2)!}{(2j+3)!}\right](J_z^2 - \frac{1}{3}J^2)$$

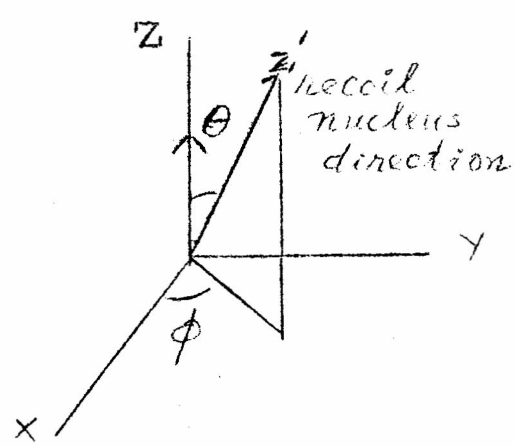
<u>Lecture 5:</u>

In this lecture, the recoil nuclear polarization and the populations of magnetic sublevels will be described by using helicity basis. This is some times convenient. The first application of helicity formalism to muon capture process is due to Bernabeu (Phys. Lett.55B (1975) 313).

Consider

$$\mu^- + N_i(J_i = 0) \rightarrow N_f(J_f = j) + \nu_\mu \qquad (10.6)$$

in the rest frame of the initial state, in which the muon is bound and the system has a definite angular momentum. In the final state, N_f and ν_μ are in free state and have spin. However experimentally the helicity of ν_μ is $-\frac{1}{2}$. Also, we have the momentum of N_f, $\vec{p} = -\vec{\nu}_\mu$. Thus, one can characterize the physical states in the helicity basis and expand them in terms of the basis states of the irreducible representations of the rotation group. The final system which is moving can be obtained from the rest system. The angular and helicity information of the final state is contained in the state vector $|\theta\phi, \lambda_f, -\frac{1}{2}\rangle$ where λ_f is the helicity of N_f, (θ, ϕ) are the polar angles of recoil nucleus direction. This state vector can be expanded in terms of definite angular momentum states



(See eqr.3.71, 3.91 and 4.200 of A.D.Martin and T.D.Spearman, <u>Elementary Particle Theory</u>, North-Holland, Amsterdam 1970)

$$|\theta\phi, \lambda_f - \tfrac{1}{2}\rangle = \sum_{J,m}' \left(\frac{2J+1}{4\pi}\right)^{1/2} \mathcal{D}^{J}_{m\lambda}(\phi,\theta,0)\,|JM, \lambda_f - \tfrac{1}{2}\rangle \tag{10.7}$$

where J is the angular momentum of the system (= ½ since initial nuclear spin is zero and muon spin is ½, L=0) and $\lambda = \lambda_f + 1/2$. The transition amplitude for the process, is given by

$$f_{\lambda m} = \langle \theta\phi\,;\lambda/T/\tfrac{1}{2}\,m\rangle = \frac{1}{\sqrt{2\pi}}\,\mathcal{D}^{1/2\,*}_{m\lambda}(\phi,\theta,0)\,\langle JM\,;\lambda_f - \tfrac{1}{2}\,|T|\,\tfrac{1}{2}\,m\rangle = \frac{1}{\sqrt{2\pi}}\,\mathcal{D}^{1/2\,*}_{m\lambda}(\phi,\theta,0)\,T_\lambda \tag{10.8}$$

where T_λ is the partial wave helicity amplitude corresponding to total angular momentum J = ½. Owing to the fixed helicity of neutrino (-½), there are only two helicity amplitudes $T_{1/2}$ and $T_{-1/2}$. The <u>final state density matrix in helicity basis</u> is given by

$$(\rho_f)_{\lambda\lambda'} = \sum_{m m'}' f_{\lambda m}\,(\rho_i)_{mm'}\,f^{*}_{\lambda' m'} \tag{10.9}$$

where $(\rho_i)_{mm'}$ is the initial state density matrix and

$$\rho_i = \tfrac{1}{2}\{1 + 2\,\vec{P}_\mu \cdot \vec{S}\} \tag{10.10}$$

Since, $\vec{P}_\mu$ is chosen along Z axis,

$$(\rho_f)_{\lambda\lambda'} = \frac{1}{2} \sum_m f_{\lambda m} \, (1 + 2m \, |\vec{P}_\mu|) \, f^*_{\lambda' m} \tag{11.1}$$

Starting from (10.8) and using the properties of rotation matrices, we can obtain

$$\sum_m{}' f_{\lambda m} \, f^*_{\lambda' m} = \frac{1}{2\pi} \, \delta_{\lambda\lambda'} \, |T_\lambda|^2 \tag{11.2}$$

$$\begin{aligned} \sum_m{}' 2m \, f_{\lambda m} \, f^*_{\lambda' m} &= \frac{1}{2\pi} \Big[\delta_{\lambda\lambda'} \, 2\lambda \, |T_\lambda|^2 \cos\theta \\ &\quad - \left(\delta_{\lambda-\lambda',1} \, T_{-\frac{1}{2}} \, T^*_{\frac{1}{2}} + \delta_{\lambda-\lambda',-1} \, T_{\frac{1}{2}} \, T^*_{-\frac{1}{2}} \right) \sin\theta \Big] \end{aligned} \tag{11.3}$$

The Intensity or transition probability is given by

$$\begin{aligned} I(\theta, \phi) = Tr \, \rho_f &= \sum_\lambda{}' (\rho_f)_{\lambda\lambda} \\ &= \sum_{\lambda, m}{}' \frac{1}{2} \, f_{\lambda m} \, (1 + 2m \, |P_\mu|) \, f^*_{\lambda m} \end{aligned} \tag{11.4}$$

Using (11.2) and (11.3)

$$\begin{aligned} I(\theta, \phi) &= \sum_\lambda{}' \frac{1}{2} \left[\frac{1}{2\pi} \, |T_\lambda|^2 + |P_\mu| \, \frac{2\lambda}{2\pi} \, |T_\lambda|^2 \cos\theta \right] \\ &= \frac{1}{4\pi} \sum_\lambda{}' \left(1 + 2 \, |P_\mu| \, \lambda \cos\theta \right) |T_\lambda|^2 \end{aligned} \tag{11.5}$$

$$I(\theta, \phi) = I(\theta) = \frac{1}{4\pi} \Big[\sum_\lambda{}' |T_\lambda|^2 + 2 \, |P_\mu| \cos\theta \sum_\lambda{}' \lambda \, |T_\lambda|^2 \Big] \tag{11.6}$$

The asymmetry coefficient in the angular distribution of recoil nucleus is given by

$$\alpha = \frac{2 \sum_{\lambda}' \lambda |T_\lambda|^2}{\sum_{\lambda} |T_\lambda|^2} \qquad (11.7)$$

