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LECTURES ON

DIOPHANTINE APPROXIMATIONS

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by

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The author is an eighteen year old student studying in the second year of the B.Sc. Mathematics class in the Madras University. The report comprises the lectures delivered at Matscience based on his research work during and after the tenure as a visiting scholar at the Australian National University under Professor Kurt Mahler, F.R.S.

C O N T E N T S

A FAREY SERIES WITH FIBONACCI NUMBERS

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The Farey series is an old and famous set of fractions associated with the integers. We here show that if we form a Farey series with Fibonacci Numbers, the properties of the series are remarkably preserved. In fact we find that with the new series we are able to observe and identify 'points of symmetry', 'intervals', 'generating fractions' and stages. The paper is divided into three parts. In part I, we define 'points of symmetry', 'intervals' and 'generating fractions' and discuss general properties of the Farey series with Fibonacci numbers. In part II we define conjugate fractions, couplets, and conjugate couplets and deal with properties associated with intervals. Part III considers the Farey series with fibonacci numbers as having been divided into stages and contains properties associated with "corresponding fractions" and "corresponding stages". A generalisation of the Farey series with Fibonacci numbers is given at the end of the third part.

The Farey series with Fibonacci numbers of order T_n (where T_n stands for the n^{th} term of the Fibonacci sequence) is the set of all possible fractions T_i/T_j , $i = 0,1,2,3,\ldots\ m-1$, $j = 1,2,3,\ldots\ n$ $(i \quad j)$ arranged in ascending order of magnitude. The last term is $\frac{1}{1}$ i.e. $\frac{T_1}{T_2}$. The first term is $\frac{0}{T_{n-1}}$. We set $T_o = 0$ so that $T_o + T_1 = T_2$, $T_1 = T_2 = 1$.

For convenience we denote a Fibonacci number by F, the Farey series of Fibonacci numbers by f.f. and the r^{th} term in the new farey series of order T_n by $f_{(r)n}$.

PART 1

Definition 1.1. Besides $\frac{1}{1}$ We define an $f_{(r)n}$ to be a point of symmetry if $f_{(r+1)n}$ and $f_{(r-1)n}$ have the same denominator. We have shown in an appendix the Farey series of all Fibonacci numbers upto 34.

Definition 1.2. We define an interval to be set of all f.f. fractions between two consecutive points of symmetry. The interval may be closed or open depending upon the inclusion or omission of the points of symmetry. A closed interval is denoted by [ ] and an open interval by ( ).

Definition 1.3. The distance between $f_{(r)k}$ and $f_{(k)n}$ is equal to r - k.

THEOREM 1.1. If $f_{(r)n}$ is a point of symmetry then it is of the form $1/T_i$. Moreover $f_{(r+k)n}$ and $f_{(r-k)n}$ have the same denominator if they do not pass beyond the next point of symmetry on either side. The converse is also true

Proof: In the f.f. series the terms are arranged in the following fashion. The terms in the last interval are of the form $\frac{T_{j-1}}{T_j}$

The terms in the interval prior to the last are of the form $\frac{T_{j-2}}{T_j}$ etc.

If there are two fractions $\frac{T_{j-1}}{T_{j-1}}$, and $\frac{T_{i-2}}{T_{j-2}}$ then their mediant $\frac{T_i}{T_j}$ lies in between them. That is if $\frac{\Gamma_{i-1}}{\Gamma_{j-1}} < \frac{\Gamma_{i-2}}{\Gamma_{j-2}}$ then $\frac{\Gamma_{i-1}}{\Gamma_{j-1}} < \frac{\Gamma_i}{\Gamma_j} < \frac{\Gamma_{i-2}}{\Gamma_{j-2}}$

if $\frac{\Gamma_{i-2}}{\Gamma_{j-2}} < \frac{\Gamma_{i-1}}{\Gamma_{j-1}}$ then $\frac{\Gamma_{i-2}}{\Gamma_{j-2}} < \frac{\Gamma_i}{\Gamma_j} < \frac{\Gamma_{i-1}}{\Gamma_{j-1}}$

This inequality can easily be established dealing with the two cases separately.

We shall adopt induction as the method of proof. Our surmise has worked for all f.f. series upto 34. Let us treat as T_{n-1}. For the next f.f. series i.e. of order T_n, fractions to be introduced are

$\frac{\Gamma_2}{\Gamma_n}, \frac{\Gamma_3}{\Gamma_n}, \ldots, \frac{\Gamma_i}{\Gamma_n}; \frac{\Gamma_{n-1}}{\Gamma_n}$. $\frac{\Gamma_i}{\Gamma_n}$ will fall in between $\frac{\Gamma_{i-1}}{\Gamma_{n-1}}$ and $\frac{\Gamma_{i-2}}{\Gamma_{n-2}}$. First assume that $\frac{\Gamma_{i-1}}{\Gamma_{n-1}} < \frac{\Gamma_{i-2}}{\Gamma_{n-2}}$

Since our assumption is valid for 34, $\frac{\Gamma_{i-1}}{\Gamma_{n-1}}$ lies just before $\frac{\Gamma_{i-2}}{\Gamma_{n-2}}$. $\frac{\Gamma_{i-3}}{\Gamma_{n-2}}$ will occur just after $\frac{\Gamma_{i-2}}{\Gamma_{n-1}}$ from our assumption regarding points of symmetry. But $\frac{\Gamma_{i-1}}{\Gamma_n}$ lies in between these two fractions. The distance of $\frac{\Gamma_{i-1}}{\Gamma_n}$ from the point of symmetry is equal to the distance $\frac{\Gamma_i}{\Gamma_n}$ from that point of symmetry. Hence this is valid for 55. Similarly it can be made to hold good for 89, ... Hence the theorem.

<u>THEOREM 1.2.</u> Whenever we have an interval $\left[\frac{1}{\Gamma_i}, \frac{1}{\Gamma_{i-1}}\right]$ the denominator of term next to $\frac{1}{\Gamma_i}$ is Γ_{i+2}. The denominator of the next term is Γ_{i+4}, then Γ_{i+6} ..... We have this till we reach the maximum for that f.f. sequence i.e. till Γ_{i+2k} does not exceed Γ_n. Then the denominator of the term after Γ_{i+2k} will be the maximum possible term not greater than Γ_n, but not equal to any of the terms formed i.e. its either Γ_{i+2k+1} or Γ_{i+2k-1} say Γ_j. The denominator of the terms after

Γ_j will be Γ_{j-2} , Γ_{j-1} , ... till we reach $\frac{1}{\Gamma_{i-1}}$ (As an example let us take $\left[\frac{1}{3}, \frac{1}{2}\right]$ in the f.f. series for 55. Then the denominator of the later terms in order are 8,21,55,34,13,5,2).

Proof: The proof of Theorem 1,2 will follow by induction on Theorem 1.1.

THEOREM 1.3. a) If $\frac{h}{k}$, $\frac{h'}{k'}$, $\frac{h''}{k''}$ are three consecutive fractions of an f.f. series then

$$\frac{h+h''}{k+k''} = \frac{h'}{k'} \quad \text{if } \frac{h'}{k'} \text{ is not a point of symmetry}$$

b) If $\frac{h'}{k'}$ is a point of symmetry say $\frac{1}{\Gamma_i}$ then

$$\frac{\Gamma_{i-2}\, h + \Gamma_{i-1}\, h''}{\Gamma_{i-2}\, k + \Gamma_{i-1}\, k''} = \frac{h'}{k'}$$

Proof: Case 1. (From 1.2) We see that

$$\frac{h}{k} = \frac{\Gamma_{i-2}}{\quad} , \frac{h'}{k'} = \frac{\Gamma_i}{\Gamma_j} , \frac{h''}{k''} = \frac{\Gamma_{i+2}}{\Gamma_{j+2}}$$

in this case $\frac{\Gamma_{i+2} + \Gamma_{i-2}}{\Gamma_{j+2} + \Gamma_{j-2}} = \frac{^*3.\Gamma_i}{3.\Gamma_j} = \frac{\Gamma_i}{\Gamma_j} = \frac{h'}{k'}$

$^*\Gamma_{n+2} - \Gamma_{n-2} = 3\ \Gamma_n$ is a property of the Fibonacci sequence.

Case 2. $\frac{h'}{k'} = \frac{\Gamma_i}{\Gamma_j}$, $\frac{h}{k} = \frac{\Gamma_{i-2}}{\Gamma_{j-2}}$ and $\frac{h''}{k''} = \frac{\Gamma_{i+1}}{\Gamma_{j+1}}$ (From 1.2)

Then $\frac{\Gamma_{i+1} + \Gamma_{I-2}}{\Gamma_{j+1} + \Gamma_{i-2}} = \frac{2\Gamma_i}{2\Gamma_j} = \frac{\Gamma_i}{\Gamma_j} = \frac{h'}{k'}$ similarly.

Case 3. $\frac{h'}{k'} = \frac{\Gamma_i}{\Gamma_j}$, $\frac{h}{k} = \frac{\Gamma_{i-2}}{\Gamma_{j-2}}$, $\frac{h''}{k''} = \frac{\Gamma_{i-1}}{T_{j-1}}$ (From 1.2)

$\therefore \frac{\Gamma_{i-1} + T_{1-2}}{\Gamma_{j-1} + \Gamma_{j-2}} = \frac{\Gamma_i}{\Gamma_j} = \frac{h'}{k'}$. Hence the result.

1.3.b. Let $\frac{h'}{k'} = \frac{1}{\Gamma_i}$. From Theorem 1.2 it follows that

$$\frac{h''}{k''} = \frac{3}{\Gamma_{i+2}} \text{ and } \frac{h}{k} = \frac{2}{\Gamma_{i+2}}$$

$$\therefore \frac{\Gamma_{i-2}\frac{h}{k} + \Gamma_{i-1}\frac{h''}{k''}}{\Gamma_{i-2}\frac{k}{k} + \Gamma_{i-2}\frac{k''}{k''}} = \frac{2\Gamma_{i-2} + 3\Gamma_{i-1}}{\Gamma_i \quad \Gamma_{i+2}} = \frac{\Gamma_{i+2}}{\Gamma_i \quad \Gamma_{i+2}} = \frac{1}{\Gamma_i}$$

Hence the theorem.

THEOREM 1.4. If $\frac{h}{k}$, and $\frac{h'}{k'}$ are two consecutive fractions of an f.f. series then $\left|\frac{h-h'}{k-k'}\right|$ f.f. $(k-k' \neq 0)$.

Proof: Since $f_{(r)n}$ is of the form $\frac{\Gamma_i}{\Gamma_j}$ it follows that $|h-h'|$ is equal to Γ_i and $|k-k'| = \Gamma_j$. Since $\frac{h}{k}$ and $\frac{h'}{k'}$ are members also, $h = \Gamma_{i_1}$, $h' = \Gamma_{i_2}$, $k = \Gamma_{j_1}$, $k' = \Gamma_{j_2}$. Further $|\Gamma_{j_1} - \Gamma_{j_2}| = \Gamma_j$ and $|\Gamma_{i_1} - \Gamma_{i_2}| = \Gamma_i$. But from the Fibonacci recurrence relation $\Gamma_n = \Gamma_{n-1} + \Gamma_{n-2}$ we see that the condition for this is $|i_1 - i_2| \leqslant 2$ and $|j_1 - j_2| \leqslant 2$ (but not zero) which follows from Theorem 1.2. Actually $\frac{h-h'}{k-k'}$ are the fractions of the same interval arranged in descending order of magnitude for increasing values of $\frac{h}{k}$.

Definition 1.4 We define the distance between $f_{(r)\,n}$ and $f_{(k)n}$ as equal to $|r-k|$.

