MATSCIENCE REPORT 94

ON A PROBLEM OF GARRETT BIRKHOFF AND RELATED TOPICS

By

V. V. RAMA RAO
(Matscience, Madras)

THE INSTITUTE OF MATHEMATICAL SCIENCES
MADRAS-600020, (INDIA)
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CONTENTS

INTRODUCTION

This report forms part of the lectures given by the author on his own research work during his stay at MATSCIENCE as a Senior Research Fellow.

Chapter 1,deals with the characterisation of the direct product of a Boolean ring and a lattice ordered group, thereby solving Birkhoff's problem no.105 "Lattice Theory" A.M.S. Col. Pub. XXV (1948) ("Is there a common abstraction which includes of Boolean rings (algebras) and lattice ordered groups as special cases?). These results which are also a generalisation of the author's paper "On a common abstraction of Boolean rings and lattice ordered groups I" Manatsheftе fur Mathematik 73 411-421 (1969), have been accepted for publication in Math. Slovaca.

In Chapter 2, "Dually residuated lattice ordered semigroups" or briefly D.R.L. semigroups have been generalised to semi-dually residuated lattice ordered semigroups ti include semi-Brouwerian algebras. Semi D.R.L. semigroups have many interesting properties of D.R.L. semigroups and these have been studied in detail. The relationship between a semi D.R.L. semigroup and a Boolean-1-algebra has been discussed with some interesting results. The results of this Chapter have appeared as a note in Math. Seminar notes Vol.6 (1978).

Chapter 3 is devoted to the study of the structure of a D.R.L. semigroup and a class of D.R.L. semigroups which can be obtained as a global sections with compact Casmiss of a sheaf of nontrivial totally ordered D.R.L. semigroups over a Boolean space, has been characterised by means of two conditions. These results have been communicated to Math. Seminar Notes and their acceptance for publication is awaited.

In the last chapter an attempt has been made to obtain the relationship between normal and distributive $*$ lattices.

The author has great pleasure in acknowledging the stimulating inspiration given to him by the Director, Professor Alladi Ramakrishnan.

15.9.1978 V.V.RAMA RAO

CHAPTER 1

ON A PROBLEM OF GARRETT BIRKHOFF

Introduction

In this chapter we shall solve Birkhoff's problem No.105 (Is there a common abstraction which includes Boolean rings (algebra) and lattice ordered groups as special cases?) by characterising the direct product B x G of a Boolean ring and a lattice ordered group G by a set of five conditions 1,2,3,4 and 5'. In [2] this author has obtained a similar characterisation by a set of conditions 1,2,3,4 and 5. While conditions 1 through 4 are identical, condition 5' is a much weaker than condition 5. Thus the present results may be treated as a generalization of the author's work in [2]. It may also be noted the proofs in [2] are greatly simplified and made much more elegant. here.

We specifically prove the following

Main Theorem: Let $A = (A; \cup, \cap, +, -)$ be an algebra of species (2,2,2,1). Then a necessary and sufficient condition that A is isomorphic to the direct product B x G of a Boolean ring B and a lattice ordered group G is

1) $(A; +)$ is a group

2) $(A; \cup, \cap)$ is a lattice

3) $a + x \cup y + b = (a+x+b) \cup (a+y+b)$
$a + x \cap y + b = (a+x+b) \cap (a+y+b)$ if $(a+b) \cap x = (a+b) \cap y$.

4) $a - (x \cup y) = b = (a-x+b) \cap (a-y+b)$
$a - (x \cap y) = b = (a-x+b) \cup (a-y+b)$ if $(a+b) \cup x = (a+b) \cup y$.

5) $a \cup 0 - a \cap 0 = a \cup -a$. for all $a, b, x, y \in A$.

Section 1: Proof of the Main Theorem is gradually developed in the following two sections. Throughout this Section A stands for an algebra of species (2,2,2,1) satisfying 1 through 4 and in the following a, b, x, y...etc denote the elements of A.

LEMMA 1.1: If $a \geq b$ then $a-b \geq 0$ and $-b+a \geq 0$

Proof: If $a \geq b$, then $a-b = a-a\cap b = (a-a)\cup(a-b)$ (by 4) $= 0\cup(a-b)$. Hence $a-b \geq 0$. Similarly $-b+a = -(a\cap b)+a = (-a+a)\cup(-b+a)$ (by 4) $= 0\cup(-b+a)$. So $-b+a \geq 0$. Q.E.D

COROLLARY 1.1: If $a \leq 0$, then $-a \geq 0$.

Proof: By Lemma 1.1, $-a = 0-a \geq 0$. Q.E.D.

LEMMA 1.2: If $a\cap b = 0$, then $a+b = a\cup b$.

Proof: If $a\cap b = 0$, then $a \geq 0$, $b \geq 0$ and $a+b = a+b\cup 0 = (a+b)\cup a$ (by 3). Similarly $a+b = (a+b)\cup b$. Therefore $a+b = a-a\cap b+b = (a-a+b)\cup(a-b+b)$ (by 4) $= b\cup a = a\cup b$. Q.E.D.

LEMMA 1.3: $a\cup b-b = a-a\cap b$.

Proof: $0 = a\cup b-(a\cup b) = (a\cup b-a)\cap(a\cup b-b)$ (by 4). So by Lemma 1.2, $(a\cup b-a)+(a\cup b-b) = (a\cup b-a)\cup(a\cup b-b) = a\cup b-a\cap b$ (by 4). Hence $a\cup b-b = a-a\cap b$. Q.E.D.

COROLLARY 1.2: If $a\cup b = 0$ then $a+b = a\cap b$.

COROLLARY 1.3: If + is commutative then $a+b = a\cup b+a\cap b$

COROLLARY 1.4: $(a-a\cap b)\cap(b-a\cap b) = 0$.

COROLLARY 1.5: $a = a\cup o+a\cap o = a\cap o+a\cup o$.

LEMMA 1.4: A is a distributive lattice.

Proof: Let $a\cup b = a\cup c$ and $a\cap b = a\cap c$. Then by Lemma 1.3, $a\cup b-b = a-a\cap b = a-a\cap c = a\cup c-c = a\cup b-c$, or $b = c$. Therefore A is a distributive lattice. Q.E.D.

Now we prove,

THEOREM 1.1: If every nonzero element of A has additive order two then $(A;+,\cap)$ is a Boolean ring.

Proof: It is clear that $(A;+)$ is a commutative group since every nonzero element has order two; and $(A;\cap)$ is a semigroup and every element of A is (multiplicatively) idempotent. In order to complete the proof, we must now show that $\cap$ distributes with respect to +; and this requires some preparation.

We now assume that every nonzero element of A has order two.

LEMMA 1.5: 0 is the least element of A.

Proof: $0 = -(a\cap o)\cap(a\cup o)$ (by Corollary 1.4) $= (a\cap o)\cap(a\cup o)=a\cap o$. Therefore $a \geq o$ for all $a \in A$. Q.E.D.

LEMMA 1.6: $a+b \leq a\cup b$.

Proof: $b+a\cap b = b-a\cap b = a\cup b-a = a\cup b-(a\cup o) = (a\cup b-a)\cap(a\cup b)$ (by) $\leq a\cup b$. By symmetry, $a+a\cap b \leq a\cup b$. Now $a+b = a\cup b+a\cap b = a\cup b-a\cap b = (a\cup b-a)\cup(a\cup b-b)$ (by 4) $= (b-a\cap b)\cup(a-a\cap b) = (b+a\cap b)\cup(a+a\cap b) \leq a\cup b$. Q.E.D.

LEMMA 1.7: $a\cup b = (a+b)\cup(a\cap b)$ and $a\cap b\cap(a+b) = 0$

Proof: $a\cup b = a+b+a\cap b \leq (a+b)\cup(a\cap b)$ (by Lemma 1.6) $\leq (a\cup b)\cup(a\cap b) = a\cup b$. So that $a\cup b = (a+b)\cup(a\cap b)$. Now $a\cup b = (a\cap b)\cup(a+b) = a\cap b+a+b+a\cap b\cap(a+b) = a\cup b+a\cap b\cap(a+b)$. Therefore $a\cap b\cap(a+b) = 0$. Q.E.D.

LEMMA 1.8: $a\cap b+a\cap c = a\cap(b+c)$

Proof: $a\cap b+a\cap c+a\cap b\cap c = a\cap b+a\cap c+(a\cap b)\cap(a\cap c) = (a\cap b)\cup(a\cap c) = a\cap(b\cup c) = a\cap[(b+c)\cup(b\cap c)] = (a\cap(b+c))\cup(a\cap b\cap c) = a\cap(b+c)+a\cap b\cap c+a\cap(b+c)\cap b\cap c = a\cap(b+c)+a\cap b\cap c$ $\therefore$ $a\cap b+a\cap c = a\cap(b+c)$. Q.E.D.

LEMMA 1.8 completes the proof of Theorem 1.

The following two theorems seem to indicate that 5 of [2] is perhaps not independent of 1 through 4. However, the author has not succeeded in establishing 5 completely.

THEOREM 1.2: The following statements are equivalent in A.

i) $(a-b)\cup(b-a) = a\cup b-(a\cap b)$

ii) If $a\cap b = 0$ then $a+b = (a-b)\cup(b-a)$

Proof: That (i) implies (ii) is obvious by Lemma 1.2. We prove that (ii) implies (i). By Corollary 1.4, $(a-a\cap b)\cap(b-a\cap b)=0$. Hence $a\cup b-a\cap b = (a\cup b-a)\cup(a\cup b-b)$ (by 4) $= (a-a\cap b)+(b-a\cap b) =$ $(a-a\cap b+a\cap b-b)\cup(b-a\cap b+a\cap b-a) = (a-b)\cup(b-a)$. Q.E.D.

THEOREM 1.3: $(a-b)\cup(b-a) = a\cup b-(a\cap b)$ in A if

i) $a\cup o-(a\cap o) = a\cup -a$.

ii) if $a\cap b = o$ then $a\cap(b+b) = 0$.

the proof of this theorem requires a Lemma and we now assume $a\cup o-a\cap o = a\cup -a$ for all a.

LEMMA 1.9: If $a\geq o$ then $a\geq -a$, $-2a\leq o$ and $2a\geq o$

Proof: If $a\geq 0$, then $a = a\cup 0-a\cap 0 = a\cup -a$ by (i), or $a\geq -a$. Also $a = (0\cup -a)-0\cap(-a)$. Therefore $-2a = -a-a = -((0\cup -a)-(0\cap -a))+$ $0\cup -a+0\cap -a = 0\cap -a-(0\cup -a)+0\cup -a+0\cap -a = 0\cap -a+0\cap -a = 0\cap(-a)\cap(0\cap -a-a)\leq 0$ By Corollary 1.1, $2a\geq 0$. Q.E.D.

Proof of Theorem 1.3: Let $a \cap b = 0$, then $a \cup b \geq a \cup -b = (a+2b-2b) \cup (-b) = (a \cup 2b-2b) \cup (-b)$ (since $a \cap (a+b) = 0$ and by Lemma 1.2) $= (a-2b) \cup (2b-2b) \cup (-b)$ (by 3) $= (a-2b) \cup 0 \cup -b \geq (a-2b) \cup -b = (a \cup b-2b)$ (by 3) $= a+b-2b = a-b$. On the same lines, we have $a \cup b \geq b-a$. Therefore $a \cup b \geq (a-b) \cup (b-a)$. Also $(a-b) \cup 0 = (a+b-2b) \cup 0 = (a \cup b-2b) \cup 0$ (by Lemma 1.2). $(a-2b) \cup (-b) \cup 0$ (by 3) $\geq (a-2b) \cup 0 = -2b$ (by 3) $= a+2b-2b = a$. Similarly $(b-a) \cup 0 \geq b$. So that $(a-b) \cup (b-a) = (a-b) \cup (b-a) \cup 0 \geq a \cup b$. Hence $a \cup b = (a-b) \cup (b-a)$. By Theorem 1.2 it now follows that $a \cup b - a \cap b = (a-b) \cup (b-a)$ always. Q.E.D.

Section 2

From now on wards, we assume that A satisfies the additional condition $5'$ also. We first prove

COROLLARY 1.6: *If a is of order two, then $a \geq 0$.*

Proof: If a is of order two, then $a = -a$ and $a \cup -a = a \geq 0$. (by 5'). the following lemma gives a complete characterisation of the elements of order two.

LEMMA 1.10: *An element a A is of order two, iff* $a = a \cup -a+0 \cap -a$

Proof: If a is of order two, than $a = -a$ and $a = a \cup 0+a \cap 0 = a \cup 0+0 \cap -a$

Now $a \cup 0+0 \cap -a+0 \cup -a+a \cap 0 = a \cup 0-a+a \cap 0 = -(a \cap 0)+a \cap 0 = 0$. Hence $a \cup 0+0 \cap -a = -(0 \cup -a+a \cap 0)$. Also $(0 \cap a) \cup (0 \cap -a) = 0 \cap (a \cup -a) = 0$ (by 5') so that by Corollary 1.2 $0 \cap a+0 \cap -a = 0 \cap -a+0 \cap a$.