If the recoil nuclear directions are averaged over then the cos term in (11.6) disappears trivially, and the transition probability is

$$\Lambda = \sum_{\lambda}' |T_\lambda|^2 \qquad (11.8)$$

The longitudinal polarization of the recoil nucleus can be obtained by finding out the average expectation value of ($\vec{J} \cdot \hat{p}$) where $\vec{J}$ and $\hat{p}$ are the spin and direction of the recoil nucleus. By eqn.(1.5)

$$\widetilde{\langle \vec{J} \cdot \hat{p} \rangle} = Tr\,[(\vec{J} \cdot \hat{p})\,\rho_f\,] \qquad (11.9)$$

Substituting for ρ_f from eqn.(11.1), writing $(\vec{J} \cdot \hat{p})_{\lambda\lambda'} = (\lambda - \frac{1}{2})\,\delta_{\lambda\lambda'}$ and using (11.2) and (11.3), we find

$$\widetilde{\langle \vec{J} \cdot \hat{p} \rangle} = \frac{1}{4\pi} \sum_{\lambda}' (\lambda - \tfrac{1}{2})\,|T_\lambda|^2 \,(1 + 2\lambda |P_\mu| \cos\varepsilon) \qquad (11.10)$$

and upon normalizing by dividing by $Tr\,\rho_f$, for <u>unpolarized</u>

muon capture

$$\langle \widetilde{\vec{J} \cdot \hat{p}} \rangle = \sum_{\lambda}' (\lambda - \tfrac{1}{2}) |T_\lambda|^2 \Big/ \sum_{\lambda} |T_\lambda|^2 \tag{12.1}$$

From (11.7) and (12.1), one immediately finds,

$$\alpha = 1 + 2j P_L \tag{12.2}$$

where j is the magnitude of J . This relation (12.2) has been independently obtained by Devanathan and Subramanian (Phys. Rev.
).

The transverse polarization can be calculated (c.f. eqn.4.204 of Martin and Spearman loc.cit.) by using the matrix element of J_x . Then the average polarization is given by

$$P_{av} = P_\mu \frac{1}{3}\sqrt{\frac{j+1}{j}} \; \frac{2\, Re\,[T_{1/2} T_{-1/2}^*]}{\sum_\lambda' |T_\lambda|^2} - P_\mu \frac{1}{3} P_L \tag{12.3}$$

Now, let us express the helicity amplitudes in terms of observables. From (12.1) and (11.8)

$$\begin{aligned} |T_{-1/2}|^2 &= -j \Lambda P_L \\ |T_{1/2}|^2 &= \Lambda + j \Lambda P_L \end{aligned} \tag{12.4}$$

and from (12.3), defining $P = P_{av}/P_\mu$,

$$2\, Re\,[T_{1/2} T_{-1/2}^*] = \sqrt{\frac{j}{j+1}} \; (3P + P_L)\Lambda \tag{12.5}$$

The density matrix in the helicity basis as given by (11.1) gives

$$\rho_{\frac{1}{2}\frac{1}{2}} = \frac{1}{2} \sum_m{}' f_{\frac{1}{2}m} (1 + 2m|P_\mu|\cos\theta) f^*_{\frac{1}{2}m}$$

$$= \frac{1}{2} \sum_m{}' \{ f_{\frac{1}{2}m} f^*_{\frac{1}{2}m} + 2|P_\mu| m f_{\frac{1}{2}m} f^*_{\frac{1}{2}m} \}$$

$$= \frac{1}{4\pi} \{ |T_{\frac{1}{2}}|^2 + |P_\mu| |T_{\frac{1}{2}}|^2 \cos\theta \}$$

$$= \frac{1}{4\pi} (1 + |P_\mu| \cos\theta) |T_{\frac{1}{2}}|^2 \qquad (12.6)$$

$$\rho_{-\frac{1}{2}-\frac{1}{2}} = \frac{1}{2} \sum_m{}' f_{-\frac{1}{2}m} (1 + 2m|P_\mu|\cos\theta) f^*_{-\frac{1}{2}m}$$

$$= \frac{1}{4\pi} \{ |T_{-\frac{1}{2}}|^2 - |P_\mu| |T_{-\frac{1}{2}}|^2 \cos\theta \}$$

$$= \frac{1}{4\pi} (1 - |P_\mu| \cos\theta) |T_{-\frac{1}{2}}|^2 \qquad (12.7)$$

$$\rho_{\frac{1}{2}-\frac{1}{2}} = \frac{1}{2} \sum_m{}' f_{\frac{1}{2}m} (1 + 2m|P_\mu|) f^*_{-\frac{1}{2}m}$$

$$= \frac{1}{2} \sum_m (f_{\frac{1}{2}m} f^*_{-\frac{1}{2}m} + 2m|P_\mu| f_{\frac{1}{2}m} f^*_{-\frac{1}{2}m})$$

$$= -\frac{1}{4\pi} |P_\mu| T_{-\frac{1}{2}} T^*_{\frac{1}{2}} \sin\theta \qquad (12.8)$$

$$\rho_{-\frac{1}{2}\frac{1}{2}} = -\frac{1}{4\pi} |P_\mu| T_{\frac{1}{2}} T^*_{-\frac{1}{2}} \sin\theta \qquad (12.9)$$

$$\rho_{-\frac{1}{2}\frac{1}{2}} + \rho_{\frac{1}{2}-\frac{1}{2}} = -\frac{2}{4\pi} |P_\mu| \,\mathrm{Re}\,[T_{\frac{1}{2}} T^*_{-\frac{1}{2}}] \sin\theta \qquad (12.10)$$

Thus, the matrix elements of the density matrix in the helicity basis, can be expressed in terms of amplitudes (12.6, 12.7, 12.8, 12.9 and 12.10) and hence in terms of observables Λ, P_L and P (12.4 and 12.5). In order to evaluate the populations of magnetic sublevels, we require the density matrix in the angular momentum basis. This can be obtained from the density matrix in the helicity basis, by the simple transformation,

$$(\rho_f)_{MM'} = \sum_{\lambda\lambda'}{}' D^{J}_{M\lambda}(\phi,\theta,0)\, \rho_{\lambda\lambda'}\, D^{J\,*}_{\lambda' M'}(\phi,\theta,0) \qquad (13.1)$$

As the azimuthal angle is not observed, averaging over the same, yields $\delta_{MM'}$ and D-matrix becomes d-function. Let us specialize for the process (7.9)*. Explicitly writing the d-matrix for spin 1. Case, the right hand side of (13.1) becomes after averaging over ϕ ,

$$\begin{bmatrix} \frac{1}{2}(1+\cos\theta) & -\frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1-\cos\theta) \\ \frac{1}{\sqrt{2}}\sin\theta & \cos\theta & -\frac{1}{\sqrt{2}}\sin\theta \\ \frac{1}{2}(1-\cos\theta) & \frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1+\cos\theta) \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho_{\frac{1}{2}\frac{1}{2}} & \rho_{\frac{1}{2}-\frac{1}{2}} \\ 0 & \rho_{-\frac{1}{2}\frac{1}{2}} & \rho_{-\frac{1}{2}-\frac{1}{2}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2}(1+\cos\theta) & \frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1-\cos\theta) \\ -\frac{1}{\sqrt{2}}\sin\theta & \cos\theta & \frac{1}{\sqrt{2}}\sin\theta \\ \frac{1}{2}(1-\cos\theta) & -\frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1+\cos\theta) \end{bmatrix} \qquad (13.2)$$

*I am very much benefitted by the discussions with Dr.J.Bernabeu during May 1975 at CERN, on this treatment.

