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LECTURES ON GENERAL RELATIVITY AND COSMOLOGY

(Basic course)

By

A. R. PRASANNA

THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS-20, (INDIA)

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by
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Preface

This report consists of the material presented in a basic course of ten lectures on general relativity and cosmology at the Institute of Mathematical Sciences, Madras. The write up consists of seven sections, five on general relativity and two on cosmology. The fundamental principles viz., the principle of equivalence and principle of covariance which form the basis of general theory of relativity are presented in section 1. In section 2, after giving a brief summary of the tensor tools to be used, we have discussed the general relativistic field equations. Section 3 and 4 constitute the first rigorous solution of field equations viz., Schwarzschild exterior and interior solution for an isolated mass and the tests of general relativity. In section 5 we have discussed the conservation laws and energy-momentum complexes in general relativity. The lectures on cosmology are presented in two sections of which the first one is descriptive covering the basic principles and a survey of different models proposed. In section 2 we have derived the Robertson-Walker metric for uniform model universe and the R(t) - Z relation is obtained.

The lecturer wishes to thank Professor Alladi Ramakrishnan for his kind interest and encouragement.

A.R.P.

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General Relativity

Cosmology

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1. The force of gravitation though the weakest of all the known types of forces is the one which extends into the depths of space and time bringing into its fold the entire universe. As is well-known it was Newton who first gave a mathematical expression to this force and he stated in his famous law that every object is attracted towards every other object by a force which is proportional directly to the product of their masses and inversely to the square of ~~the~~ distance between them.

$$F \propto \frac{m_1 m_2}{r^2} \qquad \text{or} \qquad F = G \frac{m_1 m_2}{r^2}$$

Hence according to Newton, Gravitation is an interaction between the particles themselves and this interaction has an infinite speed of propogation. Thus we have conceptually a particle theory for gravitation with an infinite velocity of propogation.

The theory of Electromagnetism on the contrary is a field theory and here the velocity of propogation of the electromagnetic interaction is a finite constant C. Thus at the turn of the century we had two theories explaining the two fundamental interactions of the Universe but in entirely different (conflicting) methods. When I used the word conflicting I mean here the fact that the laws of mechanics as described by Newton are Galilean invariant, whereas the laws of Electromagnetism of Maxwell are not.

It is at this juncture that the genius of Einstein discovered the fact that this conceptual difficulty is arose because of the then existing notion of absolute time and absolute space. It is thus he put forward his special theory of Relativity wherein he showed that it is possible to explain the laws of motion (in the absence of accelerations) and the laws of electromagnetism both the sets being invariant under the general set of Lorentz transformations which are essentially transformations in a four dimensional space-time continuum. But again in special theory the very nature of the theory excludes Gravitation from being taken into its realm. In order to include Gravitation Einstein extended the principle of relativity into all frames of reference in his General theory.

The General theory of relativity basically is built upon two principles: the principle of Equivalence and The principle of Covariance. Let us try to understand what these principles are.

The principle of Equivalence has two forms, the weak form and the strong form. In its weak form it says that the inertial mass of a body is equivalent to its gravitational mass. What does these different nomenclatures mean?

If we have a body then we can attribute three properties to it.

(1) property to resist motion - Inertial mass M_I

(2) property to attract another body - active gravitational mass M_{Ag}.

(3) property to get attracted - passive gravitational mass M_{pg}.

In fact when Einstein first propounded the principle he had the notion of only the first two. It was Bondi (1957) who first distinguished between the active and passive gravitational masses.

Though the general theory of relativity is a field theory basically the force of gravitation does follow Newton's law. Hence if we assume the law of gravitation in the form (1) then it is imperative that this force is derivable from a potential defined by

$$\varphi = -\sum_i (M^i_{AG}/r_i) \tag{1.1}$$

Now if we consider the force exerted on another body we will have

$$F = -M_{PG} \operatorname{grad} \varphi \tag{1.2}$$

Now from the second law of motion we have the motion of a body due to a force F as given by

$$F = M_I \frac{d^2 r}{dt^2} \tag{1.3}$$

It is thus we have the roles of the three types of masses associated with a body.

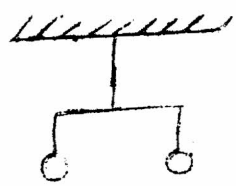
The equivalence of these three forms can be easily established. It is known from Newtonian mechanics that if a body is in motion in a gravitational field (no other forces being present) then its motion is directed by its initial position and the velocity alone i.e. we will have

$$M_I \frac{d^2 r}{dt^2} = M_{PG}(g)$$

where g is the gravitational vector field. This shows that the ratio M_I/M_g is the same for all bodies in a given field. This fact was established by the famous experiment of Galileo performed from the leaning tower of Pisa. The fact that all bodies irrespective of their masses attained the same acceleration proved the equivalence of the inertial and gravitational mass. However this fact was further confirmed through number of experiments. Eotvos (1890) famous experiment of the torsion balance proved the accuracy of the result by a part in 10^9. Same experiment repeated by Dicke (1957) has improved the accuracy to a part in 10^{10}.

Eotvos experiment.

Eotvos suspended two masses from a torsion balance as shown. When the entire apparatus is in equilibrium it is understood that the torque due to the forces G the gravitational and I the inertial cancels with that of the suspension string. Now if the apparatus is



rotated through an angle π then the sign of the associated torque component in the horizontal direction gets reversed. Since the direction of torque of the suspension string remains the same as the original one, there should be an angular deflection of the rod and masses relative to the frame of the apparatus, Eotvos however found no deflection which proved that no torque was produced.

Let us now get an expression for torque which includes the inertial and the gravitational masses.

Let i and k are the unit vectors, i from the body to the centre of the earth and k in the plane of the meridian normal to the axis of the earth. Let M_1 and m_1 and M_2, m_2 be the inertial and gravitational masses of the bodies 1 and 2. The forces acting on the bodies are the inertial force because of the centrifugal force due to earth's rotation

$$I_I = (M_1 a\omega^2 \cos\varphi)\hat{k}$$

where a is the earth's radius, ω the angular velocity and φ the latitude of the place.

$$G_1 = m_1 g\hat{i}$$

where g is the acceleration due to gravity. Let b be the length of the rod and T the torque.

$$\therefore \hat{T} = \frac{b}{2} \times (G_1 - G_2) + \frac{b}{2} \times (I_1 - I_2)$$

The resultant force acting along the direction of the suspension wire is

$$F = G_1 + G_2 + I_1 + I_2.$$

The vertical components of the torque vanishes and the horizontal component is given by

$$T_H = \frac{\hat{F}\cdot\hat{T}}{|F|} = \{(m_1 g + m_2 g)\hat{i} + (M_1 + M_2)\, a\omega^2 \cos\phi\, \hat{k}\} \cdot \{ b \times [(m_1 - m_2) g\hat{i} + (M_1 - M_2)\, a\omega^2 \cos\phi\, \hat{k}]\} / 2|F|$$

i.e.

$$T_H = g a \omega^2 \cos\phi \, M_1 M_2 (\alpha_1 - \alpha_2) \, \hat{l} \cdot \hat{i} \times \hat{k} / 2|F|$$

where

$$\alpha_1 = \frac{m_1}{M_1}, \quad \alpha_2 = \frac{m_2}{M_2}$$

Since $T_H = 0$ we have $\alpha_1 = \alpha_2$

$$\frac{m_1}{M_1} = \frac{m_2}{M_2}$$

which establishes the equivalence of inertial to the gravitational mass.

Now going back to the equation of motion in a gravitational field

$$m_I \frac{d^2 r}{dt^2} = m_p g.$$

the equivalence of these two masses directly yields the equivalence of the gravitational field with the acceleration field. This is an important consequence as this is the basis on which the equivalence of all frames of reference is established. For, let us consider an inertial frame S and a set of bodies each at such distances from the others that no acceleration exists between them. Consider another frame S' which is in uniform acceleration with respect to S. Now from the S' point of view all the masses seem to have an uniform acceleration. From the equivalence of the gravitational field to acceleration field now we can think that the masses are in an uniform gravitational field. Thus we see that an inertial frame in a gravitational field is equivalent to an accelerated frame in a gravitation free space-time. Thus it is possible through the principle of equivalence to extend the principle of relativity even to frames of reference with non-uniform motion (accelerated motion). It is through this principle that we can attribute the numerical changes that occur while transforming from one frame to another purely to changes in the gravitational field.

