MATSCIENCE REPORT 82

ON FUNDAMENTAL AND INTERPOLATING SPLINE FUNCTIONS

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INTRODUCTION

This report is the outcome of a series of lectures given at 'Matscience'. Since the introduction of 'spline' functions in 1946 by I. J. Schoenberg, many new and significant results have been unearthed, of which not a few can be traced to Schoenberg himself. Consequently, it was felt that a systematic study of the contributions of I.J. Schoenberg to the theory of splines would be beneficial to the students of mathematics. This report contains results from two of his earliest papers - 1) On Pólya Frequency Functions III: The Positivity of Translation Determinants with an Application to the Interpolation Problem by Spline Curves and 2) On Pólya Frequency Functions IV: The Fundamental Spline Functions and their Limits. We will discuss later papers in subsequent reports.

C O N T E N T S

CHAPTER I:

THE FUNDAMENTAL SPLINE FUNCTIONS:

1.1. Preliminaries

Denote by $C^n(-\infty, \infty)$ the set of functions which are continuous together with their first n derivatives on the real line. Let π_n stand for the class of polynomials of degree atmost n. Let

$$---< x_{-2} < x_{-1} < x_0 < x_1 < --- < x_i < --- \quad (1.1)$$

be a sequence of reals such that $x_i \to \pm\infty$ as $i \to \pm\infty$.

DEFINITION 1.1: By a spline function $S(x)$ of degree n having the knots (1.1), we mean a function belonging to the class $C^{n-1}(-\infty, \infty)$ such that, in each interval (x_{i-1}, x_i) it reduces to a polynomial of degree not exceeding n.

1.2. Definition and properties of fundamental splines.

We define

$$M_n(x\,;y) = (n+1)(y-x)_+^n \quad (1.2)$$

where

$$(y-x)_+^n = \begin{cases} (y-x)^n & \text{if } y \geq x \\ 0 & \text{if } y < x \end{cases} \quad (1.3)$$

Consider the divided difference of order $n+1$ of the function (1.2) with respect to the variable y and based on the values $y = x_0, x_1, \ldots, x_{n+1}$. We denote this divided difference by $M_n(x; x_0, x_1, \ldots, x_{n+1})$ and its value is given by (see [6])

$$M_n(x; x_0, \ldots, x_{n+1}) = \sum_{i=0}^{n+1} \frac{(n+1)(x_i - x)_+^n}{\omega'(x_i)} \tag{1.4}$$

where

$$\omega(x) = (x - x_0) \cdots (x - x_{n+1})$$

and

$$\omega'(x_i) = (x_i - x_0) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_{n+1}).$$

The function $M_n(x; x_0, \ldots, x_{n+1})$ is called a <u>fundamental spline function.</u> In the following we shall denote it by M(x).

THEOREM 1.2. M(x) <u>is a spline of degree n having the knots</u> $x_0, x_1, \ldots, x_{n+1}$ <u>and</u>

$$M(x) = 0 \quad \underline{\text{for}} \quad x \notin (x_0, x_{n+1}) \tag{1.5}$$

Moreover, if $f(x) \in C^{n+1}[x_0, x_{n+1}]$ then

$$f(x_0, x_1, \ldots, x_{n+1}) = \frac{1}{\lfloor n+1} \int_{x_0}^{x_{n+1}} M(x) f^{(n+1)}(x)\,dx \qquad (1.6)$$

and

$$\int_{-\infty}^{\infty} M(x)\,dx = 1 \qquad (1.7)$$

In order to prove Theorem 1.2, we need the following.

THEOREM 1.3. ([2]) Let a linear functional L given by

$$L(f) = \int_{x_0}^{x_{n+1}} [a_0(x) f(x) + \cdots + a_n(x) f^{(n)}(x)]\,dx$$

$$+ \sum_{i=0}^{j_0} b_{i0} f(x_{i0}) + \cdots + \sum_{i=1}^{j_n} b_{in} f^{(n)}(x_{in}) \qquad (1.8)$$

be defined over the class $C^{n+1}[x_0, x_{n+1}]$ where the functions $a_i(x)$ are assumed to be piecewise continuous on $[x_0, x_{n+1}]$ and the points x_{ij} lie in $[x_0, x_{n+1}]$. Suppose that

$$L(p) = 0 \text{ for all } p \in \Pi_n. \qquad (1.9)$$

then for all $f \in C^{n+1}[x_0, x_{n+1}]$

$$L(f) = \int_{x_0}^{x_{n+1}} f^{(n+1)}(t)\, K(t)\, dt \tag{1.10}$$

where

$$K(t) = \frac{1}{\underline{|n}}\, L_x[(x-t)_+^n] \tag{1.11}$$

The notation $L_x[(x-t)_+^n]$ means that the functional L is applied to $(x-t)_+^n$ considered as a function of x.

Proof of Theorem 1.2. From (1.4) we have

$$M(x) = \sum_{i=0}^{n+1} \frac{(n+1)(x_i - x)_+^n}{\omega'(x_i)}.$$

In order to prove that M(x) is a spline of degree n with the knots $x_0, ---, x_{n+1}$, we have to show that M(x) reduces to a polynomial of degree less than or equal to n in each of the intervals $(-\infty, x_0), (x_0, x_1), ---, ---, (x_n, x_{n+1}), (x_{n+1}, \infty)$ and that, $M(x) \in C^{n-1}(-\infty, \infty)$.

Now

$$M(x) = \frac{(n+1)(x_0 - x)^n}{\omega'(x_0)} + ----$$

$$+ --- + \frac{(n+1)(x_{n+1} - x)^n}{\omega'(x_{n+1})} \tag{1.12}$$

for $x \in (-\infty, x_0)$

and by (1.3) we have

$$M(x) = 0 \quad \text{for} \quad x \in (x_{n+1}, \infty) \qquad (1.13)$$

In (x_{i-1}, x_i), $i = 1, 2, ---, n+1$,

$$M(x) = \sum_{j=i}^{n+1} \frac{(n+1)(x_j - x)^n}{\omega'(x_j)} \qquad (1.14)$$

Thus $M(x) \in \Pi_n$ in each of the subintervals $(-\infty, x_0), (x_0, x_1), ---, (x_n, x_{n+1}), (x_{n+1}, \infty)$. Consequently, $M(x)$ is continuous together with its first n-1 derivatives in each of the intervals $(-\infty, x_0)$, (x_0, x_1), ------ , (x_{n+1}, ∞). So we have to check the continuity of M(x) and its derivatives upto the order n-1 only at the knots $x_0, ---, x_{n+1}$. Since

$$M^{(p)}(x) = \sum_{j=i}^{n+1} \frac{(-1)^p (n+1) n \cdots (n-p+1)(x_j - x)^{n-p}}{\omega'(x_j)}$$

$$\text{for } x \in (x_{i-1}, x_i) \text{ and } p = 0, 1, ---, n-1, \qquad (1.15)$$

the left and right limits of $M^{(p)}(x; x_0, ---, x_{n+1})$ at $x = x_i$ are given by

$$M^{(p)}(x_i+) = \sum_{j=i+1}^{n+1} \frac{(-1)^p (n+1) n --- (n-p+1)(x_j - x_i)^{n-p}}{\omega'(x_j)} \qquad (1.16)$$

and

$$M^{(p)}(x_i -) = \sum_{j=i}^{n} \frac{(-1)^p (n+1)\, n \cdots (n-p+1)(x_j - x_i)^{n-p}}{\omega'(x_j)} \qquad (1.17)$$

From (1.16) and (1.17) we see that

$$M^{(p)}(x_i +) = M^{(p)}(x_i -) \quad \text{for } p = 0, 1, \cdots, n-1$$

Hence $M(x) \in C^{n-1}(-\infty, \infty)$.

From (1.13) we see that

$$M(x) = 0 \qquad \text{for } x \geq x_{n+1} .$$

For $x \leq x_0$, from (1.12), we have

$$M(x) = \sum_{j=0}^{n+1} \frac{(n+1)(x_j - x)^n}{\omega'(x_j)} . \qquad (1.18)$$

The expression (1.18) is the divided difference of order n+1 of the function $(n+1)(y-x)^n$ i.e. (n+1)th order divided difference of a polynomial of degree n and hence mustbe equal to zero. Thus

$$M(x) = 0 \qquad \text{for } x \leq x_0 . \qquad (1.19)$$

Since the divided difference of order n+1 is an operator of the form (1.8), and satisfies the conditions of Theorem (1.3), we see that for $f \in C^{n+1}[x_0, x_{n+1}]$;

$$f(x_0, \ldots, x_{n+1}) = \int_{x_0}^{x_{n+1}} f^{(n+1)}(t)\left(\frac{1}{\lfloor n}\sum_{i=0}^{n+1}\frac{(x_i - t)_+^n}{\omega'(x_i)}\right)dt$$

$$= \frac{1}{\lfloor n+1} \int_{x_0}^{x_{n+1}} M(t)\, f^{(n+1)}(t)\, dt.$$

(1.20)

In particular, if $f(x) = x^{n+1}$, (1.20) reduces to

$$1 = \int_{x_0}^{x_{n+1}} M(t)\, dt.$$

Since M(x) vanishes outside (x_0, x_{n+1}),

$$\int_{-\infty}^{\infty} M(t)\, dt = 1.$$

This completes the proof of the theorem.

THEOREM 1.4: <u>The</u> p^{th} <u>derivative</u> $M^{(p)}(x)$ $(p = 0, 1, \ldots, n-1)$ <u>has exactly</u> p <u>distinct simple zeros in the interval</u> (x_0, x_{n+1}). <u>In particular</u>, $M(x) > 0$ <u>in</u> (x_0, x_{n+1}).

PROOF: We will prove this theorem by repeated application of the mean value theorem in suitable subintervals of $[x_0, x_{n+1}]$. First we prove that $M^{(p)}(x_0) = 0 = M^{(p)}(x_{n+1})$ for $p = 0, 1, \ldots, n-1$. From (1.4) we have

$$M(x) = \sum_{i=0}^{n+1} \frac{(n+1)(x_i - x)_+^n}{\omega'(x_i)} .$$

Since, for $n > 1$,

$$\frac{d}{dx}(x_+^n) = x_+^{n-1} ,$$

we have

$$M^{(p)}(x) = \sum_{i=0}^{n+1} \frac{(-1)^p (n+1) n \cdots (n-p+1)(x_i - x)_+^{n-p}}{\omega'(x_i)}$$

$$\text{for} \quad p = 0, 1, \cdots, n-1 . \tag{1,21}$$

Hence, for $x \geq x_{n+1}$, by (1.3) we have

$$M^{(p)}(x) = 0 , \qquad p = 0, 1, \cdots, n-1 . \tag{1.22}$$

In particular,

$$M^{(p)}(x_{n+1}) = 0 \text{ for } \quad p = 0, 1, \cdots, n-1 . \tag{1.25}$$

Now if $x \leq x_0$,

$$M^{(p)}(x) = \sum_{i=0}^{n+1} \frac{(-1)^p (n+1) n \cdots (n-p+1)(x_i - x)_+^n}{\omega'(x_i)} \tag{1.24}$$

But (1.24) represents the (n+1)th order divided difference, taken with respect to y, of the polynomial $(-1)^p(n+1)n\ldots(n-p+1)(y-x)^{n-p}$ which is a polynomial of degree $\leq n$ in y. Thus, if $x \leq x_0$, we have

$$M^{(p)}(x) = 0 \quad \text{for} \quad p = 0, 1, \ldots, n-1. \tag{1.25}$$

In particular,

$$M^{(p)}(x_0) = 0 \quad \text{for} \quad p = 0, 1, \ldots, n-1, \tag{1.26}$$

Now

$$M(x) = \frac{(n+1)(x_{n+1} - x)^n}{\omega'(x_{n+1})} \quad \text{if } x_n < x < x_{n+1} \tag{1.27}$$

Hence $M(x) > 0$ in (x_n, x_{n+1}). Let $t_0 \in (x_n, x_{n+1})$. Then $M(t_0) > 0$. We can now apply the mean value theorem to prove the dxistence of a point $t_1 \in (t_0, x_{n+1})$ and a point $t_2 \in (x_0, t_0)$ such that

$$M'(t_1) < 0 \quad \text{and} \quad M'(t_2) > 0.$$

From this we conclude that the function $M'(x)$ has at least one dhange of sign in (x_0, x_{n+1}). Since $M'(x)$ is continuousand derivable, we can apply the mean-value theorem to it and find points $t_3 \in (t_2, x_{n+1})$, $t_4 \in (t_1, t_2)$

and $t_5 \in (x_0, t_1)$ such that

$$M''(t_3) < 0, \quad M''(t_4) > 0, \quad M''(t_5) < 0.$$

Thus the function $M''(x)$ has at least two changes of sign in (x_0, x_{n+1}). Continuing in like manner, we can show that $M^{(n-1)}(x)$ has at least n-1 changes of sign in (x_0, x_{n+1}).