A direct multiplication gives the diagonal matrix elements as

$$\rho_{11} = \rho_{\frac{1}{2}\frac{1}{2}} \frac{\sin^2\theta}{2} + \rho_{-\frac{1}{2}-\frac{1}{2}} \frac{1}{4}(1-\cos\theta)^2 - (\rho_{\frac{1}{2}-\frac{1}{2}} + \rho_{-\frac{1}{2}\frac{1}{2}}) \frac{\sin\theta}{2\sqrt{2}} \times (1-\cos\theta)$$

$$\rho_{00} = \rho_{\frac{1}{2}\frac{1}{2}} \cos^2\theta + \rho_{-\frac{1}{2}-\frac{1}{2}} \frac{\sin^2\theta}{2} - (\rho_{\frac{1}{2}-\frac{1}{2}} + \rho_{-\frac{1}{2}\frac{1}{2}}) \frac{\sin\theta\cos\theta}{\sqrt{2}}$$

$$\rho_{-1-1} = \rho_{\frac{1}{2}\frac{1}{2}} \frac{\sin^2\theta}{2} + \rho_{-\frac{1}{2}-\frac{1}{2}} \frac{(1+\cos\theta)^2}{4} + (\rho_{\frac{1}{2}-\frac{1}{2}} + \rho_{-\frac{1}{2}\frac{1}{2}}) \sin\theta(1+\cos\theta)/2\sqrt{2}$$

Expressing in terms of helicity amplitudes,

$$\rho_{11} = \frac{1}{4\pi}\Big[\frac{1}{2}|T_{\frac{1}{2}}|^2 \sin^2\theta(1+P_\mu\cos\theta) + \frac{1}{4}|T_{-\frac{1}{2}}|^2(1-\cos\theta)^2 (1-P_\mu\cos\theta) + \frac{2}{2\sqrt{2}} Re[T_{\frac{1}{2}} T^*_{-\frac{1}{2}}] P_\mu \sin^2\theta(1-\cos\theta)\Big]$$

$$\rho_{00} = \frac{1}{4\pi}\Big[|T_{\frac{1}{2}}|^2\cos^2\theta(1+P_\mu\cos\theta) + \frac{1}{2}|T_{-\frac{1}{2}}|^2\sin^2\theta(1-P_\mu\cos\theta) + \frac{2}{\sqrt{2}} Re[T_{\frac{1}{2}}T^*_{-\frac{1}{2}}] P_\mu \sin^2\theta\cos\theta\Big]$$

$$\rho_{-1-1} = \frac{1}{4\pi}\Big[\frac{1}{2}|T_{\frac{1}{2}}|^2\sin^2\theta(1+P_\mu\cos\theta) + \frac{1}{4}|T_{-\frac{1}{2}}|^2(1+\cos\theta)^2 (1-P_\mu\cos\theta) - \frac{2}{2\sqrt{2}} Re[T_{\frac{1}{2}}T^*_{-\frac{1}{2}}] P_\mu \sin^2\theta(1+\cos\theta)\Big]$$

The angular integration can be carried out now. We make a convenient way by first integrating over the forward hemisphere and then backward hemisphere. This would simply mean spliting the integral $\int_{-1}^{+1}$ as $\int_{0}^{1}$ and $\int_{-1}^{0}$. Carrying out the angular integration and expressing the helicity amplitudes in terms of observables, one can obtain

Forward Result:-

$$\left.\begin{aligned} \rho_{11} &= \frac{1}{6}\left(1+\frac{3}{4}P_L\right) + \frac{1}{16}\left(1+\frac{5}{2}P+2P_L\right)P_\mu \\ \rho_{00} &= \frac{1}{6} + \frac{1}{8}\left(1+\frac{3}{2}P+2P_L\right)P_\mu \\ \rho_{-1-1} &= \frac{1}{6}\left(1-\frac{3}{4}P_L\right) + \frac{1}{16}\left(1-\frac{11}{2}P+2P_L\right)P_\mu \end{aligned}\right\} \qquad (13.2)$$

Backward Result:-

$$\left.\begin{aligned} \rho_{11} &= \frac{1}{6}\left(1-\frac{3}{4}P_L\right) - \frac{1}{16}\left(1-\frac{11}{2}P+2P_L\right)P_\mu \\ \rho_{00} &= \frac{1}{6} - \frac{1}{8}\left(1+\frac{3}{2}P+2P_L\right)P_\mu \\ \rho_{-1-1} &= \frac{1}{6}\left(1+\frac{3}{4}P_L\right) - \frac{1}{16}\left(1+\frac{5}{2}P+2P_L\right)P_\mu \end{aligned}\right\} \qquad (13.3)$$

Adding all the populations we obtain 1. The Fano's statistical tensor can be calculated using the Appendix in Lecture 4.

$$G_2^F(1) = -\frac{1}{\sqrt{6}}\left[\frac{1}{8} + \frac{P_L}{4} + \frac{9P}{16}\right]P_\mu \qquad (13.4)$$

and

$$G_2^F(1) = -G_2^B(1)$$

where B and F mean Backward and Forward.

Let us now consider the density matrix for a final nucleus whose spin is 2.

The density matrix in <u>helicity basis</u> of the complete final state (neutrino and nucleus) is given by (12.6) to (12.10), as the neutrino helicity is fixed, the projection over the nuclear space corresponds $\lambda_f = 0, -1$. Then, the density matrix in the angular momentum basis is given by, following (13.1)

$$\begin{bmatrix} d^2_{22}(\theta) & d^2_{21}(\theta) & d^2_{20}(\theta) & d^2_{2-1}(\theta) & d^2_{2-2}(\theta) \\ d^2_{12}(\theta) & - & - & - & - \\ | & & & & \\ | & & & & \\ d^2_{-22}(\theta) & d^2_{-21}(\theta) & d^2_{-20}(\theta) & d^2_{-2-1}(\theta) & d^2_{-2-2}(\theta) \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{S}_{\frac12\frac12} & \mathcal{S}_{\frac12-\frac12} & 0 \\ 0 & 0 & \mathcal{S}_{-\frac12\frac12} & \mathcal{S}_{-\frac12-\frac12} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} d^2_{22}(-\theta) & d^2_{21}(-\theta) & d^2_{20}(-\theta) & d^2_{2-1}(-\theta) & d^2_{2-2}(-\theta) \\ | & & & & \\ | & & & & \\ | & & & & \\ | & & & & \\ d^2_{-22}(-\theta) & d^2_{-21}(-\theta) & d^2_{-20}(-\theta) & d^2_{-2-1}(-\theta) & d^2_{-2-2}(-\theta) \end{bmatrix} \tag{13.5}$$

