MATSCIENCE REPORT 70

GRAMMAR OF DIRAC MATRICES

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GRAMMER OF DIRAC MATRICES

The confluence of relativity and quantum mechanics was achieved when Dirac just wrote down his famous equation in 1928. This achievement was made possible since he was able to construct four mutually anti-commuting matrices so that the Hamiltonian was consistent with the quadratic relativistic relation between energy momento and mass. Earlier, in non-relativistic quantum mechanics three mutually anti-commuting matrices were found sufficient to include the concept of intrinsic spin. It was immediately noticed by Dirac that the quantum mechanical concept of spin was also imbedded in his Hamiltonian. In the years of uninterrupted triumph that followed the birth of relativistic quantum mechanics, the study of the mathematical significance of the transition from Pauli to Dirac matrices was considered quite academic and therefore ignored. But it was obvious that it was still a live and unsolved problem since immediately after Dirac's formulation, Pauli attempted such a study and as late as 1956, Feynman himself raised the question of the relationship between spin and relativity in his famous Caltech lectures even after the total triumph of his graphical formalism in electro-dynamics.

We therefore set as our objective the understanding of the mathematical procedure of obtaining Dirac matrices from the basic Pauli set. We thought it was just the right time now to take it up since the spirit of the hour demanded a re-examination of the whole structure from the point of view of mathematical rigour and logical precision. To our strange surprise we found that the procedure which Dirac used was of such general significance that it could be extended into a grammar of anti-commuting matrices, the ramifications of which give us a better insight into various branches of theoretical physics -- relativity, complementarity, propagator formalism and the fundamental concepts of spin and mass of elementary particles. Even more surprising was the possibility of enlarging the concept of anti-commutation to ω-commutation where ω is a general root of unity.

This work was presented in a series of papers most of which were published in the "Journal of Mathematical Analysis and Applications". Since they formed a connected series, it is considered worth-while to publish them together under the title,

"Grammar of Dirac matrices".

Alladi Ramakrishnan

THEORY OF L-MATRICES

	Title and Journal
Alladi Ramakrishnan	The Dirac Hamiltonian as a member of a hierarchy of matrices, (Jour. Math. Anal. Appl., V.20, p.9-16, 1967)
	Helicity and energy as members of a hierarchy of eigenvalues, (Jour. Math. Anal. Appl., V.23, p.397-401, 1967)
	L-matrix hierarchy and higher dimensional Dirac Hamiltonian (I.I.T. Jour. Math. and Phys. Sci. (Madras), I, 190-193, 1967)
	Symmetry operations on a hierarchy of matrices, (Jour. Math. Anal. Appl., V.22, p.39-42, 1968)
Alladi Ramakrishnan and I.V.V.Raghavacharyulu.	A note on the representations of Dirac groups, Symposia in Theoretical Phys. and Maths. (Vol.8, 25, 1968) Plenum Press.
Alladi Ramakrishnan	On the relationship between the L-matrix hierarchy and Cartan spinors, (Jour. Math. Anal. Appl., V.22, p.570-576, 1968)
Alladi Ramakrishnan R.Vasudevan N.R.Ranganathan and P.S.Chandrasekaran	A generalization of the L-matrix Hierarchy, (Jour. Math. Anal. Appl., V.23, p.10-14, 1968).
Alladi Ramakrishnan	L-matrices, Quaternions and Propagators, (Jour. Math. Anal. Appl., V.23, p.250-253, 1968)
	A hierarchy of helicity operators in L-matrix theory, (I.I.T., Jour. Math. and Phys. Sci. (Madras) 1968
	On the Algebra of L-matrices, (Symposia in Theoretical Phys. and Maths. Plenum Press, Vol.9, 73-78, 1969)

Alladi Ramakrishnan and R.Vasudevan	A Hierarchy of idempotent matrices (Symposia in Theoretical Phys. and Maths., Plenum Press, Vol.9, 85-88, 1968).
Alladi Ramakrishnan	L-matrices and propagators with imaginary parameters, (Symposia in Theoretical Phys. and Maths. Plenum Press, Vol.9, 79-84, 1969)
	Generalized helicity matrices, (Jour. Math. Anal. Appl., Vol.26, 88-91, 1969).
Alladi Ramakrishnan P.S.Chandrasekaran T.S.Santhanam and A.Sundaram	Helicity matrices in the generalized Clifford Algebra, (Jour. Math. Anal. Appl., Vol.26, 275-278, 1969)
Alladi Ramakrishnan P.S.Chandrasekaran and T.S.Santhanam	L-matrices and the fundamental theorem of spinor theory, Symposia in Theoretical Phys. and Maths., (Vol.10, 1970), Plenum Press.
Alladi Ramakrishnan T.S.Santhanam and P.S.Chandrasekaran	On the representations of generalized Clifford algebras, (I.I.T. Jour. of Math. and Phys. Sci. Madras, Vol.3, 307, 1969)
Alladi Ramakrishnan P.S.Chandrasekaran N.R.Ranganathan T.S.Santhanam and R.Vasudevan	The generalized Clifford algebra and the unitary group (Jour. Math. Anal. Appl., Vol.27, 164-170, 1969)
Alladi Ramakrishnan R.Vasudevan P.S.Chandrasekaran and N.R.Ranganathan	Kemmer algebra from generalized Clifford elements, (Jour. Math. Anal. Appl., Vol.28, 108-111, 1969)
Alladi Ramakrishnan	Pageant of modern physics -- Planck to Gell-Mann, (C.P.Ramaswamy Iyer Memorial Volume, Madras)
Alladi Ramakrishnan	Unitary generalization of Pauli Matrices, Symposia in Theoretical Phys. and Maths., (Vol.10, 1970), Plenum Press.
Alladi Ramakrishnan, P.S. Chandrasekaran, N.R.Ranganathan, T.S.Santhanam, R. Vasudevan.	Idempotent matrices from a generalized Clifford Algebra, (Jour.Math. Anal.Appl.,Vol.27,563-564, 1969).

Alladi Ramakrishnan and I.V.V.Raghavacharyulu	Generalized Clifford basis and infinitesimal generators of unitary groups (Symposia in Theoretical Phys. and Maths., (Vol.10, 1970) Plenum Press)
Alladi Ramakrishnan R.Vasudevan and P.S.Chandrasekaran	Representations of Para-Fermi and generalized Clifford algebra (Jour. Math. Anal. Appl., in Press, 1970)
Alladi Ramakrishnan R.Vasudevan and P.S.Chandrasekaran	Algebras derived from polynomial conditions (Jour. Math. Anal. Appl. in Press, 1970)
Alladi Ramakrishnan	On the composition of generalized helicity matrices (Jour. Math. Anal. Appl., in Press, 1970)
Alladi Ramakrishnan	Should we revise our notions about Spin and Parity in Relativistic Quantum Theory? (Jour. Math. Phys. Sciences, Vol.3, p.213-219, 1969)
Alladi Ramakrishnan	Symmetries Associated with the Roots of the Unit Matrix (I.I.T., Jour. Math. Phys. Sci. Madras), Vol.3, 317, (1969)
Alladi Ramakrishnan R.Vasudevan and P.S.Chandrasekaran	Para-Fermi Operators and special Unitary Algebras, (Jour. Math. Anal. Appl., in Press, 1970)
Alladi Ramakrishnan and R.Vasudevan	On generalized Idempotent Matrices (Jour. Math. Anal. Appl., 1970)
Alladi Ramakrishnan	Mathematical Logic as a Guide to Physical Thought (Science Reporter, Vol.1, 24-26, 1970)
	New Generalizations of Pauli Matrices (Proceedings of the International Conference on Symmetries and Quark models, Wayne State University, Detroit, Michigan, U.S.A., June 18-20, 1969. Published by Gordon and Breach, U.S.A.)
	The Weak interaction Hamiltonian in L-matrix theory (Rand Corporation)
	On the shell-structure of an L-matrix (Jour. Math. Anal. and Appln. in Press).

THE DIRAC HAMILTONIAN AS A MEMBER OF A HIERARCHY OF MATRICES *

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THE DIRAC HAMILTONIAN AS A MEMBER OF A HIERARCHY OF MATRICES*

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'Of strange combinations out of common things'
-- Shelley

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We shall give a method of generating a hierarchy of square matrices L_m involving m independent continuous parameters $\lambda_1, \ldots, \lambda_m$ such that

$$L_m^2 = (\lambda_1^2 + \cdots + \lambda_m^2) I \tag{1}$$

as m takes values 2,3,... . We shall show that the L matrices can be expressed as a linear combination of m 'generator' matrices independent of the parameters. The L matrices fall into one of two classes, saturated or unsaturated according as m is odd or even.

One of the most interesting features of this hierarchy is that the Pauli matrices are recognised to be the generator matrices which saturate L_2, while the Dirac Hamiltonian is an unsaturated L_4.

We start by writing

$$L_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2}$$

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and requiring that

$$L_2^2 = \begin{bmatrix} a^2+bc & (a+d)b \\ (a+d)c & d^2+bc \end{bmatrix} = \begin{pmatrix} \lambda_1^2+\lambda_2^2 & 0 \\ 0 & \lambda_1^2+\lambda_2^2 \end{pmatrix} \quad (3)$$

L_2 then falls into canonical forms of two distinct types.

Type I

$$L_2 = \begin{bmatrix} 0 & \lambda_1 - i\lambda_2 \\ \lambda_1 + i\lambda_2 & 0 \end{bmatrix} \quad (4)$$

or

Type II

$$L_2 = \begin{bmatrix} \lambda_2 & \lambda_1 \\ \lambda_1 & -\lambda_2 \end{bmatrix} \quad (5)$$

Since the relation $L_2^2 = (\lambda_1^2 + \lambda_2^2) I$ is symmetric in λ_1 and λ_2 we can interchange λ_1 and λ_2 or replace λ_1 or λ_2 by $-\lambda_1$ or $-\lambda_2$, but these operations do not alter the type.