We now consider the principle in its strong form. We have just established the equivalence of inertial frames in a gravitational field to accelerated frames. In order to use it in principle we assume that it is possible to construct in the immediate neighbourhood of a point in a Riemannian space a Minkowski frame or in other words, given a curved space-time it is always possible to apply special relativity in the immediate neighbourhood of any given point in it. The principle in this form helps us to construct the equivalent concepts of pressure, density and such physical entities in a curved space-time. As these are all point functions we may just describe them even in general theory as we define them in the special theory. Thus we have a logical extension of the principle of relativity to all frames of reference through the principle of equivalence.

So much about the <u>fact</u>. Let us now go to the <u>form</u>. After all, all these physical ideas have to have mathematical forms. In special relativity our study was confined mainly to inertial frames and hence the Lorentz invariance of all the physical laws. But now since we intend studying accelerated frames it is essential to have the laws unchanged under arbitrary Gaussian transformation. In order to accomplish this Einstein introduced the principle of covariance.

<u>Principle of Covariance:</u>

According to this, the general laws of physics are expressible in such mathematical form which remains covariant under arbitrary non-singular coordinate transformations.

The major implication of such a principle is that the physical laws are represented independent of the coordinate frames of reference. Since from the principle of equivalence we have seen that the physical entities are to be same "in fact" with reference to all coordinate frames, this principle assures them to be same "in form" also. The primary motive in introducing this principle can be regarded as underlying in our desire to make full use of the idea of relativity of all kinds of motion. As emphasized by Einstein, the laws of physics are to be regarded as a codification of the results of experimental observation, which in the final analysis reduces to the determination of space-time coincidences. In order to determine such coincidences, different observers may use different systems of coordinates, according to their convenience. Obviously the physical behaviour cannot depend on the coordinate frame used by the observer. In order to exploit this independence of the coordinate system and physical behaviour the principle of covariance serves the key.

Having ascertained the principle the next question was what mathematical language should be used for its representation. At this stage Einstein found the usefulness of the language of Tensors developed mostly by Levi-Civita and Ricci. For tensors are the mathematical objects which are specially characterised by their transformation property of preserving their form under arbitrary, non-singular coordinate transformations.

Gravitation as Geometry.

In special relativity where we are concerned only with uniform rectlinear motions (absence of accelerations) the coordinate systems that could be used were all rectlinear in nature. Thus the space-time continuum was described by the Minkowski metric and the background geometry of such a 4-space was Euclidean.

On the contrary we now have accelerated motions and frames of reference wherein the bodies no-more have straight line motion. Hence for this generalisation of relativity of motion in all frames of reference, Einstein had to abandon Euclidean geometry. What could be the possible new geometry? When the motion is along a curved path we have to get at a geometry wherein curves can be studied. Gauss had studied the concept of curved 3 surfaces and its associated geometrical features and Riemann had extended Gauss's theory onto higher dimensions. It is thus Einstein chose the Riemannian geometry as the geometry underlying the space-time structure (manifold) for the study of accelerated motion. The basic feature of a Riemannian space-time is its curvature and thus Einstein associated Gravitation with the curvature of a Riemannian manifold. As against the Minkowski metric of the special relativity, we now have the Riemannian metric,

$$ds^2 = g_{ij}\, dx^i dx^j \qquad (i, j = 1, \ldots 4)$$

where g_{ij} can be functions of coordinate, to describe the features of a gravitational field.

The fact that g_{ij} is a tensor came in handy for the development of the required tensors which were to give mathematical expressions and equations for physical laws. Thus Einstein could accomplish the general theory of Relativity as an outcome of the principle of equivalence and the principle of covariance, describing gravitation as a geometrical feature (viz. curvature) of the 4 dimensional Riemannian manifold.

2. Field Equations

Before writing down the field equations, I should like to present some essential tensors that we are going to use in our study.

Basically a tensor is characterised by its transformation property. A_i is said to be a covariant vector (a vector is a tensor of rank one and a scalar is a tensor of rank 2) if it transforms such that, given two coordinate systems x^i and x'^{α}, we have

$$A'_{\alpha} = \frac{\partial x^i}{\partial x'^{\alpha}} A_i \qquad (2.1.)$$

Similarly A^i is said to be a contra-variant vector if it transforms according as

$$A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^i} A^i \qquad (2.2)$$

We have co-variant tensors, contra-variant tensors defined similarly. A mixed tensor of the form A^i_j transforms as

$$A'^{\alpha}_{\beta} = A^i_j \frac{\partial x'^{\alpha}}{\partial x^i} \frac{\partial x^j}{\partial x'^{\beta}} \qquad (2.3)$$

[Throughout we adopt Einsteins notation that a repeated index implies summation over all the values of that index].

Any geometry is normally characterized by its underlying metric which is a distance function between any 2 points of the manifold, and this is kept invariant under arbitrary non-singular transformations. Thus we have in Riemannian geometry, the metric

$$ds^2 = g_{ij}\, dx^i dx^j$$

representing an invariant interval between any two arbitrary points of the Riemannian manifold. In the study of general theory of relativity we are concerned with the 4-dimensional Riemannian manifold describing a given gravitational field in an underlying space-time continuum. An Euclidean manifold is characterised by its well-known feature of parallel transport of vectors. That is if we are given a vector V^i on an Euclidean manifold then along any arbitrary closed curve on the manifold it is possible to move the vector such that when it tours back to its original position it would have suffered no change in its direction. On the contrary such a transportation is not possible on any given curve of a Riemannian manifold.

On a Riemannian manifold the infinitesimal changes of the vector components are bilinear functions of the vector components and the components of the infinitesimal displacements, as given by

$$\delta V^i = -\Gamma^i_{kl}\, V^k \delta x^l \tag{2.5}$$

$$\delta V_k = \Gamma^i_{kl}\, V_i\, \delta x^l \tag{2.6}$$

Here Γ^i_{kl} is the affine connection and this depends purely on the metric of the given manifold. Γ^i_{kl} is not a tensor and this is known as the christoffel symbol. In general Γ^i_{jk}

could be symmetric or anti-symmetric, but f in the study of the general theory of relativity where we have the symmetric metric tensor g_{ij}, Γ^i_{jk} is always symmetric in j and k. This is given by

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (g_{lj,k} + g_{lk,j} - g_{jk,l}) \tag{2.7}$$

where a comma denotes a partial differentiation.

From the tensor transformation law it immediately follows that an ordinary (partial) differentiation of a tensor is not a tensor operation. Hence the concept of covariant differentiation is introduced. The covariant derivative of a vector is defined as follows: Let V^i be a vector at $P(x^i)$ and let it be transported parallely to the point $P(x^i + dx^i)$. Let the vector at $P(x^i + dx^i)$ be $V^i + dV^i$. The parallely transported vector at $x^i + dx^i$ is given by $V^i + \delta V^i$. The difference between these two vectors viz. $dV^i - \delta V^i$ is the covariant derivative of V^i.

$$dV^i - \delta V^i = \left(\frac{\partial V^i}{\partial x^j} + \Gamma^i_{jk} V^k \right) dx^j \tag{2.8}$$

Since dx^j itself is a vector the quantity in the paranthesis transforms as a tensor. Thus we have the covariant derivative (always denoted by ;) of V^i as

$$V^i_{;j} = \frac{\partial V^i}{\partial x^j} + \Gamma^i_{jk} V^k \tag{2.9}$$

For a covariant vector V_k, we get

$$V_{k;j} = \frac{\partial V_k}{\partial x^j} - \Gamma^i_{jk} V_i \qquad (2.10)$$

We can define the covariant derivatives of tensors analogously.

Note. When the affine connection vanishes (i.e. only when it is possible to introduce a cartesian system) the covariant derivative is same as partial derivative.

We now pose the question, whether it is possible to find curves on a Riemannian manifold along which parallel transport is possible. That is curve along which the covariant differentiation of a vector vanishes. In fact we do have such curves and they are the geodesics of the manifold. Thus we define a geodesic of a Riemannian manifold as that curve along which it is possible to transport a given vector parallely without changing its direction. Thus the equation of a geodesic is given by

$$\frac{dv^i}{dx^j} + \Gamma^i_{jk} V^k = 0 \qquad (2.11)$$

Since $V^i = \dfrac{dx^i}{ds}$, we get

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \qquad (2.12)$$

Thus having found geodesic as a counter part of a straight line it can be very easily verified that a geodesic is also an extremal between two given points of Riemannian manifold.