Carrying out the multiplication and using

$$d^j_{m'm}(\theta) = \left[\frac{(j+m')!\,(j-m')!}{(j+m)!\,(j-m)!}\right]^{\frac12} \sum_s{}' \binom{j+m}{j+m'-s}\binom{j-m}{s} \tag{13.6}$$
$$(-1)^{j-m'-s}\left(\cos\frac{\theta}{2}\right)^{2s+m'+m}\left(\sin\frac{\theta}{2}\right)^{2j-2s-m'-m}$$

and

$$d^{j}_{m'm}(-\theta) = d^{j}_{mm'}(\theta)$$

the density matrix in the angular momentum basis can be obtained. We need to consider only the diagonal elements.

$$\rho_{22} = \frac{3}{8}\sin^4\theta\,\rho_{\frac{1}{2}\frac{1}{2}} + \frac{1}{4}(1-\cos\theta)^2\sin^2\theta\,\rho_{-\frac{1}{2}-\frac{1}{2}} - \sqrt{\frac{3}{8}}\,\frac{\sin^2\theta}{2}(1-\cos\theta)$$
$$\sin\theta\left(\rho_{\frac{1}{2}-\frac{1}{2}} + \rho_{-\frac{1}{2}\frac{1}{2}}\right)$$

$$\rho_{11} = \frac{3}{2}\sin^2\theta\cos^2\theta\,\rho_{\frac{1}{2}\frac{1}{2}} + (\cos\theta - \cos^2\theta + \sin^2\theta)^2\frac{1}{4}\,\rho_{-\frac{1}{2}-\frac{1}{2}}$$
$$-\sqrt{\frac{3}{2}}\,\frac{\sin\theta\cos\theta}{2}(\cos\theta - \cos^2\theta + \sin^2\theta)\left(\rho_{\frac{1}{2}-\frac{1}{2}} + \rho_{-\frac{1}{2}\frac{1}{2}}\right)$$

$$\rho_{00} = \frac{1}{4}(2\cos^2\theta - \sin^2\theta)^2\rho_{\frac{1}{2}\frac{1}{2}} + \frac{3}{2}\cos^2\theta\sin^2\theta\,\rho_{-\frac{1}{2}-\frac{1}{2}}$$
$$-\sqrt{\frac{3}{2}}(2\cos^2\theta - \sin^2\theta)\frac{1}{2}\sin\theta\cos\theta\left(\rho_{\frac{1}{2}-\frac{1}{2}} + \rho_{-\frac{1}{2}\frac{1}{2}}\right)$$

$$\rho_{-1-1} = \frac{3}{2}\sin^2\theta\cos^2\theta\,\rho_{\frac{1}{2}\frac{1}{2}} + (\cos\theta + \cos^2\theta - \sin^2\theta)^2\frac{1}{4}\,\rho_{-\frac{1}{2}-\frac{1}{2}}$$
$$+\sqrt{\frac{3}{2}}(\cos\theta + \cos^2\theta - \sin^2\theta)\frac{1}{2}\sin\theta\cos\theta\left(\rho_{\frac{1}{2}-\frac{1}{2}} + \rho_{-\frac{1}{2}\frac{1}{2}}\right)$$

$$\rho_{-2-2} = \frac{3}{8}\sin^4\theta\,\rho_{\frac{1}{2}\frac{1}{2}} + (1+\cos\theta)^2\frac{1}{4}\sin^2\theta\,\rho_{-\frac{1}{2}-\frac{1}{2}} + \sqrt{\frac{3}{8}}\,\frac{1}{2}\sin^2\theta$$
$$(1+\cos\theta)\sin\theta\left(\rho_{\frac{1}{2}-\frac{1}{2}} + \rho_{-\frac{1}{2}\frac{1}{2}}\right) \qquad (13.7)$$

If the diagonal elements are added, one finds

$$Tr\,\rho = \frac{\Lambda}{4\pi}\left\{1 + \alpha P_\mu \cos\theta\right\} \qquad (13.8)$$

using (12.4) and (12.5). Eqn.(13.8) is a typical angular distribution with the asymmetry $\alpha = 1 + 4P_L$. Upon integrating over θ and ϕ

$$\int Tr\,\rho \;\sin\theta\, d\theta\, d\phi = \Lambda \qquad (13.9)$$

The diagonal elements in (13.7) can be made to give the populations of magnetic sublevels. Here, with some experimental arrangement in mind, let us integrate over the recoil nuclear directions in forward and backward hemisphere. The angular integrations are trivial but tedious. The results are:

Forward Population:-

$$P_2 = \frac{1}{4\pi}\Big[\frac{3}{8}|T_{1/2}|^2\Big(\frac{8}{15} + \frac{P_\mu}{6}\Big) + \frac{1}{4}|T_{-1/2}|^2\Big(\frac{3}{10} - \frac{P_\mu}{15}\Big) + \sqrt{\frac{3}{8}}\,Re[T_{1/2}T^*_{-1/2}]\,\frac{11P_\mu}{30}\Big]$$

$$P_1 = \frac{1}{4\pi}\Big[\frac{3}{2}|T_{1/2}|^2\Big(\frac{2}{15} + \frac{P_\mu}{12}\Big) + \frac{1}{4}|T_{-1/2}|^2\Big(\frac{4}{5} - \frac{17P_\mu}{60}\Big) + \sqrt{\frac{3}{2}}\,Re[T_{1/2}T^*_{-1/2}]\,13P_\mu/60\Big]$$

$$P_0 = \frac{1}{4\pi}\Big[\frac{1}{4}|T_{1/2}|^2\Big(\frac{4}{5} + \frac{P_\mu}{2}\Big) + \frac{3}{2}|T_{-1/2}|^2\Big(\frac{2}{15} - \frac{P_\mu}{12}\Big)\Big]$$

$$P_{-1} = \frac{1}{4\pi}\Big[\frac{3}{2}|T_{1/2}|^2\Big(\frac{2}{15} + \frac{P_\mu}{12}\Big) + \frac{1}{4}|T_{-1/2}|^2\Big(\frac{4}{5} - \frac{11P_\mu}{20}\Big) - \sqrt{\frac{3}{2}}\,Re[T_{1/2}T^*_{-1/2}]\,\frac{P_\mu}{20}\Big] \qquad (13.10)$$

$$P_{-2} = \frac{1}{4\pi}\Big[\frac{3}{8}|T_{1/2}|^2\Big(\frac{8}{15} + \frac{P_\mu}{6}\Big) + \frac{1}{4}|T_{-1/2}|^2\Big(\frac{13}{10} - \frac{3P_\mu}{5}\Big) - \sqrt{\frac{3}{8}}\,Re[T_{1/2}T^*_{-1/2}]\,\frac{21P_\mu}{30}\Big]$$