On the other hand, differentiation of M(x) gives

$$M^{(n-1)}(x) = (-1)^{n-1} \lfloor\!\underline{n+1} \sum_{i=0}^{n+1} \frac{(x_i - x)_+}{\omega'(x_i)} . \tag{1.28}$$

From (1.22) and (1.25) we find that

$$M^{(n-1)}(x) = 0 \text{ for } x \geq x_{n+1} \text{ and for } x \leq x_0 \tag{1.29}$$

For $x_{i-1} < x < x_i$,

$$M^{(n-1)}(x) = (-1)^{n-1} \lfloor\!\underline{n+1} \sum_{j=i}^{n+1} \frac{(x_j - x)}{\omega'(x_j)}$$

and for $x_i < x < x_{i+1}$,

$$M^{(n-1)}(x) = (-1)^{n-1} \lfloor\!\underline{n+1} \sum_{j=i+1}^{n+1} \frac{(x_j - x)}{\omega'(x_j)}$$

Hence

$$M^{(n-1)}(x_i +) = M^{(n-1)}(x_i -), \quad i = 0, 1, \ldots, n+1.$$

Thus the graph of $M^{(n-1)}(x)$ is a continuous polygonal line having vertices at $x_0, x_1, \ldots, x_{n+1}$ and vanishing outside (x_0, x_{n+1}). The graph of $M^{(n-1)}(x)$ between three consecutive knots must be represented by any one of the following six figures:

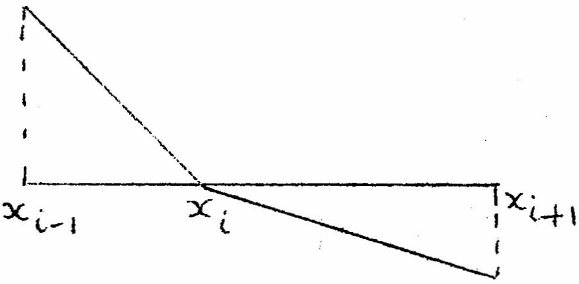


Fig 1

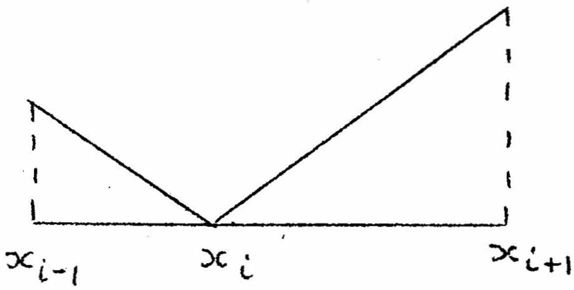


Fig 2

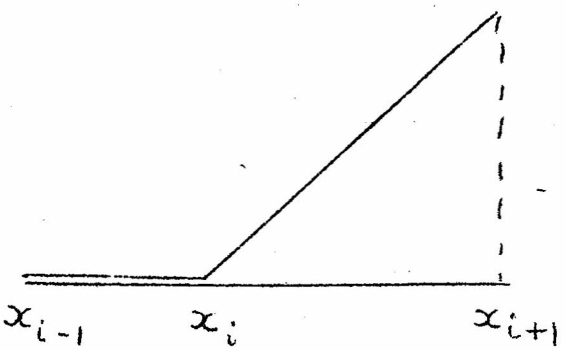


Fig 3

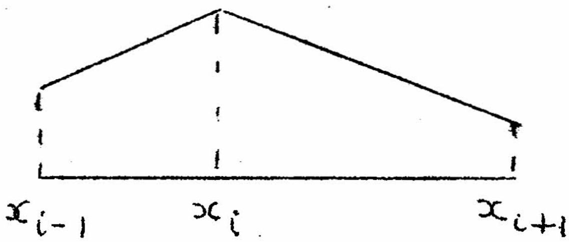


Fig 4

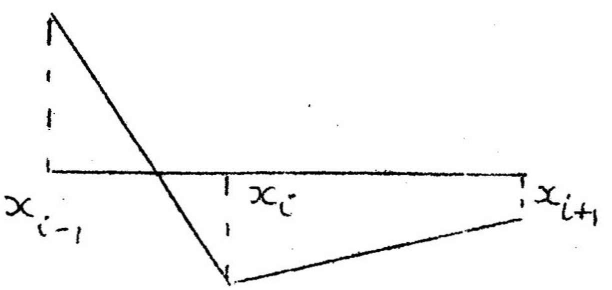


Fig 5

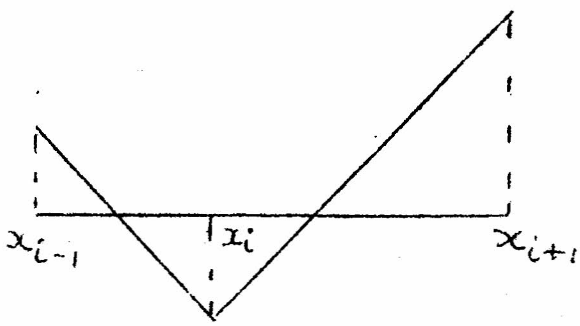


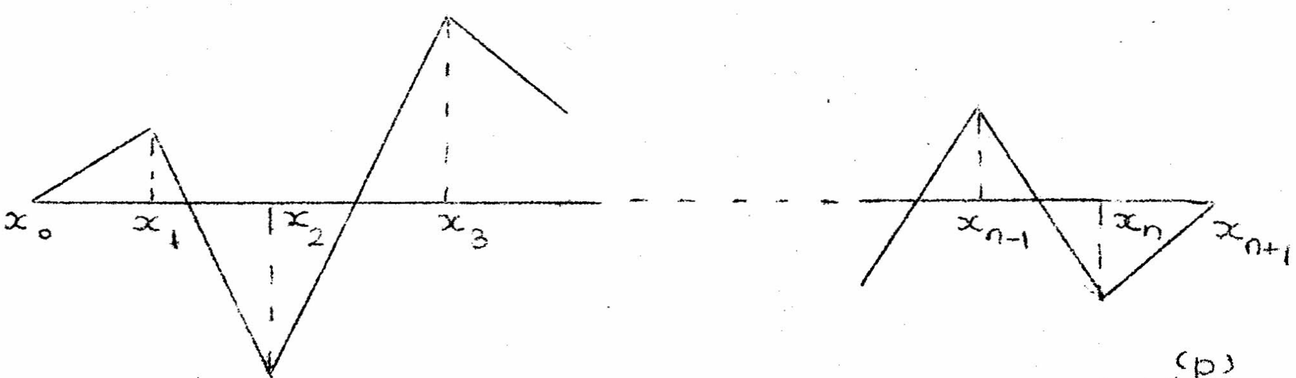
Fig 6

It is evident that $M^{(n-1)}(x)$ has the maximum number of changes of sign only when it is non-zero at $x_1, ---, x_n$ and

$$\operatorname{sign}[M^{(n-1)}(x_i)] = -\operatorname{sign}[M^{(n-1)}(x_{i+1})],$$

$$i = 1, 2, ---, n-1.$$

Thus $M^{(n-1)}(x)$ cannot have more than n-1 zeros in (x_0, x_{n+1}). We have already proved that $M^{(n-1)}(x)$ has at least n-1 zeros in (x_0, x_{n+1}). Hence we conclude that $M^{(n-1)}(x)$ has exactly n-1 zeros in (x_0, x_{n+1}) and the graph of $M^{(n-1)}(x)$ is represented by the following figure



The theorem now follows easily. We have seen above that $M^{(p)}(x)$ has at least p distinct zeros in (x_0, x_{n+1}). Suppose that $M^{(p)}(x)$ has p+1 zeros, say at $r_1, r_2, ---, r_{p+1}$. By considering only p of these zeros and by the previous argument, we are assured of the existence of points $s_1, s_2, ---, s_{p+1}$ at which $M^{p+1}(x)$ vanishes. Picking the additional zero of $M^{(p)}(x)$ in the set $\{t_i\}_1^{p+1}$ and the zero closest to it and applying Rolle's theorem to the function $M^{(p)}(x)$, we can find an additional point s_{p+2}

which does not coincide with any of $\{\delta_i\}_1^{p+1}$ and which is such that $M^{(p+1)}(\delta_{p+2}) = 0$. Continuing this process, we will arrive at a contradiction to the fact that $M^{(n-1)}(x)$ has exactly n-1 zeros. Now suppose that $M^{(p)}(x)$ has p+1 zeros situated at the points $r_1, r_2, \ldots, r_p$, i.e. one of the points $\{r_i\}_1^p$ is a double zero for $M^{(p)}(x)$. In other words, $M^{(p+1)}(x)$ has p+2 zeros and this will lead to a contradiction as before. Hence we conclude that $M^{(p)}(x)$ has exactly p distinct zeros in (x_0, x_{n+1}). From (1.27) we have

$$M(x) = \frac{(n+1)(x_{n+1} - x)^n}{\omega'(x_{n+1})} > 0 \qquad \text{in } (x_n, x_{n+1})$$

Since M(x) has no zeros in (x_0, x_{n+1}) the above relation gives

$$M(x) > 0 \qquad \text{in} \qquad (x_0, x_{n+1}).$$

Remark 1.5. A function f(x) is called a frequency function if

$$f(x) \geq 0 \quad \text{for all } x$$

and

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

From the relation (1.7) and Theorem 1.4, we see that M(x) is a frequency function.

CHAPTER II

A BASIS FOR SPLINE FUNCTIONS

2.1. Preliminaries.

In the previous chapter we had assumed that the knots $\{x_i\}_{-\infty}^{\infty}$ of the spline function were distinct, i.e.

$$\cdots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \cdots < x_i < \cdots \quad (2.1)$$

with $x_i \longrightarrow \pm\infty$ as $i \longrightarrow \pm\infty$. In this chapter, we allow some of the knots $\{x_i\}_{-\infty}^{\infty}$ to coincide. The precise mathematical meaning of multiple knots is given by the following

DEFINITION 2.1: We say that $x = x_0$ is a knot of multiplicity r for the spline function S(x) if

$$\cdots < x_{-2} < x_{-1} < x_0 = x_1 = x_2 = \cdots = x_{r-1} < x_r < \cdots \quad (2.2)$$

and

$$S(x) \in C^{n-r}(x_{-1}, x_r) \quad (2.3)$$

while

$$S(x) \in \Pi_n \text{ in each of the intervals } (x_{i-1}, x_i) \quad (2.4)$$

In particular, the case when $r = n+1$ means that there are no continuity requirements whatever, at $x = x_0$, between the two polynomials defining $S(x)$ in the adjacent intervals (x_{-1}, x_0) and (x_0, x_r). The multiplicity r of a knot is therefore restricted by the inequality

$$r \leq n+1 \tag{2.5}$$

which we assume throughout this chapter.

