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CONTRIBUTIONS TO NUMBER THEORY

(Revised and Enlarged)

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FAREY SERIES WITH FIBONACCI NUMBERS-I, II, III *

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A FAREY SERIES WITH FIBONACCI NUMBERS

Part I.

The Farey series is an old and famous set of fractions associated with the positive integers. We here show that if we form a Farey series with Fibonacci numbers, the properties of the series are remarkably preserved. In fact we find that with the new series we are able to observe and identify 'points of symmetry'.

The Farey series with Fibonacci numbers of order T_n (where T_n stands for the n-th term of the Fibonacci sequence) is the set of all possible fractions T_i/T_j, $i = 0,1,2,3,\ldots n-1$, $j = 1,2,3,\ldots n$ $(i < j)$ arranged in ascending order of magnitude. The last term is $\frac{1}{1}$ i.e. $\frac{T_1}{T_2}$. The first term is $\frac{0}{T_{n-1}}$. We set $T_o = 0$ so that $T_o + T_1 = T_2$, $T_1 = T_2 = 1$.

For convenience we denote a Fibonacci number by F, the Farey series of Fibonacci numbers by f.f. and the rth term in the new farey series of order T_n by $f_{(r)n}$.

We define an $f_{(r)n}$ to be a point of symmetry if $f_{(r+1)n}$ and $f_{(r-1)n}$ have the same denominator. We have shown in an appendix the Farey series of all Fibonacci numbers upto 34.

We define an interval to be the set of all f.f. fractions between two consecutive points of symmetry.

Proof of Symmetry

THEOREM 1.1. $f_{(r)n}$ is of the form $\frac{1}{T_i}$. Moreover $f_{(r+k)n}$ and $f_{(r-k)n}$ have the same denominator if they do not pass beyond the next point of symmetry on either side.

Proof. In the series f.f. the terms are arranged in the following fashion last interval -- $\frac{T_{i-1}}{T_j}$

interval prior to last -- $\frac{T_{i-2}}{T_j}$ etc.

If there are two fractions $\frac{T_{i-1}}{T_{j-1}}$, and $\frac{T_{i-2}}{T_{j-2}}$ then their mediant $\frac{T_i}{T_j}$ lies in between them. That is

$$\text{if} \quad \frac{T_{i-1}}{T_{j-1}} < \frac{T_{i-2}}{T_{j-2}} \quad \text{then} \quad \frac{T_{i-1}}{T_{j-1}} < \frac{T_i}{T_j} < \frac{T_{i-2}}{T_{j-2}},$$

$$\text{if} \quad \frac{T_{i-2}}{T_{j-2}} < \frac{T_{i-1}}{T_{j-1}} \quad \text{then} \quad \frac{T_{i-2}}{T_{j-2}} < \frac{T_i}{T_j} < \frac{T_{i-1}}{T_{j-1}}.$$

This inequality can easily be established dealing with the two cases separately.

We shall adopt induction as the method of proof. Our surmise has worked for all f.f. series upto 34. Let us treat as T_{n-1}. For the next f.f. series i.e. of order T_n, fractions to be introduced are $\frac{T_2}{T_n}, \frac{T_3}{T_n}, \ldots, \frac{T_i}{T_n}; \frac{T_{n-1}}{T_n}$. $\frac{T_i}{T_n}$ will fall in between $\frac{T_{i-1}}{T_{n-1}}$ and $\frac{T_{i-2}}{T_{n-2}}$. First assume that $\frac{T_{i-1}}{T_{n-1}} < \frac{T_{i-2}}{T_{n-2}}$.

Since our assumption is valid for 34, $\frac{T_{i-1}}{T_{n-1}}$ lies just before $\frac{T_{i-2}}{T_{n-2}}$. $\frac{T_{i-3}}{T_{n-2}}$ will occur just after $\frac{T_{i-2}}{T_{n-1}}$ from our assumption regarding points of symmetry. But $\frac{T_{i-1}}{T_n}$ lies in between these two fractions. The distance of $\frac{T_{i-1}}{T_n}$ from the point of symmetry $\frac{1}{T_p}$ is equal to the distance $\frac{T_i}{T_n}$ from that point of symmetry. Hence this is valid for 55. Similarly it can be made to hold good for 89, ... Hence the theorem.

THEOREM 1.2. Whenever we have an interval $\left[\frac{1}{T_i}, \frac{1}{T_{i-1}}\right]$, the denominator of term next to $\frac{1}{T_i}$ is T_{i+2}. The denominator of the next term is T_{i+4}, then T_{i+6} .... We have this till we reach the maximum for that ff sequence i.e. till T_{i+2k} does not exceed T_n. Then the term after T_{i+2k} will be the maximum possible term not greater than T_n, but not equal to any of the terms formed i.e. its either T_{i+2k+1} or T_{i+2k-1} say T_j. The term after T_j will be T_{j-2}, T_{j-4}, ... till we reach $\frac{1}{T_{i-1}}$. (As an example let us take $\frac{1}{3}$, $\frac{1}{2}$ in the ff series for 55. Then the denominator of the later terms in order are 8,21,55,34,13,5,2).

<u>Proof</u>. The proof of Theorem 1.2 will follow by induction on Theorem 1.1.

THEOREM 1.3. a) If $\frac{h}{k}$, $\frac{h'}{k'}$, $\frac{h''}{k''}$ are three consecutive fractions of an ff series then

$$\frac{h + h''}{k + k''} = \frac{h'}{k'} \quad \text{if} \quad \frac{h'}{k'} \text{ is not a point of symmetry}$$

b) If $\frac{h'}{k'}$ is a point of symmetry say $\frac{1}{T_i}$ then

$$\frac{T_{i-2}\, h + T_{i-1}\, h''}{T_{i-2}\, k + T_{i-1}\, k''} = \frac{h'}{k'}$$

Proof. Case 1. (From 1.2) We see that

$$\frac{h}{k} = \frac{T_{i-2}}{T_{j-2}} , \quad \frac{h'}{k'} = \frac{T_i}{T_j} , \quad \frac{h''}{k''} = \frac{T_{i+2}}{T_{j+2}}$$

in this case $\frac{T_{i+2} + T_{i-2}}{T_{j+2} + T_{j-2}} = \frac{{}^*3.T_i}{3.T_j} = \frac{T_i}{T_j} = \frac{h'}{k'}$.

$^*T_{h+2} - T_{n-2} = 3\, T_n$ is a property of the Fibonacci sequence.

Case 2. $\frac{h'}{k'} = \frac{T_i}{T_j}$, $\frac{h}{k} = \frac{T_{i-2}}{T_{j-2}}$, and $\frac{h''}{k''} = \frac{T_{i+2}}{T_{j+2}}$

(From I.2)

Then $\frac{T_{i+1} + T_{i-2}}{T_{j+1} + T_{i-2}} = \frac{2T_i}{2T_j} = \frac{T_i}{T_j} = \frac{h'}{k'}$ similarly.

Case 3. $\frac{h'}{k'} = \frac{T_i}{T_j}$, $\frac{h}{k} = \frac{T_{i-2}}{T_{j-2}}$, $\frac{h''}{k''} = \frac{T_{i-1}}{T_{j-1}}$ (from I.3)

$$\therefore \quad \frac{T_{i-1} + T_{i-2}}{T_{j-1} + T_{j-2}} = \frac{T_i}{T_j} = \frac{h'}{k'} .$$

Hence the result.

1.3.b. Let $\frac{h'}{k'} = \frac{1}{T_i}$. From Theorem 1.2 it follows that $\frac{h''}{k''} = \frac{3}{T_{i+2}}$ and $\frac{h}{k}$ . . $= \frac{2}{T_{i+2}}$

$$\therefore \frac{T_{i-2}h + T_{i-1}\,h''}{T_{i-2}k + T_{i-2}\,k''} = \frac{2T_{i-2} + 3T_{i-1}}{T_i\,T_{i+2}} = \frac{T_{i+2}}{T_i\,T_{i+2}} = \frac{1}{T_i}\,.$$

Hence the theorem.

THEOREM 1.4. If $\frac{h}{k}$, and $\frac{h'}{k'}$ are two consecutive fractions of an f.f. series then $\left|\frac{h-h'}{k-k'}\right| \in$ f.f. $(k-k' \neq 0)$.

Proof. Since $f_{(r)n}$ is of the form $\frac{T_i}{T_j}$ it follows that $|h-h'| = T_i$ and $|k-k'| = T_j$. Since $\frac{h}{k}$ and $\frac{h'}{k'}$ are members also, $h = T_{i_1}$, $h' = T_{i_2}$; $k = T_{j_1}$, $k' = T_{j_2}$. Further $|T_{j_1} - T_{j_2}| = T_j$ and $|T_{i_1} - T_{i_2}| = T_i$. But from the Fibonacci recurrence relation $T_n = T_{n-1} + T_{n-2}$ we see that the condition for this is $|i_1 - i_2| \leqslant 2$ and $|j_1 - j_2| \leqslant 2$, which follows from Theorem 1.2. Actually $\left|\frac{h-h'}{k-k'}\right|$ are the fractions of the same interval arranged in descending order of magnitude for increasing values of $\frac{h}{k}$.

We define the distance between $f_{(r)n}$ and $f_{(k)n}$ as equal to $|r-k|$.

We now introduce a term 'Generating Fraction'. If we have a fraction $\frac{T_i}{T_j}$ $(i \leqslant j)$. We split $\frac{T_i}{T_j}$ into

$\frac{T_{i-1} + T_{i-2}}{T_{j-1} + T_{j-2}}$. We form from this two fractions $\frac{T_{i-1}}{T_{j-1}}$ and $\frac{T_{i-2}}{T_{j-2}}$ such that $\frac{T_i}{T_j}$ is the mediant of the fractions formed. We continue this process and split the fractions obtained till we reach a state where the numerator is 1. $\frac{T_i}{T_j}$ then amounts to the Generating fraction of the others. We call $\frac{T_i}{T_j}$ as the Generating Fraction of an Interval (G.F.I.) if through this process we are able to get from the G.F.I. all the other fractions of 'that' closed interval. We can clearly see in a ff series for $T_1, T_2, \ldots T_n$, $\frac{T_i}{T_n}$ will be a G.F.I. (We also note that $\frac{T_i}{T_j}, \frac{T_{i-1}}{T_{j-1}}, \frac{T_{i-2}}{T_{j-2}}, \ldots$. belong to the same interval because the difference in the suffix of the numerator denominator and is j-i). Hence the sequence of G.F.I's is $\frac{T_1}{T_n}, \frac{T_2}{T_n}, \frac{T_3}{T_n}, \ldots \frac{T_{n-1}}{T_n}$. We now see some properties concerning G.F.I's.

THEOREM 1.5. If we form a sequence of the distance between two consecutive G.F.I's such a sequence runs thus

2,2,4,4,6,6,8,8, ...,

i.e., alternate G.F.I's are symmetrically placed about a G.F.I.

THEOREM 1.6. If we take the first G.F.I. say $f_{(g_1)n}$ then $f_{(g_1+1)n}$ and $f_{(g_1-1)n}$, have the same denominator. For $f_{(g_2)n}$ the second G.F.I. $f_{(g_2+2)n}$, and $f_{(g_2-2)n}$ have the same denominator. In general for $f_{(g_k)n}$ the k^{th} G.F.I. $f_{(g_k+k)n}$ and $f_{(g_k-k)n}$ have the same denominator.

The proofs of theorems 1.5 and 1.6 follows from 1.2.

(Note:- We can verify that for alternate G.F.I's $f_{(g_2)n}$, $f_{(g_4)n}$, $f_{(g_6)n}$ ....; $f_{(g_k+k)n}$ and $f_{(g_k-k)n}$ have the same denominator for k is even and the sequence of distance shown above is 2,2,4,4,6,6,8,8,....).

Part II

DEFINITION. We now define T_{i-2} to be the 'factor of the interval' $\left[\frac{1}{T_i} ; \frac{1}{T_{i-1}}\right]$. More precisely the factor of a closed interval is that term T_z where 'z' = suffix of denominator minus suffix of the numerator, of each fraction of that interval. It can be easily seen (Part I) that 'z' is a constant.

LEMMA 2.1. If $j_1 - i_1 = j_2 - i_2 > 0$

Then

$$T_{j_1} T_{i_2} - T_{j_2} T_{i_1} = \left|T_{j_2-j_1}\right| T_{j_1-i_1} = \left|T_{j_2-j_1}\right| T_{j_2-i_2} .$$

(Note:- The author has independently shown in one of his previous paper that the Fibonacci sequence is capable of extension to $T_{-\infty}$, and that $|T_{-n}| = T_n$. More accurately $T_{-n} = T_n (-1)^{n-1}$ n being positive and $|T_k| = T_{|k|} \;\forall\; k \in I$.

Proof. We apply Binet's formula that

$$T_n = \frac{a^n - b^n}{a - b} \quad \text{where} \quad a = \frac{1 + \sqrt{5}}{2} , \; b = \frac{1 - \sqrt{5}}{2} .$$

Then (L.H.S.) the left hand side of the expression, and (R.H.S.) the right hand side of the expression reduce as follows.

To prove

$$\frac{a^{j_1} - b^{j_1}}{a - b} \cdot \frac{a^{i_2} - b^{i_2}}{a - b} - \frac{a^{j_2} - b^{j_2}}{a - b} \cdot \frac{a^{i_1} - b^{i_1}}{a - b}$$

$$= \frac{a^{j_2-j_1} - b^{j_2-j_1}}{a - b} \cdot \frac{a^{j_1-i_1} - b^{j_1-i_1}}{a - b}$$

because $j_1 - i_1 > 0$ $T_{j_1-i_1}$ is positive and hence can be put within the sign.

To prove

$$(a^{j_1}-b^{j_1})(a^{i_2}-b^{i_2})-(a^{j_2}-b^{j_2})(a^{i_1}-b^{i_1}) = (a^{j_2-j_1}-b^{j_2-j_1})(a^{j_1-i_1}-b^{j_1-i_1})$$

L.H.S. reduces to

$$a^{j_1+i_1}-a^{j_1}.b^{i_2}+b^{j_1+i_1}-b^{j_1}a^{i_2}-a^{j_2+i_1}+a^{j_2}b^{i_1}+b^{j_2}a^{i_1}-b^{j_2-i_1}$$

$$= -a^{j_1}b^{i_2}-a^{i_2}\;b^{j_1}+a^{j_2}.b^{i_1}+b^{j_2}a^{i_1}$$

R.H.S. reduces to

$$a^{j_2-i_1}-a^{j_2-j_1}.b^{j_1-i_1}+b^{j_2-i_1}-b^{j_2-j_1}.a^{j_1-i_1}$$

This may be simplified further using $ab = -1$ and $j_1-i_1 = j_2-i_2$

R.H.S. then is

$a^{j_1}b^{i_2}+b^{j_1}a^{i_2}-a^{j_2}b^{i_1}-b^{j_2}a^{i_1}$. We can see that L.H.S. = R.H.S. Hence the Lemma.

COROLLARY. From this we may deduce that if $\frac{T_{i_1}}{T_{j_1}}$ and $\frac{T_{i_2}}{T_{j_2}}$ belong to the same interval i.e. $j_1 - i_1 = j_2 - i_2$ then

$$T_{j_1}T_{i_2}-T_{j_2}T_{i_1}\left(\frac{T_{i_1}}{T_{j_1}}<\frac{T_{i_2}}{T_{j_2}}\right)$$

$$= |T_{j_2-j_1}|\,T_{j_1-i_1} \text{ or } |T_{j_2-j_1}|\,T_{j_2-i_2}.$$

Hence $T_{j_1} T_{i_2} - T_{j_2} \quad T_{i_1}$ will be an integral multiple of $T_{j_1-i_1}$ or $T_{j_2-i_2}$ (the factor of that interval) which is the term obtained by the difference in suffixes of the numerator and denominator of each fraction of that interval.

DEFINITION. We now introduce the term 'conjugate fractions'. Two fractions h/k, and h'/k', h/k h'/k' are conjugate in an interval $\left[\frac{1}{T_i}, \frac{1}{T_{i-1}}\right]$ if the distance of h/k from $\frac{1}{T_i}$ equals the distance of $\frac{h'}{k'}$ from $\frac{1}{T_{i-1}}$ $(h/k \neq h'/k')$.

COROLLARY. Two consecutive points of symmetry are conjugate with distance zero.

THEOREM 2.2. If h/k, and h'/k' are conjugate $\frac{1}{T_i}$, $\frac{1}{T_{i-1}}$ then $kh' - hk' = T_{i-2}$.

<u>Proof</u>. Using I , we can easily see that if h/k is of the form $\frac{T_{i_1}}{T_{j_1}}$ then h'/k' is $\frac{T_{i_1-1}}{T_{j_1-1}}$ .... (1)

$\frac{1}{T_i}$, and $\frac{1}{T_{i-1}}$ are conjugate. This with (1) since $T_2 = T_1 = 1$, since the term after $\frac{1}{T_i} = \frac{T_4}{T_{i+2}}$, and the term before $\frac{1}{T_{i-1}} = \frac{1}{T_{i+1}}$. We see it confers with the statement 1 above. Proceeding in such a fashion we obtain the result 1 .

Of course we assume here that there exist atleast two terms in $\left[\frac{1}{T_i}, \frac{1}{T_{i-1}}\right]$. If there are less than two the result is immediate.