Also $a \cup 0-a \cap 0 = a \cup -a = 0 \cup -a - (0 \cap -a)$ or $a \cup 0 = 0 \cup -a - (0 \cap -a)+0 \cap a = 0 \cup -a+0 \cap a-(0 \cap -a)$, so that $a \cup 0+0 \cap -a = 0 \cup -a+0 \cap a = -(0 \cup -a+a \cap 0)$

So $a \cup 0+0 \cap -a$ is of order two Q.E.D.

LEMMA 1.11: If $a \geq 0$ and $b \geq 0$ then $a+b \geq 0$

Proof: $a+b = a+(a \cap b) \cup b = (a+a \cap b) \cup (a+b)$ (by 3). Hence $a+b \geq a+a \cap b$ and $a \geq a \cap b \geq -(a \cap b)$ since $a \cap b \geq 0$ (by Lemma 1.9). Therefore $a+a \cap b \geq 0$ and hence $a+b \geq 0$. Q.E.D.

Definition 1.1: For any $a \in A$, let $a_B = a \cup 0 + 0 \cap -a$. Then a_B is called the boolean component of a and the set A_B of all a_B for every $a \in A$ and including 0 is called the Boolean component of A.

We justify the naming of A_B as the Boolean component by the following

THEOREM 1.4: $(A_B; +, \cap)$ is a Boolean ring

Proof: Let $a, b \in A_B$, then $a \geq 0$, $b \geq 0$ by Corollary 1.6 and by Lemma 1.11 $a+b \geq 0$. Now $a+b \geq -(a+b) = b+a$ by Lemma 1.9. Similarly $b+a \geq a+b$. Hence $a+b = b+a$.

Now $0 = a+a+b+b = a+b+a+b$ so that $a+b \in A_B$. Therefore $(A_B; +)$ is a commutative group.

Now $a \cup b+b = a \cup b-b = a-a \cap b$ so that $-(a \cap b) = a+a-a \cap b =$ 0 $a+a \cup b+b \geq 0$ so that $-(a \cap b) \geq a \cap b$. Since $a \cap b \geq 0$ $a \cap b \geq -(a \cap b)$. Hence $a \cap b = -(a \cap b)$. Similarly $a \cup b = -(a \cup b)$. Then $(A_B; +, \cap)$ is a lattice. By theorem 1.1 it now follows but $(A_B, +, \cap)$ is a Boolean ring. Q.E.D.

We will prove a very important

LEMMA 1.12: If a is of order two and $x \leq 0$, then $a \cap (-x) = 0$ and $a+x = x+a$.

Proof: $x+a \cup -x \geq 0$ (by Lemma 1.1) and so $-(a \cap -x) = x+a \cup -x+a \geq 0$. Hence $-(a \cap -x) \geq a \cap -x$ (by 5'). Also $a \cap -x \geq 0$ so that $a \cap -x \geq -(a \cap -x)$ (by 5'). Thus $-(a \cap -x) = a \cap -x$ or $a \cap -x$ is of order two.

Now $(a+x) \cap 0 = a \cap -x+x$ (by 3) so that $(a+x) \cap 0 -x = a \cap -x \geq 0$. Adding x both sides. (by 3) we have $(a+x) \cap 0 = (a+x) \cap 0 -x+x \geq x$. Similarly adding $(a+x) \cap 0$ to $x-(a+x) \cap 0 = -(a -x) \geq 0$, we have $x = x-(a+x) \cap 0 +(a+x) \cap 0 \geq (a+x) \cap 0$. So that $x-(a+x) \cap 0 = 0$ or $x = (a+x) \cap 0$. Hence $a \cap -x = 0$. Therefore $a -x = a \cup -x = -x+a$, so that $a+x = x+a$. Q.E.D.

COROLLARY 1.7: For any a and b, $a_B + b_L = b_L + a_B$ where where $b = -(0 \cap -b) +0 \cap b$.

Proof: Proof of this Corollary follows from Lemma 1.12 and Corollary 1.2.

Definition 1.2: For any $a \in A$, $a_L = -(0 \cap -a) +a \cap 0$ is called the L-group component of a. The set of all a_L for every $a \in A$, to be denoted by A_L, is called the L-group component of A.

LEMMA 1.13: For any a and b A, $a \geq b$ iff $a-b \geq 0$.

Before we prove this lemma, we make the following

Remark 1.1: If $a \geq 0$ and $x \leq 0$ then $a-x \geq -x$ and $a+x \geq b+x$ for $b \geq 0$, if $a \geq b$.

Proof: by (3) we have $(a-x) \cup -x+x = a \cup 0 = a$ or $(a-x) \cup -x = a-x$. Hence $a-x \geq -x$.

If $a \geq b$ then $a+x = a \cup b+x = (a+x) \cup (b+x)$ or $a+x \geq b+x$. Q.E.D.

Proof of Lemma 1.13: If $a \geq b_L$ then by Lemma 1.1 we have $a-b \geq 0$. Conversely, let $a-b_L \geq 0$. Then $a-(0\cap b) = a-(0\cap b)+0\cap b-(0\cap -b) = a-b_L -(0\cap -b) \geq -(0\cap -b)$ by Remark 1.1. Now $a = a-(0\cap b) +0\cap b \geq -(0\cap -b)+0\cap b. = b_L$ Q.E.D.

LEMMA 1.14: $a_L+b_L\geq(a+b)_L$ and $(a+b)_L -b_L \geq a_L$ for all $a,b \in A$.

Proof: $(a+b)_B+(a+b)_L = a+b = a_B+a_L+b_B+b_L = a_B+b_B+a_L+b_L$ by Corollary 1.7. Hence $0\leq b_B+a_B+(a+b)_B= a_L+b_L-(a+b)_L$ so that by Lemma 1.13 we have $a_L+b_L\geq(a+b)_L$. Also, $0\leq b_B+a_B+(a+b)_B = -(b_B+a_B+(a+b)_B) = -(a_L+b_L-(a+b)_L) = (a+b)_L -b_L-a_L$ so that by Lemma 1.13, we have $(a+b)_L-b_L\geq a_L$.

LEMMA 1.15: $a \geq 0$ iff $a_L\geq 0$ and if $a_L\geq 0$ then $-a_L \leq 0$.

Proof: If $a\geq 0$, then $a_B +a_L= a \geq 0$ and $a_B\geq 0$ so that $a_L = a_B+a_B+a_L\geq 0$. If $a_L\geq 0$ then $a = a_B+a_L\geq 0$, by Lemma 1.11.

If $a_L\geq 0$ then $a \geq 0$ and $-a_L = 0\cap -a - 0\cap a = 0\cap -a \leq 0$. Q.E.D.

LEMMA 1.16: If $a_L\geq 0$, $b_L \geq 0$, then $(a+b)_L= a_L+b_L$, $a_L-b_L = (a-b)_L$ and $-b_L+a_L= (-b+a)_L$

Proof: Let $a_L\geq 0$ and $b_L\geq 0$. Then $-b_L\leq 0$ (by Lemma 1.15 and $a_L+b_L\geq(a+b)_L\geq 0$ by Lemma 1.14. Hence $a_L= a_L+b_L-b_L= (a_L+b_L)\cup(a+b)_L -b_L = (a_L+b_L-b_L)\cup((a+b)_L-b_L) = a_L\cup((a+b)_L-b_L)$. so that $a_L \geq(a+b)_L-b_L$. By Lemma 1.14, $(a+b)_L-b_L\geq a_L$, so that $(a+b)_L-b_L= a_L$ or $(a+b)_L = a_L+b_L$.

Also by Lemma 1.14, $a_L - b_L \geq (a-b)_L$ and $(a-b)_L + b_L \geq a_L \geq 0$. Therefore $(a-b)_L = (a-b)_L + b_L - b_L = (((a-b)_L + b_L) \cup a_L) - b_L =$ $((a-b)_L + b_L - b_L) \cup (a_L - b_L)$ (by 3) $= (a-b_L) \cup (a_L - b_L)$. Thus $(a-b)_L \geq$ $a_L - b_L$ and hence $(a-b)_L = a_L - b_L$. Once again using Lemma 1.14, we have $-b_L + a_L \geq (-b+a)_L$ and $(-b+a)_L - a_L \geq -b_L$.
Therefore $b_L + (-b+a)_L - a_L \geq 0$ and by Lemma 1.13 $b_L + (-b+a)_L \cup a_L \geq 0$ and $(-b+a)_L = -b_L + b_L + (-b+a)_L = -b_L + (b_L + (-b+a)_L) \cup a_L) =$ $\{-b_L + b_L + (-b+a)_L\} \cup \{-b_L + a_L\}$ (by (3)) $= (-b+a)_L \cup (-b_L + a_L)$.
Therefore $(-b+a)_L \geq -b_L + a_L$. Hence $(-b+a)_L = -b_L + a_L$. Q.E.D.

<u>COROLLARY 1.8</u> $a_L = (a \cup 0)_L + (a \cap 0)_L$

<u>Proof:</u> We have by Lemma 1.14, $(a \cup 0)_L + (a \cap 0)_L \geq (a \cup 0 + a \cap 0)_L = a_L$
Also by Lemma 1.14, $a_L - (a \cap 0)_L = (a \cup 0 + a \cap 0)_L - (a \cap 0)_L \geq (a \cup 0)_L \geq 0$.
Hency by (3)

$a_L = a_L - (a \cap 0)_L + (a \cap 0)_L = (a_L - (a \cap 0)_L) \cup (a \cup 0)_L + (a \cap 0)_L = (a_L - (a \cap 0)_L + (a \cap 0)_L)$ $\cup ((a \cup 0)_L + (a \cap 0)_L)$. Therefore $a_L \geq (a \cup 0)_L + (a \cap 0)_L$ and so $a_L = (a \cup 0)_L + (a \cap 0)_L$.

<u>LEMMA 1.17: $(a+b)_L = a_L + b_L$ and $(a+b)_B = a_B + b_B$</u>

<u>Proof:</u> By Corollary 1.8 we have $a_L = (a \cup 0)_L - (-(a \cap 0)_L)$.
Now $a_L + b_L = (a \cup 0)_L - (-(a \cap 0)_L) + (b \cup 0)_L - (-(b \cap 0)_L) = (a \cup 0)_L - (-(a \cap 0)_L)$ $+ (b \cup 0)_L + (-(a \cap 0)_L) - \{-(b \cap 0)_L + (-(a \cap 0)_L)\} = (a \cup 0)_L - (-(a \cap 0)_L) +$ $(b \cup 0 - (a \cap 0))_L - (-(b \cap 0)_L + -(a \cap 0)_L)$ (by Lemma 1.16) $= (a \cup 0)_L +$ $+ (-(-(a \cap 0) + b \cup 0 - (a \cap 0)))_L - ((-b \cap 0 - a \cap 0)_L)$ (by Lemma 1.16) $=$ $= (a \cup 0 - (-(a \cap 0)) + b \cup 0 - a \cap 0)_L - (-b \cap 0 - a \cap 0)_L = a \cup 0 - (-a \cap 0) + b \cup 0 - a \cap 0$ $+ a \cap 0 + b \cap 0)_L = (a \cup 0 + a \cap 0 + b \cup 0 + b \cap 0)_L = (a+b)_L$.

Now $(a+b)_B + (a+b)_L = a+b = a_B+a_L+b_B+b_L = a_B+b_B+a_L+b_L =$
$a_B+b_B+(a+b)_L$. Q.E.D.
Therefore $(a+b)_B = a_B + b_B$.

LEMMA 1.18: $(A_L\ +)$ is a subgroup of $(A\ +)$

Proof: If $a_L, b_L \in A_L$ then $a_L - b_L = a_L + (-b)_L = (a-b)_L$.
Hence $(A_L\ +)$ is a subgroup $(A, +)$.

LEMMA 1.19: $a_L \geq b_L$ iff $a_L+c_L \geq b_L+c_L$ and $c_L+a_L \geq c_L+b_L$ for all $c_L \in A_L$.

Proof: Let $a_L \geq b_L$ then $a_L - b_L \geq 0$, $-b_L+a_L \geq 0$ by Lemma 1.1 and $a_L+c_L-(b_L+c_L) = (a+c)_L-(b+c)_L$ (by Lemma 1.17) $= (a+c-(b+c))_L =$ $= (a-b)_L = a_L-b_L \geq 0$. Therefore $a_L+c_L = (a+c)_L \geq (b+c)_L = b_L+c_L$ by Lemma 1.13, the other inequality follows with the help of the following

Remark 1.2: $a \geq b_L$ iff $-b_L+a \geq 0$ for any $a, b \in A$

LEMMA 1.20: $(A_L; +, \cup, \cap)$ is a lattice ordered group.

Proof:- $(A_L; +)$ is a group by Lemma 1.18 and by Lemma 1.19 it is a partially ordered group. It is easy to see that $a \cap 0 \in A_L$ for every $a \in A$. Now if $a \in A_L$ then $a \cup 0 = a - a \cap 0 \in A_L$. Hence if $a, b \in A_L$ then $a \cup b = (a-b) \cup 0 + b \in A_L$. Similarly $a \cap b \in A_L$ and so $(A_L; \cup, \cap)$ is a lattice. Therefore $(A_L; +, \cup \cap)$ is a lattice ordered group. Q.E.D.