We now generalize our class of spline functions $S(x)$ by allowing in (2.1)

$$x_i < x_{i+n+1} \quad \text{for all } i \tag{2.6}$$

We shall assume that the knots (2.1) are located at the distinct points

$$\cdots < y_{-1} < y_0 < y_1 < y_2 < \cdots, \ y_n \to \pm\infty \text{ as } n \to \pm\infty \tag{2.7}$$

where

$$y_i \text{ is a knot of multiplicity } \alpha_i, \quad (\alpha_i \leq n+1) \tag{2.8}$$

Our discussion requires a kind of 'double book-keeping' where multiple knots (2.7) are also represented by the non-decreasing sequence $\{x_i\}$ with the correct multiplicities. To fix the ideas, we assume this representation such that

$$\cdots, x_0 = y_0, x_1 = x_2 = \cdots = x_{\alpha_1} = y_1, x_{\alpha_1+1} = y_2, \cdots \tag{2.9}$$

Thus in terms of the original sequence (2.1) the knots are

$$\left.\begin{aligned} &\cdots < x_{-\alpha_0+1} = \cdots = x_0 < x_1 = \cdots = x_{\alpha_1} \\ &< x_{\alpha_1+1} = \cdots = x_{\alpha_1+\alpha_2} < \cdots \end{aligned}\right\} \qquad (2.10)$$

Again, as in the previous chapter, we associate with the knots (2.10) the sequence of fundamental splines

$$M_j(x) = M_n(x; x_j, x_{j+1}, \ldots, x_{j+n+1}) \qquad (2.11)$$

$$(-\infty < x < \infty)$$

2.2. An example of a fundamental spline with multiple knots.

As an illustration of the change in the representation of the spline function when simple knots are replaced by multiple knots, we consider the extreme case when

$$x_0 = x_1 = \cdots = x_n < x_{n+1} = \xi \qquad (2.12)$$

From Newton's general interpolation formula written in terms of divided differences, we obtain for $f \in C^{n+1}$

$$\begin{aligned} f(\xi) = f(x_0) &+ (\xi - x_0) f(x_0, x_0) + \cdots \\ &+ \frac{(\xi - x_0)^n}{n!} f(x_0, \ldots, x_0) \\ &+ (\xi - x_0)^{n+1} f(x_0, \ldots, x_0, \xi) \end{aligned} \qquad (2.13)$$

When $x_0 = x_1 = \cdots = x_r$, the divided difference

$$f(x_0, x_1, \ldots, x_r) = f(x_0, \ldots, x_0) = \frac{f^{(r)}(x_0)}{\lfloor r} \qquad (2.14)$$

Hence (2.13) reduces to

$$f(\xi) = f(x_0) + (\xi - x_0) f'(x_0) + \cdots + \frac{(\xi - x_0)^n}{\lfloor n} f^{(n)}(x_0)$$

$$+ (\xi - x_0)^{n+1} f(x_0, \ldots, x_0, \xi) \qquad (2.15)$$

But by Taylor's theorem

$$f(\xi) = f(x_0) + (\xi - x_0) f'(x_0) + \cdots + \frac{(\xi - x_0)^n}{\lfloor n} f^{(n)}(x_0)$$

$$+ \frac{1}{\lfloor n} \int_{x_0}^{\xi} f^{(n+1)}(t) (\xi - t)^n \, dt. \qquad (2.16)$$

From (2.15) and (2.16) we have

$$(\xi - x_0)^{n+1} f(x_0, \ldots, x_0, \xi) = \frac{1}{\lfloor n} \int_{x_0}^{\xi} f^{(n+1)}(t)(\xi - t)^n \, dt$$

$$f(x_0, \ldots, x_0, \xi) = \frac{1}{\lfloor n+1} \int_{x_0}^{x_{n+1}} \frac{(n+1)(\xi - x)^n}{(\xi - x_0)^{n+1}} f^{(n+1)}(x) \, dx$$

(from (2.12)) (2.17)

We have proved in Theorem 1.2 that

$$f(x_0, ---, x_0, \xi) = \frac{1}{\lfloor n+1} \int_{x_0}^{x_{n+1}} M_n(x; x_0, ---, x_0, \xi) f^{n+1}(x) dx \tag{2.18}$$

From (2.17) and (2.18) we conclude that

$$M_n(x; x_0, ---, x_0, \xi) = \begin{cases} \frac{(n+1)(\xi - x)^n}{(\xi - x_0)^{n+1}} & \text{in } (x_0, \xi) \\ 0 & \text{outside } (x_0, \xi) \end{cases}$$

2.3 <u>The minimum number of knots necessary for the existence of a non-trivial spline</u>.

Suppose that S(x) is a spline of degree n vanishing outside an open interval. In this section we prove that if the sum of the knots (including multiplicities) does not exceed the degree of the spline plus one, then S(x) is identically equal to zero. In order to prove this result, we need the following

LEMMA 2.2. <u>Let</u> S(x) <u>be a spline of degree</u> n <u>having the knots</u>

$$--- < y_{-1} < y_0 < y_1 < y_2 < --- \quad (y_n \to \pm\infty \text{ as } n \to \pm\infty) \tag{2.19}$$

<u>where</u>

$$y_i \text{ is a knot of multiplicity } \alpha_i \tag{2.20}$$

If

S(x) vanishes outside the interval (y_1, y_N) (2.21)

then S(x) has the representation

$$S(x) = \sum_{i=0}^{\alpha_1 - 1} a_i^1 (x-y_1)_+^{n-i} + \cdots + \sum_{i=0}^{\alpha_N - 1} a_i^N (x-y_N)_+^{n-i} \tag{2.22}$$

$$(-\infty < x < \infty)$$

where

$$\sum_{i=0}^{\alpha_1 - 1} a_i^1 (x-y_1)^{n-i} + \cdots + \sum_{i=0}^{\alpha_N - 1} a_i^N (x-y_N)^{n-i} = 0$$

for all x (2.23)

PROOF: By the definition of a spline of degree n, S(x) is a polynomial of degree atmost n in (y_i, y_{i+1}), $i = 0, 1, - - - - -, N$ with $y_0 = -\infty$ and $y_{N+1} = +\infty$. Let $p_{n\ell}$ denote the polynomial to which the spline S(x) reduces in $(y_\ell, y_{\ell+1})$. Suppose that

$$p_{n1}(x) = \sum_{i=0}^{n} a_i (x-y_1)^{n-i} . \tag{2.24}$$

From (2.21)

$$p_{n0}(x) \equiv 0 \tag{2.25}$$

From (2.20), since y_1 is a knot of multiplicity α_1 for S(x),

$$S(x) \in C^{n-\alpha_1}(-\infty, y_2) . \tag{2.26}$$

Consequently,

$$p_{n1}(y_1+) = p_{no}(y_1-), \quad p'_{n1}(y_1+) = p'_{no}(y_1-),$$
$$- - - - , \quad p_{n1}^{(n-\alpha_1)}(y_1+) = p_{no}^{(n-\alpha_1)}(y_1-) .$$

Thus we have

$$a_n' = a_{n-1}' = - - - = a_{\alpha_1}' = 0$$

and $p_{n1}(x)$ reduces to

$$p_{n1}(x) = \sum_{i=0}^{\alpha_1 - 1} a_i'(x - y_1)^{n-i} , \tag{2.27}$$

Since p_{n2} and p_{n1} are polynomials of degree atmost n, $p_{n2} - p_{n1}$ is also a polynomial of degree n and has the representation

$$p_{n2}(x) - p_{n1}(x) = \sum_{i=0}^{n} a_i^2(x - y_2)^{n-i} \tag{2.28}$$

Since y_2 is a knot of multiplicity α_2 for S(x),

$$S(x) \in C^{n-\alpha_2}(y_1, y_3) \tag{2.29}$$

Hence we must have

$$p_{n2}(y_2+) = p_{n1}(y_2-),\ p'_{n2}(y_2+) = p'_{n1}(y_2-),$$
$$- - - - -,\ p_{n2}^{(n-\alpha_2)}(y_2+) = p_{n1}^{(n-\alpha_2)}(y_2-).$$

Thus

$$a_n^2 = a_{n-1}^2 = - - - = a_{\alpha_2}^2 = 0.$$

and

$$p_{n2}(x) - p_{n1}(x) = \sum_{i=0}^{\alpha_2-1} a_i^2 (x-y_2)^{n-i}. \qquad (2.30)$$

Proceeding in this manner, we get

$$p_{n\ell}(x) - p_{n,\ell-1}(x) = \sum_{i=0}^{\alpha_\ell-1} a_i^\ell (x-y_\ell)^{n-i}, \qquad (2.31)$$

$$\ell = 2, 3, ---, N.$$

Now

$$p_{nN}(x) = p_{n1}(x) + (p_{n2}(x) - p_{n1}(x)) + - - -$$
$$+ (p_{nN}(x) - p_{n,N-1}(x))$$
$$= \sum_{i=0}^{\alpha_1-1} a_i^1 (x-y_1)^{n-i} + \sum_{i=0}^{\alpha_2-1} a_i^2 (x-y_2)^{n-i} +$$
$$+ - - - + \sum_{i=0}^{\alpha_N-1} a_i^N (x-y_N)^{n-i}$$

(from (2.27) and (2.31)) (2.32)

and $p_{nN}(x)$ is the value of S(x) in (y_N, ∞). But by the hypothesis, S(x) should vanish for $x \geqslant y_N$ and hence we must have

$$p_{nN}(x) = \sum_{i=0}^{\alpha_1 - 1} a_i^1 (x-y_1)^{n-i} + \sum_{i=0}^{\alpha_2 - 1} a_i^2 (x-y_2)^{n-i}$$

$$+ \cdots + \sum_{i=0}^{\alpha_N - 1} a_i^N (x-y_N)^{n-i} = 0 \quad \text{for all } x \tag{2.33}$$

Consider the function

$$F(x) = \sum_{i=0}^{\alpha_1 - 1} a_i^1 (x-y_1)_+^{n-i} + \sum_{i=0}^{\alpha_2 - 1} a_i^2 (x-y_2)_+^{n-i}$$

$$+ \cdots + \sum_{i=0}^{\alpha_N - 1} a_i^N (x-y_N)_+^{n-i}, \quad (-\infty < x < \infty) \tag{2.34}$$

the function $(x - y_i)_+^n$ being defined as in (1.3). It is evident that the function F(x) whose equation is given by (2.34) and which satisfies the condition (2.33), vanishes outside (y_1, y_N) and reduces to $p_{n\ell}(x)$ in the interval $(y_\ell, y_{\ell+1})$, $\ell = 1, \ldots, N-1$. Also

$$F(x) \in C^{n-\alpha_\ell}(y_{\ell-1}, y_{\ell+1}),$$

i.e. y_ℓ is a knot of multiplicity α_ℓ for F(x). Hence we conclude that

$$F(x) \equiv S(x)$$

and the lemma is proved.