Hence we can see that any two conjugate fractions in $\left[\frac{1}{T_i}, \frac{1}{T_{i-1}}\right]$ are $\frac{T_{j-i+2}}{T_j}$, $\frac{T_{j-i+1}}{T_j}$.

To show

$$T_j\, T_{j-i+1} - T_{j-1}\, T_{j-i+2} = T_{i-2}.$$ This will immediately follow from Lemma 2.1.

THEOREM 2.3. a) If h/k, and h'/k' and two consecutive fractions in an f.f. series, which belong to

$\frac{1}{T_i}, \frac{1}{T_{i-1}}$ then $kh' - hk' = T_{i-2}$.

b) If h/k and h'/k' are conjugate in an interval $\frac{1}{T_i}, \frac{1}{T_{i-1}}$ $kh'-hk' = T_{i-2}$.

<u>Proof</u>. Theorems 2.3a) and 2.3b can be proved using Lemma 2.1 and Theorem 1.2.

DEFINITION. If $h/k \in \frac{1}{T_i}, \frac{1}{T_{i-1}}$ we define the couplet for h/k as the ordered pair $\left[\left(\frac{1}{T_i}, \frac{h}{k}\right), \left(\frac{h}{k}, \frac{1}{T_{i-1}}\right)\right]$.

THEOREM 2.4. In the case of couplets we find that

$$T_i\, h - k = T_p \cdot T_{i-2}$$

and

$$k - T_{i-1}\, h = T_{p+1}\, T_{i-2}$$ (where T_p is some Fibonacci number).

Proof. Let h/k be $\frac{T_{j-i+2}}{T_j}$

Then $T_i h - k$ is $T_i T_{j-i+2} - T_j = T_p T_{i-2}$ (1)

and $k - T_{i-1} h$ is $T_j - T_{i-1} T_{j-i+2} = T_{p+1} T_{i-2}$ (2)

Adding (1) and (2) we have

$$T_{i-2} T_{j-i+2} = T_{p+2} T_{i-2}$$

$$\therefore T_{j-i+2} = T_{p+2}$$

$$\text{or } j-i = p.$$

i.e. $T_i T_{j-i+2} - T_j = T_{j-i} T_{i-2}$. 3

We can establish 3 using Lemma 2.1. Hence the proof.

DEFINITION. We define $\left[\left(\frac{1}{T_i}, \frac{h}{k}\right), \left(\frac{h}{k}, \frac{1}{T_{i-1}}\right)\right]$ and $\left[\left(\frac{1}{T_i}, \frac{h'}{k'}\right) \left(\frac{h'}{k'}, \frac{1}{T_{i-1}}\right)\right]$ to be conjugate couplets if and h/k, and h'/k' are conjugate fractions of the closed interval $\left[\frac{1}{T_i}, \frac{1}{T_{i-1}}\right]$.

THEOREM 2.5. In the case of conjugate couplets

if $T_i h - k = T_p \cdot T_{i-2}$ and

$k - T_{i-1} h = T_{p+1} T_{i-2}$ then

$\underline{T_i h' - k' = T_{p-1} T_{i-2}}$ and

$\underline{k - T_i-1 h' = T_p T_{i-2}}$

Proof. We note that $(j-1)$ in the previous proof is is the difference in the suffixes of T_j and T_i. If now $h/k = \frac{T_{j-i+2}}{T_j}$ then $p = j-1$. But since h'/k' is conjugate with h/k, $h'/k' = \frac{T_{j-i+1}}{T_{j-1}}$.

Therefore the constant factor say T_q in the equation for h'/k' $\quad T_i h' - k = T_q - T_{i-2}$ is such that

$$q = j-1-i = (j-i)-1 = p-1.$$

$$\therefore \quad T_i h' - k' - T_{p-1} T_{i-2}. \quad \text{Hence}$$

$$k - T_{i-1} h' = T_p T_{i-2} \quad \text{since it follows from 2.4.}$$

THEOREM 2.6. Since we have seen that if h/k, and h'/k' are conjugate then the difference in suffixes of their numerators or denominators = 1, we find

$$\frac{h+h'}{k+k'} \in \left[\frac{1}{T_i}, \frac{1}{T_{i-1}}\right] \quad \text{and} \quad \frac{h-h'}{k-k'} \in \left[\frac{1}{T_i}, \frac{1}{T_{i-1}}\right)$$

if h/k, $h'/k' \in \frac{1}{T_i}, \frac{1}{T_{i-1}}$. We may also further note that $\frac{h+h'}{k+k'}$ are the fractions of the latter half of the interval arranged in descending order while $\frac{h-h'}{k-k'}$ are the fractions of the first half arranged in ascending order, for increasing values of h/k.

contd Proof of Theorem 2.3. Let $h/k = \frac{T_{i_1}}{T_{j_1}}$ and $h'/k' = \frac{T_{i_2}}{T_{j_2}}$ then from I.2 we have $|i_1 - i_2| = |j_1 - j_2| \leqslant 2$ but not equal to zero. $\therefore$ kh' = hk' = $T_{j_2-j_1} \quad T_{i-2}$ where $T_{(i-2)}$ is the factor of the interval. But $|T_{j_2-j_1}| = T_{|j_2-j_1|}$ and $|j_2 - j_1| = 2$ or 1. In any case

$$kh' - hk' = 1 \times T_{i-2} = T_{i-2} \text{ for } T_2 = T_1 = 1.$$

Hence the proof.

Part III

We now give a generalised result concerning 'sequence of distances'.

THEOREM 3.1.a. Points of symmetry if they are of the form $f_{(r)n}$ then $r \in \{2,3,5,8,12,17,\ldots\}$. Or the sequence of distance between two consecutive points of symmetry will be

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots.$$

an arithmetic progression with common difference 1. •

THEOREM 3.1.b. The sequence of distance for fractions with common numerator T_{2n-1} or T_{2n} is $2n-1$, $2n$, $2n+1$, ...

Proof. To prove Theorem 3.1.a we have to show that if there 'n' terms in an interval then there are $(n+1)$ terms in the next.

Let there be 'p' terms of ratio $\frac{T_i}{T_j}$. It is evident that there are $p+1$ terms of the form $\frac{T_{i+1}}{T_j}$. But these $(p+1)$ terms of the form $\frac{T_{i+1}}{T_j}$ are in an interval next to that in which the 'p' terms of the form $\frac{T_i}{T_j}$ lie. So the sequence is an AP with common difference 1. Moreover the second term is always $\frac{1}{T_n}$ (evident). Hence the result.

If we fix the numerator to be '2' and take the sequence $\frac{2}{T_n}$, $\frac{2}{T_{n-1}}$, $\frac{2}{T_{n-2}}$, ... $\frac{2}{3}$ then the sequence of distance between two consecutive such fractions is

$$3, 4, 5, \ \ldots\ .$$

From 1.2 (Part I) it follows that $\frac{2}{T_i}$ lies just before a point of symmetry say $\frac{1}{T_j}$. Since we have seen the sequence of distance concerning points of symmetry it will follow that here too the common difference is 1. The first term is 3 for there be two terms between $\frac{2}{T_n}$, and $\frac{2}{T_{n-1}}$. The inequality $\frac{2}{T_n} < \frac{1}{T_{n-2}} < \frac{3}{T_{n-2}} < \frac{2}{T_{n-1}}$ can be established. Hence the result.

In a similar fashion we find that the sequence of distance for numerator 3 is

$$3,4,5, \ldots$$

We shall give a table and the generalisation

Numerator	Sequence of distances
T_1 or T_2	1,2,3,4,5, ...
T_3 or T_4	3,4,5,6, ...
T_5 or T_6	5,6,7,8, ...
T_{2n-1} or T_{2n}	2n-1, 2n, 2n+1, 2n+2, . . .

DEFINITION. Just as we defined an interval, we now defined an interval, we now define a 'stage' as the set of f.f. fractions lying between two consecutive G.F.I's. The stage may be closed or open depending upon the inclusion or omission of the G.F.I's.

Since the sequence of distance of G.F.I's is 2,2,4,4,6,6, ... It is possible for two consecutive 'stages' to have equal number of terms. We define two stages $\left[\frac{T_{i-1}}{T_n}, \frac{T_i}{T_n}\right]$ and $\left[\frac{T_i}{T_n}, \frac{T_{i+1}}{T_n}\right]$ to be conjugate stages if the distance of $\frac{T_i}{T_n}$ from $\frac{T_{i-1}}{T_n}$ equals the distance of $\frac{T_{i+1}}{T_n}$ from $\frac{T_i}{T_n}$. That is the number of terms in two conjugate stages, are equal. We call a stage comprision of both these stages, as a 'conjugate stage'. Let us now investigate properties concerning stages. If we have conjugate stage $\left[\frac{T_{i-1}}{T_n}, \frac{T_i}{T_n}, \frac{T_{i+1}}{T_n}\right]$ then we define two fractions $\frac{h}{k}$ and $\frac{h'}{k'}$ to be 'Corresponding' if $\frac{h}{k} \in \left[\frac{T_{i-1}}{T_n}, \frac{T_i}{T_n}\right]$ and $\frac{h'}{k'} \in \left[\frac{T_i}{T_n}, \frac{T_{i+1}}{T_n}\right]$ and if the distance of $\frac{h}{k}$ from $\frac{T_{i-1}}{T_n}$ is equal to the distance of $\frac{h'}{k'}$ from $\frac{T_i}{T_n}$.

THEOREM 2.2. Two corresponding fractions have the same numerator. If $\frac{h}{k}$ and $\frac{h'}{k'}$ are corresponding fraction's then $h=h'$.

<u>Proof</u>. This will follow from 1.2 (previous Part).

Let $\frac{T_{i-1}}{T_n}$ be the maximum reached in its interval so that $\frac{T_{i-1}}{T_{n-1}}$ will be the maximum for the interval in which $\frac{T_i}{T_n}$ belongs.

The terms next to $\frac{T_{i-1}}{T_n}$ is $\frac{T_{i-2}}{T_{n-1}}$. Similarly the terms next to $\frac{T_i}{T_n}$ is $\frac{T_{i-2}}{T_{n-2}}$. But these fractions are corresponding fractions. Proceeding in such a fashion we obtain the result.

$\frac{T_{i-1}}{T_n}$ has necessarily to be the maximum in it interval. Since we have considered conjugate stages i is odd. Using 1.2 (previous part) it can be established that alternate G.F.I's are maximum in their interval and that too when suffix of numerator is even (i-1 is even). Hence the result.

DEFINITION. Since the number of terms in a stage is odd, we define $\frac{h}{k}$ to be the middle point of a stage $\left[\frac{T_{i-1}}{T_n}, \frac{T_i}{T_n}\right]$ if it is equidistant from both G.F.I's. We can deduce from this that $\frac{h}{k}$ is a point of symmetry since $\frac{T_{i-1}}{T_n}$, and $\frac{T_i}{T_n}$ have the same denominator. So the middle point of a stage is a point of symmetry.

COROLLARY. If two conjugate stages are taken these their middle points are corresponding. (This follows from the definition). But their numerators should be equal. This is so, for the middle points are points of symmetry whose numerator is 1. This agrees with the result we proved.

DEFINITION. Two fractions $\frac{h}{k}$, and $\frac{h'}{k'}$, are conjugate in a conjugate stage if the distance of $\frac{h}{k}$ from $\frac{T_{i-1}}{T_n}$ equals the distance of $\frac{h'}{k'}$ from $\frac{T_{i+1}}{T_n} < \frac{h}{k} < \frac{h'}{k'}$ and the conjugate stage being $\left[\frac{T_{i-1}}{T_n}, \frac{T_i}{T_n}, \frac{T_{i+1}}{T_n}\right]$. Taking their middle points $\frac{1}{T_p}$, $\frac{1}{T_{p+1}}$ we can see that fractions conjugate in this interval are conjugate in the conjugate stage. Further we saw that for conjugate fractions of the interval $\frac{h}{k}$, and $\frac{h'}{k'}$. $\frac{h+h'}{k+k'}$ are fractions of the latter half of the interval arranged in descending order, and $\frac{h-h'}{k-k'}$ are fractions of the first half arranged in ascending order for increasing values of $\frac{h}{k}$.

THEOREM 3.8. For conjugate fractions $\frac{h}{k}$ and $\frac{h'}{k'}$ lying in the outer half of the stage we see that $\frac{h+h'}{k+k'}$ are fractions of the interval in ascending order while $\left|\frac{h-h'}{k-k'}\right|$ are fractions of the first half in descending order for increasing values of $\frac{h}{k}$. Diagrammatically we represent it as

$\overleftrightarrow{\text{interval}}$ $\frac{h}{k}, \frac{h'}{k'}$ in outer half

$\underset{\longrightarrow\ \longleftarrow}{\text{interval}}$ $\frac{h}{k}, \frac{h'}{k'}$ in inner half.

We here only give a proof to show that $\frac{h+h'}{k+k'}$, and $\frac{h-h'}{k-k'}$ are in the inner half but do not prove the order of arrangement.

Proof. For $\frac{h}{k}$, $\frac{h'}{k'}$, the proof has been given (previous paper). The middle point of $\left[\frac{T_{i-1}}{T_n}, \frac{T_i}{T_n}\right] = \frac{1}{T_{n-i+2}}$.

Similarly middle of point of $\left[\frac{T_i}{T_n}, \frac{T_{i+1}}{T_n}\right] = \frac{1}{T_{n-i+3}}$.

That two conjugate fractions of the outer half of a conjugate stage differ in suffix by 1 can be established.

$$\text{i.e. if } \frac{h}{k} = \frac{T_{j-(n-2)-1}}{T_j} \quad \text{then } \frac{h'}{k'} = \frac{T_{j-(n-i)}}{T_{j-1}}$$

$$\frac{h+h'}{k+k'} = \frac{T_{j-(n-i)+1}}{T_{j+1}} \in I \quad \text{where } I \text{ is the Interval.}$$

$$\text{and } \frac{h-h'}{k-k'} = \frac{T_{j-(n-i)-2}}{T_{j-2}} \in I \quad '' \quad '' \quad ''$$

Hence the proof.

DEFINITION. In an f.f. series order T_n $\frac{T_i}{T_n}, \frac{T_{i+1}}{T_n}$ represents a stage. Let us take an f.f. series of order T_{n+1}. If there we take a stage $\left[\frac{T_i}{T_{n+1}}, \frac{T_{i+1}}{T_{n+1}}\right]$, then we say the two stages are corresponding stages. More generally in an f.f. series or order T_{n+k}, $\left[\frac{T_i}{T_{n+k}}, \frac{T_{i+1}}{T_{n+k}}\right]$ is corresponding with reference to $\frac{T_i}{T_n}, \frac{T_{i+1}}{T_n}$ in the f.f. series of order T_n.

We state here properties of corresponding stages. These can be proved using 1.2.

THEOREM 2.4.a) If $\left[\frac{T_i}{T_n}, \frac{T_{i+1}}{T_n}\right]$ and $\left[\frac{T_i}{T_{n+k}}, \frac{T_{i+1}}{T_{n+k}}\right]$ stages then the number of terms in both are equal.

THEOREM 3.4.c. There exists a one-one correspondence between the denominators of these stages. If the denominator of the q^{th} term of $\left[\frac{T_i}{T_n}, \frac{T_{i+1}}{T_n}\right]$ is T_j then the denominator of the q^{th} term of $\left[\frac{T_i}{T_{n+k}}, \frac{T_{i+1}}{T_{n+k}}\right]$ is T_{i+k} .

We can extend this idea further and produce one-one correspondence between $\left[\frac{T_i}{T_n}, \frac{T_{i+m}}{T_n}\right]$ and $\left[\frac{T_i}{T_{n+k}}, \frac{T_{i+m}}{T_{n+k}}\right]$ where $\left[\frac{a}{b}, \frac{c}{d}\right]$ stands for the set of fractions between $\frac{a}{b}$ and $\frac{c}{d}$ inclusive of both. A further extension would give that given two f.f. series one of order T_n and other order T_{n+k}.

THEOREM 3.5.a. The numerator of the r^{th} term of the first series equals the numerator of the r^{th} term of the second.

THEOREM 3.5.b. If the denominator of the r^{th} term of the first series is T_j, then the denominator of the r^{th} term of the second is T_{j+k}. Precisely

a) nr of $f_{(r)n}$ = nr. of $f_{(r)n+k}$

b) If dr. of $f_{(r)n} = T_j$

dr. of $f_{(r)n+k} = T_{j+k}$

where nr stands for numerator and dr for denominator. This can be proved using 1.a. We can arrive at the same result by defining corresponding intervals.

DEFINITION. Two intervals, $\left[\frac{1}{T_i}, \frac{1}{T_{i+1}}\right]$ in an f.f. series of order T_n and $\left[\frac{1}{T_{i-k}}, \frac{1}{T_{i-k}}\right]$ in an f.f. series of order T_{n+k}. are defined to be corresponding intervals.

The same one-one correspondence as in the case of corresponding stage exists for corresponding intervals. We can extend this correspondence in a similar manner to the entire f.f. series and prove that

a) nr. of $f_{(r)n}$ = nr. of $f_{(r)n+k}$

b) If dr of $f_{(r)n} = T_j$

dr of $f_{(r)n+k} = T_{j+k}$.

c) Generalised f.f. series. We defined the f.f. series in the interval $[0,1]$. We now define it in the interval $[0,\infty)$.