LEMMA 1.21: $(a \cup b)_L = a_L \cup b_L$ and $(a \cap b)_L = a_L \cap b_L$

Proof: Let $a \cap b = 0$ then $a \geq 0$, $b \geq 0$ and $0 = a \cap b \geq -a \cap -b$.

Now $a_{\varepsilon} \cap b_{\varepsilon} = (0 \cup -a + a \cap 0) \cap (0 \cup -b + b \cap 0) = (0 \cup -a) \cap (0 \cup -b) = 0 \cup (-a \cap -b) = 0$. Also observe that $a_B \cap a_L = 0 = a_B \cap b_L = b_B \cap b_L = b_B \cap a_L$ by Lemma 1.12.

So $0 = a \cap b = (a_B + a_L) \cap (b_B + b_L) = (a_B \vee a_L) \cap (b_B \cup b_L) = (a_B \cap b_B) \cup (a_L \cap b_B) \cup (a_L \cap b_B) \cup (a_L \cap b_L) = 0 \cup 0 \cup 0 \cup (a_L \cap b_L)$.
Therefore $a_L\ b_L \leq 0$. Also $a_L \cap b_L \geq 0$ so that $a_L \cap b_L = 0$.

Now considering in general $a_L \cup b_L - (a \cap b)_L = (a_L - (a \cap b)_L) \cup (b_L - (a \cap b)_L = (a - a \cap b)_L \cup (b - a \cap b)_L = (a - a \cap b)_L + (b - a \cap b)_L$ (by above, since $(a - a \cap b) \cap (b - a \cap b)$ is always 0)) $= (a - a\, b + b - a \cap b)_L = (a \cup b - a \cap b)_L = (a \cup b)_L$. Thus $a_L \cup b_L = (a \cup b)_L$.

Also $(a \cap b)_L = (a - (a \cup b) + b)_L = a_L - (a \cup b)_L + b_L = a_L - (a_L \cup b_L) + b_L = a_L \cap b_L$.

Q.E.D.

LEMMA 1.22: $(a \cup b)_B = a_B \cup b_B$ and $(a \cap b)_B = a_B \cap b_B$.

Proof: Let $a \geq 0$ and $b \geq 0$, then by Lemma 1.12,
$a_B \cap a_L = 0 = b_B \cap b_L = a_B \cap b_L = b_B \cap a_L$. Therefore $(a \cup b)_B + a_L \cup b_L = (a \cup b)_B + (a \cup b)_L = a \cup b = (a_B + a_L) \cup (b_B + b_L) = (a_B \cup a_L) \cup (b_B \cup b_L) = a_B \cup b_B \cup (a_L \cup b_L) = a_B \cup b_B + a_L \cup b_L$ since $(a_B \cup b_B) \cap (a_L \cup b_L) = 0$.
Thus $(a \cup b)_B = a_B \cup b_B$. Similarly we can show that $(a \cap b)_B = a_B \cap b_B$

Q.E.D.

Proof of the Main Theorem: By Theorem 1.3 $(A_B; +\cap)$ is a Boolean ring and by Lemma 1.15 $(A, +, \cup, \cap)$ is a lattice ordered group. Also the mapping $A \to A_B \times A_L$ defined by $a \to (a_B, a_L)$ is an isomorphism, since Lemmas 1.17, 1.21 and 1.22 assert that the operations $+, \cup, \cap$ are component-wise. Therefore A is isomorphic to $A_B \times A_L$.

Conversely if B is any Boolean ring and L any lattice ordered group then B x L is both a group and lattice. We shall show that 3 and 4 are valid in any Boolean ring.

Let $(a+b)\cap x = (a+b)\cap y$.

Then $(a+x+b)\cap(a+y+b) = (a+x+b)\cap a+(a+x+b)\cap y+(a+x+b)\cap b = a+x\cap a+b\cap a+a\cap y+x\cap y+b\cap y+a\cap b+x\cap b+b = a+(a+b)\cap x+(a+b)\cap y+x\cap y+b = a+x\cap y+b$.

On the same lines it also follows that $(a+x+b)\cup(a+y+b)=a+x\cup y+b$.

Now let $a+b+x+(a+b)\cap x = (a+b)\cup x = (a+b)\cup y = a+b+y+(a+b)\cap y$.

Then $(a+x+b)\cap(a+y+b) = a+b+(a+b)\cap x+(a+b)\cap y+x\cap y = x+a+b+y+(a+b)\cap y+(a+b)\cap y+x\cap y = a+x+y+x\cap y+b = a+x\cup y+b = a-(x\cup y)+b$.

Similarly $a-(x\cap y)+b = (a-x+b)\cup(a-y+b)$.

Therefore 3 and 4 are satisfied in BxL. Also in B $a\cup 0-a\cap 0 = a\cup 0+a\cap 0 = a = a\cup -a$. So that 5' is also true.

Hence BxL satisfies 1 through 5'.

Therefore the proof of the Main Theorem is complete. Q.E.D.

The following two theorems seem to have an interest of their own, though they follow from the Main Theorem.

<u>Theorem 1.5: If there is a least element in A, then A is a Boolean ring.</u>

<u>THEOREM 1.6: The following statements are equivalent in A.</u>

i) $(A;+,\cap)$ is a Boolean ring.

ii) $a-(x\cup y) = (a-x)\cap(a-y)$ iff $a\cup x = a\cup y$ for all $a,x,y\in A$.

iii) $a-(x\cap y) = (a-x)\cup(a-y)$ iff $a\cup x = a\cup y$ for all $a,x,y\in A$.

References

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CHAPTER 2

SEMI-DUALLY RESIDUATED LATTICE ORDERED SEMIGROUP

The concept of a D.R.L. Semigroup (Dually Residuated lattice ordered semigroup: see [4,5,6]) goes back to K.L.N.Swamy, who introduced it and obtained it as a common abstraction of Boolean rings and lattice ordered groups, thereby solving Birkhoff's problem No.105 [1] : Is there a common abstraction which includes Boolean algebras (Rings) and lattice ordered groups as special cases. D.R.L. Semigroups include Brouwerian algebras also.

It is interesting to observe that both in Boolean rings and lattice ordered groups the semilattice operation "$\cap$" is actually not independent of the rest. i.e. if B (B; + ..) is a Boolean ring with "$\cup$" and "$\cap$" as the corresponding lattice operations, then $a \cap b = a \cdot b = a + b + a \cup b$. Similarly if $G = (G; +, \cup, \cap)$ is a lattice ordered group then $a \cap b = a - (a \cup b) + b$. However this is not true in Brouwerian algebras and we call a system $B = (B; \cup, -)$ where (B; $\cup$) is a semilattice (i.e. $a \cup b$ is the least upper bound of a and b) and "$-$" is a binary operation on B such that a-b is the least element satisfying $(a-b) \cup b \geq a$; as a semi Brouwerian algebra. An implicative semilattice (see [2]) is actually the dual of a semi Brouwerian algebra.

The problem now arises as whether it is possible to obtain a common abstraction of Semi-Brouwerian algebras and D.R.l. semigroups.

In this chapter we shall solve the above problem in the affirmative, by proposing a common abstraction which we call a Semi-D.R.L. semigroup and which includes Semi-Brouwerian algebras, D.R.L. semigroups and hence Brouwerian algebras, Boolean algebras, and lattice ordered groups as special cases.

It turns out that the class of Semi-D.R.L. semigroups is fairly wider than Semi-Brouwerian algebras and D.R.L. semigroups, and to this wider class we shall extend almost all the results (algebraic as well as geometric) of semi-Brouwerian algebras and D.R.L. semigroups.

In 1, we define, a semi-D.R.L. semigroup and study the consequences of our definition. We obtain necessary and sufficient conditions for the degeneracy of a Semi-D.R.L. semigroup, into a semi-Brouwerian algebra, a lattice ordered group and their direct product. In 2, we study thd structure of a semi-D.R.L. semigroup and establish a one to one correspondence between the congruence relations and ideals, while 3 is devoted to the study of geometry of a commutative semi-D.R.L. semigroup.

<u>Definition 2.1</u>: A system $A = (A; \cup, +, -)$ is called a Semi-D.R.L. semigroup iff

(1.1) $(A; \cup)$ is a semi lattice i.e. "$\cup$" is a idempotent, commutative and associative binary operation on A (with $a \cup b$ as the least upper bound of a and b) and $(A; +)$ is a semi group with "0" such that $a + x \cup y + b = (a + x + b) \cup (a + y + b)$ for all a, b, x and $y \in A$.

(1.2) Given a and b in A there exists a least "x" such that $x + b \geq a$ and this "x" is denoted by $a - b$. Where " $\geq$ " is the partial ordering induced by " " i.e. $a \leq b$ iff $a \cup b = b$.

(1.3) $(a - b) \cup 0 + b \leq a \cup b$

(1.4) $a - a \geq 0$

(1.5) If $0 - a = 0$ and $x \leq 0$, then $(0 - x) + a + x \geq a$.

We shall illustrate the definition by means of some examples.

Example 2.1: If $G = (G; +, \cup, \cap)$ is a lattice ordered group then $(G; \cup, +, -)$ is a semi-D.R.L. Semigroup.

Example 2.2: If $B = (B; +, .)$ is a Boolean ring then the $(B; \cup, +, -)$ is a Semi-D.R.L. semigroup where $a \cup b = a + b + a.b$ $a - b = a + a.b$.

Example 2.3: If $B = (B; \cup, \cap, -)$ is a Brouwerian algebra then $(B; \cup, -)$ is a Semi-D.R.L. semigroup.

Example 2.4*: If $D = (D; \cup, \cap, +, -)$ is a D.R.L. Semigroup then $(D; \cup, +, -)$ is a Semi-D.R.L. Semigroup.

Example 2.5: If $B = (B; \cup, -)$ is a Semi Brouwerian algebra (i.e. $(B: \cup)$ is a Semi lattice and $a - b$ is a the least element satisfying $((a - b) \cup b \geq a)$ then $(B; \cup, +, -)$ is a Semi-D.R.L. Semi group (where of course $+ = \cup$).

*Examples 2.1, 2.2 and 2.3 are only special cases of example 4 but they have been clearly stated here for the convenience of the reader who is not familiar with D.R.L. Semigroups.

The following is an example of a Semi-D.R.L. Semigroup which is different from both semi Brouwerian algebras and D.R.L. Semigroups.

Example 2.6: Let B be the powerset of a two element set $\{a, b\}$. Let D[1] be the D.R.L. Semigroup of all non negative integers (see [5]). Define $f: B \times D \longrightarrow D$ by $f(0, d) = d$ and $f(c, d) = 0$ if $c \neq 0$. Define θ on $B \times D$ as follows $(a, d)\ \theta\ (b, e)$ iff $a = b$ and $f(a, d) = f(a, e)$. It easily follows that θ is an equivalence relation and define $+$, $\cup$ and $-$ on $(B \times D)\ \theta$ as follows

$[a, d] \cup [b, e] = [a \cup b, d \cup e]$, $[a, d] + [b, e] = [a \cup b, d + e]$

and $[a, d] - [b, e] = [a \cap b', f(b, d - e)]$ where $[x, y]$ is the equivalence class of (x, y). That the operations are well defined is a routine computation and with these operations $(B \times D)/\theta$ becomes a Semi-D.R.L. Semigroup. If $[x, y] \leq [a, 0]$ and $[b, 0]$ then $x = 0$ and $[0, y] < [0, y + 1]$ $a, 0$ and $b, 0$. Hence $(B \times D)/\theta$ is not a lattice.

Throughout this paper $A = (A; \cup, +, -)$ stands for a Semi-D.R.L. Semigroup and a, b, x, y.....etc. denote the elements of A.

The following lemmas (excepting lemma 2.11) and theorems (upto theorem 2.7) of D.R.L. Semigroups 4 are found to be valid in Semi-D.R.L. Semigroups also. We shall merely state them here as the proofs given in 4 are valid here also.

Lemma 2.1: $a - a = 0$ and $a - 0 = a$

Lemma 2.2: $(a - b) \cup 0 + b = a \cup b$

Lemma 2.3: $a \leq b$ implies $a - c \leq b - c$ and $c - b \leq c - a$

Lemma 2.4: $a \cup b - c = (a - c) \cup (b - c)$

1. We understand that "-" in Example 3 [5] is the following. If $a \geq b$, $a-b$ = usual difference otherwise $a-b = 0$.

Corollary 2.1: $a \geq b$ implies $a - b \geq 0$

Lemma 2.5: $a - (b + c) = (a - c) - b$

Lemma 2.6: $a \leq b$ iff $a - b \leq 0$

Lemma 2.7: $a \geq b$ implies $(a - b) + b = a$

Lemma 2.8: $(a - b) + (b - c) \geq (a - c)$

Corollary 2.2: $(a + (c - b) \geq a - (b - c)$

Lemma 2.9: $a \leq b \leq c$ implies $(c - b) + (b - a) = (c - a)$

Lemma 2.10: $(a + b) - c \leq a + (b - c)$

Lemma 2.11: $(a - b) \cup (b - a) \geq 0$

Proof: $(a - b) \cup (b - a) + a \cup b = (a - b) \cup (b - a) + a \cup (a - b) \cup (b - a) + b \geq (b - a) + a \cup (a - b) + b \geq b \cup a = a \cup b$. Hence by (1.2) $(a - b) \cup (b - a) \geq 0$ Q.E.D.