We now introduce a term 'Generating Fraction'. If we have a fraction $\frac{\Gamma_i}{\Gamma_j}$ $(i < j)$. We split $\frac{\Gamma_i}{\Gamma_j}$ into $\frac{\Gamma_{i-1} + \Gamma_{i-2}}{\Gamma_{j-1} + \Gamma_{j-2}}$. We form from this two fractions $\frac{\Gamma_{i-1}}{\Gamma_{j-1}}$ and $\frac{\Gamma_{i-2}}{\Gamma_{j-2}}$ such that $\frac{\Gamma_i}{\Gamma_j}$ is the mediant of the fractions formed. We continue this process and split the fractions detained till we reach a state where the numerator is 1. $\frac{\Gamma_i}{\Gamma_j}$ then amounts to the Generating fraction of the others. We call $\frac{\Gamma_i}{\Gamma_j}$ as the Generating Fraction of an Interval

(G.F.I) if through this process we are able to get from the G.F.I. all the other fractions of 'that' closed interval. We can clearly see in a f.f. series for $\Gamma_1, \Gamma_2, \ldots \Gamma_n$, $\frac{\Gamma_i}{\Gamma_n}$ will a G.F.I. (We also note that $\frac{\Gamma_i}{\Gamma_j}$, $\frac{\Gamma_{i-1}}{\Gamma_{j-1}}$, $\frac{\Gamma_{i-2}}{\Gamma_{j-2}}$, ... belong to the same interval because the difference in the suffix of the numerator denominator and is j-i). Hence the sequence of G.F.I's is $\frac{\Gamma_1}{\Gamma_n}, \frac{\Gamma_2}{\Gamma_n}, \frac{\Gamma_3}{\Gamma_n}, \ldots \frac{\Gamma_{n-1}}{\Gamma_n}$. We now see some properties concerning G.F.I's.

<u>THEOREM 1.5.</u> If we form a sequence of the distance between two consecutive G.F.I's such a sequence runs thus

$$2,\ 2,\ 4,\ 4,\ 6,\ 6,\ 8,\ 8,\ \ldots\ldots,$$

i.e., alternate G.F.I's are symmetrically placed about a G.F.I.

<u>THEOREM 1.6.</u> If we take the first G.F.I. say $f_{(g1)n}$ then $f_{(g_1+1)n}$ and $f_{(g_1-1)n}$, have the same denominator. For $f_{(g_2)n}$ the second G.F.I. $f_{(g_2+2)n}$, and $f_{(g_2-2)n}$ have the same denominator. In general for $f_{(g_k)n}$ the k^{th} G.F.I. $f_{(g_k+k)n}$ and $f_{(g_k-k)n}$ have the same denominator.

The proofs of theorems 1.5 and 1.6 follows from 1.2.

(NOTE:- We can verify that for alternate G.F.I's $f_{(g_2)n}$, $f_{(g_4)n}$, $f_{(g_6)n}$ ....., $f_{(g_k+k)n}$ and $f_{(g_k-k)n}$ have the same denominator for k is even and the sequence of distance shown above is 2, 2, 4, 4, 6, 6, 8, 8, ...).

<u>Definition:</u> We now define Γ_{i-2} to be the 'factor of the interval' $\frac{1}{\Gamma_i}, \frac{1}{\Gamma_{i-1}}$. More precisely the factor of a closed interval is that term Γ_z where 'z' = suffix of denominator minus suffix of the numerator, of each fraction of that interval. It can be

easily seen (Part I) that 'z' is a constant.

LEMMA 2.1. If $\left|j_1 - i_1\right| = \left|j_2 - i_2\right| > 0$

Then

$$\left|T_{j_1} T_{i_2} - T_{j_2} T_{i_1}\right| = \left|T_{j_2 - j_1}\right| \left|T_{j_1 - i_1}\right| = \left|T_{j_2 - j_1}\right| \left|T_{j_2 - i_2}\right|.$$

(Note:-

The Fibonacci sequence is capable of extension to $T_{-\infty}$, and $\left|T_{-n}\right| = \left|T_n\right|$. More accurately $T_{-n} = T_n(-1)^{n-1}$, T_n being positive and $\left|T_k\right| = T_{|k|}$

Proof: We apply Binet's formula that

$$T_n = \frac{a^n - b^n}{a - b} \qquad \text{where} \quad a = \frac{1 + \sqrt{5}}{2}, \quad b = \frac{1 - \sqrt{5}}{2}$$

Then (L.H.S.) the left hand side of the expression, and (R.H.S.) the right hand side of the expression reduce as follows.

To prove

$$\left|\frac{a^{j_1} - b^{j_1}}{a - b} \cdot \frac{a^{i_2} - b^{i_2}}{a - b} - \frac{a^{j_2} - b^{j_2}}{a - b} \cdot \frac{a^{i_1} - b^{i_1}}{a - b}\right|$$

$$= \left|\frac{a^{j_2 - j_1} - b^{j_2 - j_1}}{a - b}\right| \cdot \left|\frac{a^{j_1 - i_1} - b^{j_1 - i_1}}{a - b}\right|$$

because $j_1 - i_1 > 0$, $T_{j_1 - i_1}$ is positive and hence can be put within the sign.

To prove

$$\left|(a^{j_1} - b^{j_1})(a^{i_2} - b^{i_2}) - (a^{j_2} - b^{j_2})(a^{i_1} - b^{i_1})\right| = \left|(a^{j_2 - j_1} - b^{j_2 - j_1})(a^{j_1 - i_1} - b^{j_1 - i_1})\right|$$

L.H.S. reduces to

$$\left|a^{j_1 + i_1} - a^{j_1} . b^{i_2} + b^{j_1 + i_1} - b^{j_1} a^{i_2} - a^{j_2 + i_1} + a^{j_2} b^{i_1} + b^{j_2} a^{i_1} - b^{j_2 - i_1}\right|$$

$$= \left|-a^{j_1} b^{i_2} - a^{i_2} - b^{j_1} + a^{j_2} . b^{i_1} + b^{j_2} a^{i_1}\right|$$

R.H.S. reduces to

$$\left|a^{j_2-i_1} - a^{j_2-j_1} \,.\, b^{j_1-i_1} + b^{j_2-i_1} - b^{j_2-i_1} \,.\, a^{j_1-i_1}\right|$$

This may be simplified further using $ab = -1$ and $j_1-i_1 = j_2-i_2$

R.H.S. then is

$\left|a^{j_1} b^{i_2} + b^{j_1} a^{i_2} - a^{j_2} b^{i_1} - b^{j_2} a^{i_1}.\right|$ We can see that L.H.S. = R.H.S. Hence the Lemma.

<u>COROLLARY.</u> From this we may deduce that if $\frac{T_{i_1}}{T_{j_1}}$ and $\frac{T_{i_2}}{T_{j_2}}$ belong to the same interval i.e. $j_1-i_1 = j_2-i_2$ then

$$\left|T_{j_1} T_{i_2} - T_{j_2} T_{i_1}\right| \quad \left(\frac{T_{i_1}}{T_{j_1}} < \frac{T_{i_2}}{T_{j_2}}\right) = \left|T_{j_2-j_1}\right| \left|T_{j_1-i_1}\right| \text{ or } T_{j_2-j_1} \quad T_{j_2-i_2}$$

Hence $T_{j_1} T_{i_2} - T_{j_2} T_{i_1}$ will be an integral multiple of $T_{j_1-i_1}$ or $T_{j_2-i_2}$ (the factor of that interval) which is the term obtained by the difference in suffixes of the numerator and denominator of each fraction of that interval.

Definition: We now introduce the term 'conjugate fractions'. Two fractions h/k, and h'/k', h/k h'/k' are conjugate in an interval $\left[\frac{1}{T_i}, \frac{1}{T_{i-1}}\right]$ if the distance of h/k from $\frac{1}{T_i}$ equals the distance of h'/k' from $\frac{1}{T_{i-1}}$ $(h/k \neq h'/k')$.

COROLLARY: Two consecutive points of symmetry are conjugate with distance zero.

<u>THEOREM 2.2.</u> If h/k, and h'/k' are conjugate $\left[\frac{1}{T_i}, \frac{1}{T_{i-1}}\right]$ then $kh' - hk' = T_{i-2}$

Proof: Using I, we can easily see that if h/k is of the form $\frac{T_{i_1}}{T_{j_1}}$ then h'/k' is $\frac{T_{i_1-1}}{T_{j_1-1}}$ ..... (1)

$\frac{1}{T_i}$, and $\frac{1}{T_{i-1}}$ are conjugate. This agrees with (1) since $T_2 = T_1 = 1$, since the term after $\frac{1}{T_i} = \frac{T_4}{T_{i+2}}$, and the term before $\frac{1}{T_{i-1}} = \frac{2}{T_{i+1}}$, We see it agrees with the statement (1) above. Proceeding in such a fashion we obtain the result 1. Of course we assume here that there axist atleast two terms in $\left[\frac{1}{T_i}, \frac{1}{T_{i-1}}\right]$. If there are less than two the result is immediate.

Hence we can see that ny two conjugate to fractions in $\left[\frac{1}{T_i}, \frac{1}{T_{i-1}}\right]$ are $\frac{T_{j-i+2}}{T_j}$, $\frac{T_{j-i+1}}{T_j}$.

To show

$$\left|T_j \quad T_{j-1+1} - T_{j-1} \quad T_{j-i+2}\right| = T_{i-2}.$$

This will immediately follow from Lemma 2.1.

THEOREM 2.3. a) If h/k, and h'/k' and two consecutive fractions in an f.f. series, which belong to

$\left[\frac{1}{T_i}, \frac{1}{T_{i-1}}\right]$ then $kh' - hk' = T_{i-2}$.

b) If h/k and h'/k' are conjugate in an interval $\left[\frac{1}{T_i}, \frac{1}{T_{i-1}}\right]$ $kh' - hk' = T_{i-2}$.

Proof:- Theorem 2.3 a) and 2.3 b) can be proved using Lemma 2.1 and Theorem1.2.

Definition. If $h/k \in \left[\frac{1}{T_i}, \frac{1}{T_{i-1}}\right]$ we define the couplet for h/k as the ordered pair $\left[\left(\frac{1}{T_i}, \frac{h}{k}\right), \left(\frac{h}{k}, \frac{1}{T_{i-1}}\right)\right]$

THEOREM 2.4. In the case of couplets we find that

$$(T_i \quad h) - k = T_p \cdot T_{i-2}$$

and

$$k - T_{i-1} \quad h = T_{p+1} \quad T_{i-2}$$ (where T_p is some Fibonacci number).

Proof. Let h/k be $\dfrac{T_{j-i+2}}{T_j}$

Then $(T_i \quad h) - k$ is $T_i \quad T_{j-i+2} - T_j = T_p \quad T_{i-2}$ (1)

and $k - T_{i-i} \quad h$ is $T_j - T_{i-1} \quad T_{j-1+2} = T_{p+1} \quad T_{i-2}$ (2)

Adding (1) and (2) we have

$$T_{i-2} \quad T_{j-1+2} = T_{p+2} \quad T_{i-2}$$

$$\therefore T_{j-i+2} = T_{p+2} \quad \text{or} \quad j-i = p.$$

i.e. $T_i \quad T_{j-i+2} - T_j = T_{j-i} \quad T_{i-2}$. (3)

We can establish 3 using Lemma 2.1. Hence the proof.

Definition: We define $\left[\left(\dfrac{1}{T_i} \quad \dfrac{h}{k}\right) \left(\dfrac{h}{k}, \dfrac{1}{T_{i-1}}\right)\right]$ and $\left[\left(\dfrac{1}{T_i}, \dfrac{h'}{k'}\right), \left(\dfrac{h'}{k'}, \dfrac{1}{T_{i-1}}\right)\right]$ to be conjugate couplets if and h/k, and h'/k' are conjugate fractions of the closed interval $\left[\dfrac{1}{T_i}, \dfrac{1}{T_{i-1}}\right]$.

THEOREM 2.5. In the case of conjugate couplets

$$\text{if} \quad T_i h - k = T_p \cdot T_{i-2} \quad \text{and}$$

$$k - T_{i-1} h = T_{p+1} T_{i-2} \quad \text{then}$$

$$T_i h' - k' = T_{p-1} T_{i-2} \quad \text{and}$$

$$k - T_i -1 h' = T_p T_{i-2}$$

Proof: We note that (j-1) in the previous proof is the difference in the suffixes of T_j and T_i . If now $h/k = \dfrac{T_{j-i+2}}{T_j}$ then $p = j-1$. But since h'/k' is conjugate with h/k, h'/k' is equal to $\dfrac{T_{j-i+1}}{T_{j-1}}$.

Therefore the constant factor say T_q in the equation for h'/k', $T_i h' - k = T - T_{i-2}$ is such that

$$q = j-1-i = (j-i)-1 = p-1.$$

$\therefore T_i h' - k' - T_{p-1} T_{i-2}$. Hence $k - T_{i-1} h' = T_p T_{i-2}$ since it follows from 2.4.