DEFINITION. The f.f. series of order T_n is the set of all functions $\frac{T_i}{T_j}$ $j <$ n arranged in ascending order of magnitude i,j $>$ 0. If $i < j$ then the f.f. series is in the interval $[0,1]$. The basis properties of the f.f. series for 0,1 are retained with suitable alterations.

THEOREM 2.6.1. $f_{(r)n}$ is a point of symmetry if $f_{(r+1)n}$ and $f_{(r-1)n}$ have the same numerator (beyond $\frac{1}{1}$). If $f_{(r)n}$ is a point of symmetry then $f_{(r+k)n}$ and $f_{(r-k)n}$ have the same numerator, if each fraction does not pass beyond the next G.F.I. in either side (beyond $\frac{1}{1}$).

2) A G.F.I. is a fraction with denominator T_n.

3) An interval is the set of fractions between two consecutive points of symmetry.

4) A point of symmetry has either numerator or denominator 1.

5) Beyond $\frac{1}{1}$, any interval is given by $\left[\frac{T_{n-1}}{1}, \frac{T_n}{1}\right]$. The factor of this interval is again T_{n-2}.

6) The two basic properties

a) $\frac{h + h''}{k + k''} = \frac{h'}{k'}$ and

b) $kh' - hk' = T_{n-2}$ are retained.

7) If a) does not good for $\frac{h'}{k'}$ being a point of symmetry, then

$$\frac{h'}{k'} = \frac{T_{n-1}\ h'' + T_{n-2}\ h}{T_{n-1}\ k'' + T_{n-2}\ k} \quad \text{if } \frac{h}{k} < \frac{h'}{k'} < \frac{h''}{k''}\ ; \quad \left(\frac{h'}{k'} = \frac{T_n}{1}\right).$$

NOTE. Only in the case of $\frac{1}{2}$, and $\frac{1}{1}$ does a) hold good for $2 = 1+1$, and we do not accept 1 being split up.

ON FIBONACCI POLYNOMIALS AND THEIR GENERALISATION[Δ]

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Δ This paper was presented in a seminar at MATSCIENCE, The Institute of Mathematical Sciences, Madras-20.

ON FIBONACCI POLYNOMIALS AND THEIR GENERALISATION

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The Fibonacci Polynomials $F_n(x)$ are defined recursively by $F_0(x) = 0$, $F_1(x) = 1$, $F_n(x) = x\,F_{n-1}(x) + F_{n-2}(x)$. For complete information about Fibonacci Polynomials consult - 'A primer for Fibonacci Numbers' (Bicknell and Hoggatt). In this paper we try to answer the following questions:

I. What can be said about $\dfrac{d\,F_n(x)}{dx}$?

II. What can be said about a convolution of Fibonacci Polynomials - $\sum_{k=0}^{m} F_k(x)\,F_{m-k}(x)$?

In theorem 1 we give a simultaneous answer to these two questions. In the latter half of the paper we generalise $F_n(x)$ to $F_n(x,y)$ keeping theorem 1 in mind.

THEOREM 1. Let $f_n(x) = \sum_{k=0}^{n} F_k(x)\,F_{n-k}(x)$

then $f_n(x) = \dfrac{d\,F_n(x)}{dx}$.

<u>Proof</u>. We shall employ induction. The above theorem is true for $n = 0,1$. Now let the theorem hold for $n = 1,2,\ldots m$. Thus

$$\frac{d\,F_{m-1}(x)}{} = \sum_{k=0}^{m-1} F_k(x)\,F_{m-k-1}(x)$$

and

$$\frac{d\,F_m(x)}{dx} = \sum_{k=0}^{m-1} F_k(x)\,F_{m-k}(x)\,.$$

Now $$\frac{d\,F_{m+1}(x)}{dx} = \frac{d\left\{x\,F_m(x) + F_{m-1}(x)\right\}}{dx}$$

$$= x\,\frac{d\,F_m(x)}{dx} + F_m(x) + \frac{d\,F_{m-1}(x)}{dx}$$

Now $$\sum_{k=0}^{m+1} F_k(x)\,F_{m-k+1}(x) = \sum_{k=0}^{m+1} F_k(x)\left\{x\,F_{m-k}(x) + F_{m-k-1}(x)\right\}$$

$$= x\sum_{k=0}^{m+1} F_k(x)\,F_{m-k}(x) + \sum_{k=0}^{m+1} F_k(x)\,F_{m-k-1}(x)$$

$$= x\sum_{k=0}^{m} F_k(x)\,F_{m-k}(x) + x\,F_{m+1}(x)\,F_{-1}(x) + \sum_{k=0}^{m-1} F_k(x)\,F_{m-k-1}(x)$$

$$+ F_m(x)\,F_{-1}(x) + F_{m+1}(x)\,F_{-2}(x)$$

$$= x\,\frac{d\,F_m(x)}{dx} + \frac{d\,F_{m-1}(x)}{dx} + F_m(x)$$

by induction for the theorem works for $n = 1,2,\ldots m$ and $F_{-1}(x) = 1$, $F_{-2}(x) = -x$.

$$\therefore\ \frac{d\,F_{m+1}(x)}{dx} = \sum_{k=0}^{m+1} F_k(x)\,F_{m-k+1}(x).$$

Hence the theorem is true for $n = m+1$. Thus by mathematical induction it is true for all integral values of n.

<u>Note</u>:- Define $f_{-n}(x) = \sum_{k=0}^{n} F_{-k}(x)\,F_{-n+k}(x)$.

It is clear that $f_{-n}(x) = (-1)^n\,f_n(x) = (-1)^n\,\dfrac{d\,F_n(x)}{dx}$.

THEOREM 2. Define the r^{th} convolution sequence of polynomials by .

$$f_n^r(x) = \sum_{k=0}^{n} f_k^{r-1}(x)\, F_{n-k}(x) \qquad \left[f_n(x) = f_m'(x) \right]$$

Then $f_n^r(x) = \dfrac{d\, F_n(x)}{dx^r} \times \dfrac{1}{r!}$.

Proof. The proof of theorem 2 follows from a more general theorem on $F_n(x,y)$ given below.

Generalisation: We want a generalisation of $F_n(x)$ to $F_n(x,y)$ such that the convolution property in Theorems I and II hold. It is given as follows.

DEFINITION. Define $F_0(x,y) = 0$, $F_1(x,y) = 1$, and

$$F_n(x,y) = xF_{n-1}(x,y) + y\, F_{n-2}(x,y).$$

The following properties can be deduced easily.

a) $F_n(x,1) = F_n(x)$.

b) $F_{-n}(x,y) = \dfrac{(-1)^n\, F_n(x,y)}{y^n}$.

We shall discuss some properties of $F_n(x,y)$

THEOREM 3. $\left[\dfrac{\partial F_n(x,y)}{\partial y} \right]_{\text{at } y=1} = \dfrac{d\, F_{n-1}(x)}{dx}$.

Proof. $F_n(x,y) = x\, F_{n-1}(x,y) + y\, F_{n-2}(x,y)$.

Therefore $\dfrac{\partial F_n(x,y)}{\partial y} = x \dfrac{\partial F_{n-1}(x,y)}{\partial y} + F_{n-2}(x,y) + y \dfrac{\partial F_{n-2}(x,y)}{\partial y}$

. Let the theorem hold for $n^o = 1, 2, \ldots, m-1$. Now

$$\frac{\partial F_m(x,y)}{\partial y} = x \frac{\partial F_{m-1}(x,y)}{\partial y} + F_{m-2}(x,y) + y \frac{\partial F_{m-2}(x,y)}{\partial y}$$

Therefore $\left[\frac{\partial F_m(x,y)}{\partial y}\right]_{y=1}$ $= x \frac{d F_{m-2}(x)}{dx} + F_{m-2}(x) + \frac{d F_{m-3}(x)}{dx}$

$$= \frac{d F_{m-1}(x)}{dx} .$$

Thus the theorem is true by mathematical induction for positive integral values of 'n'.

THEOREM 4. $$\frac{\partial F_{n-1}(x,y)}{\partial x} = \frac{\partial F_n(x,y)}{\partial y}$$

Proof. Again use induction. Let the theorem hold for $n = 1,2,\ldots m-1$. By the definition of $F_n(x,y)$.

$$\frac{\partial F_m(x,y)}{\partial y} = x \frac{\partial F_{m-1}(x,y)}{\partial y} + F_{m-2}(x,y) + y \frac{\partial F_{m-2}(x,y)}{\partial y}$$

$$\frac{\partial F_{m-1}(x,y)}{\partial x} = x \frac{\partial F_{m-2}(x,y)}{\partial x} + F_{m-2}(x,y) + y \frac{\partial F_{m-3}(x,y)}{\partial x}$$

Clearly $\frac{\partial F_m(x,y)}{\partial y} = \frac{\partial F_{m-1}(x,y)}{\partial x}$.

Thus by induction the result holds for all positive integral values of 'n'.

THEOREM 5.

$$\frac{\partial F_n(x,y)}{\partial x} = \frac{\partial F_{n+1}(x,y)}{\partial y} = \sum_{k=0}^{n} F_k(x,y)F_{n-k}(x,y).$$

Proof. Let the theorem hold for $n = 0,1,\ldots m$. i.e.

i.e. $$\frac{\partial F_{m-1}(x,y)}{\partial x} = \sum_{k=0}^{m-1} F_k(x,y)\, F_{m-k-1}(x,y)$$

and $$\frac{\partial F_m(x,y)}{\partial x} = \sum_{k=0}^{m} F_k(x,y)\, F_{m-k}(x,y).$$

$$\frac{\partial F_{m+1}(x,y)}{\partial x} = x\,\frac{\partial F_m(x,y)}{\partial x} + F_m(x,y) + y\,\frac{\partial F_{m-1}(x,y)}{\partial x}$$

Now $$\sum_{k=0}^{m+1} F_k(x,y)\, F_{m-k+1}(x,y) = \sum_{k=0}^{m+1} F_k(x,y) \quad x\, F_{m-k}(x,y)$$

$$+ y\, F_{m-k-1}(x,y)$$

$$= x \sum_{k=0}^{m+1} F_k(x,y)\, F_{m-k}(x,y) + x\, F_{m+1} + x\, F_{m+1}(x,y)\, F_{-1}(x,y)$$

$$+ \sum_{k=0}^{m-1} F_k(x,y)\, F_{m-k-1}(x,y) + y\, F_m(x,y)\, F_{-1}(x,y)$$

$$+ y\, F_{m+1}(x,y)\, F_{-2}(x,y)$$

$$= x\,\frac{\partial F_m(x,y)}{\partial x} + F_m(x,y) + y\,\frac{\partial F_{m-1}(x,y)}{\partial x}$$

since the theorem is true for $n = 1, 2, \ldots m$, and $F_{-1}(x,y) = \frac{1}{y}$

$y\, F_{-2}(x,y) + F_{-1}(x,y)\, x = 0$. Hence

$$\frac{\partial F_{m+1}(x,y)}{\partial x} = \sum_{k=0}^{m+1} F_k(x,y) \quad F_{m-k+1}(x,y).$$

The theorem holds for $n = 0,1$. Hence by induction it is true for all positive integral values of 'n'.

THEOREM 6. Define $f_n^r(x,y) = \sum_{k=0}^{n} f_k^{r-1}(x,y)\ F_{n-k}(x,y)$

where $f_n(x,y) = f_n^{1}(x,y) = \sum_{k=0}^{n} F_k(x,y)\ F_{n-k}(x,y)$, $r \quad 1$.

Then $f_n^r(x,y) = \dfrac{\partial F_n(x,y)}{\partial x^r} \times \dfrac{1}{r!}$.

Proof. Let the theorem hold for $n = 0,1,2,\ldots\ m$ for a fixed 'r' . That is

$$\frac{\partial F_m(x,y)}{\partial x^r} = r! \sum_{k=0}^{m} f_k^{r-1}(x,y)\ F_{m-k}(x,y)$$

and
$$\frac{\partial F_{m-1}(x,y)}{\partial x^r} = r! \sum_{k=0}^{m-1} f_k^{r-1}(x,y)\ F_{m-k-1}(x,y).$$

Now
$$\frac{\partial F_{m+1}(x,y)}{\partial x^r} = x\ \frac{\partial F_n(x,y)}{\partial x^r} + r\ \frac{\partial F_n(x,y)}{\partial x^{r-1}} + y\ \frac{\partial F_{n-1}(x,y)}{\partial x^r}$$

and

$$r! \sum_{k=0}^{m+1} f_k^{r-1}(x,y)\ F_{m-k+1}(x,y) = r! \sum_{k=0}^{m+1} f_k^{r-1}(x,y) \quad x\ F_{m-k}(x,y)$$
$$+ y\ F_{m-k-1}(x)$$

$$= r! \sum_{k=0}^{m} f_k^{r-1}(x,y)\ F_{m-k}(x,y) + r!\ y \sum_{k=0}^{m-1} f_k(x,y)\ F_{m-k-1}(x,y)$$
$$+ r!\ x\ f_{n+1}^{r-1}(x,y)\ F_{-1}(x,y) + r!\ y\ f_y^{r-1}(x,y)\ F_{-1}(x,y)$$
$$+ r!\ y\ f_{n+1}(x,y)\ F_{-2}(x,y)$$

$$= x \frac{\partial F_n(x,y)}{\partial x^r} + y \frac{\partial F_{n-1}(x,y)}{\partial x^r} + \frac{r!}{(r-1)!} \frac{\partial F_{n+1}(x,y)}{\partial x^{r-1}}$$

$$= x \frac{\partial F_n(x,y)}{\partial x^r} + y \frac{\partial F_{n-1}(x,y)}{\partial x^r} + r \frac{\partial F_n(x,y)}{\partial x^{r-1}} = \frac{\partial F_{n+1}(x,y)}{\partial x^r}$$

Thus the theorem is true by induction for all positive integral subscribe (n) for each upto 'r'. But its true for (r+1) when n = 0,1. Thus by mathematical induction its true for al r = 1, and n 0.

NOTE. Theorem 2 follows by substituting y = 1.

We shall conclude by giving a generator matrix for $F_n(x,y)$.

Let $Q' = \begin{bmatrix} x & y \\ 1 & 0 \end{bmatrix}$

and $Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ the famous matrix.

THEOREM 7. $(Q')^{n-1} . Q = \begin{bmatrix} F_{n+1}(x,y) & F_n(x,y) \\ F_n(x,y) & F_{n-1}(x,y) \end{bmatrix}$

The proof is given by induction .

since $\left\{ \text{Det } (Q') \right\}^n = \text{Det } \left\{ (Q')^n \right\}$ we have

THEOREM 8. $F_{n+1}(x,y)\, F_{n-1}(x,y) - \left\{ F_n(x,y) \right\}^2 = (-1)\ (-y)^{n-1}.$

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Reference

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CONVOLUTION TRIANGLES OF FIBONACCI POLYNOMIALS

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Suppose we have a sequence of sets $\{A_n\}$ where A_n contains $\left[\frac{n+1}{2}\right]$ elements where $[x]$ stands for the largest integer $\le x$. Hence A_n can be written as

$$A_n = \left\{ a_{n,1}, a_{n,2}, \dots a_{n,\left[\frac{n+1}{2}\right]} \right\}.$$

These members of A_n can be successively written on rising diagonals on a triangle which has the shape of the Pascal triangle

$$\begin{array}{lll}
a_{11} & & \\
a_{21} & a_{32} & \\
a_{31} & a_{42} & a_{53} \\
a_{41} & a_{52} & \\
a_{51} & &
\end{array}$$

It can be seen easily that $f_n^r(x) = \sum_{k=0}^{n} f_k^{r-1}(x)\, F_{n-k}(x)$ is of the Fibonacci Polynomial type in the sense that if the coefficient of x^r is positive, that of x^{r-1} is zero, and x^{r-2} is non-zero. Hence if $f_n^r(x)$ is written in the form $a_0x^k + a_1x^{k-2} + \dots$ then the set $\{a_0, a_1, \dots\}$ is of the type A_n. Let us form a sequence of such sets from the first non-zero values of $f_n^r(x)$.

We call this as the r^{th} convolution tria le of Fibonacci Poly ai-als, represented thus

THEOREM. Take the n^{th} row of the th convolution trian e Let the elements in order be

$$b_{n,0}\ b_{n,1},\ b_{n,2},\ldots,\ b_{n,n}.$$

Then $b_{n,i} = {}^{n+r}C_r \cdot {}^{n}C_i$ where ${}^{n}C_k \quad \dfrac{n\,!}{k!\ (n-k)\,!}$.

Proof. The first non-zero value o $f_n^r(x)$ is when $n = r$ .. Moreover the members of n^{th} rising agonal of the r^{th} con olu-tion triangle. are the successive coeffi ents of $f^r_{n+r+1}(x)$. e theorem follow immediately from the foll ing result.

$$f^r_{n+r+1}(x) = \frac{d\ F_{n+r+1}(x)}{dx^r} \times \frac{1}{r!} = \frac{\sum_{j=0} \binom{n+r-j}{j} x^{n+r-}}{r\,!} \cdots$$

$$= \sum_{n+r-2j>0}' \frac{(n+r-2j)\,!}{r!\ (n-2j)\,!} \binom{n+r-j}{j} x^{n-2j}$$

This proves the theorem

Example:- The 1st convolution triangle i

1				
2	2			
3	6	3		
4	12	12	4	
5	20	30	20	5
.	.	.	.	.