Theorem 2.1 Any Semi-D.R.L. Semigroup can be equationally defined as an algebra with the binary operations " $\cup$ ", " + " and " − " by replacing (1.2) by the equations (1.2.1) $(a-b) + b \geq a$, (1.2.2) $x - y \leq x \cup z - y$ and (1.2.3) $(x + y) - y \leq x$.

The following theorems give the degeneracy of A into a Semi-Brouwerian algebra and a lattice ordered group.

Theorem 2.2: A is a Semi Brouwerian algebra iff $a + b = a \cup b$ for all a and b. Also if $(A; \cup, -)$ is a Semi-Brouwerian algebra then $a + b = a \cup b$.

Theorem 2.3: The following are equivalent in A

i) $(A; +)$ is a group

ii) $a - b = a - c \Longrightarrow b = c$.

Definition 2.2: An element x is said to be invertible iff there exists a, $y \in A$ such that $x + y = y + x = 0$

Theorem 2.4 The set of all invertible elements of A is a lattice ordered group.

Definition 2.3: If there is an element $1 \in A$ such that $(1 - a) + a = a + (1-a) = 1 + 1$ for all a, then A is said to have unity 1.

Theorem 2.5: In A with 1, i) 1 is unique ii) A is a lattice ordered group iff $1 = 0$.

Theorem 2.6: If A contains a least element x, then $x = 0$. Dually if A (with 1) contains a greatest element x, then $x = 1$.

Theorem 2.7: If A contains an element which is strictly less than "0" then it contains an infinity of elements.

The following theorem gives the degeneracy of A into the direct product of a Semi-Brouwerian algebra and a lattice ordered group.

Theorem 2.8: A is the direct product of a Semi-Brouwerian algebra and a lattice ordered group iff $a - (b + b) = (a-b) + (0-b)$ for all $a, b \in A$.

We shall first prove a lemma which we need in the proof of the above theorem.

<u>Lemma 2.12</u>: <u>If $0 - a = 0$, and x [illegible] 0 then $a + x = x + a$.</u>

<u>Proof</u>: $(0 - x) + a \geq 0 - x$ since $a \geq 0$. Let $y + a \geq 0 - x$. Then $0 - (0 - x) \geq 0 - (y + a)$ (by lemma 2.3) $= (0 - a) - y = 0 - y$. Now by corollary to lemma 2.8, $x > 0 - (0 - x) \geq 0 - y$. Hence $y \geq 0 - (0 - y) \geq 0 - x$. Thus $(0 - x)$ is the least element such that $(0 - x) + a \geq 0 - x$. Hence by (1.2) $0 - x = (0 - x) - a$. Now $(0 - x) \cup a = \{(0 - x) - a\} \cup 0 + a$ (by lemma 2.2)$=(0-x)+a$, so that $a + (0 - x) \geq a \cup (0 - x) \geq (0 - x) + a$. Or $x + a = x + a + (0 - x) + x \geq x + (0 - x) + a + x = a + x$. By (1.5) we have $(0 - x) + a + x \geq$ [illegible] Hence $x + (0 - x) + a + x \geq x + a$ or $a + x = x + a$.

The proof of this theorem is divided into the following lemmas and from now on, we assume $a - (b + b) = (a - b) + (0 - b)$ for all $a, b \in A$.

<u>Lemma 2.13</u>: <u>Let $A_B = \{a \in A \mid a + a = a\}$. Then A_B is a Semi Brouwerian algebra.</u>

<u>Proof</u>: If $a \in A_B$, then $0 = 2a - 2a = (2a - a) - a = (2a - a) - a = 0 - a$. [illegible] $= 0 - a$. Hence $a \geq 0 - (0 - a) = 0$. Similarly if $a \geq 0$, and $0 - a = 0$, then $0 = 2a - 2a = (2a - a) + (0 - a) = 2a - a$. Hence $2a \leq a$. Since $a \geq 0$, $2a \geq a$. Hence $a \in A_B$.

Let $a, b \in A_B$. Then $a + b \geq 0$ and $0 - (a + b) = (0 - b) - a = 0 - a = 0$. Hence $a + b \in A_B$. Also $a - b \geq 0 - b = 0$, so that $a \cup b = (a - b) + b$. Now $a + b = (a + b) \cup b = a \cup b + b = \{(a - b) + b\} + b = (a - b) + b = a \cup b$. Hence $a \cup b \in A_B$ so that $0 - (a \cup b) = 0 - ((a - b) + b) = (0 - b) - (a - b) = 0 - (a - b)$. Hence $a - b \in A_B$. Thus A_B is closed under the operations $\cup$ and $-$ and so A_B is a Semi Brouwerian algebra.

Q.E.D.

Lemma 2.14: <u>Let $A_L = \{2a - a \mid a \in A_L\}$. Then A_L is a lattice ordered group.</u>

<u>Proof:</u> By lemma 2.10, $2a - a \leq a$ and so $(0 - a) \leq 0 - (2a - a)$ and $0 - (2a - a) \leq (a - 2a)$ (since $a - (b - c) \leq a + (c - b)$ for all a, b, c) $= 0 - a$ (by lemma 2.5). Hence $0 - a = 0 - (2a - a)$.
Now $0 = 2a - 2a = (2a - a) + (0 - a) = (2a - a) + 0 - (2a - a)$ $\geq 0 - \{0 - (2a - a)\} + \{0 - (2a - a)\} \geq 0$ so that
$0 - \{0 - (2a - a)\} + 0 - (2a - a) = 0$. Now $0 = 2\{0 - (2a - a)\}$ $- 2\{0 - (2a - a)\} = 2\{0 - (2a - a)\} - \{0 - (2a - a)\} + 0 - \{0 - (2a - a)\}$. Adding $0 - (2a - a)$ both sides, we have
$0 - (2a - a) = [2\{0 - (2a - a)\} - \{0 - (2a - a)\}]$. $[0 - \{0 - (2a - a)\}] +$ [illegible] $-$
$\{0 - (2a - a)\} + 0 = 2\{0 - (2a - a)\} - \{0 - (2a - a)\}$.
Hence $0 - (2a - a) \in A_L$ [and $2a - a = (2a - a) + 0 = (2a - a)$ $+ 2\{0 - (2a - a)\} - 2\{0 - (2a - a)\} = (2a - a) + [2\{0 - (2a - a)\}$ $- \{0 - (2a - a)\}] + 0 - \{0 - (2a - a)\} = (2a - a) + \{0 - (2a - a)\}$ $+ 0 - \{0 - (2a - a)\} = 0 - \{0 - (2a - a)\}$. Hence every $(2a - a)$ is invertible and so A_L is a lattice ordered group by theorem 4.

Q.E.D.

Lemma 2.15: Let $a_B = a - (2a - a)$ then $a_B \in A_B$

Proof: $a + \{0 - (2a - a)\} + (2a - a) \geq a$. Let $y + (2a - a) \geq a$. Then $y = y + (2a - a) + \{0 - (2a - a)\} \geq a + \{0 - (2a - a)\}$. Thus $a + \{0 - (2a - a)\}$ is the least element such that $a + \{0 - (2a - a)\} + 2a - a \geq a$. Hence $a + \{0 - (2a - a)\} = a - (2a - a)$. Now $0 = 2a - 2a = (2a - a) - a = [0 - \{0 - (2a - a)\}] - a = 0 - [a + \{0 - (2a - a)\}]$. Also $a + \{0 - (2a - a)\} \geq 0$ since $a - (2a - a) \geq 0$. Hencd $a_B + a_B = a_B$ or $a_B \in A_B$. Q.E.D.

Lemma 2.16: Lét $a_L = 2a - a$, then $b_B + a_L = a_L + b_B$ for all a, b A.

Proof. Since every element of A_L can be written as the sum of a negative element and an inverse of a negative element, the lemma follows by lemma 2.12. Q.E.D.

Lemma 2.11: $(a + b)_B = a_B + b_B$ and $(a + b)_L = a_L + b_L$.

Proof: If x is invertible, then $(a + x) - x = a$; For if $y + x \geq a + x$, then $y = y + x + t \geq a + x + t = a$ where t is the inverse of x. Hence a is the least element satisfying $a + x \geq a+x$. Thus $a = (a + x) - x$.

Now for any a, $a = a_B + a_L$. Let $a = x + y$ where $x \in A_B$ and $y \in A_L$. $- a_L = 0 - (2a - a) = 0 - a = 0 - (y + x) = (0 - x) - y = 0 - y = - y$. Hence $a_L = y$ and so $a_B = a - a_L = a - y = x$. Hence the representation of a as the sum of a_B and a_L is unique. Now $a_B = b_B + a_L + b_L = a_B + a_L + b_B + b_L = a + b = (a + b)_B + (a + b)_L$. Hence $a_B + b_B = (a + b)_B$ and $a_L + b_L = (a + b)_L$. Q.E.D.

<u>Lemma 2.18:</u> <u>$a \geq b$ iff $a_B \geq b_B$ and $a_L \geq b_L$</u>

<u>Proof:</u> If $a \geq b$ then $-b_L = 0 - (2b - b) = 0 - b \geq 0 - a = 0 - (2a - a) = -a_L$ or $a_L \geq b_L$. Also $a_B = (a \cup b)_B = \{(a - b) + b\}_B = (a - b)_B + b_B \geq b_B$. If $a_L \geq b_L$ and $a_B \geq b_B$ then $a = a_B + a_L \geq b_B + a_L = b_B + b_L = b$. Q.E.D.

<u>Lemma 2.19:</u> <u>$a_B = (a \cup 0)_B$</u>

<u>Proof:</u> If x is invertible, then $(2x - x) + x = 2x$ and so $2x - x = (2x - x) + x + y = x + x + y = x$ where y is the inverse of x. Hence $2x - x = x$ or $x - (2x - x) = x_B = 0$. Now $0 - a$ is invertible for all a and so is $0 \cup (0 - a)$. Hence $\{0 \cup (0 - a)\}_B = 0$ and so $(a \cup 0 - a)_B = \{0 \cup (0 - a)\}_B = 0$. Hence $0 = (a \cup 0 - a)_B \geq (a \cup 0)_B - a_B \geq 0$ so that $(a \cup 0)_B \leq a_B$. By lemma 2.18, $(a \cup 0)_B \geq a_B$ and so $(a \cup 0)_B = a_B$. Q.E.D.

<u>Lemma 2.20:</u> <u>$(a \cup b)_B = a_B \cup b_B$ and $(a \cup b)_L = a_L \cup b_L$ for all $a, b \in A$.</u>

<u>Proof:</u> $a_B = a_B + (0 - b)_B = \{a + (0 - b)\}_B \geq (a - b)_B$. Now $(a \cup b)_B = \{(a - b) \cup 0 + b\}_B = \{(a - b) \cup 0\}_B + b_B = (a-b)_B + b_B = (a - b)_B \cup b_B \leq a_B \cup b_B$ and so $(a \cup b)_B = a_B \cup b_B$ by lemma 2.18.

Also $a_L - b_L = a_L + (0 - b)_L = \{a + (0 - b)\}_L \geq (a - b)_L$. Also $(a - b)_L + b_L = \{(a - b) + b\}_L \geq a_L$. Hence $(a - b)_L \geq a_L - b_L$. Thus $a_L - b_L = (a - b)_L$.

Now $(a \cup b)_L = \{(a - b) \cup 0 + b\}_L = \{(a - b) \cup 0\}_L + b_L =$ $(a_L - b_L) \cup 0 + b_L = a_L \cup b_L$ since $(a \cup 0)_L = a_L \cup 0$ for all a.

Q.E.D.

Lemma 2.21: $(a - b)_B = a_B - b_B$ for all a, b

Proof: $a_B - b_B = (a - a_L) - (b - b_L) = [a - \{(a - b)_L + b_L\}] - (b - b_L) = \{(a - b_L) - (a - b)_L\} - (b - b_L) = (a - b_L) - \{(b - b_L + (a - b)_L\} = (a - b_L) - \{(a - b)_L + (b - b_L)\} = \{(a - b_L) - (b - b_L)\} - (a - b)_L = a - \{(b - b_L) + b_L\} - (a - b)_L = (a - b) - (a - b)_L = (a - b)_B.$

Q.E.D.

The mapping $a \to (a_B, a_L)$ is an isomorphism from A onto $A_B \times A_L$ as is shown by lemmas 2.17, 2.20 and 2.21. Thus A is isomorphic to $A_B \times A_L$. The converse is obvious and so the proof of theorem 2.8 is complete.

The following theorem of Swamy (Theorem 3.4 6) follows as a corollary to Theorem 2.8.

Corollary 2.3: A D.R.L. Semigroup is the direct product of a Brouwerian algebra and a lattice group iff $a - (b + b) = (a - b) + (0 - b)$.

The following theorem characterises the direct product of a Semi D.R.L. Semigroup with "0" as least element and a commutative lattice ordered group, among the class of Semi-D.R.L. Semigroups.

Theorem 2.9: A is the direct product of a Semi-D.R.L. Semigroup with 0 as least element and a commutative lattice ordered group iff $0 - (x + y) = (0 - x) + (0 - y)$ for all x, y.