THEOREM 2.3: Suppose that $S(x)$ is a spline of degree n having the knots

$$\cdots < y_{-1} < y_0 < y_1 < y_2 < \cdots \quad (y_n \to \pm\infty \text{ as } n \to \pm\infty) \tag{2.35}$$

where

$$y_i \text{ is a knot of multiplicity } \alpha_i \tag{2.36}$$

If

$$u = \sum_{i=1}^{N} \alpha_i \leq n+1 \tag{2.37}$$

and

$$S(x) = 0 \text{ every-where outside the interval } (y_1, y_N) \tag{2.38}$$

then

$$S(x) = 0 \text{ for all } x.$$

PROOF: From Lemma 2.2 we know that the spline $S(x)$ has the representation

$$S(x) = \sum_{i=0}^{\alpha_1 - 1} a_i^1 (x - y_1)_+^{n-i} + \sum_{i=0}^{\alpha_2 - 1} a_i^2 (x - y_2)_+^{n-i} +$$

$$+ \cdots + \sum_{i=0}^{\alpha_N - 1} a_i^N (x - y_N)_+^{n-i} \quad (-\infty < x < \infty) \tag{2.39}$$

ere

$$\sum_{i=0}^{\alpha_1-1} a_i^1 (x-y_1)^{n-i} + \sum_{i=0}^{\alpha_2-1} a_i^2 (x-y_2)^{n-i} + \cdots$$

$$+ \cdots + \sum_{i=0}^{\alpha_N-1} a_i^N (x-y_N)^{n-i} = 0 \quad \text{for all } x. \tag{2.40}$$

ase (i) $\quad u = \sum_{i=1}^{N} \alpha_i = n+1$.

n order to prove that $S(x) \equiv 0$, we will show that the dentity (2.40) implies that each $a_i^j = 0$. It is sufficient o prove that the polynomials

$$\left.\begin{array}{l}
\dfrac{(x-y_1)^n}{\lfloor n}, \; \dfrac{(x-y_1)^{n-1}}{\lfloor n-1}, \; \cdots, \; \dfrac{(x-y_1)^{n-\alpha_1+1}}{\lfloor n-\alpha_1+1}, \\
- \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \\
\dfrac{(x-y_N)^n}{\lfloor n}, \; \dfrac{(x-y_N)^{n-1}}{\lfloor n-1}, \; \cdots, \; \dfrac{(x-y_N)^{n-\alpha_N+1}}{\lfloor n-\alpha_N+1}
\end{array}\right\} \tag{2.41}$$

are linearly independent. Now

$$\frac{(x-y_1)^n}{\lfloor n} = \frac{1}{\lfloor n} x^n - \frac{y_1}{\lfloor 1 \lfloor n-1} x^{n-1} + \cdots$$

$$+ \frac{(-1)^{n-1} y_1^{n-1}}{\lfloor n-1 \lfloor 1} x + \frac{(-1)^n y_1^n}{\lfloor n},$$

$$\frac{(x-y_1)^{n-1}}{\lfloor\underline{n-1}} = \frac{1}{\lfloor\underline{n-1}}\,x^{n-1} - \frac{y_1}{\lfloor\underline{1}\;\lfloor\underline{n-2}}\,x^{n-2} + \cdots$$

$$+ (-1)^{n-2}\frac{y_1^{n-2}}{\lfloor\underline{n-2}\;\lfloor\underline{1}}\,x + (-1)^{n-1}\frac{y_1^{n-1}}{\lfloor\underline{n-1}},$$

— — — — — — — — — — —

$$\frac{(x-y_1)^{n-\alpha_1+1}}{\lfloor\underline{n-\alpha_1+1}} = \frac{1}{\lfloor\underline{n-\alpha_1+1}}\,x^{n-\alpha_1+1} - \frac{y_1}{\lfloor\underline{1}\;\lfloor\underline{n-\alpha_1}}\,x^{n-\alpha_1}$$

$$+ \cdots + (-1)^{n-\alpha_1}\frac{y_1^{n-\alpha_1}}{\lfloor\underline{n-\alpha_1}\;\lfloor\underline{1}}\,x + (-1)^{n-\alpha_1+1}\frac{y_1^{n-\alpha_1+1}}{\lfloor\underline{n-\alpha_1+1}}$$

— — — — — — — — — — —

— — — — — — — — — — —

— — — — — — — — — — —

$$\frac{(x-y_N)^{n}}{\lfloor\underline{n}} = \frac{1}{\lfloor\underline{n}}\,x^{n} - \frac{y_N}{\lfloor\underline{1}\;\lfloor\underline{n-1}}\,x^{n-1} + \cdots$$

$$+ (-1)^{n-1}\frac{y_N^{n-1}}{\lfloor\underline{n-1}\;\lfloor\underline{1}}\,x + (-1)^{n}\frac{y_N^{n}}{\lfloor\underline{n}},$$

$$\frac{(x-y_N)^{n-1}}{\lfloor\underline{n-1}} = \frac{1}{\lfloor\underline{n-1}}\,x^{n-1} - \frac{y_N}{\lfloor\underline{1}\;\lfloor\underline{n-2}}\,x^{n-2} + \cdots$$

$$+ (-1)^{n-2}\frac{y_N^{n-2}}{\lfloor\underline{n-2}\;\lfloor\underline{1}}\,x + (-1)^{n-1}\frac{y_N^{n-1}}{\lfloor\underline{n-1}},$$

— — — — — — — — — — — — —

$$\frac{(x-y_N)^{n-\alpha_N+1}}{\lfloor\underline{n-\alpha_N+1}} = \frac{1}{\lfloor\underline{n-\alpha_N+1}}\, x^{n-\alpha_N+1} - \frac{y_N}{\lfloor\underline{1}\;\lfloor\underline{n-\alpha_N}}\, x^{n-\alpha_N} + \cdots + \frac{(-1)^{n-\alpha_N}\, y_N^{n-\alpha_N}}{\lfloor\underline{n-\alpha_N}\;\lfloor\underline{1}}\, x + \frac{(-1)^{n-\alpha_N+1}\, y_N^{n-\alpha_N+1}}{\lfloor\underline{n-\alpha_N+1}}.$$

e find that the (n+1) x (n+1) determinant of the coefficients of $1, x, \ldots x^n$ in the expansion of the polynomials (2.41) is he following

$$) = \left|\begin{array}{ccccc}
\frac{(-1)^n y_1^n}{\lfloor\underline{n}} & \frac{(-1)^{n-1} y_1^{n-1}}{\lfloor\underline{n-1}\;\lfloor\underline{1}} & \cdots & \frac{-y_1}{\lfloor\underline{1}\;\lfloor\underline{n-1}} & \frac{1}{\lfloor\underline{n}} \\
\frac{(-1)^{n-1} y_1^{n-1}}{\lfloor\underline{n-1}} & \frac{(-1)^{n-2} y_1^{n-2}}{\lfloor\underline{n-2}\;\lfloor\underline{1}} & \cdots & \frac{1}{\lfloor\underline{n-1}} & 0 \\
- & - & - & - & - \\
\frac{(-1)^{n-\alpha_1+1} y_1^{n-\alpha_1+1}}{\lfloor\underline{n-\alpha_1+1}} & \frac{(-1)^{n-\alpha_1} y_1^{n-\alpha_1}}{\lfloor\underline{n-\alpha_1}\;\lfloor\underline{1}} & \cdots & 0 & 0 \\
- & - & - & - & - \\
- & - & - & - & - \\
- & - & - & - & - \\
\frac{(-1)^n y_N^n}{\lfloor\underline{n}} & \frac{(-1)^{n-1} y_N^{n-1}}{\lfloor\underline{n-1}\;\lfloor\underline{1}} & \cdots & \frac{-y_N}{\lfloor\underline{1}\;\lfloor\underline{n-1}} & \frac{1}{\lfloor\underline{n}} \\
\frac{(-1)^{n-1} y_N^{n-1}}{\lfloor\underline{n-1}} & \frac{(-1)^{n-2} y_N^{n-2}}{\lfloor\underline{n-2}\;\lfloor\underline{1}} & \cdots & \frac{1}{\lfloor\underline{n-1}} & 0 \\
- & - & - & - & - \\
\frac{(-1)^{n-\alpha_N+1} y_N^{n-\alpha_N+1}}{\lfloor\underline{n-\alpha_N+1}} & \frac{(-1)^{n-\alpha_N} y_N^{n-\alpha_N}}{\lfloor\underline{n-\alpha_N}\;\lfloor\underline{1}} & \cdots & 0 & 0
\end{array}\right|$$

Suppose that

$$D = 0 \tag{3.42}$$

Then there exist a set of non-zero numbers $A_0, A_1, ---, A_n$

such that

$$\frac{A_0\, y_1^{n}}{\underline{|n}} + \frac{A_1\, y_1^{n-1}}{\underline{|n-1}} + --- + A_{n-1} y_1 + A_n = 0$$

$$A_0 \frac{y_1^{n-1}}{\underline{|n-1}} + A_1 \frac{y_1^{n-2}}{\underline{|n-2}} + --- + A_{n-2} y_1 + A_{n-1} = 0$$

$$- - - - - - - - - - - -$$

$$A_0 \frac{y_1^{n-\alpha_1+1}}{\underline{|n-\alpha_1+1}} + A_1 \frac{y_1^{n-\alpha_1}}{\underline{|n-\alpha_1}} + --- + A_{n-\alpha_1} y_1 + A_{n-\alpha_1+1} = 0$$

$$- - - - - - - - - - - -$$

$$- - - - - - - - - - - -$$

$$- - - - - - - - - - - -$$

$$A_0 \frac{y_N^{n}}{\underline{|n}} + A_1 \frac{y_N^{n-1}}{\underline{|n-1}} + --- + A_{n-1} y_N + A_n = 0$$

$$A_0 \frac{y_N^{n-1}}{\underline{|n-1}} + A_1 \frac{y_N^{n-2}}{\underline{|n-2}} + --- + A_{n-2} y_N + A_{n-1} = 0$$

$$- - - - - - - - - - - -$$

$$A_0 \frac{y_N^{n-\alpha_N+1}}{\underline{|n-\alpha_N+1}} + A_1 \frac{y_N^{n-\alpha_N}}{\underline{|n-\alpha_N}} + --- + A_{n-\alpha_N+1} = 0 \tag{2.43}$$

consider the polynomial

$$P(x) = A_0 \frac{x^n}{n!} + A_1 \frac{x^{n-1}}{(n-1)!} + \cdots + A_{n-1} x + A_n \quad (2.44)$$

From the equations (2.43) we see that y_i $(i = 1, \ldots, N)$ is an α_i - fold zero of $P(x)$. So the polynomial P(x) which is a polynomial of degree n has $\alpha_1 + \alpha_2 + \cdots + \alpha_N = n+1$ zeros which is impossible. Hence

$$D \neq 0 \qquad (2.45)$$

Consider the polynomial

$$\mathcal{P}(x) = \sum_{i=0}^{\alpha_1 - 1} b_i^1 \frac{(x-y_1)^{n-i}}{(n-i)!} + \sum_{i=0}^{\alpha_2 - 1} b_i^2 \frac{(x-y_2)^{n-i}}{(n-i)!} + \cdots + \sum_{i=0}^{\alpha_N - 1} b_i^N \frac{(x-y_N)^{n-i}}{(n-i)!} \qquad (2.46)$$