Note 2,2 = 2(1,1), 3,6,3, = 3(1,2,1), 4,1 12,4 = 4(1,3,3,1)

Each row has a common factor.

Reference

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FIBONACCI NUMBERS AND HOGGATT'S ARRAYS

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Hoggatt [1], [2] proposed two arrays of numbers given below in connection with problem on Fibonacci numbers, and stated that the row sums in each array was a Fibonacci number.

1) Hoggatt Steps

1	1						
1	2						
2	4	1	1				
5	9	3	4				
13	22	7	11	[illegible]	1		
34	56	16	27	[illegible]	6		
89	145	38	65	1[illegible]	22	1	1
.	.	.	.		.	.	.

2) Array of hope

1					
1	1				
2	2	1			
5	4	3	1		
13	9	7	4	1	
34	22	16	11	5	1
.	.	.	.	.	.

.

Let ${}^{n}s_{r}$ represent the nth row th column element in [illegible] first array and ${}^{n}h_{r}$ the correspondin element in the second [illegible] Hoggatt steps is formed by the relation

$$(1) \quad {}^{n}s_{r+1} = {}^{n}s_{r} + {}^{n-1}s_{r+1} \qquad r \text{ being odd.}$$

and

$${}^{n}s_{r} + {}^{n+1}s_{r+1} = {}^{n+2}s_{r+1} \qquad r \text{ being even}$$

The members in the Hoggatt steps [illegible]e the same as those the Array of hope with a shift given by

$$(2) \quad {}^{n}s_{r} = {}^{n}h_{r} \qquad \text{if } r \text{ is odd}$$

$${}^{n}s_{r} = {}^{n+1}h_{r} \qquad \text{if } r \text{ is even}$$

The Array of Hope is formed with the rel[illegible]ion

$$(3) \quad {}^{n}h_{r} = {}^{n-1}h_{r} + {}^{n-1}h_{r-1}$$

We shall now state a number of id[illegible]tities which give va[illegible]ous methods of forming Fibanacci numbers wit the above arrays.

As stated by Hoggatt

$$(4) \quad {}^{n}h_{1} + {}^{n}h_{2} + {}^{n}h_{3} + \cdots \qquad F_{2n-1}$$

and

$${}^{n}s_{1} + {}^{n}s_{2} + {}^{n}s_{3} + \cdots \qquad F_{2n}$$

It is indeed remarkable, that we get Fibonacci numbers inspite of the transformation given in (2). We now strengthen (4) by stating

$$(5)\quad n_{h_1} + n_{h_3} + n_{h_5} + \cdots = F_{2n-2}$$

$$n_{h_2} + n_{h_4} + n_{h_6} + \cdots = F_{2n-3}$$

and

$$(6)\quad n_{s_1} + n_{s_3} + n_{s_5} + \cdots = F_{2n-2}$$

$$n_{s_2} + n_{s_4} + n_{s_6} + \cdots = F_{2n-1}$$

(5) and (6) clearly $\Rightarrow$ 4.

Consider the Pascal triangle given by

$$^{n}C_{r} = {}^{n-1}C_{r} + {}^{n-1}C_{r-1}, \quad ^{n}C_{r} = \frac{n!}{r!(n-r)!} \quad n, r \geq 0 \quad n \quad r$$

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1
.	.	.	.	.	.

We find that the following 5 identities hold

(7) $n_{s_2} - n_{c_0} = F_{2n-2} \qquad n>1$

(8) $n_{s_3} - {}^{n-2}c_1 = F_{2n-5} \qquad n>3$

(9) $n_{s_4} - {}^{n-3}c_2 = F_{2n-4} \qquad n>4$

(10) $n_{s_5} - {}^{n-4}c_3 = F_{2n-7} - 1 \qquad n>5$

(11) $n_{s_6} - {}^{n-5}c_4 = F_{2n-6} - (n-3) \qquad n>6$

Identities 7,8,9,10,11, also hold for n_{h_r} with the transformation (2) being introduced.

We first note that both in Hoggatt steps and in the Array of Hope the first column contained Fibonacci numbers

$F_{-1}, F_3, F_3, F_5, F_7, \ldots\ldots$.

We here form two new arrays with the first column having $F_2, F_4, F_6, F_8, F_{10}, \ldots\ldots$. We call these arrays as Hoggatt steps (2) and array of Hope (2).

Let us denote the rth column, nth row elements by $n_{s'_r}$ and $n_{h'_r}$ respectively.

Hoggatt Steps (2)

1	1				
3	4				
8	12	1	1		
21	33	5	6		
55	88	17	23	1	1
144	232	50	73	7	8
o	o	o	o	o	o

Array of Hope (2)

1					
3	1				
8	4	1			
21	12	5	1		
55	[illegible]	17	6	1	
144	[illegible]	50	23	7	1
o	o	o	o	o	o

Relation (2) holds here

(2') ${}^{n}s'_{r} = {}^{n}h'_{r}$ if r is odd

${}^{n}s'_{r} = {}^{n+1}h'_{r}$ if r is even

We have the Pascal identities for these arrays also.

(12) ${}^{n}h'_{2} + {}^{n}c_{0} = F_{2n-1}$ $\quad n > 2$

(13) ${}^{n}h'_{3} + {}^{n-1}c_{1} = F_{2n-2}$ $\quad n > 3$

(14) consider the sequence of numbers which when added to give Fibonacci numbers

$$4_{h'_4} + 5 = F_5 \ , \ 5_{h'_4} + 7 = F_7 \ , \ 6_{h'_4} + 11 = F_9 \ \cdots$$

In general

$${}^{n}h'_4 + U_n = F_{2n-3} \qquad \text{where } U_n \text{ is nth term of}$$

4,7,11,16,22, ...

It is interesting that $V_n = U_n - U_{n-1}$ is the sequence 3,4,4,5, 7,8, ...

Identities 12, 13, and 14 hold for ${}^{n}S'_r$ also, with the transformation introduced by (2)'.

It is true that in the two arrays introduced by us the row sums are not Fibonacci numbers. But we could get Fibonacci numbers by another method.

Let $H_n = {}^{n}h'_1 + {}^{n}h'_2 + \cdots {}^{n}h'_n$

Then

(15) $$H_n - 2H_{n-1} = F_{2n-1} \qquad n > 1$$

In fact if $H_n^o = {}^{n}h'_1 + {}^{n}h'_3 + {}^{n}h'_5 + \cdots$ and

$$H_n^e = {}^{n}h'_2 + {}^{n}h'_4 + {}^{n}h'_6 + \cdots \quad \text{then}$$

(16) $$H_n^o - 2H_{n-1}^o = F_{2n-2}$$

$$H_n^e - 2H_{n-1}^e = F_{2n-3} \qquad n \geq 2$$

which is similar to (5) and (6).

Further if

$$S_n = {}^nS'_1 + {}^nS'_2 + {}^nS'_3 + \dots {}^nS'_n \quad \text{and}$$

$$S_n^o = {}^nS'_1 + {}^nS'_3 + {}^nS'_5 + \dots$$

$$S_n^e = {}^nS'_2 + {}^nS'_4 + {}^nS'_6 + \dots$$

(17) $$S_n - 2S_{n-1} = F_{2n}$$

$$S_n^o - 2S_{n-1}^o = F_{2n-2} \qquad n \geq 1$$

$$S_n^e - 2S_{n-1}^e = F_{2n-1}$$

Let us now extend the Array of Hope (2) in a two dimensional pla and produce values of ${}^n\Lambda'_\gamma$, n, γ being integers

5	-8	12	-17	23	30	38
-2	-3	4	-5	6	-7	8
1	-1	1	-1	1	-1	1
1	0	0	0	0	0	0
2	1	0	0	0	0	0
5	3	1	0	0	0	0
13	8	4	1	0	0	0
34	21	12	5	1	0	0
89	55	33	17	6	1	0
233	144	88	50	3	7	1

It is evident that

$$(18)\quad {}^{n}h'_{1} + {}^{n+1}h'_{2} + {}^{n+2}h'_{3} + \cdots {}^{n+r}h'_{r+1} = {}^{n+r+1}h'_{r+1} - {}^{n}h_{0}$$

If $n < 0$, and $r = -n$ then (18) implies that the sum of numbers in every descending diagonal of the first quadrant is a Fibonacci Numbers. But surprisingly these numbers on the descending diagonals are the numbers on the rows of the Array of Hope (2) with alternate (-) signs. Hence transforming the numbers in the first quadrant a triangle we get

-1				
-3	1			
-8	4	-1		
-21	12	-5	1	
-55	33	-17	6	-1
- -	- -	- - -	- -	- -

This gives us the following result

$$\left| {}^{n}h'_{1} - {}^{n}h'_{2} + {}^{n}h'_{3} - {}^{n}h'_{4} + \cdots \right| = F_{2n-1} \qquad n > 0$$

This is analogous to (4).

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MAGIC SQUARES AND MATRIX TRANSFORMATIONS*

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MAGIC SQUARES AND MATRIX TRANSFORMATIONS

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INTRODUCTION

Magic squares have been a sort of Mathematical recreatio and for centuries people have known its mystic properties. In this paper we shall give a mathematical transformation which shall transform the normal $(n \times n)$ array of consecutive integers into a magic square. In the second part of this paper we shall study this transformation on matfices. Throughout we consider 'n' as an odd positive integer.

Part I. (Magic squares)

DEFINITION. We define N to be the n x n matrix (squar

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ n+1 & n+2 & \cdots & \cdots & 2n \\ \cdots & & & & \\ n(n-1)+1 & \cdots & \cdots & \cdots & n^2 \end{bmatrix}$$

DEFINITION 1.2. An element of the ith row jth column of a matrix A is denoted by (i,j) C A.

DEFINITION 1.3. If $(x_1, x_2, \ldots x_n)$ is an 'ntuple' we define the operator τ as

$$\tau(x_1, x_2, \ldots x_n) = (x_n, x_1, \ldots x_{n-1})$$

$$\tau^k(\vec{x}) = \tau(\tau \ldots \tau(\vec{x}) \ldots) \text{ k times.}$$

DEFINITION 1.4. Consider the s are N(n x n). Take th ith row elements, i.e. (i,1, (i,2).....(i,

Let $\tau^{k}(x_1, x_2, \cdot\cdot \quad n) = (x_1, x \quad , \cdot\cdot\cdot\ x_n)$

and $\tau^{-k}(1, 2, 3, \cdot\cdot\cdot n) = (x_1', \quad ', \cdot\cdot x_n')$

Form a new matrix under the mapping

$$(i, j) \longrightarrow (x_j, x_j')$$

and call it μ. N = M. We say M is a ma ic square of order (n : n)

THEOREM 1.1. The matrix M has the same elements of rearranged.

Proof. Since every element of N as been mapped in M ur theorem is proved if we show that no tw elements of M are pped to the same element in M. Let (i_1, j_1) nd (i_2, j_2) $\in$ N be ma ed to (i^*, j^*) $\in$ M. Then by Definition 1.4 we have

$$j_1 - \left(\frac{n+1}{2} - i_1\right) \equiv i^* \pmod{n} \qquad (1. \;)$$

$$j_1 + \left(\frac{n+1}{2} - i_1\right) \equiv j^* \pmod{n} \qquad (1.$$

$$j_2 - \left(\frac{n+1}{2} - i_2\right) \equiv i^* \pmod{n} \qquad (1.$$

$$j_2 + \left(\frac{n+1}{2} - i_2\right) \equiv j^* \pmod{n} \qquad (1.$$

Now (11 + (1.2) gives

$$2j_1 \equiv i^* + j^* \pmod{n} \qquad (1.$$

and (13) + (1.4) gives

$$2j_2 \equiv i^* + j^* \pmod{n} \qquad (1\ [illegible])$$

$$2j_1 \equiv 2j_2 \pmod{n} \text{ or } j_1 \equiv j_2 \pmod{\ } \text{ for } n \text{ is odd} \qquad (1\ [illegible])$$

|||y $$i_1 \equiv i_2 \pmod{n} \qquad (1\ [illegible])$$

But i_1, i_2, j_1, j_2 are all $\leq$ n and [illegible] . Hence $i_1 = i_2$ [illegible]nd $j_1 = j_2$. This means the theorem is [illegible]ue.

THEOREM 1.2. The elements of t[illegible] ith row of M form a [illegible]et of incongruent residues modulo n.

The elements of the ith column [illegible] M form a set of incon[illegible]uent residues modulo n.

<u>Proof</u>. Let (i, j_1) and (i, j_2) [illegible] M and let them belo[illegible] to the same residue class modulo n. Let [illegible]$_3, j_3)$, $(i_4, j_4) \in N$ [illegible] be mapped to (i, j_1) $(i, j_2) \in M$ respect[illegible]ely. Then

$$j_3 - \left(\frac{n+1}{2} - i_3\right) \equiv i \pmod{n} \qquad (1.\ [illegible])$$

$$j_3 + \left(\frac{n+1}{2} - i_3\right) \equiv j_1 \pmod{n} \qquad (1.\ \ )$$

$$j_4 - \left(\frac{n+1}{2} - i_4\right) \equiv i \pmod{n} \qquad (1.\ \ )$$

$$j_4 + \left(\frac{n+1}{2} - i_4\right) \equiv j_2 \pmod{n} \qquad (1.\ \ )$$

Since (i, j_1) and (i, j_2) belong to the s[illegible] residue class $j_3 =$ [illegible] .
Hence (1.9) - (1.11) gives

$$i_3 - i_4 \equiv 0 \pmod{n} \text{ and } i_3, [illegible] \leq n \qquad (1.[illegible])$$

so $i_3 = i_4$ and $j_3 = j_4$. This implies t the same element c N is mapped to two elements of N. This is ontradiction. Hence he first statment of Theorem 1.2 is correct.

The second statement might be prov by replacing (i, j_1) d (i, j_2) by (i_1, j) and (i_2, j) respectivel Hence the theorem proved.

THEOREM 1.3. No two elements of t same row in M belon to the same row in N.

No two elements of the same columr f M belong to the ne row in N.

Proof. Let (i_3, j_3), (i_4, j_4) $\in$ N e mapped to (i, j_1) a (i, j_2) $\in$ M. This gives rise to equation .9), (1.10), (1.11) (1.12). If (i, j_1) and (i, j_2) belong to e same row in N. Th $i_3 = i_4$. Now (1.9) - (1.11) gives

$$j_3 \equiv j_4 \pmod{n} \quad \text{and} \quad j_3, j_4 \leq \qquad (1.14$$

which is again a contradiction. Hence the eorem.

THEOREM 1.4. The sum of the numbe in any row, column o diagonal of M is the constant $\frac{n(n^2 + 1)}{2}$

Proof. Take the ith row of M co isting of elements

$$(i,1) \; (i,2) \; \ldots\ldots \; (i,n)$$

Let $(i,j) = a_j \, n + b_j$ where $< b_j < n$.

Theorem (1.2) says that the b s are 1,2,3,....n i some order, and Theorem 1.3 says that the j's are 0,1,2,....n- in some order.

Hence $\sum_{j=1}^{n} a_j n + b_j = \frac{n(n^2 \cdot 1)}{2}$.

The same result holds for the c umns also. As far as he diagonals of M are concerned, they a the elements of the iddle row and middle column of N. Hence the um here is also $\frac{n(n^2 - 1)}{}$ which proves the theorem.

DEFINITION 1.5. Let (i,j) and i*,j*) be elements of n (n x n) matrix. If i + i* = j + j* = + 1 we say they are corresponding elements.

THEOREM 1.5. Let (i,j), (i*,j* ∈ N and let them be rres- ponding. If (i,j) and i*,j*) are ma d to (p,q) and (p*,q in M then (p,q) and p*,q*) are correspor ing elements and conv sely.

<u>Proof.</u> We have the following fo congurences

$$j - \left(\frac{n+1}{2} - i\right) \equiv p \pmod n \qquad (1.\ 5)$$

$$j + \left(\frac{n+1}{2} - i\right) \equiv q \pmod n \qquad (1.\ 6)$$

$$j^* - \left(\frac{n+1}{2} - i^*\right) \equiv p^* \pmod n \qquad (1.\ 7)$$

$$j^* + \left(\frac{n+1}{2} - i^*\right) \equiv q^* \pmod n \qquad (1.\ 8)$$

Now (1.15) + (1.17) gives

$$(j+j^*) + (i+i^*) - (n+1) \quad p+p^* \ (\ \text{ode } n) \qquad (1.\ \ )$$

But $i+i^* = j+j^* = n+1$. Hence $p+p^* = n$. This implies $q+q^*$ $n+1$. Hence (p,q) and (p^*,q^*) are correspondi elements in M. The converse is also true since there are a any corresponding el ents in M as in N.