Backward population:-

$$P_2 = \frac{1}{4\pi}\Big[\frac{3}{8}|T_{1/2}|^2\Big(\frac{8}{15}-\frac{P_\mu}{6}\Big)+\frac{1}{4}|T_{-1/2}|^2\Big(\frac{13}{10}+\frac{9P_\mu}{15}\Big)+\sqrt{\frac{3}{8}}\,Re\,[T_{1/2}T^{*}_{-1/2}]\frac{21P_\mu}{30}\Big]$$

$$P_1 = \frac{1}{4\pi}\Big[\frac{3}{2}|T_{1/2}|^2\Big(\frac{2}{15}-\frac{P_\mu}{12}\Big)+\frac{1}{4}|T_{-1/2}|^2\Big(\frac{4}{5}+\frac{11P_\mu}{20}\Big)+\sqrt{\frac{3}{2}}\,Re\,[T_{1/2}T^{*}_{-1/2}]\frac{P_\mu}{20}\Big]$$

$$P_0 = \frac{1}{4\pi}\Big[\frac{1}{4}|T_{1/2}|^2\Big(\frac{4}{5}-\frac{P_\mu}{2}\Big)+\frac{3}{2}|T_{-1/2}|^2\Big(\frac{2}{15}+\frac{P_\mu}{12}\Big)\Big]$$

$$P_{-1} = \frac{1}{4\pi}\Big[\frac{3}{2}|T_{1/2}|^2\Big(\frac{2}{15}-\frac{P_\mu}{2}\Big)+\frac{1}{4}|T_{-1/2}|^2\Big(\frac{4}{5}+\frac{17P_\mu}{60}\Big)-\sqrt{\frac{3}{2}}\,Re\,[T_{1/2}T^{*}_{-1/2}]\frac{13P_\mu}{60}\Big]$$

$$P_{-2} = \frac{1}{4\pi}\Big[\frac{3}{8}|T_{1/2}|^2\Big(\frac{8}{15}-\frac{P_\mu}{6}\Big)+\frac{1}{4}|T_{-1/2}|^2\Big(\frac{3}{10}+\frac{P_\mu}{15}\Big)-\sqrt{\frac{3}{8}}\,Re\,[T_{1/2}T^{*}_{-1/2}]\,11P_\mu/30\Big] \qquad (14.1)$$

Again summing over the populations and integrating over

$$\sum\!\!\!\!\int' P^{F,B}_{2,1,0,-1,-2}\, d\phi = \Lambda \qquad (14.2)$$

which is same as (13.9), thus providing a check on the intermediate steps. With these populations, one can calculate the forward and backward alignment.

Let us obtain the recoil nuclear polarization in forward and backward hemisphere by using the formula

$$P(J) = \frac{1}{J}\,\frac{\Sigma'\, m P_m}{\Sigma' P_m} \qquad (14.3)$$

and remembering, for forward or backward cases, $\Sigma' P_m$ is always the <u>total population</u>. Therefore, it is straight forward from ecns.(13.10) and (14.1), to obtain

$$P^{forward}(2) = \frac{P_L}{4} + \frac{P_\mu}{2} \tag{14.4}$$

$$P^{backward}(2) = -\frac{P_L}{4} + \frac{PP_\mu}{2} \tag{14.5}$$

where P_L and P refer to longitudinal and average polarizations. In this way, <u>the average polarization in one hemisphere is half of the total average polarization and one fourth of longitudinal polarization whose sign changes for each hemisphere.</u> Summing (14.4) and (14.5), one obtains a well known identity. The fact that P_L in (14.4) and (14.5) comes with opposite signs in forward and backward hemisphere is an anticipated feature which can be understood with the kinematics. When the recoil nucleus is along z'-axis, its polarization is longitudinal $\langle \tilde{J}_{z'} \rangle$ and when the average polarization over the entire sphere is evaluated, z'-axis loses its identity, owing to integration over θ and the only direction is Z-axis, or P_μ direction. On the other hand, when averaging is carried over a hemisphere, the polarization of recoil nucleus consists of two parts, a longitudinal component and a partially averaged polarization along Z-axis.

The next step will be to evaluate the tensor polarization for spin 2 final nucleus in muon capture. This can be calculated simply from the populations obtained in eqns.(13.10) and (14.1). Let us first give the definition of tensor polarization in terms of populations, as

$$\langle T_2^0 \rangle = \sum_m{}' \, [2] \, c(222; m\,o\,m) P_m \qquad (14.6)$$

Expanding the summation and using the explicit values of Clebsch-Gordon coefficients, one finds

$$\langle T_2^0 \rangle = \sqrt{\frac{5}{14}} \left[2(P_2 + P_{-2}) - (P_1 + P_{-1}) - 2P_0 \right] \qquad (14.7)$$

Substituting for P_2, P_1, P_0, P_{-1} and P_{-2} from (13.10) we immediately find

$$\langle T_2^0 \rangle^f = -\frac{P_\mu}{8} \sqrt{\frac{15}{14}} (1 + 4P_L + 3P) \qquad (14.8)$$

where of course P_L and P are the longitudinal and average polarization respectively. (Note all the populations are normalized such that $\sum_m{}' P_m = 1$). Eqn.(14.8) is the same as eqn.(34) of our earlier result. (Nucl. Phys. A262 (1976) 433), provided we make the identification about the definition of P_2 and P as P_L or P of this notation = $\frac{1}{2}$(P_L or P_H) of earlier result. The backward tensor polarization can be evaluated in the similar way and one can find

$$\langle T_2^0 \rangle^F = -\langle T_2^0 \rangle^B \qquad (14.9)$$

Eqn.(14.8) can also be written as

$$\langle T_2^0 \rangle^f = -\frac{P_\mu}{8}\sqrt{\frac{5}{14}}\,(\alpha + 3P) \tag{14.10}$$

where is the asymmetry in the angular distribution of recoil nucleus. Eqns.(14.8) and (14.10) are independent of nuclear model and muon capture coupling constants. It is a general relation among tensor, longitudinal and average polarization of the recoil nucleus.

With the help of (13.2) and (13.3) and using (14.6) the tensor polarization of spin 1 final nucleus can also be evaluated. The procedure is straight forward and the final result is

$$\langle T_2^0 \rangle^f = -\frac{1}{\sqrt{16}}\,\frac{1}{2\sqrt{2}}\,(2 + 4P_L + 9P)P_\mu \tag{15.1}$$

and

$$\langle T_2^0 \rangle^f = -\langle T_2^0 \rangle^b \tag{15.2}$$

I would like to comment upon two important observations.

(1) With (14.6) and its generalization for $\langle T_K^\mu \rangle$ and with the populations, all the tensor moments can be evaluated. Then, the density matrix is completely determined using eqn.(2.5), given a suitable representation for T_K^μ .

(2) Relations (14.8) and (15.1) are independent of nuclear models and weak interaction coupling constants.

<u>Lecture 5:</u>

In the course of lectures so far given, we have seen how the density matrix of the final nucleus or particle can be constructed either directly from the interaction Hamiltonian or from the helicity basis and how the polarizations could be evaluated. Needless to say that this scheme can be generally applied to almost all direct interactions.