Let us consider a Riemannian space-time with the metric

$$ds^2 = g_{ij}\, dx^i dx^j$$

An extremal is defined through the variational principle

$$\delta \int_A^B ds = 0 \qquad (2.13)$$

i.e. $\delta \int_A^B L\, d\tau = 0$. with (2.14)

$$L^2 = g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} \qquad (2.15)$$

Using Euler-Lagrange equations for the extremal we get after changing the parameter from τ to s, the equations of geodesics

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \qquad (2.16)$$

We have seen above that the parallel transport of a vector along any closed curve is not possible on a Riemannian manifold. Hence the descripency that occurs during the transportation should be mainly due to the curvature of the manifold. Consider a vector ξ^i at P_1 and transport it along a closed rectangle say $P_1(u, v)$, $P_2(u+\Delta u, v)$, $P_3(u+\Delta u, v+\Delta v)$ $P_4(u, v+\Delta v)$ and come back to P_1. Now the new vector ξ^{*i} will not coincide with ξ^i and the difference $\Delta\xi^i = \xi^{*i} - \xi^i$ will be of the order of $\Delta u \Delta v$. From the definitions of infinitesimal displacement it can be seen that

$$\lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \frac{\Delta \xi^i}{\Delta u \Delta v} = R^i_{jkl}\, \xi^j \frac{\partial x^k}{\partial u} \frac{\partial x^l}{\partial v} \qquad (2.17)$$

where $R^i_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \frac{\partial \Gamma^i_{jk}}{\partial x^k} + \Gamma^i_{lm}\Gamma^m_{jk} - \Gamma^i_{km}\Gamma^m_{jl}$ (2.18)

is called the Riemann-Christoffal curvature tensor. This tensor plays the fundamental role in the study of general relativity. As can be easily seen it is made up of the metric tensor g_{ij} and its first and second derivatives. The vanishing of the curvature tensor is a necessary and sufficient criterion for a space-time to be flat. If R^i_{jkl} vanishes in one coordinate system it vanishes in all coordinate system and this will make it possible to introduce a set of Cartesian coordinates, for the parallel transport of the vector ξ^i along the closed curve is now possible.

We now give some tensors derived from R^i_{jkl} and g_{ij} which play important roles in the ensuing theory.

$$R_{hijk} = g_{hm} R^m_{ijk} = \frac{1}{2}\left\{ g_{ij,hk} + g_{hk,ij} - g_{hj,ik} - g_{ik,hj} \right\} + g_{lm}\left\{ \Gamma^l_{ij}\Gamma^m_{hk} - \Gamma^l_{hj}\Gamma^m_{ik} \right\} \quad (2.19)$$

is symmetric in the pairs hi, jk, whereas antisymmetric in its indices h,i and j,k. This tensor R_{hijk} in general has 256 components for a 4 dimensional Riemannian manifold. But because of the above mentioned symmetry we get only 21 independent components. Further this tensor satisfies the following cyclic relation

$$R_{hijk} + R_{hjki} + R_{hkij} = 0 \quad (2.20)$$

and hence we will be left with only 20 independent components.

Contraction of R^i_{jkl} by setting i=l, leads us to the Ricci tensor,

$$R_{jk} = \Gamma^i_{jk,i} - \Gamma^i_{ji,k} + \Gamma^i_{jk}\Gamma^l_{il} - \Gamma^i_{jl}\Gamma^l_{ki} , \tag{2.21}$$

and a further contraction with g^{jk} gives us the scalar of curvature

$$R = g^{jk} R_{jk}.$$

The Weyl tensor, or the Conformal curvature tensor is defined as

$$C_{hijk} = R_{hijk} - \frac{1}{2}(g_{ij}R_{hk} + g_{hk}R_{ij} - g_{ik}R_{hj} - g_{hj}R_{ik}) + \frac{R}{6}(g_{ij}g_{hk} - g_{ik}g_{hj}) \tag{2.23}$$

Vanishing of C_{hijk} for a given space-time means that the space-time is conformally flat, i.e. g_{ij} the metric tensor of such a space time can be expressed as

$$g_{ij} = f(x^i)\eta_{ij} \tag{2.24}$$

where f is a function known as the conformal factor.

Having thus understood the various tensors that we are going to deal with we now pass on to get the field equations of general relativity.

We have in Newtonian physics the law of gravitation described by the Poisson equation

$$\nabla^2\varphi = 4\pi G\rho \tag{2.25}$$

where φ is the gravitational potential and ρ is the density of material distribution. In a 4-dimensional frame we have the same law generalised to yield

$$\Box \varphi = 4\pi G \rho \tag{2.26}$$

Thus we see that we have the second derivatives of the potential being set equal to the density of the system. Hence it would be natural to assume a similar form of equations in the general theory. In a given Riemannian manifold described by a metric of the form

$$ds^2 = g_{ij}\, dx^i\, dx^j ,$$

We know that g_{ij}'s are the metric potentials and these are ten independent components of these, as against only one potential φ of the Newtonian physics. This immediately suggests that there should be ten field equations as against a single Poisson equation. But in the approximation (weak fields) our general tensor equation must reduce to the D'alembertian form and hence the equations to be constructed should be linear in second derivatives of g_{ij} .

In fact there exist three possible generally covariant second-order differential expressions that a symmetric second rank tensor g_{ij} can satisfy. R_{hijk} , C_{hijk} , and $\alpha R_{ij} + \beta R g_{ij} + \Lambda g_{ij}$, (α, β, Λ constants). But in the absence of matter $R_{hijk} = 0$, makes g_{ij} an absolute object and the space-time becomes flat. $C_{hijk} = 0$ again leads to only conformally flat space-times and further these two sets have twenty independent equations and thus are to be rejected for field equations. Hence we are left with the set of equations

$$\alpha R_{ij} + \beta R g_{ij} + \Lambda g_{ij} = T_{ij} \tag{2.27}$$

as the possible set. These are in fact ten in number. Since this is a tensor equation it holds good in all frames of reference. If all the ten equations are to be treated as independent then we will have for a given set of solutions g_{ij} any number of orbitrary equivalent sets g'_{ij} s obtained just by coordinate transformations. To remove this ambiguity it is necessary to have four identities in the set of 10 equations so that this would leave us with four arbitrary functions which can be characterised through the particular coordinate system. From the analogy of the poisson equation we must now have on the right hand side of the field equations a tensor which represents the physics of the field. Hence considering T_{ij} to be the energy-momentum tensor we will have to satisfy the conservation law, which from special relativity can be generalised to be

$$T_i{}^j{}_{;j} = 0 \tag{2.28}$$

Incidentally this puts the restriction on the left hand side that

$$(\alpha R_i{}^j + \beta R \delta_i^j + \Lambda \delta_i^j)_{;j} = 0 \tag{2.29}$$

It can be seen from contracted Bianchi identities that we must have now $\beta = -\frac{\alpha}{2}$. Hence we get the field equations

$$R_{ij} - \frac{1}{2} R g_{ij} + \Lambda g_{ij} = \kappa T_{ij} \tag{2.30}$$

where κ is a suitable constant. On comparison with Newtonian equations for the weak field we get $\kappa = -\frac{8\pi G}{c^2}$.

It is possible to obtain this set of field equations also through a variation principle in which we use the gravitational Lagrangian as given by $\sqrt{-g}R$ where R is the scalar of curvature. In this approach the third term on the left hand side comes in as a constant of integration. Having thus established the field equations we now go into the study of the well-known solution of Schwarzschild.

3. Schwarzschild's Solutions

The first rigorous solution of field equations was obtained by Schwarzschild in 1916. As a simple situation he considered the gravitational field due to some material distribution under purely gravitational forces. The solutions are such that the exterior solution describes the field outside the distribution and the interior the field inside, and the two solutions are matched properly across the boundary of the distribution. Equivalently one can consider the solutions as those equivalent to that of Laplace's equation (empty region) and Poisson's equation (inside a distribution). Since the entire material distribution is governed only by gravitational force it is imperative that one assumes the distribution to be spherically symmetric. Since the field is entirely due to this distribution without loss of generality one can assume the field to be spherically symmetric. Thus in order to obtain the Schwarzschild's solutions we start from the general spherically symmetric metric in canonical coordinates as given by

$$ds^2 = e^{\nu} dt^2 - r^2(d\theta^2 + \sin^2\theta \, d\phi^2) - e^{\lambda} dr^2 \tag{3.1}$$

where ν and λ can be functions of r and t only. But our interest is in a static distribution of matter and hence the potentials will be independent of t. Essentially our problem will be now to compute the components of the Ricci tensor and the scalar of curvature and use them in the field equations to be solved. The non-vanishing components of the Ricci-tensor and the scalar of curvature R for the metric (3.1) are given by

$$
\begin{aligned}
R_{11} &= \frac{\nu''}{2} - \frac{\lambda'}{r} - \frac{\nu'\lambda'}{4} + \frac{\nu'^2}{4} \\
R_{22} &= R_{33}/\sin^2\theta = e^{-\mu}\left[1 + \frac{r}{2}(\nu' - \lambda')\right] - 1 \qquad (3.2) \\
R_{44} &= e^{\nu-\lambda}\left\{-\frac{\nu''}{2} - \frac{\nu'}{r} + \frac{\lambda'\nu'}{4} - \frac{\nu'^2}{4}\right\}
\end{aligned}
$$

and

$$R = -e^{-\lambda}\left\{\nu'' - \frac{\nu'\lambda'}{2} + \frac{\nu'^2}{2} + \frac{2(\nu' - \lambda')}{r} + \frac{2}{r^2}\right\} + \frac{2}{r^2}$$

The field equations corresponding to the empty region are given by (putting $\lambda = 0$)

$$R_i^j - \frac{1}{2} R \delta_i^j = 0 \qquad (3.3)$$

which is same as

$$R_{ij} = 0 \qquad (3.4)$$

Using (3.2) in (3.4) we get the set of ordinary differential equations for ν and λ which when solved yields the solution

$$e^{\nu} = e^{-\lambda} = \left(1 - \frac{\alpha}{r}\right) \qquad (3.5)$$

where α is a constant of integration. In order to evaluate α we make use of the condition that the field varies as 1/r in Newtonian approximation.