To generate L_3, we can adopt the same procedure as we did to find L_2. We can define

$$L_3 = \begin{pmatrix} 0 & L_2 - i\lambda_3 I \\ L_2 + i\lambda_3 I & 0 \end{pmatrix} \quad (6)$$

or

$$L_3 = \begin{pmatrix} \lambda_3 I & L_2 \\ L_2 & -\lambda_3 I \end{pmatrix} \quad (7)$$

or

$$L_3 = \begin{pmatrix} L_2 & \lambda_3 I \\ \lambda_3 I & -L_2 \end{pmatrix} \tag{8}$$

or

$$L_3 = \begin{pmatrix} \lambda_3 & \lambda_1 - i\lambda_2 \\ \lambda_1 + i\lambda_2 & -\lambda_3 \end{pmatrix} \tag{9}$$

We notice that while in the first three cases L_3 has double the dimension of L_2, in the last case the dimension of L_3 remains the same as that of L_2. In general L_{m+1} can be generated from L_m as follows

$$L_{m+1} \doteq \begin{pmatrix} 0 & L_m - i\lambda_{m+1} I \\ L_m + i\lambda_{m+1} I & 0 \end{pmatrix} \tag{10}$$

or

$$L_{m+1} = \begin{pmatrix} \lambda_{m+1} I & L_m \\ L_m & -\lambda_{m+1} I \end{pmatrix} \text{ or } \begin{pmatrix} L_m & \lambda_{m+1} I \\ \lambda_{m+1} I & -L_m \end{pmatrix} \tag{11}$$

In all these three cases, the dimension of L_{m+1} is double that of L_m. However, if L_m is of the form

$$\begin{pmatrix} 0 & L_{m-1} - i\lambda_m \\ L_{m-1} + i\lambda_m & 0 \end{pmatrix} \tag{12}$$

then L_{m+1} can be generated with the same dimension as L_m by defining

$$L_{m+1} = \begin{pmatrix} \lambda_{m+1} I & L_{m-1} - i\lambda_m \\ L_{m-1} + i\lambda_m & -\lambda_{m+1} I \end{pmatrix} \tag{13}$$

Thus the most 'economical' way of building L_{m+1} is to 'saturate' the L_m if L_m is 'unsaturated', i.e., having zeros on the diagonal. Therefore, L_m is of type II while L_{m+1} is saturated. We thus have the table connecting the number of parameters, dimension of the matrix and type.

Matrix	Number of parameter	Dimension	Character
L_1	1	1	Saturated
L_2	2	2	Unsaturated
.	3	2	Saturated
.	4	$2^2 = 4$	Unsaturated
.	5	$2^2 = 4$	Saturated
	⋮		⋮
L_{2n}	2n	2^n	Unsaturated
L_{2n+1}	2n+1	2^n	Saturated

Thus the saturated L matrices involve an odd number of parameters. L_{2n+1} can be obtained from L_{2n-1} bg performing a σ-operation on it defined as

$$L_{2n+1} = \sigma(L_{2n-1})$$

i.e.

$$L_{2n+1} = \begin{pmatrix} \lambda_{2n+1} I & L_{2n-1} - i\lambda_{2n} I \\ L_{2n-1} + i\lambda_{2n} I & -\lambda_{2n+1} I \end{pmatrix} \tag{14}$$

The σ-operation involves the addition of two parameters and the doubling of the dimension.

We shall study these matrices by writing

$$L_{2n+1} = \sum_{i=1}^{2n+1} \lambda_i \mathcal{L}_i^{2n+1} \tag{15}$$

where $\mathcal{L}_i^{2n+1}$ are (2n+1) 'generator matrices' independent of λ_i. Then if

$$L_{2n-1} = \sum_{i=1}^{2n+1} \lambda_i \mathcal{L}_i^{2n+1} \tag{16}$$

we have

$$\mathcal{L}_i^{2n+1} = \begin{pmatrix} 0 & \mathcal{L}_i^{2n-1} \\ \mathcal{L}_i^{2n-1} & 0 \end{pmatrix} \qquad i = 1, 2, \ldots, 2n-1 \tag{17}$$

and

$$\mathcal{L}_{2n}^{2n+1} = \begin{pmatrix} 0 & -i\ I \\ i\ I & 0 \end{pmatrix} \; ; \; \mathcal{L}_{2n+1}^{2n+1} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{18}$$

Thus the σ-operation on L_{2n-1} consists of generating $\mathcal{L}_i^{2n+1}$ from $\mathcal{L}_i^{2n-1}$ and adding two matrices $\mathcal{L}_{2n}^{2n+1}$ and $\mathcal{L}_{2n+1}^{2n+1}$.

We now summarise the results we have obtained as follows:

1. A saturated L matrix involves $(2n+1)$, i.e., an odd number of parameters. Its dimension is 2^n. It can be expressed as a linear combination of $(2n+1)$ matrices $\mathcal{L}_i^{2n+1}$, $i = 1,..,(2n+1)$ with λ_i as their coefficients, respectively.

2. An L matrix involving $2n$(even) parameters is unsaturated. Its dimension is 2^n and it can be expressed as a linear combination of $2n$ matrices, i.e., a set obtained by omitting one of the $2n+1$ matrices.

There are $(2^n)^2$ independent matrices of dimension 2^n and these can be generated either from the $2n+1$ matrices which saturate L or the $2n$ matrices as follows:

The $2n+1$ matrices have the important feature that their product is 'idempotent'. More precisely,

$$\mathcal{L}_1^{2n+1}\mathcal{L}_2^{2n+1}\cdots\mathcal{L}_{2n+1}^{2n+1} = i^n I \tag{19}$$

Hence to generate all the independent matrices we form products of $2,3,\ldots,n$ matrices. The product of $(n+r)$ matrices is just equal to the product of $(n-r+1)$ matrices and a numerical factor, and so no independent matrix can be generated by taking products of more than n matrices out of the $2n+1$. The number of independent matrices are

$$\binom{2n+1}{0} + \binom{2n+1}{1} + \cdots + \binom{2n+1}{n} = 2^{2n} \tag{20}$$

The $2n+1$ matrices anticommute with each other. The idempotent property implies that $2n+1$ is the maximum of anti-commuting matrices in a set of 2^n independent matrices.

If on the other hand we had taken $2n$ matrices, we can form products of 2,3,4...2n matrices and we obtain

$$\binom{2n}{0} + \binom{2n}{1} + \cdots + \binom{2n}{2n} = 2^{2n} \tag{21}$$

independent matrices.

The following two properties of the L matrices are immediately noticed

1. If A is a nonsingular matrix then ALA^{-1} is also an L matrix since

$$\begin{aligned} (ALA^{-1})(ALA^{-1}) &= (AL^2A^{-1}) \\ &= A\lambda^2 I A^{-1} \\ &= \lambda^2 I \end{aligned} \tag{22}$$

2. If $\underline{\Lambda}$ is a diagonal matrix with half its diagonal elements equal to $\pm\lambda$ then the matrix

$$U = L + \Lambda I \tag{23}$$

satisfies the equation

$$LU = U\Lambda \tag{24}$$

i.e, the columns of the matrix U are eigenvectors of L with eigenvalues $\pm\lambda$.

PAULI MATRICES AND THE DIRAC HAMILTONIAN

Starting with L_1, which is the number

$$L_1 = \lambda_1 \tag{25}$$

L_3 is obtained by a σ-operation as

$$L_3 = \begin{pmatrix} \lambda_3 & \lambda_1 - i\lambda_2 \\ \lambda_1 + i\lambda_2 & -\lambda_3 \end{pmatrix} = \sum_i \lambda_i \mathcal{L}_i^3 \tag{26}$$

If σ_x, σ_y and σ_z are the Pauli matrices defined as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \tag{27}$$

we recognise that

$$\sigma_x = \mathcal{L}_1^3; \quad \sigma_y = \mathcal{L}_2^3; \quad \sigma_z = \mathcal{L}_3^3 \tag{28}$$

Performing a σ-operation on L_3 we obtain L_5.

$$L_5 = \sum \lambda_i \mathcal{L}_i^5 \tag{29}$$

If $\alpha_x, \alpha_y, \alpha_z, \beta$ are the Dirac matrices and γ_5 the product of the four gamma matrices defined as

$$\alpha_x = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} ; \quad \alpha_y = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} ; \quad \alpha_z = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} I & \\ & -I \end{pmatrix} ; \quad \gamma_5 = \begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix} \tag{30}$$

we recognise that

$$\mathcal{L}_1 = \sigma_x ; \quad \mathcal{L}_2 = \alpha_y ; \quad \mathcal{L}_3 = \alpha_z$$

$$\mathcal{L}_4 = -\beta\gamma_5 ; \quad \mathcal{L}_5 = \beta . \tag{31}$$

Thus

$$L_5 u = \left(\sum_{i=1}^{5} \lambda_i \mathcal{L}_i^{\,5} \right) u = \pm \lambda u \tag{32}$$

is an eigenvector equation for the saturated matrix L_5. From (24) we immediately write the U matrix solution for L_5 as

$$\begin{pmatrix} \lambda + \lambda_5 & 0 & \lambda_3 - i\lambda_4 & \lambda_1 - i\lambda_2 \\ 0 & \lambda + \lambda_5 & \lambda_1 + i\lambda_2 & -\lambda_3 - i\lambda_4 \\ \lambda_3 + i\lambda_4 & \lambda_1 - i\lambda_2 & -\lambda - \lambda_5 & 0 \\ \lambda_1 + i\lambda_2 & -\lambda_3 + i\lambda_4 & 0 & -\lambda - \lambda_5 \end{pmatrix} \tag{33}$$

where each solamn is the eigenvector of L_5, corresponding to the eigenvalue $+\lambda$ for the first two columns and $-\lambda$ for the last two columns and

$$\lambda = +\left(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2+\lambda_5^2\right)^{1/2} \tag{34}$$

If we omit L_4 and λ_4 and we obtain the equation for an unsaturated L_4.

$$L_4 u = \left(\lambda_1 \gamma_1 + \lambda_2 \gamma_2 + \lambda_3 \gamma_3 + \lambda_5 \gamma_5\right) u = \lambda u \tag{35}$$

If we write

$$\lambda_1 = p_x \; ; \quad \lambda_2 = p_y \; ; \quad \lambda_3 = p_z \; ; \quad \lambda_5 = m \; , \quad \lambda = E \tag{36}$$

where p_x, p_y, p_z are the components of momenta, m the mass and E the energy, we obtain the eigenvalue equation for the Dirac Hamiltonian.