In this lecture, let us consider the applicability of the density matrix methods to a two-step process. This has been extensively dealt with for gamma-gamma angular correlations or beta-gamma angular correlations by H.Frauenfelder and R.M.Steffen; S.R.De Groot, H.A.Tolhoek and W.J.Huiskamp, <u>α, β, γ Ray Spectroscopy</u>, Vol.2, Ed.K.Siegbahn, North-Holland Pub. Co. 1965. We confine to muon capture and subsequent gamma emission and the quantity of interest is gamma-neutrino angular correlations. Although gamma-neutrino angular correlation, at first glance, seems almost impossible to measure by conventional coincidence techniques, an ingenious and indirect method has been suggested by Prof.L.Grenacs of Louvain University in 1968 (Nucl. Instrum. Methods 58 (1968) 164) and had been successfully carried out by G.H.Miller (Phys. Rev. Lett. 29 (1972) 1194). The new technique is to observe the Doppler broadening of the gamma ray due to nuclear recoil which is due to neutrino emission. The reaction of interest is

$$\mu^- + {}^{28}Si(0^+) \rightarrow {}^{28}Al(1^+; 2.202\ MeV) + \nu_\mu$$
$$\hookrightarrow {}^{28}Al(0^+; 0.973\ MeV) + \gamma \quad (15.3)$$

The procedure is to construct the density matrix for intermediate state, retaining the angular identity of neutrino and then construct the density matrix of ${}^{28}Al(0^+)$ the final nucleus. If we denote the density matrix for the intermediate state as $(\rho^{\mu\cdot c})_{M_f M_f'}$ where $\mu\cdot c$ stand for muon capture and the interaction Hamiltonian for gamma decay by H_γ, then the final state density matrix is given by

$$(\rho_f)_{M_F M_F} = \sum_{M_f, M_f'}' \langle J_F M_F | H_\gamma | J_f M_f \rangle (\rho^{\mu\cdot c})_{M_f M_f'} \langle J_F M_F | H_\gamma | J_f M_f' \rangle^{*} \quad (15.4)$$

The final state density matrix can be used to give the angular distribution of gamma rays and by 'angular momentum algebra', one can obtain, the angular correlations. For the sake of illustration, let us consider unpolarized muons and neglect the nucleon momentum dependent terms in the muon capture Hamiltonian (5.1). Expressing the Hamiltonian (5.1) in terms of spherical tensors,

$$\mathcal{H} = \sum_{i=1}^{3} C_i \sum_{\lambda, m_\lambda} t_\lambda^{m_\lambda}(i) \quad (15.5)$$

where C_i's muon capture coupling constants and using our earlier description for the density matrix

$$\langle J_f M_f | \rho_f | J_f M_f' \rangle = \sum_{i,j} \sum_{M_i M_i'} C_i C_j \langle J_f M_f | \sum_{\lambda m_\lambda} t_\lambda^{m_\lambda}(i) | J_i M_i \rangle (\rho_I)_{M_i M_i'} \langle J_f M_f' | t_{\lambda'}^{m_{\lambda}'}(j) | J_i M_i' \rangle^* \tag{15.6}$$

along with the equation

$$(\rho_I)_{M_i M_i'} = \frac{1}{2J_i + 1} \delta_{M_i M_i'} \tag{15.7}$$

the density matrix for the intermediate state can be obtained. Let us outline the procedure for i = 1.j = 1. This term will be $M_1 M_1^*$

$$M_1 = \langle J_f M_f | \sum_{i=1}^{A} e^{-i \vec{\hat{\nu}} \cdot \vec{r}_i} \phi(r_i) \tau_i | J_i M_i \rangle$$

Expanding the exponential in terms of partial waves

$$M_1 = 4\pi (i)^{-J_f} Y_{J_f}^{-M_f}(\hat{\nu})(-1)^{M_f} \langle J_f M_f | \sum_{i=1}^{A} Y_{J_f}^{M_f}(\hat{r}_i) \phi_\mu(r_i) j_{J_f}(\nu r_i) | 00 \rangle$$

$$= 4\pi (i)^{-J_f} Y_{J_f}^{-M_f}(\hat{\nu})(-1)^{M_f} \langle J_f \| \sum_{i=1}^{A} Y_{J_f}(\hat{r}_i) \phi_\mu(r_i) j_{J_f}(\nu r_i) \| 0 \rangle$$

by Wigner-Eckart theorem. Similarly M_1^* can be obtained.

Then, using the property of spherical harmonics,

$$(\mathcal{S}_f)_{M_f M_f'} \overset{i=1\,j=1}{=} G_V^2\, \mathcal{J}(J_f 0 J_f ; J_f 0 J_f) \sum_J (-1)^{M_f} C(J_f J_f J ; -M_f M_f') \, C(J_f J_f J ; 000) \frac{[J_f]^2}{\sqrt{4\pi}\,[J]} Y_J^{(M_f' - M_f)}(\hat{\nu}) \tag{15.8}$$

Then the other terms can be evaluated to give

$$\begin{aligned}(\mathcal{S}_f)_{M_f M_f'} = {} & G_V^2\, \mathcal{J}(J_f 0 J_f ; J_f 0 J_f) \sum_J (-1)^{M_f} C(J_f J_f J ; -M_f M_f') \\ & C(J_f J_f J ; 000) \frac{[J_f]^2}{\sqrt{4\pi}\,[J]} Y_J^{M_f' - M_f}(\hat{\nu}) + G_A^2 \sum_{\ell\ell'} (i)^{\ell' - \ell} \\ & (-1)^{\ell' - J_f} [J_f]^2\, \mathcal{J}(\ell 1 J_f ; \ell' 1 J_f) \sum_J (-1)^{M_f} C(J_f J_f J ; -M_f M_f') \\ & C(\ell \ell' J ; 000) \frac{[\ell][\ell']}{\sqrt{4\pi}\,[J]} W(J_f 1 J \ell' ; \ell J_f) Y_J^{(M_f' - M_f)}(\hat{\nu}) \\ & + (G_P^2 - 2 G_P G_A) \sum_{\ell\ell'} (i)^{\ell' - \ell} \frac{[\ell][\ell']}{[J_f]^2} C(1 \ell J_f ; 000) \\ & C(1 \ell' J_f ; 000)\, \mathcal{J}(\ell 1 J_f ; \ell' 1 J_f) \sum_J (-1)^{M_f} C(J_f J_f J ; -M_f M_f') \\ & C(J_f J_f J ; 000) \frac{[J_f]^2}{\sqrt{4\pi}\,[J]} Y_J^{(M_f' - M_f)}(\hat{\nu})\end{aligned} \tag{15.9}$$

where

$$\mathcal{J}(\ell 1 J_f ; \ell' 1 J_f) = 16\pi^2 \langle J_f \| \sum_{i=1}^{A} (Y_\ell(\hat{r}_i) \times \sigma_i)_{J_f} j_\ell(\nu r_i) \phi_\mu(r_i) \| 0 \rangle$$

$$\langle J_f \| \sum_{i=1}^{A} (Y_{\ell'}(\hat{r}_i) \times \sigma_i)_{J_f} j_{\ell'}(\nu r_i) \phi_\mu(r_i) \| 0 \rangle$$