Suppose we consider a particle at rest in the gravitational field described by the potentials (3.5) then the force acting on it is obtained through the geodesic equations to be

$$\frac{d^2 r}{dt^2} + \Gamma_{44}^{i} = 0 \tag{3.6}$$

which when substituted for Γ_{44}^{i} becomes

$$\frac{d^2 r}{dt^2} = \frac{-\alpha}{2r^2}\left(1 - \frac{\alpha}{r}\right) \tag{3.7}$$

which in the first approximation is $\frac{-\alpha}{2r^2}$. Since we want the force to be Newtonian in this approximation we have

$$-\frac{\alpha}{2r^2} = -\frac{GM}{r^2} \tag{3.8}$$

where G is the gravitational constant and M is the total mass of the distribution. Hence we have the solution for the exterior field of a mass M as given by

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{3.9}$$

also written as

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{3.10}$$

where $m = \frac{GM}{c^2}$

As apparent from the form of the metric apart from the singularity at $r = 0$, there is another singularity at $r=2m$. The nature of this singularity is still disputed. One school of thought, categorises it as a mere coordinate singularity by which they mean that the singularity exists purely because of the choice of the coordinate system.

Eddington (1924) and Finkelstein (1958) have transformed the Schwarzschild solution to a non-singular form as given by

$$ds^2 = -dx^2 - dy^2 - dz^2 + d\bar{t}^2 - \frac{2m}{r}(d\bar{t} - dr)^2 \quad (3.11)$$

through transformations of the form

$$t = \bar{t} + 2m \log (r - 2m) \quad (3.12)$$

But it is apparent that the singularity at r=2m is now showing up in the transformation and hence this singularity is not a removable one. This supports the view of the other school of thought which propounds that the singularity r = 2m is a physical one. We shall deal with a more detailed study of this singularity at a later stage

Tests of General relativity

The most important application of the exterior solution of Schwarzschild obtained above was found in obtaining tests for general relativity. Important of various tests proposed are the "advance of perihelion" of Mercury and the bending of light ray near the solar limb. We shall now deduce mathematical expressions for these dynamical phenomena through Schwarzschild solution.

1. Advance of perihelion.

It was long back observed that the planet mercury as it revolves round the sun in its orbit, suffers a shift of the orbit which is referred to as the advance of perihelion. In fact this effect exists for other planets too, but mercury being closest to the Sun the effect is maximum. The observed precision is 5599".74 ± 0".41 per century and of this except for about 43" of arc/century remaining was explained in Newtonian physics as perturbation in the orbital motion caused by the influence of other planets. Einstein's theory could now account for this 43" of arc accurately. This is derived here through the dynamical equation for the orbit obtained through the geodesic of a test particle. That is the planet is treated as a test particle in the gravitational field of the Sun. Using the Schwarzschild solution, if we can obtain the geodesics as follows.

The general equations of motion of a test particle in a gravitational field is given by the set

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \tag{3.13}$$

Since we are interested in only the motion in the plane $\theta = \pi/2$, we get from (3. 13) using the metric (3. 10)

$$\frac{d^2 r}{ds^2} + \frac{\lambda'}{2}\left(\frac{dr}{ds}\right)^2 - r e^{-\lambda}\left(\frac{d\varphi}{ds}\right)^2 + \frac{e^{\nu-\lambda}}{2}\nu'\left(\frac{dt}{ds}\right)^2 = 0 \tag{3.14}$$

$$\frac{d^2\varphi}{ds^2} + \frac{2}{r}\frac{d\varphi}{ds}\frac{dr}{ds} = 0 \tag{3.15}$$

$$\frac{d^2t}{ds^2} + \nu'\frac{dr}{ds}\frac{dt}{ds} = 0 \tag{3.16}$$

where $e^{\nu} = e^{-\lambda} = (1 - 2m/r)$ (3.17)

Integrating (3.15) and (3.16) we get

$$r^2\frac{d\phi}{ds} = h, \quad e^{\nu}\frac{dt}{ds} = k, \tag{3.18}$$

where h and k are constants. Instead of trying to integrate (3.14) we can use the metric (3.) as the other integral so that using (3.18) we get after some minor computations the equation of the orbit

$$\frac{d^2u}{d\varphi^2} + u = 3mu^2 + \frac{m}{h^2} \tag{3.19}$$

where $u = \frac{1}{r}$. It can be observed that without the term this equation is the usual Newtonian orbit and hence $3mu^2$ is the relativistic correction. Hence using the solution for Newtonian equation and then solving this equation (3.19) by method of perturbation we get the solution

$$u = \frac{m}{h^2}\left[1 + e\cos(\phi - \omega - \delta\omega)\right] \tag{3.20}$$

where e is the eccentricity of the orbit and ω is the longitude of the perihelion.

$$\delta\omega = \frac{3m^2}{h^2}\varphi \tag{3.21}$$

$$\therefore \frac{\delta\omega}{\varphi} = \frac{3m}{a(1-e^2)} \tag{3.22}$$

where 'a' is the semi major axis, of the orbit. Hence we find that for every revolution, i.e. $\varphi = 2\pi$, there is a precession of the amount $\delta\omega = 6\pi m/a(1-e^2)$. In the case of planet mercury the calculated precession is $42.9''$ which matches well with the observed effect of $43.5''$.

Recently Dicke has argued that in calculating the precession one should consider the oblateness of the Sun and this brings in a change of about 8" of arc/century less than previously thought 42". Dicke tries to fit in his figures with those obtained from his scalar-tensor theory of gravitation. As the evidence for ascertaining the oblateness of the solar disc is not sufficient, Dicke's scalar-tensor theory does not replace the general theory of Relativity.

2. Bending of light ray.

As the second best test of general relativity we have the gravitational deflection of light ray. It is propounded that a light ray from a distnat star when passes the Sun gets bent towards the Solar limb because of the strong gravitational field. In fact Newton's theory offers an explanation for this beinding, by considering the mass equivalent of energy but the figure that Newton's theory yields is only half the observed effect. General theory of relativity offers an explanation just by considering that the normal path of the light ray is a null geodesic, and the figure we get here matches with the observed figures very well.

We start with the equation for a null geodesic as given by $ds = 0$, and using this in (3.18) we find that $h = \infty$ and thus we get from (3.19)

$$\frac{d^2 u}{d\varphi^2} + u = 3 m u^2 \tag{3.23}$$

Again using the perturbation technique we can solve this to get

$$u = \frac{\cos\varphi}{R} + \frac{m}{R^2}\left(\cos^2\phi + 2\sin^2\phi\right) \tag{3.24}$$

where R is solar radius. Transforming from (r, φ) to (x, y) system we get

$$x = R - \frac{m}{R}\frac{x^2 + 2y^2}{\sqrt{x^2+y^2}} \simeq R - \frac{m}{R}(\pm 2y) \tag{3.25}$$

which gives the deflection angle $\theta = \frac{4m}{R} \simeq 1''.75$.
Campbell and Trumper have given a recent figure of their observation of deflection to be $1''.72 \pm 0.11''$ – $1''.82 \pm 0.15''$.

Recently the principle of bending of light rays was applied to the electromagnetic radiation at radio frequency for the case of these radio stars. Particularly the radio star 3C 273 was used to investigate this effect when Sun occults it and by using interferometric techniques two groups Seielstad et al (1970) and Muhleman et al (1970) both from Caltech have arrived at figures which are in good agreement with general relativity

4. Schwarzschild's interior solution

In the last section we obtained the solution of Einstein's equations for the exterior field of a matter distribution. Now we shall consider the field inside the matter distribution. Here it is now essential to assume some properties of the fluid distribution to be known. Mathematically simplest of the fluid distributions is a perfect fluid distribution which is defined as one incapable of having transverse stresses. In order to formulate the equations we have to know the structure of the Energy momentum tensor T_i^{α} which appears on the right hand side of the field equations.