If on the other hand we omit L_4, we obtain another unsaturated equation

$$\left(\vec{\alpha}\cdot\vec{p} - \beta\gamma_5 m\right) u = E u \tag{37}$$

If the Dirac equation can be written in the form

$$\left(\not{p} - m\right) u_D = 0 \tag{38}$$

with

$$\not{p} = \gamma^{\mu} p_{\mu} = \gamma\cdot p \; ; \quad \vec{\gamma} = \beta\vec{\alpha} \; ; \quad \gamma_0 = \beta \; ; \; \mu = 0,1,2,3. \tag{39}$$

then the other unsatuated equation can be written as

$$(\not{p} - m)\, u_A = 0 \tag{40}$$

with

$$\not{p} = \gamma^{\mu} \gamma_5 \, p_{\mu} \tag{41}$$

Solving this equation, we can get the spinor solutions u_A.

We shall now obtain the relation between u_A and u_D. The two spinor solutions u_D for positive energy are given either in

Form I:

$$u_D^{I} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p_z \\ p_x + i p_y \end{pmatrix}, \quad \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ p_x - i p_y \\ -p_z \end{pmatrix} \tag{42}$$

or

Form II:

$$u_D^{II} = \frac{1}{\sqrt{E-m}} \begin{pmatrix} p_z \\ p_x + i p_y \\ E-m \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{E-m}} \begin{pmatrix} p_x - i p_y \\ -p_z \\ 0 \\ E-m \end{pmatrix} \tag{43}$$

Form II is obtained by operating on Form I with $(-i\gamma_5)$ and replacing m by -m. Form B is unsuitable since in the rest system, the normalization factor $1/(E-m)^{1/2}$ becomes infinite while the spinor vanishes. On the other hand form B is suitable for negative energy.

In a similar way, we get the two solutions u_A for positive energy in

Form I:

$$u_A^{I} = \frac{1}{\sqrt{E}}\begin{pmatrix} E \\ 0 \\ p_z+im \\ p_x+ip_y \end{pmatrix}, \quad \frac{1}{\sqrt{E}}\begin{pmatrix} 0 \\ E \\ p_x-ip_y \\ -p_z+im \end{pmatrix} \tag{44}$$

or

Form II:

$$u_A^{II} = \frac{1}{\sqrt{E}}\begin{pmatrix} p_z-im \\ p_x+ip_y \\ E \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{E}}\begin{pmatrix} p_x-ip_y \\ -p_z-im \\ 0 \\ E \end{pmatrix} \tag{45}$$

We note that interesting feature that both these forms are suitable for the rest system and have the same normalization factor. We also notice that

$$u_A^{I} + i\, u_A^{II} = \sqrt{\frac{E+m}{E}}\; (1+\gamma_5)\, u_D^{I} \tag{46}$$

and $(1+\gamma_5)\, u_D^{I}$ is a solution of the other unsaturated equation can be seen from the observation that[1]

$$\left(\frac{1+\gamma_5}{\sqrt{2}}\right)\left(\vec{\alpha}\cdot\vec{p} + \beta m\right)\left(\frac{1-\gamma_5}{\sqrt{2}}\right) = \left(\vec{\alpha}\cdot\vec{p} - \beta\gamma_5 m\right) \tag{47}$$

It should be noticed that $(1+\gamma_5)/\sqrt{2}$ is nonsingular and has $(1-\gamma_5)/\sqrt{2}$ as its inverse.

[1] This argument is due to Santhanam.

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HELICITY AND ENERGY AS MEMBERS OF A HIERARCHY OF EIGENVALUES**

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** This paper was published in the 'Journal of Mathematical Analysis and Applications', Vol.20, No.2, p.397-401, 1967 Since the reprints of this paper have gone out of stock, type-script copies are now made.

HELICITY AND ENERGY AS MEMBERS OF A HIERARCHY OF EIGENVALUES

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In a previous contribution to this Journal[1], a hierarchy of matrices L_{2n+1} $(n = 0,1,\ldots)$ was generated through what was defined by the author as a σ-operation which takes L_{2n-1} to L_{2n+1} as follows:

$$L_{2n+1} = \sigma(L_{2n-1}) = \begin{pmatrix} \lambda_{2n} I & L_{2n-1} - i\lambda_{2n+1} I \\ L_{2n-1} + i\lambda_{2n+1} I & -\lambda_{2n} I \end{pmatrix} \qquad (1)$$

L_{2n+1} contains $2n+1$ parameters $\lambda_1,\ldots,\lambda_{2n+1}$ and is of dimension 2^n. The unit matrix I in (1) is of dimension 2^{n-1}, the same as that of L_{2n-1}. The matrix L_{2n+1} has 2^n independent eigenvectors, but only two eigenvalues $\pm\Lambda_n$ given by

$$\begin{aligned}\Lambda_n &= +\left(\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_{2n+1}^2\right)^{1/2} \\ &= +\left(\Lambda_{n-1}^2 + \lambda_{2n}^2 + \lambda_{2n+1}^2\right)^{1/2}\end{aligned} \qquad (2)$$

where

$$\Lambda_{n-1} = +(\lambda_1^2 + \cdots + \lambda_{2n-1}^2)^{1/2} \qquad (3)$$

$\pm \Lambda_{n-1}$ are the eigenvalues of the 2^{n-1} dimensional matrix L_{2n-1} and equally of the 2^n dimensional matrix

$$\begin{pmatrix} L_{2n-1} & 0 \\ 0 & L_{2n-1} \end{pmatrix} \qquad (4)$$

Similarly we find a sequence of eigenvalues

$$\pm\Lambda_n \; ; \; \pm\Lambda_{n-1} \; ; \; \cdots \; ; \Lambda_1 \qquad (5)$$

corresponding to the sequence of n matrices of the same dimension 2^n,

$$L_{2n+1} \; ; \begin{pmatrix} L_{2n-1} & 0 \\ 0 & L_{2n-1} \end{pmatrix} ; \cdots ; \begin{pmatrix} L_3 & & & & \\ & L_3 & & 0 & \\ & & \ddots & & \\ & 0 & & L_3 & \\ & & & & L_3 \end{pmatrix} \qquad (6)$$

L_{2m+1} occurring 2^{n-m} times on the diagonal. These n matrices commute with each other and hence the 2^n eigenvectors of the 2^n dimensional matrix L_{2n+1} can be obtained as the simultaneous eigenvectors of this sequence.

It is to be noted that though the square of L_{2n+1} is a unit matrix with the sum of squares of $(2n + 1)$ parameters as its coefficient, the eigenvectors of L_{2n+1} can be distinguished by only n eigenvalues,

$$\pm \Lambda_k, \quad (k = 1,2,\ldots n) \text{ defined in (2).}$$

Taking the particular case of L_3 and L_5, we note that the eigenvectors of L_5 are distinguished by the eigenvalues $\pm \Lambda_1$ and $\pm \Lambda_2$. In the case of the Dirac equation, we identify that

$$\lambda_1 = p_x \ ; \ \lambda_2 = p_y \ ; \ \lambda_3 = p_z$$
$$L_3 = \sigma \cdot \lambda = \sigma \cdot p \tag{7}$$

$$\Lambda_1 = +\left(p_x^2 + p_y^2 + p_z^2 \right)^{1/2} = p \tag{8}$$

Setting $\lambda_5 = m$ and $\lambda_4 = 0$ in L_5, we identify

$$L_5(\lambda_5 = m \ ; \ \lambda_4 = 0) = H \tag{9}$$

the Dirac Hamiltonian. Thus Λ_2 is identified to be the energy, while

$$\pm \Lambda_1 = \text{helicity}. \Lambda_1 ; \tag{10}$$

since helicity is the eigenvalue of the operator $(\sigma.p)/\Lambda_1$ and is equal to ± 1. Hence, we find the helicity multiplied by a positive numerical factor Λ_1 and energy are members of the same hierarchy of eigenvalues $\pm \Lambda_i$, corresponding to L_{2i+1}.

This implies that we have interpreted helicity in the light of the Clifford algebraic structure of the L matrices as contrasted with the structure of a Lie algebra associated with angular momentum.

APPENDIX

Pauli and Dirac Spinors as the Members of a Hierarchy of Eigenvectors

We now obtain the eigenvectors of L_{2n+1} in terms of those of L_3.

Let $u(2n+1)$[1] be an eigenvector of L_{2n+1} with dimension 2^n. We shall show that $u(2n+1)$ can be written as

$$u(2n+1) = \begin{pmatrix} a\,u(2n-1) \\ b\,u(2n-1) \end{pmatrix} \tag{A(1)}$$

by just 'solving' for a and b. Substituting the above form of $u(2n+1)$ in the eigenvalue equation

[1] In the case of the eigenvectors we do not indicate the number of parameters by a suffix but only inside brackets so that there will be no confusion with the usual practice of denoting the components of eigenvector though a suffix.

$$L_{2n+1}\, u^{\pm}(2n+1) = \pm\Lambda_n\, u^{\pm}(2n+1) \qquad A(2)$$

we obtain an eigenvalue equation for the two dimensional vector with components a, b :

$$\begin{pmatrix} \lambda_{2n+1} & \Lambda_{n-1} - i\lambda_{2n} \\ \Lambda_n + i\lambda_{2n} & -\lambda_{2n+1} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\Lambda_n\begin{pmatrix} a \\ b \end{pmatrix} \qquad A(3)$$

yielding two solutions

$$\frac{a}{b} = \frac{+\Lambda_{n-1} - i\lambda_{2n}}{\pm\Lambda_n - \lambda_{2n+1}} = \frac{\pm\Lambda_n + \lambda_{2n+1}}{\Lambda_{n-1} + i\lambda_{2n}} \qquad A(4)$$

Similarly we have two more solutions if we replace Λ_{n-1} by $-\Lambda_{n-1}$ in Eq.(3).