(15.10)

which can be evaluated once a nuclear model is assumed. For example, if a particle-hole description of excited states and the concept of Isobaric Analogue States, are used, then,

$$\mathcal{J}(\ell 1 J_f ; \ell' 1 J_f) = 16\pi^2 \sum_{ph} \sum_{p'h'} \frac{[j_p][j_{p'}]}{[J_f]^2} X^{J_f}_{ph} X^{J_f}_{p'h'}$$

$$\langle j_p \| (Y_\ell(\hat{r}) \times \sigma_1)_{J_f} \| j_h \rangle \langle j_{p'} \| (Y_{\ell'}(\hat{r}) \times \sigma_1)_{J_f} \| j_{h'} \rangle$$

$$\int_0^\infty R_{n_p \ell_p}(r) j_\ell(\nu r) R_{n_h \ell_h}(r) r^2 dr$$

$$\int_0^\infty R_{n_{p'} \ell_{p'}}(r) j_\ell(\nu r) R_{n_{h'} \ell_{h'}}(r) r^2 dr \langle \phi^2_\mu(r) \rangle_{av} \qquad (16.1)$$

where, the nuclear operator is given TDA Fock space, $p\,(h)$ stand for the set of particle (hole) quantum numbers $n_p \ell_p j_p \; (n_h \ell_h j_h)$ and the muon wave function is averaged over the nuclear volume. $X^{J_f}_{ph}$ are the particle hole amplitudes normalized so that

$$\sum_{p,h}' | X^{J_f}_{p,h} |^2 = 1 \qquad (16.2)$$

Although eqn.(16.3) seems lengthy, it can be analytically evaluated. The inclusion of nucleon velocity dependent terms in (5.1) will involve radial integrals involving derivatives of radial wave functions (due to the use of Gradient formula) which can again be analytically evaluated, so long as one use harmonic oscillator basis. In radial Hartree-Fock scheme, one expands radial HF orbital in terms of H.O orbitals and so, again analytic evaluation is possible.

Having constructed the density matrix of the intermediate state, eqn.(15.4) can be used to obtain the final density matrix. The gamma decay matrix element can be evaluated by using $H_\gamma = \vec{j}_N \cdot \vec{A}_p$ with eqn.(8.7) for $\vec{A}_p$. Designating the multipolarity of the gamma ray by L

$$(\rho_F)_{M_F M_F} = \sum_{M_f M_f'} (\rho^{\mu \cdot c})_{M_f M_f'} \langle J_F M_F | \vec{j}_N \cdot \vec{A}_p(L) | J_f M_f \rangle \langle J_F M_F | \vec{j}_N \cdot \vec{A}_p(L) | J_f M_f' \rangle^* \tag{16.3}$$

The multipole expansion of $\vec{A}_p$ will be (M.E.Rose loc. cit)

$$\vec{A}_p = \sqrt{2\pi} \sum_{L,M}{}' (i)^L \sqrt{2L+1}\, \mathcal{D}^L_{MP}(\phi, 0, 0) \left\{ \vec{A}_L^M(\mathcal{M}) + ip\, \vec{A}_L^M(\mathcal{E}) \right\} \tag{16.4}$$

where $\mathcal{D}^L_{MP}$ is the rotation matrix

$p = 1$ for right circularly polarized gamma

$p = -1$ for left circularly polarized gamma

$\mathcal{M}, \mathcal{E}$ stand for Magnetic or Electric transition.

For a given L, either $A_L^M(\mathcal{M})$ or $A_L^M(\mathcal{E})$ can occur but not both owing to parity selection rule. In some special cases wherein the emitting state is not a pure parity state, but parity admixtured state, the admixture of wrong parity could occur when one considers parity violating internucleon potential either by conventional weak interaction theory or by gauge theories, then one can have both $A_L^M(\mathcal{M})$ and $A_L^M(\mathcal{E})$ and and this leads to circular polarization of gamma owing to the interference of Electric and magnetic transitions. See R.J.Blinstovle, 'Fundamental interactions in nucleus'.

In our case we have either $\mathcal{M}$ or $\mathcal{E}$ type transition, and L is fixed.

$$\vec{A}_p^L(L) = \sqrt{2\pi}(i)^L [L] \sum_M{}' D^L_{MP}(\phi,\theta,0)\left[\vec{A}_L^M(\mathcal{M}) + i p \vec{A}_L^M(\mathcal{E})\right] \tag{16.5}$$

Define

$$a(\mathcal{M}) = \sqrt{2\pi}(i)^L \, [L]$$
$$a(\mathcal{E}) = \sqrt{2\pi}(i)^{L+1} p[L]$$

Then

$$\vec{A}_p^L = \sum_{\tau=\mathcal{M},\mathcal{E}}{}' \sum_M{}' a(\tau) \vec{A}_L^p(\tau) D^L_{MP}(\phi,\theta,0) \tag{16.6}$$

Substituting (16.6) in (16.3)

$$(\rho)_{M_F M_F} = |a(\tau)|^2 \sum'_{M_f M_f'} \sum'_{MM'} (\rho^{\mu.c})_{M_f M_f'} D^L_{Mp}(\phi,\theta,0) D^{L*}_{M'p}(\phi,\theta,0)$$

$$\langle J_F M_F | \vec{j}_N \cdot \vec{A}^M_L(\tau) | J_f M_f \rangle \langle J_F M_F | \vec{j}_N \cdot \vec{A}^{M'}_L(\tau) | J_f M_f' \rangle^* \tag{16.7}$$

Applying Wigner-Eckart theorem and using

$$D^L_{Mp}(\phi,\theta,0) D^{L*}_{M'p}(\phi,\theta,0) = (-1)^{M'-p} D^L_{Mp}(\phi,\theta,0) D^L_{-M'-p}(\phi,\theta,0)$$

$$= (-1)^{M'-p} \sum_{\nu=0}^{2L}{}' c(LL\nu; M-M' m_\nu) c(LL\nu; p-p) D^{\nu}_{(M-M'),0}(\phi,\theta,0)$$

$$= (-1)^{M'-p} \sum_{\gamma=0}^{2L}{}' c(LL\gamma; M-M' M_\gamma) c(LL\gamma; p-p) \frac{\sqrt{4\pi}}{[\gamma]} Y^{M_\gamma *}_\gamma(\theta,\phi) \tag{16.8}$$

and after expressing C.G. coefficients as a Racah, we get

$$\sum'_{M_F} (\rho_F)_{M_F M_F} = |a(\tau)|^2 \sum'_{M_f M_f'} (\rho^{\mu.c})_{M_f M_f'} \sum_{\gamma=0}^{2L}{}' (-1)^p (-1)^{J_f - J_F}$$

$$c(LL\gamma; p-p0) \frac{\sqrt{4\pi}}{[\gamma]} |\langle J_F \| L(\tau) \| J_f \rangle|^2 \frac{[J_F]^2 [\gamma]}{[J_f]}$$

$$W(J_f L J_f L; J_F \gamma) c(J_f \gamma J_f; M_f M_\gamma M_f') Y^{M_\gamma *}_\gamma(\theta_\gamma, \phi_\gamma) \tag{16.9}$$

In the last step, (ϕ,θ) has been written as $(\phi_\gamma,\theta_\gamma)$ as they refer to gamma direction. A direct substitution of $(\rho^{\mu.c})_{M_f M_f'}$

from eq .(15.9) into (16.9) will give the desired result.