THEOREM 1.6. Let (i,j) and i^*,j be elements of M. I ; (p,q) and $p^*,q^*)$ be their corresponding lements. Then

$$(i,j) - (p,q) = (p^*,q^*) - (i \quad j^*)$$

or $$(i,j) + (i^*,j^*) = (p,q) + (p^*, \quad) = n^2 + 1.$$

<u>Proof.</u> Let (i_1,j_1) (i_1^*,j_1^*) $(p_1 \quad)$, $p_1^*,q_1^*)$ $\in$ N be ma ed to (i,j) (i^*,j^*) (p,q) (p^*,q^*) respecti ly. Then (i_1,j_1) and (p_1,q_1) are corresponding. Further $(i_1 \quad)$ and (p_1,q_1) are $(i_1-1)n+j_1$ and $(p_1-1)+q_1$ respectively. imilarly (i_1^*,j_1^*) and (p_1^*,q_1^*) are also corresponding and are al respectively to $(i_1^* - 1)n+j_1^*$, $(p_1^* - 1)n+q_1^*$.

then $$(i_1,j_1) + (p_1,q_1) = n^2 + 1$$

$$(i_1^*,j_1^*) + (p_1^*,q_1^*) = n^2 + 1.$$

Thus

$(i,j) + (p,q) = (i^*,j^*) + (p^*,q^*) = n^2 +$, which proves Theor 1.6.

DEFINITION 1.6. If $0 < K <$. We define D_K to b the Kth off diagonal of M consisting of ments $(1,K+1)$ $(2,k+2$ ... $(n-k,n)$ $(n-k+1,1)$ $(n-k-1,2,2)$ .... (n,k) We call these $D_{k,1}$, $_{,2}$, ... $D_{k,n}$ respectively.

THEOREM 7. If the residues mod o n of $D_{k,1} \ldots . D_{k,n}$ are taken in order they are

$$\tau^{n'k}(1,2,\ldots n) \quad \text{where} \quad n' = \frac{1}{} \quad \text{or} \quad 2n' + 1 = n.$$

Proof. Let j n-k and take he element $D_{k,j} = (\quad +j) \in M$. We would have to show that

$$D_{k,j} \equiv j-n'k \pmod{n} \qquad .20)$$

Let $(i_1 j_1)$ be the element mapped to $I_{,j}$. We have

$$j_1 - (\frac{n+1}{2} - i_1) \equiv j \pmod{n} \qquad .21)$$

$$j_1 + (\frac{n+1}{2} - i_1) \equiv k+j \pmod{n} \qquad .22)$$

giving $$2j_1 \equiv (n+1)-k \pmod{n}$$

$$2j_1 \equiv 2j+k \pmod{n}.$$

Now $2D_{k,j} \equiv n(i_1-1) + 2j + k \equiv 2(\quad n'k) \pmod{n}$ where $n \quad = \frac{n-1}{2}$.

Hence $D_{k,j} \equiv j-n'k \pmod{n}$ for n i odd. As for the case $n > j > n-k$ the proof is similar. nce the theorem

THEOREM 1.8. In the Kth off d gonal $D_k \in M$ all th elements belong to the same row in N.

We know that $D_{k,1}, D_{k,2}, \ldots . D_{k,}$ are the elements $(1, \quad 1)$ $(2,k+2) \ldots (n-k,k)$ $(n-k+1,1) \ldots (n,$. It can be shown th if $(i,j) \in N$ is mapped to $(1,k+1)$ then $(\quad j+1)$ is mapped to $(2, \quad 2)$, $(i,j+2)$ to $(3,k+3), \ldots$ where the $j+p$'s are reduced modulo n, ince the following congruences are simultaneou y satisfies

$$j + k - (\frac{n+1}{2} - i) \equiv k + 1 \quad (d\ n)$$

$$j + k + (\frac{n+1}{2} - 1) \equiv 2k + 1 \ (d\ n) \quad k = 1,2,3,\ldots .$$

COROLLARY. As a corollary to the ove theorem we may de ce the following result. If we Define D_k^c be the centre of the K^{th} off diagonal if it lies on D_k and on he diagonal (1,n),(n.) then

$$(1,k+1) + (n-k,k) = 2\,D_k^c$$

$$(2,k+2) + (n-k-1, k-1) = 2$$

.... for one part f D_k and

$$(n-k+1,1) + (n,k) = 2\,D_k^c$$

$$(n-k+2) + (n-k,k-1) = 2D_k^c$$

.... in the other par

Proof. The proof of this corollary s given by combining the statements of Theorem 1.8 and 1.9.

Part II: MATRIX TRANSFORMATIONS

We shall now investigate some inte ting invariants unde the transformation on N, generalised now t any matrix. Hence g en a matrix A of dimension (n x n) n being d we form μ.A -

DEFINITION 2.1. Four members of a matrix A $(i_1,j_1)(i_2, j_2)$ (i_3,j_3) and (i_4,j_4) preserve symmetry S_1 if

$$i_1 - i_3 \equiv j_2 - j_4 \pmod n \qquad (2.1)$$

$$i_2 - i_4 \equiv j_3 - j_1 \pmod n \qquad (2.2)$$

THEOREM 2.1. If four elements of matrix A satisfy S_1 then when mapped to $\mu.A$, they still satisfy S_1 and viceversa.

__Proof.__ We prove the converse first. Let (i_k^*, j_k^*) $k = 1,2,3,4 \in A$ and be mapped to (i_k, j_k) $k = 1,2,3,4 \in \mu.A$. Further let (i_k, j_k) $k = 1,2,3,4$, preserve S_1. Then we have

$$j_k^* - (\tfrac{n+1}{2} - i_k^*) \equiv i_k \pmod n \qquad (2.3)$$

$$j_k^* + (\tfrac{n+1}{2} - i_k^*) \equiv j_k \pmod n)\ k = 1,2,3,4. \qquad (2.4)$$

Now

$$i_1 - i_3 \equiv j_1^* + i_1^* - j_3^* - i_3^* \pmod n \qquad (2.5)$$

and

$$j_2 - j_4 \equiv j_2^* - i_2^* - j_4^* + i_4^* \pmod n \qquad (2.6)$$

But (2.1) we have

$$(j_1^* - j_3^*) + (i_1^* - i_3^*) \equiv (j_2^* - j_4^*) - (i_4^* - i_2^*) \pmod n \qquad (2.8)$$

Now (2.7) and (2.8) give

$$2(i_1^*-i_3^*) \equiv 2(j_2^*-j_4^*) \pmod n \Rightarrow i_1^* - i_3^* \equiv j_2^* - j_4^* \pmod n$$

and

$$2(i_2^*-i_4^*) \equiv 2(j_3^*-j_1^*) \pmod n \Rightarrow i_1^* - i_4^* \equiv j_3^* - j_1^* \pmod n$$

for 'n' is odd. Hence S_1 is preserv . The statement of Theorem 2.1 is true for there are just many elements in μ.. preserving S_1 as there are in A.

DEFINITION 2.2. Four elements ($, j_k)$ $k = 1, 2, 3, 4$, in matrix A(n x n) are said to preserve if

$$i_1 + i_3 \equiv i_2 + i_4 \quad (\text{mod} \quad) \qquad (2 \; 9)$$

$$j_1 + j_3 \equiv j_2 + j_4 \quad (\text{mod} \quad) \qquad (2 \; .0)$$

THEOREM 2.2. If four elements o a Matrix A satisfy S then when mapped to μ.A they still sat y S_2 and vice vers

Proof. Let (i_k^*, j_k^*) $k = 1, 2, 3, 4$ A be mapped respect ely to $(i_k^*, j_k^*) \in \mu$.A. Further let (i_k, j_k) $= 1, 2, 3, 4$ preserve . Then

$$j_k^* - \left(\frac{n+1}{2} - i_k^*\right) \equiv i_k \quad (\text{mod} \qquad (2 \; 1)$$

$$j_k^* + \left(\frac{n+1}{2} - i_k^*\right) \equiv j_k \quad (\text{mod} \qquad (2. \; 2)$$

Now (2.11) and (2.12) together with (2.9 give

$$(i_1^* + i_3^*) + (j_1^* + j_3^*) \equiv (i_2^* + i_4^*) \quad (j_2^* + j_4^*) \quad (\text{mod } n) \qquad (2. \; 3)$$

and (2.11) and (2.12) together with (2.1(give

$$(i_1^* + i_3^*) - (j_1^* + j_3^*) \equiv (i_2^*, i_4^*) - (\quad + j_4^*) \quad (\text{mod } n) \qquad (2. \quad)$$

which is

$$i_1^* + i_3^* \equiv i_2^* + i_4^* \quad (\text{mod}$$

$$j_1^* + j_3^* \equiv j_2^* + j_4^* \quad (\text{mod} \quad r$$

This is precisely S_2. The statement of Theorem 2.2. is obvio ly true for there are just as many elements n μ.A preserving as there are in A.

DEFINITION 2.3. Four elements of matrix $A(n \times n)$ $(i_k \ \ _k)$ $k = 1,2,3,4$ preserve S_3 if

$$i_1 - i_3 \equiv i_2 - i_4 \quad (\text{mod} \qquad (2. \ 5)$$

$$j_1 - j_3 \equiv j_2 - j_4 \quad (\text{mod} \qquad (2. \ 5)$$

THEOREM 2.3. If four elements of matrix A preserve S_3 then when mapp-ed to μ.A, they still pr erve S_3 and convers y.

<u>Remark</u>. Let us conceive of figur inside a m trix. Le us say four elements of a matrix (i_k, j_k) f a figure (quadrila te l or rhombus) if their corresponding posit ons on a lattice form a quadrilateral or rhombus. Squares with v tices (i_k, j_k) $k = 1, \ 3,4,$ satisfy the relation

$$i_1 - i_3 = j_2 - j_4$$

$$i_2 - j_4 = j_3 - j_1 \quad \text{which} \quad S_1. \qquad (2.\ )$$

A parallellogram satisfies

$$i_1 + i_3 = i_1 + i_4$$

$$j_1 + j_3 = j_2 + j_4 \quad \text{which} \quad S_2 \qquad (2.1$$

In general we may say that our transforma on maps <u>mostly</u> parall ram to parallelograms, squares to squares etc.

DEFINITION 2.5. An alternative de nition for correspond g elements could be given

Let Δ be the distance function ch that

$$\Delta\Big((i_1,j_1),\ (i_2,j_2)\Big) = \langle i\ \ j_1\rangle - \langle i_2,j_2\rangle$$

where $\langle a,b\rangle = (a-1)n+b$

We say two elements $(i_1,j_1)\ (i_2,j_2) \in A$ are corresponding if

$$\Delta\Big((i_1,j_1),\ (\tfrac{n+1}{2},\ \tfrac{n+1}{2})\Big) = \Delta\Big((\tfrac{n+1}{2},\ \tfrac{n+}{2}\ \ ,\ (i_2,j_2)\Big)\ . \qquad (2.\ \ )$$

THEOREM 2.5. Definition 1.5 $\Longleftrightarrow$ finition 2.5.

Proof.

$$\Delta\Big((i_1,j_1),\ (\tfrac{n+1}{2},\ \tfrac{n+1}{2})\Big) = (\ \ -1)n+j_1 - \tfrac{n(n-1)}{2} - n$$

$$\Delta\Big((\tfrac{n+1}{2},\ \tfrac{n+1}{2}),\ (i_2,j_2)\Big) = \ \ \tfrac{\ -1)}{} + \tfrac{n+1}{2} - (i_2-1)n\ \ \ j_2.$$

Clearly equation (2.19) holds if $i_1 + i_2\ \ \ j_1 + j_2 = n+1$.

Hence Definition 1.5 $\Longrightarrow$ Definit n 2.5.

Now let equation (2.19) hold, so that

$$n(i_1 + i_2 - 2) + j_1 + j_2 = n^2\ \ 1 \qquad (2.$$

let $i_1 + i_2 \neq n + 1$ but $i_1 + i_2 = n\ \ 1 + \delta,\ \ \delta > 0$.

This implies that (2.20) reduces to

$$n(n - 1 + \delta) + j_1 + j_2 = \ ^2 + 1 \qquad (2.$$

This is false for $j_1 + j_2 > 1$ an $n(n-1+\delta) > n^2$ since > 0.

Now let $i_1 + i_2 = n + 1 + \delta,\ \ \delta\ \ 0$.

Now $j_1 + j_2 < 2n-1$ and $\delta <$ hich implies that the left hand side $< n^2 - 1$. Hence $\delta = 0$ $i_1 + i_2 = n + 1$.

Hence $j_1 + j_2 = n + 1$.

Thus Definition 2.5 $\Longrightarrow$ Definiti 1.5, and the theore is proved.

ON A GENERALISATION OF THE E ER FUNCTION*

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ON A GENERALISATION OF THE EULER FUNCTION*

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****

The Euler ϕ function is the first function antrant to number theory comes across. In this paper we study a generalisation of the Euler function and discuss some interesting properties of 'error functions' defined through this generalisation.

DEFINITION. 1** Let $\phi(n,x)$ represent the number of $m \leqslant x$, $(m,n) = 1$ $m, n \in Z^+$. This implies that

$$\phi(n,x) = \phi((n)\,[x]) = \sum_{\substack{m \leqslant x \\ (m,n) = 1}} 1 \tag{1}$$

where $[x]$ stands for the largest integer $\leqslant x$.

Further if $n \in Z^+$ and $n = \sum_{i=1}^{r} p_i^{\alpha_i}$ where the p_is are distinct primes and $\alpha_i \in Z^+$ then

$$\phi(n,x) = [x] - \sum_{0<i\leq r}{}^{'} \left[\frac{x}{p_i}\right] + \sum_{0<i<j\leq r}{}^{'} \left[\frac{x}{p_i p_j}\right] \cdots (-1)^r \sum{}^{'} \left[\frac{x}{p_1 \cdots p_r}\right]. \tag{2}$$

DEFINITION 2. If $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, $\psi(n) = 2^r$, $\psi(n) = 1$ if $n = 1$. Definition 2 together with (2) gives

$$\frac{x}{n}\phi(n) - \psi(n) < \phi(n,x) < \frac{x}{n}\phi(n) + \psi(n) \tag{3}$$

$$\text{and} \quad \psi(n) \leq \tau(n) = O(n^{\epsilon}) \quad \forall \epsilon > 0 \ [1] \tag{3'}$$

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* Work done, while the author participated in the International Summer Institute in Number Theory (197[illegible]) at Ann Arbor, Michigan.

** the generalisation in Definition 1 is attributed to Cesaro, but the considerations of this paper are different.

Further it is clear from (2) and (3) that

$$\lim_{x\to\infty} \frac{\phi(n,x)}{\phi(n)} \Big/ \frac{x}{n} \longrightarrow 1 \tag{4}$$

We shall now need two more properties of $\phi(n,x)$ to prove the theorems that follow. Since $(m,n) = 1$ $m \leq n$ $m,n \in Z+$ implies $(n-m,n) = 1$ we have

$$\phi(n,x) + \phi(n,n-x) = \phi(n) + 1 \quad \text{if } (n,x) = 1 \tag{5}$$

and

$$\phi(n,x) + \phi(n,n-x) = \phi(n) \quad \text{otherwise} \tag{6}$$

Let us now define the error function. Since (4) holds one may get a rough estimate of $\phi(n,x)$ by assuming $\phi(n,x) = \frac{x}{n}\phi(n)$. Let us make an estimate of the error made in this assumption, by the following definition.

DEFINITION 3. $e(n,x) = \frac{x}{n}\phi(n) - \phi(n,x)$

THEOREM 1. a) $\displaystyle\sum_{x=0}^{n} e(n,x) = -\frac{\phi(n)}{2}$ $\quad n \geq 2$

b) $\displaystyle\sum_{x=0}^{N} e(n,x) = -\frac{\phi(n)}{2} \times \left[\frac{N}{n}\right] + O(n^{1+\epsilon})\ \forall \epsilon>0$

where $[x]$ stands for the greatest integer $\leq x$.

Proof. for $x \in Z^+$ $x \leq n$ we have

$$e(n,x) + e(n,n-x) = \frac{x}{n}\phi(n) - \phi(n,x) + \frac{n-x}{n}\phi(n) - \phi(n,n-x)$$

$$= \phi(n) - \phi(n,x) - \phi(n,n-x)$$

$$= 0 \text{ if } (x,n) > 1$$

$$= -1 \text{ if } (x,n) = 1.$$

Now it is clear that

$$\sum_{x=1}^{n}{}' e(n,x) = \sum_{\substack{x \leq n \\ (x,n)=1}}{}' e(n,x) = - \frac{\phi(n)}{2}$$

Part b can be proved using (3)'' and the property

$$e(n, \lambda n + x) = e(n,x) \quad \text{if } \lambda \in Z^{+} \tag{A}$$

THEOREM 2.

a) $\displaystyle\int_0^n e(n,x)\,dx = 0$ b) $\displaystyle\int_0^N e(n,x)\,dx = O(n^{1+\epsilon}) \;\forall \epsilon > 0$

<u>Proof.</u>

$$\int_0^n e(n,x)\,dx = \int_0^n \frac{x}{n}\phi(n) - \phi(n,x)\,dx$$

$$= \int_0^n \frac{x}{n}\phi(n)\,dx - \int_0^n \phi(n,x)\,dx$$

$$= S_1 - S_2 \text{ respectively.}$$

Clearly $S_1 = \dfrac{n\,\phi(n)}{2}$. We need an estimate of S_2. Since $\phi(n,x)$ is a step function with jumps at every $m \leq n$, $(m,n) = 1$. We have

$$\int_0^n \phi(n,x)\,dx = \sum_{r=1}^{\phi(n)}{}' (n_r - n_{r-1}) \quad \phi(n, n_{r-1})$$

where $n_0, n_1, \ldots n_{\phi(n)-1}$ are the integers $\leq n$ coprime to n, (in ascending order) and $n_{\phi(n)} = n$.