We have said earlier that the principle of equivalence in its strong form plays an essential role in the development of general relativity. Presently we see that it helps in formulating the energy-momentum tensor in general relativity as a generalisation from that of special relativity. Because of this principle we can assume that at each point of the Riemannian manifold we can introduce locally Euclidean character with Minkowski metric. Since pressure and density are point functions we have at any given point, where we have a cartesian frame of reference, the energy-momentum tensor $T_0^{\mu\nu}$ for a perfect fluid distribution is given by

$$T_0^{\mu\nu} = \begin{pmatrix} p & & 0 \\ & p & \\ 0 & & p \; \rho \end{pmatrix} \qquad (4.1)$$

where p represents the pressure (being isotropic, same in all the three directions) and ρ the density. Since this is a tensor we can obtain the equivalent expression in any frame through the tensor transformation law,

$$T^{ij} = T_0^{\mu\nu} \frac{\partial x^i}{\partial x_0^\mu} \frac{\partial x^j}{\partial x_0^\nu} \tag{4.2}$$

where the subscript o indicates the proper coordinates. Similarly we have for the metric potential g^{ij},

$$g^{ij} = g_0^{\mu\nu} \frac{\partial x^i}{\partial x_0^\mu} \frac{\partial x^j}{\partial x_0^\nu} \tag{4.3}$$

But $g_0^{\mu\nu} = \eta^{\mu\nu}$ and hence using (4.3) and (4.1) in (4.2) we get

$$T^{ij} = -pg^{ij} + (p+\rho) \frac{dx^i}{dx_0^4} \frac{dx^j}{dx_0^4} \tag{4.2a}$$

Considering the 4-velocity $\dfrac{dx^i}{ds}$ we have

$$\frac{dx^i}{ds} = \frac{\partial x^i}{\partial x_0^1} \frac{dx_0^1}{ds} + - - - - + \frac{\partial x^i}{\partial x_0^4} \frac{dx_0^4}{ds} \tag{4.4}$$

As the fluid is at rest with respect to proper coordinate frame we have the spatial components of velocity zero. Hence we get

$$\frac{dx^i}{ds} = \frac{\partial x^i}{\partial x_0^4} \tag{4.5}$$

Using (4.5) in (4.3) we have now the general expression for a perfect fluid distribution as given by

$$T^{ij} = (\rho+p) v^i v^j - pg^{ij} \tag{4.6}$$

where $v^i = dx^i/ds$.

In order to obtain the Schwarzschild interior solution we take the form of T^{ij} as given by (4.6) and further we say that there is no fluid motion. This makes $V^i = (0,0,0,1)$ and since we are interested in a spherically symmetric distribution we assume the metric as given in (3. 1). Considering R_i^j and R as in (3. 2) (3.) we get the field equations to be

$$8\pi T_1^1 = -p = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} \tag{4.7}$$

$$8\pi T_2^2 = 8\pi T_3^3 = -p = e^{-\lambda}\left\{\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu'-\lambda'}{2r}\right\} \tag{4.8}$$

$$8\pi T_4^4 = \rho = e^{-\lambda}\left\{\frac{\lambda'}{r} - \frac{1}{r^2}\right\} + \frac{1}{r^2} \tag{4.9}$$

Equating (4.7) and (4.8) we get after some simple manipulations

$$\frac{dp}{dr} + \frac{\nu'}{2}(p+\rho) = 0 \tag{4.10}$$

which is actually the Relativistic analogue of the Newtonian equation $\frac{dp}{dr} + \rho\frac{d\psi}{dr} = 0$

Schwarzschild was interested in a distribution which is homogeneous and this leads to have the density constant everywhere. Thus putting ρ = constant, we get from (4.9)

$$(re^{-\lambda})' = -\rho r^2 + 1 \tag{4.11}$$

or $$e^{-\lambda} = -\frac{\rho r^2}{3} + 1 + \frac{c}{r} \tag{4.12}$$

In order to remove singularity at r = 0 we make c = 0.
Putting $\rho = 3/R^2$, R a constant we have

$$e^{-\lambda} = \left(1 - r^2/R^2\right) \tag{4.13}$$

From (4.10) we get

$$(p+\rho)\, e^{\nu/2} = k \text{ (constant)} \tag{4.14}$$

Using again (4.7), (4.9) and (4.13) in (4.14) we are lead to

$$e^{\nu/2}\left[\frac{\nu'}{2}\left(1-\frac{r^2}{R^2}\right)+\frac{r}{R^2}\right] = \frac{kr}{2} \tag{4.15}$$

Solving which we get

$$e^{\nu} = \left[\frac{kR^2}{2} + B\left(1-\frac{r^2}{R^2}\right)^{1/2}\right]^2 \tag{4.16}$$

In order to evaluate the arbitrary constants K and B we resort to boundary conditions. It is imperative that the interior solution should match with the exterior solution (3.10) at the boundary of the fluid distribution say r = a, and further to obtain continuity of pressure we take p=0 at r=a. These two conditions yield us the result

$$B = \frac{1}{2}, \quad k = -\frac{6B}{R^2}\left(1-\frac{a^2}{R^2}\right)^{1/2} \tag{4.17}$$

With these we now arrive at the complete solution for the interior field of a homogeneous fluid sphere as given by

$$ds^2 = -\left(1-\frac{r^2}{R^2}\right)^{-1} dr^2 - r^2 d\Omega^2 + \left\{A + B\left(1-\frac{r^2}{R^2}\right)^{1/2}\right\}^2 dt^2 \tag{4.18}$$

where $A = -\frac{3}{2}\left(1-\frac{a^2}{R^2}\right)^{\frac{1}{2}}$; $B = \frac{1}{2}$

with the pressure and density,

$$p = \frac{3}{R^2}\left\{\left(1-\frac{a^2}{R^2}\right)^{\frac{1}{2}} - \left(1-\frac{r^2}{R^2}\right)^{\frac{1}{2}}\right\} \Big/ \left\{-\frac{3}{2}\left(1-\frac{a^2}{R^2}\right)^{\frac{1}{2}} + \frac{1}{2}\left(1-\frac{r^2}{R^2}\right)^{\frac{1}{2}}\right\} \tag{4.19}$$

$$\rho = 3/R^2 \tag{4.20}$$

where R is a constant.

Thus we have a complete picture of the gravitational field due to a spherically symmetric perfect fluid distribution the exterior of it being described by (3.10) and the interior by (4.18) and both being continuous across the boundary of the distribution $r = a$.

5. Conservation laws and Energy momentum complexes

The concepts of Energy and Momentum and their conservation play a fundamental role in all branches of physics. As any dynamical system where the forces can be derived from a potential is defined as a conservative system, the generalisation of the laws of conservation from Newtonian physics to Relativistic physics is but an easy step.

In Newtonian physics it is well known that the conservation of mass is expressed by the well-known equation of continuity viz.,

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho u) = 0 \tag{5.1}$$

where ρ is the density and u the velocity.

In special relativity the conservation laws (charge conservation) followed from Maxwell's equations and here since the total stress, energy of the system was expressed by the second rank tensor T^{ij}, we have the conservation laws as expressed by

$$T^{ij}{}_{,j} = 0 \tag{5.2}$$

Coming to general relativity we have known already from the principles of equivalence and covariance that the laws of special relativity are to be extended in a covariant way to include all frames of reference. This suggests that the generalised conservation laws in general relativity are given by

$$T^{ij}{}_{;j} = 0 \tag{5.3}$$

where now T^{ij} represents the stress-energy-momentum tensor.

It may be recalled now that in the development of field equations we were enforced for these four identities

$$T^{ij}{}_{;j} = 0$$

which serves now as the conservation laws.

In order to get the familiar divergence form of these equations one has to make use of the canonical pseudo-tensor of energy-momentum t_i^j first used by Einstein and with this we get

$$\left\{ \sqrt{-g}\,\left(T_i^{\,j} + t_i^{\,j} \right) \right\}_{,j} = 0 \tag{5.4}$$

These represent the local conservation laws and thus one can define the total energy-momentum of the system as given by the 4-vector

$$P_i = \iiint\limits_{x^4 = \text{Const}} \sqrt{-g}\,\left(T_i^4 + t_i^4 \right) dx^1\, dx^2\, dx^3 \tag{5.5}$$

It can immediately be noticed that the expressions for the energy and momenta obtained through this are not covariant as t_i^j is not a tensor. With this in view various alternate proposals have been made.

