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Proceedings of the
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on
NUMERICAL ANALYSIS AND COMBINATORIAL METHODS

held at
The Institute of Engineers (India), Mysore Centre
Bangalore-1
(March 9 - 12, 1973)

Convener

Professor Alladi Ramakrishnan
Director, MATSCIENCE, Madras

Edited by

Dr. G. N. Keshava Murthy
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THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS

FOREWORD

The conference on "Numerical Analysis and Combinatorial Methods", conducted by MATSCIENCE, was held at the Institution of Engineers, Bangalore, from March 9 to 12, 1973. The conference was inaugurated by the Hon'ble Thiru Sri C.Subramaniam, Union Minister of Industrial Development, Science and Technology and the patron of the Institute.

Professor Alladi Ramakrishnan, Director of Matscience, welcoming the chief guest and the participants of the conference said that the object of the conference was to bring together people who had done creative work and also those who were interested in knowing that and offering their criticisms. Nearly 40 participants to the conference discussed various aspects of numerical analysis and combinatorial methods.

Thiru C.Subramaniam, pointed out that science and technology should play a crucial role in meeting the basic needs of the millions of people in the country. The minister spoke about the importance of mathematical science in providing momentum for the growth of science and technology and said that mathematics was all pervading. Further he said that mathematics was being used even in fields like medicine for improving the efficiency of medical services. He stressed the importance of self-reliance for industrial growth and pointed out the various efforts which are being made by the Government in the form of an approach paper on the science and technology plan for the country which was being scrutinised by the scientific community and which was expected to be finalised by September 1973.

Professor K.R.Unni proposed a vote of thanks and expressed the hope that the importance of pure mathematics would be fully recognised in the realm of scientific research.

The organizers of the conference wish to thank all those participants who have contributed to the success of the conference and who have submitted their manuscripts in due course. They are also thankful to the Institution of Engineers (India), Bangalore, for making their lecture hall available.

The Editor expresses his grateful thanks to Drs.K.R.Unni, N.R.Ranganathan, T.S.Santhanam, V.Radhakrishnan, K.H.Mariwalla and K.Srinivasa Rao for actively taking part in the organisation of the conference. He also wishes to place on record, the help rendered by Mr.D.John Vincent and Mr.R.Jayaraman in the organisation of the conference, and the other members of the administrative staff for having successfully brought out this report.

Dr.G.N.Keshava Murthy
Editor

C O N T E N T S

A MATRIX DECOMPOSITION THEOREM*

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***

We state and prove the following theorem relating to matrix decomposition:

THEOREM: Any matrix M of dimension n x n can be written in a unique manner as

$$M = \sum_{k,\ell=0}^{n-1} a_{k\ell} B^k C^\ell$$

where the quantities occurring on the right hand side are defined as follows:

B is the diagonal matrix

$$\begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{n-1} \end{pmatrix}$$

ω being the primitive n-th root of unity. C is the

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primitive cyclic matrix of dimension n x n

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \cdots & 0 \\ & & 0 & 1 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

The n^2 matrices $B^k C^\ell$ represent all possible products of all possible powers of B and C which are roots of the unit matrix.

$a_{k\ell}$ $(k = 0, 1, \cdots n-1;\ \ell = 0, 1, \cdots n-1)$ are the elements of a matrix A defined uniquely by the relation

$$R = SA \qquad A = S^{-1}R$$

where S is the matrix obtained by writing the elements

of $I, B, B^2, \ldots, B^{n-1}$ as the elements of its columns

$$S = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{pmatrix}$$

$$S^{-1} = \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{n-1} & \omega^{n-2} & \cdots & \omega \\ 1 & \omega^{2(n-1)} & \omega^{2(n-2)} & \cdots & \omega^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \omega^{(n-1)^2} & \omega^{(n-1)(n-2)} & \cdots & \omega^{n-1} \end{pmatrix}$$

R is the matrix obtained by rearranging the elements of M according to the following prescription:

M can be written uniquely as

$$M = D_0 + D_1 C + D_2 C^2 + \cdots + D_{n-1} C^{n-1}$$

where $D_0, D_1, D_2, \cdots$ are diagonal matrices with their elements corresponding to the elements of M in the positions of the nonzero elements of $I, C, C^2, \cdots C^{n-1}$ respectively.

R is now defined as the matrix with elements of its $(k+1)$-th column being chosen as the elements of $\vec{\ }$ of D_k $(k = 0, 1, \cdots n-1)$.

PROOF: The proof of the theorem follows immediately on observing that any diagonal matrix can be expressed as a linear combination of n linearly independent diagonal matrices and without loss of generality can be expressed as

$$D = \sum_{0}^{n-1} a_k B^k$$

If the elements of D are arranged as a column vector $\vec{k}$, then we can write

$$\vec{k} = S \vec{a}$$

where the elements of $\vec{a}$ are a_k $(k = 0, 1, \ldots n-1)$ and S is the matrix with the elements of its $(k+1)$-th column being chosen as the elements of the diagonal matrix B^k.

Applying these considerations to $D_0, D_1, \ldots D_{n-1}$ we obtain

$$R = SA$$

The matrix S has been known in matrix theory as the Sylvester matrix but here it has been obtained by a rearrangement of the diagonal matrices, a procedure which has not been used in matrix literature till now. The decomposition theorem promises to be useful since the matrices B and C have the interesting property

$$CB = \omega BC \ , \qquad B^n = C^n = I$$

Thus this theorem amounts to a 'soft completion of the face' of the generalised Clifford Algebra which has been revealed to us only in recent years.

A GENERALISATION OF SEIDEL-EQUIVALENCE OF GRAPHS AND PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

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ABSTRACT An equivalence between two strongly regular graphs was defined by Seidel, which is generalised here and application of this generalised equivalence is given to partially balanced incomplete block design.

0. SUMMARY

Bose [2] introduced the concept of a strongly regular graph and showed that every such graph is isomorphic with an association scheme of a partially balanced incomplete block design (PBIBD) with two associate classes when treatments of the design are identified with vertices of the graph, a pair of first associates are identified with a pair of adjacent vertices and a pair of second associates are identified with a pair of non-adjacent vertices of the graph. Seidel [6] defined an equivalence relation between two strongly regular graphs G and G^* on the same set of vertices, which was exploited by Bhagwandas and Shrikhande [1] to find the necessary

condition for the existence of Pseudo-L_2 (4) and Pseudo-T (8) association schemes. A generalisation of Seidel-Equivalence of graphs is proposed with some application to PBIBD with two associate classes. The detailed results will be published elsewhere. It is to be noted that for applications of this equivalence, regularity or strong-regulairty is not necessary.

1. INTRODUCTION AND PRELIMINARIES

A finite undirected graph G on the set V of v vertices having no loops or multiple edges is called regular, if each vertex is adjacent to exactly n_1 other vertices and hence non-adjacent to the remaining $n_2 = v - 1 - n_1$ vertices. A regular graph has been called strongly regular by Bose [2] if any two adjacent (non-adjacent) vertices are adjacent to exactly p_{11}^1 (respectively p_{11}^2) vertices and then $(v, n_1, p_{11}^1, p_{11}^2)$ are the parameters of such a graph G.

The adjacency matrix of a finite undirected graph G without loops or multiple edges is a symmetric v x v matrix A with 0's along the main diagonal and 1 or 0 in position (i, j), $i \neq j$ according as vertices i and j and adjacent or non-adjacent.

Now, we define Seidel-equivalence of strongly regular graphs. Let G and G^* be two essentially different strongly

regular graphs on the same set V of v vertices and having parameters $(v, n_1, p^1_{11}, p^2_{11})$ and let A and A^* be the corresponding adjacency matrices. Then following Seidel [6] we say that G and G^* (equivalently, A and A^*) are Seidel-equivalent, if there exists a partition (V_1, V_2) of V, where V_1 and V_2 both are non-empty sets and the following is true :

If m, n ($m \neq n$) both belong either to V_1 or to V_2, then they are adjacent in G^*, if and only if, they are adjacent in G; and if one belongs to V_1 and the other to V_2, then they are adjacent in G^*, if and only if, they are non-adjacent in G. Without loss of generality, assume that $V_1 = (1,2,\ldots, a)$ and $V_2 = (a + 1, a + 2, \ldots., v)$. If,

$$A = \begin{pmatrix} C & D \\ D' & E \end{pmatrix} \qquad \ldots \qquad (1.1)$$

where C and E are square matrices of orders a and v−a respectively is the adjacency matrix of G, then it is obvious that

$$A^* = \begin{pmatrix} C & \overline{D} \\ \overline{D} & E \end{pmatrix} \qquad \ldots \qquad (1.2)$$

where $\overline{D}$ is obtained from D by interchanging 0 and 1, is the adjacency matrix of G^*.

2. GENERALISATION OF SEIDEL EQUIVALENCE

Given a graph G on V as set of vertices, partition V into r subsets, say $V_1, V_2, \ldots V_r$ and define another graph G^* on V satisfying the following:

(i) Two vertices m. n ($m \neq n$) belonging to the same set V_i ($i = 1, 2, \ldots r$) are adjacent in G^* if and only if they are adjacent in G.

(ii) Two vertices m. n ($m \neq n$), one belonging to V_i and another to V_j ($i \neq j$, $i, j = 1, 2, \ldots r$) are adjacent in G^* if and only if they are non-adjacent in G.

Without loss of generality, assume that $V_1 = (1, 2, \ldots a_1$. $V_2 = (a_1 + 1, a_1 + 2, \ldots, a_1 + a_2), \ldots, V_r = (a_1 + a_2 + \ldots + a_{r-1} + 1, \ldots, a_1 + a_2 + \ldots + a_r = v)$; $|V_i| = a_i$, where $|X|$ denotes the cardinality of the set X.

If A is the incidence matrix of G, then A can be written as:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{pmatrix} \qquad \ldots \quad (2.1)$$

where A_{ij} of order $a_i \times a_j$ is incidence, submatrix of the set V_i versus V_j ($A_{ji} = A_{ij}$ because of symmetry), then incidence matrix A^* of G^* is of the form,

$$A = \begin{pmatrix} A_{11} & \overline{A}_{12} & \cdots & \overline{A}_{1r} \\ \overline{A}_{21} & A_{22} & \cdots & \overline{A}_{2r} \\ \overline{A}_{r1} & \overline{A}_{r2} & \cdots & A_{rr} \end{pmatrix} \qquad \ldots \quad (2.2)$$

where $\overline{A}_{ij}$ is obtained from A_{ij} by interchanging 0 and 1.

Such a G^* will be said to be Generalised - Seidel Equivalent to G. In particular, if sets V_i's are equicardinal, i.e. $|V_i| = m$, for every i, then one can see that

$$A + A^* = \begin{pmatrix} 0 & J_n & \dots & J_n \\ J_n & 0 & \dots & J_n \\ J_n & J_n & \dots & 0 \end{pmatrix} \qquad \dots \qquad (2.3)$$

$$= (J_r - I_r) \times J_n \qquad \dots \qquad (2.4)$$

where J_r is a is a square matrix of order r consisting of only 1's, I_r is the identity matrix of order r; X denotes the Kronecker produce ot matrices.

Addition in L.H.S. of (2.3) is congruent mddulo 2 i.e. $1 + 1 = 0 + 0 = 0$, $1 + 0 = 0 + 1 = 1$.

It may be remarked here, that matrix (2.4) is similar to the adjacency matrix of a tournament. For further reference, see Moon [4] .

REMARK :

The structure of G^* heavily depends on the mode of partitioning of V, e.g.

Suppose G is three copies of K_2; (Figure 1) where K_n is the complete graph on n vertices.

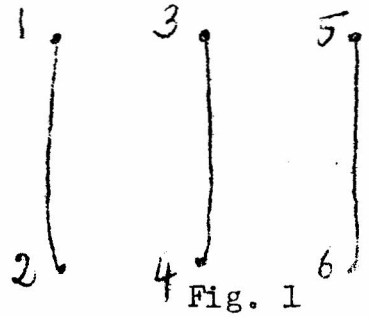

Fig. 1

Partition V = (1, 2, 3, 4, 5, 6)) into three sets as V_1 = (1, 2), V_2 = (3, 4), V_3 = (5, 6).

Corresponding G–S equivalent graph G_1^* is given below (Figure 2) which is regular with valence 4

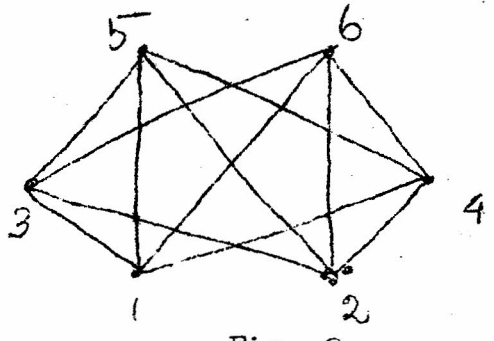

Fig. 2

but for the partitioning.

V_1 = (1, 4), V_2 = (2, 5), = (3, 6), we get G–S equivalent graph G_2^* (Figure 3) which is regular with valence .

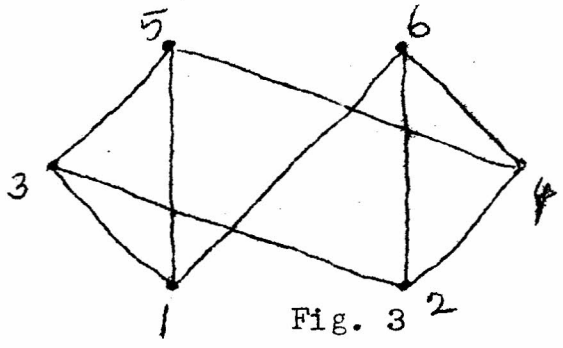

Fig. 3

3. G-S EQUIVALENCE AND PBIBD WITH TWO ASSOCIATE CLASSES

(a) Following Bose and Shrimamoto [3] we define Group Divisible (GD) design as follows :

There are $v = mn$ treatments divided into m groups of n treatments each, such that any two treatments of the same group are first associates while two treatments from different groups are second associates.

It is easy to check that we get a strongly regular graph G with parameters,

$v = mn, n_1 = n - 1, p_{11}^1 = n - 2$ and $p_{11}^2 = 0$. Since $p_{11}^2 = 0$, it is easy to prove that the corresponding graph is disconnected and is union of m complete graphs K_n.

Now, if we partition V into m classes according to groups and form $V_1, V_2 \ldots, V_m$ where graphs corresponding to V_1 is a complete graph on n vertices K_n, then G-S equivalent of G is the graph K_{mn},

(b) E.A. Nordhaus [5] studied the class of strongly regular graphs with $n_1 = 2p^1_{11} = 2R$ and showed that this class includes the complete tripartite graph $K_{R,R,R}$ and the line graph of the complete graph K_{R+2}.

Now, G-S equivalent of $K_{R,R,R}$ with partitioning same as that of complete tripartite graph is a null graph of 3R vertices ; complementary graph of $K_{R,R,R}$ will have G-S equivalent as K_{3R}.

In both the cases (a) and (b), the resulting G-S equivalent graph is a complete graph which is simplest to study and satisfies many nice properties.

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INFORMATION THEORETIC TECHNIQUES IN NUMBER THEORY

By

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Synopsis with references

The system N of positive integers (natural numers) is generally considered as the simplest mathematical system from which other mathematical systems are developed or generated in stages. So, Kronecker considered that natural numers were God-sent and other systems should be constructed from them.

The system I which is generally constructed very simply from the system N by introduction of the zero, the additive inverse (negative) of natural number is the system I of integers. Number Theory mainly concerns with this system. deals with several interesting properties and results of additive and multiplicative properties of integers, [Cf. Hardy and Wright, 1960].

+(delivered on March 15, 1973, in Matscience, The Institute of Mathematical Sciences, Adyar, Madras, INDIA)

The system I of integers is further extended by introduction of the multiplicative inverses of all the integers except the zero. The system, thus obtained is known as the system F of rationals. So far as algebraic operations are considered, the system F may be looked upton as maximal.

Now, if numer of the system F are represented as decimal fractions, it is found that they are finite or a recurring decimals. But from the study of elementary geometry and algebra, some numbers like $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{3}$, (i.e. surds) ..... ϕ e π are obtained and their exact values are not expressible as finite or recurring decimals. The numers represented by non-finite and non-recurring decimals are known as irrational numers. The set of all numers, rational and irrational, are known as the system R of reals. The system R is a very important mathematical system with which nearly all branches of mathematics are related directly or indirectly.

In addition to varicus discussions of properties and results connected with integers, the number theory concerns with two classes of problems connected with the system, R, viz.(i) those of the approximations of reals by rationals, their order etc. (ii) those of the frequency of occurrence of digits in decimal (or in the scale of r) representation.

Some of the interesting results of the problems of type (i) 'An algebraic irrational can not be approximated to any order'. 'After Thue, Siegal and Roth, no algebraic irrational is approximable of order greater than 2 i.e. $|\xi - \frac{p}{q}| < \frac{1}{2\sqrt{2}q^2}$

Detailed discussion of problems of type (i) may be seen in the standard books of the subjects [Cf. Hardy and Wright, 1960 ; Dickson, 1928, Levaque, 1954]. Discussions of the problems of type (ii) may be seen also in leterature [Cf. Hardy and Wright, 1960; Niven, 1956]. The present discussion mainly concerns with those of type (ii).

Let us consider a particular digit, i.e. 6, and a real number η . If ξ be rational, represented by finite decimal, then zeroes will be added after the significant digit after the decimal. Let r_n be the number of times b occurring in the n places after decimal in the decimal representation of ξ . Thus, r_n/n is the frequency of occurrence of digit b, in the n places after the decimal in the decimal representation of ξ Then, on the assumption of statistical regularity [Cf. Crammer, 1949 ; Dutta and Pal, 1963], the probability of occurrence of the digit, b, in the decimal representation of ξ may be taken as $\lim_{n\to\infty} r_n/n$. All the discussion is valid for representation of the number ξ in the scale of notation r.

Thus, in the r - iadic representation of numbers, the occurrence of a particular digit may be taken as an independent random event. Now, it is well-known that in a probabilistic scheme, entropy may be defined always [Cf. Shamnar, 1948, Khinchin 1957]. So, entropy of a real may be introduced simply.

A number is said to be simply normal in the scale r if when represented in the scale r, the occurrence of each of the digits are equally probable. There are two definitions for normal numbers, viz, (i) A number is normal in scale r, if it is simply normal in the scale r, r^2, r^3, (2) a number ξ is normal in scale r, if ξ is simply normal in scale $\eta, \eta\xi$ in scale $r^2, \eta^2\xi$ in scale r^3 and so on. Of various theorems about normalcy let the following two important theorems may be quoted:

1. Almost all numbers are normal [Borel Theorem; Cf. Hardy and Wright, 1960; Uspenesky 1937]

2. Two definitions of normal number are equivalent [Pillai Theorem, Cf. Pillai, 1940, Niven, 1956].

By introduction of entropy for reals, almost all known theorems and notions about simple normal, and normal numbers may be proved and introduced simply. The proof of Pillai theorem become very simple. By use of a known result of functional analysis [Cf. Gelfond and Shilov 1960] and of the central statistical theorem of Glivenko-Glivenko-Centelli [Cf. Loeve, 1960; Dutta, 1966], a sharper result than the

Borel theorem may be proved. In addition to these known theorems and results, some new notions, like a standard numb degree of normalcy may be used. These discussions may be seen in a paper of Dutta and Sen (1967).