Proof: As in theorem 8, we divide the proof of this theorem into he following lemmas and assume $0 - (x + y) = (0 - x) + (0 - y)$.

Lemma 2.22. Let $A_S = \{a \in A \mid 0 - a = 0\}$. Then A_S is a Semi-D.R.L. Semigroup with 0 as least element

Proof: If $a, b \in A_S$, then $0 - (a + b) = (0 - b) - a = 0 - a = 0$. Hence $a * b \;\; A_S$. Also $a \geq 0 - (0 - a) = 0$ and so $a - b \geq (0 - b) = 0$ Hence $0 - (a - b) \leq 0$. Also $a \leq a + b$ and so $0 \geq a - (a + b) = (a - b) - a$ or $a - b \leq a$ so that $0 - a \leq 0 - (a - b)$. Hence $0 - (a - b) = 0$ or $a - b \in A_S$. Also $a \cup b = (a - b) + b \;\; A_S$, so that A_S is a Semi-D.R.L. Semigroup with "0" as least element.

Q.E.D.

Lemma 2.23: Let $A_L = \{a \in A \mid 0 - (0 - a) = a\}$. Then A_L is a commutative lattice ordered group.

Proof: Let $a \in A_L$. Now $a + (0 - a) = \{0 - (0 - a)\} + (0 - a) = 0 - \{(0 - a) + a\} = (0 - a) - (0 - a) = 0$. Also $(0 - a) + a = (0 - a) + 0 - (0 - a) = 0 - \{a + (0 - a)\} = \{0 - (0 - a)\} - a = 0$. Hence every element of A_L is invertible and so A_L is a lattice ordered group.

Now $b + (0 - a) = \{0 - (0 - b)\} - a = 0 - \{a + (0 - b)\} = (0 - a) + 0 - (0 - b) = (0 - a) + b$. Hence A_L is commutative.

Q.E.D.

For any a, let $a_L = 0 - (0 - a)$ and $a_S = (0 - a) + a$ then

<u>Lemma 2.24: $(a + b)_L = a_L + b_L$ and $(a + b)_S = a_S + b_S$ for all a and b.</u>

<u>Proof</u>. $0 - a_L + a_L = 0 - \{0 - (0 - a)\} + 0 - (0 - a) = 0 - \{0 - (0 - a) + (0 - a)\} = 0 - (0 - a) - \{0 - (0 - a)\} = 0.$

Now $0 = 0 - ((0 - a_L) + a_L) = \{0 - (0 - a_L)\} + (0 - a_L)$. Adding a_L both sides, $a_L = 0 - (0 - a_L) + (0 - a_L) + a_L = 0 - (0 - a_L)$. Hence $a \in A_L$. And since $0 - \{(0 - a) + a\} = (0 - a) - (0 - a) = 0$, $(0 - a) + a \in A_S$. Now $a_L + a_S = \{0 - (0 - a)\} + (0 - a) + a = 0 - \{(0 - a) + a\} + a = (0 - a) - (0 - a) + a = a$ and by lemma 2.12 $a_L + a_S = a_S + a_L$.

Now if $a = x + y$ where $x \in A_S$ and $y \in A_L$, then $0 - a = 0 - (y + x) = (0 - x) - y = 0 - y$. Hence $0 - (0 - a) = 0 - (0 - y) = y$, and $(0 - a) + a = 0 - y + y + x = x$. Hence the representation of a as the sum of a_L and a_S is unique.

Also $a_S + b_S + a_L + b_L = a_S + a_L + b_S + b_L = a + b = (a + b)_S + (a + b)_L$. Hence $(a + b)_L = a_L + b_L$ and $(a + b)_S = a_S + b_S$. Q.E.D.

<u>Lemma 2.25: $a \geq b$ iff $a_S \geq b_S$ and $a_L \geq b_L$</u>

<u>Proof</u>: If $a \geq b$, then $0 - b \geq 0 - a$ and so $a_L = 0 - (0 - a) \geq 0 - (0 - b) = b_L$. Also $a_S = (a \cup b)_S = \{(a - b) \cup 0 + b\}_S = \{(a - b) \cup 0\}_S + b_S \geq b_S$. Also if $a_S \geq b_S$ and $a_L \geq b_L$ then $a = a_S + a_L \geq b_S + a_L \geq b_S + b_L = b$. Q.E.D.

<u>Lemma 2.26</u>: $(a \cup b)_S = a_S \cup b_S$ and $(a \cup b)_L = a_L \cup b_L$

<u>Proof</u>: $(a \cup b)_S = (0 - a \cup b) + a \cup b = \{(0 - a \cup b) + a\} \cup \{(0 - a \cup b) + b\} \leq \{(0 - a) + a\} \cup \{(0 - b) + b\} = a_S \cup b_S$. By the above lemma, $(a \cup b)_S \geq a_S \cup b_S$ and hence $(a \cup b)_S = a_S \cup b_S$.

Also let $a_L \cup 0 = x_L$. Then $x_L + a_S \geq (a_L + a_S) \cup 0 = a \cup 0$. Thus $x_L = (x_L)_L + (a_S)_L \geq (a \cup 0)_L$. Hence $(a \cup 0)_L = a_L \cup 0$.

Now $(a - b)_L + b_L = \{(a - b) + b\}_L \geq a_L$ so that $(a-b)_L \geq a_L - b_L$. Also $a_L - b_L = a_L + (0 - b)_L = \{a + (0 - b)\}_L \geq (a - b)_L$. Hence $(a - b)_L = a_L - b_L$.

Also $(a \cup b)_L = \{(a - b) \cup 0 + b\}_L = \{(a - b) \cup 0\}_L + b_L = (a - b)_L \cup 0 + b_L = (a_L - b_L) \cup 0 + b_L = a_L \cup b_L$ Q.E.D.

<u>Lemma 2.27</u>: $(a - b)_S = a_S - b_S$.

<u>Proof</u>: $a_S - b_S = (a - a_L) - (b - b_L) = [a - \{(a - b)_L + b_L\}] - (b - b_L) = (a - b_L) - (a - b)_L - (b - b_L) = a - b_L - \{b - b_L + (a - b)_L\} = a - b_L - \{(a - b)_L + b - b_L\} = \{(a - b_L) - (b - b_L)\} - (a - b)_L = [a - \{(b - b_L + b_L)\}] - (a-b)_L = (a - b) - (a - b)_L = (a - b)_S$.

The mapping $a \to (a_S, a_L)$ from A onto $A_S \times A_L$ is an isomorphism as is shown by lemmas 2.24, 2.26 and 2.27 and so A is isomorphic to $A_S \times A_L$. The converse is obvious. Thus the proof of Theorem 2.9 is complete. Q.E.D.

<u>Corollary 2.4:</u> A D.R.L. Semigroup is the direct product of a D.R.L. Semigroup with "0" as least element and a commutative lattice ordered group iff $0 - (x + y) = (0 - x) + (0 - y)$ for all x and y (Theorem 3.6 [6]).

Definition 2.4 A subset S of A is called an ideal iff i) a, b S imply a + b $\in$ S. ii) a $\in$ S and $b * 0 \leq a * 0$ imply b $\in$ S iii) x $\in$ S implies (a + x) − a S for all a.

Theorem 2.10: Any ideal S of A is a convex sub Semi D.R.L. Semigroup in the sense that i) It is closed under the operations " + " " $\cup$ " and " − " ii) a, b S and $a \leq x \leq b$ implies x S.

Proof: Let a, b $\in$ S. Since $(a * 0) * 0 = a * 0$, $a * 0 \in S$. Similarly $b * 0 \in S$ and $a * 0 + b * 0 = a \cup (0 - a) + b \cup (0 - b) \geq \{a + (0 - b)\} \cup 0 \geq (a - b) \cup 0$ and so by (ii) $(a - b) \cup 0 \in S$. Similarly $(b - a) \cup 0 \in S$ and $a \cup b = (a - b) \cup 0 + b$ S by (i), so that $a * b = (a - b) \cup (b - a) \in S$. Now $(a - b) * 0 \leq a * b$ and hence $a - b \in S$. Hence S is a Sub Semi-D.R.L. Semigroup.

Let $a \leq x \leq b$. Then $x * 0 \leq b \cup (0 - a) = (b * 0) \cup (a * 0) \in S$ and so $x \in S$. Thus S is a comvex-Sub-Semi-D.R.L. Semigroup of A.

Q.E.D.

Theorem 2.11: There is a one to one correspondence between the ideals and congruence relations of A.

Proof: Let S be an ideal in A. Define $a \equiv b$ iff $a * b \in S$ i.e. iff a − b, b − a $\in$ S. First we shall show that $\equiv$ is a congruence relation.

That "$\equiv$"is reflexive and symmetric is obvious. If $a \equiv b$, and $b \equiv c$, then a − b , b − a, b − c, c − b $\in$ S and $(a - c) \cup 0 \leq \{(a - b) + (b - c)\} \cup 0 \in S$ and so $(a - c) \cup 0 \in S$ by ii. Similarly $(c - a) \cup 0 \in S$ and hence $a * c = (a - c) \cup (c - a) \in S$ or $a \equiv c$. Hence "$\equiv$" is an equivalence relation.

Also $a - (x + a) = 0 - x$ and $(x + a) * a = \{(x + a) - a\} \cup \{a - (x + a)\} \leq x \cup (0 - x) = x * 0$. Hence $(x + a) * a \in S$ for all a, if $x \in S$, or $(x + a) - a \in S$ if $x \in S$.

If $a \equiv b$, then $\{(a - b) + (b - c)\} - (b - c) \in S$ and $\{(a - b) + (b - c)\} - (b - c) \geq (a - c) - (b - c)$ which implies $\{(a - c) - (b - c)\} \cup 0 \in S$ by (ii). Similarly $\{(b - c) - (a - c)\} \cup 0 \in S$ and hence $(b - c) * (a - c) \in S$ or $a - c \equiv b - c$.

Also, $\{(c - a) + (a - b)\} - (c - a) \geq (c - b) - (c - a)$ and since $\{(c - a) + (a - b)\} - (c - a) \in S$ by (iii) $\{(c - b) - (c - a)\} \cup 0 \in S$. Similarly $\{(c - a) - (c - b)\} \cup 0 \in S$ and hence $(c - a) * (c - b) \in S$ or $c - a \equiv c - b$.

Thus if $a \equiv b$, and $c \equiv d$, then $a - c \equiv b - c \equiv b - d$.

Also, $(a + c) - (b + c) = \{(a + c) - c\} - b \leq a - b$. Hence $\{(a + c) - (b + c)\} \cup 0 \in S$ by (ii) if $a - b \in S$. Similarly $\{(b + c) - (a + c)\} \cup 0 \in S$ if $b - a \in S$ and so $(a + c) * (b + c) \in S$ or $a + c \equiv b + c$ if $a * b \in S$.

On the same lines, $(c + a) - (c + b) = \{(c + a) - b\} - c \leq \{c + (a - b)\} - c \in S$ by (iii) if $a - b \in S$. Hence $\{(c + a) - (c + b)\} \cup 0 \in S$ if $a - b \in S$. Similarly $\{(c + b) - (c + a)\} \cup 0 \in S$ if $b - a \in S$, so that $(c + a) * (c + b) \in S$, if $a * b \in S$ or $c + a \equiv c + b$ if $a \equiv b$.

Thus if $a \equiv b$ and $c \equiv d$, then $a + c \equiv b + c \equiv b + d$.

Finally, $(a \cup c) * (b \cup d) = (a \cup c - b \cup d) \leq (a - b) \cup (b - a) \cup (c - d) \cup (d - c) = (a * b) \cup (c * d)$ so that $a \cup c * b \cup d \in S$ if $a * b$, $c * d \in S$ or $a \cup c \equiv b \cup d$ if $a \equiv b$. $c \equiv d$.

Therefore "$\equiv$" is a congruence relation.

Let "θ" be a congruence relation and L be the set of all $x \in A$ such that $x \equiv 0(\theta)$. It is easy to see that if $x, y \in L$ then $x + y \in L$ and $(a + x) - a \in L$ for all a. Let $y * 0 \leq x * 0$ and $x \in L$ then $x \equiv 0(\theta)$ so that $0 - x \equiv 0(\theta)$ so that $0 - x \in L$ hence $x * 0 = x \cup (0 - x) \in L$. Now $0 \equiv x * 0 = (x * 0) \cup (y * 0) \equiv 0 \cup (y * 0) = y * 0$. Hence $y * 0 \in L$. Also $0 \equiv y * 0 = \{y \cup (0 - y)\} \cup y \equiv 0 \cup y$ so that $y \cup 0 \in L$ similarly $0 \cup (0 - y) \in L$. Now $y = 0 + y \equiv (0 - y) \cup 0 + y \equiv y \cup 0$ so that $y \equiv 0$ or $y \in L$. Hence L is an ideal.

Now let R be the congruence relation obtained by defining $a \equiv b(R)$ iff $a-b$, $b - a \in L$. Then $a \equiv b(R) \Leftrightarrow a * b \in L \Leftrightarrow a * b \equiv 0(\theta)$ so that $(a-b) \cup 0 \equiv 0(\theta)$. Now $a \cup b = (a - b) \cup 0 + b$ $0 + b \equiv b(\theta)$. Similarly $a \cup b \equiv a(\theta)$ or $a \equiv b(\theta)$, on the same lines if $a \equiv b(\theta)$ then $(a - b) \cup 0 \equiv 0(\theta)$ and $(b - a) \cup 0 \equiv 0(\theta)$ or $a * b \equiv 0(\theta)$ or $a * b \in L$ so that $a \equiv b(R)$.