The various double index complexes, successively formulated by Einstein (1916), Landau-Lifshitz (1951) and Moller (1958) ~~[illegible] (19[illegible])~~ have a common feature in them that all of them have a three index super-potential as their fundamental element.

These super-potentials are built up of g_{ij}, and its derivatives and they are given as follows:

$$\mathcal{U}_i^{[kl]} = \begin{cases} \frac{1}{2\varkappa}(-g)^{-\frac{1}{2}}\, g_{in}\, \partial_m \left[(-g)(g^{kn}g^{lm} - g^{ln}g^{km})\right] & \text{Einstein} \\ \frac{1}{2\varkappa}(-g)^{\frac{1}{2}}\, g^{km} g^{ln} \left[\partial_m g_{in} - \partial_n g_{im}\right], & \text{Møller} \end{cases} \qquad (5.6)$$

$$\mathcal{U}^{i[kl]} = \frac{1}{2\varkappa}\, \partial_m \left[(-g)(g^{ik}g^{lm} - g^{il}g^{km})\right] \quad \text{Landau Lifshitz.} \qquad (5.7)$$

As is apparent all these super-potentials are antisymmetric in k and l and thus the conservation laws are now obtained automatically for the complexes

$$\mathcal{T}_i^{k} = \mathcal{U}_i^{kl}{}_{,l} \quad , \quad \mathcal{T}^{ik} = \mathcal{U}^{ikl}{}_{,l} \qquad (5.8)$$

as given by

$$\mathcal{T}_i^{k}{}_{,k} = 0 \, , \quad \mathcal{T}^{ik}{}_{,k} = 0 \qquad (5.9)$$

In general when nothing special is stipulated about the then the conservation laws given above are called the strong conservation laws. On the other hand when we choose g_{ij} s such that they satisfy the field equat' s of Einstein

$$R_i^{j} - \tfrac{1}{2} R\, \delta_i^{j} = 8\pi T_i^{j} \qquad (5.10)$$

then the conservation laws are weak conservation laws.

We can introduce canonical pseudo-tensors t_i^k and t^{ik} of Einstein and Landau-Lifshitz and then the energy momentum complex will be split up as

$$\tau_i^k = \sqrt{-g}\,(T_i^k + t_i^k) \tag{5.11}$$

$$\tau^{ik} = -g\,(T^{ik} + t^{ik}) \tag{5.12}$$

As Cattaneo (1965) points out, Einstein's and Landau-Lifshitz complexes present the exceptional property of t_i^k and t^{ik} not depending on the second derivatives of g_{ij}. Though Møller's complex does not exhibit the same, it has another notable property of giving rise to an energy density invariant under spatial transformation which is not true of the other two complexes. Further L-L COMPLEX being symmetric in i and k yields a conservation law for angular momentum as well. However none of these complexes constitute a geometrical object and hence it is difficult to attribute a local physical content to them and their continuity equation.

Møller in 1961 revising the complex given earlier seems to have remedied the lack of covariance inherent in the other complexes. As a principal step of the new formulation Møller argues that the metric potentials g_{ij} themselves are not true fundamental field variables as for instance the field equations for a fermion field in presence of a gravitational field cannot be described in terms of these variables alone. One needs in that case the field described by Fetrads. These fetrads are point functions and they have sixteen components defined through the relations

$$\lambda_{(a)}{}^{i} \lambda^{(a)j} = g^{ij} , \qquad \lambda_{(a)}{}^{i} = g_{ij} \lambda^{j}{}_{(a)} \tag{5.13}$$

The process of constructing a new e-m complex is now based on the possibility of decomposing the gravitational Lagrangian $\sqrt{-g}\,R$ into a divergence type term which does not influence the variation of action and into a term $\hat{L}$ constructed by means of $\lambda^{i}_{(a)}$ and their covariant derivatives. This results in giving a scalar density invariant under coordinate transformations.

Using the $\lambda^{i}_{(a)}$ defined above, Møller constructs the tensor γ_{ikl}

$$\gamma_{ikl} = \overset{(a)}{\lambda_{i}} \lambda_{(a)k;l} \tag{5.14}$$

which is a local fetrad tensor such that its components along the directions of the fetrad are the Ricci coefficients of rotation

$$\gamma_{(abc)} = \gamma_{ikl} \lambda^{i}_{(a)} \lambda^{k}_{(b)} \lambda_{(c)}{}^{l} \tag{5.15}$$

Though the $\lambda^{i}_{(a)}$'s defined above determine g_{ij}'s uniquely, for a given $g_{ik}(x)$ there is a six fold freedom in the choice of the fetrads $\lambda^{i}_{(a)}$ corresponding to the arbitrary independent rotations of the four unit vectors of the Fetrad. Møller seems to remove this arbitrariness through imposing certain supplementary conditions on the $\lambda^{i}_{(a)}$'s which he assumes will single out suitably oriented Tetrads, given by

$$\gamma^{\ell}{}_{ik;\ell} + \gamma_{ik\ell}\,\phi^{\ell} = 0 \tag{5.16}$$

where ϕ^{ℓ} is defined as

$$\varphi^{\ell} = g^{\ell k}\varphi_k\,, \quad \varphi_k = \gamma^{i}{}_{ki}\,, \tag{5.17}$$

Now the super-potential which Møller constructs out of these $\lambda^{i}_{(a)}$ is given by

$$\mathcal{U}_i^{\;k\ell} = -\mathcal{U}_i^{\;\ell k} = \pm\frac{\sqrt{-g}}{2}\Big\{\lambda^{k}_{(a)}\lambda^{(a)\ell}{}_{;i} - \lambda^{\ell}_{(a)}\lambda^{(a)k}{}_{;i} + 2\big(\delta_i^{k}\lambda^{(a)\ell} - \delta_i^{\ell}\lambda^{(a)k}\big)\lambda^{s}_{(a);s}\Big\} \tag{5.18}$$

and the energy-momentum complex $\mathcal{T}_i^{\;k}$ is given by

$$\mathcal{T}_i^{\;k} = \mathcal{U}_i^{\;k\ell}{}_{,\ell} \tag{5.19}$$

Having thus defined various energy momentum complexes we now have to ask the question whether we are anywhere near defining the energy-momentum of a system uniquely. The answer is we are far from it. We now in the following show that the general field equations of Einstein admit quite a many solutions which all represent curved space-times and valid physical systems whereas the fourth component of the energy-momentum complex vanishes identically for these solutions. It was indicated that when the pseudo-tensor $t_i^{\;j}$ was introduced, it was supposed to represent the stress-energy tensor of the gravitational field produced by the corresponding distribution T^{ij}

We now consider the class of solutions having homogeneous metric potentials. We define (Prasanna 1970) the metric potentials g_{ij} as homogeneous if they are functions of the time variable alone.

$$\text{i.e. } ds^2 = g_{ij}(x^4)\,dx^i dx^j \,, \tag{5.20}$$

From the conservation laws defined above we have for the energy momentum complex,

$$\sqrt{-g}\,(T_i^{\,j} + t_i^{\,j}) = \mathcal{U}_i^{\,jk}{}_{,k} \tag{5.21}$$

$$\sqrt{-g}\,(T_i^{\,4} + t_i^{\,4}) = \mathcal{U}_i^{\,41}{}_{,1} + \mathcal{U}_i^{\,42}{}_{,2} + \mathcal{U}_i^{\,43}{}_{,3} = 0 \tag{5.22}$$

for $\mathcal{U}_i^{\,41}, \ldots$ are independent of x^1, x^2, x^3. Thus the fourth component of energy momentum complex which is supposed to represent the total energy and momentum vanishes identically for the entire class of metrics with homogeneous metric potentials. In this class we have the well known metrics as that of Friedmann, Narlikar-Karmarkar...(Prasanna 1969).

Since almost all physically meaningful metrics have their metric potentials a function of both space and time variables it is of interest to study whether there are any metrics with inhomogeneous metric potentials, having this fourth component of energy-momentum complex identically zero. The first such metric was that given by Einstein and Rosen for cylindrical gravitational waves. We (Prasanna 1970) have recently shown that the space-time described by

$$ds^2 = 4e^{(x-t)}(dt^2 - dx^2) - (x-t)(dy^2 + dz^2)$$

representing plane-electromagnetic wave propagating in the x direction has

$$T_i^{\,4} + t_i^{\,4} = 0$$

This example further adds to the view that it has not been possible to get an invariant definition of energy and momentum in general relativity. The fundamental difficulty seems to us to be in the way we have defined our conservation laws. However the problem is still wide open and there is a good scope for detailed investigations regarding these conservation laws.