In a recent paper, [Dutta and Mukherjee, 1973], degre of normalcy and associated notions and also results connecte with these notions have been discussed in some details. Mor over, applying methods similar to those based on Boltzmann hypothesis in statistical physics [Boltzmann 1923], some interesting results have been obtained.

In another two papers [Mukherjee 1972, 73] how the entropy may be introduced with some advantages in disuussion of normalcy of K - tuples of numbers have been shown.

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ON CONVERGENCE AND UNIQUENESS FOR ITERATIVE PROCESSES FOR SOME OPERATOR EQUATIONS IN ABSTRACT SPACES :

By

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The work done by the academic members of the Centre of Advanced Study in Applied Mathematics, Calcutta University on Numerical Analysis and Combinatorial Methods, has been reported very briefly as a separate article.

The present discussion consists of mainly two parts. In the first part, the work recently completed and communicated by Dutta and his associates, is reported briefly. In the second part, a new problem, appearing to be important for application of logical developments of several branches of mathematical physics has been discussed and formulated conveniently as a problem of multinorm spaces. The problem is not yet completely solved and is under investigation.

Part I. Convergence and Uniqueness of Iterative Processes for some Operator Equations

In applied mathematics except in some of the modern developments, the problems are formulated mostly in terms of operator equations (differential equations, integral equations, integro-differential equations) in some abstract spaces.

But in the majority of cases, particularly when the problems are really taken from real life, the determination of the complete analytical of the equations is not possible and in many cases, even if possible, the solutions are not in an useful form. So, approximate evaluations of the solution are necessary. Iterative methods are most frequently used for approximate evaluations. But, iterative methods are also used in the proof of some important basic theorems of analysis. So, discussions of convergence, rapidity of convergence uniqueness of several iterative process are important in mathematics (pure and applied).

Let the operator equation in an abstract space, S,

$u = Au + f, \qquad u, f \in S$ and $A : S \longrightarrow S$ ...(1)

Thus, the iterative process may be sketched as

$$u_1 = Au_o + f \qquad \ldots(2)$$

$$u_2 = Au_1 + f \qquad \ldots(3)$$

$$u_3 = Au_2 + f \qquad \ldots(4)$$

$$\ldots\ldots\ldots\ldots$$

$$u_n = Au_{n-1} + f \qquad \ldots(5)$$

where u_o is an arbitrary element from the domain of definitic of A (Generally u_o is chosen suitably for a process. Now, if $u_n \longrightarrow u$ as $n \to \infty$ then evidently we have

$$u = Au + f$$

i.e. u is a solution of equation (1). Thus, the convergence of an iterative method is important.

But in practice, iterative process can not be continued infinitely and has to be stopped after some steps. So, the rapidity of convergence and the estimation of errors for stopping the process at any stage is important. Also the uniqueness of the solution u is important, particularly for applications in physics and other similar branches of science.

In general discussions by functional analysis, these discussions lead to the fixed point theorems. In literatures, there are numerous fixed point theorems. Naturally all these fixed point theorems are not connected with actual problems taken from real life. Some may be due to intellectual inertia, a tendency pointed by klein in the preface of his famous book, 'Elementary Mathematics from Higher Standpoints, Part II- Geometry' (now available in Dover Series). Some may be looked upon as due to the tendency of analysing the necessary and the sufficient conditions of the fixed point theorems. It should be mentioned in this connection that in some branches of abstract mathematics like algebraic topology, not only fixed point theorems but also different classes of fixed theorems are discussed [Cf. Pontyagrin, L.S. Foundations of Combinatorial Topology, Graylock Press, 1955, Lefscheton, S : Algebraic Topology, A.M.S. Colloquium Publications, XXVII, N.Y, 1942]

Dilation Principle :

Mostly fixed point theorems are connected with contraction - principles. But, in a recent paper a dilation principle has been formulated as follows :

Let (S, d) be a complete metric space and D_A be the domain of definition of A, an operator in S, where

$$A : D_A \longrightarrow D_A \qquad \ldots(6$$

So that $d(Ax, Ay) \geq \lambda d(x, y)$, $\lambda > 1$ $\qquad \ldots(7$

Then, A is said to satisfy a dilation principle

Now, it is easy to see that A is one-to-one and so invertible. If D_A be a closed and compact set in S, then, D_A is a complete space. In this case, fixed point theorem is very easy and straight forward.

A Fixed-Point Theorem in a weak Metric Space :

In the same recent paper (1), a fixed point theorem ha been formulated suitably for a weak metric and then proved. The metric space, is a set of points, of which with every pa a non-negative real numer (to be referred to as weak distan satisfying only two postulates, viz, that the weak distance of a point from itself is zero and that of triangle ineuqlai is associated.(3)

A fixed-point Theorem for an Operator equation in Banach Spa

In the same recent paper (1), the following theorem is proved :

If A be a dilation operator of a Banach spaces and B be a completely continuous operator of X into itself so that BS is bounded, then there exists at least one solution of the equation,

$$A u = Bu + f \qquad u, f \in S \qquad \ldots(8)$$

Fixed-Point Theorem in Banach-Space

In another recent paper [4] the following theorem is proved :

If A be a dilation operator of a Banach space S to itself i.e.

$$\|Ax - Ay\| \geq \lambda \|x - y\|, \qquad x, y \in x \qquad \ldots(9)$$

and B is a constraction of S so that

$$\|Bx - By\| \leq q \|x - y\| \qquad \ldots(10)$$

then, the operator equation

$$ABu + u = Au + f \qquad (f \in S) \qquad \ldots(11)$$

has a unique solution provided $0 < 1 + \frac{1}{\lambda} < 1$ ..(12)

Its relation with iterative method for the operator equation (11) is simply evident.

Part II : Approximation Theory in Multinormed Space

For applications in the problem of mathematical physics including rational mechanics, particularly in logical developments[5] the function are to be approximated in such a way, that not only the functions but also some of (even in some cases all) its derivatives are to be approximated simultaneously.

So, the approximation theory is to be formulated in multi-normed spaces [6]. Attempts are now being made to discu the approximation theory in multi-normed spaces.

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A VERY BRIEF REPORT ON WORKS DONE BY THE ACADEMIC MEMBER OF THE CENTRE OF ADVANCED STUDY IN APPLIED MATHEMATICS CALCUTTA UNIVERSITY IN

NUMERICAL ANALYSIS AND COMBINATORIAL METHODS

(Reported in the Conference by Dr.M.Dutta, Centre of Advanced in Applied Mathematics, Calcutta University)

...

COMBINATORIAL METHODS:

Partition of Numbers: (M. Dutta)

A partition of numbers in which any part can be repeated upto d times has been introduced and its simple algebraic properties have been investigated. A rough approximate value for this partition has been calculated by a Taubarian theorem. Evidently the formulae for partition of unequal parts and unrestricted partition have been obtained when $d = 1$ and $d \to \infty$ respectively.

Some simple congresent relations and some relations between this new partition with upto d repetitions otherwise unrestricted and usual unrestricted partitions have been established. Two new recurrence relations for partitions of integers into m parts with utmost d repetitions have been deduced.

From some identities between modular functions, congruence relations have also been obtained.

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Graph Theory and its Application in Finite Automata, Net Works, and Logical Circuits in Computers: (T.Mukherjee, P.K.Sarkar, M.Mazumder; Supervisor:- Professor P.K.Ghosh).

Under the supervision of Professor P.K.Ghosh, T.Mukherjee and P.K.Sarkar worked on Graph Theory and its applications. Some problems of minimizations of Boolean functions, intimately connected with simplification and design of logical circuits used in computers, have been studied by a graph-theoretic method developed by them. This graph theoretic method has been further extended for a generalised study of Boolean functions. Also, algorithms for determination of all the cut points of a network (known as articulation points in literatures of graph theory) have been developed. The set of maximum compatible states of an incompletely specified sequential machines has been determined by a method, developed by them for determination of all the cliques of a graph. A new matrix representation of a finite deterministic sequential machine (finite automate) has been suggested.

Sm. M. Majumdar has solved a problem of minimization of a Boolean function by a graph-theoretic method.

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Numerical Analysis:

Technique : (Professor P.K.Ghosh & N.K. Basu)

(P.K.Ghosh) : A much simpler method for detection and evaluation of certain type of complex roots of algebraic (polinomial) equations Graeffe's root-squaring method has been sketched. Estimations of errors in a certain type of quadrative formulae has been discussed Significances of solutions of linear equations by the method of least squares were studied.

(N.K.Basu): Problems of evaluation of definite integrals, linear differential equations and integral equations associated with eigenvalued problems have been solved approximately by a series of derivatives of the Chebyshev polynomials of the first kind. The error-estimates, obtained in the approximate solutions, have been verified numerically and it is found that the results are of some order of accuracy and even sometimes better than those of other similar existing methods.

Approximate values of integrals near a simple pole using Chebyshev absciosa and the estimate of error for a Chebysev quadrature method proposed by Fillipi have been calculated. Errors in the representation of a function by Chebyshev polynomials of the second kind and approximate expression of a bivariate function as a double Chebysev series expansion have also been obtained. A generalised bivariate Hermite's interpolation formula and polynomial approximation to integral transforms have been investigated.

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Theoretical Investigations: Convergence of Iterative Process:

Iterative processes are most frequently used method for approximate evaluation of operator equations in abstract spaces. Theoretical investigations mainly concern with convergence of iterative processes, uniqueness of the solution and error estimation.

(Sen, R.N.): The questions of convergence, uniqueness, error estimations have been discussed for a bounded linear operator in ℓ_2 space and also in a Banach space, unbounden linear operator in separable Hilbert space, general operator equations in supermetric, linear metric and partially ordered linear topological spaces. Newton-Kantorovitch method has been modified. A general unified theory which yields Newton-Kantorovitch method in Banach spaces and some other of its important variants, has been developed.

(Chatterjea, S.K.): In most of discussion of convergence and uniqueness for iterative processes from the standpoint of functional analysis, fixed point theorems play the central role. Some theorem

fixed points particularly those for a sequence of mappings with contractive iterates.

(Das, M.K.): The following theorems have been proved:

I : If A is an endromorphism of a Banach space X so that $(I-A)^{-1}$ exists, and if B is a mapping of X into itself so that $\| Bx - By \| < k \| x-y \|$, $0 < k < \| \frac{I}{I-A} \|$, $x, y \in X$ then there exists an element $u \in X$ such that $Au + Bu = u$

II : If A and B are mapping of X into itself so that

(i) $T = (I-A)^{-1}$ exists and $\| Tx - Ty \| < n \| x-y \|$

n being a finite real numbers,

(ii) B is completely continuous and BX is bounded, then there exist at least one element u so that $Au + Bu = u$

III : If (X,d) be a complete metric space and if A and B are onto continuous mapping of X into Cx, the family of all bounded closed subsets of X, so that

$$H(Ax, By) \leq \alpha [d(x, By) + d(Ax, y)] + \lambda d(x, y)$$

H being the Hausdorff metric in (X,d) and $0 < \alpha < 1$, $\lambda > 0$, $0 < \frac{\alpha + \lambda}{1-\alpha} < 1$ then A and B has a common fixed point.

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SOME TECHNIQUES IN WEIGHTED SPACES

By

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ABSTRACT

The classical theorem of Weierstrass guarantees that a continuous function f, defined on a closed interval (a,b) can be uniformly approximated by means of a polynomial of sufficiently large degree n. In other words, the class P_n, of all polynomials P_n, constitutes a class of approximants to f e C(a,b) and P_n is said to be dense in C(a,b). The question naturally arises, as to what would happen if (a,b) is replaced by the entire real line R. Are polynomials the only functions which will serve as approximants? Are there other classes of functions which will perform the task equally well? These questions led to the formulation of the Bernstein himself, as well as by Mergelyan (6), Koosis (5) and Akutowicz (1). Bernstein (2) has provided a set of necessary and sufficient conditions for the class of all polynomials P to be dense in the weighted space $C_r(R)$, which is the Banach space of continuous functions F defined on R, with $\frac{f(x)}{\gamma(x)} \to 0$ as $|x| \to \infty$ and $\|f\|_\gamma = \sup_{x \in R} \frac{|f(x)|}{\gamma(x)}$

where $\gamma \geq 4$ is a continuous function and

$x^n / \gamma(x) \to 0$ as $|x| \to \infty$. γ

is called a weight and its presence in the denominator leads to the concept of weighted approximation. Later, Koosis has investigated the conditions under which P is dense in

$C_\gamma(R)$ or $C_\gamma(R_\delta)$

where $R_\delta = \bigcup_{n=-\infty}^{\infty} [n-\delta, n+\delta]$

where $0 < \delta < 1/2$. . It has been found that entire functions of finite exponential type are also dense in $\mathcal{L}_\gamma(R)$

or $C_\gamma(R_\delta)$

under suitable conditions, using the techniques of Mergelyan ([3]). Analogously, the necessary and sufficient conditions for P or the class of entire functions of finite exponential type to be dense in $L^p_\gamma(R)$ the weighted space of Lebesque measurable functions of order p, $1 \leq p < \infty$, have also been established, exploiting the methods of Akutowics ([3]).

The case when P is not dense in $C_r(R)$ has also been studied by Mergelyan. The condition under which P is dense in the class of entire functions of zero exponential type has been furnished by Hacatryan (4). An analogous condition

for $L^{P}_{r}(R)$ has been obtained ([3]). The case when r is not a continuous function has also been discursed and the necessary and sufficient conditons for the density to occur in such a case have been established in (3), following the techniques of Pollard (7).

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ZERO-ONE-NON-LINEAR PROGRAMMING

By

R.K. Gupta and Kanti Swarup

ABSTRACT

The paper considers a class of optimization problems. The problems are non-linear programming problems: maximise cx subject to $f_i(x) \leqslant b_i$ with the additional constraint that every component of x is 0 or 1 i.e. x is an extreme point of the n-dimensional unit cube defined by $I_n x \leqslant 1$, $x \geqslant 0$. Extreme point (1) (2) method for solving such problem is presented. One numerical example is also given.

INTRODUCTION

We define the "Zero-One Integer Programming Problem"- to be the integer programming problem in which all the variable are restricted to the values 0 or 1.

Most of the practical problems belong to zero-one linear programming problems are such as:

a. The scheduling of products subject to capacity constraints.

b. The machine sequencing problem. (4,5,6)

c. Make or buy problems in which the decision is of the "either or" type and in which a linear criterion function is to be optimised falls neatly within the scope of the problems treated.

Here we deal with the maximisation of a linear function subject to some constraints which are non-linear with the additional restriction that all the variables are required to be either zero or one.

Mathematical formulation

(P-I) Max: $c\,X$

Subject to

$$f_i(x) \leqslant b_i \qquad i=1,2,\ldots m$$

and $x_j = 0$ or 1 for all j

where c is 1xn, and x is nx1 vector.

Any problem of the type (P-1) can be formulated as an Extreme Point Mathematical Programming Problem (1), (2) as follows.

(P-II) Maximize cx

subject to $f_i(x) \leqslant b_i \qquad i=1,2,\ldots m$

and x is an extreme point of

$$I_n\, x \leqslant 1 \qquad x \geqslant 0.$$

where I_n is the n x n identity matrix and 1 is a nx1 sum vector consisting entirely ones, and 0 is nx1. Further problem (P-II) is equivalent to the following problem:

(P-III) Maximize cx
$x \in s$
subject to

$f_i(x) \leqslant b_i$

where s is the finite set given by $s=\{x:$ x is a extreme point of $I_n\, x \leqslant 1,\ x \geqslant 0$

Hence an iterative procedure which has the following three characteristics will constitute an algorithm for solving problem (P-I).

(A) At the i-th iteration the i-th best extreme point for Max cx, are found

$x \in s$

(B) At the i-th iteration, the points found in (A) are tested and to see whether they satisfy $f_i(x) \leq b_i$

and

(C) If the test in (B) indicates that a point found in (A) satisfies $f_i(x) \leq b_i$, then algorithm terminates.

Since 's' is a finite set, the steps (A) and (B) and (C) gurantee convergence in a finite number of steps as long as the extreme points found in (A) are never repeated.

Notation

Let $u_1 >, u_2 > u_3 > \ldots, > u_n$, be the value of the objective function assumed at the extreme points of the convex set defined by $I_n x \leq 1, x \geq 0$ and let

$X_i = \{x;\ cx = u_i,\ x$ is an extreme point of $I_n x \leq 1,\ x \geq 0\}$

The set X_i can also be described as the i-th best extreme point solution of the problem:

(P-IV) Max cx

subject to

$I_n x \leq 1,\ x \geq 0.$

The fundamental idea underlying in our algorithm can be described in the following way.

Any feasible solution of Problem (P-III) is also a feasible solution of problem (P-IV) i.e. is an element of X_r for some r, $1 \leqslant r \leqslant N$. Hence we seek smallest value of r such that X_r contains an element which satisfies $f_i(x) \leqslant b_i$. This is achieved by finding and testing the element of X_r in $f_i(x) \leqslant b_i$, beginning with $\bigwedge^{\nu=1}$ and continuing with r=2,3, etcetera until an x_r is found which contains an element which satisfied $f_i(x) \leqslant b_i$ or some indication of no solution is obtained.

The theory of the simplex method (7) is sufficient to justify the theory needed to explain how the sets X_r are found The set X_1 is found by applying the simplex algorithm to problem (P-IV) and to find the sets for all optimal extreme point solutions. In fact these solutions can be obtained without large calculations because of the very special structure of the constraint set. In finding the element of X_{r+1}, $r \geqslant 1$, we use the basic principle that an element of X_{r+1} is adjacent to some element of $\bigcup_{k=1}^{\nu} X_k$ Thus when the elements of X_r does not satisfy $f_u(x) \leqslant b_i$, then X_{r+1} are generated, the next step will be to find the value of the objective function at each extreme point of I_n $x \leqslant 1$, $x \geqslant 0$ which is (I) adjacent to an element of X_r and (II) provides a value of the objective function which is less than u_r.

To explain the algorithm with problem (P-IV) let x' be the n - component, non-negative slack variable for $I_{n} x \leq 1$, $x \geq 0$, thus problem (P-IV) reduces to

(P-V) $\quad \text{Max } (c, 0)\begin{pmatrix} x \\ x' \end{pmatrix} = c' x + 0x'$

subject to

$$D t = 1, \quad t > 0$$

where $t = \begin{pmatrix} x \\ x' \end{pmatrix}$, $t_j = x_j$

$t_{n+j} = x_j \qquad J = 1, 2, \ldots, n.$

and $D = (I, I)$. A column n+j (j=1,2, ...n), Corresponds to slack variable x_j.

Any basic feasible solution $t = \begin{pmatrix} x \\ x' \end{pmatrix}$ of (P-V) will be the required solution of our problem if it is satisfies to $f_i(x) \leq b_i$.

As problem has special structure we have the following simplifications in the simplex tableau obtained for (P-V) at various stages (7) i.e., $y_j = y_{n+j} = e_j$, the unit vector with j^{th} component one.

$$\theta_j \quad j = \theta_{n+j} = 1$$

where $\theta_j = \min\left\{ \frac{X_{Di}}{y_{ij}}, \; y_{ij} > 0 \right\} = 1 \quad \forall j$

$z_j - c_j = -c_j$ if x_j is non-basic

$z_{n+j} - c_{n+j} = c_j$ if x'_j is non-basic

Notice that here we have taken an advantage of the special structure of the constraints of prob (P-V). We being with any basic feasible solution to Prob (P-V), the basis is I and one only one of x_j and x_j' for each j=1,2,...,n is in the basis and value of each basic variable is one i.e., every basic feasible solution Prob (P-V) is non-degenerate and every extreme point solution of Prob (P-V) is expressed by a unique basis.