Hence the ideals correspond one to one to its congruence relations.

<u>Corollary 2.5</u>: There is a one to one correspondence between the congruence relations and ideals of a commutative D.R.L. Semigroup (Th. 1.2 [6]).

In this section we show that a commutative Semi-D.R.L. Semigroup is an autometrized algebra, A system $B = (B; \leq, +, *)$ is called an autometrized algebra iff (i) $(B; +)$ is a binary commutative algebra with a zero element "0" (ii) $\leq$ is an anti-symmetric reflexive ordering on B(iii) $* : B \times B \to B$ is a mapping satisfying (iv) $a * b \geq 0$ with equality iff $a = b$ (v) $a * b = b * a$. (vi) $a * c \leq a * b + b * c$. (for details see [7]).

In this section we assume that $(A; +)$ is a commutative Semigroup.

Define $a . b = a \cup b - a * b$. We refer "." as the multiplication in A.

Lemma 2.28. $a.b \leq a$ and b

Proof: $a + (a - b) \cup (b - a) = (a - b) \cup 0 \cup (b - a) + a = \{(a - b) \cup 0 + a\} \cup \{(b - a) + a\} \geq a \cup b$. Hence $a \geq a \cup b - (a - b) \cup (b - a) = a * b$. Similarly $b \geq a.b$. Q.E.D.

Lemma 2.29: $a \cup b - a.b = a * b$

Proof: $a \cup b - a.b = (a - ab) \cup (b - ab) \geq (a - b) \cup (b - a)$. Also $a * b + ab = ab + a * b = (a \cup b - a * b) + a * b \geq a \cup b$. Hence $a * b \geq a \cup b - ab$. Thus $a \cup b - ab = a * b$. Q.E.D.

Theorem 2.12: A is an autometrized algebra with "*" as a distance function.

Proof: Define $d: A \times A \to A$ by $d(a,b) = a * b$. Then by lemma 11, $a * b \geq 0$. If $a * b = 0$ then $a - b \leq a * b = 0$ so that $a \leq b$. Similarly $b \leq a$ and so $a = b$. Also $a * b + b * c \geq \{(a - b) + (b - c)\} \cup \{(c - b) + (b - a)\} = a * c$. Hence A is an autometrized algebra. Q.E.D.

From the above theorem it follows that, commutative lattice ordered groups, Brouwerian algebras commutative D.R.L. Semigroups and Semi Brouwerian algebras are all autometrized algebras. We refer to $*$ as the metric on A and $a * b$ as the distance between a and b.

In a commutative lattice ordered group, the metric $*$ is invariant under the group translations. That this property characterises lattice ordered groups among the class of Semi-D.R.L. Semigroups is established by the following.

Theorem 2.13: A, with 1 is a lattice ordered group iff the distance is invariant under translations i.e. $a * b = (a + x) = (b + x)$ for all x.

Proof. $1 * 0 = 1 * (1.1) = (1 + 1) * \{(1 - 1) + 1\} = (1 + 1) * 1 = 1 * 1 = 0$. Hence $1 = 0$. Hence by theorem 5, A is a lattice ordered group.

Definition. A is said to be symmetric iff $a * b = (a + b) * ab$. Our definition is slightly different from the one gives in [7] but is more general.

Theorem 2.14: A is symmetric iff A is a Semi-Brouwerian algebra.

Proof: $a - a.0 = a * a.0 = (a + 0) * a.0 = a * 0 = a \cup 0 - a.0$. Hence $a = (a - a.0) + a.0 = (a \cup 0 - a.0) + a . 0 = a \cup 0$ so that $a \geq 0$ for all a. Hence $a + b \geq a \cup b$. Also $(a + a) * a.a = a * a = 0$ which implies $a + a = a.a = a$ so that $a \cup b = a \cup b + a \cup b \geq a + b$. Hence $a + b = a \cup b$ for all a and b and so A is a Semi-Brouwerian algebra. Q.E.D.

Theorem 2.15: A is a Boolean ring iff $*$ is associative

The proof of this theorem requires the following two lemmas.

Lemma 2.30: If $a \geq b \cup c$ and $a - b \geq a - c$ then $c \geq b$.

Proof: $a * (a * 0) = (a * a) * 0 = 0$ so that $a = a * 0$ or $a \geq 0$ for all $a \in A$. Hence $a - b \leq a$. Also if $a \geq b$ then $a - (a - b) = a * (a * b) = (a * a) * b = 0 * b = b$. Now if $a \geq b \cup c$ and $a - b \geq a - c$ then $c = a - (a - c) \geq a - (a - b) = b$. Q.E.D.

Lemma 2.31. $a.b$ is the greatest lower bound of a and b.

Proof: Let $x \leq a$ and b. Then $a - b \leq a - x$ and $b - a \leq b - x$. Hence $a \cup b - x = (a - x) \cup (b - x)$ $(a - b) \cup (b - a) = a \cup b - a.b$ so that by the above lemma $a.b \geq x$. Q.E.D.

Proof of Theorem 2.15: Now by the above two lemmas, it follows that A is a lattice and so A is a D.R.L. Semigroup. Hence by Theorem 5, [4] A is a Boolean ring. Q.E.D.

Theorem 2.16: If $a \cup (a - a) \geq 1$ in A with 1, then the following are equivalent.

i) $a * b = a * c \Rightarrow b = c$ ii) A is regular and the group of isometries of A is simply transitive. iii) A is a Boolean algebra.

(An autometrized algebra is said to be regular iff $a * 0 = a$ for all the elements a. A mapping σ from an autometrized algebra into itself is called an isometry iff $a * b = \sigma(a) * \sigma(b)$ for all a, b. The group of isometries is called simply transitive iff given x and y there exists an element a such that $x = a * y$ and $y = a * x$ see [7]).

Proof: $i \Rightarrow (iii)$. If $a \geq b$, then $a * \{a - (a - b)\} = a - \{a - (a - b)\} = a - b = a * b$. Hence $a - (a - b) = b$. If $a \geq b \cup c$ and $a - b \geq a - c$ then $c = a - (a - c) \geq a - (a - b) = b$. Hence $c \geq b$. If $x \leq a$ and b, then $a \cup b - x = (a - x) \cup (b - x) \geq (a - b) \cup (b - a) = a \cup b - a.b$. Hence $x \leq a.b$. Thus $a.b.$ is the greatest lower bound of a and b, or Λ is a lattice. Hence Λ is a Boolean algebra by Theorem 2.2 [5].

Q.E.D.

ii i follows from Theorem 13 [7]

iii i and ii obvious.

Theorem 2.17: The following are equivalent in Λ.

i) $a.b.$ is the greatest lower bound of a and b.

ii) $a \leq b$ implies $a.c \leq b.c$

iii) $a - (b.c) = (a - b) \cup (a - c)$ for all a, b, c

iv) $a \cup (b.c) = (a \cup b) . (a \cup c)$

v) $a \geq b \cup c$ and $a - b \geq a - c$ imply $c \geq b$

v) $a \geq b$ implies $a - (a - b) = b$

vi) $a \geq b$ iff $a.b. = b$

Proof. That $i \Rightarrow (ii)$, and (iv) is obvious.

$i \Rightarrow (v)$ we have $a.b. \geq b$ since ab is the greatest lower bound and b is a lower bound of a, but $a.b \leq b$. Hence $ab = b$. Similarly $ac = c$. Now $c = ac = a \cup c - (a * c) = a - (a - c) \geq a-(a - b) = a.b = b$.

$(ii) \Rightarrow (i)$ Let $x \leq a$ and b. Then $x = x.x \leq x.b \leq x$. Hence $x.b = x$. Now $x = x.b \leq a.b$. Hence $a.b.$ is the greatest lower bound.

( ) $\Rightarrow$(i) If x $\geq$ a, and b, then $x \cup (a.b) = (x \cup a).(x \cup b) = a.b$ therefore $x \geq a.b$

(v') $\Rightarrow$ (v) If $a \geq b \cup c$ and $a - b \geq a - c$ then $c = a - (a - c) \geq a - (a-b) = b$.

(v) (v) $a - \{a - (a - b)\} = a - b$. Hence $a - (a - b) = b$.

(v) and (v) are equivalent Q.E.D.

The importance of the above theorem is reflected in the following

Theorem 2.18: Let L = (L; $\cup$,-) be a system such that (L; $\cup$) is a given semi lattice and "-" is a binary operation on L satisfying

1. $a \cup b - c = (a - c) \cup (b - c)$

   $a - (b \cup c) = (a - b) \, . \, (a - c)$ where in general for x,y L

   $x.y = x \cup y - \{(x - y) \cup (y - x)\}$

2. $a \leq b \Rightarrow c - b = (c - a) - (b - a)$ for all c.

3. Given a and b in L there exists c $\ni$ $p - c = (p - b) - a = (p-a)-b$ for all p, then (L; $\cup$) is a lattice and a commutative boolean-L-algebra if $a - (b \cdot c) = (a - b) \cup (a - c)$

Proof: In view of the above theorem it is sufficient to show that a.b is the greatest lower bound of a and b.

Let $x \leq a,b$ then $x - a \, .b = (x - a) \cup (x - b) \leq 0$ therefore $x \leq a.b$

Remark 2.6: If $(a - b) \, . \, (b - a) \leq 0$, then A is a Boolean-L-algebra (see [3]) iff any one of the above statements holds in A.

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CHARACTERIZATION OF A CLASS OF DUALLY RESIDUATED LATTICE ORDERED SEMI GROUPS BY SHEAVES

INTRODUCTION:

The notion of a Dually residuated lattice ordered semigroup or briefly D.R.L. semigroup goes back to K.L.N.Swamy (see [2,3,4,5]) who introduced it and obtained it as a common abstraction of Boolean rings and lattice ordered groups, thereby solving Birkoff's problem no.105 [1] (Is there a common abstraction which includes Boolean rings (algebras) and Lattice ordered groups as special cases?) in a general way. In this paper we study the structure of a D.R.L. semigroup and characterise the class of all D.R.L. semigroups which can be obtained as the 'global sections with compact open supports of a Hausdorff Sheaf ($\sim$, Π, x) of nontrivial totally ordered D.R.L. semigroups over a base space x which is locally Boolean' by a set of three conditions 1, 2 and 3.

These three conditions have a useful interpretation in the particular case when the D.R.L. semigroup is a lattice ordered group. Condition (1) is always true in any arbitrary lattice ordered group, but not so conditions (2) and (3). In fact Banaschewski has shown in [6] that in a lattice ordered group G condition (2) is satisfied iff G has a realization. Banaschewski has also established condition (3) in a complete lattice ordered group and that "its lattice of filets is a Boolean ring" in view of our results, these results of Banaschewski may be interpreted as follows:

<u>An l-group G has a realisation iff it can be embedded in the l-group of all global sections of a Sheaf of non-trivial simply ordered l-groups.</u>

<u>Every complete l-group G is isomorphic to the l group of all global sections with compact open supports of Sheaf of non-trivial simply ordered l-groups over a locally Boolean base space.</u>

Thus, the results presented in this paper may also be looked upon as a generalisation of Banaschewski's results in [6] in a Sheaf theoretic sense.

This chapter is divided into two sections. In the first section, we study the properties of ultra filters in a D.R.L. semigroup satisfying conditions 1 and 2, while section (2) is devoted to establish the Sheaf theoretic characterization of a D.R.L. semi group with the extra condition (3).

<u>DEFINITION 3.1:</u> <u>A system A =(A, $\cup$, $\cap$, + , −) is said to be a dually residuated lattice ordered semigroups (or briefly D.R.L. semigroup) iff</u>

(1.1) (A, $\cup$, $\cap$) is a lattice

(1.2) (A, +) is a semigroup with 0 satisfying

$$a + x \cup y + b = (a + x + b) \cup (a + y + b)$$
$$a + x \cap y + b = (x + x + b) \cap (a + y + b)$$

(1.3) Given a and b in A there is a least x such that

$x + b \geq a$ (this x is denoted by a-b)

(1.4) $(a - b) \cup 0 + b \leq a \cup b$ for all a , b , x , y A

(1.5) $a - a \geq 0$.

For further results on D.R.L. semigroups, which we will be using, without making an explicit mention of them, we refer to [2,3,4,5]

Throughout this paper A stands for a D.R.L. semigroup and a,b,x,y,..... etc. denote the elements of A. We assume that A satisfies the following two conditions

1. $a \cap b - c = (a - c) \cap (b - c)$
2. $x > 0$ implies $x \cap (a + x) - a > 0$.

We denote by A^+ the set of all $x \in A$ such that $x \geq 0$.

DEFINITION 3.2: A filter M in A^+ is said to be normal $x \in M$ implies $(a + x) - a \in M$, for every a.

LEMMA 3.1: If M is an ultra filter in A^+, then M is normal.