We notice that we have obtained the eigenvector of an L_3 matrix with λ_3 replaced by λ_{2n+1}, λ_1 by $\pm\Lambda_1$ and λ_2 by λ_{2n}. By iterating this procedure to obtain the eigenvectors of L_{2n-1}, L_{2n-3} ... and L_5, we determine the entire set of 2^n eigenvectors of L_{2n+1} as the simultaneous eigenvector of n matrices with a sequence of eigenvalues

$$\pm\Lambda_n,\ \pm\Lambda_{n-1},\ ---,\ \pm\Lambda_1$$

Considering the particular case of L_5 and L_3 we have as the eigenvectors of L_3

$$u^{\pm}(3) = \begin{pmatrix} \lambda_1 - i\lambda_2 \\ \pm\Lambda_1 - \lambda_3 \end{pmatrix} \qquad \text{A(5)}$$

corresponding to the two eigenvalues $\pm\ \Lambda_1$ with $\Lambda_1 = \left(\lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2\right)^{1/2}$. The four eigenvectors of L_3 are

$$\begin{pmatrix} (\pm\Lambda_1 - i\lambda_4)\, u^{+}(3) \\ (\pm\Lambda_2 - \lambda_5)\, u^{+}(3) \end{pmatrix} \ \& \ \begin{pmatrix} (\pm\Lambda_1 - i\lambda_4)\, u^{-}(3) \\ (\pm\Lambda_2 - \lambda_5)\, u^{-}(3) \end{pmatrix}$$

Writing $\lambda_1 = p_x$, $\lambda_2 = p_y$, $\lambda_3 = p_z$, $\Lambda_1 = p$, we identify $u^{\pm}(3)$ as the Pauli spinors and setting $\lambda_5 = m$, $\lambda_4 = 0$, $\Lambda_2 = E$, we obtain the Dirac spinors as the simultaneous eigenstates of energy and helicity.

This can be compared with the conventional methods of obtaining these solutions for the Dirac equation[2)].

The procedure we have adopted to resolve the 'degeneracy' in the eigenvectors of L_{2n+1} amounts to generating the independent eigenvectors of dimension 2^n, starting from the two independent eigenvectors of the two dimensional L_3. Complementary to this method of generating eigenvectors 'inside-out' we can devise a method 'outside-in', starting from the most general form of the eigenvector of L_{2n+1} and requiring it to be the simultaneous eigenvector of L_{2n-1}, L_{2n-3}, ... and L_3.

If $\omega(2n-1)$ be an arbitrary 'vector' of dimension 2^{n-1} and $\omega'(2n-1)$ a 'vector' defined as

$$\omega'(2n-1) = \frac{L_{2n-1} - i\lambda_{2n} I}{\Lambda_n - \lambda_{2n+1}}\, \omega(2n-1) \tag{A(7)}$$

then the 2^n dimensional vector

$$u(2n+1) = \begin{pmatrix} \omega'(2n-1) \\ \omega(2n-1) \end{pmatrix} \tag{A(8)}$$

is an eigenvector of L_{2n+1}. This is not in general an eigenvector of $L_{2n+1}(2m+1)$, where

$$L_{2n+1}(2m+1) = \begin{pmatrix} L_{2m+1} & & & \\ & L_{2m+1} & & \\ & & \ddots & \\ & & & L_{2m+1} \end{pmatrix} \tag{A(9)}$$

with L_{2m+1} being repeated 2^{n-m} times on the diagonal.

If we require this to be an eigenvector of L^{2n-1}_{2n+1}, $\omega(2n-1)$ must be chosen to be an eigen-vector of L_{2n-1} and hence must be written as $u(2n-1)$, where

$$u(2n-1) = \begin{pmatrix} \omega'(2n-3) \\ \omega(2n-3) \end{pmatrix} \;;\; \omega'(2n-3) = \left(\frac{L_{2n-3} - i\lambda_{2n-2}}{\Lambda_{n-1} - \lambda_{2n-1}} \right) \omega(2n-3). \qquad A(10)$$

ω_{2n-3} being an arbitrary vector of 2^{n-2} dimensions. In such a case we can replace the matrix L_{2n-1} by its eigenvalue Λ_{n-1}. The successive application of this producedure leads to the resolution of the degeneracies and we arrive at the same expression for $u(2n+1)$ obtained earlier.

Instead of taking $\omega(2n-1)$ arbitrary and defining $\omega'(2n-1)$ in terms of it, we can as well take $\omega'(2n-1)$ arbitrary and write

$$\omega(2n-1) = \frac{L_{2n-1} + i\lambda_{2n} I}{\Lambda_n + \lambda_{2n+1}} \omega'(2n-1) \qquad A(11)$$

As before if we write $\omega.p$ for L_3 and set $\lambda_4 = 0$, $\lambda_5 = m$ we recognize the famous 'bispinor' form of the solutions of the Dirac equation within the larger framework of our L-matrix hierarchy.

If corresponding to the positive eigenvalues, $+\ _n$, we choose the arbitrary vector $\omega'(2n-1)$ as any one of the set of

2^{n-1} basic vectors of dimension 2^{n-1} :

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots\cdots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \qquad A(12)$$

we obtain a set of 2^{n-1} independent eigenvectors of dimension 2^n. As regards the negative eigenvalues, $-\Lambda_n$, if we choose the vector $\omega(2n-1)$ to be one of the above set A(12), we obtain the other 2^{n-1} independent eigenvectors of dimension 2^n. The two sets together are recognized to be just the 2^n columns of the matrix

$$L_{2n+1} + [\Lambda]\, I$$

where $[\Lambda]$ is a 2^n dimensional diagonal matrix with the first 2^{n-1} terms on the diagonal being equal to the eigenvalue $+\Lambda_n$ and the rest of the 2^{n-1} terms being equal to $-\Lambda_n$, consistent with a result established in an earlier paper[1].

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L-MATRIX HIERARCHY AND THE HIGHER DIMENSIONAL DIRAC HAMILTONIAN

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L-Matrix Hierarchy and the Higher Dimensional Dirac Hamiltonian

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The Dirac Hamiltonian was recognised to be the member of the L-matrix hierarchy defined by the prescription

$$L_{2n+1}^2 = (\lambda_1^2 + \cdots + \lambda_{2n+1}^2) I \tag{1}$$

$$L_{2n} = L_{2n+1} (\lambda_{2n} \text{ or } \lambda_{2n+1} = 0) \tag{2}$$

where the parameters $\lambda_1, \lambda_2, \ldots, \lambda_{2n+1}$ are pure real or imaginary and $\pm\Lambda_n$ are the eigenvalues of L_{2n+1}. The usual representation of the Dirac Hamiltonian is obtained by setting in L_5,

$$\lambda_1 = p_x, \lambda_2 = p_y, \lambda_3 = p_z, \lambda_4 = 0, \lambda_5 = m \tag{3}$$

and

$$\Lambda_2 = E$$

Since L_4 is of dimension 4 x 4, it has four independent eigenvectors though it has only two eigenvalues $\pm\Lambda_2$. The degeneracy is resolved by requiring the eigenvector of L_4 to be the simultaneous eigenvectors of the matrix

$$\begin{pmatrix} L_3 & 0 \\ 0 & L_3 \end{pmatrix}$$

where the 2 x 2 matrix L_3 occurs twice on the diagonal. The eigenvalues of the above matrix are

$$\mp\Lambda_1 = \pm\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)^{1/2} = \pm\left(p_x^2+p_y^2+p_z^2\right)^{1/2} \qquad (4)$$

$$= \pm |p|$$

$$= \text{helicity x } p$$

where p is the modulus of the momentum. Thus we obtain the simultaneous eigenvectors of energy and helicity.

We now discuss the possibility of obtaining higher dimensional Dirac Hamiltonians from the hierarchy.

The characteristic feature of the hierarchy is the relationship between the dimension of the L-matrix and the parameters imbedded in it. If $2n+1$ parameters are to be imbedded we need the $2^n \times 2^n$ dimensional matrix L_{2n+1}. However, if $2n+1-m$ parameters are set equal to zero, we have a $2^n \times 2^n$ dimensional matrix with m parameters. This can obviously be a higher dimensional representation of an L-matrix with m parameters.

Thus in L_7 if we set

$$\lambda_1 = p_x,\ \lambda_2 = p_y,\ \lambda_3 = p_z,\ \lambda_4 = \lambda_5 = \lambda_6 = 0,\ \lambda_7 = m$$

we obtain a Dirac Hamiltonian of dimension 8 x 8.

From L-matrix theory we know that the eigenvectors of can be chosen to be the simultaneous eigenvectors of

$$L_3^7 = \begin{pmatrix} L_3 & & & \\ & L_3 & & \\ & & L_3 & \\ & & & L_3 \end{pmatrix}, \quad L_5^7 = \begin{pmatrix} L_5 & \\ & L_5 \end{pmatrix} \qquad (5)$$

where L_3 is of dimension 2 x 2 and L_5 of dimension 4 x 4. In the case when $\lambda_4 = \lambda_5 = 0$, L_5 takes the particular form

$$L_5 = \begin{pmatrix} 0 & L_3 \\ L_3 & 0 \end{pmatrix}$$

In the notation of the Dirac Hamiltonian

$$L_3 = \vec{\sigma}\cdot\vec{p} = \sigma_1 p_x + \sigma_2 p_y + \sigma_3 p_z \qquad (6)$$

$\sigma_1, \sigma_2, \sigma_3$ being the Pauli matrices. Thus the eigenvectors of L_7 are chosen as the simultaneous eigenvectors of the helicity operator and L_5^7. To interpret the latter, we write

$$L_5^7 = \begin{pmatrix} L_5 & 0 \\ 0 & L_5 \end{pmatrix} = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix} L_3^7 \qquad (7)$$

Thus the simultaneous eigenvectors of L_2^7 and L_5^7 are recognised to be the simultaneous eigenvectors of L_5^7 and the 'chirality' operator,

$$\begin{pmatrix} -i\gamma_5 & 0 \\ 0 & -i\gamma_5 \end{pmatrix} \quad \text{where} \quad -i\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (8)$$

the γ_5 notation being used to facilitate comparison with the familiar notation in elementary particle physics.