S_2 now reduces to

$$S_2 = \sum_{r=1}^{\phi(n)} (n_r)\phi(n, n_{r-1}) - n_{r-1}\phi(r \quad r-1)$$

$$= \sum_{r=1}^{\phi(n)} \left\{ n_r \phi(n, n_r) - n_{r-1}\phi(n, n_{r-1}) \right\} - \sum_{r=1}^{\phi(n_1)} n_r \phi(n, n_{r-1}, n_r$$

where $\phi(n, a, b)$ is the number

$a < m \leq b, \ (m, n) = 1.$

$$= n\phi(n) - 1 \ \frac{-n\phi(n)}{2} + 1$$

$$S_2 = \frac{n\phi(n)}{2} = \int_0^n \phi(n, x)\, dx$$

Hence $S_1 - S_2 = 0$ which proves theorem a. Part b follows from (3) A

We shall now use the error functi to show

THEOREM 3. $\lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m} \frac{\phi(n}{n} \longrightarrow \frac{6}{\pi^2}$

Proof. $e(n,m) = \frac{m}{n} \phi(n) - \phi(n.$

Hence

$$m \sum_{n=1}^{m} \frac{\phi(n)}{n} = \sum_{n=1}^{m} e(n,m) + \sum_{=1}^{n} \phi(n,m).$$

$$= S_3 + S_4 .$$

We shall first evaluate S_4. If represents the numb of fractions in a Farey sequence of orde m then it is a clas ical result that

$$F_m = \sum_{n=1}^{m} \phi(n)$$

If we denote by $\phi(n,a,b)$ the number of $a < m \leq b$ such tha $(m,n) = 1$ then

$\phi(n,m) = \phi(n) + \phi(n,n,$[illegible] . The S_4 is

$$S_4 = \sum_{n=1}^{m}{}' \phi(n) + \sum_{n=1}^{m}{}' \phi(n,n,m) = F_{[illegible]} + \sum_{n=1}^{m}{}' \phi(n,n,m).$$

Now $\phi(n,n,m)$ counts the number of $n < m' \leq m$ $(m',n$ [illegible] 1

Or $\frac{n}{m'} < 1$ and is in its lowest order. This is a Farey fraction of order $\leq m$. Now every Farey fraction $\frac{a}{b}$, $b \leq m$ is counted by $\phi(a,a,n)$. This clearly implies that

$$\sum_{n=1}^{m}{}' \phi(n,n,m) = F_m \qquad)$$

Hence $S_4 = 2F_m = \frac{6m^2}{\pi^2} + O(m \log m)$ [1]

Let us now estimate S_3. Since $|e(n,$[illegible]$| \leq \psi(n)$ and $\psi(m) \leq$ [illegible] $m)$ by Definition (2), we have that

$$|e(n,m)| = O(n^{\epsilon}) \text{ for every } \epsilon > 0 \text{ since}$$

$$\tau(m) = O(n^{\epsilon}) \text{ for every } \epsilon > 0 \quad [1]$$

Hence $S_4 = \sum_{n=1}^{m}{}' e(n,m) = O(m^{1+\epsilon})$ $\forall \epsilon > 0$.

Thus

$$m \sum_{n=1}^{m}{}' \frac{\phi(n)}{n} = O(m^{1+\epsilon}) + \frac{6m}{\pi^2} + O(m \log m) .$$

or $$\frac{1}{m} \sum_{n=1}^{m}{}' \frac{\phi(n)}{n} = \frac{O(m^{1+\epsilon})}{m^2} + \frac{6}{\pi^2} + O\left(\frac{\log m}{m}\right)$$

Hence $$\lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m}{}' \frac{\phi(n)}{n} = \frac{6}{\pi^2}$$

(Note $\frac{6}{\pi^2}$ is the famous limit $\frac{1}{\sum_{n=1}^{\infty}{}' \frac{1}{n^2}}$).

Let us now consider an error func on of three variables
From (2) it is evident that

$$\lim_{\substack{a \to \infty \\ \text{or } b \to \infty}} \frac{\phi(n, a+b)}{\phi(n,a) + \phi(n,b)} \longrightarrow . \quad n \in Z^+$$

Thus an approximate estimate of $\phi(n,a+$ would be $\phi(n,a+b =$ $\phi(n,a) + \phi(n,b)$. The following err function gives us a estimate of this error.

DEFINITION 4. Let $e(n,m,i) = \phi\ n,m+i) - \phi(n,m) -$ $\phi(n,i)$ $n,mi \in Z^+$

THEOREM 4. a) $\sum_{i=0}^{n} e(n,m,i)$ $n\ e\ (n,m)$

b) $\sum_{i=0}^{N} e(n,m,i) = \left[\frac{N}{n}\right] ne(n,m) + 0($

Proof. a) $\sum_{i=0}^{n} e(n,m,i) = \sum_{i=}^{n-} e(n,m,i)$ since

$e(n,m,n) = 0$, for $(n',n) = 1$ $n' < n$ gives $(n'+n,n) = 1$.

Now
$$e(n,m,i) = \sum_{i=0}^{n-1} \phi(n,m+i - \phi(n,m) - \phi(n,i)$$

$$= \sum_{i=0}^{n-1} \phi(n,i) + m\ \phi(n) - n\ \phi(n,m) - \sum_{=0}^{-1} \phi(n,i)$$

$$= n \left\{ \frac{m}{n} \phi(n) - \phi(n, \right\} = n\ e(n,m).$$

Part b follows from (A) for $e(n,m,i) = e(n\ \ ) + e(n,i) - e(n,m+i)$

THEOREM 5. $\int_0^n e(n,m,x)\,dx = ne(n,m)$

b) $\int_0^N e(n,m,x)\,dx = \left[\frac{N}{n}\right] ne(n,m) + O(n)$

where $e(n,m,x) = \phi(n,m,x) - \phi(n,m) - \phi(n,x)$, $m \in Z^+$ m n.

<u>Proof.</u> $\int_0^n e(n,m,x)\,dx = \int_0^n \phi(n,m+x) - \phi(n,m) - \phi(n,x)\,dx.$

$$= S_5 - S_6 - S_7.$$

Since $\phi(n,n+x) = \phi(n) + \phi(n,x)$ we havd

$$S_5 = \int_0^n (n,m+x)\,dx = \int_0^n \phi(n,x)\,dx + \int_0^m \phi(n)\,dx = S_7 + m\phi(n)$$

$$S_6 = \int_0^n \phi(n,m)\,dx = n\,\phi(n,m).$$

Hence $S_5 - S_6 - S_7 = n\,\phi(n) - n\,\phi(n,m) = n\,e(n,m).$

Note that the summation over the discrete as well as over the continuous variable of $e(n,m,x)$ gives the same quantity. That is

$$\sum_{i=0}^{n} e(n,m,i) = \int_0^n e(n,m,x)\,dx = ne(n,m).$$

Let us now find upper and lower bounds for the average value of the error functions,

$e(n,m,x)$ and $e(n,x)$ where x is fixed.

THEOREM 6. $\displaystyle \lim_{m \to \infty} \sup \frac{1}{m} \sum_{n=1}^{m+i} e(n,\ ,i) \leq i - \frac{i\,\phi(i!)}{i!}$

$$\lim_{m \to \infty} \inf \frac{1}{m} \sum_{n=1}^{m+i} e(n,\ ,i) \geq -i + \frac{1}{2}\left[\frac{i}{2}\right]$$

Proof. We have

$$\sum_{n=1}^{m+i}{}' e(n,m,i) = \sum_{n=1}^{m+i}{}' (\phi(n,m+i) - \phi(n,m) - \phi(n,i))$$

$$= \sum_{n=1}^{m+i}{}' \phi(n,m+i) - \sum_{n=1}^{m}{}' \phi(n,m) - \sum_{n=m+1}^{m+i}{}' \phi(n,m) - \sum_{n=1}^{m+i}{}' \phi(n,i)$$

$$= 2F_{m+i} - 2F_m - \sum_{n=m+1}^{m+i}{}' \phi(n,m) - \sum_{n=1}^{m+i}{}' \phi(n,i)$$

since $\sum_{n=1}^{n'}{}' \phi(n,n') = 2F_{n'} = 2\sum_{n=}^{n'} \phi(n)$

By (7) and by (5,6) we have

$$\sum_{n=m+1}^{m+i} \phi(n,m) = \sum_{n=m+1}^{m+i}{}' \phi(n) - \sum_{n=m+1} (n,m,n)$$

$$= F_{m+i} - F_m - C_m \quad \text{where } C_m = o(m)$$

Hence

$$\sum_{n=1}^{m+i}{}' e(n,m,i) = F_{m+i} - F_m + C_m - \sum_{1}^{m+i}{}' \phi(n,i) .$$

$$= F_{m+i} - F_m + C_m - (n,m,i) \qquad (8)$$

Let us find bounds for $f(n,m,i)$

$$f(n,m,i) = \sum_{n=1}^{m+i}{}' (n,i) = \sum_{\substack{n\le m+i \\ n \text{ odd}}}{}' \phi(n,i) - \sum_{\substack{n\le m+i \\ n \text{ even}}}{}' (n,i)$$

$$\le (m+i)\left(i - \frac{1}{2}\left[\frac{i}{2}\right]\right.$$

Now $\phi(n,i) = i$ if $(n,i!) = 1$. The m ber of $1 \leq n \leq m+$ as m ∞ such that $(n,i!) = 1$ is as ~~totic to~~ $\leq$ or $\frac{\phi(i!)}{i!}(\frac{m}{i})$

Hence $\lim_{m \to \infty} \frac{f(n,m,i)}{m}$ $\lim_{m \to \infty} \frac{\sum'_{\substack{n \leq m+i \\ (n,i!)=}} \phi(n,i)}{n} \geq \frac{i\phi(i!)}{i!}$

substituting the bounds of $f(n,m,i)$ in (we get

$$\lim_{m \to \infty} \inf \frac{1}{m} \sum_{n=1}^{m+i\,'} e(n,m,i) \geq \frac{c_m}{m} - \frac{m+i}{m}\left\{ i - \frac{1}{2}\left[\frac{i}{2}\right]\right\}$$

$$\geq -i + \frac{1}{2}\left[\frac{i}{2}\right]$$

and

$$\lim_{m \to \infty} \sup \frac{1}{m} \sum_{n=1}^{m+i\,'} e(n,m,i) \leq \frac{\sum_{n=m+}^{m+i}}{m} + \frac{c_m}{m} - \frac{i\phi(i!)}{i!}\frac{m}{}$$

$$\leq i - \frac{i\phi(i!)}{i!}$$

This completes the theorem.

THEOREM 7. $$\lim_{m \to \infty} \sup \frac{1}{m} \sum_{n=1}^{m+i\,'} e(n,i) \leq \frac{6i}{\pi^2} - \frac{i\phi(i!)}{i!}$$

$$\lim_{m \to \infty} \inf \frac{1}{m} \sum_{n=1}^{m+i\,'} e(n,i) \geq \frac{6i}{\pi^2} - i + \frac{1}{2}\left[\frac{i}{2}\right].$$

Proof. $e(n,i) = \frac{i}{n}\,\phi(n) - \phi(n,i)$.

Thus $$\sum_{n=1}^{m+i_1^{\circ}} e(n,i) = i \sum_{n=1}^{m+i_1} \frac{\phi(n)}{n} - \sum_{n=1}^{m+i_1^{\circ}} \phi(n,i)$$

$$= i \sum_{n=1}^{m+i_1} \frac{\phi(n)}{n} - f(n,m,i)$$

This gives

$$\frac{1}{m} \sum_{n=1}^{m+i_1} e(n,i) = \frac{i}{m} \sum_{n=1}^{m+i_1} \frac{\phi(n)}{n} - \frac{f(n,m,i)}{m}$$

Thus

$$\lim_{m \to \infty} \sup \frac{1}{m} \sum_{n=1}^{m+i_1} e(n,i) \leqslant \frac{\epsilon}{\pi} - \frac{i\,\phi(i!)}{i!}$$

and

$$\lim_{m \to \infty} \sup \frac{1}{m} \sum_{n=1}^{m+i_1^{\circ}} e(n,i) \geqslant \frac{\epsilon}{\pi} - i - \frac{1}{2}\left[\frac{i}{2}\right]$$

Using the bounds of $f(n,m,i)$ and eorem 8.

Let us now conclude by considering ne final question. Do there exists numbers 'n' such that

$$e(n,a,b) \leqslant 0 \quad \text{or} \quad e(n,a,b) \geqslant 0 \quad \forall \quad a,b \in Z^+ \ ?$$

THEOREM 8. A necessary and suffic nt condition for

$$\phi(n,a+b) \leqslant \phi(n,a) + \phi(n,b) \quad \forall \ a,b \quad Z^+$$

which is the same as saying

$$e(n,a,b) \leqslant 0 \quad \forall \quad a,b \in Z^+ \quad \text{is}$$

that 'n' should be a power of a prime (p^m .

Proof. We prove necessity first.

Let $e(n,a,b) \leqslant 0 \ \forall \ a,b \in Z+$ and let $n \ \prod_{i=1}^{r} p_i^{\alpha_i}$.

$\alpha_i \in Z+$ $r \geqslant 2$ p_is distinct primes. F any $i,j, i \neq j, i,j \leqslant$

since $(p_i, p_j) = 1$ we have a solution to the equation

$$\left| p_i x - p_j y \right| = 1 \quad \text{in } x, y \in Z^+$$

Let us assume without loss of generality that $p_i x < p_j y$. Clearly $p_i x$, $p_j y$. are consecutive and not coprime to n.

Now consider any $m \in Z^+$

$$m \equiv 0 \pmod{p_1, p_2, \dots p_r} \qquad (9)$$

and the two intervals $(0, p_i y)$, $(m-2, m+p_j y-2)$.

Clearly for any $0 < \epsilon < p_j y-2 (\epsilon, n) = 1$ there exist an $m < m + \epsilon < m + p_j y-2$ $(m+\epsilon) = 1$ by (9). Moreover $(n, m-1) = 1$. This implies that

$$\phi(n, p_i y) \quad \phi(n, m-2, m+ \quad _j y-2)$$

If $e(n,a,b) \not> 0$ then

$$e(n,a,b) = \phi(n, a+b) - \phi(n, a) - \phi(n, b) \quad 0$$

$$\text{or} \quad \phi(n, a, a+b) - \phi(n, b) > 0$$

Now for $a = m-2$, $b = p_j y$, we have seen that this is true. Hence $e(n,a,b) > 0$ for some $a, b \in Z+$ (and hence infinitely many times since the solutions to (10) are infinite). This is a contradiction and so necess ity is proved.

Now let $n = p^m$ p being a prim

$$\phi(n, a+b) = a + b - \left[\frac{a + b}{p}\right]$$

$$\phi(n, a) = a - \left[\frac{a}{p}\right]$$

$$\phi(n, b) = b - \left[\frac{b}{p}\right]$$

Clearly $e(n,a,b) \leq 0 \; \forall \; a,b \in Z^+$ since $\left[\frac{a+b}{p}\right] \geq \left[\frac{a}{p}\right] + \left[\frac{b}{p}\right]$.

This completes the proof.

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2. On the Distribution of Pseudo Random Numbers Generated by a linear congruential method, H.Neiderrieter, Maths. of Comp. p.p.

Acknowledgement

Professor H.Neiderrieter [2] has shown that

$$\sup_{\alpha \leq 1} \left| \frac{A^{\alpha}(K)}{\phi(K)} - \alpha \right| = O(K^{\epsilon-1}) \; \forall \epsilon > 0 \text{ where } A^{\alpha}(K)$$

represents the number of farey fractions in $[0, \alpha)$ with denominator $\leq K$.

On discussing with him we observed that his result is the same as

$$e(n, x_\alpha) = O(n^{\epsilon}) \; \forall \epsilon > 0 \text{ where } x_\alpha = n\alpha$$

This is given in (3)'.

A NEW LOGARITHMIC FUNCTION[Δ]

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ABSTRACT:- The purpose of this contribution is to introduce a new function in number theory and to discuss properties of numbers defined through this function. Our definition could be extended to all rationals but we confine ourselves to natural numbers. Our function is closely linked with the number of partitions of an integer into primes and we hope that this would prove such a study interesting.

Notations:-

I - set of positive integers

P - set of primes

C - set of composite integers.

We shall use the word 'Result', for minor theorems, and Lemma for results used for the proof of theorems.

A NEW LOGARITHMIC FUNCTION

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INTRODUCTION: In this contribution we introduce a new function in number theory associated with the canonical representation of an integer as product of primes and study its properties.

DEFINITION 1. If $n \in I$, and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_i \neq p_j$, and $p_i \in P$, then $A(n) = \alpha_1 p_1 + \alpha_2 p_2 + \ldots \alpha_r p_r$, and $\bar{A}(n) = n - A(n)$.

Result 1. (i) If $n \in I$, $A(n) \leq n$; (ii) $p \in P$, $A(p) = p$; (iii) $A(1) = 0$, (iv) $A(m.n) = A(m) + A(n)$, $m, n \in I$.

The proofs for the above zare quite easy.

DEFINITION 2. If $n \in I$, then 'n' is said to be highly logarithmic if $A(n) > A(m)$, for all $m < n$, $m \in C$.