Now

$$
\mathcal{P}(x) = \text{const.} \times
\begin{bmatrix}
b_0^1 \dfrac{y_1^{n}}{\lfloor n} & b_0^1 \dfrac{y_1^{n-1}}{\lfloor n-1} & - - - & b_0^1 \dfrac{y_1}{\lfloor 1} & b_0^1 \\
b_1^1 \dfrac{y_1^{n-1}}{\lfloor n-1} & b_1^1 \dfrac{y_1^{n-2}}{\lfloor n-2} & - - - & b_1^1 & 0 \\
- & - & - & - & - \\
b_{\alpha_1-1}^1 \dfrac{y_1^{n-\alpha_1+1}}{\lfloor n-\alpha_1+1} & b_{\alpha_1-1}^1 \dfrac{y_1^{n-\alpha_1}}{\lfloor n-\alpha_1} & - - - & 0 & 0 \\
- & - & - & - & - \\
- & - & - & - & - \\
- & - & - & - & - \\
b_0^N \dfrac{y_N^{n}}{\lfloor n} & b_0^N \dfrac{y_N^{n-1}}{\lfloor n-1} & - - - & b_0^N \dfrac{y_N}{\lfloor 1} & b_0^N \\
b_1^N \dfrac{y_N^{n-1}}{\lfloor n-1} & b_1^N \dfrac{y_N^{n-2}}{\lfloor n-2} & - - - & b_1^N & 0 \\
- & - & - & - & - \\
b_{\alpha_N-1}^N \dfrac{y_N^{n-\alpha_N+1}}{\lfloor n-\alpha_N+1} & b_{\alpha_N-1}^N \dfrac{y_N^{n-\alpha_N}}{\lfloor n-\alpha_N} & - - - & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\ x \\ \\ x^{\alpha_1-1} \\ \\ \\ \\ \\ \\ \\ x^{n}
\end{bmatrix}
$$

Suppose that

$$\mathcal{P}(x) \equiv 0 .$$

Then

$$
\left.
\begin{aligned}
& b_0^1 \frac{y_1^{n}}{|\underline{n}} + b_1^1 \frac{y_1^{n-1}}{|\underline{n-1}} + \cdots + b_{\alpha_1-1}^1 \frac{y_1^{n-\alpha_1+1}}{|\underline{n-\alpha_1+1}} + \cdots \\
& + b_0^N y_N^{n} + b_1^N \frac{y_N^{n-1}}{|\underline{n-1}} + \cdots + b_{\alpha_N-1}^N \frac{y_N^{n-\alpha_N+1}}{|\underline{n-\alpha_N+1}} = 0, \\[1em]
& b_0^1 \frac{y_1^{n-1}}{|\underline{n-1}} + b_1^1 \frac{y_1^{n-2}}{|\underline{n-2}} + \cdots + b_{\alpha_1-1}^1 \frac{y_1^{n-\alpha_1}}{|\underline{n-\alpha_1}} + \cdots \\
& \cdots + b_0^N \frac{y_N^{n-1}}{|\underline{n-1}} + b_1^N \frac{y_N^{n-2}}{|\underline{n-2}} + \cdots + b_{\alpha_N-1}^N \frac{y_N^{n-\alpha_N}}{|\underline{n-\alpha_N}} = 0, \\[1em]
& - - - - - - - - - - - \\
& - - - - - - - - - - - \\
& - - - - - - - - - - - \\[1em]
& b_0^1 \frac{y_1}{|\underline{1}} + b_1^1 + \cdots + 0 + \cdots + b_0^N \frac{y_N}{|\underline{1}} \\
& + b_1^N + \cdots + 0 = 0, \\[1em]
& b_0^1 + 0 + \cdots + 0 + \cdots + b_0^N + 0 + \cdots + 0 = 0
\end{aligned}
\right\}
\qquad (2.47)
$$

The determinant of coefficients of the system (2.47) of equations is the same as the determinant D with its rows and columns interchanged. Since $D \neq 0$ the only solution to the system (2.47) of equations is

$$b_0^1 = b_1^1 = \cdots = b_{\alpha_1 - 1}^1 = \cdots = b_0^N = b_1^N = \cdots = b_{\alpha_N - 1}^N = 0 \quad (2.48)$$

We have proved that $\Phi(x) = 0$ implies that (2.48) holds. Hence the polynomials $(x - y_1)^n, (x - y_1)^{n-1}, \cdots,$

$$\cdots, (x - y_1)^{n - \alpha_1 + 1}, \cdots, (x - y_N)^n,$$

$$(x - y_N)^{n-1}, \cdots, (x - y_N)^{n - \alpha_N + 1}$$

are linearly independent. This completes the proof.

<u>Case (ii)</u> $\quad u = \sum_{i=1}^{N} \alpha_i < n + 1.$

Let $\sum_{i=1}^{N} \alpha_i = r.$

By (2.38), since $S(x) \equiv 0$ for $x > y_N$ adding knots beyond y_N does not affect the behaviour of $S(x)$. By adding $n+1-r$ fictitious simple knots beyond y_N, $S(x)$ will have a total of $n+1$ knots while $S(x)$ will be zero to the left and right of these knots. By the previous case we know that $S(x)$ must vanish identically.

2.4 The fundamental spline functions form a basis.

In this section we prove that every spline function can be expressed as a linear combination of fundamental splines. We prove this result in two stages, first proving it for splines vanishing outside an open interval and then extending it to the most general case. We need the following

LEMMA 2.4. *If*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_N = n + 2 \tag{2.49}$$

then the fundamental spline function

$$M_1(x) = M_n(x; \overbrace{y_1, \ldots, y_1}^{\alpha_1}, \ldots, \overbrace{y_N, \ldots, y_N}^{\alpha_N}) \tag{2.50}$$

belongs to the continuity class $C^{n-\alpha_i}$ *in the neighbourhood of the point* $x = y_i$ *and does not belong to any higher continuity class in that neighbourhood.*

Proof. We have already seen that $M_1(x)$ is a spline of degree n, having the knots $y_1, \ldots, y_N$ with multiplicities $\alpha_1, \ldots, \alpha_N$. Also $M_1(x)$ vanishes outside (y_1, y_N). Hence from Lemma 2.2, $M_1(x)$ must be represented by

$$M_1(x) = \sum_{i=0}^{\alpha_1 - 1} a_i^1 (x - y_1)_+^{n-i} + \cdots + \sum_{i=0}^{\alpha_N - 1} a_i^N (x - y_N)_+^{n-i}$$

$$(-\infty < x < \infty) \tag{2.51}$$

th

$$\sum a_i^1 (x-y_1)^{n-i} + \cdots + \sum_{i=0}^{\alpha_N - 1} a_i^N (x-y_N)^{n-i} = 0$$

for all x. (2.52)

ppose that there exist coefficients b_i^j such that $M_1(x)$ n also be represented as

$$M_1(x) = \sum_{i=0}^{\alpha_1 - 1} b_i^1 (x-y_1)_+^{n-i} + \cdots + \sum_{i=0}^{\alpha_N - 1} b_i^N (x-y_N)_+^{n-i}$$

$$(-\infty < x < \infty) \qquad (2.53)$$

th

$$\sum^{-1} b_i^1 (x-y_1)^{n-i} + \cdots + \sum_{i=0}^{\alpha_N - 1} b_i^N (x-y_N)^{n-i} = 0$$

for all x (2.54)

nce the functions $(x-y_1)_+^n, \cdots, (x-y_1)_+^{n-\alpha_1+1}, \cdots$

$\cdots, (x-y_N)_+^n, \cdots, (x-y_N)_+^{n-\alpha_N+1}$ are linearly independent, have

$a_0^1 = b_0^1, \cdots, a_{\alpha_1-1}^1 = b_{\alpha_1-1}^1, \cdots, a_{\alpha_N-1}^N = b_{\alpha_N-1}^N$

e., the representation (2.51) of $M_1(x)$ is unique.

ppose that in the equation (2.51)

$$a_{\alpha_1-1}^1 = 0 \qquad (2.55)$$

en Y_1 is a knot of multiplicity $\alpha_1 - 1$ for $M_1(x)$.

(x) now has knots of multiplicities $\alpha_1 - 1, \alpha_2, \cdots, \alpha_N$ the points $y_1, \cdots, y_N$ and the sum of the multipli-ties of the knots of $M_1(x)$ is

$$\alpha_1 - 1 + \alpha_2 + \cdots + \alpha_N = n+1 \qquad \text{(from (2.49))}$$

In such a case, by Therem 2.3, $M_1(x) \equiv 0$ which is a contradiction (See Theorem 1.4). Hence

$$a^{1}_{\alpha_1 - 1} \neq 0 \qquad (2.56)$$

Similarly we can prove that each of $a^{2}_{\alpha_2 - 1}, \ldots, a^{N}_{\alpha_N - 1}$ is non-zero. Thus the identity (2.52) expresses the linear dependence of the functions $(x-y_1)^n, (x-y_1)^{n-\alpha_1+1}, \ldots, (x-y_N)^n, \ldots, (x-y_N)^{n-\alpha_N+1}$ and, up to a constant factor, only one such relation exists. Since

$$\begin{aligned} M_1(x) = {} & a^1_0 (x-y_1)^n_+ + a^1_1 (x-y_1)^{n-1}_+ + \cdots + a^1_{\alpha_1-1}(x-y_1)^{n-\alpha_1+1}_+ \\ & + \cdots \\ & + a^{k-1}_0 (x-y_{k-1})^n_+ + a^{k-1}_1 (x-y_{k-1})^{n-1}_+ + \cdots \\ & + a^{k-1}_{\alpha_{k-1}-1}(x-y_{k-1})^{n-\alpha_{k-1}+1}_+ + a^k_0 (x-y_k)^n_+ \\ & + a^k_1 (x-y_k)^{n-1}_+ + \cdots + a^k_{\alpha_k-1}(x-y_k)^{n-\alpha_k+1}_+ \\ & + a^{k+1}_0 (x-y_{k+1})^n_+ + a^{k+1}_1 (x-y_{k+1})^{n-1}_+ + \cdots \\ & + \cdots \\ & + a^N_0 (x-y_N)^n_+ + a^N_1 (x-y_N)^{n-1}_+ + \cdots + a^N_{\alpha_N-1}(x-y_N)^{n-\alpha_N+1}_+ . \end{aligned}$$

$M_1(x)$ reduces to

$$\begin{aligned} M_1(x) = {} & a^1_0 (x-y_1)^n + a^1_1 (x-y_1)^{n-1} + \cdots \\ & + a^1_{\alpha_1-1}(x-y_1)^{n-\alpha_1+1} + \cdots \\ & + \cdots \\ & + a^{k-1}_0 (x-y_{k-1})^n + \cdots + a^{k-1}_{\alpha_{k-1}-1}(x-y_{k-1})^{n-\alpha_{k-1}+1} \end{aligned}$$

for $y_{k-1} < x < y_k$

and

$$M_1(x) = a_0^1 (x-y_1)^n + a_1^1 (x-y_1)^{n-1} + \cdots + a_{\alpha_1-1}^1 (x-y_1)^{n-\alpha_1+1}$$

$$+ \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad -$$

$$+ a_0^{k-1} (x-y_{k-1})^n + \cdots + a_{\alpha_{k-1}-1}^{k-1} (x-y_{k-1})^{n-\alpha_{k-1}+1}$$

$$+ a_0^k (x-y_k)^n + \cdots + a_{\alpha_k-1}^k (x-y_k)^{n-\alpha_k+1}$$

for $y_k < x < y_{k+1}$

Denote by $P_{k-1}(x)$ and $P_k(x)$ the polynomials to which $M_1(x)$ reduces in the internals (y_{k-1}, y_k) and (y_k, y_{k+1}) respectively. Now.