The eight independent eigenvectors can be explicitly written as

$$U(\pm) = \begin{pmatrix} (\Lambda_1 - i\lambda_4)\,\chi_\pm \\ (\pm\Lambda_2 - \lambda_5)\,\chi_\pm \end{pmatrix}, \quad \chi_\pm = \begin{pmatrix} u_\pm \\ u_\pm \end{pmatrix}, \qquad (9)$$

$$u_\pm = \begin{pmatrix} \lambda_1 - i\lambda_2 \\ \pm\Lambda \; - i\lambda \end{pmatrix}$$

$\lambda_\pm$ correspond to the two eigenvalues of helicity, $\pm\chi$ to the eigenvalues of chirality and $U(\pm)$ to those of energy.

The eigenvalues Λ_1, Λ_2 and Λ_3 in the L-matrix notation are related to the physically interesting eigenvalues as follows:

$$\begin{aligned}
\pm\Lambda_1 &= \text{helicity } |\Lambda_1| \\
\pm\Lambda_2 &= \text{chirality } |\Lambda_2| \\
\pm\Lambda_3 &= \pm\text{energy}
\end{aligned} \tag{10}$$

$$|\Lambda_1| = |\Lambda_2| = p, \quad |\Lambda_3| = +(p^2+m)^{1/2} = |E|$$

It is interesting to note the distinction between the matrices $(1\pm\gamma_5)$ and $(1\pm i\gamma_5)$. The matrix $(1+\gamma_5)$ is non-singular and transforms the eigenvector of an L_5 matrix with $\lambda_4 = 0$, $\lambda_5 = m$ to an eigenvector of an L_5 matrix with $\lambda_4 = m$ and $\lambda_5 = 0$. The matrix $(1\pm i\gamma_5)$ is singular and is a projection operator; operating on the eigenvector of H it yields the eigenvector of γ_5 which is the solution of the second order Dirac equation and not of the equation of first order.

We can construct any higher dimensional Dirac Hamiltonian from L_{2n+1} by setting

$$\begin{aligned}
\lambda_1 = p_x,\ \lambda_2 = p_y,\ \lambda_3 = p_z,\ \lambda_4 = \lambda_5 = \cdots = \lambda_{2n-1} = 0 \\
\lambda_{2n} = m,\ \lambda_{2n+1} = 0
\end{aligned} \tag{11}$$

The 2^n dimensional eigenvector can be chosen to be the simultaneous eigenvector of the matrices of dimension $2^n \times 2^n$

$$L^{2n+1}_{3},\ L^{2n+1}_{5},\ \cdots,\ L^{2n+1}_{2n-1},\ L^{2n+1}_{2n+1}$$

where

$$L^{2n+1}_{2n+1} \equiv L_{2n+1}$$

$$L^{2n+1}_{2m+1} \equiv \begin{pmatrix} L_{2m+1} & & & \\ & L_{2m+1} & & \\ & & \ddots & \\ & & & L_{2m+1} \end{pmatrix} \tag{12}$$

and

$$L_{2m+1} = \begin{pmatrix} 0 & L_{2m-1} \\ L_{2m-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_{2m-1} \\ I_{2m-1} & 0 \end{pmatrix} \begin{pmatrix} L_{2m-1} & 0 \\ 0 & L_{2m-1} \end{pmatrix} \tag{13}$$

where I_{2m-1} is a unit matrix of dimension $2^{m-1} \times 2^{m-1}$. Writing, for obvious reasons

$$i\gamma_5(m) = \begin{pmatrix} 0 & I_{2m+1} \\ I_{2m+1} & 0 \end{pmatrix}$$

we can obtain the eigenvectors of the higher dimensional Hamiltonian L_{2n+1} to be the simultaneous eigenvectors of the following matrices

$$\begin{pmatrix} L_3 & & & \\ & L_3 & & \\ & & \ddots & \\ & & & L_3 \end{pmatrix}, \begin{pmatrix} i\gamma_5(1) & & & \\ & i\gamma_5(1) & & \\ & & \ddots & \\ & & & i\gamma_5(1) \end{pmatrix}, \ldots, \tag{15}$$

$$\begin{pmatrix} i\gamma_5(n-2) & 0 \\ 0 & i\gamma_5(n-2) \end{pmatrix}.$$

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SYMMETRY OPERATIONS ON A HIERARCHY OF MATRICES

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SYMMETRY OPERATORS ON A HIERARCHY OF MATRICES

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INTRODUCTION

In a sequence of contributions[1,2,3] it was shown that the Dirac Hamiltonian with its four parameters P_x, P_y, P_z, m and eigenvalues $\pm E$, is just a member of a hierarchy of matrices $L_{(2n+1)}$, characterized by the following properties

1. L_{2n+1} involves (2n+1) parameters $\lambda_1, \ldots\ldots\ldots, \lambda_{2n+1}$

2. L_{2n+1} is of dimension 2^n and

$$L^2_{2n+1} = (\lambda_1^2 + \ldots\ldots\ldots + \lambda^2_{2n+1})\ I$$

where I is a unit matrix of dimension 2^n

3. The eigenvalues of L_{2n+1} are $\pm$ n where

$$\Lambda_n = \sqrt{\lambda_1^2 + \ldots\ldots\ldots + \lambda^2_{2n+1}}$$

4. The set of n quantities $\pm\Lambda_1, \pm\Lambda_2, \ldots\ldots, \pm\Lambda_n$ are the eigenvalues of the n matrices $L_3, L_5, \ldots\ldots L_{2n+1}$ and also of the n matrices L^{2n+1}_{2m+1} (m = 1,.....n) with the same dimension 2^n

$$L^{2n+1}_{2n+1} = L_{2n+1} \;, \quad L^{2n+1}_{2n-1} = \begin{bmatrix} L_{2n-1} & 0 \\ 0 & L_{2n-1} \end{bmatrix} , \ldots$$

$$\ldots \quad L^{2n+1}_{2m-1} = \begin{bmatrix} L_{2m+1} & & \\ & L_{2m+1} & \\ & & \ddots & \\ & & & L_{2m+1} \end{bmatrix} , \quad L_3^{2n+1} = \begin{bmatrix} L_3 & & \\ & L_3 & \\ & & \ddots & \\ & & & L_3 \end{bmatrix} \qquad (1)$$

L^{2n+1}_{2m+1} is a matrix with L_{2m+1} repeated (n-m) times on the diagonal.

5. L_{2n+1} has 2^n independent eigenvectors, each of which is a simultaneous eigenvector of the above set.

The Dirac Hamiltonian is identified to be L_5 setting $\lambda_1 = P_x$, $\lambda_2 = P_y$, $\lambda_3 = P_z$, $\lambda_4 = 0$ and $\lambda_5 = m$. Since symmetry operations on the Dirac equation deal with reversal of these parameters and the eigenvalue E their generalization to the hierarchy consists in studying the reversal of the parameters $\lambda_1, \ldots\ldots, \lambda_{2n+1}$ and the eigenvalues $\pm\Lambda_1, \pm\Lambda_2 \ldots\ldots, \pm\Lambda_n$

SYMMETRY OPERATIONS

Our starting point is the realization that the L_{2n+1} is expressible as a linear combination of (2n+1) matrices

$$L_{2n+1} = \lambda_1 \alpha_1^{2n+1} + \lambda_2 \alpha_2^{2n+1} + \ldots.. \qquad (2)$$

The condition that L^2 is a multiple of the unit matrix implies that the α matrices anticommute with one another. There is no matrix that anticommutes with all of them.

Let u be an eigenvector of the set defined by (1), corresponding to a particular choice of signs of the eigenvalues from the set $\pm \Lambda_1, \ldots\ldots, \pm \Lambda_n$. We note that $\alpha_i^{2n+1} u$ is an eigenvector of an L matrix in which λ_i is reversed and corresponds to a reversed eigenvalue. In familiar language, we say that α_i reverses the signs of λ_i and $\pm \Lambda_n$. Thus we tabulte the matrices and the parameters and eigenvalues reversed by them.

<u>Matrix.</u>	<u>Reversal of parameters</u>	<u>Reversal of eigenvalues</u>
α_i^{2n+1}	λ_i	$\pm \Lambda_n$
$\alpha_i^{2n+1} \alpha_j^{2n+1}$	λ_i, λ_j	-
$\alpha_i^{2n+1} \alpha_j^{2n+1} \alpha_k^{2n+1}$	$\lambda_i, \lambda_j, \lambda_k$	$\pm \Lambda_n$

(3)

Thus a product of an odd number of α's reverses the eigenvalues and an even number of α's does not. If we are interested only in the reversal of the eigenvalue and not the parameters, we can set

$$\lambda_3 = \lambda_5 = \ldots\ldots\ldots = \lambda_{2n+1} = 0 \tag{4}$$

In such a case, the n-matrices $\alpha_3^{2n+1}, \alpha_5^{2n+1} \ldots\ldots \alpha_{2n+1}^{2n+1}$ reverse only the sign of the eigenvalues

$$\pm\Lambda_1, \ldots\ldots\ldots, \pm\Lambda_n$$

where α_{2m+1}^{2n+1} is defined as

$$\alpha_{2m+1}^{2n+1} = \begin{bmatrix} \alpha_{2m+1}^{2m+1} & & & \\ & \alpha_{2m+1}^{2m+1} & & \\ & & \ddots & \\ & & & \alpha_{2m+1}^{2m+1} \end{bmatrix} \tag{5}$$

α_{2m+1}^{2m+1} being repeated (n-m) times on the diagonal.

Thus when we set $\lambda_3 = \lambda_5 = \ldots\ldots = \lambda_{2n+1} = 0$ we find that the set of n matrices

$$\alpha_3^{2n+1}, \alpha_5^{2n+1} \ldots\ldots \alpha_{2n+1}^{2n+1}$$

are 'shift' operators which take an eigenvector of L_{2n+1} to another with one eigenvalue reversed. Similarly products of two such α matrices 'shift' one eigenvector to another with two eigenvalues reversed. Thus we have

$$\binom{n}{1} + \binom{n}{2} + \cdots\cdots\cdots + \binom{n}{n} = 2^n - 1 \qquad (6)$$

'shift' operators which take one eigenvector characterized by a set of eigenvalues to another vector with another set of eigenvalues. Each set of eigenvalues belong to a set of commuting matrices having simultaneous eigenvectors. Thus there seems to be a possibility of associating a Lie Algebra with these commuting matrices and 'shift' operators.