THEOREM 1. If $p \in P$, then p, and $2p$, are highly logarithmic. If $n \in I$, and 'n' is not of the form p or $2p$, $p \in P$, then 'n' is not highly logarithmic, for all $n > N$.

Proof. Let $p \in P$, and $m \in C$, and $m < p$. Then $A(m) < m < p = A(p)$. Hence 'p' is highly logarithmic.

Now consider $2p$. Let $m \in C$, and $m < 2p$. Let $m = \lambda q$, where λ, $m \in I$, and $q \in P$.

Now $A(m) = A(\lambda) + q \leq \lambda \pm q < 2 \pm p$ for $\lambda q < 2p$. Therefore $2p$ is highly logarithmic.

We shall now prove the converse for the case $n = 3p$. The proof for other canonical representation proceeds on similar lines.

Let $n = 3p$, where $p \in P$. Now consider a $q \in P$ such that $A(2q) > A(3p) \Rightarrow 2 + q > 3 \quad p$ or $q < p + 1$. Now if we prove that $2q < 3p$, for some $q \in P$, our theorem is proved . This implies that we would have to produce a $q \in P$ satisfying $p + \underset{\lambda}{1} \overset{<}{q} < \frac{3p}{2}$ which is true for all $n > N$. Hence our theorem is proved.

DEFINITION 3. If $n \in I$, let $\omega(n)$ represent the number of $m < n$, $m \in C$ such that $A(m) < A(n)$.

<u>Result 2.</u> $\lim_{n \to \infty} \omega(n) = \infty, n \in I$.

<u>Proof;</u> Let $c(m)$ represent the number of composite integers $< m$.

Therefore for all $m < A(m)$, $m \in C$, $A(m) < m < A(n)$. This implies that $\omega(n) > c(A(n))$. Hence $\lim_{n \to \infty} \omega(n) = \infty$ for $\lim_{n \to \infty} A(n) = \infty$.

<u>Result 3;</u> If 'n' is highly logarithmic $n \in I$, i.e.

$$A(n) > A(m) \quad \text{for all } m < n, \; m \in C \quad \text{then}$$
$$\omega(n) > \omega(m) \quad \text{for all } m < n, \; m \in C.$$

<u>Proof.</u> Since n is highly logarithmic we have

$A(n) > A(m)$ for all $m < n$, $m \in C \Rightarrow A(n) = c(n)$. But if

$m \in C$, then $\omega(m) \leq c(m) < c(n)$, for $m < n$, and $m \in C$. Therefore $\omega(m) < c(n) = (n)$, for all $m < n$, $m \in I$. This proves our result.

<u>THEOREM 2.</u> If $\omega(n) > \omega(n')$, for all $m' \in C$, $m' < n$, $n \in I$, then $A(n) > A(m')$, for all $m' \in C$, $m' < n$, $n \in I$, atleast for all $n > N$.

<u>Proof.</u> Let 'n' be not highly logarithmic. We prove for the case $n = 3p$. The proof for other canonical representations is similar.

Let q be the largest prime such that $2q < 3p$. If p_n represents the nth prime, Huxley[2] has shown that

$$P_{n+1} - P_n = O(n^{7/12})$$

This implies that

$$3p - 2q = O(q^{7/12}) \tag{1}$$

$$c(n,m) \leq |n - m| \tag{2}$$

Now let $q' \in P$ such that $2q' < 3p$ and $q' + 2 > 3 + p$

$$p+1 < q' < 3p/2 \tag{3}$$

We shall need an estimate of the number of such q'. Since $\pi(x) \sim \frac{x}{\log x}$ we have the following two conditions satisfied.

(a) If $0 < C < 1$, then there exists an $N' \in I$, such that for all $n > N'$, $\pi(x) > C \frac{x}{\log x}$ and b) If $C > L$, there exists an $N'' \in I$ such that $\pi(x) < C' \frac{x}{\log x}$ for all $x > N'$. Let now C, and C' be such that $1 < \frac{c'}{c} < 1.25$.

Now if $p + 1$, and $\frac{3p}{2}$ are made greater than the maximum of N' and N'' then the number q' satisfying (2) will be

$$> c\frac{3p}{2\log(3p/2)} - c'\frac{p}{\log p} > c_1\frac{p}{2\log p} \quad \text{where } c_1 > 0$$

Therefore there exist $C_1 \frac{p}{2\log p}$ composite integers $< 3p$, whose 'A' functions are greater that $p + 3$.

Now clearly $\omega(3p) < c(3p) - \frac{c_1 p}{2\log p} < c(2q) + C(2q, 3p)$

$$- C_1\frac{p}{2\log p} < c(2q) + |2q - 3p| - \frac{C_1 p}{2\log p}.$$

Therefore $\omega(3p) - c(2q) < 3p - 2q - C_1\frac{p}{2\log p}$. But $c(2q) = \omega(2q)$. So $\omega(3p) - \omega(2q) < 3p - 2q - C_1\frac{p}{2\log p}$.

It can be shown from (1) that $3p - 2q < C_1\frac{p}{2\log p}$ for all $n^* > N$.

Therefore we have $\omega(3p) < (2q)$ which is a contradiction. Hence the theorem is true.

Result 4. If $n \in I$, and $n > 2$, there exist at least one $m \in I$, such that $A(m) = n$. If $F(n)$ represents the number of such integers then $F(n)$ is the number of partitions of 'n' into primes.

Proof. Now for all $n > 2$, 'n' can be written as

$n = \alpha_1 p_1 + \cdots + \alpha_r p_r$, where the p_1 are distinct primes. This implies that $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, satisfies the condition $A(m) = n$. Further the number of representations of 'n' as

$n = \alpha_1 p_1 + \cdots + \alpha_r p_r$, is $F(n)$ and it is the number of partitions of 'n' into primex. In this context we would like to mention that an estimate of $F(n)$ has been made by Roth and Szakaras [ ] They have proved that

$$F(n) \sim \exp\left\{\sqrt{\frac{2n}{3\log n}}\,\pi\left(1+O\left(\frac{\log\log n}{\log n}\right)\right)\right\} \quad \text{as } n \to \infty \qquad (4)$$

DEFINITION 4. If $n \in I$, then n is said to be a logarithmic number if $n = K.\ A(n)$ and $K \in I$,

THEOREM 3. There exist infinitely many logarithmic numbers.

Proof. Let n be a logarithmic number, $\Rightarrow n = K.A(n)$,

$$K \in I \Rightarrow A(K) = \bar{A}(A(n)) \ldots. (5)$$

Now let $m \in I$, and consider the number

$$m A^{-1}(\pi(n)) = m \qquad (6)$$

$A(m') = m$, and $m \mid m' \Rightarrow m'$ is a logarithmic number, where "m'" is composite. We here call, m' as the logarithmic number obtained through $m \Rightarrow$ that $A(m') = m$.

Further equation (7), is similar to equation (6) with the change $A(n) = m$, or $n = m'$. This implies that a logarithmic number n' obtained through 'n' say, has $A(n) = A(n')$. But the logarithmic numbers obtained through n', n'' are different, for the 'A' function for any integer is unique.

Lastly the logarithmic number obtained through a prime 'p' is $p.A^{-1}(\bar{A}p) = p\bar{A}^{-1}(0) = p$ itself ! Hence in our definition we considered 'n' to be composite $\Rightarrow A(n)$ is composite. But since the number of composite integers is infinite, the number of logarithmic numbers is infinite.

Result 5. If $f(n)$ represents the number of logarithmic numbers obtained through 'n', $n \in C$ then $f(n) = F(\bar{A}(n))$, and $\underset{n \to \infty}{Lt}\, f(n) = \infty \Rightarrow$ Theorem 3.

Proof. A logarithmic number obtained through 'n' is of the form $n\,A^{-1}(\bar{A}(n))$. Therefore the number of such logarithmic numbers will be equal to the number of integers $m = A^{-1}(\bar{A}(m))$ or $A(m) = \bar{A}(n)$, which is precisely $F(\bar{A}(n))$. Therefor $f(n) = F(\bar{A}(n))$.

Further $f(n) = F(\bar{A}(n)) \geqslant F[n/2 - 2]$ where $[\ ]$ stands for the greatest integer function. We know by the theorem of Roth and Szakaras that $\underset{n \to \infty}{Lt}\, F[n/2 - 2] = \infty = \underset{n \to \infty}{Lt}\, f(n) \Rightarrow$ Theorem 3

(for $\bar{A}(n) \geqslant \frac{n}{2} - 2 \Leftarrow$ Theorem 4).

LEMMA 1. If $a \in I$, then for all $n > N$, there exist at least one $n_i^\circ \in \mathcal{L}_n$ such that $n_i^\circ \equiv 1 \pmod{a}$.

Proof. There exist infinitely many primes $\{p_j\}_{j=1}^{\infty}$ such that $p_j^\circ \equiv 1 \pmod{a}$. Let p_1 and p_2 be the smallest of them. We now see that for all $n > N$, n can be partitioned as $n = \lambda_1 p_1 + \lambda_2 p_2$. For if $n \equiv K \pmod{p_2}$, we need a λ_1 to satisfy the congruence $\lambda_1 p_1 \equiv K \pmod{p_2}$ or $n - \lambda_1 p_1 \equiv 0 \pmod{p_2}$ or $= \lambda_2 p_2$. Now the above congruence has a solution if $(p_1, p_2) | K$. But $(p_1, p_2) = 1$ and $1 | K \Rightarrow n = \lambda_1 p_1 + \lambda_2 p_2 \Rightarrow$ Now consider $n_1 = p_1^{\lambda_1} p_2^{\lambda_2}$. $A(n_i^\circ) = n \Rightarrow n_i^\circ \in \mathcal{L}_n$. Further $n_i^\circ \equiv 1 \pmod{a}$. Hence the lemma.

LEMMA 2. If $(a, b) = 1$, $a, b \in I$. a, b, then for all n there exists an $n_i^\circ \in \mathcal{L}_n$ such that $n_i \equiv \pmod{a}$.

Proof. Lemma 1 shows that for all $m > N$, there exists an $m_j^\circ \in \mathcal{L}_m$ such that $m_j = 1 \pmod{a}$. Now consider $n_i = m_j \times b$

$n_i \equiv b \pmod{a}$.

Now lemma 2 is true where $n = m + A(b)$ and $N' = N + A(b)$.

Proof of Theorem 4. Any logarithmic number is of the form $m.\bar{A}^{-1}\ \bar{A}(m)$ where $m \in C$. Now choose $m \equiv 1 \pmod{a}$. There exist infinitely many such 'm'.

Now consider $\mathcal{L}_{\bar{A}(m)}$, since $\bar{A}(m) \geq \frac{m}{2} - 2$. We have that

$\overline{A}(m) > N'$, for all $m > N \Rightarrow$ there exists an $m' \in \mathcal{L}_{\overline{A}(m)}$ such that $m' \equiv b \pmod a$. Therefore $m.m' = n \equiv b \pmod a$, and $m.m'$ is a logarithmic number. Further $m \nmid m'$ $m,m' \in \mathcal{L}_m$ and $\mathcal{L}_m \cap \mathcal{L}_n = \emptyset$ if $n \neq m$. Hence the theorem.

THEOREM 5. Let $\{l_n\}_{n=1}^{\infty}$ denote the sequence of logarithmic Numbers. Then $\lim_{n \to \infty} l_n / l_{n-1} = 1$

Proof. The theorem is proved if we produce a subsequence l_{n_i} such that $\lim_{i \to \infty} \frac{l_{n_i}}{l_{n_{i-1}}} = 1$. Consider the sequence $2p(p-2)$ $p \in P$, $p-2 \in P$. For this we assume the following. The number of pairs $p, p-2 \in P$ is infinite. Further if $\pi(x, 2)$ represents the number of such pairs $\leq x$

$$\pi(x,2) \sim \frac{cx}{(\log x)^2}$$

where c is a constant [3].

This implies that

$$\lim_{i \to \infty} \frac{p_i}{p_{i-1}} \to 1$$

where p_i represents $\max(p_i, p_i - 2)$, $p_i, p_i - 2 \in P$ where $(p_i, p_i + 2)$ is the ith pairs of such primes.

Now $2p_i(p_i - 2)$ is a logarithmic number. Clearly if $n_{j_i} = 2p_i(p_i - 2)$ then

$$\lim_{i \to \infty} \frac{n_{j_i}}{n_{j_i - 1}} \to 1$$

Normally the limit $\frac{A(n)}{n}$ does not exist as $n \to \infty$. But we shall give below a result in the case when 'n' is a logarithmic number.

Result 6. If 'n' is a logarithmic number

$$\underset{n\to\infty}{Lt} \frac{A(n)}{n} = 0$$

Proof. Since n is logarithmic, $n = K.A(n)$, where $A(k) = \bar{A}\big(A(n)\big)$.

Therefore $\frac{A(n)}{n} = \frac{1}{K} = \frac{1}{A^{-1}(\bar{A}(A(n)))}$ and $\bar{A}(A(n)) \geq \frac{A(n)}{2} - 2$.

Therefore $0 \leq \underset{n\to\infty}{Lt} \frac{A(n)}{n} \leq \underset{n\to\infty}{Lt} \frac{1}{\frac{A(n)}{2} - 2} \leq \underset{A(n)\to\infty}{Lt} 1/\frac{A(n)}{2} - 2 < \epsilon$

Hence the result.

DEFINITION 5. If $n \in I$, we define $\mathcal{L}_n$ to be the set of all $n_i \in \mathcal{L}_n$ such that $A(n_i) = n$, $i = 1, 2, 3, \ldots F(n)$, arranged in ascending order.

Deduction. $\mathcal{L}_n \cap \mathcal{L}_m = \phi$ if $n \neq m$. This shall be used in the proof of the theorem given below.

THEOREM 4. If $a, b \in I$, $a > b$, and $(a, b) = 1$ then an arthmetical progression with first term b and common difference 'a' contains infinitely many logarithmic numbers.

We need two Lemmas to prove our theorem.

This means the theorem is proved.

DEFINITION 6. If $n_i \in \alpha_n$, let $E(n_i) = i-1$. or $E(m')$ represents the number of $m < m'$ such that $A(m) = A(m')$.

THEOREM 6. If we take the sequence $n_{n+1} \in \alpha_n$ then
$$\sum \frac{1}{n_{n+1}} < \infty.$$

Proof. We shall show that $n_{n+1} > n(\log n)^2$ as $n \to \infty$, thus proving our theorem.

Let $A(n_{n+1}) = n = p_1 + p_2 + \cdots$. Now if one of the primes say $p_1 < n - (\log n)^2$, then $p_2, p_3, \cdots > p_2 + p_3 + \cdots > (\log n)^2$

Therefore $n_{n+1} > n(\log n)^2$.

Now let $p_1 > n - (\log n)^2$, so that we do not know whether $p_1, p_2 \cdots > n(\log n)^2$ or not. Clearly the number of $n_i \in \alpha_n$, such that $A(n_i) = p_1 + \cdots$, will be the number of partitions into primes of $(n - p_1)$ for the $n - (\log n)^2 < p_1 < n$. It can be easily computed that the number of such p_1 is $(\log n)$.

Now the total number of integers obtained by partitioning $(n - p_1)$ is

$$F(n-p_1) < F((\log n)^2) \sim \exp\left\{\sqrt{\frac{2(\log n)^2}{6\log\log n}}\,\pi\left(1 + O\frac{\log\log\log n}{2\log\log n}\right)\right\}$$

Further the number of such $p_i = (\log n)$. Therefore the maximum number of n_i, such that $n_i = p_1 + p_2 + \cdots$ is

$$(\log n)\exp\left\{\sqrt{\frac{2(\log n)^2}{6\log\log n}}\,\pi\left(1 + O\frac{\log\log\log n}{\log\log n}\right)\right\} = Q$$

Clearly the number of n_i such that $n_i > n(\log n)^2 < Q$

$\text{i.e. } i < Q$ But if n_{n+1} is taken, we clearly see that n + 1 $> Q$

for n+1 = $\exp(\log(n+1)) \therefore n_{n+1} > n(\log n)^2$ if n $\to \infty$.

Therefore $\sum_n \frac{1}{n_{n+1}} < \infty$,

Remark. In fact we have proved not only for the case but n_m where $m > \lambda \cdot n$, $\lambda > 0$. Hence the following theorem.

THEOREM 6. If $n_m \in \mathcal{L}_m$, then $\sum_n \frac{1}{n_m} < \infty$ if $m > \lambda \cdot n$, $\lambda > 0$

Proof is the same.

Result 7. If n $\in$ I, then $n_{F(n)}$ is as follows:-

$$n_{F(n)} = 3^{n/3} \text{ if } n \equiv 0 \pmod 3$$
$$n_{F(n)} = 3^{(n-2)/3} \cdot 2 \quad \text{if } n \equiv 3 \pmod 3$$
$$n_{F(n)} = 3^{(n-4)/3} \cdot 2^2 \quad \text{if } n \equiv 1 \pmod 3$$

Proof. If $n = p_1 + p_2 + \cdots$ then $n_1 = p_1 p_2 \cdots$.
If any one of the p_i is partitioned as $p_i = \lambda_i \cdot 3 + \lambda_2 \cdot 2$
then $n_j' = p_1 p_2 \cdots 3^{\lambda_1} 2^{\lambda_2} \geq n_1$ and $n_j \in \mathcal{L}_n$

Furbher 3 + 3 = 6 = 2 + 2 + 2 , and $3^2 > 2^3$. This implies all the statements of Result 7.