Proof: Let $x \in M$ and suppose $(a + x) - a \notin M$ for some a. Since M is ultra there exists $c \in M$ such that $\{(a+x)-a\} \cap c = 0$. Now $x \cap c \cap (a + x \cap c) - a = x \cap c \cap \{(a + x) - a\} \cap \{(a + c) - a\} = 0$ which is a contradiction to condition (2) since $x \cap c > 0$.

LEMMA 3.2 If M is an ultra filter in A^+ such that $(a + x) - a \in M$ for some a , then $x \in M$.

Let $(a + x) - a \in M$ and if possible let $x \notin M$ once again there exists $c \in M$ such that $x \cap c = 0$ and since $c \in M$,

$(a + c) - a \in M$ (by above) and

$\{(a + x) - a\} \cap \{(a + c) - a\} = (a + x \cap c) - a = (a + 0) - a = 0$

which is again a contradiction since $\{(a + c) - a\} \cap \{(a + x) - a\} \in M$ and hence is nonzero. Q.E.D.

The following lemma is a partial converse to the above lemma.

<u>LEMMA 3.3: If $B = (T; \cup, \cap, +, -)$ is a D.R.L. semigroup satisfying (1) and if every ultra filter in B^+ is normal, then (2) is satisfied</u>.

<u>Proof</u>: If $x \in B$ and $x > 0$, then there exists an ultrafilter M in B^+ such that $x \in M$. Since by hypothesis M is normal, so $(a + x) - a \in M$ for every a. Hence $x \cap \{(a+x) - a\} > 0$. Q.E.D.

<u>DEFINITION 3.3: A subset L of a D.R.L. semigroup B is said to be an ℓ-ideal in B, if (1.3.1) $a, b \in L$ implies $a + b \in L$ (1.3.2) $b \in L$ and $a * 0 \leq b * 0$ then $a \in L$ where $a * 0 = a \cup (0-a)$ (1.3.3.) if $x \in L$ then $(a + x) - a \in L$ for every $a, b, x, y \in B$.</u>

<u>LEMMA 3.4: Let $K_M = \{x \in A : x * 0 \notin M\}$ where M is an ultra filter in A^+. K_M is an ℓ-ideal.</u>

<u>Proof</u>: First, observe that $x \in K_M$ iff $x * 0 \in K_M$ and $x \in K_M$ and $z * 0 \leq x * 0$, then $z \in K_M$. Let $x, y \in K_M$, so that $x * 0 \notin M$, and $y * 0 \notin M$, since M is ultra, there exists $c, d \in M$ such that $(x * 0) \cap c = 0 = (y * 0) \cap d$. Hence $(x * 0 + y * 0) \cap c \cap d = 0$ so that $x * 0 + y * 0 \notin M$ and hence $x * 0 + y * 0 \in K_M$. Similarly $y * 0 + x * 0 \in K_M$ and also $x * 0 + y * 0 + y * 0 + x * 0 \in K_M$.

Now $(x+y) * 0 = (x+y) \cup \{0-(x+y)\} = (x+y) \cup \{(0-y)-x\} \leq (x*0+y*0) \cup \{(y*0 + x*0)\} \leq x*0 + y*0 + y*0 + x*0 \in K_M$ hence $x+y \in K_M$

in order to complete the proof we now have to check whether 1.3.2 is satisfied in K_M. Let $x \in K_M$ then $x * 0 \notin M$ and so is $(a+x*0)-a$ if $(a+x)-a \notin K_M$ then $\{(a+x)-a\} * 0 \in M$ and since $\{(a+x)-a\} * 0$ $(a+x*0)-a$, since the latter belongs to M, we get a contradiction. Therefore $(a+x)-a \in K_M$ and so K_M is an $-\ell$-ideal. q.e.d.

<u>Remark 3.1: It is easy to see that K_M is infact a minimal prime-ℓ-ideal.</u>

<u>LEMMA 3.5: $A/K_M \cap A/K_N = \emptyset$ for $M \neq N$, where A/K_M and A/K_N are the quotient D.R.L. semigroups modulo K_M and K_N respectively.</u>

<u>Proof</u>: Suppose $\overline{x}(K_M) = \overline{y}(K_N)$ where $\overline{x}(K_M)$ and $\overline{y}(K_N)$ are the congruence classes of x and y modulo K_M and K_N respectively. Let $s \in K_M$. Now $x-(x+s) = (x-s)-x \leq (x+(0-s))-x \in K_M$ so that by 1.3.2 $\{x-(x+s)\} \cup 0 \in K_M$, also $(x+s)-x \in K_M$ so that

$x * (x+s) = \{(x+s)-x\} \cup \{(x-(x+s)\} \in K_M$. Hence $x+s \in \overline{x}(K_M)$ so that $x+s \in \overline{y}(K_N)$, therefore $(x+s)-y \in K_N$. Now $\{(x+s)-x\} \leq \leq \{(x+s)-y\} + (y-x) \in K_N$. So that $\{(x+s)-x\} \cup 0 \in K_N$ by 1.3.2. Also $x-(x+s) = (x-s)-x \leq (x+(0-s))-x \in K_N$ since $0-s \in K_N$. Therefore $\{x-(x+s)\} \cup 0 \in K_N$ and hence $(x+s) * X \in K_N$ so that $(x+s)-x \in K_N$. Since K_N is normal, we have $s \in K_N$. Therefore $K_M \subseteq K_N$ and so $N \subseteq M$. Therefore $N = M$ q.e.d.

In this section we construct a Sheaf $\langle \mathcal{S}, \pi, X \rangle$ of non-trivial simply ordered D.R.L. semigroups over a locally Boolean base space X . For this purpose we assume that A satisfies the following.

(3) <u>given $a, x \in A^+$ with $(x)^* \subseteq (a)^*$ there is a positive u</u> satisfying $(x * u) \cap a = 0$ and $(u)^* = (a)^*$ where in general $(z)^* = \{ y \in A \mid z \cap y = 0 \}$.

For the elementary results on Sheafs, which we will be using, without explicitly mentioning them, we refer to [7] .

We denote by X(A) the set of all ultra filters in A^+ and $\eta(a) = \{ M \in X(A) \mid a \in M \}$ the following theorem gives a 'locally Boolean space' structure for X(A).

<u>THEOREM 3.1:</u> $\mathcal{B} = \{ \eta(a) \mid a \in A^+ \}$ <u>is a Clopen base for a Hausdorff topology $\mathcal{T}$ on X(A) and each $\eta(a)$ is $\mathcal{T}$-compact.</u>

Clearly $\bigcup_{a \in A^+} \eta(a) = X(A)$ and $\eta(a) \cap \eta(b) = \eta(a \cap b)$. Hence $\mathcal{B}$ forms a base for a topology $\mathcal{T}$ on X(A). We will now show that each $\eta(a)$ is $\mathcal{T}$-closed. Let $M \in X(A) - \eta(a)$ then $a \notin M$ and so there exists $c \in M$ such that $a \cap c = 0$. Hence $M \in \eta(c) \subseteq X(A) - \eta(a)$. Therefore $\eta(a)$ is $\mathcal{T}$-closed. Also if $M, N \in X(A)$ and $M \neq N$, then there exists $a \in M$ such that $a \notin N$ and hence there exists $c \in N$ such that $a \cap c = 0$. Now $\eta(c)$ is a n.h.d. of N and is disjoint with $\eta(a)$. Thus X(A) is a Hausdorff space.

In order to obtain the second part of the lemma, we shall first show that every basic closed subset of $\eta(a)$ is of the form $\eta(c)$ for some c. Now let F be a basic closed subset of $\eta(a)$

i.e. $F = \eta(a) - \eta(a \cap b)$ for some b. We have $(a)^* \subseteq (a \cap b)^*$ and so by (3) there exists u satisfying $(u)^* = (a \cap b)^*$ and $(a-u) \cap a \cap b = 0$. It is easy to see that $\eta((a-u) \cup (a \cap b)) = \eta(a)$ and so $\eta(a) - \eta(a \cap b) = \eta((a-u) - (a \cap b))$ which establishes our claim.

Now let $\{\eta(b) \mid b \in I\}$ be a family of basic closed subsets of $\eta(a)$ with the finite intersection property where I is a subset of A^+. Since $\{\eta(b) \mid b \in I\}$ satisfies finite intersection property, so is I and consider the filter L generated by I. Let M be an ultrafilter such that $L \subseteq M$. Since $b \in M$ for every $b \in I$, we have $M \in \eta(b)$ for every b which means $\bigcap_{b \in I} \eta(b) \neq \emptyset$. Therefore $\eta(a)$ is compact for each a. q.e.d.

<u>Remark 3.2: There is no difficulty in establishing that every compact open subset of $X(A)$ is of the form $\eta(a)$ for some a.</u>

We will now construct a Sheaf $\langle \mathcal{S}, \pi, X \rangle$ with $X = X(A)$ as follows. Put $\mathcal{S}(A) = \bigcup_{M \in X(A)} A/K_M$ and define $\pi: \mathcal{S}(A) \longrightarrow X(A)$ by $\pi(s) = M$ where $s = \bar{x}(K_M)$. For each $x \in A$, let $\hat{x}: X(A) \longrightarrow \mathcal{S}(A)$ be defined by $\hat{x}(M) = \bar{x}(K_M)$. Then clearly

$$\bigcup_{a \in A^+,\, x \in A} \hat{x}(\eta(a)) = \mathcal{S}(A) \text{ so that } \{\hat{x}(\eta(a)) \mid x \in A,\ a \in A^+\}$$

form a subbase for a topology on $\mathcal{S}(A)$.

<u>THEOREM 3.2</u> $\langle \mathcal{S}(A), \pi, X(A) \rangle$ <u>is a Hausdorff Sheaf of non-trivial simply ordered D.R.L. semigroups and</u> $x \longrightarrow \hat{x}$ <u>is an</u> embedding of A <u>into</u> $\Gamma(x, \mathcal{S})$ <u>where</u> $\Gamma(X, \mathcal{S})$ <u>is the set of all global sections.</u>

<u>Proof</u>: First, we shall show that each $\hat{x}$ is open-continuous and $\pi \circ \hat{x}$ = identity. Since $\hat{x}^{-1}\{\hat{y}(\eta(a))\} = \{M \in \eta(a) \mid x * y \notin M\} = \eta(a) \cap \{X(A) - \eta(x*y)\}$ which is clopen. Hence $\hat{x}$ is continuous. Clearly each $\hat{x}$ is open and $\pi \circ x$ = identity for every x. Secondly, we shall establish π as a local homeomorphism.

Let $s = \bar{x}(K_M) \in \mathcal{Y}(A)$. Then $s \in x(\eta(A))$ for some $a \in M$. Now $\hat{x} \mid \eta(a) \,.\, \eta(a) \longrightarrow \hat{x}(\eta(a))^{-1}$ is open, continuous and is a bijection. Also $\pi / \hat{x}(\eta(a)) = \{\hat{x} \mid \eta(a)\}^{-1}$ so that π is a local homeomorphism. Lastly, put $\theta : \mathcal{Y}(A) + \mathcal{Y}(A) \longrightarrow \mathcal{Y}(A)$ be defined by $\theta(s,t) = s \,\theta\, t$ where $\theta : \cup, \cap$, + or −. We shall show that θ is continuous. Let $(s,t) \in \mathcal{Y}(A) + \mathcal{Y}(A)$ so that $s = \hat{y}(M)$ and $t = \hat{z}(M)$ and $M = \pi(s) = \pi(t)$ and let $\hat{x}(\eta(a))$ be a n.h.d. of $s \cap t$. Now $M \in \eta(a)$ and since $\hat{x}(M) = \widehat{u \cap z}(M)$ it follows that there is a n.h.d. $\eta(b)$ of M on which $\hat{x} = \widehat{y \,\theta\, z}$. Now put $U = (\hat{y}(\eta(b \cap a)), \hat{z}(\eta(b \cap a))$ is a n.h.d. of (s,t) with $\theta(U) \subseteq \hat{x}\,\eta(a)$. Therefore θ is continuous. Since $\bigcap_{M \in X(A)} K_M = \{0\}$ it readily follows that $x \longrightarrow \hat{x}$ is an embedding.

That A/K_M for each M is nontrivial and simply ordered follows from the primeness of K_M. Thus $\langle \mathcal{Y}, \pi, X \rangle$ is a sheaf of nontrivial simply ordered D.R.L. semigroups and the proof of the theorem is complete.

We shall now show that sections with compact open supports, characterise A by the following:

<u>THEOREM 3.3</u>: <u>The mapping $x \longrightarrow \hat{x}$ is a subjection of A onto $\Gamma(x, \mathcal{Y})$-global sections with compact open supports.</u>

The proof of this theorem is established in the following three lemmas.

LEMMA 3.6. Given a, z A^+ with $\eta(a) \subseteq \eta(z)$, there exists $u \in A^+$ such that

$$\hat{u} = \hat{z} \text{ on } (a)$$
$$= \hat{o} \text{ outside}$$

Proof: Since $\eta(a) \subseteq \eta(z)$ we have $(z)^* \subseteq (a)^*$. Hence by (3) there exists u A^+ such that $(z*u) \cap a = 0$ and $(u)^* = (a)^*$. Now if $M \in \eta(a)$ for any M $X(A)$, then $a \in M$ so that $(z*u) \in K_M$ from which we get $\hat{z} = \hat{u}$ on M. If $M \notin \eta(a)$ then $c \cap a = 0$ for some $c \in M$ and hence $u \cap c = 0$, so that $u \in K_M$ and so $\hat{u} = \hat{0}$, at M.

q.e.d.