APPLICATION TO THE DIRAC EQUATION

We also recognize that in the case of Dirac equation, setting

$$\alpha_1^5 = -\beta\gamma_x, \quad \alpha_2^5 = -\beta\gamma_y, \quad \alpha_3^5 = -\beta\gamma_z$$

$$\alpha_4^5 = -\beta\gamma_5, \quad \alpha_5^5 = \beta$$

and

$$\lambda_1 = p_x, \ \lambda_2 = p_y, \ \lambda_3 = p_z, \ \lambda_4 = 0, \ \lambda_5 = m \qquad (7)$$

that

$$\alpha_1^5 \alpha_4^5 = \gamma_5\gamma_x \quad \text{reverses} \quad p_x$$

$$\alpha_2^5 \alpha_4^5 = \gamma_5\gamma_y \quad \text{reverses} \quad p_y$$

$$\alpha_3^5 \alpha_4^5 = \gamma_5\gamma_z \quad \text{reverses} \quad p_z$$

$$\alpha_5^5 \alpha_4^5 = \gamma_5 \quad \text{reverses} \quad m$$

$$\alpha_4^5 = -\beta\gamma_5 \quad \text{reverses} \quad E \qquad (8)$$

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A NOTE ON THE REPRESENTATIONS OF DIRAC GROUPS

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A NOTE ON THE REPRESENTATIONS OF DIRAC GROUPS *

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1. INTRODUCTION

In a recent note[1] (hereafter referred to as (R) by one of us a hierarchy of matrices L_m, m representing the number of parameters occuring in L_m is introduced which contains the Dirac Hamiltonian as a particular case. These matrices can be expressed as linear combinations of matrix representations of Clifford elements[2] satisfying anticommutation relations the parameters being the coefficients. In obtaining the hierarchy of matrices L_m in a systematic way, a σ-operation is defined corresponding to the introduction of two additional parameters.

In this paper, we study the group theoretical significance of the hierarchy of matrices and the -operation by introducing a group called Dirac group G(m). For that, given m Clifford elements $\{\gamma_m\}$ hereafter called Dirac operators, augment $\{\gamma_m\}$ by two more elements E and $\bar{E}$. Now making use of $\{E, \bar{E}, \{\gamma_m\}\}$ as generating elements, we set up the group[3,5,6,7] G(m) whose generating relations are $\gamma_i \gamma_j = -I\gamma_j = -I\gamma_j\gamma_i$ and $-1 = \bar{E}$.

Now obviously the well known theory of group representations can be used in setting up the matrix representations of Clifford elements satisfying the usual anticommutation relations through G(m). In particular, we make use of the

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Mackey theory[4] of induced representations in setting up the representations of the Dirac group G(m). Since the Dirac group is solvable[3], the Mackay theory reduces to the little group technique and enables us to obtain directly all the irreducible representations of the group in steps. The σ-operation is found to be identical with this process.

2. THE DIRAC GROUP

As in paper (K) let us introduce a hierarchy of square matrices L_m involving m independent continuous parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that

$$L_m^2 = (\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_m^2) I. \tag{1}$$

Obviously, L_m should be linear in each one of the parameters λ_i and can be taken as

$$L_m = \mathcal{L}_1\lambda_1 + \mathcal{L}_2\lambda_2 + \ldots + \mathcal{L}_m\lambda_m \tag{2}$$

where $\mathcal{L}_i$ are the 'generator matrices' independent of the parameters. Note that the dimension of the matrices $\mathcal{L}_i$ is not yet specified. Now imposing the condition (1) we note that the $\mathcal{L}_i$ satisfy the anticommutation relations

$$(\mathcal{L}_i\mathcal{L}_j + \mathcal{L}_j\mathcal{L}_i) = 2\delta_{ij} \tag{3}$$

From the nature of the relations (3) it is obvious that $\mathcal{L}_i$ can be looked upon as m Clifford elements γ_i and from (1) and (2) it is obvious that we are considering their matrix representations. If m = 4, these $\mathcal{L}_i$ are known as Dirac matrices. Hence, hereafter, we call $\mathcal{L}_i$, in the general case, also, Dirac matrices, which when irreducible, are matrix representations of the Clifford elements. [Hence Dirac matrices are matrix representation of the Clifford elements for the case m = 4.] To motivate the study of the Clifford elements in the general case when m ≠ 4 we give another instance[7] from physics where they are of importance.

In quantum field theory one frequently encounters a set of operators $a_1, a_2, \ldots, a_n ; b_1, b_2, \ldots b_n$ such that

$$\begin{aligned} [a_i, a_j]_+ = [b_i, b_j]_+ = 0 \\ [a_i, b_j]_+ = \delta_{ij} \end{aligned} \tag{4}$$

where a's and b's are called annihilation and creation operators. If we now form the operators $q_i = a_i + b_i$ and $p_i = -i(a_i - b_i)$ then we have

$$\begin{aligned} [q_i, q_j]_+ = [p_i, p_j]_+ = 2\delta_{ij} \\ [q_i, p_j]_+ = 0 \end{aligned} \tag{5}$$

Now, let $q_i = \gamma_{2i-1}$, $p_i = \gamma_{2i}$. Then collectively $\{\gamma_i\}$ where $i = 1, 2, \ldots, 2n$ satisfy the anticommutation relations (2). Hence the study of the representation theory of Clifford elements satisfying anti-commutation relations is of basic importance in physics.

In the paper (R) the representation matrices of the Clifford elements are obtained by matrix methods in a systematic way by introducing a σ-operation. This note has motivated us to study in detail the mathematical significance of the σ-operation.

Consider the set $\{\underline{\gamma}_i\}$ of m Clifford elements and augment them by E and $\bar{E}$. Now define a group with the generating elements $\{E, \bar{E}, \{\underline{\gamma}_i\}\}$ satisfying the following generating relations

$$\underline{\gamma}_i^2 = \bar{E}^2 = E$$

$$\underline{\gamma}_i \underline{\gamma}_j = \bar{E}\ \underline{\gamma}_j \underline{\gamma}_i$$

and $$\bar{E}\ \underline{\gamma}_i = \underline{\gamma}_i \bar{E}$$

Obviously, E is the identity element of G(m) and as the order of each element is 2, all the elements of the group are given by $\bar{E}^{j_0} \underline{\gamma}_1^{j_1} \underline{\gamma}_2^{j_2} \cdots \underline{\gamma}_m^{j_m}$ where each j is either zero or unity. Its order is 2^{m+1}. When m = 4 the $\underline{\gamma}_i(\neq \alpha)$ correspond to the Dirac matrices when $\bar{E} = -1$. We call G(m) the Dirac group.

G(m) has $1+2^m$ when m is even, or $2+2^m$ when m is odd, conjugate classes given by E, $\bar{E}$, $(A, \bar{E}A)$ for all $A \in G(m)$ and $A \neq E, \bar{E}$ when m is even and $E, \bar{E}, \underline{\gamma}_1 \underline{\gamma}_2 \cdots \underline{\gamma}_m$ $(=\delta_m)$, $\bar{E}\delta_m$, $(A, \bar{E}A)$ for all $A \in G(m)$ and $A \neq E, \bar{E}, \delta_m, \bar{E}\delta_m$ when m is odd respectively.

Further, we can establish the following properties regarding G(m) very easily.

(1) Every subgroup of G(m) different from E contains the normal subgroup $G_0 = \{E, \bar{E}\}$ and hence G_0 is the minimal

normal subgroup.

$G(p)$ is the proper maximal normal subgroup of $G(p+1)$ and $G(p+1)/G(p)$ is a factor group of order 2. Hence, considering the composition series $G(m) \supset G(m-1) \supset \cdots \supset G_0 \supset E$ where each factor group is of order 2, it follows that G(m) is a solvable group.

4. REPRESENTATIONS OF THE DIRAC GROUP

To set up the matrix representations of G(m) let us apply the Mackey technique of induced representations to reduce G(m) with respect to the normal subgroup G_0. The one-dimensional irreducible representations of G_0 are given by $\Gamma_{\pm}$; $\bar{E} \to \pm 1$ when specified through the generating element $\bar{E}$ of G_0. The orbits of the representations of G_0 relative to G(m) are Γ_+ and Γ_- and their stability groups are the same and are given by G(m). So the representations of G(m) fall into two classes those in which $\bar{E} \to I$ and $\bar{E} \to -I$ as the element $\bar{E}$ commutes with all the elements of G(m) and $\bar{E}^2 = E$.

Class 1. Consider the induced representation $\Gamma_+ \uparrow G(m)$ of G(m) induced from the representation Γ_+ of G_0 with respect to any coset decomposition of G(m) with respect to G_0. From equation (2) it follows that every matrix corresponding to an arbitrary element of G(m) commutes with all other matrices corresponding to other elements of it. Hence $\Gamma_+ \uparrow G(m)$ is completely reducible, to one dimensional representations in which, as the order of each γ_i is two the matrices corresponding to γ_i are given by $\gamma_i \to \pm 1$. These representations are 2^m in number which is also the order of the factor group $G(m)/G_0$. Hence these are the only possible representations of G(m) in which $\bar{E} \to I = 1$.

Class 2. As there are $1+2^m$ $(2+2^m)$ conjugate classes of $G(m)$ when m is even (odd) there exists one (two) more non-equivalent irreducible representation(s) of $G(m)$ of dimension greater than one. When $m=2n$ (say) we designate it by $\Delta^-(2n)$ thereby indicating that in it $\bar{E} \to -I$. Now making use of the completeness relation we find that the dimension of $\Delta^-(2n)$ is 2^n.