NUMERICAL EXAMPLE

Max $3x_1 + 6x_2 + 9x_3 + x_4$.

subject to $x_1^2 + 5x_1x_2 + 6x_3^2 + x_4^2 \leqslant 10$

and $x_j = 0$ or 1 for j=1,2,3,4.

i.e., $x= (x_1, x_2, x_3 x_4)$ is an extreme point of

$$x_1 + \quad +x_1' \quad =1$$

$$x_2 + \quad +x_2' \quad =1$$

$$x_3 + \quad +x_3' \quad =1$$

$$x_4 + \quad +x_4' \quad =1$$

$x_j \geqslant 0$ and $x_j' \geqslant 0$ for all j=1,2,3,4.

TABLEAU No.1.

c_B	Vectprs in Basis	c_j / x_B	3 x_1	6 x_2	9 x_3	1 x_4	0 x_1'	0 x_2'	0 x_3'	0 x_4'
3	x_1	1	1	0	0	0	1	0	0	0
6	x_2	1	0	1	0	0	0	1	0	0
9	x_3	1	0	0	1	0	0	0	1	0
1	x_4	1	0	0	0	1	0	0	0	1
	Z=19									
	z_j-c_j		0	0	0	0	3	6	9	1
	$Z-(z_j-c_j)$		-	-	-	-	16	13	10	18

TABLEAU No.2

c_B	Vectprs in Basis	c_j / x_B	3 x_1	6 x_2	9 x_3	1 x_4	0 x_1'	0 x_2'	0 x_3'	0 x_4'
3	x_1	1	1	0	0	0	1	0	0	0
6	x_2	1	0	1	0	0	0	1	0	0
9	x_3	1	0	0	1	0	0	0	1	0
0	x_4	1	0	0	0	1	0	0	0	1

Z=18									
z_j-c_j	-	-	-	(-1)	3	6	6	9	0
$Z-(z_j-c_j)$					15	12	9		

TABLEAU No.3

c_B	Vectprs in Basis	c_j / x_B	3 x_1	6 x_2	9 x_3	1 x_4	0 x_1'	0 x_2'	0 x_3'	0 x_4'
0	x_1'	1	1	0	0	0	1	0	0	0
6	x_2	1	0	1	0	0	0	1	0	0
9	x_3	1	0	0	1	0	0	0	1	0
1	x_4	1	0	0	0	1	0	0	0	1
	Z=16									
	z_j-c_j		(-3)	0	0	(-1)	0	6	9	0
	$Z-(z_j-c_j)$		-	-	-	-	-	9	6	-

Thus we obtained a solution $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$, which satisfies $x_1^2+5x_1x_2+6x_3^2+x_4^2 \leqslant 10$. Therefore it is an optimal solution of the problem.

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CERTAIN TOTIENT FUNCTIONS AND RAMANUJAN'S SUMS

By

J. Hanumanthachari

ABSTRACT

The main aim of this paper is to study the sum

$$C(L,M,N) = \sum_{\substack{\chi \pmod N \\ (\chi,N)=1 \\ (M-\chi,N)=1}} \exp\left(\frac{2\pi i L\chi}{N}\right)$$

the particular cases of which reduce to Ramanujan's sum [2, Theorem 271], Nagell's totient function and Euler's totient function. Using a well known combinatorial lemma [2, Thorem 260], theorem 268 [2] and lemma 1, a formula for C(L,M,N) is obtained. Using lemma 2, Holder identity for C(L,M,N) is obtained. Also certain multiplicative properties are studied.

Lemma 1: If $d \mid N$ and $(j,d) = 1$, then there are $\frac{\phi(N)}{\phi(d)}$ numbers congruent to $j \pmod d$ but incongruent mod N and relatively prime to N. These numbers are given by

$$a_k + (t_k + sd)\ \frac{\gamma(N)}{\gamma(d)} \quad \text{where}$$

(i) $a_k \pmod{\frac{\phi(N)}{\phi(d)}}$, $(a_k, \frac{\phi(N)}{\phi(d)}) = 1$

(ii) t_k is given $t_k \frac{\phi(N)}{\phi(d)} \equiv j - a_k \pmod{d}$

(iii) $s = 1, 2, \ldots, \left(\frac{N}{d}\right) \frac{\phi(d)}{\phi(N)}$

$\phi(N)$ being the Euler's totient function [2]

Lemma 2:

Let $d \mid N$ and $(j,d) = 1$, $(m - j, d) = 1$ then there are $\frac{\theta(M,N)}{\theta(M,d)}$ number of x's congruent to j (mod d) but incongruent modulo N and $(x,N) = 1$, $(M-x,N) = 1$, $\theta(M,N)$ being Nagell's totient function [1].

Formula for C(1,M,N):

$$C(L,M,N) = \begin{cases} \frac{N}{\gamma(N)} \sum_{d \mid \gamma(N),\ (d,M)=1} (-1)^{\omega(d)} \sum_{\delta \mid \frac{\gamma(N)}{d}} \mu\left(\frac{\gamma(N)}{d\delta}\right) J_{L,M,N}\left(j, \frac{\gamma(N)}{d}, \delta\right) & \text{if } \frac{N}{\gamma(N)} \mid L \\ 0, & \text{if } \frac{N}{\gamma(N)} \nmid L \end{cases}$$

where $\omega(n)$ = Number of distinct prime factors of N, $\gamma(N)$ = product of distinct prime factors of N, and

$$J_{L,M,N}\left(\frac{\gamma(N)}{d}, \delta\right) = \sum_{a \pmod{\delta}} \exp\left(\frac{2\pi i L}{N}\left(\frac{a}{\delta}+y\right)\frac{\gamma(N)}{d}\right),$$

Y is given by $\left(\frac{a}{\delta} + Y\right) \frac{\gamma(N)}{d} \equiv M \pmod{d}$

Holder identity for $C(L,M,N)$: If $d \mid (L,N)$,

$$C(L,M,N) = \frac{\theta(M,N)}{\theta\left(M, \frac{N}{d}\right)} C\left(\frac{L}{d}, M, \frac{N}{d}\right)$$

Certain Multiplicative properties:

1. If $(L_2, N) = 1$, then $C(L_1L_2, M, N) = C(L_1, L_2M, N)$.

2. If $(N_1, N_2) = 1$, then

$$C(L_1, M_1, N_1)\ C\ (L_2, M_2, N_2) = C(L_1N_2 + L_2N_1,$$

$$m_1N_2 + m_2N_1, N_1N_2).$$

where

$$M_1 \equiv N_2 m_1 \pmod{N_1}, \quad M_2 \equiv N_1 m_2 \pmod{N_2}.$$

Cor. (1): If $L_1 = L_2$ then

$$C(L,M_1,N_1)\ C(L,M_2,N_2) = C(L, M_1N_2 + M_2N_1, N_1N_2).$$

Cor. (2): If $(L_2,N_1) = (L_1,N_2) = (N_1,N_2) = 1$, then

$$C(L_1,M_1,N_1)\ C(L_2,M_2,N_2) = C(L_1L_2, m_1N_2 + m_2N_1, N_1N_2), \text{ where}$$

$$M_1 \equiv L_2 m_1' \pmod{N_1}, \quad M_2 \equiv L_1 m_2' \pmod{N_2}.$$

Also the author obtained the formulae and corresponding results for the same

$$C^{s}(L_1,L_2,\ldots,L_s,M_1,M_2,\ldots,M_s,N) = \sum \exp\left(\frac{2\pi i}{N}\sum_{1}^{s} L_j x_j\right)$$

$$x_j \pmod N,\ j=1,\ldots,s$$

$$((x_j),N)=1$$

$$((M_j - x_j),N)=1$$

$$C(s,L_1,L_2,\ldots,L_s,M_1,\ldots,M_s,N) = \sum \exp\left(\frac{2\pi i}{N}\sum_{1}^{s} L_j (x+j)\right)$$

$$x \pmod N$$

$$(x+j,N)=1,\ j=1,2,\ldots,s$$

$$(M_j - \overline{x+j},N)=1,\ j=1,2,\ldots,s$$

The details of this will be dealt with later.

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FINITE DIFFERENCE METHOD FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction: Consider the second order Volterra functional differential equation of the form

$$x''(t) = g(t)\,F(x(s)) + f(t), \qquad a \le t \le b \tag{1}$$

together with the boundary conditions:

$$\left.\begin{aligned} x(t) &= \phi(t), && \text{if } t \in [\alpha, a] \\ x(b) &= B \end{aligned}\right\} \tag{2}$$

Here $F: C([\alpha, b] \to R)$ is a Volterra functional, i.e. $F(x)$ depends on $x(s)$ for $s \in [\alpha, t]$, but is independent of $x(s)$ for $s > t$; $\phi \in C^1([\alpha, a] \to R)$ is a specified initial function, where $\alpha = \min_{a \le t \le b} s$. Functions $g(t)$ and $f(t)$ are continuous with $g(t) \ge 0$ on $[a,b]$ and a, b, B are real finite constants.

In recent years, several sufficient conditions have been obtained to prove the existence of a unique solution of the boundary value problem (1), (2) (See Grimm and Schmitt [1], [2], Kamenskii [3], [4] and Schmitt [5]).

Theorem. Let K_1 and K_2 be given positive constants such that $|\phi(t)| \le K_1$, $|\phi'(t)| \le K_2$ for all $t \in [\alpha, a]$, and F satisfies a uniform Lipschitz condition of the type:

(3) $|F(x(S)) - F(y(S))| \leq L|x(S) - y(S)|$, $|x(S)| \leq K_1$, $|y(S)| \leq K_1$

where L is the Lipschitz constant. Then, boundary value problem (1), (2) has atmost one solution x(t) with

$$|x(t)| \leq K_1, \quad |x'(t)| \leq K_2 \quad \text{, provided that}$$

$$1/8\, K L (b-a)^2 < 1 \tag{4}$$

where $K = \underset{a \leq t \leq b}{\text{Max}}\, g(t)$.

(For proof see (1)

In this note we have used an ordinary Finite-Difference method to show that boundary value problem (1), (2) may be solved numerically but it is not our intent to provide efficient numerical methods.

The finite difference method for solving two-point boundary value problem converts the differential equation into a finite set of algebraic or transcendental equations. The solution of the set of algebraic or transcendental equations yields approximations to the solution of the original differential equation at the discrete set of points. If the original differential equation is linear the finite difference equations will be linear algebraic equations. If the differential equation is nonlinear the resulting finite difference equations will be nonlinear algebraic or transcendental equations. The resulting nonlinear equations may then be solved by any numerical technique available for solving such equations, for example the Newton - Raphson method.

2. Finite - Difference Method An approximate solution of the boundary value problem (1), (2) can be obtained by standard finite difference methods. Divide the interval $[a,b]$ into N_b equally spaced intervals of length h_b, where $N_b - 1$ is the number of

interval points.

$$h_p = \frac{(b-a)}{N_p}, \quad N_p = 2^p, \quad p = 1, 2, \ldots$$

The discrete points are given by

$$t_{j,p} = a + j h_p, \quad j = 0, 1, \ldots N_p$$

where $t_{0,p} = a$ and $t_{N_p,p} = b$.

The second derivative at $t_{j,p}$ may be approximated by the second central central difference,

$$\left(\frac{d^2x}{dt^2}\right)_{t_{j,p}} = \frac{x(t_{j+1,p}) - 2x(t_{j,p}) + x(t_{j-1,p})}{h_p^2} + O(h_p^2)$$

The discrete approximations to (1), is then

5) $$x(t_{j+1,p}) - 2x(t_{j,p}) + x(t_{j-1,p}) = h_p^2 \left[g(t_{j,p}) F(x(s)|_{t_{j,p}}) + f(t_{j,p})\right] j = 1, 2 \ldots N_p - 1.$$

In general F(x(s)) requires the value of x(s) at some point other than the disdrete point $t_{j,p}$ say at the point s^*, then we have either $s^* < a$, then $\phi(s^*)$ is used, or $s^* > a$ a then $x(s)|_{t^*}$ is used where

$$t^* = \max \{ t_{j,p} < s^* \}$$

The two-point boundary value problem given by (1), (2), may now be approximated by the $N_p - 1$ equations of (5), where the boundary conditions are included. If the differential equation (1) is linear (5) gives $N_p - 1$ linear algebraic equations, and in matrix from the equations can be expressed as

$$AX = B \qquad (6)$$

where X is the column matrix with elements $x_p(t_1), x_p(t_2), \ldots, x_p(t_{N-1})$ and B is also a column matrix with elements dependent on $x_p(t_0)$, $x_p(t_N)$, $f(t_j)$ and P_p. The matrix A is irreducible and monotone (recalling $g(t) \geq 0, t \in [a,b]$), so there exists a solution of the equation (6) or the convergence follows which can be proved by Gauss-Seidel or Jacobi iterative methods.

Now if the differential equation (1) is non linear (5) gives N_p-1, non linear algebraic or transcendental equations which can be solved by Newton-Raphson method and if the conditions of Kantorovich theorem (see [6]) are satisfied then the convergence is guaranteed.

3. Numerical Examples: Here, we present two numerical examples.

Example 1: Consider the linear Volterra-differential equation

$$x''(t) = x(t^2) \qquad 0 \leq t \leq 1 \tag{7}$$

with the boundary conditions:

$$\begin{aligned} x(0) &= 0 \\ x(1) &= 1 \end{aligned} \tag{8}$$

There exists a unique solution of this boundary value problem (7) (8) by Theorem 1. The results are given in Table 1.

Table 1

p	x (1/8)	x_p(1/4)	x_p(3/8)	x_p(1/2)	x_p(5/8)	x_p(3/4)	x_p(7/8)
1				.5			
2		.235294		.470588		.720588	
3	.116748	.233497	.350245	.468818	.591039	.718732	.853751
4	.116086	.232172	.348938	.467518	.589727	.717390	.853074
5	.115812	.231708	.348397	.466980	.589184	.716921	.852794
6.	.115580	.231333	.347940	.466525	.588727	.716544	.852558

Example 2: Consider the nonlinear volterra-differential equation

$$x''(t) = \left[x\left(\frac{t}{(t+2)^2} \right) \right] (1+2t)^2 \qquad (9)$$

with the boundary conditions:

$$x(0) = 1$$
$$x(1) = e \qquad (10)$$

It is easily verified that the exact solution of the boundary value problem (9), (10), is $x(t) = e^t$. The results are given in Table 2.

Table 2

p	1	2	3	4	Exact
$x_p(1/8)$		1.	1.154053	1.144304	1.133148
$x_p(1/4)$		1.335820	1.323731	1.306479	1.284025
$x_p(3/8)$			1.509034	1.487307	1.454991
$x_p(1/2)$	1.734141	1.734141	1.709962	1.688949	1.648721
$x_p(5/8)$			1.938606	1.913737	1.868246
$x_p(3/4)$		2.194961	2.182874	2.159729	2.117000
$x_p(7/8)$			2.442768	2.427659	2.393875

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ON CERTAIN NUMERICAL AND COMBINATORIAL ASPECTS OF CLIFFORD ALGEBRA, ITS GENERALISATIONS AND ASSOCIATED STRUCTURES

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Abstract:-

Certain interesting numerical aspects of Clifford algebra, its generalizations and associated structures are discussed.

Clifford algebra, its generalizations, associated structures and their applications have been systematically studied at Matscience by Alladi Ramakrishnan and his collaborators in recent years [1] . This article elaborates on certain interesting numerical and combinatorial features of those structures.

Discovery of quaternions by Hamilton was the first generalization of the commutative number systems - algebra of real numbers and complex numbers. The non-commutative (associative) algebra Q of Hamilton's quaternions has four basis elements $(1, e_1, e_2, e_3)$ obeying

$$e_1^2 = e_2^2 = e_3^2 = 1 \qquad (1)$$
$$e_1 e_2 = e_3, \quad e_2 e_3 = e_1, \quad e_3 e_1 = e_2,$$
$$e_1 1 = 1 e_1, \quad e_2 1 = 1 e_2, \quad e_3 1 = 1 e_3$$
$$e_1 e_2 = -e_2 e_1, \quad e_2 e_3 = -e_3 e_2, \quad e_1 e_3 = -e_3 e_1$$

and any quaternion is

$$q = q_0 1 + q_1 e_1 + q_2 e_2 + q_3 e_3 \qquad (2)$$

If $q, q' \in Q$ then

$$\begin{aligned} qq' = {} & (q_0 q_0' - q_1 q_1' - q_2 q_2' - q_3 q_3')\,1 \\ & + (q_0 q_1' + q_1 q_0' + q_2 q_3' - q_3 q_2')\,e_1 \\ & + (q_0 q_2' + q_2 q_0' + q_3 q_1' - q_1 q_3')\,e_2 \\ & + (q_0 q_3' + q_3 q_0' + q_1 q_2' - q_2 q_1')\,e_3 \end{aligned} \tag{3}$$

Obviously $qq' \neq q'q$. So these cannot be represented by ordinary complex numbers. The sub-algebras with basis elements $(1, e_1)$, $(1, e_2)$, $(1, e_3)$ are commutative and are all isomorphic to the algebra of complex numbers. Since associative law of multiplication holds one can think of matrix representation of Q. To determine the irreducible representations of the basis elements $(1, e_1, e_2, e_3)$ and hence of Q first it is observed from (1) that these elements do form a group apart from a factor of -1. So it is possible to construct a group with eight elements given by $Q \equiv \{1, e_1, e_2, e_3, -1, -e_1, -e_2, -e_3\}$ - called the quaternion group. From its multiplication table it is easy to see that this group has five conjugate classes namely $\{1\}$, $\{-1\}$, $\{e_1, -e_1\}$, $\{e_2, -e_2\}$, $\{e_3, -e_3\}$ and hence there should be five inequivalent irreducible representations of dimensions say $d(1)$, $d(2)$, $d(3)$, $d(4)$, $d(5)$ obeying

$$\sum_{i=1}^{5} d(i)^2 = 8 \tag{4}$$

in view of Burnside's theorems. One can guess easily four one dimension 1 representations given by

	1	e_1	e_2	$e_3=e_1e_2$	-1	$-e_1$	$-e_2$	$-e_3$
Γ_1:	1	1	1	1	1	1	1	1
Γ_2:	1	1	-1	-1	1	1	-1	-1
Γ_3:	1	-1	1	-1	1	-1	1	-1
Γ_4:	1	-1	-1	1	1	-1	-1	1

(5)

which are obtained by considering the fact that the subgroup $\{1, -1\} \cong \mathbb{Z}_2$ (the cyclic group of order 2) is a normal divisor of $\mathbb{Q}$ and the quotient group $\mathbb{Q}/\{1,-1\} \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2$. Thus homomorphism of $\mathbb{Q}$ with $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ gives rise to the above un-faithful representations. From (4) it is now clear that there should exist one and only one more representation of dimension 2. This faithful representation is furnished in terms of Pauli matrices as

$$1 = e_0 = I_2, \quad e_1 = -i\sigma_1, \quad e_2 = -i\sigma_2, \quad e_3 = -i\sigma_3 \tag{6}$$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Thus any quaternion $q = q_0 1 + q_1 e_1 + q_2 e_2 + q_3 e_3$ has the faithful irreducible representation

$$q = \begin{pmatrix} q_0 - iq_3 & -iq_1 - q_2 \\ -iq_1 + q_2 & q_0 + iq_3 \end{pmatrix} \tag{7}$$

All the four matrices $(I_2, \sigma_1, \sigma_2, \sigma_3)$ are linearly independent and hence the algebra of quaternions with four linearly independent basis elements is isomorphic to the total 2 x 2 matrix algebra over the field of complex numbers. Thus $\mathcal{Q} \cong M_2(\mathbb{C})$. From (6) it is seen that Pauli σ-matrices obey

$$\begin{aligned} &a)\ \sigma_i^2 = I_2, \quad \forall i = 1,2,3 \\ &b)\ \sigma_i \sigma_j = -\sigma_j \sigma_i, \quad \forall i, j = 1,2,3. \\ &\quad (i \neq j) \\ &c)\ \sigma_1 \sigma_2 \sigma_3 = i I_2 \end{aligned} \tag{8}$$

(8,C) gives $\sigma_1 \sigma_2 = i\sigma_3$, $\sigma_2 \sigma_3 = i\sigma_1$, $\sigma_3 \sigma_1 = i\sigma_2$

It is precisely to avoid the occurrence of i, extraneous to the basis of $\mathcal{Q}$, Hamilton's quaternion basis elements are normalized as $e_i^2 = -1$ instead of $e_i^2 = +1$, $\forall i = 1,2,3$.