$$P_k(x) - P_{k-1}(x) = a_0^k (x-y_k)^n + \cdots + a_{\alpha_k-1}^k (x-y_k)^{n-\alpha_k+1}$$

and

$$P_k^{(j)}(y_k) - P_{k-1}^{(j)}(y_k) = 0 \quad \text{for } j = 0, 1, 2, \ldots, n-\alpha_k$$

But

$$P_k^{(n-\alpha_k+1)}(y_k) - P_{k-1}^{(n-\alpha_k+1)}(y_k) = \underline{|n-\alpha_k+1}\; a_{\alpha_k-1}^k$$

In other words

$$M_1^{(j)}(y_k+) = M_1^{(j)}(y_k-) \text{ for } j = 0,1,2,\ldots,n-\alpha_k$$
$$\text{and } k = 1,2,\ldots,N$$

and

$$M_1^{(n-\alpha_k+1)}(y_k+) - M_1^{(n-\alpha_k+1)}(y_k-)$$
$$= \underline{|n-\alpha_k+1}\; a^k_{\alpha_k-1}, \quad k = 1,2,\ldots,N$$

and we have earlier proved that

$$a^k_{\alpha_k-1} \neq 0 \quad \text{for} \quad k = 1,2,\ldots,N$$

Hence we conclude that there exists $\varepsilon > 0$ such that

$$M_1(x) \in C^{n-\alpha_k}(y_k-\varepsilon, y_k+\varepsilon)$$

and

$$M_1(x) \notin C^{n-\alpha_k+1}(y_k-\varepsilon, y_k+\varepsilon)$$

<u>Theorem 2.5</u> <u>Let</u> $S(x)$ <u>be a spline of degree</u> n <u>having the knots</u>

$$\cdots < y_{-1} < y_0 < y_1 < y_2 < \cdots \tag{2.57}$$

<u>where</u>

$$y_i \text{ is a knot of multiplicity } \alpha_i \tag{2.58}$$

(i) <u>If</u>

$$u = \sum_{i=1}^{N} \alpha_i \geq n+2 \tag{2.59}$$

and

$S(x) = 0$ everywhere outside the interval (y_1, y_N)

(2.60)

then $S(x)$ can be uniquely represented in the form

$$S(x) = \sum_{i=1}^{u-(n+1)} c_j M_j(x). \tag{2.61}$$

(ii) If

$S(x) = 0$ wherever $x < y_1$ (2.62)

then $S(x)$ can be uniquely represented in the form

$$S(x) = \sum_{i=1}^{\infty} c_j M_j(x) \tag{2.63}$$

Proof. (i) Since y_1 is a knot of multiplicity α_1 for $S(x)$, we have

$$S(x) \in C^{n-\alpha_1}(-\infty, y_2)$$

Since $S(x)$ vanishes to the left of y_1 and $S(x)$ is a polynomial of degree atmost n in , in a right neighbourhood of , we must have have

$$S(x) = (x - y_1)^{n-\alpha_1+1} \phi(x) \tag{2.64}$$

where $\phi(x)$ is a polynomial. We are now working under the assumption (2.59), i.e.,

$$\alpha_1 + \alpha_2 + \cdots + \alpha_N \geq n+2$$

Thus

$$M_1(x) = M_n(x; \overbrace{y_1, \cdots, y_1}^{\alpha_1}, \overbrace{y_2, \cdots, y_2}^{\alpha_2}, \cdots) \quad (2.65)$$

where, within the brackets, as many y_i's appear as are necessary to make $\alpha_1 + \alpha_2 + \cdots = n+2$. $M_1(x)$ can therefore be represented as

$$\left.\begin{aligned} M_1(x) = {} & a_0^1 (x-y_1)_+^n + \cdots + a_{\alpha_1-1}^1 (x-y_1)_+^{n-\alpha+1} \\ & + a_0^2 (x-y_2)_+^n + \cdots \\ & + a_{\alpha_2-1}^2 (x-y_2)_+^{n-\alpha_2+1} + \cdots \\ & + \cdots \end{aligned}\right\} (2.66)$$

the summation terminating when all the y_i appearing in (2.65) have been exhausted with appropriate multiplicities. For $x \in (y_1, y_2)$, $M_1(x)$ then reduces to

$$M_1(x) = a_0'(x-y_1)^n + - - - + a'_{\alpha_1-1}(x-y_1)^{n-\alpha_1+1}$$

$$= a'_{\alpha_1-1}(x-y_1)^{n-\alpha_1+1} + \text{higher powers of } (x-y_1)$$

(2.67)

We can therefore determine C_1 uniquely so that

$$S_1(x) = S(x) - c_1 M_1(x) = (x-y_1)^{n-\alpha_1+2}\phi_1(x), \quad (2.68)$$

$$(y_1 < x < y_2)$$

where $\phi_1(x)$ is a polynomial. In (2.11) we had defined $M_j(x)$ as

$$M_j(x) = M_n(x;\, x_j,\, x_{j+1},\, ---,\, x_{j+n+1})$$

Where some of the x_i's may coincide. Thus

$$M_{j+1}(x) = M_n(x;\, x_{j+1},\, x_{j+2},\, ---,\, x_{j+n+2}).$$

Since

$$M_1(x) = M_n(x;\, \underbrace{\overbrace{y_1, ---, y_1}^{\alpha_1}, \overbrace{y_2, ---, y_2}^{\alpha_2}, ---}_{n+2}),$$

$M_2(x)$ will stand for

$$M_2(x) = M_n(x;\, \overbrace{y_1, ---, y_1}^{\alpha_1-1}, \overbrace{y_2, ---, y_2}^{\alpha_2}, ---) \quad (2.69)$$

and $M_2(x)$ has the representation

$$M_2(x) = b_0^1(x-y_1)_+^n + \cdots + b_{\alpha_1-2}^1(x-y_1)_+^{n-\alpha_1+2}$$

$$+ b_0^2(x-y_2)_+^n + \cdots\cdots$$

$$+ b_{\alpha_2-1}^2(x-y_2)_+^{n-\alpha_2+1} + \cdots\cdots$$

the summation terminating when the y_i's appearing in (2.69) are exhausted. For $x \in (y_1, y_2)$

$$M_2(x) = b_0^1(x-y_1)^n + \cdots + b_{\alpha_1-2}^1(x-y_1)^{n-\alpha_1+2}$$

$$= b_{\alpha_1-2}^1(x-y_1)^{n-\alpha_1+2} + \text{higher powers of } (x-y_1).$$

Once again we can find a unique C_2 such that

$$S_2(x) = S(x) - c_1 M_1(x) - c_2 M_2(x) = (x-y_1)^{n-\alpha_1+3}\phi_2(x)$$

when $y_1 < x < y_2$ and where $\phi_2(x)$ is a polynomial.

Two cases are possible.

Case(i)

$$\alpha_2 + \alpha_3 + \cdots + \alpha_N \le n+1.$$

We have seen that

$$M_1(x) = M_n(x;\ \underbrace{\overbrace{y_1, \ldots, y_1}^{\alpha_1}, \overbrace{y_2, \ldots, y_2}^{\alpha_2}, \ldots}_{n+2})$$

and

$$M_2(x) = M_n(x;\ \underbrace{\overbrace{y_1, \ldots, y_1}^{\alpha_1 - 1};\ \overbrace{y_2, \ldots, y_2}^{\alpha_2}, \ldots}_{n+2})$$

Suppose that in this process the final fundamental spline is $M_t(x)$. Then $M_t(x)$ must necessarily contain y_1 as an argument since $\alpha_2 + \alpha_3 + \cdots + \alpha_N \leq n+1$ and we are taking divided differences of order n + 1, i.e., we need n + 2 arguments for each divided difference. Hence $M_t(x)$ will be of the form

$$M_t(x) = M_n(x;\ \overbrace{y_1, \ldots, y_1}^{\alpha_1 - t + 1},\ \overbrace{y_2, \ldots, y_2}^{\alpha_2}, \ldots,\ \overbrace{y_N, \ldots, y_N}^{\alpha_N})$$

where

$$\alpha_1 - t + 1 + \alpha_2 + \alpha_3 + \cdots + \alpha_N = n + 2 \qquad (2.70)$$

i.e.,

$$t = \alpha_1 + \alpha_2 + \cdots + \alpha_N - (n+1) \leq \alpha_1$$

we have already proved the existence of unique C_1, C_2 such that

$$S_1(x) = S(x) - c_1 M_1(x) = (x - y_1)^{n - \alpha_1 + 2} \phi_1(x)$$

$$(y_1 < x < y_2)$$

and

$$S_2(x) = S(x) - c_1 M_1(x) - c_2 M_2(x)$$

$$= (x-y_1)^{n-\alpha_1+3}\,\phi_2(x) \qquad (y_1 < x < y_2)$$

Proceeding thus we can find unique $c_3, c_4, \ldots, c_t$ such that

$$S_t(x) = S(x) - c_1 M_1(x) - \cdots - c_t M_t(x)$$

$$= (x - y_1)^{n-\alpha_1+t+1}\,\phi_t(x) \qquad (y_1 < x < y_2) \qquad (2.71)$$

Since each of S(x), M_1 (x), - - - , $M_t(x)$ vanish outside (y_1, y_N),

$$S_t(x) = 0 \quad \text{for} \quad x \notin (y_1, y_N) \qquad (2.72)$$

Also S_t (x) is a spline of degree n having the knots $y_1, \ldots, y_N$. From (2.71) and (2.72) we conclude that

$$S_t(x) \in C^{n-\alpha_1+t}(-\infty, y_2)$$

i.e. y_1 is a knot of multiplicity $\alpha_1 - t$ for S_t (x).
The other knots $y_2, \ldots, y_N$ of S_t (x) are of multiplicities $\alpha_2, \ldots, \alpha_N$. Thus the sum of the multiplicities of the knots of S_t (x) is

$$\alpha_1 - t + \alpha_2 + \cdots + \alpha_N = n+1 \quad \text{(from (2.70))}$$

Using Theorem 2.3, we conclude that

$$S_t(x) \equiv 0 ,$$

i.e., from (2.71),

$$S(x) = c_1 M_1(x) + \cdots + c_t M_t(x) \text{ for all } x.$$

In other words

$$S(x) = \sum_{i=1}^{t} c_i M_i(x)$$

where

$$t = \alpha_1 + \alpha_2 + \cdots + \alpha_N - (n+1)$$

$$= u - (n+1)$$

and the c_i have been determined uniquely.

<u>Case (ii)</u> $\alpha_2 + \alpha_3 + \cdots + \alpha_N > n+1$.