THEOREM 2.6. Since we have seen that if h/k, and h'/k' are conjugate then the difference in suffixes of their numerators or denominators = 1 we find

$$\frac{h+h'}{k+k'} \left| \frac{1}{T_i} \quad \frac{1}{T_{i-1}} \right| \quad \text{and} \quad \left| \frac{h-h'}{k-k'} \right| \in \left[\frac{1}{T_i} , \frac{1}{T_{i-1}} \right]$$

if h/k, h'/k' $\dfrac{1}{T_i}$, $\dfrac{1}{T_{i-1}}$. We may also further note that $\dfrac{h+h'}{k+k'}$

are the fractions of the latter half of the interval arranged in descending order while $\frac{h - h'}{k - k'}$ are the fractions of the first half arranged in ascending order, for increasing values of h/k.

PART III

We now give a generalised result concerning 'sequence of distances'.

THEOREM 3.1.a. Points of symmetry if they are of the form $f_{(r)n}$ then r 2,3,5,8,12,17, ... Or the sequence of distance between two consecutive points of symmetry will be

1, 2, 3, 4, 5, 6, ...

an Arithmetic progression with common difference 1.

THEOREM 3.1.b. The sequence of distance for fractions with common numerator T_{2n-1} or T_{2n} is 2n-1, 2n, 2n+1, ...

Proof: To prove Theorem 3.1.a. we have to show that if there 'n' terms in an interval then there are (n+1) terms in the next.

Let there be 'p' terms of ratio $\frac{T_i}{T_j}$. It is evident that there are p+1 terms of the form $\frac{T_{i+1}}{T_j}$. But these (p+1) terms of the form $\frac{T_{i+1}}{T_j}$ are in an interval next to that in which the 'p' terms of the form $\frac{T_i}{T_j}$ lie. So the sequence is an AP with common difference 1. Moreover the second term is always $\frac{1}{T_n}$ (evident). Hence the result.

If we fix the numerator to be '2' and take the sequence $\frac{2}{T_n}$, $\frac{2}{T_{n-1}}$, $\frac{2}{T_{n-2}}$, ... $\frac{2}{3}$ then the sequence of distance between two consecutive such fractions is

3, 4, 5, ...

From 1.2 (Part I) it follows that $\frac{2}{T_i}$ lies just before a point of symmetry say $\frac{1}{T_j}$. Since we have seen the sequence of distance concerning points of symmetry it will follow that here too the common difference is 1. The first term is 3 for there be two terms between $\frac{2}{T_n}$, and $\frac{2}{T_{n-1}}$. The inequality $\frac{2}{T_n} < \frac{1}{T_{n-2}} < \frac{3}{T_{n-2}} < \frac{2}{T_{n-1}}$ can be established. Hence the result.

In a similar fashion we find that the sequence of distance for numerator 3 is

3, 4, 5, ...

We shall give a table and the generalisation

numerator	Sequence of distances
T_1 or T_2	1, 2, 3, 4, 5, ...
T_3 or T_4	3, 4, 5, 6, ...
T_5 or T_6	5, 6, 7, 8, ...
T_{2n-1} or T_{2n}	2n-1, 2n, 2_{n+1}, 2_{n+2} ......

Definition: Just as we defined an interval, we now defined an interval, we now define a 'stage' as the set of f.f. fractions lying between two consecutive G.F.I's. The stage may be closed or open depending upon the inclusion or omission of the G.F.I's.

Since the sequence of distance of G.F.I's is 2, 2, 4, 4, 6, 6,.. It is possible for two consecutive 'stages' to have equal number of terms. We define two stages

$$\left[\frac{T_{i-1}}{T_n}, \frac{T_i}{T_n}\right] \text{ and } \left[\frac{T_i}{T_n}, \frac{T_{i+1}}{T_n}\right]$$ to be conjugate

stages if the distance of $\frac{T_j}{T_n}$ from $\frac{T_{i-1}}{T_n}$ equals the distance of $\frac{T_{i+1}}{T_n}$ from $\frac{T_i}{T_n}$. That is the number of terms in two conjugate stages, are equal. We call a stage comprision of both these stages, as a '~~conjugate~~ complex stage'. Let us now investigate properties concerning stages. If we have ~~conjugate~~ complex stage $\left[\frac{T_{i-1}}{T_n}, \frac{T_i}{T_n}, \frac{T_{i+1}}{T_n}\right]$ then we define two fractions $\frac{h}{k}$ and $\frac{h'}{k'}$ to be 'corresponding if

$\frac{h}{k} \in \left[\frac{T_{i-1}}{T_n}, \frac{T_i}{T_n}\right]$ and $\frac{h'}{k'} \in \left[\frac{T_i}{T_n}, \frac{T_{i+1}}{T_n}\right]$ and if the distance of $\frac{h}{k}$ from $\frac{T_{i-1}}{T_n}$ is equal to the distance of $\frac{h'}{k'}$ from $\frac{T_i}{T_n}$.

THEOREM 3.2. Two corresponding fractions have the same numerator. If $\frac{h}{k}$ and $\frac{h'}{k'}$ are corresponding fraction's then $h = h'$.

Proof: This will follow from 1.2 (previous Part).

Let $\frac{T_{i-1}}{T_n}$ be the maximum reached in its interval so that $\frac{T_{i-1}}{T_{n-1}}$ will be the maximum for the interval in which $\frac{T_i}{T_n}$ belongs. The terms next to $\frac{T_{i-1}}{T_n}$ is $\frac{T_{i-2}}{T_{n-1}}$. Similarly the terms next to $\frac{T_i}{T_n}$ is $\frac{T_{i-2}}{T_{n-2}}$. But these fractions are corresponding fractions. Proceeding in such a fashion we obtain the result.

$\frac{T_{i-1}}{T_n}$ has necessarily to be the maximum in it interval. Since we have considered conjugate stages i is odd. Using 1.2 (previous part) it can be established that alternate G.F.I's are maximum in their interval and that too when suffix of numerator is even (i-1 is even). Hence the result.

Definition. Since the number of terms in a stage is odd, we define $\frac{h}{k}$ to be the middle point of a stage $\left[\frac{T_{i-1}}{T_n}, \frac{T_i}{T_n}\right]$ if it is equidistant from both G.F.I's. We can deduce from this that $\frac{h}{k}$ is a point of symmetry since $\frac{T_{i-1}}{T_n}$, and $\frac{T_i}{T_n}$ have the same denominator. So the middle point of a stage is a point of symmetry.

COROLLARY. If two conjugate stages are taken these their middle points are corresponding. (This follows from the definition). But their numerators should be equal. This is so, for the middle points are points of symmetry whose numerator is 1. This agrees with the result we proved.

Definition. Two fractions $\frac{h}{k}$, and $\frac{h'}{k'}$ are conjugate in a complex stage if the distance of $\frac{h}{k}$ from $\frac{\Gamma_{i-1}}{\Gamma_n}$ equals the distance of $\frac{h'}{k'}$ from $\frac{\Gamma_{i+1}}{\Gamma_n}$, $\frac{h}{k} < \frac{h'}{k'}$ and the complex stage being $\left[\frac{\Gamma_{i-1}}{\Gamma_n}, \frac{\Gamma_i}{\Gamma_n}, \frac{\Gamma_{i+1}}{\Gamma_n}\right]$. Taking their middle points $\left[\frac{1}{\Gamma_p}, \frac{1}{\Gamma_{p+1}}\right]$ we can see that fractions conjugate in this interval are conjugate in the conjugate stage. Further we saw that for conjugate fractions of the interval $\frac{h}{k}$, and $\frac{h'}{k'}$. $\frac{h+h'}{k+k'}$ are fractions of the latter half of the interval arranged in descending order, and $\frac{h-h'}{k-k'}$ are fractions of the first half arranged in ascending order for increasing values of $\frac{h}{k}$.

THEOREM 3.3. For conjugate fractions $\frac{h}{k}$ and $\frac{h'}{k'}$ lying in the outer half of the stage we see that $\frac{h+h'}{k+k'}$ are fractions of the interval in ascending order while $\left|\frac{h-h'}{k-k'}\right|$ are fractions of the first half in descending order for increasing values of $\frac{h}{k}$.

We here only give a proof to show that $\frac{h + h'}{k + k'}$, and $\frac{h - h'}{k - k'}$ are in the inner half but do not prove the order or arrangement.

Proof. For $\frac{h}{k}$, $\frac{h'}{k'}$, the proof has been given (previous paper). The middle point of $\left[\frac{T_{i-1}}{T_n}, \frac{T_i}{T_n}\right] = \frac{1}{T_{n-i+2}}$.

Similarly middle point of $\left[\frac{T_i}{T_n}, \frac{T_{i+1}}{T_n}\right] = \frac{1}{T_{n-i+3}}$.

That two conjugate fractions of the outer half of a conjugate stage differ in suffix by 1 can be established.

i.e. if $\frac{h}{k} = \frac{T_{j-(n-2)-1}}{T_j}$ then $\frac{h'}{k'} = \frac{T_{j-(n-i)}}{T_{j-1}}$

$\frac{h + h'}{k + k'} = \frac{T_{j-(n-i)+1}}{T_{j+1}} \in$ I where I is the Interval.

and $\frac{h - h'}{k - k'} = \frac{T_{j-(n-i)-2}}{T_{j-2}} \in$ I where I is the Interval.

Hence the proof.

Definition. In an f.f. series order T_n, $\left[\frac{T_i}{T_n}, \frac{T_{i+1}}{T_n}\right]$ represents a stage. Let us take an f.f. series of order T_{n+1}. If there we take a stage $\left[\frac{T_i}{T_{n+1}}, \frac{T_{i+1}}{T_{n+1}}\right]$, then we say the two stages are corresponding stages. More generally in an f.f. series

and an f.f. series of order $n+k$, $\left[\frac{T_i}{T_n}, \frac{T_{i+1}}{T_n}\right]$, $\left[\frac{T_i}{T_{n+k}}, \frac{T_{i+1}}{T_{n+k}}\right]$ are corresponding stages.
of order T_{n_k}. We state here properties of corresponding stages. These can be proved using 1.2.

THEOREM 3.4.a) If $\left[\frac{T_i}{T_n}, \frac{T_{i+1}}{T_n}\right]$ and $\left[\frac{T_i}{T_{n+k}}, \frac{T_{i+1}}{T_{n+k}}\right]$ stages then the number of terms in both are equal.

Theorem 3.4.b. There exists a one-one correspondence between the denominators of these stages. If the denominator of the q^{th} term of $\left[\frac{T_i}{T_n}, \frac{T_{i+1}}{T_n}\right]$ is T_j then the denominator of the q^{th} term of $\frac{T_i}{T_{n+k}}, \frac{T_{i+1}}{T_{n+k}}$ is T_{i+k}.

We can extend this ideas further and produce one-one correspondence between $\frac{T_i}{T_n}, \frac{T_{i+m}}{T_n}$ and $\frac{T_i}{T_{n+k}}, \frac{T_{i+m}}{T_{n+k}}$ where $\frac{a}{b}$ and $\frac{c}{d}$ stands for the set of fractions between $\frac{a}{b}$ and $\frac{c}{d}$ inclusive of both. A further extension would give that given two f.f. series one of order T_n and other order T_{n+k}.

THEOREM 3.5.a. The numerator of the r^{th} term of the first series equals the numerator of the r^{th} term of the second.

THEOREM 3.5.b. If the denominator of the r^{th} term of the first series is T_j, then the denominator of the r^{th} term of the second is T_{j+k}. Precisely

a) nr of $f_{(r)n}$ = nr. of $f_{(r)n+k}$

b) If dr. of $f_{(r)n} = T_j$

dr. of $f_{(r)n+k} = T_{j+k}$

where nr stands for numerator and dr. for denominator. This can be proved using 1.2. We can arrive at the same result by defining corresponding intervals.

Definition: Two intervals, $\left[\frac{1}{T_i}, \frac{1}{T_{i+1}}\right]$ in an f.f. series or order T_n and $\left[\frac{1}{T_{i+k}}, \frac{1}{T_{i+k}}\right]$ in an f.f. series of order T_{n+k} are defined to be corresponding intervals.