Let us now pass on to the consideration of more general algebraic systems developed by W.K.Clifford, following Hamilton. These are the so called Clifford algebras. The relations (8) a,b can be generalized to read as

$$\begin{aligned} &a)\ e_i^2 = 1, \quad \forall i = 1, 2, \ldots, n. \\ &b)\ e_i e_j = -e_j e_i, \quad \forall i, j = 1, 2, \ldots, n. \\ &\quad i \neq j \end{aligned} \tag{9}$$

((8)C can be derived from 8(a) and (b) for the case of matrix representations). e_i's are called generators and by taking all possible products of all possible powers of them we get 2^n elements

$$\mathcal{G} = \left\{ \prod_{i=1}^{n} e_i^{k_i} \;\middle|\; k_i = 0, 1 \right\} \tag{10}$$

Product of any two elements of this set $\mathcal{G}_n$ is given by

$$\left\{ \prod_{i=1}^{n} e_i^{k_i} \right\} \left\{ \prod_{i=1}^{n} e_i^{l_i} \right\} = (-1)^{\left\{ \sum_{(i<j)=1} l_i k_j \right\}} \times \left\{ \prod_{i=1}^{n} e_i^{(k_i + l_i)} \right\} \tag{11}$$

with $(k_i + l_i) \bmod 2 = k_i + l_i$, $\forall\ i = 1 \cdots n$. Thus product of the elements of the set $\mathcal{G}$ being algebraically closed the set $\mathcal{G}$ forms a basis for an algebra denoted by $C_n^{(2)}$, a Clifford algebra, whose elements are linear combinations of the 2^n elements of $\mathcal{G}$ with coefficients as complex numbers. This algebra can now be generalised into a generalized Clifford algebra $C_n^{(m)}$ where m^{th} root of unity replaces the square root in the defining equations of the generalising relations (9). Now this algebra $C_n^{(m)}$ has generators $\{ e_i \mid i = 1 \cdots n \}$ obeying

$$\begin{aligned} &\text{a)} \quad e_i^{\,m} = 1, \quad \forall i = 1, \cdots, n. \\ &\text{b)} \quad e_i e_j = \omega\, e_j e_i \;, \quad \forall\ i, j = 1, \cdots, n \\ &\qquad\qquad i < j \end{aligned} \tag{12}$$

where $\omega = \exp(2\pi i / m)$ (or any primitive m^{th} root of unity). There are many ways of looking at the algebra $C_n^{(m)}$ generated by (12). It can be thought of as a direct generalization of $C_n^{(2)}$ replacing the square root by an m^{th} root. It can be derived from

considerations of linearization of m^{th} order generalization of Laplacian differential equations in n variables (due to $\left(\sum_{i=1}^{m} \lambda_i e_i\right)^m = \left(\sum_{i=1}^{m} \lambda_i^m\right) I$) or it can be obtained from certain group theoretical considerations namely projective representations of the finite Abelian group $Z_m \otimes \cdots \otimes Z_m$ (n copies). The first way of generalization has been studied extensively at Matscience in recent years by Alladi Ramakrishnan and his collaborators [1] and the other types of approaches have been studied in detail by H.Weyl [5], K.Yamazaki, K.Morinaga and T.NoNo, A.O.Morris and I.Popovicci and Tustoi and others [2]. Just like we found that the quaternion algebra has only one irreducible faithful representations (of dimension 2), for Clifford algebra a theorem of Pauli says that $C^{(2)}_{2\nu}$ has only one irreducible faithful representation of dimension 2^ν and $C^{(2)}_{2\nu+1}$ has two such irreducible faithful representations each of dimension 2^ν. By generalizing the approach adapted for studying the representation theory of quaternion algebra we shall study the representation theory of (12), i.e. of $C^{(m)}_n$, of which quaternion algebra and Clifford algebras are special cases. This study through the construction of the analogous generalization of quaternion group presents very interesting numerical and combinatorial problems involving the various properties, like oddity and primitivity of the concerned intergers n and m.

Explicitly writing the elements of the base of the algebra $C^{(m)}_n$ we have

$$\left\{ \prod_{i=1}^{n} e_i^{k_i} \;\middle|\; k_i = 0, 1, \cdots, m_i - 1 \right\} \tag{13}$$

which contains m^n elements. The generalization of the product rule (11) becomes in this case.

$$\left\{ \prod_{i=1}^{n} e_i^{k_i} \right\} \left\{ \prod_{i=1}^{n} e_i^{l_i} \right\} = \omega^{-\left\{ \sum_{\substack{i<j \\ i,j=1}}^{n} l_i k_j \right\}} \left\{ \prod_{i=1}^{n} e_i^{(k_i + l_i)_m} \right\} \tag{14}$$

with $(k_i + l_i)_m \bmod m = (k_i + l_i)$, $\forall i = 1, 2, \cdots, n$. It is seen that as k_i and l_j vary over all their possible values $(0, 1, \cdots, m-1)$, $\forall i, j = 1, \cdots, n$ the product of any two elements (14) of the base gets a phase factor which is a power of ω. Thus these m^n elements of the base of $C_n^{(m)}$ form a group only upto phase factors which are powers of ω. So this suggests that if we augment the base with all m^n elements, multiplied respectively by the $(m-1)$ powers of ω, $(\omega, \omega^2, \cdots, \omega^{m-1})$ we would get a group with $m^n(m-1) + m^n = m^{n+1}$ elements. We can describe this group abstractly by the set of elements

$$G_n^{(m)} = \left\{ \prod_{i=0}^{n} e_i^{k_i} \;\middle|\; \begin{array}{l} k_i = 0, 1, \cdots, m-1 \\ \forall i = 0, 1, \cdots, n \end{array} \right\} \tag{15}$$

where the n+1 generators $\{e_i \mid i = 0, 1, \cdots, n\}$ of the group $G_n^{(m)}$ called by us [3,4] as Generalized Clifford

group, obey

$$e_i e_j = e_0 e_j e_i \quad , \forall\, i = 1, 2, \cdots, n.$$
$$i<j$$
$$e_i^m = 1, \quad \forall\, i = 1, 2, \cdots, n.$$
$$e_i e_0 = e_0 e_i, \quad \forall\, i = 1, 2, \cdots, n. \tag{16}$$

Since the relations (12) are particular cases of (16) with the study of representations of (16) or $G_n^{(m)}$ provides the representations of (12). Now we shall begin counting the conjugate classes in $G_n^{(m)}$ which is the main subject of this paper.

First let m = a prime number. Let us find the elements in the conjugate class of $\left\{ \prod_{i=0}^{n} e_i^{k_i} \right\}$. Forming the conjugate element of $\left\{ \prod_{i=0}^{n} e_i^{k_i} \right\}$ with $\left\{ \prod_{i=0}^{n} e_i^{l_i} \right\}$ we get

$$\left\{\prod_{i=0}^{n} e_i^{l_i}\right\}\left\{\prod_{i=0}^{n} e_i^{k_i}\right\}\left\{\prod_{i=0}^{n} e_i^{l_i}\right\}^{-1}$$
$$= \left(\omega^{-\left\{\sum_{i=1}^{n}\left(\sum_{j=1}^{i-1} l_i k_j - \sum_{j=i+1}^{n} l_i k_j\right)\right\}}\right) \times \left\{\prod_{i=0}^{n} e_i^{k_i}\right\} \tag{17}$$
$$= \omega^{+\tilde{l}\,T\,k} \left\{\prod_{i=0}^{n} e_i^{k_i}\right\}$$

where $\tilde{l} = (l_1, l_2, \cdots, l_n)$, $\tilde{k} = (k_1, \cdots k_n)$ § and

§ $\tilde{a}$ = transpose of a.

$$T = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \cdots & 1 \\ \vdots & & & & \\ -1 & -1 & -1 & \cdots & 1 \\ -1 & -1 & -1 & \cdots & 0 \end{bmatrix} \tag{18}$$

Now there are two possible properties for the chosen element $\left\{ \prod_{i=1}^{n} e_i^{k_i} \right\}$ Either it should happen that $\ell^{\sim} T k = 0 \mod m, \forall \ell^{\sim}$ or $\exists\ \ell^{\sim}$ such that $\ell^{\sim} T k \neq \neq 0 \mod m$ In the first case all the elements $\left(\prod_{i=0}^{n} e_i^{\ell_i} \right)$ commute with $\left(\prod_{i=0}^{n} e_i^{k_i} \right)$ making it a class in itself. In the latter case the existence of $\ell^{\sim} T k \neq 0 \mod m$ implies that the class of $\left(\prod_{i=0}^{n} e_i^{k_i} \right)$ is given by the set of m elements

$$\left\{ e_0^{k_0} \left(\prod_{i=1}^{n} e_i^{k_i} \right) \middle| \; k_0 = 0, 1, \cdots, m-1 \right\} \tag{19}$$

which follows from the assumed property of m, that it is a prime number.

Now we require to know how many self conjugate elements are there in $G_n^{(m)}$. We shall prove that when $n = 2\nu$ there are no non-trivial self-conjugate elements (i.e. except the powers of e_0) and when $n = 2\nu + 1$ there are m^2 self conjugate elements including the m powers of e_0, irrespective of whether m is prime or nonprime and even or odd. We come to this conclusion as follows. The conditions that the element $\left\{ \prod_{i=0}^{n} e_i^{k_i} \right\}$

commutes with all other elements implies the necessary and sufficient condition that it commutes with all the generators $\{e_i \mid i = 1 - - n\}$ using (17) this condition becomes a set of n-equations to be satisfied by $(k_1, k_2, - -, k_n)$, namely

$$\begin{bmatrix} 0 & 1 & 1 & - & - & 1 & 1 \\ -1 & 0 & 1 & - & - & 1 & 1 \\ \vdots & & & & & & \\ -1 & -1 & -1 & - & - & 0 & 1 \\ -1 & -1 & -1 & - & - & -1 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{mod. } m \qquad (20)$$

or $T k = 0 \mod m$

Only elements with $(k_1, - - , k_n)$ satisfying this equation will be self conjugate elements.

a) When $n = 2\nu$, $\text{rank } T = 2\nu$ and $\det T \neq 0$.

Hence T^{-1} exists and the only solution possible for $(k_1, - -, k_n)$ is the trivial solution $k_i = 0 \ (\text{mod. } m) \ \forall \ i = 1 - - n$.

So only self conjugate elements are $\{e_0^{k_0} \mid k_0 = 0, 1, - - m-1\}$.

b) When $n = 2\nu + 1$ rank $T = 2\nu$ and $\det T \neq 0$, and hence a non-trivial solution is possible. Since (dim T - rank T) = 1 there can be only one undetermined parameter in the solution for $(k_1, - -, k_n)$ and since the integers k_i's are bounded as $0 \leq k_i \leq m-1, \forall i$, this free parameter can take only m possible values. Thus we get that there are m self conjugate elements of the type $\left(\prod_{i=1}^{n} e_i^{k_i}\right)$ and hence there should be totally m^2 self conjugate elements since each of these m elements can be multiplied by m elements $\{e_0^{k_0} \mid k_0 = 0, 1, - -, m-1\}$ resulting product being self conjugate. Morris has given the solution for

k_i's corresponding to the self conjugate elements as

$\{k_1 \cdots k_n\} = \{\underbrace{\ell\ \ell \cdots \ell}_{\nu}\ \underbrace{(m-\ell)\cdots(m-\ell)}_{\nu}\ \ell\}$ where $\ell = 0, 1, \ldots, m-1$.

Now we are in a position to count the number of classes in $G_n^{(m)}$ when m is prime and $n = 2\nu$ or $2\nu+1$.

a) When $n = 2\nu$: As proved above there are m self conjugate elements and the rest are all partitioned into classes each having an equal number of m elements. So the total number of classes in $G_{2\nu}^{(m)}$ is

$$N_{2\nu}^{(m)} = m + \left(\frac{m^{2\nu+1} - m}{m}\right) = m^{2\nu} + (m-1) \qquad (21)$$

b) When $n = 2\nu+1$: There are m^2 self-conjugate elements and the rest are all partitioned into classes each with m elements. So

$$N_{2\nu+1}^{(m)} = m^2 + \left(\frac{m^{2\nu+2} - m^2}{m}\right)$$

$$= m^{2\nu+1} + m(m-1) \qquad (22)$$

These imply that $G_{2\nu}^{(m)}$ has $m^{2\nu} + (m-1)$ irreducible inequivalent representations. Denoting by $d(i)$ the dimension of the i^{th} representation we have by Burnside's theorem

$$\sum_{i=1}^{N_{2\nu}^{(m)}} d(i)^2 = m^{2\nu+1} \qquad (23)$$

Solution of this equation for $\{d(i)\}$ are obtained as

$$d(i) = 1 \;, \forall\, i = 1 \cdots m^{2\nu}$$
$$d(i) = m^{\nu}, \; \forall\, i = m^{2\nu}+1, \cdots, m^{2\nu}+(m-1)$$

(24)

i.e. there are $m^{2\nu}$ one-dimensional representations and (m-1), representations of dimension m^{ν} each when m = 2 this just repeats Pauli's theorem for $G_{1\,2\nu}^{(2)}$ which applies for $C_{2\nu}^{(2)}$ since there is only one nontrivial square root of unity giving rise to the algebraic relationships (12) → (9). When $n = 2\nu+1$, the solutions of

$$\sum_{i=1}^{N_{2\nu+1}^{(m)}} d(i)^2 = m^{2\nu+2} \qquad (25)$$

are guessed at

$$d(i) = 1 \;, \forall\, i = 1 \cdots m^{2\nu+1}$$
$$d(i) = m^{\nu}, \; \forall\, i = m^{2\nu+1}+1, \cdots, m^{2\nu+1}+(m)(m-1)$$

(26)

i.e. there are $m^{2\nu+1}$ one dimensional representations and m(m-1) m^{ν}-dimensional representations. When m=2 this coincides with Pauli's theorem that $G_{1\,2\nu+1}^{(2)}$ (and hence $C_{2\nu+1}^{(2)}$) has 2 , 2^{ν}-dimensional irreducible representations. We shall not elaborate on the actual representation theory of these

groups and algebras which have been dealt with in detail in our papers [3,4] .

Now we shall look into the case of m being a non-prime integer. The analysis of class structure is now a little more difficult but very interesting.

Let $n = 2\nu$. Consider an element $g_{(K)} = \left\{ \prod_{i=1}^{2\nu} e_i^{k_i} \right\}$

When $\exists\, k_i \neq 0 : g_{(k)}$ is not a self-conjugate element. Let $\mathbb{D} \equiv \{1 = d_1 < d_2 < \cdots < d_{\tau(m)} = m\}$ be the set of all divisors of m in ascending order. $\tau(m)$ denotes the number of divisors of m. Then $\forall\, g_{(K)}.\ \exists\, d'_{(k)} \in \mathbb{D}$ such that

$$g_{(k)}^{d'_{(K)}} = \pm 1 \tag{27}$$

$$k_i d'_{(K)} = 0 \mod m \ , \ \forall\, i = 1 \cdots n.$$

proof: If $k_i = mr/d'_{(k)}$, $1 \leq r \leq d'_{(k)}$, $\forall\, i = 1 \cdots n$ then obviously $k_i d'_{(k)} = 0 \mod m \ \forall\, i = 1 \cdots n$. Hence $d'_{(k)}$ can be chosen such that $m/d'_{(K)}$ is a common factor of $(k_1, \cdots, k_n)$

Then

$$g_{(k)}^{d'_{(K)}} = \left\{ \prod_{i=1}^{2\nu} e_i^{k_i} \right\}^{d'_{(k)}} = e_0^{\frac{1}{2} d'_{(K)}(d'_{(K)}-1)K} \tag{28}$$

where

$$K = (k_1 \cdots k_{2\nu}) \begin{bmatrix} 0 & -1 & \cdots & -1 \\ 0 & 0 & -1 & \cdots -1 \\ \vdots & & & \\ 0 & 0 & 0 & \cdots -1 \\ 0 & 0 & 0 & \cdots 0 \end{bmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_{2\nu} \end{pmatrix} \tag{29}$$

If $d'_{(K)} = 2S+1$ then

$$g_{(K)}^{d'_{(K)}} = e_0^{\frac{1}{2}(2S+1)2SK} = e_0^{S d'_{(K)} K} = 1$$

$$\because \; e_0^{d'_{(K)} K} = 1 \tag{30}$$

If $d'_{(K)} = 2S$ then

$$g_{(K)}^{d'_{(K)}} = e_0^{\frac{1}{2} 2S(2S-1)K} = e_0^{\frac{1}{2} d'_{(K)} K (2S-1)} =$$

$$e_0^{-\frac{1}{2} d'_{(K)} K} = \pm 1 \tag{31}$$

For a given element $g_{(K)}$ let us choose a $d_{(K)} \in \mathbb{D}$ such that $m / d_{(K)}$ is the highest common factor of $(k_1, \ldots, k_n)$ Then $d_{(K)}$ becomes the minimum integer solution for d obtaining the equation $g_{(K)}^{d} = \pm 1$. Then

$$g_{(K)}^{d_{(K)}} = \pm 1 \rightarrow g_{(j)} \left(g_{(K)}^{d(K)} \right) g_{(j)}^{-1} = \left(g_{(j)} g_{(K)} g_{(j)}^{-1} \right)^{d(K)} \tag{32}$$

$$= \pm 1$$

or

$$g_{(j)} g_{(K)} g_{(j)}^{-1} = e_0^{r(j) q_{(K)}} g_{(K)} \quad \text{for any } (j) \tag{33}$$

where

$$q_{(K)} = \frac{m}{d_{(K)}} \; ; \; 0 \leq r_{(j)} \leq d_{(K)} - 1$$

when we vary $g_{(j)} \in G^{(m)}_{2N}$ over all elements we obtain a set $\{e_0^{r_{(j)}} q_{(K)} g_{(K)}\}$ which are in a class. If it happens that is nonprime then there is a possibility that $\{e_0^{r_{(j)}} q_k\}$ may not contain all the $d_{(K)}$-th roots of unity which arises when $(r_{(j)}, d_{(K)}) > 1, \forall r_{(j)}$. In that case let $l_{(K)}$ be the greatest common divisor of $\{(r_{(j)}, d_{(K)}) | \text{ all } r_{(j)}\}$. Then the set will contain only all the $(d_{(K)}/l_{(K)})$th roots of unity and this would imply that $g_{(K)}^{(d_{(K)}/l_{(K)})}$ commutes with all $g_{(j)} \in G^{(m)}_{2N}$. By Schur's lemma this means that in an irreducible representation of $G^{(m)}_{2N}$

$$g_{(K)}^{(d_{(K)}/l_{(K)})} \simeq 1 \tag{34}$$

which is in contradiction to the fact that $d_{(K)}$ is the minimum divisor of m satisfying this condition. Hence the set $\{e_0^{r_{(j)} d_{(K)}} g_{(K)}\}$ contains $d_{(K)}$ distinct elements corresponding to $0 \leq r_{(j)} \leq (d_{(K)} - 1)$. This shows that the set of m elements $\{e_0^{k_0} g_{(K)} | k_0 = 0, \cdots m-1\}$ are partitioned into $q_{(K)} = m/d_{(K)}$ classes each containing $d_{(K)}$ elements. Explicitly writing the classes are given by

$$\left\{ \left(e_0^{k_0' + r q_{(K)}} g_{(K)} \;\middle|\; 0 \leq r \leq d_{(K)} - 1 \right) \;\middle|\; 0 \leq k_0' \leq q_{(K)} - 1 \right\}$$