In the previous case we had to stop with

$$M_t(x) = M_n(x; \overbrace{y_1, \cdots, y_1}^{\alpha_1 - t+1}, \overbrace{y_2, \cdots, y_2}^{\alpha_2}, \cdots, \overbrace{y_N, \cdots, y_N}^{\alpha_N})$$

since $\alpha_2 + \cdots + \alpha_N$ was $\leq n+1$ and we needed

$n + 2$ arguments for taking divided differences. Now, however, the least value that $\alpha_2 + \dots + \alpha_N$ can take is $n + 2$ and hence we can proceed until we reach an $M_i(x)$ which does not contain y_1 as an argument and yet which has $n + 2$ arguments, the last α_N of which are appearing α_N times. Once we go to an $M_i(x)$ which does not have y_1 as an argument, that particular $M_i(x)$ will vanish to the left of y_2. Hence the only $M_i(x)$ which give non-trivial contribution in (y_1, y_2) are $M_1(x), \dots, M_{\alpha_1}(x)$.
We have earlier proved the existence of unique c_1, c_2 such that

$$S_1(x) = S(x) - c_1 M_1(x)$$

$$= (x - y_1)^{n-\alpha_1+2} \phi_1(x) \quad \text{in } (y_1, y_2)$$

and

$$S_2(x) = S(x) - c_1 M_1(x) - c_2 M_2(x)$$

$$= (x - y_1)^{n-\alpha_1+3} \phi_2(x) \quad \text{in } (y_1, y_2).$$

Proceeding further, we can prove the existence of unique $c_3, c_4, \dots, c_{\alpha_1 - 1}$.

such that

$$S_{\alpha_1-1}(x) = S(x) - c_1 M_1(x) - \cdots - c_{\alpha_1-1} M_{\alpha_1-1}(x)$$

$$= (x-y_1)^n \phi(x) \text{ in } (y_1, y_2)$$

Since each of $S(x)$, $M_1(x)$, ---, $M_{\alpha_1-1}(x)$ is a polynomial of degree atmost n, $\phi(x)$ must necessarily be a constant. Now

$$M_{\alpha_1}(x) = M_n(x; y_1, \overbrace{y_2, \cdots, y_2}, \cdots).$$

Hence $M_{\alpha_1}(x)$ is of the form

$$M_{\alpha_1}(x) = m_0^1 (x-y_1)_+^n + m_0^2 (x-y_2)_+^n + \cdots$$

For $x \in (y_1, y_2)$,

$$M_{\alpha_1}(x) = m_0^1 (x-y_1)^n$$

and we can find a unique c_{α_1} such that

$$S_{\alpha_1}(x) = S_{\alpha_1-1}(x) - c_{\alpha_1} M_{\alpha_1}(x) = 0 \text{ for } x \in (y_1, y_2) \tag{2.73}$$

Since

$$S_{\alpha_1}(x) = S(x) - c_1 M_1(x) - \cdots - c_{\alpha_1} M_{\alpha_1}(x)$$

and each of $S(x), M_1(x), \ldots, M_{\alpha_1}(x)$ vanishes to the left of y_1,

$$S_{\alpha_1}(x) = 0 \quad \text{for} \quad x \le y_1 \qquad (2.74)$$

From (2.73) and (2.74), we find that

$$S_{\alpha_1}(x) = 0 \quad \text{for} \quad x < y_2 \; (2.75)$$

$M_{\alpha_1+1}(x)$ does not contain y_1 as an argument and stands for

$$M_{\alpha_1+1}(x) = M_n(x;\, \overbrace{y_2, \ldots, y_2}^{\alpha_2}, y_3, \ldots)$$

It has the representation

$$M_{\alpha_1+1}(x) = a_0^{\alpha_1+1}(x-y_2)_+^n + \cdots + a_{\alpha_2-1}^{\alpha_1+1}(x-y_2)_+^{n-\alpha_2+1}$$

$$+ \cdots$$

$I_n(y_2, y_3)$

$$M_{\alpha_1+1}(x) = a_0^{\alpha_1+1}(x-y_2)^{n-1} + \cdots + a_{\alpha_2-1}^{\alpha_1+1}(x-y_2)^{n-\alpha_2+1}$$

From the expression for $S_{\alpha_1}(x)$ in a right neighbourhood of y_2 we can factor out $(x-y_2)^{n-\alpha_2}$ and find a unique c_{α_1+1} such that

$$S_{\alpha_1+1}(x) = S_{\alpha_1}(x) - c_{\alpha_1+1} M_{\alpha_1+1}(x)$$

$$= (x-y_2)^{n-\alpha_1+2} \phi_{\alpha_1+1}(x)$$

$$\text{in } (y_2, y_3),$$

$\phi_{\alpha_1+1}(x)$ being a polynomial. We can proceed just as before until we reach an M_t (x) beyond which we will no longer get the necessary n + 2 arguments for taking divided differences. The final M_t (x) will be of the form

$$M_t(x) = M_n(x; \overbrace{y_j, \dots, y_j}^{\alpha_j - l + 1}, \overbrace{y_{j+1}, \dots, y_{j+1}}^{\alpha_j + 1}, \dots, \overbrace{y_N, \dots, y_N}^{\alpha_N})$$

where

$$t = \alpha_1 + \alpha_2 + \alpha_{j-1} + l$$

and

$$\alpha_j - l + 1 + \alpha_{j+1} + \dots + \alpha_N = n+2 .$$

In other words

$$\ell = \alpha_j + \alpha_{j+1} + \cdots + \alpha_N - n - 1 \tag{2.76}$$

We can now write

$$S_t(x) = S(x) - c_1 M_1(x) - \cdots - c_t M_t(x)$$

where

$$t = \alpha_1 + \cdots + \alpha_{j-1} + \ell$$

$$= \alpha_1 + \cdots + \alpha_{j-1} + \alpha_j + \cdots + \alpha_{N-n+1}$$

from (2.76)

$$= u - (n + 1)$$

Hence

$$S_{u-(n+1)}(x) = S(x) - \sum_{i=1}^{u-(n+1)} c_i M_i(x)$$

Our proof will be complete if we show that

$$S_{u-(n+1)}(x) = 0 \qquad \text{for all } x.$$

To this end, let the integers r, α and β be defined by the relations

$$\alpha_{r+1} + \cdots + \alpha_N < n+2 \leq \alpha_r + \alpha_{r+1} + \cdots + \alpha_N$$

$$(2 \leq r < N)$$

$$\alpha + \alpha_{r+1} + \cdots + \alpha_N = n+2, \qquad (0 < \alpha \leq n)$$

$$\alpha + \beta = \alpha_r .$$

The functions $S(x)$, $S_1(x)$, $\ldots$, $S_{\alpha_1 - 1}(x)$ are near the point $X = y_1$ of the continuity classes $C^{n-\alpha_1}$, $C^{n-\alpha_1+1}$, $\ldots$, C^{n-1} respectively. Likewise $S_{\alpha_1}(x)$, $\ldots$, $S_{\alpha_1+\alpha_2-1}(x)$ vanish for $x < y_2$ and are near $X = y_2$ of the continuity classes $C^{n-\alpha_2}$, $C^{n-\alpha_2+1}$, $\ldots$, C^{n-1} respectively. Finally,

$$S_{\alpha_1+\alpha_2+\cdots+\alpha_{r-1}}(x), \quad S_{\alpha_1+\alpha_2+\cdots+\alpha_{r-1}+1}(x),$$

$$\cdots\cdots, \quad S_{\alpha_1+\alpha_2+\cdots+\alpha_{r-1}+\beta+1}(x)$$

vanish for $x < y_r$ and are near $X = y_r$ of the continuity classes

$$C^{n-\alpha_{r-1}+1}, \quad C^{n-\alpha_r+1}, \quad ----$$

$$-----, \quad C^{n-\alpha+1}$$ respectively.

Observe now that by our definition of r, α and β we have

$$\alpha_1 + --- + \alpha_{r-1} + \beta + 1 = u - (n+1).$$

Also that $S_{u-(n+1)}(x)$ is a spline function

which may be thought of as having at $x = y_r$ a knot of multiplicity $\alpha - 1$. Thus the total sum of the multiplicities of its knots is $(\alpha - 1) + \alpha_{r+1} + --- + \alpha_N = n+1$.

As $S_{u-(n+1)}(x)$ vanishes outside (y_1, y_N), we may apply the result of Theorem 2.3 and conclude that

$$S_{u-(n+1)}(x) = 0 \qquad \text{for all } x$$

and this completes the proof of part (i) of the theorem.

(ii) The previous construction applies with the difference that it continues indefinitely resulting in (2.63).

The following corollary is a restatement of Theorem 2.5. In this form, the theorem has been applied to obtain solutions to the problem of interpolation in [4].

Corollary 2.6 Let $x_1, x_2, \ldots, x_k$, $(k > n+1)$ be non-decreasing reals such that

$$x_i < x_{i+n+1} \qquad (1 \le i, \ i+n+1 \le k).$$

This implies that almost n of the x_i may coalesce at a point. Then the fundamental spline functions.

$$M_j(x) = M_n(x; x_j, x_{j+1}, \ldots, x_{j+n+1})$$
$$(j = 1, 2, \ldots, k-(n+1))$$

are linearly independent. Every spline function $S(x)$ of degree n, having the points $x_1, \ldots, x_k$ as knots, with multiplicities as they occur in this sequence and such that

$$S(x) = 0 \quad \text{everywhere outside } (x_1, x_k)$$

may be uniquely represented in the form

$$S(x) = \sum_{i=1}^{k-(n+1)} c_i M_i(x).$$

Theorem 2.7 Let S (x) be a spline of degree n having the knots

$$\cdots < y_{-1} < y_0 < y_1 < y_2 < \cdots \tag{2.77}$$

Where

y_i *is a knot of multiplicity* α_i. (2.78)

Then S (x) can be uniquely represented in the form

$$S(x) = \sum_{i=-\infty}^{\infty} c_i M_i(x) \qquad (2.79)$$

In order to prove Theorem 2.7, are read the following

LEMMA 2.8 *The fundamental spline functions*

$$M_{-n}(x),\ M_{-n+1}(x),\ ---,\ M_0(x) \qquad (2.80)$$

are linearly independent in the interval

$$(x_0, x_1) = (y_0, y_1)$$

and therefore form in this interval a basis for $\widetilde{\Pi}_n$.

Proof. Let us assume that

$$S(x) \equiv \sum_{i=-n}^{0} c_i M_i(x) \text{ in } (y_0, y_1) \qquad (2.81)$$

and let us show that this implies that

$$c_i = 0 \qquad (i = -n, -n+1, ---, 0) \qquad (2.82)$$

Consider the restriction of S (x) to the interval $(y_0, +\infty)$ By (2.81) this restriction is a spline function vanishing everywhere outside the interval (x_1, x_{n+1}) provided that we define it as being equal to zero if $x \leq y_0$. It may therefore be thought of as a spline of degree n having utmost n + 1 knots, namely $x_1, x_2, \ldots, x_{n+1}$; it must therefore vanish everywhere by Theorem 2.3. Thus (2.81) implies that S (x) = 0 in the interval (y_0, ∞). Similarly it is shown that $S(x) = 0$ in $(-\infty, y_1)$. Thus $S(x) = 0$ for all x and now (2.82) follows from the unicity in Theorem 2.5. This completes a proof of the lemma.