The same one-one correspondence as in the case of corresponding stage exists for corresponding intervals. We can extend this correspondence in a similar manner to the entire f.f. series and prove that

a) nr. of $f_{(r)n}$ = nr. of $f_{(r)n+k}$.

b) If dr of $f_{(r)n} = T_j$

dr of $f_{(r)n+k} = T_{j+k}$.

a) <u>Generalised f.f. series.</u> We defined the f.f. series in the interval $[0,1]$. We now define it in the interval $[0,\infty)$.

Definition: The f.f. series of order T_n is the set of all functions $\frac{T_i}{T_j}$ $j \leq n$ arranged in ascending order of magnitude $i,j > 0$. If $i < j$ then the f.f. series is in the interval $[0, 1]$. The basis properties of the f.f. series for 0,1 are retained with suitable alterations.

THEOREM 3.6.1. $f_{(r)n}$ is a point of symmetry if $f_{(r+1)n}$ and $f_{(r-1)n}$ have the same numerator (beyond $\frac{1}{1}$). If $f_{(r)n}$ is a point of symmetry then $f_{(r+k)n}$ and $f_{(r-k)n}$ have the same numerator, if each fraction does not pass beyond the next G.F.I. in either side (beyond $\frac{1}{1}$).

THEOREM 3.6.2. A G.F.I. is a fraction with denominator T_n .

THEOREM 3.6.3. An interval is the set of fractions between two consecutive points of symmetry.

THEREOM 3.6.4. A point of symmetry has either numerator or denominator 1

THEOREM 3.6.5. Beyond $\frac{1}{1}$, any interval is given by $\left[\frac{T_{n-1}}{1}, \frac{T_n}{1}\right]$ The factor of this interval is again T_{n-2} .

THEOREM 3.6.6. The two basic properties

a) $\dfrac{h + h''}{k + k''} = \dfrac{h'}{k'}$ and

b) $kh' - hk' = T_{n-2}$ are retained.

THEOREM 3.6.7. If a) does not good for $\frac{h'}{k'}$ being a point of symmetry, then

$$\frac{h'}{k'} = \frac{T_{n-1}\,h'' + T_{n-2}\,h}{T_{n-1}\,k'' + T_{n-2}\,k} \text{ if } \frac{h}{k} \quad \frac{h'}{k'} \quad \frac{h''}{k''} \; ; \quad \frac{h'}{k'} = \frac{T_n}{1}$$

NOTE: Only in the case of $\frac{1}{2}$, and $\frac{1}{1}$ does a) hold good for 2 = 1 + 1, and we do not accept 1 being split up.

f.f. series of order 5 $\frac{0}{3}, \frac{1}{5}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{1}{1}$

f.f. series of order 8 $\frac{0}{5}, \frac{1}{8}, \frac{1}{5}, \frac{2}{8}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{1}{1}$

f.f. series of order 13 $\frac{0}{8}, \frac{1}{13}, \frac{1}{8}, \frac{2}{13}, \frac{1}{5}, \frac{3}{13}, \frac{2}{8}, \frac{1}{3}, \frac{3}{8}, \frac{5}{13}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \frac{5}{8}, \frac{2}{3}, \frac{1}{1}$.

f.f. series of order 21. $\frac{0}{13}, \frac{1}{21}, \frac{1}{13}, \frac{2}{21}, \frac{1}{8}, \frac{3}{21}, \frac{2}{13}, \frac{1}{5}, \frac{3}{13}, \frac{5}{21}, \frac{2}{8}, \frac{1}{3}, \frac{3}{8}, \frac{8}{21}, \frac{5}{13}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \frac{13}{21}, \frac{5}{8}, \frac{2}{3}, \frac{1}{1}$

f.f. series of order 34 $\frac{0}{21}, \frac{1}{34}, \frac{1}{21}, \frac{2}{34}, \frac{1}{13}, \frac{3}{34}, \frac{2}{21}, \frac{1}{8}; \frac{3}{21}, \frac{5}{34}, \frac{2}{13}, \frac{1}{5}, \frac{3}{13}, \frac{8}{34}, \frac{5}{21}, \frac{2}{8}, \frac{1}{3}, \frac{3}{8}, \frac{8}{21}, \frac{13}{34}, \frac{5}{13}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \frac{21}{34}, \frac{13}{21}, \frac{5}{8}, \frac{2}{3}, \frac{1}{1}$

APPROXIMATION OF IRRATIONALS WITH FAREY FIBONACCI FRACTIONS

Krishnaswami Alladi

The author in [1] had defined the Farey sequence of Fibonacci Numbers as follows:

A Farey sequence of Fibonacci Numbers of order f_n = is the set of all possible fractions $\frac{f_i}{f_j}$ $j \leq n$ put in ascending order of magnitude { $n, i, j \geq 0$, are positive integers; f_n denotes the n-th Fibonacci Number . $\frac{0}{f_{n-1}}$ is the first fraction. This set is denoted by $f.f_n$. }

We also defined an "INTERVAL" in $f.f_n$ to consist of all fractions in $f.f_n$ between fractions of the form $\left[\frac{1}{f_i}, \frac{1}{f_{i-1}}\right]$ $i \leq n$ $[f_{j-1}, f_j]$ $j > 0$, j a positive integer. Two symmetry properties were established:

1) Let $\frac{h}{k}, \frac{h'}{k'}, \frac{h''}{k''}$ be three consecutive fractions in $f.f_n$ all greater than 1. Further let $f_{i-1} < \frac{h}{k} < \frac{h'}{k'} < \frac{h''}{k''} < f_i$.

then

a) $\frac{h+h''}{k+k''} = \frac{h'}{k'}$,

b) $kh' - hk' = f_{i-2}$.

1') Let $\frac{h}{k}, \frac{h'}{k'}, \frac{h''}{k''}$ be three consecutive fractions in $f.f_n$ than 1. Further let $\frac{1}{f_i} < \frac{h}{k} < \frac{h'}{k'} < \frac{h''}{k''} < \frac{1}{f_{i-1}}$. Then

a) $\frac{h+h''}{k+k''} = \frac{h'}{k'}$

b) $kh' - hk' = f_{i-2}$.

Many other relations of symmetries besides these are proved in [1] . 1(a), 1(b) are similar to properties which are preserved by the Farey sequence also. Actually instead of arranging Fibonacci fractions, in ascending order, we had arranged fractions of the sequence $U_n = U_{n-1} + U_{n-2}$ $U_1 > U_0 \geq 0$ integers, still some of the properties will remain. However with the Fibonacci Sequence we get more symmetries.

The problem we discuss in this paper is the approximation of irrationals with Farey Fibonacci Fractions. We prove some theorem on best approximations.

DEFINITION: Consider any $f.f_n$. Form a new ordered set $ff_{n,1}$ consisting of all reationals in $f.f_n$, together with mediants of consecutive rationals in $f.f_n$. Define recursively $f.f_{n,r+1}$ as all the rationals in $f.f_{n,r}$ together with mediants of consecutive rationals in $f.f_{n,r}$. The first rational in $f.f_{n,r+1}$ is rewritten as $\frac{0}{f_{n+r}}$. We now define

$$F.F_n = \bigcup_{r=1}^{\infty} f.f_{n,r} .$$

PROPOSITIONS: $F.F_n$ is dense in $(0,\infty)$ in the sense that its closure gives the interval $(0,\infty)$. This implies that every irrational 'θ' can be approximated by a sequence of rationals $\frac{h}{k}$ in $F.F_n$. Without loss of generality we consider only the case $\theta > 0$, for $\theta < 0$ can be approximated by $\frac{-h}{k}$ where $\frac{h}{k}$ belong to $F.F_n$. They are all quite obvious, and can be easily seen from 1) and 1^{*}).

We now begin with a theorem on best approximation.

THEOREM 1. a) Let θ be an irrational >1, say $f_{i-1}<\theta<f_i$. Then there exist infinitely many rationals $\frac{h}{k} \in F.F_n$ for each 'n' such that

$$\left| \theta - \frac{h}{k} \right| < \frac{f_{i-2}}{\sqrt{5}k^2} .$$

b) Let θ be an irrational < 1 say $\frac{1}{f_i} < \theta < \frac{1}{f_{i-1}}$.

Then there exist infinitely many $\frac{h}{k} \in F.F_n$ for every 'n' such that

$$\left| \theta - \frac{h}{k} \right| < \frac{f_{i-2}}{\sqrt{5}k^2} .$$

Moreover the constant $\sqrt{5}$ is the best possible in the sense that the assertion fails if $\sqrt{5}$ is replaced by a bigger constant.

Proof. We prove only Theorem 1a). The proof of 1b) is similar. In proving the theorem we follow the proof of Hurwitz theorem as tiven in Niven's book 2 .

We need the well known lemma

LEMMA. It is impossible to find integers x,y such that the two inequalities simultaneously hold.

$$\frac{1}{xy} \geq \frac{1}{\sqrt{5}} \left[\frac{1}{x^2} + \frac{1}{y^2} \right] ; \quad \frac{1}{x(x+y)} \geq \frac{1}{\sqrt{5}} \left[\frac{1}{x^2} + \frac{1}{(x+y)^2} \right] .$$

We don't give the proof of the lemma as it is known.

Now let 'θ' lie between two consecutive fractions of $f.f_{n.r}$ i.e. $\frac{a}{b} < \theta < \frac{c}{d}$. It is clear that

$$\frac{a}{b} < \frac{a+b}{b+d} < \frac{c}{d} \text{ and } \frac{a+c}{b+d} \quad f.f_{n,r+1} .$$

Now we shall show that at least one of these fractions say $\frac{h}{k}$ satisfies $\left| \theta - \frac{h}{k} \right| < \frac{f_{i-2}}{\sqrt{5}k^2}$.

Case 1. Let $\frac{a}{b} < \frac{a+c}{b+d} < \theta < \frac{c}{d}$. and let

$$\theta - \frac{a}{b} \geq \frac{1}{\sqrt{5}b^2} ; \quad \theta - \frac{a+c}{b+d} \geq \frac{1}{\sqrt{5}(b+d)^2} ; \quad \frac{c}{d} - \theta \geq \frac{1}{\sqrt{5}d^2} .$$

These three inequalities give rise to

$$\frac{c}{d} - \frac{a}{b} \geq \frac{f_{i-2}}{\sqrt{5}} \left[\frac{1}{b^2} + \frac{1}{d^2}\right] \text{ and } \frac{c}{d} - \frac{a+c}{b+d} \geq \frac{f_{i-2}}{\sqrt{5}} \left[\frac{1}{d^2} + \frac{1}{(b+d)^2}\right].$$

Now by properties 1a) and 1b) we get

$$\frac{1}{bd} \geq \frac{1}{\sqrt{5}} \left[\frac{1}{b^2} + \frac{1}{d^2}\right]; \quad \frac{1}{d(b+d)} \geq \frac{1}{\sqrt{5}} \left[\frac{1}{d^2} + \frac{1}{(b+d)^2}\right]$$

which is a contradiction according to the lemma.

Proceed similarly for the case $\frac{a}{b} < \theta < \frac{a+c}{b+d} < \frac{c}{d}$. Hence at least one of the three fractions say $\frac{h}{k}$ gives

$$\left| \theta - \frac{h}{k} \right| < \frac{f_{i-2}}{\sqrt{5}k^2}$$

One can very easily see that there are infinitely many of them in $F.F_n$ from the Propositions given and from the very definition of $F.F_n$.

It is easy to see that $\sqrt{5}$ is the best possible constant. Consider the case $\theta = \frac{1+\sqrt{5}}{2}$. Now $f_{i-2} = 1$, and so we have

$$\left| \theta - \frac{h}{k} \right| < \frac{1}{\sqrt{5}k^2}$$

for infinitely many $\frac{h}{k}$ in $F.F_n$. We can't obviously improve $\sqrt{5}$. This follows from the classical theorem of Hurwitz 2 .

<u>Note</u>. A) The counter example $\frac{1+\sqrt{5}}{2}$ which Hurwitz gave is actually the $\lim_{n \to \infty} \frac{f_{n+1}}{f_n}$.

B) In the interval $\left[\frac{1}{3}, 3\right]$ $F.F_n$ provides the same approximation as do the Farey Fractions, for $f_{i-2} = 1$.