Let $N_{(K)}$ denote the total number of elements with $k_0 = 0$,

Obeying the equation $g_{(k)}^{d(k)} = \pm 1$ with d_k as the minimum such an integer $d_k \in \mathbb{D}$. Then the total number of classes in $G_{2\nu}^{(m)}$ is

$$N_{2\nu}^{(m)} = \sum_{k=1}^{\tau(m)} N_k \left(\frac{m}{d_k}\right) \tag{35}$$

Obviously $N_1 = 1$, $N_{\tau(m)} = \left\{ m^{2\nu} - \sum_{k=1}^{\tau(m)-1} N_k \right\}$ For others

$$N_k = d_k^{2\nu} - \sum_{s<k} N_s U_s \tag{36}$$

where

$$U_s = \begin{cases} 1 & \text{if } d_s \mid d_k \\ 0 & \text{if } d_s \nmid d_k \end{cases} \tag{37}$$

These follow from the observation that the condition

$$k_i d_k = 0 \cdot \mathrm{mod}\, m \,, \quad \forall\, i = 1, \cdots, 2\nu.$$

is satisfied whenever $0 \leq k_i / q_k \leq d_k - 1 = 1 \cdots 2\nu$

and thus there are d_k solutions for this condition. Substituting (36) in (35) we have

$$N_{2\nu}^{(m)} = \sum_{k=1}^{\tau(m)} \left(\frac{m}{d_k}\right) \left\{ d_k^{2\nu} - \sum_{s<k} N_s U_s \right\} \tag{38}$$

Now the validity of the following identity is gaurenteed by number theoretic considerations

$$N_{2\nu}^{(m)} = \sum_{l=1}^{m} (l, m)^{2\nu} \tag{39}$$

When $m = 2\nu+1$ there are totally m^2 self conjugate elements given by $\{ e_0^{k_0} \eta^{k_\eta} / 0 \le k_0, k_\eta \le m-1 \}$ where η denotes the nontrivial self conjugate element

$$\eta = \left\{ \prod_{i=1}^{n} e_i^{k_i} \;\middle|\; \begin{matrix} k_1 \cdots k_\nu = 1 \\ k_{\nu+1} \cdots k_{2\nu} = m-1, \end{matrix} \; k_{2\nu+1} = 1 \right\}$$

In any irreducible representation $\eta \simeq I$ due to Schur's lemma and this implies that any one of the e_i's i=1...n can be replaced by a product of powers of the other $^{2\nu}\wedge e_i$'s ~~of them~~. Thus the entire set of $m^{2\nu+2}$ elements of $G_{2\nu+1}^{(m)}$ can be considered as a direct sum of m subsets each involving only $\{ e_i \mid i = 0, 1, \ldots, 2\nu \}$ Each of these subsets giving rise to $N_{2\nu}^{(m)}$ classes the total number of classes in $G_{2\nu+1}^{(m)}$ would become $m N_{2\nu}^{(m)}$ Thus

$$N_{2\nu+1}^{(m)} = m N_{2\nu}^{(m)} \tag{40}$$

It is easy to convince oneself that the following scheme of irreducible dimensions of representations satisfy the required conditions

$$\text{I}\ G_{2\nu}^{(m)} \quad \left\{ \begin{array}{l} \text{Corresponding to each } \ell = 1 \ldots m \text{ there} \\ \text{are respectively } (\ell, m)^{2\nu} \text{ representations} \\ \text{of dimensions } \left(m/(\ell, m) \right)^{\nu} \text{ each.} \end{array} \right.$$

$$\text{II}\ G_{2\nu+1}^{(m)} \quad \left\{ \begin{array}{l} \text{Corresponding to each } \ell = 1 \ldots m \text{ there} \\ \text{are respectively } m\,(\ell, m)^{2\nu}, \\ \left\{ = \frac{m}{(\ell, m)} (\ell, m)^{2\nu+1} \right\}, \text{representations of} \\ \text{dimensions } \left(m/(\ell, m) \right)^{\nu} \text{ each.} \end{array} \right. \tag{41}$$

Putting m as equal to a prime number we get back the results in (24) and (26). Construction of all these representations is very interesting which can be obtained from the papers [3,4].

Thus so far we have considered the beautiful and rich numerically interesting algebraic structure arising out of generalizations of Clifford algebra. I feel that it would be very interesting and rewarding to consider a graph theoretical understanding of these groups. This may be dealt with by me later elsewhere.

I wish to thank Professor Alladi Ramakrishnan and Dr.N.R.Ranganathan for their kind encouragement and useful discussions.

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THE ROLE OF HAMMERSTEIN INTEGRAL EQUATION IN PHYSICAL PROBLEMS

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The non-linear integral equation of the type

$$\psi(x) + \int_0^1 K(x,y)\, f(y,\psi(y))\, dy = 0 \qquad (1)$$

is called Hammerstein equation[1]. In many physical problems, especially dealing with condensation, or phase transition, one must study the solution of this type of equation.

Consider a plasma of negatively charged particles (electrons) in a uniform positive background (Jellium model), or a system of N particles moving in a field described by a self-consistent potential, confined to a volume of Ω. Then $\frac{N}{\Omega} = \rho_o$, is the density. In the equilibrium, the toral energy will be minimum, but there will be constantly varying local fluctations of density. Let the value of the density at a point within Ω, at time t be denoted $D(\vec{r}, t)$. Then define the fluctuation as,

$$\rho(\vec{r}, t) = D(\vec{r}, t) - \rho_o \qquad (2)$$

The static equilibrium is attained if

$$\frac{\partial \rho}{\partial t} = 0 \qquad (3)$$

This density fluctuation $\rho(\vec{r}, t)$, will set about preassure gradient. $\nabla_{\vec{r}} P(\vec{r}, t)$ which will bring restoring forces into play, to restore the density to ρ_o, i.e., make $\rho = 0$. The potential that is set up (self consistently determined) $V(\vec{r}, t)$ will be related (we consider conservative systems), to the preassure gradient through a

relation of the type

$$\nabla_r P(\vec{r}, t) = B\, \rho(\vec{r}, t)\, \nabla_r V(\vec{r}, t) \qquad (4)$$

and $$\nabla_r P(\vec{r}, t) = A\, \nabla_r \rho(\vec{r}),$$

where A and B are taken for simplicity as constants for the system under consideration. Then

$$\nabla_r \rho(\vec{r}) = \frac{B}{A}\, \rho(\vec{r})\, \nabla_r V(\vec{r})$$

and hence

$$\rho(\vec{r}, t) = C\, e^{\frac{B}{A} V(\vec{r}, t)} \qquad (5)$$

Since $V(\vec{r})$ is a self-consistent potential,, we can write

$$V(\vec{r}) = \int K(\vec{r}, \vec{r}')\, \rho(\vec{r}')\, d\vec{r}' \qquad (6)$$

from (5) and (6) we get

$$V(\vec{r}) = C \int K(\vec{r}, \vec{r}')\, e^{\frac{B}{A} V(\vec{r}')}\, d\vec{r}' \qquad (7)$$

It is very difficult to solve these equations without any approximations and simplifications. Assume

$$K(\vec{r}, \vec{r}') = K(|\vec{r} - \vec{r}'|)$$

and $$\frac{B}{A} V(\vec{r}') = \varphi(\vec{r}')$$

Then

$$\varphi(\vec{r}) = \lambda \int K^n(\vec{R})\, e^{\varphi(\vec{r}')}\, d\vec{r}'$$

where $$K^n(\vec{R}) = \frac{K(|\vec{r} - \vec{r}'|)}{4\pi\, K(\vec{R})\, |\vec{R}|^2\, d\vec{R}}$$

and $$\lambda = \frac{CA}{B}\, 4\pi \int K(\vec{R})\, |\vec{R}|^2\, d\vec{R}$$

One can see that

$$\int_\Omega K^n(\vec{R})\, |\vec{R}|\, d\vec{R} = 1 \qquad \text{(normalisation)}.$$

The problem is to find out for what value of λ one can, obtain various types of solutions. There can exist a special value of $\lambda = \lambda_0$. For values of λ, greater than λ_0, there will be no real solution and for $\lambda < \lambda_0$, real solution or solutions appear. λ_0 is called a Bifurcation point for the equation.

The usual procedure, now adopts numerical approximations

$$\lambda = \lambda_0 - \epsilon$$

$$\phi(\vec{r}) = \phi_0(\vec{r}, \lambda_0) + \psi(\vec{r})$$

and then assume

$$\psi(\vec{r}) = \epsilon \psi_1(\vec{r}) + \epsilon^2 \psi_2(\vec{r}) + \ldots\ldots$$

and proceed by the well-known method of successive approximation.

For example of we are looking for a solid phase, then $\psi(r)$ should be periodic in a lattice structure. Then we can take

$$\psi(\vec{r}) = \sum_p a_p e^{ip \cdot r}$$

and obtain the conditions on a_p's to get the proper solution.

To summarise we have indicated how in physical problems, we came across difficult non-linear integral equations and have to turn for numerical Analytical methods to achieve physically, meaninfful results.

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DIOPHANTINE EQUATIONS AND PARTITION FUNCTIONS

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The ordered partition functions play an important role in many physical applications[1]. But what may be the most pertinant in this conference is the degeneracy of the solutions of some particular types of diophantine equations which we have solved using the theory of partitions[2].

Using the standard concepts of roots, positive roots, simple roots, weights, dominant weights, highest weight etc.[3], the inner multiplicity $M(m)$ of a weight m belonging to the irreducible representation of a classical group G with highest weight Λ is given by Kostant's formula[4]

$$M^{\Lambda}(m) = \sum_{S \in W} \delta_S \, P\left[S(\Lambda + R_0) - (m + R_0)\right] \qquad (1)$$

$$R_0 = \frac{1}{2} \sum_{i=1}^{N} \beta_i$$

where W is the Weyl group (finite) (which is the permutation group in $(\ell+1)$ dimensions when $G = SU(\ell+1)$. $\delta_S = \pm 1$ depends on whether the permutation is even or odd respectively P(M) is the ordered partition function which is equal to the number of ways one can write M as

$$M = \sum_{\mu=1}^{N} a_\mu \beta_\mu \qquad (2)$$

where a_μ's are non-negative integers and β_μ are the positive roots $\sum^{+}$ of the group. By definition

$$P(0) = 1$$

$$P(M) = 0 \quad M < 0$$

But since M is defined in -dimensional space of simple roots (ℓ = rank) we have

$$M = \sum_{i=1}^{\ell} K_i \alpha_i \tag{3}$$

where K_i are given non-negative integers and α_i is the system of simple roots. We again know that the Σ system can be expressed in terms of Π through the relation

$$\beta_\mu = \tilde{C}_{\mu i} \alpha_i \tag{4}$$

where the explicit form of the rectangular matrices c for various classical groups has been worked out in reference 1. So, P(M) is given by the degeneracy in the solutions of the diophantine equation

$$K = C\, a \tag{5}$$

where K and the matrix C are given. Since all K's and a's are integers, equation (5) may be recognised as simply the projection of positive root lattice E_N on the simple root lattice E_ℓ.

We have solved Eq.(5) for the degeneracy of solutions using the method of generating functions. It is known that the number of partitions[2] of an integer n

$$n = k_1 p_1 + k_2 p_2 + \cdots\cdots + k_n p_n \tag{6}$$

where p's are integers (k_i in the multiplicity of the occurrance of p_i) is given by the coefficient of of the generating function

$$f(x) = \prod_{i=1}^{n} \left(1 - x^{k_i}\right)^{-1} \tag{7}$$

In the same way to solve (5) we define the generating function

$$f(x_1, \cdots x_\ell) = \prod_{\mu=1}^{N} \left(1 - x_1^{c_{1\mu}} \cdots x_\ell^{c_{\ell\mu}}\right)^{-1} \qquad (8)$$

$$|x|<1$$

The degeneracy of solutions of (5) is given by the coefficient of $x_1^{k_1} \cdots x_\ell^{k_\ell}$ of $f(x_1, \cdots x_\ell)$ which is simply the Taylor coefficient

$$\frac{1}{k_1!} \cdots \frac{1}{k_\ell!} \frac{\partial^{k_1}}{\partial x_1^{k_1}} \frac{\partial^{k_2}}{\partial x_2^{k_2}} \cdots \frac{\partial^{k_\ell}}{\partial x_\ell^{k_\ell}} f(x_1, \cdots x_\ell) \Big|_{x_1 = x_2 = \cdots = 0} \qquad (9)$$

Various recursion relations for the multiplicity have been worked out in Ref.1. In fact, the above analysis has been extended to non-compact groups by Biedenharn et. al.[5]

It has further been shown by us that the generating function $f(x_1, \cdots x_\ell)$ is given by

$$f(x_1, \cdots x_\ell)\, x_1 \cdots x_\ell = \frac{1}{\Delta} \qquad (10)$$

where Δ is the characteristic of the unit representation when x_i's are given by

$$x_i = \exp\left[-2\pi i (\alpha_i, \phi)\right] \qquad (11)$$

where the ϕ's are the coordinates of the maximal Toroid T_ℓ and (α_i, ϕ) stands for Kartan-Killing form.

The character of the group G for the representation Λ is given by Weyl's formula

$$X^{\Lambda}(\phi_1, \ldots, \phi_\ell) = \frac{X^{\Lambda}}{\Delta} \tag{12}$$

where the characteristic X is given by

$$X^{\Lambda} = \sum_{S \in W} \delta_S \exp 2\pi i (\Lambda + R_0, \phi) \tag{13}$$

and $\Delta = X^{(0)}$. We refer the reader for further details to Ref.(1).

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SOME NUMERICAL METHODS IN NUCLEAR PHYSICS

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In this article, I will be concerned with some of the numerical methods of I have used in the course of my studies in nuclear physics. All my research publications todate contain invariably a section pertaining to numerical results, their comparison with existing experimental data and a discussion based on the comparison. The numerical calculations were done in the initial stages of the problems on the IBM 1620 computer, at the Fundamental Engineering Research Establishment, Guindy and on the IBM 1130 computer at the Department of Physics, University of Madras and in the final stages of the problems on the CDC 3600-160 A installation at the Tata Institute of Fundamental Research, Bombay. I do not wish to go into the aspects of computer programming here or into the details of the physics of the physics of the problems but only wish to project the specific methods I employed, out of a host of methods which exist, for

i) Numerical integration

ii) Matrix diagonalization and

iii) Numerical solution of ordinary differential equations.

1. Numerical integration

The age-old, simple, Simpson's rule for performing the integration:

$$I = \int_a^b f(x)\,dx$$

is given by:

$$I = \frac{h}{3}\left\{f(a) + 4\left[f(a+h) + f(a+3h) + \cdots + f(b-h)\right] + 2\left[f(a+2h) + f(a+4h) + \cdots + f(b-2h)\right] + f(b)\right\}$$

where h is the width of the intervals into which the area under the curve f(x) is divided. It is a fact that the error in this approximation can be made as small as possible by making h small. But it should also be remembered that the 'round-off' error in the numerical procedure increases as h decreases. To avoid this difficulty, the following method is adopted:

To start with, we choose $h = \frac{b-a}{2}$ i.e. we divide the area under the curve into 2 equal strips as shown in Fig.1. In this case, Simpson's rule gives:

$$I = \frac{h}{3}\left[f(a) + 4f(a+h) + f(b)\right]$$

where $h = \frac{b-a}{2}$. In the next stage, we choose the interval $h' = \frac{b-a}{4}$, i.e. We take half of the interval we chose to start with, thus providing twice the former number of strips, as shown in Fig.2.

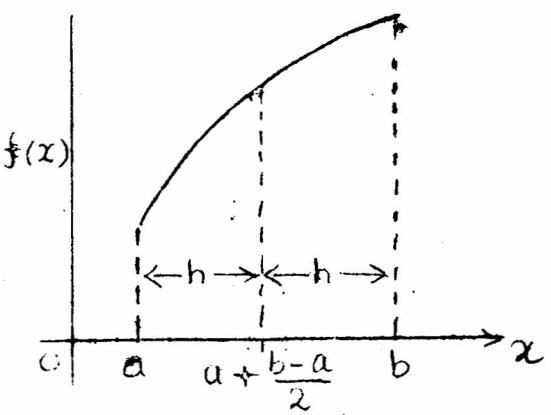

Fig.1

The Simpson's rule for this case is:

$$I_{New} = \frac{h'}{3}\left\{f(a) + 4\left[f(a+h') + f(a+3h')\right] + 2f(a+2h') + f(b)\right\}$$

where $h' = \frac{h}{2} = \frac{b-a}{4}$. The new value of the integral, called I_{New}, ought to be a better approximation since the error in the process is definitely reduced by reducing the width of the interval. We now should compare the two values of the integral I and I_{New}; if they are sufficiently close we accept I_{New} as the value of the integral. Otherwise, we rename I_{New} as I, divide the interval by 2 again (i.e. $h' = \frac{h'}{2}$) and obtain the value of the integral using Simpson's rule naming this value as (the new)I_{New}. Once again we compare the successive values of the integral, obtained with h' and $\frac{h'}{2}$ to check for their closeness. This iterative procedure is continued to achieve convergence or until the successive values are as close as we please, depending upon the demands on the accuracy of the result required.