When $m=2n+1$ (say), $\gamma_1 \gamma_2 \cdots \gamma_m = \delta_m$ commutes with all the elements of $G(m)$. Hence the matrix corresponding to δ_m in an irreducible representation of $G(m)$ is either kI or $-kI$ since $\delta_m^2 = \bar{E}^n$ and $k = i(1)$ if n is odd/even. Hence, only two non-equivalent irreducible representations of dimension greater than one can exist and they should necessarily be of the same order. We designate them by $\Delta_\pm^-(2n+1)$ thereby indicating that in them $\delta_m \to \pm kI$ and $\bar{E} \to -I$. Now making use of the completeness relation we find that the dimension of $\Delta_\pm^-(2n+1)$ is 2^n .

By the Mackey theory of induced representations $\Delta^-(2n)$ or $\Delta^-_\pm(2n+1)$ should be obtained by reducing the induced representation $\Gamma_-(m) = \Gamma_- \uparrow G(m)$ whose dimension is same as the order of G/G_0 and is given by 2^m . Now, obviously, $\Delta^-(2n)$ and $\Delta^-_\pm(2n+1)$ are each contained 2^n times in $\Gamma_-(2n)$ and $\Gamma_-(2n+1)$ respectively.

5. EXPLICIT FORMS OF $\Delta^{-}(2n)$ AND $\Delta^{-}_{\pm}(2n+1)$

Consider the composition series

$$G(m) \supset G(m-1) \supset \cdots \supset G_0$$

terminating in G_0 . In general it is difficult to reduce $\Gamma_{-}(m)$ directly. Hence, we apply the Mackey in steps through the composition series and obtain explicit forms of $\Delta^{-}(2n)$ and $\Delta^{-}_{\pm}(2n+1)$.

Now the orbits containing the representations $\Delta^{-}(i)$ $(\Delta^{-}_{\pm}(i))$ of $G(i)$ when i is even (odd) relative to $G(i+1)$ are given by $\{\Delta^{-}(i)\}$, $(\{\Delta^{-}_{+}(i), \Delta^{-}_{-}(i)\})$ For when i is even, there exists only one nonequivalent irreducible representation of $G(i)$ and $\underline{Y}_{i+1}$ should transform $\Delta^{-}(i)$ into itself as $\underline{Y}_{i+1}$ commutes with $\overline{E}$. When i is odd $\underline{Y}_{i+1}\,\delta_i\,\underline{Y}_{i+1} = \bar{E}\,\delta_i$ and $\delta_i \to \pm k\,\underline{I}$ in $\Delta^{-}_{\pm}(i)$, the two represantations ~~of~~ $\Delta^{-}_{\pm}(i)$ are transformed into each other by $\underline{Y}_{i+1}$.

When i is odd the dimension of the induced representation $\Delta^{-}_{+}(i) \uparrow G(i+1)$ is $2 \times 2^{(i-1)/2} = 2^{(i+1)/2}$ which is also the dimension of $\Delta^{-}(i+1)$. Further as Y_{i+1} commutes with $\overline{E}$, in the induced representation $\Delta^{-}_{+}(i) \uparrow G(i+1)$ the $\bar{E} \to -I \otimes I_2$. Hence $\Delta^{-}_{+}(i) \uparrow G(i+1)$ is irreducible and must be equivalent to the representation $\Delta^{-}(i+1)$. Now making use of the coset decomposition $G(i+1) = G(i) + Y_{i+1}\,G(i)$ the induced representation which is now identified with $\Delta^-(i+1)$ is given by

$$\Delta^-(i+1): \quad \gamma_r \to \alpha_r \otimes \sigma_1 \quad r=1,\dots,i$$
$$\gamma_{i+1} \to I \otimes \sigma_2 \qquad (6)$$

where $\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and I is a unit matrix of dimension $2^{(i-1)/2}$. Note that the above representations $\Delta^-(i+1)$ are specified by giving the matrix representations of the generating elements of $G(i+1)$. When i is even from the above discussion, without loss of generality, $\Delta^-(i)$ can be taken as

$$\Delta^-(i): \quad \gamma_r \to \alpha_r \otimes \sigma_1 \quad r=1,\dots,i-1$$
$$\gamma_i \to I \otimes \sigma_2 \qquad (7)$$

when $i \geq 2$. Since the γ's anticommute with each other the transform of $\Delta^-(i)$ by γ_{i+1} is given by

$$\gamma_{i+1}\, \Delta^-(i)\, \gamma_{i+1}: \quad \gamma_r \to \alpha_r \otimes (-\sigma_1) \qquad (8)$$
$$\gamma_i \to I \otimes (-\sigma_2).$$

As γ_{i+1} commutes with $\bar{E}$ and there exists only one non-equivalent irreducible representation of $G(i)$, with $\bar{E} \to -I$, $\underline{\gamma}_{i+1} \Delta^-(i) \underline{\gamma}_{i+1}$ must be equivalent to $\Delta^-(i)$. Hence by Schur's lemma there exists a nonsingular matrix S such that

$$S \Delta^-(i) = \underline{\gamma}_{i+1} \Delta^-(i) \underline{\gamma}_{i+1} S.$$

Obviously, S should be of the form $I \otimes \sigma_3$ and σ_3 should anticommute with σ_1 and σ_2. Hence σ_3 is given by $\pm \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Now ~~in~~ the two nonequivalent representations of $G(i+1)$

$$
\begin{aligned}
\underline{\gamma}_\nu &\to \mathcal{L}_\gamma \otimes \sigma_1 \\
\underline{\gamma}_i &\to I \otimes \sigma_2 \\
\underline{\gamma}_{i+1} &\to \pm I \otimes \sigma_3
\end{aligned}
\qquad (9)
$$

are of dimension $2^{i/2}$ and in them $\bar{E} \to -I \otimes I_2$. As $G(i+1)$ when i is even has only two non-equivalent irreducible representations $\Delta^-_\pm(i+1)$ of dimension $2^{i/2}$ they must be equivalent to the above and hence we identify $\Delta^-_\pm(i+1)$ with them. To complete induction we note that when $i = 0$ the relevant representation of G_0 is given by $\bar{E} \to -1$ and when $i = 1$ the representations $\Delta^-_\pm(1)$ of G_1 are given by $\bar{E} \to -1, \gamma_1 \to \pm 1$. Note that only the representation $\Delta^-_+(1)$: $\gamma_1 \to 1$ need be considered in setting up the representations for different $G(m)$'s.

Now obviously from (6) and (9) the representations $\Delta^-(2n)$, $(\Delta^-_\pm(2n+1))$ are given by

$$\gamma_1 \to \sigma_1 \otimes \sigma_1 \cdots \otimes \sigma_1 = \mathcal{L}_1$$

$$\gamma_{2j} \to \underbrace{I \otimes \cdots \otimes I}_{j-1 \text{ times}} \otimes \sigma_2 \otimes \sigma_1 \cdots \sigma_1 = \mathcal{L}_{2j}$$

$$\gamma_{2j+1} \to I \otimes \cdots \otimes I \otimes \sigma_3 \otimes \sigma_1 \cdots \sigma_1 = \mathcal{L}_{2j+1}$$

with n terms in each product. When m is odd γ_m should be taken with ± signs to obtain $\Delta^-_\pm(2n+1)$.

6. THE σ-OPERATION

Now we recover the σ-operation and find it identical with the above induction procedure. For that we write

$$L_m = \sum_{i=1}^{m} \mathcal{L}_i \lambda_i$$

when m is even and

$$L^\pm_m = \sum_{i=1}^{m-1} \mathcal{L}_i \lambda_i \pm \mathcal{L}_m \lambda_m.$$

when m is odd.

Now when i is odd multiplying $\mathcal{L}'_{\gamma}s$ in (6) by λ_{γ} and adding we obtain

$$L_{i+1} = \sum_{\gamma=1}^{i} \lambda_{\gamma} \mathcal{L}_{\gamma} \otimes \sigma_1 + \lambda_{i+1} I \otimes \sigma_2$$

$$= L_i^{+} \otimes \sigma_1 + \lambda_{i+1} I \otimes \sigma_2$$

Since $i+1$ is even from (9) adding") [illegible] $\pm \lambda_{i+2} I \times \sigma_3$ to be above we obtain

$$L^{\pm}_{i+2} = L_i^{+} \otimes \sigma_1 + I \otimes (\lambda_{i+1} \sigma_2 \pm \lambda_{i+2} \sigma_3)$$

This is the abstract form for the σ operation. Obviously, the form of $L^{\pm}_{i+2}$ depends on the matrices used for the σ's which anticommute with each other and the square of each is identity. For example if we use the matrix representation

$$\sigma_1 \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 \to \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{11}$$

used in section 4 we obtain

$$L^{\pm}_{i+2} = \begin{bmatrix} L_i^{+} & (\lambda_{i+1} \mp i \lambda_{i+2}) I \\ (\lambda_{i+1} \pm i \lambda_{i+2}) I & -L_i^{+} \end{bmatrix}$$

But if we use

$$\sigma_1 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \to \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

obtained from (11) by a simple permutation of σ's we obtain taking $r = 2n-1$

$$L^{\pm}_{2n+1} = \begin{bmatrix} \pm \lambda_{2n+1} I & L^{+}_{2n-1} - i\lambda_{2n} I \\ L^{+}_{2n-1} + i\lambda_{2n} I & \mp \lambda_{2n+1} I \end{bmatrix}$$

the σ-operation used in (R).

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ON THE RELATIONSHIP BETWEEN THE L-MATRIX HIERARCHY AND CARTAN SPINORS

(Revised and enlarged version)

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ON THE RELATIONSHIP BETWEEN THE L-MATRIX HIERARCHY AND CARTAN SPINORS

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In a series of papers the structure of a hierarchy of L-matrices was studied by the author through the eigen values and eigenvectors of such matrices characterised by the property,

$$L^2_{2n+1} = (\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_{2n+1}^2) I = \Lambda_n^2 I ,$$

the λ's and Λ being either pure real or pure imaginary, and I a unit matrix of the same dimension as L_{2n+1}. The matrix L_{2n+1} involves (2n+1) real parameters (since purely imaginary parameters can be written as i times a real parameter) and is of dimension $2^n \times 2^n$. We now observe that this theory has a close relationship to E. Cartan's theory of spinors. Before establishing such a relationship it is necessary to understand why Cartan in his celebrated formulation did not include eigenvalues and eigenvector in description of the structure. On the other hand, he used the condition of isotropy defined by

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 0$$

and associated with each isotropic vector with components $(x_1, x_2, \ldots, x_n)$ a certain matrix X, the columns of which are defined as spinors. The striking relationship between the L-matrix theory and Cartan's theory of spinors can be established as soon as we realize that the is ropy condition can be rewritten as

$$x_1^2 + x_2^2 + \dots + x_{2n}^2 = - x_{2n+1}^2$$

requiring the total number of components of Cartan's vector to be odd. We then make the following identification

$$x_1 = \lambda_1 , \quad x_2 = \lambda_2 , \dots , x_{2n} = \lambda_{2n}$$

but $\lambda_{2n+1} = 0$

and $\Lambda_n = i X_{2n+1}$.