In theorem 1 the constant $\sqrt{5}$ was seen as the best possible over the interval $(0,\infty)$. That is if $\sqrt{5}$ were replaced by a larger constant the theorems do not hold for all irrationals 'θ' > 0. Now our question is the following:- Is $\sqrt{5}$ the best possible constant for every "INTERVAL" (f_{i-1}, f_i) ? The answer is in the affirmative in the sense that if $\sqrt{5}$ is replaced by a larger constant the theorem fails to hold in all "INTERVALS" (f_{i-1}, f_i) and $\left[\frac{1}{f_i}, \frac{1}{f_{i-1}}\right]$, $i = 2, \ldots, \omega$. We now state and prove our final pair of theorems which are much stronger than Theorem 1.

THEOREM 2a. Consider any "INTERVAL" $[f_{i-1}, f_i]$. Let θ be an irrational which belongs to this interval. Then for any 'n' there exists infinitely many $\frac{h}{k} \in F.F_n$ such that

$$\left| \theta - \frac{h}{k} \right| < \frac{f_{i-2}}{\sqrt{5}k^2} .$$

The constant $\sqrt{5}$ is the best possible in the sense that if $\sqrt{5}$ are replaced by a larger constant the assertion fails for each 'n' for some θ which belongs to the interval.

Proof. The existence has already been established in Theorem 1. We shall concentrate on the bound $\frac{f_{i-2}}{\sqrt{5}}$. If we show that $\frac{f_{i-2}}{\sqrt{5}}$ is the best possible constant when n = 1, it proves the theorem for using properties 1a, 1b a one can show $F.F_{n+1}^* \subset F.F_n^*$ for any n.

$F.F_n^* = \{x \in F.F_n \mid x \geq 1\}$

Consider the interval $\left[\frac{f_{i-1}}{1}, \frac{f_i}{1}\right]$ and call 's' the set $\left[\frac{f_{i-1}}{1}, \frac{f_i}{1}\right]$. Let $s_1 = \left[\frac{f_{i-1}}{1}, \frac{f_{i-1}+f_i}{1\ 1}, \frac{f_i}{1}\right]$. Defined recursively let s_{r+1} consist of all fractions in s_r, together with the mediants of consecutive fractions in s_r.

Let $S = \bigcup_{r=1}^{\infty} s_r$. Similarly let $s' = \left[\frac{1}{1}, \frac{2}{1}\right]$ and $s'_{1+1} = \left[\frac{1}{1}, \frac{1+2}{1+1}, \frac{2}{1}\right]$. Define s'_{r+1} as all fractions in s'_r together with mediants of consecutive fractions in s'_r. Nlw let

$$S' = \bigcup_{r=1}^{\infty} s'_r .$$

What we are interested here is S and not $F.F_n$. If we compare the sets s_r, and s_r', the following can easily be seen.

(i) A one-one onto map can be established between s_r and s_r' as follows.

Map

$$\frac{\nu 1 + \mu .2}{\nu 1 + \mu .2} + \frac{\nu f_{i-2} 1 + \mu f_i}{\nu .1 + \mu 2} .$$

We call two such numbers corresponding numbers.

(ii) The map says that to every $\frac{p}{q} \in s_r$ there exists a unique $\frac{p'}{q} \in s_r'$ and conversely.

(iii) The distance between the consecutive numbers in s_r is f_{i-2} times the distance between consecutive numbers in s_r'.

Now let $\theta_0 = \frac{1+\sqrt{5}}{2}$ and $\theta_1 = \frac{1+\sqrt{5}}{2} - 1$. Clearly $f_{i-1}+f_{i-2}\theta_1=\theta'$ is in s_r. Now if there exist infinitely many $\frac{h_r}{k_r}$ in S with

$$\left| \theta' - \frac{h_r}{k_r} \right| < \frac{f_{i-2}}{\alpha k_r^2}, \quad \alpha > \sqrt{5} \qquad r = 1,2,\ldots$$

then (i), (ii),(iii) would imply that there exists infinitely many corresponding numbers $\frac{h_r'}{k_r}$ in S' with

$$\left| \theta' - \frac{h_r'}{k_r} \right| < \frac{1}{\alpha k_r^2} \text{ with } \alpha > \sqrt{5} ,$$

which is a contradiction according to Hurwitz theorem. Hence the theorem fails for θ' if $\sqrt{5}$ is replaced by a bigger constant.

<u>THEOREM 2b</u>. Let $\frac{1}{f_i} < \theta < \frac{1}{f_{i-1}}$ be an irrational. Then there exists infinitely many $\frac{h}{k}$ in $F.F_n$ for all $n \geq i$ such that

$$\theta - \frac{h}{k} < \frac{f_{i-2}}{\sqrt{5}k^2} .$$

The constant here again in the best possible in the same sense as Theorem 2a.

<u>Proof</u>. The existence is already known. We just prove the converse for $F.F_i$. It automatically follows for the other cases. Now let

$$s'' = \left[\frac{1}{f_i} , \frac{1}{f_{i-1}} \right] .$$

Let $s''_1 = \left[\frac{1}{f_i} , \frac{1+1}{f_{i-1}+f_i} . \frac{1}{f_{i-1}} \right]$. Define recursively s''_{r+1} as all fractions in s''_r together with mediants of consecutive fractions in s''_r. Now

$$S'' = \bigcup_{r=1}^{\infty} s''_r .$$

Clearly a one-one onto map exists between S'' and S.

Map $\frac{h}{k} \rightarrow \frac{k}{h}$.

Consider the irrational $\frac{1}{\theta'} = \theta''$. Let there exist infinitely many $\frac{h_r}{k_r} \in F.F_i$ with

$$\left| \theta'' - \frac{h_r}{k_r} \right| < \frac{f_{i-2}}{k_r^2} .$$

Now if $\theta'' = \frac{h_r}{k_r} + \frac{\delta f_{i-2}}{k_r^2}$ then $|\delta| < \frac{1}{\alpha}$. Now this gives that

$$\left| \theta' - \frac{k_r}{h_r} \right| < \frac{1}{\left[h_r + \frac{\delta f_{i-2}}{k_r} \right] (h_r)} \quad \text{(mere computation).}$$

for infinitely many $\frac{k_r}{h_r}$ in S. Now choose any $\beta > \sqrt{5}$ with $\sqrt{5} < \beta < \alpha$. Then we get

$$\left| \theta' - \frac{k_r}{h_r} \right| < \frac{1}{\beta h_r^2} \quad \text{for infinitely many } \frac{k_r}{h_r} \text{ in } S$$

i.e. for all $r > r_0$ $(\beta > \sqrt{5})$.

This is a contradiction according to theorem 2a and so theorem 2b is proved.

I would like to thank Professor B.H.Neumann, and Professor K. Mahler for an award of a visiting scholarship during the tenure of which this work was done.

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A THEOREM ON THE RATIONAL APPROXIMATION OF IRRATIONALS

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In this paper we prove a theorem on the best approximation of quadratic irrationals in a specified interval through rationals which belong to a certain set. We need the following prerequisites.

Let $\frac{\ell}{k}$, $\frac{\ell'}{k'}$ be two consecutive rationals in a Farey sequence of some order say n. This implies that

$$k\ell' - \ell k' = 1 \tag{1}$$

Let $\frac{h}{k}$ and $\frac{h'}{k'}$ be two rationals and let

$$kh' - hk' = \lambda , \qquad \lambda \quad 0. \tag{2}$$

Let us now form two sets as follows. Let s be the set $\left(\frac{h}{k}, \frac{h'}{k'}\right)$. Let $s_1 = \left(\frac{h}{k}, \frac{h+h'}{k+k'}, \frac{h'}{k'}\right)$. Define recursively s_{r+1} all the rationals in s_r together with mediants of consecutive rationals in s_r. Now let

$$S = \bigcup_{r=1}^{\infty} s_r .$$

Similarly let $s' = \left(\frac{\ell}{k}, \frac{\ell'}{k'}\right)$ and $s'_1 = \left(\frac{\ell}{k}, \frac{\ell+\ell'}{k+k'}, \frac{\ell'}{k'}\right)$. Now define s'_{r+1} as all the rationals in s'_r together with mediants of consecutive rationals in s'_r .

$$S' = \bigcup_{r=1}^{\infty} s'_r .$$

(For elementary properties of mediants and Farey sequence consult [1]).

It is clear from the properties of Farey sequences that every rational p/q which $\frac{\ell}{k} < \frac{p}{q} < \frac{\ell'}{k'}$ with (p, q) = 1 belongs to S'.

We now give two more definitions. Let θ be a quadratic irrationality. Define M(θ) as

(a) there exists infinitely many solutions to

$$\left|\theta - \frac{a}{b}\right| < \frac{1}{M(\theta)\, b^2} .$$

(b) If $M(\theta)$ is replaced by $M(\theta) + \epsilon$ for any ϵ $\quad 0$, then (a) fails. It is known from the theory of Diophantine Approximations that $M(\theta) \geqslant \quad 5$ for all irrationals θ. If θ is equivalent to $\frac{1 \pm \ 5}{2}$ then $M(\theta) = \sqrt{5}$. Otherwise $M(\theta) \quad 8$. For definition of equivalence see [2] .

Finally we define two numbers $\quad \alpha \quad$, and $\frac{h}{k} \quad \beta \quad \frac{h'}{k'}$ to be corresponding if $\beta = \frac{h}{k} + \lambda\left(\alpha - \frac{\ell}{k}\right)$. If α is corresponding to β and α equivalent to Υ we say β is quasi-equivalent to Υ.

<u>THEOREM</u>. Let θ' be a quadratic irrational with $\frac{\ell}{k} < \theta' < \frac{\ell'}{k'}$. Let θ be the corresponding irrational to θ'. Then there exist infinitely many rationals $\frac{p_r}{q_r}$, $r = 1, 2, \ldots$ in S with

$$\left|\theta - \frac{p_r}{q_r}\right| < \frac{\lambda}{M(\theta')\, q_r^2} , \quad r = 1, 2, \ldots .$$

Moreover the number $M(\theta')$ is the best possible in the sense that the assertion fails if $M(\theta')$ is replaced by $M(\theta') + \epsilon$ for any $\epsilon \quad 0$.

<u>Proof</u>. Consider the sets s_r and s_r' . The number of rationals in these two sets are equal. Establish a one-to-one onto map as follows:

Map $\frac{v\ell + u\ell'}{vk + uk'} \longrightarrow \frac{vh+uh'}{vk+uk'}$. It can be easily seen that this map preserves the corresponding nature of numbers. That is $\frac{v \quad + u \quad}{vk+uk'}$ and $\frac{vh+uh'}{vk+uk'}$ are corresponding numbers. This follows from the properties of medians.

The following propositions hold and can be proved from the definition of the map and from (1) to (2).

(i) The length of the set S is the length of the set S' multiplied by λ.

(ii) The distance between any two elements of S is the distance between their corresponding elements in S' multiplied by λ.

(iii) To every fraction p/q in S there corresponds a unique p'/q in S' and conversely.

Now by the very definition of $M(\theta)$ we see that there exist infinitely many $\frac{a_r'}{b_r}$ in S', $r = 1, 2, \ldots$ with

$$\left| \theta' - \frac{a_r'}{b_r'} \right| < \frac{1}{M(\theta') b_r^2} .$$

This is possible only because S' consists of all reduced rationals p/q with $\frac{\ell}{k} < \frac{p}{q} < \frac{\ell'}{k'}$, $q \geq \max(k, k')$

Now let θ ~~in S~~ be the corresponding number to θ'. Let $\frac{a_r}{b_r}$ in S be the corresponding rational to $\frac{a_r'}{b_r}$, $r = 1, 2, \ldots$. Then from (i), (ii) and (iii) we deduce that

$$\left| \theta - \frac{a_r}{b_r} \right| = \lambda \left| \theta' - \frac{a_r'}{b_r} \right| < \frac{\lambda}{M(\theta') b_r^2} , \; r = 1, 2, \ldots .$$

This proves the first part of the theorem.

To prove the second assume the contrary. Let there exist infinitely many $\frac{a_r}{b_r}$ in S, $r = 1, 2, \ldots$,

$$\boxed{\frac{a_r'}{b_r} \Big| < \frac{1}{(M(\theta')+\epsilon) b_r^2}} \qquad \left| \theta - \frac{a_r}{b_r} \right| \quad \frac{\lambda}{M(\theta')+\epsilon) b_r^2} , \quad r = 1, 2, \ldots .$$

This is contradiction according to the definition of $M(\theta)$. This proves the second part of the theorem.