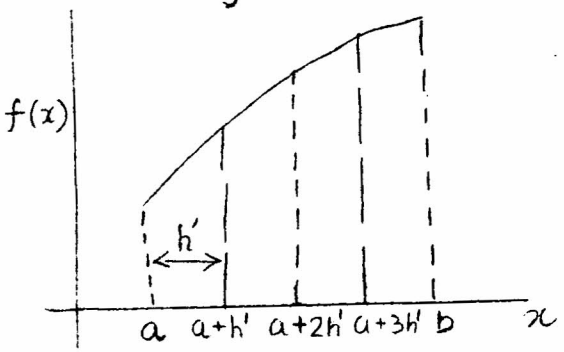

Fig.2

It should be noted that it is unnecessary to evaluate the complete Simpson's rule formula, every time we start with a new value for the interval h. For, note that the end points will never change, and the mid-point in Fig.1 is needed again in Fig.2, only in the former case the value of the function at that point is to be multiplied by 4 in the Simpson's rule, while in the latter case this multiplicative factor is 2. Thus, the attraction of the Simpson's

rule method of numerical integration is that we can programme to compute the value of the function at as many ordinates as necessary to get the required accuracy without recomputing the value of the function at any one ordinate.

While calculating the differential cross sections for photoproduction of pions from complex nuclei, we encountered integrals of the form

$$\langle j_\lambda(kr)\rangle_{n_f l_f, n_i l_i} = \int_a^\infty R_{n_f l_f}(r)\, J_\lambda(kr)\, R_{n_i l_i}(\hat{r})\, dr$$

where $R_{nl}(r)$ is the radial wave function and $J_\lambda(kr)$ the spherical Bessel function of order λ and the lower limit of integration is:

(i) a = 0 for volume production of pions; or

(ii) a = r_0 for surface production of pions,

r_0 being called as the cut-off parameter and is conventionally chosen to be the root mean square radius of the nucleus determined by electron scattering experiments. When the cross section is calculated in the harmonic oscillator basis, the radial wave functions, $R_{nl}(r)$ are Hermite polynomials and further if $a = 0$, then the integrals, which we denote by $\langle J_\lambda(kr)\rangle^{(V)}_{n_f l_f, n_i l_i}$ can be evaluated analytically. In the harmonic oscillator basis, when $a \neq 0$, the radial integral denoted by $\langle J_\lambda(kr)\rangle^{(S)}_{n_f l_f, n_i l_i}$ can be written as:

$$\langle j_\lambda(kr)\rangle^{(S)}_{n_f l_f, n_i l_i} = \langle j_\lambda(kr)\rangle^{(V)}_{n_f l_f, n_i l_i} - \int_0^a R_{n_f l_f}(r)\, j_\lambda(kr)\, R_{n_i l_i}(r)\, r.$$

The second term on the right hand side of the above expression cannot be evaluated analytically and we have to resort to numerical integration, the method for which has been detailed above.

ii) Matrix Diagonalisation:

In the absence of a central force of attraction, the nucleon in a nucleus is considered to be moving independently within a nuclear potential due to the remaining nucleons. Making the basic assumption that the potential is of the infinite square well type and taking spin-orbit coupling into account, one readily finds an ordered sequence of energy levels, each level being characterised by a set of quantum numbers. Each energy level accommodates a certain number of nucleons, allowed by the Pauli exclusion principle. Given the number of nucleons in a nucleus, the ground state configuration is found by filling the energy levels in sequence with the allowed number of nucleons in each level starting with the lowest level. Let us consider the case of a nucleus which has in its ground state, nucleons occupying the three lowest energy levels and two nucleons (to be specific) in the fourth level. To calculate the excited states of the nucleus, we consider the nucleons in occupied levels to constitute an inert core with the extra-core nucleons in the fourth, fifth, etc., unoccupied levels, characterized by sets of quantum numbers nli , etc., respectively. (Note: each n stands for a set of quantum numbers). One can in principle, choose any number of unoccupied levels, but in practice, a restriction on the number of levels is essential to

keep the dimensionalities of the matrices to be diagonalised manageable on a computer. Let us say, that we restrict ourselves to the first three unoccupied levels for the two extra-core nucleons. Then these two nucleons can have the following allowed configurations:

$$(n_1)^2, (n_2)^2, (n_3)^2 ; (n_1 n_2), (n_1 n_3), (n_2 n_3)$$

where the first three configuration imply that the two nucleons are in any one of the three unoccupied levels, while the next three configurations imply that each of the two nucleons occupies a different unoccupied level. Having chosen the configurations, we now choose a nucleon-nucleon potential to take into account the interaction of the extra-core nucleons with each other. The interaction of an extra-core nucleon with the core is taken care of, later on, through what are called single particle energies of neighbouring nuclei. If we call $V(r)$ as the nucleon-nucleon potential $(n_i n_j)$ as a general uncoupled configuration for the two extra-core nucleons, while $|(n_i n_j) J T\rangle$ denotes a coupled angular momentum state of the two-extra-core nucleons, J and T being respectively the coupled values of the total angular momentum and total isotopic spin; then we should evaluate the following two-body matrix elements;

$$\langle (n'_i n'_j) J' T' | V(r) | (n_i n_j) J T \rangle$$

These can be evaluated using standard techniques of angular momentum. So, we have for each allowed J and T values, a set of allowed configurations, which fixes the number of matrix elements and hence the dimension of the matrix. For example, when J = 0 and T = 0, the allowed configurations are $(n_1)^2, (n_2)^2$ and $(n_3)^2$, and therefore we have a three-dimensional matrix. These matrices are real, symmetric matrices. The eigenvalues and eiger vectors of these matrices correspond to the excited states of the nucleus and their wave functions respectively. So, this is how, we are led to matrix diagonalisation in nuclear physics to get the excited states of a nucleus and their corresponding wave functions.

To illustrate the matrix diagonalisation procedure, let us first consider, a real, symmetric, two-dimensional matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

In Jacobi's method for diagonalisation of a real symmetric matrix, a similarity transformation is performed on A, with the rotation matrix:

$$T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

to eliminate the off-diagonal term in A, by selecting the angle of rotation θ. To find θ, we should write out explicitly,

$$B = T^{-1} A T$$

where $T^{-1} = \tilde{T}$, and then set the off-diagonal element of B, viz. b_{12}, equal to zero. The necessary value of θ to make b_{12} zero is found to be

$$\tan 2\theta = \frac{2a_{12}}{a_{11} - a_{22}}$$

Using this value of θ, the diagonal matrix elements of B, b_{11} and b_{22}, are calculated and these are the eigenvalues of A. Thus A is diagonalized.

If $[A]$ is a higher dimensional real, symmetric matrix with a_{ij} as the largest off-diagonal element, then a rotation matrix $[T_1]$ whose dimension is the same as that of $[A]$ and with matrix elements.

$$T_{ii} = T_{jj} = \cos\theta$$

$$T_{ij} = -T_{ji} = -\sin\theta$$

$$T_{nn} = 1 \quad \text{for} \quad n \neq i \text{ and } j.$$

and all other matrix elements zero is chosen to perform the first similarity transformation on A and the angle of rotation θ determined by

$$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}}$$

to eliminate the largest off-diagonal element a_{ij}. The matrix

$$[A_1] = [T_1]^{-1}[A][T_1]$$

is thus determined.

We now repeat the procedure, assuming $[A_1]$ is now $[A]$ and choosing a matrix $[T_2]$ to eliminate the largest off-diagonal element of $[A_1]$. This procedure is continued until a matrix $[A_m]$ is created, after a series of m plane rotations are made on $[A]$ successively to eliminate the remaining off-diagonal elements. The principal diagonal elements of $[A_m]$ given by

$$[A_m] = [T_m]^{-1} \cdots [T_2]^{-1} [T_1]^{-1} A [T_1] [T_2] \cdots [T_m]$$

represent the eigenvalues of the matrix $[A]$.

It should be noted that although any one rotation may set an off-diagonal element to zero, it does cause an element previously reduced to zero to attain a non-zero value. In other words the number of plane rotations is by necessity greater than the number of off-diagonal non-zero elements of the matrix $[A]$. Fortunately each rotation makes the maximum valued off-diagonal element decrease, in absolute value and therefore, matrix $[A]$ converges to a diagonal matrix.

Obviously, if we multiply the successive $[T]$ matrices we get a square matrix $[V]$ given by

$$[V] = [T_1] [T_2] \cdots [T_m]$$

whose columns represent the eigenvectors. The first column of $[V]$ corresponds to the first eigenvalue. (Or the first diagonal element of $[A_m]$) of A, the second column of $[V]$ corresponds to the second eigenvalue of A, and so on.

iii) Numerical solution of ordinary differential equations:

Just as there are methods of finding to any desired degree of accuracy, the roots of any algebraic or transcendental equation having numerical coefficients, so likewise there are methods of finding to any desired degree of accuracy the numerical solution of any ordinary differential equation having numerical coefficients and given initial conditions.

Starting with initial values, the solutions are thence constructed by short steps ahead for equal intervals each step usually being checked by some method before proceeding to the next step. The available methods for solving differential equations of the first order are called:

a) Euler's method and its modification

b) Picard's method of successive approximations, and

c) use of approximating polynomials.

The determination of the first few values of the function is the most important and usually the most laborius part in the numerical solution of a different*ial* equation. It is the most important because the first few values must be accurate to the number of significant figures described in the solution and it is the most laborious because the first few values are sometimes not easily found to the desired accuracy.

One of the methods of starting the solution is by the use of Taylor's series.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots\cdots$$

Other methods of starting the solution are:

a) Milne's formulae

b) Modified Euler method, and

c) Runge-Kutta method.

After the first five consecutive starting values have been found, the numerical solution of the differential equation is continued as far as desired by the use of Newton's backward interpolation formula. This part of the solution is mostly smooth sailing.

Any differential equation of the second or higher order can be reduced to a system of first order equations by the introduction of auxiliary variables. Thus, the second order equation:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = f(x)$$

can be reduced to two first order equations by setting:

$$y' = \frac{dy}{dx}.$$

Then

$$y'' = \frac{dy'}{dx} = f(x) - Py' - Qy$$

These first order equations can be solved by the methods mentioned earlier on.

Second order differential equations with the first derivative absent are of particular interest in Physics and their numerical solutions can be found by a shorter method called the Runge-Kutta method, which we will detail here. This method advised by Runge

around 1894, has been extended by Kutta a few years later. Reliable starting values can be found by the Runge-Kutta method in many cases. In this method, the increments of the function are calculated once for all by means of a definite set of formulae. The derivation of the formulae is somewhat a lengthy process and will not be given here.

Let $\frac{dy}{dt} = f(x, y)$, represent any first order equation and let h denote the interval between equidistant values of x. If the initial values are then the first increment in y is computed from the formulae given below:

$$k_1 = f(x_0, y_0)\,h$$
$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)h$$
$$k_3 = f\left(x_0 + \frac{h}{2}, y_0 + k_3\right)h$$
$$k_4 = f(x_0 + h, y_0 + k_3)\,h$$

and

$$\Delta y_1 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Then,

$$x_1 = x_0 + h \quad \text{and} \quad y_1 = y_0 + \Delta y_1$$

The increment in y, for the second interval, is computed in a similar manner by means of the formulae:

$$k_1 = f(x_1, y_1)\,h,$$
$$k_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)h,$$
$$k_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)h,$$
$$k_4 = f(x_1 + h, y_1 + k_3)\,h$$

and

$$\Delta y_2 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Then, $x_2 = x_1 + h$

and $y_2 = y_1 + \Delta y_2$

It will be noticed that any change in the formulae for the different intervals is in the values of x and y to be substituted. Thus, to find Δy in the n^{th} interval, we should substitute x_{n-1}, y_{n-1} in the expressions for k_1, k_2, k_3 and k_4.

In the special case where $\frac{dy}{dx} = f(x)$ only, the Runge-Kutta method reduces to Simpson's rule. For, in this case

$$k_1 = f(x_0)h,$$
$$k_2 = f\left(x_0 + \frac{h}{2}\right)h,$$
$$k_3 = f\left(x_0 + \frac{h}{2}\right)h$$
$$k_4 = f(x_0 + h)h$$

so that

$$\Delta y = \frac{1}{6}h\left[f(x_0) + 2f\left(x_0 + \frac{h}{2}\right) + 2f_3\left(x_0 + \frac{h}{2}\right) + f(1 +$$
$$= \frac{(h/2)}{3}\left[f(x_0) + 4f\left(x_0 + \frac{h}{2}\right) + f(x_0 + h)\right]$$

which is the same result as would be obtained by applying Simpson's rule to the interval from x_0 to $x_0 + h$, if we take equal subintervals of width h/2.

Finally, for the special second-order differential equation with the first derivative absent, viz:

$$y'' = f(x, y)$$

the increments in y and y' are found from the following formulae:

$$k_1 = f(x_n, y_n)\,h,$$

$$k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}y_n' + \frac{h}{8}k_1\right)h,$$

$$k_3 = f\left(x_n + h, y_n + hy_n' + \frac{h}{2}k_2\right)h,$$

$$\Delta y = h\left[y_n' + \frac{1}{6}(k_1 + 2k_2)\right], \text{ and}$$

$$\Delta y' = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where $n = 0, 1, 2, 3, \ldots$

The inherent error in the Rung-Kutta method is not easy to estimate, but it is of the order of h^5 and is therefore of the same order as that in Simpson's rule.

The radial Schrodinger equation

$$\left\{-\frac{\hbar^2}{2M}\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V(r)\right\}u(r) = E\,u(r)$$

subject to the normalisation condition

$$\int_0^\infty |u(r)|^2 dr = 1$$

is a typical case of a second order differential equation with the first derivative absent. The single particle wave functions $u(r)$ in a potential well $V(r)$ can be classified according to the quantum numbers n, l and j.

When the central potential is the harmonic oscillator potential:

$$V(r) = \frac{1}{2}M\omega^2 r^2$$

the Schrodinger equation can be solved analytically and the eigen states which are called the single particle states are Hermite polynomials of the form:

$$R_{nl}(r) = N_{nl}\, v_{nl}(r)\, r^{l+1} \exp\left(-r^2/2b^2\right)$$

where $b = \sqrt{\hbar/M\omega}$ is called the size parameter of the nucleus, N_{nl} is the normalization constant

$$N_{nl} = \left[\frac{2^{l+n+2}}{n!\sqrt{\pi}\, b^3}\,(2l+2n+1)!!\right]^{1/2} \frac{1}{b^l}$$

and $v_{nl}(r)$ are the polynomials

$$v_{nl}(r) = \sum_{k=0}^{n} (-1)^{k-n} 2^k \frac{n!}{k!(n-k)!} \frac{\left(r^2/b^2\right)^k}{(2l+2k+1)!!}$$

The corresponding energy levels are given by:

$$E_{nl} = \left[2n + l + \frac{3}{2}\right]\hbar\omega$$

with $n = 0, 1, 2, \ldots\ldots$

These radial wave functions are particularly convenient to handle and therefore most calculations are made in the harmonic oscillator basis, though the harmonic oscillator potential is not a realistic one.

The shell model potential in which a nucleon moves is derived from the interactions of the nucleon with its nearest neighbours. Therefore, the potential will be essentially constant

within the nucleus but it must diminish in magnitude in the surface where some of the nearest neighbours are missing, and it must be zero outside the nucleus. A popular choice of the radial shape of the potential is the Saxon-Woods form which has a radial shape similar to the nuclear density distribution.

The realistic shell model potential, including a spin-orbit term is of the form:

$$V(r) = -V_N f(r) + \left(\frac{\hbar}{m_\pi c}\right)^2 V_{so} \frac{df}{dr} + V_c(r)$$

where $f(r) = \left[1 + \exp\left(\frac{r-R}{a}\right)\right]^{-1}$ is the Saxon-Woods form factor; V_N, V_{so} and V_c are respectively the nuclear, spin-orbit and Coulomb potentials.

The Schrodinger equation has now to be solved numerically using the Runge-Kutta method to obtain the single particle wave functions. So, the radial wave functions in the Saxon-Woods basis are numerically obtained.

The cross sections for pion photoproduction from complex nuclei, when evaluated in the Saxon-Woods basis, would involve radial integrals of the form:

$$\left\langle j_l(kr) \right\rangle_{n_f l_f j_f, n_i l_i j_i} = \int_0^\infty u_{n_f l_f j_f}(r)\, j_l(kr)\, u_{n_i l_i j_i}(r)\, dr$$

Since the integrand is known at a discrete set of points as close as we please, the radial integral has to be evaluated numerically using the simple Simpson's rule given at the very beginning of this article.

In conclusion, it must be stated that though the numerical methods are laborious, they have the redeeming feature of providing a means of obtaining solutions to problems which cannot be solved otherwise. Further for one who has gained some experience of programming numerical methods for a computer there always seems to be a great deal of satisfaction gained when a program runs correctly!.

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GRAPHS AND DIGRAPHS WITH PRESCRIBED PARTITIONS

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In this paper we present an algorithm for obtaining generating functions for graphs and digraphs with prescribed partitions. This algorithm makes use of lemma of Burnside. Homeomorphically irreducible graphs, k-regular graphs and Eulerian graphs can be enumerated by using this algorithm as these are graphs with some restrictions on the degrees of their points. These details will appear elsewhere.

We refer to [1] for basic definitions and [2] for a statement and a proof of Burside's lemma. We define the weight functions for a graph s and a digraph s' as

$$W(s) = 0\ x^q(d_1,d_2,d_3,\ldots,d_n)$$
$$= 0\ W'(s)$$

and

$$W(s') = \theta\ x^q(s_1,t_1),(s_2,t_2),(s_3,t_3),\ldots,(s_n,t_n))$$
$$= \theta\ W'(s)$$

respectively where 0 is an unordering operator as in [3], θ is the lexicographic operator introduced in [4], $(d_1,d_2,\ldots,d_n)$ is the degree sequence of a graph with n points and $((s_1,t_1),(s_2,t_2),\ldots,(s_n,t_n))$ is the degree sequence giving the out-degree

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*Presented by M.R. Sridharan.

and in-degree of the n points of a diagraph. The symmetric group S_n induces permutation groups π and ψ acting on the graphs and digraphs. This algorithm gives the contributions of these permutation groups. These contributions depend on the cycle structures of the individual permutations of π and ψ, which in turn depend on the cycle structures of the permutations of S_n. Let $\alpha \in S_n$ have k cycles $C_1, C_2, \ldots, C_k$. The contributions of the permutations induced by cycles C_i and pair of cycles C_i and C_j are denoted by (C_i) and (C_i, C_j). These contributions are arranged as sets in the following manner

Set 1 : $(C_1), (C_1, C_2), (C_1, C_3), \ldots, (C_1, C_k)$

Set 2 : $(C_2), (C_2, C_3), (C_2, C_4), \ldots, (C_2, C_k)$

$\vdots$

Set (k-1): $(C_{k-1}), (C_{k-1}, C_k)$

Set k : (C_k).

Even though we have taken the weight of a graph as an n-type with a factor x^q, these contributions give k-tuples with a factor x^q as the weights of graphs. This is because of the fact that every point in the same cycle gets the same degree. After writing these contributions as indicated earlier, we add the k-tuples coordinate wise and multiply the factors of the type x^q algebraically. First the subsets within the sets are multiplied and then the set contributions are multiplied. Applying the operators O and θ for graphs and digraphs respectively, summing over these expressions for every $\alpha \in S_n$ and dividing by $n!$ we obtain the desired generating functions.

(a) Contributions for graphs

We now give (C_i) and (C_i,C_j) for cycles of $\alpha \in S_n$ with different lenths. In writing these contributions, we indicate only the i-th coordinate for (C_i) and the i-th and the j-th coordinates for (C_i,C_j).