Proof of Theorem 2.6 Let

$$S(x) = P(x) \quad \text{if } y_0 < x < y_1 \quad (P \in \Pi_n)$$

By Lemma 2.8 we can write

$$P(x) = \sum_{i=-n}^{0} c_i M_i(x)$$

which implies that

$$S^*(x) = S(x) - \sum_{i=-n}^{0} c_i M_i(x) \quad (2.83)$$

is a spline of degree n vanishing in the interval

$(y_0, y_1) = (x_0, x_1)$. We may therefore write

$$S^*(x) = S_1(x) + S_0(x) \qquad (2.84)$$

where S_1 and S_0 are splines of degree n vanishing in the intervals $(-\infty, x_1)$ and $(x_0, +\infty)$ respectively. Using Theorem 2.5, we may therefore write uniquely

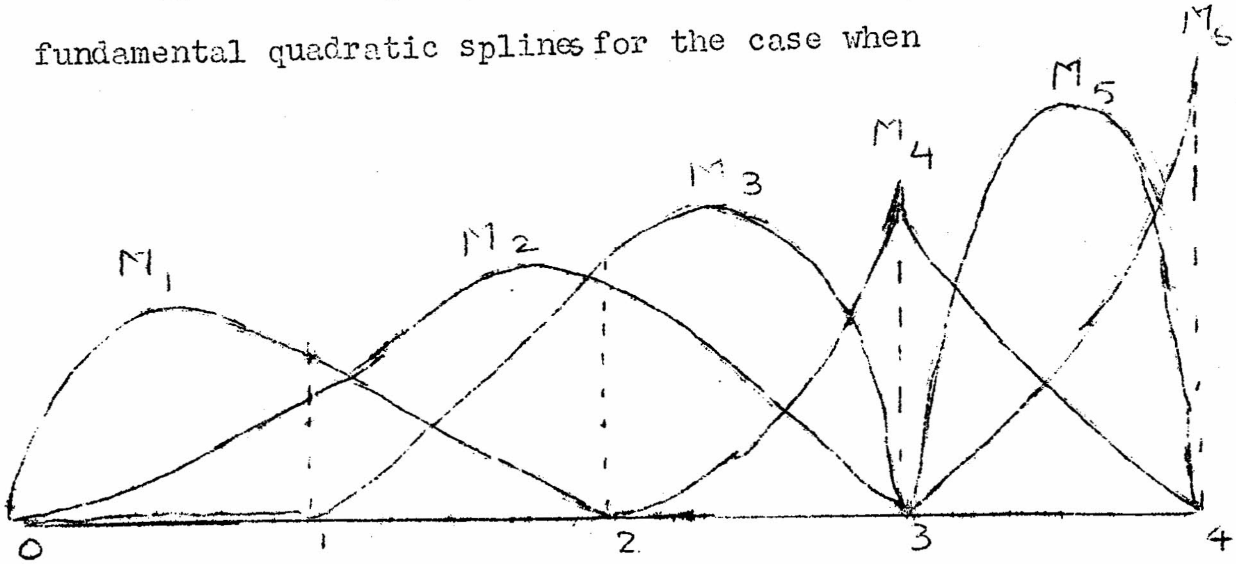
$$S_1(x) = \sum_{i=1}^{\infty} c_i M_i(x)$$

$$S_0(x) = \sum_{i=-\infty}^{-(n+1)} c_i M_i(x) .$$

Now (2.83) and (2.84) imply the desired representations (2.79) and h the theorem.

2.5 Examples to illustrate the results of § 2.4.

The following figure shows a sketch of five fundamental quadratic splines for the case when



In terms of expression (2.11) we may write.

$$M_1(x) = M_2(x; 0, 0, 1, 2)$$
$$M_2(x) = M_2(x; 0, 1, 2, 3)$$
$$M_3(x) = M_2(x; 1, 2, 3, 3)$$
$$M_4(x) = M_2(x; 2, 3, 3, 4)$$
$$M_5(x) = M_2(x; 3, 3, 4, 4),$$
$$M_6(x) = M_2(x; 3, 4, 4, 4)$$

All arcs between consecutive integral values of x are paraleolic and are easily found explicitly. The above figure illustrates nicely Lemma 2.4. By Theorem 2.5 we know that

$$S(x) = \sum_{i=1}^{6} c_i \, M_i(x)$$

represents the most general quadratic spline function in the interval $[0, 4]$ satisfying the following conditions:

1) It has the knots 1, 2, 3,
2) It belongs to the continuity classes C^1, C^1, C^0 near the points x = 1, 2, 3 respectively.
3) S (0) = 0.

As illustrations of Theorem 2.5 and Theorem 2.7 we mention the following identities. The knots being integral $(x_i = i)$ and writing

$$M_j(x) = M_m(x; j, j+1, \ldots, j+m+1)$$

we find that

$$x_+^m = \sum_{j=0}^{\infty} (j+1)(j+2) \cdots (j+m)\, M_j(x)$$

and

$$x^m = \sum_{j=-\infty}^{\infty} (j+1)(j+2) \cdots (j+m)\, M_j(x).$$

CHAPTER III

AN INTERPOLATING SPLINE FUNCTION

In this chapter we prove a theorem which solves the following interpolation problem:

Problem 3.1 Let

$$x_1 < x_2 < \cdots < x_{n+k+1} \tag{3.1}$$

be a given set of $n+k+1$ abscissae. When can we interpolate in the points (3.1) an arbitrary given set of ordinates $y_1, y_2, \ldots, y_{n+k+1}$ by means of a spline function of degree n having the knots

$$\xi_1 < \xi_2 < \cdots < \xi_k \ ? \tag{3.2}$$

We will first prove a representation theorem for splines of degree n with the knots (3.2).

Theorem 3.2. The function

$$F(x) = P_n(x) + \sum_{j=1}^{k} A_j (x - \xi_j)_+^n \tag{3.3}$$

Where $P_n(x)$ is an arbitrary polynomial of degree utmost n and A_j are arbitrary parameters, represents uniquely the most general spline curve of degree n having the knots (3.2)'

Proof. For $j = 0, 1, \ldots, k$, let $P_{nj}(x)$ be the polynomial that gives the value of $F(x)$ in the interval (ξ_j, ξ_{j+1}) where ξ_0 stands for $-\infty$ and ξ_{k+1} for $+\infty$. It follows from the definition of a spline of degree n that $P_{nj}(x) - P_{n,j-1}(x)$ is a polynomial of degree n having an n - fold zero at $x = \xi_j$. That means

$$P_{n,j}(x) - P_{n,j-1}(x) = A_j (x - \xi_j)^n \quad (3.4)$$

for some A_j. Thus

$$\begin{aligned} P_{nl}(x) &= P_{n0}(x) + (P_{n1}(x) - P_{n0}(x)) \\ &\quad + (P_{n2}(x) - P_{n1}(x)) + \cdots \\ &\quad + \cdots + (P_{n,l}(x) - P_{n,l-1}(x)) \\ &= P_{n0}(x) + \sum_{j=1}^{l} A_j (x - \xi_j)^n \end{aligned} \quad (3.5)$$

Taking $P_n(x) = P_{n0}(x)$ we see that (3.5) is equivalent to (3.3).

It remains to be shown that the representation given in (3.3) is unique. By our definition of the polynomials P_{nj}, $P_{n0}(x) = F(x)$ for $x < \xi_1$. But for $x < \xi_1$, (3.3) reduces to $F(x) = P_n(x)$. Hence, in (3.3), $P_n(x)$ stands for the polynomial to which the spline F (x) reduces in $(-\infty, \xi_1)$. Further, n - fold differentiation of (3.4) gives

$$P^{(n)}_{n,j}(x) - P^{(n)}_{n,j-1}(x) = A_j \lfloor n .$$

Taking $x = \xi_j$, $j = 1, 2, \ldots, n$, this may be written as

$$A_j = \frac{1}{\lfloor n} \left[F^{(n)}(\xi_j + 0) - F^{(n)}(\xi_j - 0) \right]$$

where the terms in the square brackets are the right and left n^{th} derivatives respectively. Thus, given the spline F (x), the expression (3.3) for the spline is completely and uniquely determined.

It is easy to see that an expression of the form (3.3) defines a spline function of degree n with the knots $\xi_1, \ldots, \xi_k$ and this completes the proof of Theorem 3.2.

Theorem 3.3 We can interpolate in $n+k+1$ given abscissae (3.1) arbitrarily given ordinates by a spline curve of degree n and knots (3.2) if and only if the inequalities

$$\left.\begin{array}{l} x_1 < \xi_1 < x_{n+2} \\ x_2 < \xi_2 < x_{n+3} \\ - - - - - - - \\ x_k < \xi_k < x_{n+k+1} \end{array}\right\} \qquad (3.6)$$

are satisfied .

In the proof of Theorem 3.3, we will make use of the following

Theorem 3.4 [3] Let ℓ and m be natural integers. If

$$x_1 < x_2 < \cdots < x_\ell, \quad y_1 < y_2 < \cdots, y_\ell,$$

then

$$D = \det \| (x_i - y_j)_+^m \|_1^\ell > 0$$

iff the ineuqalities

$$x_{i-m-1} < y_i < x_i, \qquad i = 1, \ldots, \ell$$

hold.

Proof of Theorem 3.3. As a first step towards the solution of Problem 3.1 we observe the following: If we choose in any way n + 1 fixed abscissae $y_1, \ldots, y_{n+1}$ such that

$$y_1 < y_2 < \cdots < y_{n+1} < x_1, \quad y_{n+1} < \xi_1, \tag{3.7}$$

then the polynomial $P_n(x)$ of (3.3) may be uniquely written as

$$P_n(x) = \sum_{i=1}^{n+1} c_i (x - y_i)^n$$

for appropriate values of the c_i. But then we also have

$$P_n(x) = \sum_{i=1}^{n+1} c_i (x - y_i)_+^n$$

$$\text{if} \quad x \geq y_{n+1}.$$

If we now write

$$y_{n+2} = \xi_1, \; y_{n+3} = \xi_2, \ldots, y_{n+k+1} = \xi_k \tag{3.8}$$

and correspondingly define $c_{i+n+1} = A_i \; (i = 1, \ldots, k)$ we see that the arbitrary spline curve of knots (3.2) may also be written in the form

$$F(x) = \sum_{i=1}^{n+k+1} c_i (x - y_i)_+^n$$

in the range $x \geq y_{n+1}$.

We conclude that the interpolation problem 3.1 has a solution if and only if the determinant of order $n+k+1$

$$\det \| (x_i - y_j)_+^n \| \neq 0. \tag{3.9}$$

Using Theorem 3.4, we see that (3.9) holds if and only if

$$x_{i-n-1} < y_i < x_i, \quad i = 1, 2, \ldots, n+k+1. \tag{3.10}$$

In view of (3.1) and (3.7) we see that the first $n+1$ inequalities (3.10) are automatically satisfied. Returning by (3.8) to our old notation we find the remaining k inequalities (3.10) to be

$$x_1 < \xi_1 < x_{n+2}$$

$$x_2 < \xi_2 < x_{n+3}$$

$$- - - - - - -$$

$$x_k < \xi_k < x_{n+k+1}$$

This concludes the proof of the theorem.

The ineuqlities (3.6) may be described in words: The first k interpolation points $x_1, \ldots, x_k$ precede the knots $\xi_1, \ldots, \xi_k$ respectively which in turn precede the last k interpolation points $x_{n+2}, \ldots, x_{n+k+1}$ respectively. Notice that all these conditions are satisfied automatically in case the knots $\xi_1, \ldots, \xi_k$ are chosen among the $n + k - 1$ interpolation points $x_2, x_3, \ldots, x_{n+k}$. We may rephrase this last result as follows:

Theorem 3.5 Let there be given a set of $N + 1$ abscissae

$$x_0 < x_1 < \cdots < x_N \tag{3.11}$$

and $N + 1$ arbitrary corresponding ordinates y_i $(i = 0, 1, \ldots, N)$. As is well-known, we may interpolate them uniquely by a polynomial of degree almost N. This corresponds to the classical case $k = 0$ of no knots whatever. Also we may choose any combination of k $(1 \leq k \leq N - 1)$ among the interior abscissae.

$$x_1, x_2, \ldots, x_{N-1} \tag{3.12}$$

to serve as the knots $\xi_1, \ldots, \xi_k$ of an

<u>interpolating spline curve $y = F(x)$ in which case again we may interpolate uniquely by a spline curve of degree</u>

$$n = N - k \tag{3.13}$$

<u>Remark 3.6</u> In the extreme case of $k = N - 1$ all points (3.12) are knots of F (x) which by (3.13) is a spline curve of degree $n = N - (N - 1) = 1$. Therefore $y = F(x)$ is the ordinary polygonal line outaimed by linear interpolation between each pair of consecutive points. Theorem 3.5 furnishes a sequence of interpolation procedures bridging the gap between ordinary polynomial interpolation and the linear polygonal interpolation, each additional knot lowering the degree of the spline curve by one unit.

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