COROLLARY. Let θ ~~in S~~ be quasi-equivalent to $\frac{1+\sqrt{5}}{2}$. Then there exist infinitely many $\frac{a_r}{b_r}$, $r = 1, 2, \ldots$ in S with

$$\left| \theta - \frac{a_r}{b_r} \right| < \frac{1}{\sqrt{5}b_r^2} .$$

The constant $\sqrt{5}$ cannot be improved.

If θ is not quasi-equivalent to $\frac{1+\sqrt{5}}{2}$ then there exist infinitely many $\frac{a_r}{b_r}$ in S, $r = 1,2, \ldots$ with

$$\left| \theta - \frac{a_r}{b_r} \right| < \frac{1}{\sqrt{8}b_r^2} .$$

if θ is quasi-equivalent to $1 + \sqrt{2}$, $\sqrt{8}$ cannot be improved. Proceeding indefinitely we get the Markoff chain of constants.

<u>Proof.</u> The corollary follows from the definition of quasi-equivalence and from the theorem we have proved. For information about Markoff constants consult [2].

I would like to thank Professors Kurt Mahler and B.H.Neumann for the award of a visiting scholarship during the tenure of which this work was done.

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ON THE DISTRIBUTION OF RATIONALS IN A CERTAIN SET

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In this paper we shall prove the uniform distribution of rationals belonging to a certain set.

Let $\alpha_1, \alpha_2, \ldots \alpha_n \ldots$ be a sequence of real numbers belonging to an interval (α, β). Let α', β' be real numbers satisfying $\alpha \leq \alpha' < \beta' \leq \beta$. Let

$$\varphi_n(\alpha', \beta') = \sum_{\substack{\alpha_i,\ i \leq n \\ \alpha' \leq \alpha_i \leq \beta'}} 1$$

Further let

$$\mathcal{D}_n(\alpha', \beta') = \left| \frac{\varphi_n(\alpha', \beta')}{n} - \frac{\beta' - \alpha'}{\beta - \alpha} \right|$$

If $\mathcal{D}(\alpha', \beta') \longrightarrow 0$ as $n \longrightarrow \infty$ for every $\alpha \leq \alpha' < \beta' \leq \beta$ we say that the sequence α_i is uniformly distributed in (α, β).

Let $(k, k') = 1$ be two integers and $\frac{h}{k} < \frac{h'}{k'}$. Further let $kh' - hk' = \lambda > 0$. Let $s = \left(\frac{h}{k}, \frac{h'}{k'}\right)$ and $s_1 = \left(\frac{h}{k}, \frac{h+h'}{k+k'}, \frac{h'}{k'}\right)$. Define s_{r+1} as all the rationals in s_r together with mediants of consecutive rationals in s_r. Finally let

$$S = \bigcup_{r=1}^{\infty} s_r .$$

We write the rationals in s_r as they occur. We don't bring them down to their lowest form.

Now partition S as $S = \bigcup_{r=1}^{\infty} s_{(r)}$ with $s_{(i)} \cap s_{(j)} = \phi$ if $i \neq j$. Where all the elements of $s_{(r)}$ have the same denominator. Further if $\frac{p}{q} \in s_{(i)}$ and $\frac{x}{y} \in s_{(j)}$ and $i < j$ then $q < y$. We now arrange the rationals in $s_{(i)}$ in ascending order. Now form a sequence of rationals writing first the rationals in $s_{(1)}$, then those in $s_{(2)}$ ..... . Let this sequence be $\{U_i\}_{i=1}^{\infty}$. It is clear that $\frac{h}{k} \leq U_i \leq \frac{h'}{k'}$, $i = 1,2,3, \ldots$.

<u>THEOREM</u>. The sequence $\{U_i\}_{i=1}^{\infty}$ is uniformly distributed $\left(\frac{h}{k}, \frac{h'}{k'}\right)$.

<u>Proof</u>. Since $(k,k') = 1$ there exist integers ℓ, ℓ' such that

$$k\ell' - \ell k' = 1, \quad \frac{\ell}{k} < \frac{\ell'}{k'}$$

This implies that ℓ/k and ℓ'/k' are consecutive rationals in the Farey sequence of order $\max(k,k')$. Now let $s' = \left(\frac{\ell}{k}, \frac{\ell'}{k'}\right)$. Let $s_1' = \left(\frac{\ell}{k}, \frac{\ell+\ell'}{k+k'}, \frac{\ell'}{k'}\right)$. Define s'_{r+1} as all the rationals in s'_r together with mediants of consecutive rationals in s'_r. Let

$$S' = \bigcup_{r=1}^{\infty} s'_r \qquad ([illegible])$$

Partition as before $S' = \bigcup_{r=1}^{\infty} s'_{(r)}$; $s'_{(i)} \cap s'_{(j)} = \phi$ if $i \neq j$. The rationals in $s'_{(r)}$ have the same denominator and are in ascending order. Further if $p'/q' \in s'_{(i)}$ and $x'/y' \in s'_{(j)}$ and $i < j$ then $q' < y'$. Now write the rationals in $s'_{(1)}$ first, then those in $S'_{(q)}$ .... forming a sequence $\{V_i\}_{i=1}^{\infty}$. Clearly $\ell/k \leq V_i \leq \ell'/k'$, $i = 1, 2, \ldots$. We will first prove that the V_i s are uniformly distributed in $(\ell/k, \ell'/k')$.

From the properties of Farey sequences it is clear that every p/q with $\ell/k < p/q < \ell'/k'$, $(p,q) = 1$, $q > \max(k,k')$ belongs to S and vice versa $(p/q \neq \ell/k$, or $\ell'/k')$. For properties of Farey sequences consult [1] .

Now let .

$$\phi(n,x) = \sum_{\substack{a \leq x \\ (a,n)=1}} 1, \quad \phi(n,n) = \phi(n)$$

and $$e(n,x) = \frac{x}{n}\phi(n) - \phi(n,x).$$

It has been established in [2] that $e(n,x) = O(n^{\varepsilon})$ for any $\varepsilon > 0$. Now let Den $(S') = \{y/x/y \in S'\}$.

Now let $q > \max(k,k')$. The number of rationals p'/q', in S' with $q' \leq q$ is

$$N = \sum_{\substack{q' \in Den(S') \\ \min(k,k') \leq q' \leq q}} \phi\left(q', \frac{\ell'}{k'}q'\right) - \phi\left(q', \frac{\ell}{k}q'\right)$$

$$= \left(\frac{\ell'}{k'} - \frac{\ell}{k}\right) \sum_{\substack{q' \in Den(S') \\ \min(k,k) \leq q' \leq q}} \phi(q') + O(q'^{\varepsilon})$$

The number of rationals p'/q' in S with $q' \leq q$ and $\frac{\ell}{k} \leq \gamma' < p'/q' < \delta' \leq \ell'/k'$ is

$$\varphi_N(\gamma', \delta') = \sum_{\substack{q' \in \mathcal{D}en(S') \\ \min(k,k') \leq q' \leq q}}' \phi(q', \delta' q') - \phi(q', \gamma' q')$$

$$= (\delta' - \gamma') \sum_{\substack{q' \in \mathcal{D}en(S') \\ \min(k,k') \leq q' \leq q}}' \phi(q') + O(q'^{\epsilon})$$

Now

$$\mathcal{D}_N(\gamma', \delta') = \left| \frac{\varphi_N(\gamma', \delta')}{N} - \frac{\delta' - \gamma'}{\frac{\ell'}{k'} - \frac{\ell}{k}} \right|$$

which implies that $\mathcal{D}_N(\gamma', \delta') \to 0$ as $N \to \infty$ for every (γ', δ') because $\phi(q') > \frac{cq'}{\log \log q'}$ [1]. This means the sequence $\{V_i\}_{i=1}^{\infty}$ is uniformly distributed in $(\ell/k, \ell'/k')$.

Define $\ell/k \leq x' \leq \ell'/k'$ and $\frac{h}{k} \leq x \leq \frac{h'}{k'}$ to be corresponding numbers if

$$x = \frac{h}{k} + \lambda(x' - \frac{\ell}{k}).$$

From this we deduce that if $\frac{h}{k} \leq x < y \leq \frac{h'}{k'}$ and $\frac{\ell}{k} \leq x' < y' \leq \frac{\ell'}{k'}$ and x, x', y, y' are corresponding numbers then

$$y - x = \lambda(y' - x').$$

Now compare the sets s_r and s_r'. Establish a one-one onto map

Map $\dfrac{v\ell + u\ell'}{vk + uk'} \longrightarrow \dfrac{vh + uh'}{vk + uk'}$ (4)

It is easy to see from the properties of mediants and from (1) and (2) and (3) that $\dfrac{v\ell + u\ell'}{vk + uk'}$, and $\dfrac{vh + uh'}{vk + uk'}$, are corresponding numbers. Further (4) implies that U_i and V_i are corresponding numbers. Now let $\frac{\ell}{k} \leq \alpha' < \beta' < \frac{\ell'}{k'}$ and $\frac{h}{k} \leq \alpha < \beta \leq \frac{h'}{k'}$ and α, α' and β, β' corresponding numbers. Now (3) implies that

$$\alpha' \leq V_i \leq \beta' \tag{5}$$

if and only if

$$\alpha \leq U_i \leq \beta.$$

Now let

$$\Psi_n(\alpha', \beta') = \sum_{\substack{V_i \in (\alpha, \beta) \\ i \leq n}}' 1$$

and

$$\Psi_n(\alpha, \beta) = \sum_{\substack{U_i \in (\alpha, \beta) \\ i \leq n}}' 1$$

Clearly (5) implies that

$$\Psi_n(\alpha', \beta') = \Psi_n(\alpha, \beta) \tag{6}$$

Now (6) gives that

$$\mathcal{D}(\alpha', \beta') = \left| \frac{\Psi_n(\alpha', \beta')}{n} - \frac{\beta' - \alpha'}{\frac{\ell'}{k'} - \frac{\ell}{k}} \right| = \left| \frac{\Psi_n(\alpha, \beta)}{n} - \frac{\beta - \alpha}{\frac{h'}{k'} - \frac{h}{k}} \right|$$

$$= D_n(\alpha, \beta).$$

But $\mathcal{L}_n(\alpha',\beta') \longrightarrow 0$ as $n \longrightarrow \infty$ which implies that $D_n(\alpha,\beta) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\frac{h}{k} \leq \alpha < \beta \leq \frac{h'}{k'}$. This proves that the sequence $\{U_i\}_{i=1}^{\infty}$ is uniformly distributed in $(\frac{h}{k}, \frac{h'}{k'})$.

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CONTINUED FRACTIONS OF HIGHER TYPE

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In this paper we prove theorems connected with continued fractions which are generalisations of the results already known. We shall follow in general the line of proofs given in [1].

Let λ be a given positive integer. Throughout the paper we will be referring to the same λ.

Let $a_0, a_1, a_2, \ldots a_n$ be a set of integers $a_i > 0$, $i \geqslant 1$. Then we define a continued fraction of type λ to be

$$a_0 + \cfrac{\lambda}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, \cfrac{1}{a_n}}}}\ .$$

which is denoted by $\langle a_0, a_1, \ldots\ldots a_n \rangle_\lambda$. The above definition could be extended to $\langle a_0, a_1, a_2 \ldots\ldots \rangle_\lambda$ as

$$a_0 + \cfrac{\lambda}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

Let θ be an irrational number. We define the continued fraction of type λ or λ-fraction of θ as

$$\langle\theta\rangle_\lambda = [\theta]_\lambda + \frac{\lambda}{\theta'} = a_0 + \cfrac{\lambda}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

where $[\theta]_\lambda$ is the largest integer $\leq \theta$, $\equiv 1 \pmod{\lambda}$ and $\langle a_1, a_2, \ldots \rangle$ is the standard continued fraction of θ'. Thus the λ-fraction of θ is unique.

Now let

$$\langle\theta\rangle_\lambda = \langle a_0, a_1, a_2, \ldots\ldots \rangle_\lambda .$$

Let $\frac{p_n}{q_n} = \langle a_0, a_1, \ldots\ldots a_n \rangle_\lambda$. p_n/q_n written as it occur .
$\frac{p_n}{q_n}$ is called a λ-principal convergent to θ.