(i) An odd cycle C_i of length $p = 2a + 1$ gives

$$(C_i) = \sum_{i=o}^{a} \binom{a}{i} x^{p_i(2i)}$$

(ii) An even cycle C_i of length $p = 2a$ gives

$$(C_i) = \sum_{i=o}^{a-1} \sum_{i=o}^{1} \binom{a-1}{i} \binom{1}{j} x^{a(2i+j)} (2i+j)$$

(iii) Two cycles C_i and C_j of lengths p and q respectively give

$$(C_i,C_j) = \sum_{i=o}^{d} \binom{d}{i} x^{mi} (p_1 i, q_1 i)$$

where d and m are the greatest common divisor and the least common multiple of p and q, $p_1 = \frac{m}{p}$ and $q_1 = \frac{m}{q}$.

(b) Contributions for digraphs

In this case the contribution of C_i is a pair (p_i,q_i) meaning that the points of C_i have out-degree p_i and in-degree q_i. Similarly two cycles C_i and C_j contribute $((p_i,q_i),p_j,q_j))$

(i) A Cycle C_j of length p gives

$$(C_i) = \sum_{i=0}^{p-1} \binom{p-1}{i} x^{pi} (i,i)$$

(ii) Two cycles C_i and C_j of lengths p and q give

$$(C_i, C_j) = \sum_{i=0}^{d} \sum_{j=0}^{d} \binom{d}{i} \binom{d}{j} x^{m(i+j)} ((p_1 i, p_1 j), (q_1 j, q_1 i)).$$

The advantage in this algorithm is that the computations are kept to a minimum by deleting the unwanted terms at each stage of the multiplication. A modification of this method enabled the authors to count isographs of various types [5].

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SOME RESULTS ON MULTIPLICATIVE INCIDENCE FUNCTIONS

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1. This notion of the incidence functions and incidence algebras over locally finite partially ordered sets introduced by L.Weisner[8] and G.C.Rota[4], paved a way for a new study of combinatorial theory from a unified point of view. D.A.Smith in his papers papers[5,6,7] studied some properties of incidence functions, applications of Dirichlet's Cauchy's unitary and Lucas products etc., axiomatic characterisation of incidence algebras of functions on lattices, Brauer-Rademacher and other related identities.

In this paper the author interprets the results of T.M. Apostol[1] on completely multiplicative or linear functions of a single argument as results incidence functions over a locally distributive, locally finite, $(S, \leq)$ with proofs.

2. Definitions and Notations.

(1) An incidence function is a function $f : S \times S \longrightarrow K$, an arbitrary field, with the restriction that $f(x,y) = 0$ unless $x \leq y$.

These incidence functions form an associative K-algebra (with Kronecker δ-function as identity) with respect to point-wise edition, scalar multiplication and the Dirichlet product defined by

$$(f*g)(x,y) = \sum_{x \leq z \leq y} f(x,z)\, g(z,y)$$

We denote the totality of these incidence functions on $S \times S$ by $\mathcal{I}(S)$. We also define the point-wise product

$$(f*g)(x,y) = f(x,y)\, g(x,y).$$

(2) An incidence function is said to be multiplicative if $f(x,y)\ f(x',y') = f(x\vee x',\ y\vee y')$, whenever $y\wedge y' = x\wedge x'$. We shall denote the totality of these multiplicative incidence functions by $\mathcal{M}(S)$.

(3) An incidence function is said to be completely multiplicative if (i) $f(x,z)\ f(z,y) = f(x,y)$, for $x \leq z \leq y$ and

(ii) $f(x,y) = f(x\vee z,\ y\vee z)$ for $x \leq y$ and $x\wedge z = y\wedge z$. We shall denote the totality of all these completely multiplicative functions on $S \times S$ by $\mathcal{L}(S)$.

(4) An incidence function is said to be invertible if it has an inverse with respect to Dirichlet product and $\mu(x,y)$ denotes the Mobius analogue on $S \times S$ which is the inverse of $E(x,y)$,

$$E(x,y) = \begin{cases} 1 & \text{for all } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

(5) We denote by $\mathcal{BA}(S)$ the class of all intervals in $S \times S$ each of which forms a Boolean Algebra.

3. Interpretations of the results of Apostol.

Result 1. $f \in \mathcal{M}(S)$, $[x,y] \in \mathcal{BA}(S) \Longrightarrow f^{-1}(x,y) = (\mu \times f)(x,y)$.

Result 2. $f \in \mathcal{M}(S) \Rightarrow f \in \mathcal{L}(S)$ iff $f^{-1}(x,y) = 0$ for every chain $[x,y]$ of more than two elements.

Result 3. (i) $g,h \in \mathcal{I}(S)$. $f \in \mathcal{L}(S) \Longrightarrow f\times(g*h) = (f*g)*(f \times h)$

and (ii) $f \in \mathcal{M}(S)$ and $\delta = (f \times \mu) * (f \times E) \Longrightarrow f \in \mathcal{L}(S)$.

Result 4. $f \in \mathcal{M}(S) \Longrightarrow f \in \mathcal{L}(S)$ iff $f*g^{-1} = (f*g)^{-1}$ for every invertible $g \in \mathcal{I}(S)$.

Result 5. (i) If $G \in \mathcal{J}(S)$ and $g = G*\mu$ then for every $f \in \mathcal{L}(S)$, $f \times g = (f \times g)*f^{-1}$, and

(ii) if $f \in \mathcal{M}(S)$ and if $\exists\ G \in \mathcal{L}(S)$ such that $G(x_1,x_2) = \alpha \neq$ a root of unity, whenever x_2 covers x_1 and $f \times (G*\mu) = (f \times G)*f^{-1}$, then $f \in \mathcal{L}(S)$.

4. Proofs.

Result 1. $[x,y] \in \mathcal{BA}(S) \Rightarrow [x,y] = \prod_{i=1}^{n} [x_i,y_i]$

(direct product) x_i, y_i being join irreducible, where each $[x_i,y_i]$ is of length utmost 2 and a Boolean algebra[2].

Also if $x_i \neq y_i$ $[x_i,y_i]$ is of length 2 and so

$$0 = \delta(x_i,y_i) = (f * f^{-1})(x_i,y_i)$$
$$= f^{-1}(x_i,y_i) + f(x_i,y_i)$$

noting that $f \in \mathcal{M}(S) \Rightarrow f(1,1) \neq 1$ and $f \in \mathcal{M}(S) \Rightarrow f^{-1} \in \mathcal{M}(S)$.

Hence, $f^{-1}(x,y) = \prod_{i=1}^{n} f^{-1}(x_i,y_i) = (-1)^r \prod_{i=1}^{n} f(x_i,y_i)$

$$= \mu(x,y)\ f\ (x,y)$$
$$= (\mu \times f)\ (x,y)$$

whenever r is the number of chains of length 2 among the $[x_i,y_i]$.

LEMMA. $f \in \mathcal{L}(S)$ iff for every interval $[x,y]$, $f^{-1}(x,y) = (\mu \times f)\ (x,y)$.

This is indirectly involved and proved in Theorem 2, McCarthy[3].

Result 2. If $f \in \mathcal{L}(S)$, then $f^{-1} = \mu*f$ by the above lemma. When $[x,y]$ is chain of more than two elements $\mu(x,y) = 0$, so that

$f^{-1}(x,y) = \mu(x,y)\ f(x,y) = 0$. Conversely, let $f \in \mathcal{M}(S)$ and $f^{-1}(x,y) = 0$ whenever $[xmy]$ is a chain of more than two elements Take any interval $[x,y]$. Either $[x,y] \in \mathcal{BA}(S)$ or $[x,y] \notin \mathcal{BA}(S)$ If $x,y \in \mathcal{BA}(S)$ then by Result 1 $f^{-1}(x,y) = (\mu * f)(x,y)$. Otherwise,

$$[x,y] = \prod_{i=1}^{m} [x_i, y_i] \prod_{j=1}^{n} [x'_j, y'_j]$$

as a direct product of chains (irreducible representation) where $[x_i,y_i]$ are chains of more than two elements and $[x'_j,y'_j]$ are the other chains involved.

$$f^{-1}(x,y) = \prod f^{-1}(x_i,y_i) \prod f^{-1}(x'_j,y'_j) = 0.$$

Using the hypothesis and noting that when $[x,y] \notin \mathcal{BA}(S)$, the first part of the direct product is not empty. Hence $f^{-1}(x,y) = (\mu * f)(x,y)$ using the definition of μ. Therefore $f^{-1} = \mu * f$ for all x,y in $(S,\leq)$ and so by the above Lemma $f \in \mathcal{L}(S)$.

Result 3. Let $f \in \mathcal{L}(S)$, $g,h \in \mathcal{G}(S)$, then for $x \leq t \leq y$, $f(x,y) = f(x,t)\ f(t,y)$ and so

$$\begin{aligned}(f \times (g*h))(x,y) &= f(x,y) \sum_{x \leq t \leq y} g(x,t)\ h(t,y) \\ &= \sum_{x \leq t \leq y} f(x,t) f(t,y)\ g(x,t)\ h(t,y) \\ &= ((f \times g) * (f \times h))(x,y).\end{aligned}$$

Conversely, if $\delta = f \times (E * \mu)$

$$= (f \times E) * (f \times \mu) = f * (f \times \mu),$$

we have $f^{-1} = f \times \mu$ which implies by the above Lemma that $f \in \mathcal{L}(S)$.

Result 4. If $g \in \mathcal{L}(S)$ and g is invertible $g(x,x) \neq 0$, $f \in \mathcal{L}(S) \Longrightarrow \delta = (f \times (g * g^{-1}) = (f \times g) * (f \times g^{-1})$ so that $f \times g^{-1} = (f \times g)^{-1}$. Conversely, if $f \times g^{-1} = (f \times g)^{-1}$ for every invertible g then taking $g = E$ we obtain $f^{-1} = (f \times E)^{-1} = f \times E^{-1} = f \times \mu$ and so $f \in \mathcal{L}(S)$.

Result 5. Let $g \in \mathcal{G}(S)$ and $g = G * \mu$ then if $f \in \mathcal{L}(S)$ $f \times g = f \times (G * \mu) = (f \times G) * (f \times \mu) = (f \times G) * f^{-1}$.

Conversely, let $G(x_1,x_2) = \alpha \neq$ a root of unity whenever x_2 covers x_1, and G is a linear function such that $f \times g = (f \times g) * f^{-1}$ where $g = G * \mu$.

Let $[x,y]$ be a chain consisting of three elements say

$$[x,y] = \{x,t,y\}, \quad x < t < y.$$

We note that $f(x,x) = 1$, $G(x,x) = 1$, $f^{-1}(y,y) = 1$ and using $f * g^{-1} = \delta$, that $f^{-1}(t,y) = -f(t,y)$ $f^{-1}(t,y) = -f(x,t)$ $f^{-1}(t,y) - f(x,y) = f(x,t)\, f(t,y) - f(x,y)$.

Also using the definition of μ

$$g(x,y) = (\mu * G)(x,y) = G(x,y) - G(x,t)$$

Hence from

$$(f \times G)(x,y) = ((f \times G) * f^{-1})(x,y).$$

We obtain $f^{-1}(x,y)$ $G(x,t) - 1 = 0$.

But t convers x and so $G(x,t) \neq 1$ so that $f^{-1}(x,y) = 0$ for $[x,y]$ consisting of three elements.

We shall now assume that $f^{-1}(x,y) = 0$ for chains consisting of 3,4,5 etc...(n-1) elements and prove on similar lines that $f^{-1}(x,y) = 0$ for a chain of n elements.

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TRACES OF GENERALISED CLIFFORD ELEMENTS

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Clifford algebra, its generalisations and various applications to physics and to Matrix Theory have been extensively studied by Professor Alladi Ramakrishnan[1] and his collaborators. I shall here present some new results on the traces of linear combinations of Clifford elements and Generalised Clifford Elements.

Several formulae exist for the traces of linear combinations of Pauli and Dirac matrices which are especially useful in work on perturbation theory to higher orders. In particular, Caianiello[2] and his collaborators have identified them in terms of pfaffians. I shall now try to update some of these results in the light of the recent work on Generalised Clifford Algebra and indicate a possible generalisation of the concept of the pfaffian.

I. Trace Properties of Products of Line r Combinations of Clifford Elements.

The set of elements satisfying

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \qquad i, j = 1, 2, \cdots, n \tag{1}$$

form the base elements of the Clifford algebra . The set of elements consisting of all possible products of all possible powers[1] (a.p.p. of a.p.p.) of the above

$$A_{\rho_1 \cdots \rho_n} = \alpha_1^{\rho_1} \alpha_2^{\rho_2} \cdots \alpha_n^{\rho_n} \tag{2}$$

where α is an integer modulo 2 constitute 2^n linearly independent elements with a product defined by eq.(1) and form the algebra C_n^2. When n is odd, the last base element can be expressed as the product

$$\alpha_{2\nu+1} = i^{\nu} \alpha_1 \alpha_2 \cdots \alpha_{2\nu} \tag{3}$$

The representations of these algebras and methods of obtaining them from the basic Pauli matrices have been given by Professor Alladi Ramakrishnan[1].

Consider now the linear combination

$$L_n^{(\mu)} = \sum_{i=1}^{n} a_i^{(\mu)} \alpha_i \tag{4}$$

and the product of K such terms

$$P_n^{(K)} = L_n^{(1)} L_n^{(2)} \cdots L_n^{(K)} \tag{5}$$

We shall attempt to find expression for traces of $P_n^{(K)}$

$$T_n^{(K)} = T_{\nu} P_n^{(K)} \tag{6}$$

First we note that from eqs.(1) and (3) it is clear that all the elements of C_n^2 except the unit element are traceless. Now different cases rise according as n and k are odd or even.

Let us first consider the case when n is even, say $n = 2\nu$. Then, for K odd, say $K = 2K+1$, we have, using eq.(1)

$$T_{2\nu}^{(2K+1)} = 0 \tag{7}$$

and for K even, say K = 2k

$$T_{2\nu}^{(2K)} = \cdot\, m^{\nu}(1\,2\cdots 2K) \tag{8}$$

where (1,2.....2k) is a pfaffian defined by

$$(1\,2\cdots 2K) = \sum_{P} (-1)^{P}(i_1 i_2)(i_3 i_4)\cdots(i_{2K-1} i_{2K}) \tag{9}$$

where the sum is over all permutations $i_1\, i_2 \cdots i_{2K}$ of 1,2,.....2k such that $i_1 < i_3 < \cdots < i_{2K-1}$ and $i_1 < i_2\,,\ i_3 < i_4, \cdots\ i_{2K-1} < i_{2K}$ and p is the parity of the permutation with respect to the original ordering (1,2 .....2k). The bracket (1 2) denotes the scalar product

$$(1\,2) = a^{(1)} a^{(2)} = a_1^{(1)} a_1^{(2)} + \cdots + a_{2\nu}^{(1)} a_{2\nu}^{(2)} \tag{10}$$

This result is well known in the case of Dirac and Pauli matrices and we find it is time for the elements of $C_{2\nu}^{2}$.

Now consider the case when n is odd, say $n=2\nu+1$. Then, for K even, say K = 2k, we again have by virtue of eqs.(1) and (3)

$$T_{2\nu+1}^{(2K)} = \gamma m^{\nu}(1\,2\cdots 2K \tag{11}$$

However, for K odd, the corresponding eq.(7) is in general not true. The case of special interest is thus when the number of generators n id odd, the number of factors K is odd and $L^{(r)}$ contain all the $2\nu+1$ generators in the linear combination. In this case we find

$$T_{2\nu+1}^{(2K+1)} = 0 \tag{12}$$

when $K < \nu$ and

$$T_{2\nu+1}^{(2\nu+1)} = m^{\nu} i^{\nu} \{1\,2 \cdots 2\nu+1\} \tag{13}$$

where $\{i_1\, i_2 \cdots\cdots i_{2\nu+1}\}$ is the determinant

$$\{i_1 i_2 \cdots i_{2\nu+1}\} \equiv \mathrm{Det} \begin{vmatrix} a_1^{(i_1)} a_1^{(i_2)} \cdots a_1^{(i_{2\nu+1})} \\ \cdots\cdots\cdots \\ a_{2\nu+1}^{(i_1)} \cdots a_{2\nu+1}^{(i_{2\nu+1})} \end{vmatrix} \tag{14}$$

Now the trace of 2k+1 factors can be reduced recursively by using the above equation to yield

$$T_{2\nu+1}^{(2K+1)} = m^{\nu} i^{\nu} (1\,2 \cdots 2K+1)_{2\nu+1} \tag{15}$$

where the new bracket is expanded in terms of determinants and pfaffians

$$(1\,2 \cdots 2K+1)_{2\nu+1} = \sum_{P} (-1)^{P} \{i_1 i_2 \cdots i_{2\nu+1}\} (i_{2\nu+2} \cdots i_{2K+1}) \tag{16}$$

where the summation is over all permutations $i_1\, i_2 \cdots i_{2K+1}$ of 1 2 .....2k+1 such that $i_1 < i_2 < \cdots < i_{2\nu+1}$, $i_{2\nu+2} < i_{2\nu+3} \cdots < i_{2K+1}$ and $i_1 < i_{2\nu+2}$ and the signature

factor $(-1)^P$ is the parity of the permutation with respect to the original ordering 1 2.....2k+1. Eq.(16) can also be written (with a fewer number of terms especially when $k >> \nu$) as

$$(1\ 2\ \ldots\ 2K+1\}_{2\nu+1} = \{1\ 2\ \ldots\ 2\nu+1\}(2\nu+2\ \ldots\ 2K+1) + \sum_P (-1)^P (i_1 i_2)(i_3\ \ldots\ i_{2K+1})_{2\nu+1} \tag{17}$$

where the sum is over all permutation $(i_1\ i_2)$ of 1 2...2K+1 such that $i_1 < i_2$ keeping $i_3 < i_4 < \cdots < i_{2K+1}$ and the parity is with respect to the original permutation 1 2....2k+1. Several intermediate forms are also possible for expanding the new bracket. In general, it can be expanded as

$$(1\ 2\ \ldots\ 2K+1\} = \sum_P (-1)^P (1\ 2\ \ldots s\ i_{s+1}\ \ldots\ i_{2\nu+1}\}(i_{2\nu+2}\ \ldots\ i_{2K+1} + \sum_P (-1)^P (i_1 i_2)(i_3 i_4\ \ldots\ i_{2K+1})_{2\nu+1} \tag{18}$$

where the first sum is over all permutations $i_{s+1}\ i_{s+2}$ of $s+1, s+2, \ldots\ 2K+1$ such that $i_{s+1} < \cdots < i_{2\nu+1}$ $i_{2\nu+2} < \cdots < i_{2K+1}$ and the second sum is over all permutations of $(i_1\ i_2)$ of 1 2....s such that $i_1 < i_2$ and keeping $i_3 < i_4 < \cdots < i_{2K+1}$. The number of terms N in this last expansion is

$$N = \binom{2K+1-s}{2K-2\nu} + \binom{s}{2} \tag{19}$$

It would be interesting to work out what value of s would make N minimum for any given value of k and ν .

All the results given above can now be combined in the equation

$$T_n^{(K)} = m^{\nu} \{ 1\,2 \cdots K \}_{2\nu+1} \qquad (20)$$

where the bracket now means

$$\{ 1\,2 \cdots K \}_{2\nu+1} = i^{-\nu} (1\,2 \cdots K)$$

when K is even and is expanded as in eq.(18) when K is odd.