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'DUALITY' AMONG A CERTAIN CLASS OF FUNCTIONS*

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Introduction

The purpose of this contribution is to introduce a concept called "Duality" among functions defined on integers and to study the class of functions which possess "Duality". We are led to this concept of "Duality" which is applicable to a wide class of functions, by initially considering a relation between $A(n)$ and $\omega(n)$ two functions introduced by the author earlier 1 .

If n is an integers > 0, and $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ then $A(n) = \sum_{i=1}^{r} \alpha_i p_i$. We defined 'n' to be a highly logarithmic number if $A(n) > A(m)$, for all $m < n$, n and m being integers and m being composite. We also defined (n) to be the number of $m < n$, n,m being integers, m being composite such that $A(m) < A(n)$.

It was then shown by the author 1 that

(1) $A(n) > A(m)\ \forall\ m < n,\ m,\ n \in I$, m, being composite
$\omega(n) > \omega(m)$ " " " " "

and (2) $\omega(n) > \omega(m)$ " " " " "
(2) $A(n) > A(m)$ " " " " "

$\forall\ n > N$. That is, the numbers satisfying (1) (highly logarithmic) coincide with the numbers satisfying (2) after a certain stage 'N', where inequalities (1) and (2) are defined over the same set of Integers - composite integers.

NOTE: I is the set of positive integers.

DUALITY

DEFINITION 1. We define $\mathcal{C}$ to be the class of all functions, such that if $\phi \in \mathcal{C}$ then

a) $\phi(n) \geqslant 0$; $n \in I$, $n \geqslant 0$ $\phi: I \longrightarrow R^+$

b) There exists an infinite sequence of integers in increasing order $A = \{a_1, a_2, a_3, \ldots\}$ so that

3) If $\{n_j\}_{j=1}^{\infty}$ in increasing order are all the integers satisfying $\phi(n_j) > \phi(a_i)$ for all $a_i < n_j$

d) If $\psi(n)$ equals the number of $a_i < n_j$ such that $\phi(n) > \phi(a_i)$ then

$$\phi(n_j) > \phi(a_i) \; \forall a_i < n_j \Rightarrow \psi(n_j) > \psi(a_i) \; \forall a_i < n_j$$

and $\psi(n_j) > \psi(a_i) \; \forall a_i < n_j \Rightarrow \phi(n_j) > \phi(a_i) \; \forall a_i < n_j$

at least $\forall$ $n_j > N$. and

e) the sequence $\{a_i\}$ and $\{n_i\}$ do not coincide even after a certain stage.

DEFINITION 2. The integers $n_1, n_2, n_3, \ldots$ are said to be 'highly ϕ like', and ψ is said to be the dual of ϕ. ϕ, said to possess the property of 'duality'.

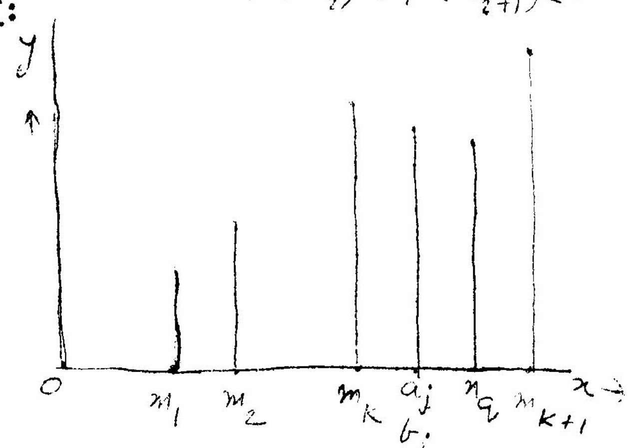
<u>NOTE</u>: 1) In the case of $A(n)$, ϕ was equal to A, $\psi = \omega$ and $a_1, a_2, \ldots$ the set of composite integers, and $n_1, n_2, \ldots$ 'highly A like' i.e. highly logarithmic numbers.

2) Whenever we refer to a sequence of integers, it is always positive and in increasing order $\longrightarrow$ to infinity.

DEFINITION 3. If ϕ, and ψ are functions ψ is said to be a dual of ϕ if $\psi(n)$ represents the number of $a_i < n$ such that $\phi(a_i) < \phi(n)$ for some sequence $\{a_i\}$. The sequence $\{a_i\}$ is called an associated (integers) sequence. We also use the notation ϕ^* instead of ψ.

THEOREM 1. If ϕ is a $\xleftrightarrow{\text{nogap}}$ function such that $\phi(n) \geq 0$ for all $n \geq 0$, $n \in I$, and there exists an infinite sequence of integers $m_1', m_2', \ldots$ for which $\phi(m_1) < \phi(m_2) < \ldots < \phi(m_i) < \phi(m_{i+1}) < \ldots$ and $\phi(m_i) \to \infty$ as $i \to \infty$ then $\phi \in \mathcal{E}$.

<u>Proof</u>:



We shall now show how to form the highly like numbers and the associated sequence. Choose an integer m_1 such that $\phi(m_1) > \phi(m)$ $\forall\, m < m_1$. We here choose m_2 as the integer next to m_1 for which $\phi(m_2) > \phi(m_1)$, and m_3 in a similar manner, ... to m_k, ... This is possible for $\phi(m_i) \to \infty$ as $i \to \infty$.

$\Rightarrow$ that there exists integers 'n' satisfying $\phi(n) > \phi(m)$ for all $m < n$, (1)

Case 1: Let the m_i s be not all consecutive. This implies that there exists an interval (m_k, m_{k+1}) for which m_{k+1} $m_k + 1$. In this interval denote by b_i, the greatest value of the ϕ function, less than $\phi(m_k)$. If there exists more that one than choose the first. Choose similar $\{b_i\}$ s in every other interval whose distance is not one unit. If m_p, and m_{p+1} consecutive for sme $p < k$ choose m_p. Now if $\phi(b_i) > \phi(b_j)\ \forall\ j < i$, then choose m_k itself as b_{i-1} call the b_i s and the m_p s as the sequence $\{a_i\}$, and let the value of a_i in (m_k, m_{k+1}) be a_j. Clearly the $\{m_i\}_{i=1}^{\infty}$ satisfy the inequality $\phi(m_i) > \phi(a_k)\ \forall a_k < m_i$. This implies that $\psi(m_i) > \psi(a_k)\ \forall a_k < m_i$ where ψ is the dual of ϕ with the associated sequence $\{a_j\}$. Now let the sequence of integers n_q be all the integers which satisfy the inequality

$\psi(n_q) > \psi(a_k)\ \forall a_k < n_q$ where $n_q > 0$ and $n_q \in I$ and let this sequence $\{n_q\}$ not coincide with $\{m_i\}$

Then let $m_k < n_q < m_{k+1}$. But we know that since $m_k < a_j < m_{k+1}$, $\psi(n_q) \leq j-2$ for there exist $j-2$, $a_i < a_{j-1}$ and $\phi(a_j) \geq \phi(n_q)$. But $\phi(m_k) > \phi(a_i)$ $\forall$ $a_i < m_k \Rightarrow \psi(m_k) = j-2$.

But we know $\psi(n_q) > \psi(a_i)$ for all $a_i < a_j$, but when $a_i = a_{j-1}$, $\psi(n_q) \leq \psi(a_{j-1}) = \underline{\overset{\text{omit}}{\psi}(m_k)}$. This is a contradiction. Therefore $\{'n'_q\}$ coincides with the m_ks. Further the sequence $\{a_i\}$ does not coincide with m_i. Therefore where the 'm_ks' are not consecutive integers.

<u>Case 2:</u> If the m_k's are consecutive integers, then choose any arbitrary sequence of integers $a_1, a_2 a_3, \ldots$ Since. Now the sequence $b_1, b_2, \ldots$ $\phi(\overset{a}{m}_k) < \phi(\overset{a}{m}_{k+1})$ (where $b_1, b_2, b_3, \ldots$ are the integers $1, 2, 3, \ldots$) are the integers satisfying $\phi(n) > \phi(a_i)$ $\forall$ $a_i < n$ then they also satisfy the inequality of the dual of ϕ. Hence $\phi \in \mathcal{C}$

THEOREM 2. If $f(n) \geq 0, n \geq 0$ $n \in I$, and if for $m_1, m_2, m_3 \ldots$ an infinite sequence of integers, $f(m_1) < f(m_2) \ldots$ then $f \in \mathcal{C}$.

<u>Proof:</u> There exists three cases. If we consider the sequence of intervals (m_1, m_2) (m_2, m_3) ... then one the three conditions must be satisfied.

1) There exist no interval (m_i, m_{i+1}) in which the functional value at n (m_i, m_{i+1}) i.e. $f(n)$ $f(m_i)$.

2) There exist a finite number of intervals (m_i, m_{i+1}) which the functional value at $n \in (m_i, m_{i+1})$ i.e. i.e. $f(n) > f(m_i)$.

3) There exist infinitely many intervals $(m_i + m_{i+1})$ in which the functional value of $n \in (m_i, m_{i+1})$ i.e. $f(n) > f(m_i)$.

Case 1. This is the same as the condition (1) in Theorem 1. Hence $f \in \mathcal{C}$.

Case 2. If we now leave these finite number of intervals and start with some (m_j, m_{j+1}), and rename $m_j, m_{j+1}, m_{j+2} \ldots\ldots$ as $n_1, n_2, \ldots\ldots$ then it falls now under the same condition (1) as Theorem 1. Hence $f \in \mathcal{C}$.

Case 3: There now exist infinitely many m_i, such that for atleast one $n \in (m_i, m_{i+1})$ $f(n) > f(m_i)$. This shows that it is possible to choose a subsequence m_{i_1}, such that there exists at least one $n \in (m_{i_j}, m_{i_{j+1}})$ such that $f(n) > f(m_{i_j})$. We now choose an n_j, where n_j is the first integer for which $f(n_j) > f(m_{i_j})$.

It can now be seen that the $\{n_j\}$ and $\{m_{i_j}\}$s form the highly 'f' like numbers and $f \in \mathcal{C}$ (for the n_j, and m_{i_j} do not coincide). The other conditions for Duality are also satisfied. Hence $f \in \mathcal{C}$. // What are the types of functions that do not belong to $\mathcal{C}$?

THEOREM 3a: If $f(n) \geq 0$, $n \geq 0$, $n \in I$, and $f(n) = C$, for all $n > m$, then $f \notin \mathcal{C}$

Proof: If $f \in \mathcal{C}$ then there exist an infinite sequence of integers a_i, such that $f(n) > f(a_i)$ $\forall$ $a_i < n$, is satisfied by an infinitude of 'highly' f like' numbers. But since the a_i's are integers, $f(a_i) = c$, $\forall$ $i > k$. Therefore if $n > m$, then for all $n > N'$ $f(n) = f(a_i)$. Hence there cannot exist an infinite of 'highly 'f' like' numbers. Hence

THEOREM 3b: (Analogous). If $f(n) \geq 0$, $n \geq 0$, $n \in I$, and $f(n)$ is a strictly decreasing sequence for all $n > m$, then

THEOREM 4: If $f \in \mathcal{C}$ then $f^* \in \mathcal{C}$ where f^* is the dual of f.

Proof: Since $f \in \mathcal{C}$, there exists an infinite sequence of integers $\{a_i\}_{i=1}^{\infty}$ and $\{n_j\}_{j=1}^{\infty}$ such that $f(n_j) > f(a_i)$ for all $a_i < n_j$

Let a_K be the last of the $a_i < n_j$. This implies that $f^*(n_j) = K$. But $n_j \to \infty \Rightarrow n_j \to \infty$, and so $f^*(n_i) = K \to \infty$ since $a_i \to \infty$ as $i \to \infty$. Therefore $f^*(n_j)$ for the increasing subsequence . Hence by theorem 1 $f^* \in \mathcal{C}$.

THEOREM 5: If $f \in \mathcal{C}$ $f: I \to R^+$ and if f^* is the dual of f then $f^*(n) \leq n$, and $f^*(n) \neq n$ for infinitely many values.

Proof: Let $f^*(n) \neq n$ for all n. Then the sequence $a_i = 1, 2, 3, \ldots$, and the highly f like numbers are also $1, 2, 3, \ldots$. Since the two sequence coincide $f \notin \mathcal{C}$. Hence $f^*(n)$ is not equal to 'n' for every n.

Let $f^*(n) \neq n$ for a finite n. This implies that the associated sequence a_1 is not the sequence 1, 2, 3, ... and so $f^*(n+1) \neq n+1$. Hence by induction the $f^*(n) \neq n$ for infinitely many n.

THEOREM 6: If f is a function $f : I \longrightarrow I$ that $f(n) \leq n$ then there exists a $g \in$ such that $g^* = f$, where g^* is a dual of g.

Proof: We shall construct the function 'g' as follows.

Take any arbitrary value for $g(0)$ say K. We know that $f(1) = 0$, or $f(1) = 0$. If $f(1) = 0$, take $g(1) = k' \lessgtr k$. If $f(1) = 1$, then $g(1) = k' > k$. If general let us have n values for which $g^*(n)$ is precisely $f(n)$, where $g^*(n)$ represents the number of $m < n$, $m \in I$ such that $g(m) < g(n)$. We take values of g^* in such a way that $g^*(n) \neq g^*(n')$ when $n \neq n'$. Now let $f(n+1) = l$ where $l \in I$. Now take among $g(0)$, $g(1)$, ... $g(n)$, the 'l' smallest values and let $g(M)$ be the greatest of these. Let $g(L)$ be the least of the remaining values. This implies that $g(M) < g(L)$ and there exist no $0 < M' < n$ satisfying $g(M) < g(M') < g(L)$. Now put $g(n+1) = \dfrac{g(M) + g(L)}{2}$.

This will imply that $g^*(n+1) = l = f(n+1)$, and $g(n+1) \neq g(m)$ for any $m < n$. We have formed the functional value at $(n+1)$ when $f(n+1) = l < n+1$. If $f(n+1) = n+1$ then put $g(n+1) = g(M)+1$ where $g(M)$ is maximum of $g(0)$, $g(1)$, ... $g(n)$.

Hence by induction we have formed the functional value at $(n+1)$. But we have formed the functional value at 0, and 1. Hence $g^* = f$.

THEOREM 7: If $f : I \longrightarrow I$ $f(n) \leqslant n$, $f(n) \neq n$ for infinitely many and 'f' contains an increasing subsequent then there exists a $g \in \mathcal{C}$ such that $g^* = f$.

Proof. Form the function 'g' as in theorem 6. Since $g(n) \neq g(n')$ when $n' = n$, and $g^* = f$, and f contains an increasing subsequence. Hence by theorem 2 $g \in \mathcal{C}$, for the associated sequence is $1, 2, 3, \ldots$ and this does not coincide with the highly 'g' like numbers since $f(n) \neq n$ for infinitely many n.

THEOREM 8: If $f \in \mathcal{C}$ then f^* satisfies the following inequality: $n - m \geq f^*(n) - f^*(m)$ where $n > m$, $f(n) > f(m)$ for infinitely many pairs n, m (f^* is the dual of f).

Proof: Since $f \in \mathcal{C}$, there exists an infinite sequence of integers $a_1, a_2, \ldots a_i \ldots$ in increasing order, which is the associated sequence for f^*, and another infinite sequence of integers $n_1, n_2, n_3, \ldots n_j, \ldots$ in ascending order such that

$$f(n_j) > f(a_i) \quad \text{for all } a_i \quad n_j$$

Now take two numbers in the sequence n_j namely n_{j_1}, n_{j_2} where n_{j_2} n_{j_1}. Let a_{i_1} be the greatest of the a_i's $< n_{j_1}$, and a_{i_2} be the greatest of the a_is less than n_{j_2}.

Now $f^*(n_{j_2}) = i_2$, and $f^*(n_{j_1}) = i_1$.

$\therefore\ n_{j_2} - n_{j_1} \geqslant i_2 - i_1 = f^*(n_{j_2}) - f^*(n_{j_1}) \geqslant 0$ for the a_is are a sequence of integers, and $n_{j_2} > n_{j_1}$. Since the sequence n_j is infinite, the number of pairs n_{j_1}, n_{j_2} is also infinite.

REMARK: 1) In Definition1, we stated that the sequence $a_1, a_2, \ldots$ and $m_1, m_2, \ldots$ should not coincide even after a certain stage. If we permit this coincidence we can easily that a function has an increasing subsequence $m_1, m_2, \ldots$ then $m_1, m_2, \ldots$ become the 'highly ϕ like' numbers for $A = \{m_1, m_2, \ldots\}$ where the m_is satisfy

$$\phi(m_i) > \phi(m) \ \forall\ m < m_i \quad m \in I.$$

REFERENCE

A new logarithmic function, Krishnaswami Alladi (to appear)

nss/10.4.73.

ERRATA

1. In the paper 'Fibonacci Polynomials and their generalization' the r^{th} derivatives and partial derivatives are represented by

$$\frac{d}{dx^r} \text{ and } \frac{\partial}{\partial x^r} \qquad \text{respectively}$$

Please change them to $\frac{d^r}{dx^r}$ and $\frac{\partial^r}{\partial x^r}$ respectively.

2. Also change the subscript 'n' to 'm' in the first few lines of Page 7 in the same paper.