<u>Result 1</u>. $p_n = a_n p_{n-1} + p_{n-2}$

$$q_n = a_n q_{n-1} + q_{n-2} \qquad n \geqslant 2.$$

<u>Proof</u>. We let $\frac{p_n'}{q_n'} = \langle a_1, a_2, \ldots a_n \rangle$. This implies that

$$\frac{p_n}{q_n} = a_0 + \frac{\lambda}{p_n'/q_n'} .$$

This implies that

$$p_n = a_0 p_n' + \lambda q_n'$$

and

$$q_n = p_n'. \tag{1}$$

Now since

$$q_n' = a_n q_{n-1}' + q_{n-2}'$$

$$p_n' = a_n p_{n-1}' + p_{n-2}' \qquad n \geqslant 2$$

We deduced the result 1 from (1) .

<u>Result 2</u>. $\left| p_n q_{n-1} - p_{n-1} q_n \right| = \lambda$. This is easily proved from Result 1. It is also clear from result 1 that p_n/q_n for odd n form a strictly decreasing sequence converging to θ and p_n/q_n for even n a strictly increasing sequence converging to θ. Now consider the ordered set $s = \left(\frac{1}{1}, \frac{1+\lambda}{1+1} \right)$. Let $s_1 = \left(\frac{1}{1}, \frac{1+1+\lambda}{1+1}, \frac{1+\lambda}{1} \right)$ In general let s_{r+1} consist of all rationals in s_r together with mediants of consecutive rationals in s_r. Finally set

$$S = \bigcup_{r=1}^{\infty} s_r \qquad (2)$$

Now let the θ we considered satisfy $1 < \theta < 1 + \lambda$. We say a fraction p/q is a λ-best approximation to θ if

(i) $p/q \in S$.

(ii) $\left| q\theta - p \right| < \frac{\lambda}{q}$.

(iii) $\left| q\theta - p \right| < \left| q'\theta - p' \right|$ for all $q' < q$, $p'/q' \in S$.

<u>THEOREM 1</u>. Every λ-best approximation to θ is a λ-principal convergent to θ. Every λ-principal convergent to θ is a λ-best approximation to θ.

<u>Proof</u>. We first have to prove that every λ-principal convergent of θ belongs to S. We know

$$\theta = \langle a_0, a_1, \ldots\ldots, \rangle_\lambda.$$

Let $\theta' = \langle a_0, a_1, \ldots\ldots \rangle$.

Now let $p_n/q_n = \langle a_0, a_1, \ldots\ldots a_n \rangle_\lambda$

and $p_n'/q_n' = \langle a_0, a_1, \ldots\ldots a_n \rangle$.

Now since $\left| p_n' q_{n-1}' - p_{n-1}' q_n' \right| = 1$, p_n'/q_n' is in lowest form.

Now if $s' = \left(\frac{a_0}{1}, \frac{a_0+1}{1}\right)$ and $s_1' = \left(\frac{a_0}{1}, \frac{a_0+a_0+1}{1+1}, \frac{a_0+1}{1+1}\right)$ and s_{r+1}' consists of all rationals in s_r' together with mediants of consecutive rationals in s_r', then

$$p_n'/q_n' \in S'$$

where $$S' = \bigcup_{r=1}^{\infty} s_r'$$

This is evident from the property of Farey sequences. Now let α and β be rationals such that

$$a_0 < \alpha < a_0 + 1$$

$$1 < \beta < 1 + \lambda \qquad (3)$$

Further let $\beta = 1 + \lambda(\alpha - a_0)$. We say α, and β are corresponding numbers. From the properties of mediants it is clear that $\alpha \in S'$ if and only if $\beta \in S$. For properties of Farey sequences and mediants consult [2]. (Note that α and β have the same denominator)

Now let $\frac{p_n''}{q_n''} = \langle a_1, a_2, \ldots a_n \rangle$

This implies that

$$\frac{p_n}{q_n} = a_0 + \frac{\lambda}{p_n''/q_n''} \qquad (4)$$

and $$\frac{p_n'}{q_n'} = a_0 + \frac{1}{p_n''/q_n''} \qquad (5)$$

Now (4) and (5) imply that

$$\frac{p_n}{q_n} = a_0 + \lambda\left(\frac{p_n'}{q_n'} - a_0\right)$$

Now by (3) we deduce that $\frac{p_n}{q_n} \in S$ for $a_0 = 1$

Now let $p/q \in S$ be a λ-best approximation to θ. Let $\frac{p}{q} < \frac{p_0}{q_0} = \frac{a_0}{1}$. Then $|\theta - a_0| < |\theta - p/q| < |\theta q - p|$ a contradiction

If $p/q > p_1/q_1$ then
$$|p/q - \theta| \quad > \quad |p/q - p_1/q_1| > \frac{\lambda}{q\, q_1} \tag{6}$$

because $\left|\frac{p}{q} - \frac{p_1}{q_1}\right| = \lambda \left|\frac{p'}{q'} - \frac{p_1'}{q_1'}\right| \geqslant \frac{\lambda}{q'\, q_1'}$ where p'/q' and $\frac{p_1'}{q_1'}$ are in S' and are corresponding numbers to p/q and p_1/q_1 respectively. Now (6) gives

$$|q\theta - p| \quad \frac{\lambda}{q_1} = \frac{1}{a_1} > |\theta - a_0| \ .$$

Which is again a contradiction. Now finally let p/q between $\frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_{n+1}}{q_{n+1}}$ but not equal to any of these fractions. Then

$$\frac{\lambda}{q\, q_{n-1}} < \left|\frac{p}{q} - \frac{p_{n-1}}{q_{n-1}}\right| < \left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| = \frac{\lambda}{q_n\, q_{n-1}} \ .$$

Hence $q_n < q$. On the other hand by arguments similar to (6)

$$\frac{\lambda}{q\, q_{n+1}} \leqslant \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p}{q}\right| < \left|\theta - \frac{p}{q}\right| \tag{7}$$

which gives

$$|q_n \theta - p_n| < \frac{\lambda}{q_{n+1}} < |q\theta - p| \ .$$

by (7). This is a contradiction establishing the first part of the theorem. It is clear from the properties of convergents that p_n/q_n, $n = 1, 2, \ldots$ is a sequence of λ-best approximations i.e.

$$|q_n \theta - p_n| < |q_{n-1} \theta - p_{n-1}| .$$

This completes the proof.

THEOREM 2. If $\frac{p}{q} \in S$ is a fraction such that

$$\left|\theta - \frac{p}{q}\right| < \frac{\lambda}{2q^2}$$

then p/q is a λ-principal convergent to θ.

Proof. By Theorem 1 it will suffice to prove that p/q is a λ-best approximation to θ. Let $p'/q' \in S$ be such that

$$\frac{p'}{q'} \neq \frac{p}{q}$$

$$|q'\theta - p'| < |q\theta - p| < \frac{\lambda}{2q}.$$

Then

$$\left|\frac{p'}{q'} - \frac{p}{q}\right| \geqslant \frac{\lambda}{qq'}$$

because $\left|\frac{p'}{q'} - \frac{p}{q}\right| = \lambda \left|\frac{x'}{y'} - \frac{x}{y}\right| \geqslant \frac{\lambda}{yy'} = \frac{\lambda}{qq'}$ (8)

where $\frac{x}{y}, \frac{x'}{y'} \in S$ and are corresponding numbers to p/q and p'/q' respectively. Now (8) implies that

$$\frac{\lambda}{q\,q'} < \left|\frac{p'}{q'} - \frac{p}{q}\right| < \left|\theta - \frac{p'}{q'}\right| + \left|\theta - \frac{p}{q}\right| \leq \frac{\lambda}{2qq'} + \frac{\lambda}{2q^2} =$$

$$= \frac{\lambda(q+q^1)}{2q^1q^2}.$$

From this we conclude that $q^{1'} > q$. This implies by (iii) that p/q is a λ-best approximation to θ and so a λ-principal convergent by Theorem 1.

We define $\frac{p_{n,r}}{q_{n,r}}$ to be a convergent to θ where

$$\frac{p_{n,r}}{q_{n,r}} = \frac{rp_{n+1} - p_n}{rq_{n+1} + q_n}.$$

THEOREM 8. If p/q is rational with

$$\left|\theta - p/q\right| < \frac{\lambda}{q^2}$$

then p/q is a convergent to θ.

Proof. Let $\theta < p/q$. If p/q is not a convergent then there exist two successive convergents $\frac{P}{Q}$ and $\frac{P'}{Q'}$ such that

$$\theta < \frac{P}{Q} < \frac{p}{q} < \frac{P'}{Q'}$$

and $P'Q - PQ' = \lambda$ because

$$p_{n,r}\, q_{n,r+1} - p_{n,r+1}\, q_{n,r} = \lambda.$$

Thus

$$\frac{\lambda}{q^2} > \frac{p}{q} - \theta > \frac{p}{q} - \frac{P}{Q} \geqslant \frac{\lambda}{qQ}$$

by arguments similar to (8). Again

$$\frac{\lambda}{q'Q'} \leqslant \frac{P'}{Q'} - \frac{p}{q} < \frac{P'}{Q'} - \frac{P}{Q} = \frac{\lambda}{QQ'}.$$

The estimates are contradictory proving the theorem for $\theta < p/q$. The proof for $\theta > p/q$ can also be given. Actually all the results will hold for θ lying between $n\lambda + 1$ and $(n+1)\lambda+1$. In that case we would have to construct the set S for $n\lambda + 1$ and $(n+1)\lambda+1$ instead of from 1, and $1+\lambda$ as we did.

References.

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ON SETS GENERATED BY ARITHMETIC PROGRESSIONS

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In this paper we deal with sets generated by the multiples of irrational numbers by members of Arithmetical progressions. We shall discuss some necessary and sufficient conditions on these irrationals to produce complementary subsets of Arithmetical Progressions. We shall essentially generalise the following well known theorem.

If $N_\alpha \{[n\alpha]\}$ $n = 1,2,3,\ldots$ then a necessary and sufficient condition for

$$N_\alpha \cup N_\beta = Z = \{1,2,3,\ldots\}$$

and $$N_\alpha \cap N_\beta = \emptyset \text{ is}$$

α,β irrational and $\alpha^{-1} + \beta^{-1} = 1$.

where $[x]$ represents the largest integer less than or equal to x. For complete information consult [1]. Skolem and Bang [2], [3] have discussed in detail about the set N_α. In this paper we not only generalise the above theorem but prove an analogou result for the nearest integer function. This theorem does <u>not</u> hold for the set of integers.

Let a and b be positive integers. Consider the Arithmetic progressions $an+b$, $n = 1,2,3,\ldots$. For real $x \geq b$

$$[x]_b^a = \max\{an+b \leq x\}.$$

Let $\lambda,\mu,\lambda',\mu' > 0$ be integers. Consider two irrationals α and β. Let A denote the arithmetical progression

$$A = \{\lambda' n + \mu'\} \qquad n = 1,2,3,\ldots .$$

Now let

$$A_\alpha = [(\lambda n + \mu)\alpha]_{\lambda'}^{\mu'} \qquad n = 1,2,3,\ldots .$$
$$A_\beta = [(\lambda n + \mu)\beta]_{\lambda'}^{\mu'} \qquad n = 1,2,3,\ldots .$$

Further let $\lambda, \mu, \lambda', \mu'$ satisfy

$$\mu'\lambda = \lambda'\mu. \tag{C_1}$$

THEOREM 1. A necessary and sufficient condition for

$$A_\alpha \cup A_\beta = A$$

and

$$A_\alpha \cap A_\beta = \phi \tag{1.1}$$

is

$$\alpha^{-1} + \beta^{-1} = \frac{\lambda}{\lambda'} \tag{1.2}$$

Proof. Let $h \in A$. Denote by $\mu(\alpha,h)$

$$\mu(\alpha,h) = \sum_{[(\lambda n+\mu)\alpha]_{\lambda'}^{\mu'} \leq h} 1 = n \tag{1.3}$$

Now (1.3) implies

$$\left[(\lambda n + \mu)\alpha\right]_{\lambda'}^{\mu'} \leq h < \left[\{\lambda(n+1) + \mu\}\alpha\right]_{\lambda'}^{\mu'} \tag{1.4}$$

Because $h \in A$ we refine inequality (1.4) to

$$\begin{aligned}(\lambda n + \mu)\alpha - \lambda' &< h \\ h + \lambda' &< \{\lambda(n+1) + \mu\}\alpha\end{aligned} \tag{1.5}$$

Clearly from (1.5) we deduce that

$$\frac{(h+\lambda')\alpha^{-1} - \mu}{\lambda} - 1 < \mu(\alpha,h) < \frac{(h+\lambda')\alpha^{-1} - \mu}{\lambda} . \tag{1.6}$$

If conditions (1.1) hold then we have

$$\lim_{h \to \infty} \frac{\mu(\alpha,h)}{h} + \frac{\mu(\beta,h)}{h} = \frac{1}{\lambda'}$$

But by (1.6) we get $\lim\limits_{h \to \infty} \frac{\mu(\alpha,h)}{h} = \frac{\alpha^{-1}}{\lambda}$

which gives

$$\alpha^{-1} + \beta^{-1} = \frac{\lambda}{\lambda'}.$$

This proves half of the theorem.