II. Generalised Clifford Elements and their Trace Properties:

The set of elements satisfying the ordered commutation relations

$$e_i^m = 1 \quad \text{and} \quad e_i e_j = \omega e_j e_i \qquad (21)$$

where $i < j$ and $i, j = 1, 2, \cdots, n$ and ω is a primitive mth root of unity form the base elements of the generalised Clifford algebra C_n^m, the ordinary Clifford algebra being obtained as a special case when m = 2. The set of elements $e_1^{\nu_1} e_2^{\nu_2} \cdots e_n^{\nu_n}$ consisting of m^n elements which are linearly independent with ν integer mod m form the algebra C_n^m with product defined by eq.(21). It has been shown[1] that when n is even, $C_{2\nu}^m$ is isomorphic to the matrix ring of dimension m^{ν} and when n is odd, $C_{2\nu+1}^m$ reduces to m copies of $C_{2\nu}^m$ consisting of elements $\omega^i e_1^{\nu_1} e_2^{\nu_2} \cdots e_{2\nu}^{\nu_{2\nu}}$, $i = 0, 1, \cdots m-1$. Even in this case, the last generator is obtained as a product of the other 2ν as

$$e_{2\nu+1} = \omega^{*} e_1^{m-1} e_{\nu+1} e_2^{m-1} e_{\nu+2} \cdots e_\nu^{m-1} e_{2\nu} \quad (22)$$

All the elements of C_n^m except the unit element are traceless and for even n, the unit element is obtained iff $r_1 \cdots r_{2\nu} = 0 \bmod m$

Let us now consider the case $n = 2\nu$. Define the linear combination

$$L_n^{(\mu)} = \sum_{i=1}^{2\nu} a_i^{(\mu)} e_i \quad (23)$$

$$T_n^{(K)} = T_r \rho_n^{(K)} = T_r L_n^{(1)} \cdots L_n^{(K)} \quad (24)$$

Then

$$T_{2\nu}^{(K)} = \sum_P \prod_{i=1}^{K} a_{\mu_i}^{(i)} T_r \prod_{i=1}^{K} e_{\mu_i} \quad (25)$$

where the summation is over the permutation of the sub-indices. This can be rewritten as

$$T_{2\nu}^{(K)} = T_r \sum C_{\ell_1 \ell_2 \cdots \ell_{2\nu}} e_1^{\ell_1} \cdots e_{2\nu}^{\ell_{2\nu}} \quad (26)$$

where ℓ_i satisfy

$$\ell_1 + \ell_2 + \cdots + \ell_{2\nu} = K \quad (27)$$

The only terms in the expression which give a non-vanishing trace occur when

$$\ell_1, \ell_2, \cdots, \ell_{2\nu} = 0 \mod m \tag{28}$$

Thus

$$T_{2\nu}^{(K)} = m^{\nu} \sum_{\{\ell\}} C_{\ell_1 \ell_2 \cdots \ell_{2\nu}} \tag{29}$$

where ℓ_i satisfy eqs.(27) and (28) and

$$C_{\ell_1 \cdots \ell_{2\nu}} = \sum_P (\omega^*)^{T_P} \left[\prod_{i_1=1}^{\ell_1} a_{i_1}^{(i_1)} \prod_{i_2=\ell_1+1}^{\ell_1+\ell_2} a_{i_2}^{(i_2)} \cdots \prod_{K=\ell_{2\nu}+1}^{K} a_{2\nu}^{(i_{2\nu})} \right] \tag{30}$$

where the summation is over all permutation of the subindices and T_P is the number of adjacent transpositions required to take it to the completely ordered form.

From eqs. (27) and (28), it immediately follows that

$$T_{2\nu}^{(K)} = 0 \tag{31}$$

when $K \neq 0$ mod m. However, for $K = 0$ mod m, the trace qill be non-vanishing. In this case the above expression for $T_{2\nu}^{(K)}$ can be recast in a pfaffian-like form and a partitioning problem avoided. Thus

$$T = m^{\nu} (1\,2 \cdots K)_m \tag{32}$$

where

$$(1\,2 \cdots K)_m = \sum_P (\Delta P) \prod_{j=0}^{K-1} \left(i_{jm+1}\, i_{jm+2} \cdots i_{jm+m} \right) \tag{33}$$

with

$$i_{jm+1} < - - - < i_{jm+m}$$

and

$$i_1 < i_{m+1} - - - < i_{k-m+1}$$

and

$$(12 - - - m)_m = \sum_{i=1}^{2\nu} a_i^{(1)} a_i^{(2)} - - - a_i^{(m)} \qquad (34)$$

The factor ΔP is a function of ω and is not uniquely fixed but can be calculated for each of the terms in the final expansion on the basis of the prescription given in eq.(30). This is because the e_i obey an ordered commutation relation and not a cyclic one. In this sense, eq.(32) can only be interpreted as giving the various terms in the expansion while the explicit ω factor has to be worked out in each case. However, eq.(33) may be considered as a generalisation of the well known pfaffian.

The case when $n = 2\nu+1$ reduces to the above form when the number of factors K is a multiple of m. However, in general, for arbitrary K we have the result

$$T_{2\nu+1}^{(k)} = \sum_{\{l\}} C_{l_1 l_2 - - - l_{2\nu+1}}$$

where $\{l\}$ is chosen such that

$$l_1 + l_2 + - - - + l_{2\nu+1} = k$$

and

$$l_1 - l_{2\nu+1} = 0 \bmod m, \quad l_2 - l_{2\nu+1} = 0 \bmod m$$

$$- - - - , \quad l_\nu - l_{2\nu+1} = 0 \bmod m .$$

and

$$\ell_{\nu+1} + \ell_{2\nu+1} = 0 \bmod m, \quad \ell_{\nu+2} + \ell_{2\nu+1} = 0 \bmod m,$$

$$\ldots, \quad \ell_{2\nu} + \ell_{2\nu+1} = 0 \bmod m.$$

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SPLINES IN HILBERT SPACES

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Polynomial splines, introduced by I.J.Schoenberg in 1946([7]) consist of pieces of polynomials joined together at certain partition points of a closed interval of the real line. The solution to certain extremal problems in some special Hilbert spaces is given by polynomial splines (see[8],[9]). The study of splines in abstract Hilbert spaces, initiated by Marc Atteia in 1965([2]), pursued by Auselone and Laurent([1]), Jerome and Schumaker([5]) and Atteia[3][4], was motivated by the above-mentioned property of polynomial splines.

Definition:

Suppose that H and H' are two real Hilbert spaces and that f is a continuous linear transformation of H onto H'. Let M be a closed subspace of H with orthogonal complement $M^{\perp}$ in H. If $m \in M$, we set

$$\Phi_m = m + M^{\perp}.$$

An element $\phi_0 \in \Phi_m$ satisfying

$$\| f(\phi_0) \|_{H'} = \min_{\phi \in \Phi_m} \| f(\phi) \|_{H'}$$

is called an interpolating spline of Φ_m relative to f.

Consider a real Hilbert space X. Choose two closed subspaces A and B of X such that the sum of their orthogonal complements $A^{\perp} + B^{\perp}$ is closed in X. For $a \in A$ and $b \in B$, we set

$$\Phi_a = a + A^{\perp}$$

and

$$\Psi_b = b + B^{\perp}$$

Let Y and Z be two Hilbert spaces isomorphic to A and B respectively. with isomorphisms I_A and I_B respectively. We construct two continuous linear transformations T and τ of X onto Y and Z respectively by setting $T = I_A P_A$ and $\tau = I_B P_B$ where P_A and P_B are the projection operators mapping X onto A and B respectively. The following theorem proves the existence of two classes of interpolating splines and gives the criterion for uniqueness of the spline.

THEOREM 1: <u>There exist two sets S_a and Σ_b of interpolating splines satisfying</u>

$$\| \tau s_a \|_Z = \min_{\phi \in \Phi_a} \| \tau \phi \|_Z \text{ for all } s_a \in S_a$$

and

$$\| T \sigma_b \|_Y = \min_{\psi \in \Psi_b} \| T \psi \|_Y \text{ for all } \sigma_b \in \Sigma_b$$

<u>A necessary and sufficient condition for S_a and Σ_b to reduce to a single element each is that $A^{\perp} \cap B^{\perp} = \{\theta_X\}$ where θ_X is the zero element</u> of X.

To obtain a generalized smoothing spline, we proceed as follows: Consider the product spaces $G = Z \times A$ and $H = Y \times B$. If $g_1 = (z_1, a_1)$, $g_2 = (z_2, a_2)$ are any two elements of G, we define an inner product in G by setting

$$\langle g_1, g_2 \rangle_G = \langle z_1, z_2 \rangle_Z + \rho \langle a_1, a_2 \rangle_X , \qquad \rho > 0$$

With this inner product, G is a Hilbert space. Similarly, H can be made a Hilbert space by defining a suitable inner product in it. The transformation L and Q given by $Lx = (Tx, P_A x)$ and $Qx = (Tx, P_B x)$ are continuous and linear and map X into G and H respectively. We have

THEOREM 2: *If* $g = (\theta_Z, a) \in G$ *and* $h = (\theta_Y, b) \in H$, *there exist two sets* $\hat{S}_g$ *and* $\hat{\Sigma}_h$ *of interpolating splines satisfying*

$$\| L\hat{s}_g - g \|_G = \min_{x \in X} \| Lx - g \|_G \qquad \text{for all } \hat{s}_g \in \hat{S}_g$$

and

$$\| Q\hat{\sigma}_h - h \|_H = \min_{x \in X} \| Qx - h \|_H \qquad \text{for all } \hat{\sigma}_h \in \hat{\Sigma}_h .$$

Further, the sets $\hat{S}_g$ *and* $\hat{\Sigma}_h$ *reduce to a single element each if and only if* $A^{\perp} \cap B^{\perp} = \{\theta_X\}$.

We now define *a smoothing spline* as follows:

Definition: *An element of* $\hat{S}_g$ *is a smoothing spline relative to* L *and* $g = (\theta_Z, a)$ *and an element of* $\hat{\Sigma}_h$ *is a smoothing spline relative to* Q *and* $h = (\theta_Y, b)$.

If we now set

$$S = \bigcup_{a \in A} S_a \quad \text{and} \quad \Sigma = \bigcup_{b \in B} \Sigma_b ,$$

$$\hat{S} = \bigcup_{g \in \Theta_Z \times A} \hat{S}_g \quad \text{and} \quad \hat{\Sigma} = \bigcup_{h \in \Theta_Y \times B} \hat{\Sigma}_h ,$$

then

$$S \equiv \hat{S} \quad \text{and} \quad \Sigma \equiv \hat{\Sigma}$$

and the two classes of splines are related in the following manner:

$$\tau^* \tau S = (L^* L S) \cap (Q^* Q \Sigma) = T^* T \Sigma .$$

where τ^*, L^*, Q^* and T^* are the adjoint operators of τ, L, Q and T respectively.

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A NEW LOGARITHMIC FUNCTION

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We shall now summarise some of the results obtained by us, bearing relation to the following function.

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, let $A(n) = \left(\sum_{i=1}^{r} \alpha_i p_i \right)$

One can immediately deduce the following results.

DEFINITION 1. $A(1) = 0$, $A(m,n) = A(m) + A(n)$,

$A(m^n) = n\,A(m)$, $m, n \in z^+$.

DEFINITION 2. The number of solutions to the equation

$A(x) = n$, $x \in Z^+$ where n is a fixed integer greater than zero, is the number of partitions of n into primes.

DEFINITION 3. $A(n) \leq n$, $A(p) = p$ if p is a prime

DEFINITION:- A number n is said to be highly logarithmic if $A(n) > A(m)$ for all $m < n$, $m,n \in z^+$

THEOREM 1. If p is prime p, and $2p$ are highly logarithmic. Conversely if $A(n) > A(m)$ for all $m < n$, $m,n \in z^+$, then $n = p$ or $n = 2p$, p being a prime, for all $n > N$.

DEFINITION: Let $\omega(n)$ be the number of $m < n$, m being composite such that $A(m) < A(n)$.

THEOREM 2. $A(n) > A(m)$ for all $m < n$ implies

$\omega(n) > \omega(m)$ for all $m < n$

Conversely $\omega(n) > \omega(m)$ for all $m < n$ implies

$A(n) > A(m)$ for all $m < n$

NOTE. The result of the above theorem has been generalised by us to a concept of Duality.

We now make an attempt to generalise primes through the function A(n).

DEFINITION:- A number n is a logarithmic Number

$$\text{if} \quad n \equiv 0 \pmod{A(n)} \qquad (1)$$
$$n \neq A(n).$$

The logarithmic numbers possess similar properties to primes and are a generalisation of primes.

THEOREM 3. The number of solutions to (1) are infinite

THEOREM 4. If a,b are positive integers, then an arithmatical progression with first term 'a' and common difference 'b' contains infinitely many logarithmic numbers.

THEOREM 5. If we accept the conjecture of prime pairs [1] and $\{1_n\}$ is the sequence of logarithmic numbers then

$$\lim_{n \to \infty} \frac{1_n}{1_{n+1}} = 1$$

THEOREM 6. If 1_n is the sequence of logarithmic numbers

$$\lim_{n \to \infty} \frac{A(1_n)}{1_n} = 0$$

THEOREM 7. The number of solutions to

$$n \equiv 0 \pmod{n}$$
$$A(n) = m$$

is the number of partitions of $m - A(m)$ into primes.

NOTE. We have also 'observed' logarithmic number pairs example 70,72 like prime pairs 71,73,etc. We do not in this summary go through the details of the proofs of the above theorems.

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'DUALITY' AMONG A CERTAIN CLASS OF FUNCTIONS

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Introduction

The purpose of this contribution is to introduce a concept called 'Duality' among functions defined on integers and to study the class of functions which possess 'Duality'. We are led to this concept of 'Duality' which is applicable to a wide class of functions, by initially considering a relation between $A(n)$ and $\omega(n)$ two functions introduced by the author earlier [1].

If n is an integer ≥ 0, and $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ then $A(n) = \sum_{i=1}^{r} \alpha_i p_i$. We define 'n' to be a highly logarithmic number if $A(n) > A(m)$, for all $m < n$, n and m being integers and m being composite. We also defined $\omega(n)$ to be number of m n, n,m being integers, m being composite such that $A(m) < A(n)$.

It was then shown by the author [1] that

(1) $A(n) > A(m)$ $\forall$ $m < n$, m, $n \in I$, m, being composite

$\Rightarrow$ $\omega(n) > \omega(m)$ " " "

and (2) $\omega(n) > \omega(m)$ " " "

$\Rightarrow$ $A(n) > A(m)$ " " "

$\forall$ $n > N$. That is, the numbers satisfying (1) (highly logarithmic) coincide with the numbers satisfying (2) after a certain stage 'N', where inequalities (1) and (2) are defined over the same set of Integers - composite integers.

NOTE: I is the set of positive integers.

DUALITY

DEFINITION 1. We define $\mathcal{C}$ to be the class of all functions, such that if $\phi \in \mathcal{C}$ then

(a) $\phi(n) \geq 0, n, \in I, n \geq 0 \quad \phi : I \longrightarrow R^+$

(b) There exists an infinite sequence of integers in increasing order $A = \{ a_1, a_2, a_3, \ldots \}$ so that

(c) If $\{ n_j \}_{j=1}^{\infty}$ in increasing order all the integers satisfying $\phi(n_j) > \phi(a_i)$ for all $a_i < n_j$

(d) If $\psi(n)$ equals the number of $a_i < n_j$ such that $\phi(n) > \phi(a_i)$ then

$$\phi(n_j) > \phi(a_i) \; \forall a_i < n_j \Rightarrow \psi(n_j) > \psi(a_i) \; \forall a_i < n_j$$

and $$\psi(n_j) > \psi(a_i) \; \forall a_i < n_j \Rightarrow \phi(n_j) > \phi(a_i) \; \forall a_i < n_j$$

at least $\forall n_j > N$, and

(e) the sequence $\{ a_i \}$ and $\{ n_i \}$ do not coincide even after a certain stage.

DEFINITION 2. The integers $n_1, n_2, n_3, \ldots$ are said to be 'highly like', and ψ is said to be the dual of ϕ. ϕ, said to possess the property of 'duality'.

NOTE. 1) In the case of $A(n)$, ϕ was equal to A, $\psi = \omega$ and $a_1, a_2, \ldots$ the set of composite integers, and $n_1, n_2, \ldots$ 'highly A like' i.e. highly logarithmic numbers.

2) Whenever we refer to a sequence of integers, it is always positive and in increasing order $\longrightarrow$ to infinity.

DEFINITION 3. If ϕ, ψ are functions ψ is said to be a dual of ϕ if $\psi(n)$ represents the number of $a_i < n$ such that $\phi(a_i) < \phi(n)$ for some sequence $\{a_i\}$. The sequence $\{a_i\}$ is called an associated (integers) sequence. We also use the notation ϕ^* instead of ψ.

We now state some theorems but we do not go through the details of the proofs.

THEOREM 1. If ϕ is a function such that $\phi(n) \geq 0$ for all $n > 0$, $n \in I$, and there exists an infinite sequence of integers $m_1, m_2, \ldots$ for which $\phi(m_1) < \phi(m_2) \ldots \phi(m_i) < \phi(m_{i+1}) \ldots$ and $\phi(m_i) \to \infty$ as $i \to \infty$ then $\phi \in \mathcal{C}$.

THEOREM 2. If $f(n) \geq 0, n \geq 0$ $n \in I$, and if for m_1, m_2, m_3 .... an infinite sequence of integers, $f(m_1) < f(m_2) \ldots$..then $f \in \mathcal{C}$.

THEOREM 3a. If $f(n) \geq 0, n \geq 0$, $n \in I$, and $f(n) = C$, for all $n > m$, then $f \notin \mathcal{C}$.

THEOREM 3b. (Analogous). If $f(n) \geq 0$, $n \geq 0$, $n \in I$, and $f(n)$ is a strictly decreasing sequence for all $n > m$, then $f \notin \mathcal{C}$.

THEOREM 4. If $f \in \mathcal{C}$ with a dual f^* then $f^* \in \mathcal{C}$.

THEOREM 5. If $f \in \mathcal{C}$ with a dual f^* then $f^*(n) \leq n$, and $f^*(n) \neq n$ for infinitely many values.

THEOREM 6. If f is a function $f: I \to I$ that $f(n) \leq n$ then there exists a g such that $g^* = f$, where g^* is a dual of g.

THEOREM 7. If $f: I \to I$ $f(n) \leq n$, for infinitely many and 'f' contains an increasing subsequence then there exists a $g \in \mathcal{C}$ such that $g^* = f$.

THEOREM 8. If $f \in \mathcal{C}$ with a dual f^* then f^* satisfies the following inequality: $n - m \geq f^*(n) - f^*(m)$ where $n > m$, $f(n) > f(m)$ where $n > m$, $f(n) > f(m)$ for infinitely many pairs n,m.

REMARK 1. In Definition 1, we stated that the sequence $a_1, a_2, \ldots$ and $m_1, m_2, \ldots$ should not coincide even after a certain stage. If we permit this coincidence we can easily that a function has an increasing subsequence $m_1, m_2, \ldots$ then $m_1, m_2, \ldots$ become the 'highly ϕ like' numbers for $A = \{ m_1, m_2, \ldots \}$ where the m_i's satisfy

$$\phi(m_i) > \phi(n) \quad \forall \; n < m_i \qquad m \in I.$$

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