Cosmology

1. Cosmology is the study of the universe as regards to its structure, origin and evolution. Though early astronomors studied the planets and stars there were not many attempts to construct theories as to the evolution or models as to the structure of the universe. With Newtons famous law of universal gravitation astronomors for the first time thought about the distant stars and star clusters that would have their influence on the motion of bodies. Though there was an early attempt to have a Newtonian cosmology before the advent of relativity, successful Newtonian cosmology developed only in 1934 when Milne and McCrea showed that with proper interpretation of Newtonian terms Newtonian cosmology was in many respects equivalent to relativistic cosmology.

I have just used the phrase 'Relativistic Cosmology'. Let us understand now what it is. In these lectures I shall only be talking about relativistic cosmology. With the advent of the general theory of relativyty wherein gravitation was identified with the curvature of space-time there appeared an indication that the theory describing gravitational interaction which is universal should directly yield a possible structure of the universe. After all we describe the gravitation field of a certain matter distribution by a Riemannian metric which directly infers about the structure of the underlying space-time. Thus we have Relativistic Cosmology as that branch of study wherein we can describe the possible structure of the universe by means of a Riemannian metric and since it is possible to have different Riemannian manifolds with different physical characteristics we can have different models of the universe.

In developing the models of the Universe two ideas that play a fundamental role are the principle of uniformity and the principle of extrapolation. While the principle of uniformity assumes that the overall picture of the universe presents the same view for all observers, the principle of extrapolation assumes that the results obtained locally can be assumed to hold good at distant regions too. The former is the basis for theoretical cosmology whereas the latter is the basis for the experimental or observational cosmology.

In the first two or three lectures I shall deal with the theoretical cosmology wherein I shall present to you the theoretical models of relativistic cosmology. Later I shall consider the observational aspects and their conclusions regarding models.

Relativistic cosmological models are built upon two fundamental assumptions, viz., the cosmological principle and the Weyl's postulate. The cosmological principle states that the universe presents the same picture globally (except for local irregularities) for all observers at all points of space. This essentially is the assumption of homogeneity and isotropy. The cosmological principle when further extended to all times will be known as the perfect cosmological principle.

The Weyl's postulate states that the world-lines of galaxies form a non-interacting congruence of time-like geodesics which are hyper-surface orthogonal. That is the world lines are all time like geodesics such that they are orthogonal to the surface x^4 = const.

The earliest of the cosmological models were those of Einstein and de Sitter. Einstein after identifying gravitation with the geometry of space-time was lead to the natural conclusion that the field equations provide a space-time structure whenever a material distribution is given. It follows that, if the average distribution of matter in the universe is put into the field equations, the average space-time structure for the whole cosmos may be deduced. With this Einstein constructed a static model of the universe, in which the matter is distributed fairly uniformly throughout space-time giving an isotropic and homogeneous picture of the universe, closed by a spherical hyper surface, Within an year of Einstein's proposal of such a universe, de-sitter proposed another model based on general relativity, but not a static one. According to de-sitter the most remote celestial objects are moving away from us. With such a view de-sitter constructed his model of an expanding universe as a consequence of which the density decreases and approaches a non vanishing limit. Thus the idea of an expanding universe evolved from de-sitter model very soon got established from observational astronomy. For, when the spectrum of distant galaxies were analysed, they found a shift towards the red end. This was explained by Doppler's effect as a consequence of the recession of galaxies into distant space. In 1929, Edwin Hubble, who made a very detailed spectroscopic analysis of the galaxies found that the red shift was proportional to the distance of the galaxy. With this observation, he used Doppler's principle and obtained an elegant mathematical expression relating the red shift and the velocity of the moving object. He found that greater the distance of a galaxy, its velocity of recession is also great. This study of Hubble set a new ground for constructing the models.

1. <u>Friedmann's Model</u>:

In 1922 Friedman proposed a model of the universe on the basis of general relativity. According to this, universe can either expand or contract. It is possible therefore, that prior to the presently observed expansion, universe may have been undergoing contraction. When sufficiently small volume was reached, the mass became unstable and started expanding. In such a case it may cease to expand after some years and start contracting again. Thus there was neither a beginning for the universe nor an end but an indefinite cycle of expansion and contraction. The difficulty in this oscillating model is as follows. From daily experience we know that anything oscillating cannot continue indefinitely and in such a case what would be the final form?

2. <u>Big bang model</u>:

A Leimaitre thought of considering the origin of the universe and he started with the assumption of a primeval atom of very high density and temperature. In this model supported by G.Gamow, Lemaitre said that this highly condensed mass became unstable once and started expanding which is often referred to as "the big bang". As it expanded, the expanded material aggregated into separate portions which were referred to as protogalaxies; and inside these protogalaxies there was condensation of matter into the form of stars. The outward expansion is still continuing as a result of which the galaxies are receding. One of the interesting features of this theory is the fact that gravity (which is attractive in ordinary experience) offers a repulsive effect in case of galaxies which are very widely separated, and speeds up the expansion. According to this model there should be a continuous decrease in the density of the universe. When we speak of an origin for the universe, it is quite relevant to ask the age of the universe. Assuming that galaxies have been moving at a constant speed since the time universe started to expand, one may crudely consider the age as equivalent to the distance of a galaxy divided by its velocity. Since by Hubble's formula $H = v/D$, one may gauge the age of the universe as approximately H^{-1} which gives value of 10^{10} years. The main difficulties of this model are as follows: In the first place, where did the primeval atom come from? And secondly if the galaxies are moving away further will this continue indefinite

3. <u>Steady State theory</u>:

This theory was put forward by Bondi, Gold and Hoyle. The main object of this model was to eliminate the difficulties of "Singularity" and "age" of the big bang model. Though the model in the final form is same, the approaches of Bondi, Gold and Hoyle were quite different. Bondi and Gold started their theory on the basis of "perfect cosmological principle". According to this principle there is no preferred place or time in this universe. At any epoch, for any observer universe looks alike. Now, if an expanding universe has to be in a steady state then there must be creation of matter, for otherwise density would decrease. Hence they said that as the universe expands, new matter gets created which in turn condenses to form stars and galaxies. These newly formed galaxies take the position of older ones as they move away. This solves the problem of singularities. Since the universe is expected to be in a steady state, there can be no 'beginning' or "end" and this solves the question of age, for the age may be considered to be infinite. The difficulty with the approach of Bondi and Gold is that they do not give the mechanism of creation.

Hoyle tried to solve this difficulty in his approach. He assumed no cosmological principle. He started with a continuous creation of matter in the universe. He said that the creation of matter is same as any natural physical process. The main feature of Hoyle's approach is the "coupling" of the creation of matter to the expansion of the universe. This coupling is of such a nature that if one can know the rate of expansion, the rate of creation may be deduced. Since it is possible to arrive at the rate of expansion through observation, the rate of creation was calculated and it is found that matter is getting created at a rate of 1 atom per century in a volume corresponding to that of the "Imperial State building". Though this figure appears so insignificant, in the universal scale, it works out to be enormous. Now how does creation take place? To answer this Hoyle assumed of a "Creation field" made out of a reservoir of negative energy and in this field he said that energy is getting converted to matter. The idea of negative energy is important in the fact that the storage of energy go on increasing as the energy is removed out of the reservoir. Thus Hoyle brought out a natural correspondence to the creation and expansion, as a consequence of which universe has to be in a steady state. Thus in a steady state model the density of matter should be the same in the past, present and future. Here is a point for the experimentalist to put into test the theory. If by observations one can calculate the number of galaxies in a particular region in the past one can know the density then and compare with the density today. As the light from distant galaxies which is reaching us today here is the one started long back there, when we observe a galaxy at a distance of billion light years we are not seeing it as it is today, but as it was billion years ago. Thus one can take sections of the universe and count the number of galaxies and thus arrive at the idea of average density over different periods of time. Professor Ryle of Cambridge in 1962 made a study of density distribution in the case of radio sources and his figures showed a decrease in the average density as time elapsed. This indeed was a very big blow to the steady state model. Along with this another difficulty with the steady model was about the formation of galaxies. This model was unable to give any satisfactory explanation for galaxy formation.

4. Unsteady Universe?

In view of these difficulties, Hoyle and Narlikar in 1964, made fresh attempts to set a model of the universe. They abandoned the supposed homogeneity of the universe with gentle creation going on everywhere, and said that creation takes place only in certain regions of space called the "pockets of creation". The new picture runs as follows: As we observe very heavy particles (baryons) in nature, the "C field" must be able to create such particles and this in turn requires a certain minimum strength of the field. This they called as the 'threshold'. So creation takes place only when the 'C field' strength is above the threshold. Now where can such a threshold exist? They said that in the neighbourhood of very heavy masses where the gravitational field is intense the C field will also be stronger. And so the creation may take place only in some immediate neighbourhood of the massive objects. Here as the creation goes on, because of the negative energy of the C field, there exists a repulsive force which tend to expand the pocket of creation. So as it expands the C field strength falls and this stops any further creation, and region becomesevacuated as a bubble. This bubble expands to a maximum radius and then begins to contract. During contraction once again the C field strength increasesand the pocket beginsto operate again. Thus they said that though in total the universe expands, there are certain bubbles which expand and contract. This gives an inhomogeneous picture of the universe with cluster of galaxies which may themselves be oscillating bubbles. The important feature of this model is that since the pockets of creation die out as the bubble expands, we should see more pockets in the past than today.