Conversely let (1.2) hold. Proceeding from inequality (1.6) we get by virtue of (1.2).

$$\frac{(h+\lambda')\frac{\lambda}{\lambda'} - 2\mu}{\lambda} - 2 < \mu(\alpha,h) + \mu(\beta,h) < \frac{(h+\lambda')\frac{\lambda}{\lambda'} - 2\mu}{\lambda} .$$

Now if N_h denotes $\mu(\alpha,h) + \mu(\beta,h)$ then

$$(h-\lambda') - \frac{2\mu\lambda'}{\lambda} + \mu' < \lambda' N_h + \mu' < h + \lambda - \frac{2\mu\lambda'}{\lambda} + \mu' \quad (1.7)$$

Now because $\mu'\lambda = 2\lambda'\mu$ we refine (1.7) to

$$h - \lambda' < \lambda' N_h + \mu' < h + \lambda' \qquad (1.8)$$

and $h \in A$. This implies $\lambda' N_h + \mu' = h$ for every h. This in turn shows that condition (1.1) holds. This completes the proof of the theorem.

In theorem 1 we considered a mapping of one arithmetical progression onto another. We now map two arithmetical progression onto one and prove a more general form of Theorem 1.

Consider the three arithmetic progressions

$$A = \{\lambda n + \mu\} \qquad n = 1,2,3,\ldots..$$

$$\{\lambda_1 n + \mu_1\} \qquad n = 1,2,3,\ldots..$$

$$\{\lambda_2 n + \mu_2\} \qquad n = 1,2,3,\ldots..$$

Let α, and β be irrationals and define

$$A_1(\alpha) = \Big[(\lambda_1 n + \mu_1)\alpha\Big]_{\lambda}^{\mu} \qquad n = 1,2,3,\ldots.$$

$$A_2(\beta) = \Big[(\lambda_2 n + \mu_2)\beta\Big]_{\lambda}^{\mu} \qquad n = 1,2,3,\ldots.$$

Further let $\lambda, \mu, \lambda_1, \lambda_2, \mu_1, \mu_2 > 0$ (integers) satisfy

$$\frac{\mu}{\lambda} = \frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2} \qquad (C_2)$$

THEOREM 2. A necessary and sufficient condition for

$$A_1(\alpha) \cup A_2(\beta) = A$$

and

$$A_1(\alpha) \cap A_2(\beta) = \phi \qquad (2.1)$$

is

$$\frac{\alpha^{-1}}{\lambda_1} + \frac{\beta^{-1}}{\lambda_2} = \frac{1}{\lambda} \qquad (2.2)$$

Proof. Let $h \in A$. Denote by $\mu_1(\alpha, h)$

$$\mu_1(\alpha,h) = \sum_{[(\lambda_1 m+\mu_1)\alpha]^{\mu}_{\lambda} \leq h} 1 = n.$$

Using arguments similar to (1.3),(1.4) and (1.5) we get

$$\frac{(h + \lambda)\alpha^{-1} - \mu_1}{\lambda_1} - 1 < \mu_1(\alpha, h) < \frac{(h + \lambda)\alpha^{-1} - \mu_1}{\lambda_1} \qquad (2.3)$$

and

$$\frac{(h + \lambda)\beta^{-1} - \mu_2}{\lambda_2} - 1 < \mu_2(\beta, h) < \frac{(h + \lambda)\beta^{-1} - \mu_2}{\lambda_2}. \qquad (2.4)$$

where $\mu_2(\beta, h) = \sum_{[(\lambda_2 m+\mu_2)\beta]^{\mu}_{\lambda} \leq h} 1$

If conditions (2.1) hold then

$$\lim_{h \to \infty} \frac{\mu_1(\alpha, h)}{h} + \frac{\mu_2(\alpha, h)}{h} = \frac{1}{\lambda}.$$

Because we have by (2.3) and (2.4)

$$\lim_{h \to \infty} \frac{\mu_1(\alpha,h)}{h} = \frac{\alpha^{-1}}{\lambda_1} \text{ and } \lim_{h \to \infty} \frac{\mu_2(\beta,h)}{h} = \frac{\beta^{-1}}{\lambda_2}$$

we get $\dfrac{\alpha^{-1}}{\lambda_1} + \dfrac{\beta^{-1}}{\lambda_2} = \dfrac{1}{\lambda}$.

Conversely let (2.2) hold. Proceeding from (2.3) and (2.4) we get

$$\frac{\lambda + \lambda}{\lambda} - \frac{\mu_1}{\lambda_1} - \frac{\mu_2}{\lambda_2} - 2 < \mu_1(\alpha,h) + \mu_2(\beta,h) < \frac{h+\lambda}{\lambda} - \frac{\mu_1}{\lambda_1} - \frac{\mu_2}{\lambda_2}.$$

Now if $N_h = \mu_1(\alpha,h) + \mu_2(\beta,h)$ then

$$h-\lambda - \lambda\left(\frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2}\right) + \mu < \lambda N_h + \mu < h+\lambda - \lambda\left(\frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2}\right) + \mu \quad (2.5)$$

Now putting (2.5) and (C_2) together we get

$$h - \lambda < \lambda N_h + \mu < h + \lambda \qquad (2.6)$$

Now from (2.6) we deduce that $\lambda N_h + N_h + \mu = h$, for $h \in A$. This means condition (2.1) holds. This completes the proof of the theorem.

We shall prove some theorems connected with the nearest integer function. If $an+b$, $n = 1,2,3,\ldots$. $a,b > 0$ integers is an arithmetical progression then for real $x \geq b$ $N(x)_a^b$ represents the 'nearest' member to x of the arithmetical progression. We shall now straightaway prove the general theorem which is similar to theorem 2. Consider the three arithmetical progressions

$$A = \{\lambda n + \mu\} \qquad n = 1,2,3,\ldots$$
$$\{\lambda_1 n + \mu_1\} \qquad n = 1,2,3,\ldots$$
$$\{\lambda_2 n + \mu_2\} \qquad n = 1,2,3,\ldots$$

Let α,β be irrational numbers and let

$$A_1(\alpha) = N\left[(\lambda_1 n + \mu_1)\alpha\right]^{\mu}_{\lambda} \qquad n = 1,2,3,\ldots\ldots$$

$$A_2(\beta) = N\left[(\lambda_2 n + \mu_2)\beta\right]^{\mu}_{\lambda} \qquad n = 1,2,3,\ldots\ldots$$

Further let $\lambda,\mu,\lambda_1,\mu_1,\lambda_2,\mu_2 > 0$ satisfy

$$\frac{\mu}{\lambda} = \frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2} + \frac{1}{2} \qquad (C_3)$$

THEOREM 2. A necessary an sufficient condition for

$$A_1(\alpha) \bigcup A_2(\beta) = A$$
$$A_1(\alpha) \bigcap A_2(\beta) = \phi \qquad (3.1)$$

is
$$\frac{\alpha^{-1}}{\lambda_1} + \frac{\beta^{-1}}{\lambda_2} = \frac{1}{\lambda} \qquad (3.2)$$

Proof. Let $h \in A$. Denote by $\mu_1(\alpha,h)$ and $\mu_2(\beta,h)$

$$\mu_1(\alpha,h) = \sum_{N\left[(\lambda_1 m+\mu_1)\alpha\right]^{\mu}_{\lambda} \leq h} 1 = h_\alpha .$$

$$\mu_2(\beta,h) = \sum_{N\left[(\lambda_2 m+\mu_2)\beta\right]^{\mu}_{\lambda} \leq h} 1 = h_\beta$$

Then we get

$$N\left[(\lambda_1 h_\alpha+\mu_1)\alpha\right]^{\mu}_{\lambda} \leq h < N\left[\left\{\lambda_1(h_\alpha+1) + \mu_1\right\}\alpha\right]^{\mu}_{\lambda} \qquad (3.3)$$

$$N\left[(\lambda_2 h_\beta+\mu_2)\alpha\right]^{\mu}_{\lambda} \leq h < N\left[\left\{\lambda_2(h_\beta+1) + \mu_2\right\}\alpha\right]^{\mu}_{\lambda} \qquad (3.4)$$

Because $h \in A$, (3.3) and (3.4) can be strengthened to

$$(\lambda_1 h_\alpha + \mu_1)\alpha - \frac{\lambda}{2} < h \quad \text{and}$$

$$h + \lambda < (\lambda_1(h_\alpha + 1) + \mu_1)\alpha + \frac{\lambda}{2} \qquad (3.5)$$

and similarly for β. Inequalities (3.5) give rise to

$$\frac{(h + \frac{\lambda}{2})\alpha^{-1} - \mu_1}{\lambda_1} - 1 < \mu_1(\alpha,h) < \frac{(h + \frac{\lambda}{2})\alpha^{-1} - \mu_1}{\lambda_1} \qquad (3.6)$$

$$\frac{(h + \frac{\lambda}{2})\beta^{-1} - \mu_2}{\lambda_2} - 1 < \mu_2(\beta,h) < \frac{(h + \frac{\lambda}{2})\beta^{-1} - \mu_2}{\lambda_2} \qquad (3.7)$$

If (3.1) holds then we have

$$\lim_{h \to \infty} \frac{\mu_1(\alpha,h)}{h} + \frac{\mu_2(\alpha,h)}{h} = \frac{1}{\lambda}$$

and using (3.6) and (3.7) we get

$$\frac{\alpha^{-1}}{\lambda_1} + \frac{\beta^{-1}}{\lambda_2} = \frac{1}{\lambda}.$$

Conversely let (3.2) hold. Then proceeding from (3.6) and (3.7) we get

$$\frac{\lambda + \frac{\lambda}{2}}{\lambda} - \frac{\mu_1}{\lambda_1} - \frac{\mu_2}{\lambda_2} - 2 < \mu_1(\alpha,h)+\mu_2(\beta,h) < \frac{\lambda+\frac{\lambda}{2}}{\lambda} - \frac{\mu_1}{\lambda_1} - \frac{\mu_2}{\lambda_2}.$$

Now if $N_h = \mu_1(\alpha,h) + \mu_2(\beta,h)$ then

$$h + \frac{\lambda}{2} - \lambda\left(\frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2}\right) + \mu - 2\lambda < \lambda N_h + \mu < h + \frac{\lambda}{2} - \lambda\left(\frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2}\right) + \mu \qquad (3.8)$$

Now using (C_3) we sharpen (3.8) to

$$h - \lambda < \lambda N_h + \mu < h + \lambda \qquad (3.9)$$

and because $h \in A$, $\lambda N_h + \mu = h$. This proves that (3.1) should hold establishing the theorem.

We would like to draw attention to the fact the condition (C_3) is not satisfied when all the three progressions refer to the set of positive integers, i.e. when $\lambda_1 = \lambda_2 = \lambda = 1$ $\mu = \mu_1 = \mu_2 = 0$. This might be the reason why Skolem and Bang did not consider their

Theorems with the greatest integer function replaced by the nearest integer function. Further it is also possible to see on examining our proofs that the theorems won't hold if α, β are replaced by $\alpha_1, \alpha_2, \ldots, \alpha_r$ and consequently 'r' arithmetical progressions. Two irrationals give the best partition.

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