A typical term in $\left(\rho^{\mu \cdot c}\right)_{M_f M_f'}$ will be taken for illustration. The G_V^2 term in (15.9) will give

$$|a(\tau)|^2 \sum_{M_f M_f'}' G_V^2 \, \mathcal{J}(J_f 0 J_f ; J_f 0 J_f) \sum_J' (-)^{M_f} \frac{[J_f]^2}{\sqrt{4\pi}\,[J]} Y_J^{M_\nu}(\hat{\nu})$$

$$C(J_f J_f J ; -M_f M_f' M_\nu)\, C(J_f J_f J ; 000) \sum_{\gamma=0}^{2L}' (-1)^p (-1)^{J_f - J_F}$$

$$C(L L \gamma ; p\, -p\, 0) \frac{\sqrt{4\pi}\,[J_F]^2}{[J_f]} |\langle J_F \| L(\tau) \| J_f \rangle|^2 (-1)^{J_f - M_f}$$

$$W(J_f L J_f L ; J_F \gamma)\, C(J_f J_f \gamma ; M_f\, -M_f'\, -M_\gamma)\, Y_\gamma^{M_\gamma}(\hat{\gamma})^*$$

The two C.G. coefficients when summed over M_f give $\delta_{\gamma J}\,\delta_{M_\gamma, M_\nu}$ and then the two spherical harmonics when summed over M_ν give $P_J(\cos\theta_{\gamma\nu})$. So

$$G_V^2 \, \mathcal{J}(J_f 0 J_f ; J_f 0 J_f) \sum_{J=0}^{2J_f}' (-1)^p (-1)^{J - J_F} \frac{[J_f]^2 \, [J_F]^2}{4\pi}$$

$$|\langle J_F \| L(\tau) \| J_f \rangle|^2 \, C(J_f J_f J ; 000)\, C(L L J ; p\, -p\, 0)$$

$$W(J_f L J_f L ; J_F J)\; P_J(\cos\theta_{\gamma\nu})$$

As the circular polarization is not observed, we can sum over p. In a similar way, the remaining terms in $\left(\rho^{\mu \cdot c}\right)_{M_f M_f'}$ can be considered and the final result is

$$
\begin{aligned}
W_L(\theta_{\gamma\nu}) = {} & -|a(\tau)|^2 \, |\langle J_F \| L(\tau) \| J_f \rangle|^2 \Big[G_V^2 \, J(J_f\, 0\, J_f ; J_f\, 0\, J_f) \\
& \frac{[J_f]^2 [J_F]}{4\pi} \sum_J{}' (-1)^{J-J_F} \, C(J_f J_f J ; 000) \, C(L L J ; 1 -1\, 0) \\
& W(J_f L J_f L ; J_F J) \{1 + (-1)^J\} P_J(\cos\theta_{\gamma\nu}) + G_A^2 \sum_{\ell\ell'}{}' \\
& (i)^{\ell'-\ell} (-1)^{\ell'-J_f} [J_f]^2 \, J(\ell\, 1\, J_f ; \ell'\, 1\, J_f) \, \frac{[\ell][\ell'][J_F]^2}{4\pi} \\
& \sum_J{}' (-1)^{J-J_F} C(\ell \ell' J ; 000) \, C(L L J ; 1 -1\, 0) \, W(J_f\, 1\, J\, \ell' ; \ell\, J_f) \\
& W(J_f L J_f L ; J_F J) \{1 + (-1)^J\} P_J(\cos\theta_{\gamma\nu}) + (G_P^2 - 2 G_P G_A) \\
& \sum_{\ell\ell'}{}' (i)^{\ell'-\ell} \, C(1\, \ell\, J_f ; 000) \, C(1\, \ell'\, J_f ; 000) \, J(\ell\, 1\, J_f ; \ell'\, 1\, J_f) \\
& \frac{[\ell][\ell'][J_F]^2}{4\pi} \sum_J{}' (-1)^{J-J_F} \, C(J_f J_f J ; 000) \\
& C(L L J ; 1 -1\, 0) \, W(J_f L J_f L ; J_F J) \{1 + (-1)^J\} P_J(\cos\theta_{\gamma\nu}) \Big]
\end{aligned}
\tag{16.10}
$$

This gives the Intensity of gamma radiation as a function of angle between gamma direction and neutrino direction. For a complete expression taking into account the nucleon-velocity dependent terms, see R.Parthasarathy and V.N.Sridhar, Phys. Rev. C. July issue 1978. However one can get some idea of the angular correlation by considering only the S-wave neutrino. This is known as Fujii-Primakoff approximation. So by setting

$\ell = \ell' = 0$ in (16.10) and expanding the sum over J, the J=0 part will give the angular independent intensity, J=1 part will not contribute while J=2 part will give the angular dependent intensity. So

$$W(\theta_{\gamma\nu}) = W(0; J=0) + W(J=2)\, P_2(\cos\theta_{\gamma\nu})$$

$$= W(0; J=0)\left\{1 + \alpha\, P_2(\cos\theta_{\gamma\nu})\right\} \tag{17.1}$$

where $$\alpha = \frac{W(J=2)}{W(J=0)} \tag{17.2}$$

In Fujii-Primakoff approximation, the nuclear matrix elements get cancelled in (17.2) and

$$\alpha = \frac{-\frac{1}{3}\left(G_P^2 - 2G_P G_A\right)}{G_A^2 + \frac{1}{3}\left(G_P^2 - 2G_P G_A\right)} \tag{17.3}$$

Under the same approximation, the longitudinal polarization is given by (L.Wolfenstein, Nuo. Cimento. 13 (1959) 319)

$$P_L = \frac{-2G_A^2}{3G_A^2 + G_P^2 - 2G_P G_A} \tag{17.4}$$

From (17.3) and (17.4)

$$-\alpha = 1 + \frac{3}{2} P_L \tag{17.5}$$

This relation hold good when the nucleon velocity dependent terms are taken into account and when all partial waves of the neutrino are considered. Thus the relation (17.5) is rigorous and independent of the details of nuclear structure and muon capture coupling constants. Bernabeu (loc. cit) has shown that the limits for P_L admitted by Time-reversal invariance are 0 and -1. Eqn.(17.5) then gives

$$-1 \leq \alpha \leq 0.5 \tag{17.6}$$

and any deviation of α from this limit is an indication of Time-reversal symmetry violation. The experiment of Miller (loc. cit) gives

$$\alpha = (0.15 \pm 0.25)$$

The complete expression for α including nucleon-momentum dependent terms and higher partial waves for neutrino has been evaluated by Parthasarathy and Sridhar (ICTP-preprint IC/78/15) by using the particle-hole model of Donnelly and Walker (Ann. Phys. 60(1970) 209) wherein the residual nucleon-nucleon interaction is taken to be of Serber-Yukawa type. This model is able to describe the inelastic electron scattering by ^{28}Si very well. When compared with experimental result of α, we find

$$2 g_A < g_P < 15 g_A$$

We have undertaken the study of gamma-neutrino angular correlation for a polarized muon capture. This would give $P_1(\cos\theta)$ term as well.