There were attempts to identify the quasi stellar galaxies with such pockets. However this aspect has not been satisfactorily clarified as the question of quasars being local or cosmological is still argued. Apart from this there is another observation which seems to contradict the steady state model. This is the observation of micro-wave-background radiation at $3^{\circ}K$.

Gamow around 1948, had predicted that as a consequence of the early "big bang" there should have been a large emission of radiation which at the passage of time would have cooled down to micro-wave radiation. This prediction was further emphasised by Dicke and Peebles in 1961 and they said that it should be possible to find micro-wave background radiation at $3^{\circ}K$. In 1958, Penzias and Wilson had recorded background radiation at $7^{\circ}K$ which when analysed for noises reduced to $3^{\circ}K$ and at that time no source of this radiation was known. After Dicke's hypothesis it was more or less concluded that this $3^{\circ}K$ background radiation must be the fossil of the early big bang. Steady state theory offers no explanation for this observation and hence if the cosmological origin of this $3^{\circ}K$ micro-wave background radiation is established then the evolutionary theory will be in better status than the steady state theory.

2. Uniform model universes

We shall now consider the actual construction of relativistic models based on the two fundamental principles mentioned in the last lecture, viz., the cosmological principle and the Weyl's postulate. We first note that as a consequence of Weyl's postulate we can define a cosmic time (which is same for all observers) and this restricts for us the general form of the Riemannian metric to be geodesic form, i.e. as given by

$$ds^2 = dt^2 - g_{\alpha\beta}\, dx^\alpha dx^\beta \tag{2.1}$$

where α, β can take values 1,2,3. From the cosmological principle we have that the large scale picture of the universe should be same for all the observers at any given time and hence this imposes the conditions of isotropy and homogeneity. With this it is natural to assume that the Universe is spherically symmetric and thus we start from the general form of the metric as given by

$$ds^2 = dt^2 - e^\mu (dr^2 + r^2 d\Omega^2) \tag{2.2}$$

where μ can be a function of r and t. Having started with a metric of this form, our task now is just to assume a perfect fluid distribution in the universe and solve the general relativistic field equations. Hence considering

$$T_i^{\,j} = (p+\rho) v_i v^j - p\delta_i^j \tag{2.3}$$

where p and ρ denotes the pressure and density of the fluid distribution, we get from the field equations

$$R_i^j - \tfrac{1}{2} R\,\delta_i^j = -(T_i^j + \Lambda \delta_i^j) \qquad (\chi = 1) \tag{2.4}$$

the set of equations,

$$
\begin{aligned}
-(\ddot\mu + \tfrac{3}{4}\dot\mu^2) + e^{-\mu}\left(\frac{\mu'}{r} + \frac{\mu'^2}{4}\right) + \Lambda &= -(p+\rho)V_1V^1 + p \\
-(\ddot\mu + \tfrac{3}{4}\dot\mu^2) + e^{-\mu}\left(\mu'' + \frac{\mu'^2}{4} + \frac{2\mu'}{r}\right) + \Lambda &= -(p+\rho)V_4V^4 + p \\
-(\ddot\mu + \tfrac{3}{4}\dot\mu^2) + e^{-\mu}\left(\frac{\mu''}{2} + \frac{\mu'}{r}\right) + \Lambda &= p \\
e^{-\mu}\dot\mu' &= -(p+\rho)V_1V^4.
\end{aligned}
\tag{2.5}
$$

We now observe that as the fluid distribution is made up of the galaxies, each observer is always in motion along with the fluid particles and thus with respect to the observer the fluid is at rest. Hence we have to use the co-moving system of coordinates which makes the 4-velocity to be $(0,0,0,V^4)$. Using this in (2.5) we get from the last of the set

$$\dot\mu' = 0 \tag{2.6}$$

i.e.

$$\mu = f(r) + g(t) \tag{2.7}$$

From the first and third of (2.5) we now get

$$\mu'' - \frac{\mu'}{r} - \frac{\mu'^2}{2} = 0 \tag{2.8}$$

which gives

$$f = -2\log\left(1 + \frac{kr^2}{4}\right) \tag{2.9}$$

where k is a constant of integration. Thus we get

$$ds^2 = dt^2 - \frac{e^{g}}{\left(1+\frac{kr^2}{4}\right)^2}\left(dr^2 + r^2 d\Omega^2\right) \tag{2.10}$$

This is also normally written in the form

$$ds^2 = dt^2 - \frac{R^2(t)}{\left(1+kr^2/4\right)^2}\left(dr^2 + r^2 d\Omega^2\right) \tag{2.11}$$

where R is any arbitrary function of t. The pressure and density of such an universe are given by

$$p = -\frac{k}{R^2} - \frac{2\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} + \Lambda \tag{2.12}$$

$$\rho = \frac{3k}{R^2} + \frac{2\dot{R}^2}{R^2} - \Lambda \tag{2.13}$$

This model is known as Robertson-Walker model of the universe.

The constant 'k' is supposed to be the space curvature and the space is said to be spherical, Euclidean or hyperbolic according as k = +1, 0, or -1.

As an important feature of any cosmological model it is essential to account for the red shift of distant sources. In order to achieve this we shall now obtain an expression connecting the red shift 'z' with the function R appearing in the above R-W. model. Consider a source at $r = r_e$ and an observer at r=0. Let a light signal be emitted from r_e at time t = te and observed at 'O' at time t = t_0. As the light propogates along a null geodesic we will have $ds = 0$, i.e.

$$dt^2 - \frac{R^2}{(1+kr^2/4)^2}\, dr^2 = 0 \tag{2.14}$$

θ, and φ being fixed. Hence we have the light velocity

$$\frac{dr}{dt} = -(1+kr^2/4)/R \tag{2.15}$$

as the light is emitted from a distant point to the origin. Considering two rays emitted with an interval of time dt we have now

$$\int_{t_e}^{t_0} \frac{dt}{R} = \int_0^{r_e} \left(1+\frac{kr^2}{4}\right) dr \tag{2.16}$$

$$\int_{t_e+dt_e}^{t_0+dt_0} \frac{dt}{R} = \int_0^{r_e} \left(1+\frac{kr^2}{4}\right) dr \tag{2.17}$$

From these two we get

$$\int_{t_e+dt_e}^{t_e} \frac{dt}{R} + \int_{t_e}^{t_0} \frac{dt}{R} + \int_{t_0}^{t_0+dt_0} \frac{dt}{R} = \int_{t_e}^{t_0} \frac{dt}{R} \tag{2.18}$$

i.e. $$\frac{dt_e}{R(t_e)} = \frac{dt_0}{R(t_0)} \tag{2.19}$$

Considering the wave length of the radiation to be λ at the time of emission and $\lambda + d\lambda$ at the time of observation we get from (2.19)

$$\frac{\lambda + d\lambda}{\lambda} = \frac{R(t_0)}{R(t_e)} \tag{2.20}$$

i.e.
$$1 + Z = \frac{R(t_0)}{R(t_e)} \tag{2.21}$$

where 'Z' is the shift in the spectral line. Thus we have for a given universe (characterised by $\dot{R}(t)$) a direct relation between the spectral shift and the function R. Since observationally only red shift is recorded we have $Z > 0$ and thus R(t) is a monotonic increasing function of t.

At this stage I should like to make an important observation. It is known that in general the function 'R' appearing in metric is arbitrary and only in two special cases of $p=0$ and $\rho + p = 0$ an exact equation for R has been solved. On the other hand it will be interesting to solve for R using a general equation of state. As a first step in this approach we are trying to solve the equation obtained by considering $dp/d\rho$ = K(constant). Physically it would mean that if a pressure-density curve is obtained then the slope of it is constant. In such cases $dp/d\rho = K$ supplies us with an equation for R as a function of t solving which one can get a class of models with K as a parameter. In fact a more interesting situation is one where we can consider

$p=\rho^{\gamma}$, γ the ratio of specific heats. This again provides a general equation for R the solution of which supplies a class of metrics with γ as a parameter. In each such model we can use the relation (2.21) and this enables us to make a proper comparison between observation and theory.

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