LEMMA 3.7: Let $x \in A$ and $a \in A^+$ with $\eta(a) \subseteq \eta(x * 0)$, then $y \in A$ can be so chosen that $\hat{y} = \hat{x}$ on $\eta(a)$

$$= \hat{0} \text{ outside}$$

Proof: If $x^+ = 0 \cup x$ and $x^- = 0 \cup (o-x)$ then $x + o \cup (0-x) = x \cup (x + (0-x)) \geq x \cup 0$ and so $x = x + 0 = \{x + 0 \cup (0-x)\} - \{o \cup (o - x)\} \geq x \cup 0 - 0 \cup (0 - x)$. Also $x \cup o - o \cup (0-x) = x \cup o - x \cup (0 - x) + (o - x \cap o) + x \cap o \geq x \cup o + x \cap o = x$, so that $x = x^+ - x^-$. Since $\eta(a \cap x^+) \subseteq \eta(x^+)$ and $\eta(a \cap \bar{x}) \subseteq \eta(\bar{x})$ it follows from the above lemma that there exist $u, v \in A^+$ such that $\hat{u} = \hat{x}^+$ on $\eta(a \cap x^+)$ and $\hat{v} = \hat{x}^-$ on $\eta(a \cap \bar{x})$

$$= \hat{o} \text{ outside} \qquad\qquad = \hat{o} \text{ on outside.}$$

Now $y = u-v$ serves our purpose. For if $M \notin \eta(a)$ then $M \notin \eta(a \cap x^+)$ and $M \notin \eta(a \cap \bar{x})$, so that $\hat{u} = \hat{o}$, $\hat{v} = \hat{o}$ at M and so $\hat{y} = \hat{o}$. Now let $M \in \eta(a)$ then either $M \in \eta(a \cap x^+)$ or $M \in \eta(a \cap x^-)$ (but not both) if $M \in \eta(a \cap x^+)$ then $x^-(M) = \hat{o}(M)$ so that $\hat{y} = \hat{u} = \hat{x}^+ - \hat{x}^- = \hat{x}$ at M a similar argument gives $\hat{y} = \hat{x}$ if $M \in \eta(a \cap \bar{x})$ q.e.d.

<u>LEMMA 3.2: If $f \in \Gamma_0(X, \mathcal{Y})$ then $f = \hat{x}$ for some $x \in A$</u>

<u>Proof</u>: Let V be the support of f. Then for each $M \in V$, there exists $x_M \in A$ such that $f(M) = \hat{x}_M(M)$ so that $f = \hat{x}_M$ on some compact open n.h.d., $\eta(a_M)$., since $f = \hat{x}_M$ on $\eta(a_M) \cap (\hat{x}_M \neq 0)$ and support of $\hat{x}_M = \eta(x \neq 0)$, we have $M \in \eta(a_M) \cap (\hat{x}_M \neq 0) \subseteq V$.

. It follows that V is open in X. Now for each $M \in V$ there exists a compact open n.h.d. V_M of M with $V_M \subseteq V$. The family $\{V_M \mid M \in V\}$ is an open cover for V so that by the partition property, we can choose $x_1, x_2, \ldots, x_n \in L$, $a_1, a_2, \ldots a_n \in A^+$ such that $V = \bigcup_{i=1}^{n} \eta(a_i)$ and $f = \hat{x}_i$ on $\eta(a_i)$ $i = 1, 2, \ldots n$.

and $\eta(a_i) \subseteq \eta(x_i \neq 0)$. By the above lemma we have $u_i \in A$ such that $\hat{u}_i = \hat{x}_i$ on $\eta(a_i)$
$= \hat{0}$ outside

Now putting $x = u_1 + u_2 + \cdots + u_n$, we have $f = \hat{x}$

q.e.d.

<u>Remark 3.3:</u> If $\langle \mathcal{Y}, \Pi, X \rangle$ <u>is any sheaf of nontrivial simply ordered O.R.L. semigroups with $\Gamma(X, \mathcal{Y})$ as the set of all sections</u> then conditions (1), (2) and (3) are satisfied in $\Gamma(X, \mathcal{Y})$.

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This work was done by the author in collaboration with C.Jayaram of Matscience.

CHEPTER 4

REMARKS ON NORMAL AND DISTRIBUTIVE * LATTICES*

Introduction.

The notions of Normal and Distributive * lattices were introduced by W.H.Cornish and T.P.Speed respectively in [1,2]. It turns out that a normal lattice is a generalisation of Stone lattice, while the class of distributive * lattice includes that of distributive pseudocomplemented lattices and hence there is some underlying relationship between these two classes, which has not been fully studied. However, the following result of W.H.Cornish is significant. "Every Normal Distributive * lattice is sectionally stone".

In this chapter, we shall make a much deeper study into the internal relationship of normal and distributive * lattices and obtain some interesting results which we present them as remarks. These remarks will in turn, exhibit the unlikelyhood of the generalisation of the results on Stone lattices or distributive pseudocomplemented lattices to Normal and Distributive * lattices.

In a distributive * - lattice L, $\forall x \in L$ there exists, an $x' \in L$ such that $x \cap x' = 0$ and $x \cup x'$ is dense. The following remark establishes that the association $x \longrightarrow x'$ can never be a mapping without L being pseudocomplemented.

<u>Remark 4.1</u>: A distributive * lattice L is pseudocomplemented iff $\forall x \in L$, $\exists$ a unique $x' \ni x \cup x'$ is dense and $x \cap x' = 0$.

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Proof: Suppose there exists a unique x' such that $x \cap x' = 0$ and $x \cup x'$ is dense, in L. We shall show that x' is the pseudocomplement of x. Let $x \cap y = 0$. Then $x \cap (y \cup x') = 0$ and $x \cup (y \cup x')$ is a dense element. Therefore $y \cup x'$ an element satisfying the conditions and hence by the uniqueness of x', we have $x' = y \cup x'$ or $y \leq x'$. Thus x' is the largest element, which is disjoint with x and hence is the pseudocomplement. Q.E.D

. . . Eventhough the association $x \longrightarrow x'$ cannot be a mapping in a distributive $*$ lattice, the following remark will establish that out of the many x'/s not even one can be complemented.

Remark 4.2: A distributive $*$ lattice L is pseudocomplemented (and hence a stone lattice) iff $\forall x \in L$ there exists a complemented x' such that $x \cap x' = 0$ and $x \cup x'$ is dense.

Proof: Suppose x' is a complemented element such that $x \cap x' = 0$ and $x \cup x'$ is dense, in a distributive $*$ lattice L. We shall show that x' is the pseudocomplement of x.

Let $x \cap y = 0$ and z be the complement of x' i.e. $x' \cup z = 1$ and $x' \cap z = 0$. Now $x \cap y \cap z = 0$ and $x' \cap y \cap z = 0$ so that $(x \cup x') \cap y \cap z = 0$. Since $x \cup x'$ is dense we get $y \cap z = 0$. So that $x' = 0 \cup x' = (y \cap z) \cup x'$ $= (y \cup x') \cap (z \cup x') = (y \cup x') \cup 1 = y \cup x'$ or $y \leq x'$. Thus x' is the pseudocomplement of x and since it is complemented L is a stone lattice. Q.E.D

Eventhough Cornish has obtained that every normal distributive * lattice is sectionally stone the following remark tells us how far a distributive * lattice can be treated as a normal lattice.

Remark 4.3. Every distributive * lattice L is a generalised normal lattice in the following sense: given $x, y \in L$ such that $x \cap y = 0$ then $\exists\, x_1, y_1 \ni x_1 \cup y_1$ is dense and $x \cap x_1 = 0, y \cap y_1 = 0$ $x \leq y_1$, $y \leq x_1$.

Proof: Let L be a distributive * lattice and let $x \cap y = 0$ for $x, y \in L$. Take $x_1 = y \cup x'$ and $y_1 = x \cup y'$. Then $x_1 \cup y_1 = y \cup x' \cup x \cup y'$ is dense and $y \leq x_1$, $x \leq y_1$ and $x \cap x_1 = x \cap (y \cup x') = (x \cap y) \cup (x \cap x') = 0$ and $y \cap y_1 = y \cap (x \cup y') = (y \cap x) \cup (y \cap y') = 0$. Q.E.D.

Remark 4.4: A normal lattice M is pseudocomplemented and hence is a stone lattice iff $x \cap y = 0$ implies there exist x_1, y_1 $1 = x_1 \cup y_1$ and $x \cap x_1 = 0 = y \cap y_1$ and $(x)^* = (y_1)^*$

Proof: Suppose M is a normal lattice such that $x \cap y = 0$ implies there exist $x_1, y_1 \in M \ni x \cap x_1 = 0 = y \cap y_1$ and $(x)^* = (y_1)^*$ for every $x, y \in M$. We shall show that M is pseudo-complemented. Suppose $x \in M$ is such that $(x)^* = (0)$. Then define the pseudocomplement x^* of x as $x^* = 0$. If $(x)^* \neq (0)$, then there exist $y \in m$ $x \cap y = 0$. Then by hypothesis there exist $x_1, y_1 \ni 1 = x_1 \cup y_1$ and $x \cap x_1 = 0 = y \cap y_1$. We shall show that x_1 is the pseudocomplement of x. Let $x \cap z = 0$. Then $z \in (x)^*$ and since $(x)^* = (y_1)^*$ we have $z \cap y_1 = 0$. Hence

$z = 1 \cap z = (x_1 \cup y_1) \cap z = (x_1 \cap z) \cup (y_1 \cap z) = x_1 \cap z.$
Therefore $z \leq x_1$. Hence x is the pseudocomplement of x. It now follows that y_1 is the pseudocomplement of x_1 since y_1 is the complement of x_1. Q.E.D.

Remark 4.5 Every complete normal lattice is pseudocomplemented

Proof: Let M be a complete normal lattice. If $x \in M$ is such that $(x)^* = (0)$ then we define the pseudocomplement x^* of x, as $x^* = 0$. If x is such that $(x)^* \neq (0)$ then there exist a $y \in M \ni x \cap y = 0$ and since M is normal, we have $x_1, y_1 \in M$ such that $x \cap x_1 = 0 = y \cap y_1$ and $1 = x_1 \cup y_1$. Consider the set N of all x_1's such that $x \cap x_1 = 0$ and $1 = x_1 \cup y_1$ and let x^* be the l.u.b of N. Then obviously x^* is the pseudocomplement of x.

Q.E.D.

Remark 4.6: Every complete distributive * lattice is a pseudocomplemented.

Proof: Obvious.

Remark 4.7: The following hold in a sectionally stone lattice.

i) $(x \cup y)^{**} = (x)^{**} \cup (y)^{**}$

ii) $(x \cap y)^{**} = (x)^{**} \cap (y)^{**}$

iii) $(x \cup y)^* = (x)^* \cap (y)^*$

iv) $(x \cap y)^* = (x)^* \cup (y)^*$

Proof: Let M be a sectionally stone lattice. Since every sectionally stone lattice is normal, we have (iv). (iii) is true in any distributive lattice. We shall prove (ii).

$(x \cap y)^{**} = \{(x)^* \cup (y)^*\}^*$ (since M is normal) $= (x)^{**} \cap (y)^{**}$

Finally, let us prove (i). It is obvious that $(x)^{**} \cup (y)^{**} \subseteq (x \cup y)^{**}$. Let $c \in (x \quad y)^{**}$. Since M is sectionally stone, we have $c = t_1 \cup t_2$ for $t_1 \in (x)^*$, $t_2 \in (x)^{**}$. Similarly $c = s_1 \cup s_2$ for $s_1 \in (y)^*$, $s_2 \in (y)^{**}$. Therefore $t_1 \cup t_2 = s_1 \cup s_2$, so that $t_1 = t_1 \cap (t_1 \cup t_2) = t_1 \cap (s_1 \cup s_2) = (t_1 \cap s_1) \cup (t_1 \cap s_2)$. Also $t_1 \cap s_1 \in (x)^* \cap (y)^*$ and $t_1 \cap s_1 \leq c$ and $c \in ((x)^* \cap (y)^*)^*$. Hence $t_1 \cap s_1 = 0$ so that $t_1 = t_1 \cap s_2 \leq s_2$. Thus $c = t_1 \cup t_2 \leq s_2 \cup t_2$. It is obvious that $s_2 \cup t_2 \leq c$ so that $c = t_2 \cup s_2 \quad (x)^{**} \cup (y)^{**}$. Hence $(x \cup y)^{**} \subseteq (x)^{**} \cup (y)^{**}$ so that $(x \cup y)^{**} = (x)^{**} \cup (y)^{**}$

Q.E.D.

An interesting corollary is the following

<u>Corollary 4.1:</u> Let M be a sectionally stone lattice and let $C(M) = \{ x \in M \mid (x) = (x)^{**} \}$. C(M) is a Boolean ring and $D(M) = \{ x \in M \mid (x)^* = 0 \}$. D(M) is a distributive lattice.

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