Thus in Cartan's theory we are dealing with the case of "unsaturated" L matrices, i.e. one of the $2n+1$ parameters of the L-matrix is set equal to zero. The $2n+1$ components of the vector are therefore the $2n$ parameters and $i\Lambda_n$ where Λ_n is an eigenvalue. To identify the X matrix of Cartan composed of spinors, we proceed as follows: We write as postulated in the L-matrix theory

$$L_{2n+1} = \sum_{i=1}^{2n+1} \lambda_i \mathcal{L}^i_{2n+1}$$

The unsaturated L_{2n} matrix is obtained by setting $\lambda_{2n+1} = 0$

i.e. dropping the α^{2n+1}_{2n+1} in the linear combination. If L_{2n+1} is obtained by a "σ-operation" on L_{2n-1} and defined as

$$L_{2n+1} = \begin{bmatrix} \lambda_{2n+1} I & L_{2n-1} - i\lambda_{2n} I \\ L_{2n-1} + i\lambda_{2n} I & -\lambda_{2n+1} I \end{bmatrix}.$$

~~then, we immediately recognise~~ that

$$\alpha^{2n+1}_{2n+1} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

where I is a unit matrix of dimension $2^{n-1} \times 2^{n-1}$. In earlier work, it was proved that the U matrix with its columns representing the eigenvectors of L_{2n+1} can be written as

$$U = L_{2n+1} + \Lambda_n \alpha^{2n+1}_{2n+1}$$

The Cartan matrix X is then identified to be:

$$X = \sum_{i=1}^{2n} x_i \alpha^{i}_{2n+1} + x_n \alpha^{2n+1}_{2n+1}$$

$$= L_{2n} + i\Lambda_n \quad n$$

This is just the matrix obtained by replacing λ_{2n+1} by $i\Lambda_n$

in the matrix L_{2n+1} with the condition that

$$\Lambda_n^2 = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_{2n}^2$$

i.e. Λ_n is the eigen-value of an unsaturated matrix with $\lambda_{2n+1} \neq 0$. Thus the Cartan theory of spinors is the theory of unsaturated L-matrices, the eigen-vectors being identified as the spinors and the parameters together with $i\Lambda_n$ as the components of a vector.

Actually, an unsaturated matrix can be defined more generally if we realize that the σ -operation on L_{2n+1} can be done in one of three ways, i.e., by requiring that any one of the three parameters λ_1, λ_2, λ_3, in the matrix

$$L_3 = \begin{bmatrix} \lambda_3 & \lambda_1 - i\lambda_2 \\ \lambda_1 + i\lambda_2 & -\lambda_3 \end{bmatrix}$$

can be replaced by the matrix L_{2n-1} and the other two parameters re-labelled as λ_{2n} and λ_{2n+1} attaching the unit matrix to them. Thus, the matrix $\mathcal{L}_{2n+1}^{2n+1}$ can have one of the three following forms:

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \begin{bmatrix} D & I \\ I & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & -iI \\ iI & 0 \end{bmatrix}$$

The form of the eigenvector of L_{2n+1} depends on the way in which the σ -operation is performed. If ω is an arbitrary vector of dimensión 2^{n-1} and ω' defined by

$$\omega' = \left[\frac{L_{2n-1} - i\,\lambda_{2n} I}{\Lambda_n - \lambda_{2n+1}} \right] \omega$$

Then $\binom{\omega'}{\omega}$ is an eigenvector of L_{2n+1} obtained by replacing λ_1 by L_{2n-1} in L_3 . If on the other hand we replace λ_2 by L_{2n-1} in L_3 , then ω' should be defined as:

$$\omega' = \left[\frac{\lambda_{2n} I - i\,L_{2n-1}}{\Lambda_n - \lambda_{2n+1}} \right] \omega$$

If L_{2n+1} is obtained by replacing λ_3 by L_{2n-1} in L_3 then ω' should be defined by

$$\omega' = \left[\frac{\lambda_{2n} - i\,\lambda_{2n+1}}{\Lambda_n I - L_{2n+1}} \right] \omega$$

where $\dfrac{1}{\Lambda_n I - L_{2n+1}}$ is the inverse $\Lambda_n I - L_{2n+1}$.

For the unsaturated case of L_3 we get

$$\omega' = \frac{\lambda_1 - i\lambda_2}{\Lambda_2}\,\omega = \frac{\sqrt{\lambda_1 - i\lambda_2}}{\sqrt{\lambda_1 + i\lambda_2}}\,\omega$$

which is the famous ratio of the components of the Cartan spinors.

We also obtain the more general result that if ' is an eigenvector of L_{2n-2} and then $\binom{\omega'}{\omega}$ is an eigenvector of L_{2n} if $\frac{\omega'}{\omega} = \frac{\Lambda_{n-1} - i\lambda_{2n}}{\Lambda_n} = \sqrt{\frac{\Lambda_{n-1} - i\lambda_{2n}}{\Lambda_{n-1} + i\lambda_{2n}}}$

We shall now show that this structure of the eigenvectors of L_{2n} enables us to associate a Lie Algebra with a set of matrices suitably built out of unsaturated L-matrices and their "saturator" α -matrices.

A saturated L-matrix expressed in terms of the -matrices as

$$L_{2n+1} = \sum_{i=1}^{2n+1} \lambda_i \alpha_{2n+1}^{i}$$

can be "desaturated" by setting $\lambda_{2n+1} = 0$ i.e. by omitting α_{2n+1}^{2n+1} in the linear combination to obtain the unsaturated matrix L_{2n} i.e.

$$L_{2n} = \sum_{i=1}^{2n} \lambda_i \alpha_{2n+1}^{i}$$

For obvious reasons, we call α_{2n+1}^{2n+1} as the "saturator" of L_{2n} .

We now define a sequence of n unsaturated matrices of dimension $2^n \times 2^n$

$$L_{2n}^{(n)} = L_{2n} \, , \quad L_{2n}^{(n-1)} = \begin{bmatrix} L_{2n-2} & 0 \\ 0 & L_{2n-2} \end{bmatrix} , \ldots$$

$$L_{2n}^{(1)} = \begin{bmatrix} L_2 & & 0 \\ & L_2 & \\ 0 & & \ddots \; L_2 \end{bmatrix}$$

The simaltaneous eigenvectors of these n-operators are completely determined through the Cartan ratio.

$$\frac{\Lambda_{m-1} - i\lambda_{2m}}{\Lambda_m}$$

between the two halves of the eigenvector of $L_{2m} (m \leq n)$

The 2^n simaltaneous eigenvector is characterized by the set of ei...

$$\pm \ldots \pm \Lambda_{n-1} , \ldots \ldots , \pm \Lambda_1$$

If we def... the matrices

$$\mathbb{L}_n^n = \mathcal{L}_{2n+1}^{2n+1} ,$$

$$\mathbb{L}_n^{n-1} = \begin{bmatrix} \mathcal{L}_{2n-1}^{2n-1} & 0 \\ 0 & \mathcal{L}_{2n-1}^{2n-1} \end{bmatrix} \left(\mathcal{L}_{2n+1}^{2n} \right) ,$$

$$\vdots \qquad \vdots$$

$$\mathbb{L}_n^m = \begin{bmatrix} \mathcal{L}_{2m+1}^{2m+1} & & 0 \\ & \mathcal{L}_{2m+1}^{2m+1} & \\ 0 & & \ddots \; \mathcal{L}_{2m+1}^{2m+1} \end{bmatrix} \begin{bmatrix} \mathcal{L}_{2m+3}^{2m+2} & & 0 \\ & \mathcal{L}_{2m+3}^{2m+2} & \\ 0 & & \ddots \; \mathcal{L}_{2m+3}^{2m+2} \end{bmatrix}$$

3.

where $\mathcal{L}^{2m+1}_{2m+1}$ is repeated 2^{n-m} times, $\mathcal{L}^{2m}_{2m+3}$ is repeated 2^{n-m-1} times on the diagonal. We observe that $\mathbb{L}^{m}_{n}$ operating on an eigenvector reverses Λ_m without affecting the other parameters or eigenvalues.

Products of r of these $\mathbb{L}$ - matrices reverse r eigenvalues in the eigenvector. Thus the total number of shift operators is equal to

$$\binom{n}{1}+\binom{n}{2}+\cdots\cdots+\binom{n}{n}\neq 2^{n}$$

Thus there are n commuting operators $\mathbb{L}_{2^{i-1}}$ $(i=1,2,\ldots n)$ and $2^{n}-1$ shift operators in the Cartan canonical form of the Lie algebra.

This leads to an "unprecedented" interpretation of σ_x, σ_y, σ_z as the members of a Lie algebra in which <u>σ_z is a shift operator</u> and

$$\lambda_1 \sigma_x + \lambda_2 \sigma_y = L_2 ,$$

is the matrix which has two eigenvectors

$$\begin{pmatrix} \lambda_1 - i\lambda_2 \\ \Lambda_1 \end{pmatrix} \text{ and } \begin{pmatrix} \lambda_1 - i\lambda_2 \\ -\Lambda_1 \end{pmatrix}$$

corresponding to two eigen-values $\pm \Lambda_1$ where

$$\Lambda_1 = + \sqrt{\lambda_1^2 + \lambda_2^2}$$

The number of "commuting generators" is one. The number of shift operators is also one, since $2^n - 1 = 1$ for n=1.

In the conventional structure of the Lie algebra of the Pauli spin matrices σ_x, σ_y, σ_z; "the commuting generator" is σ_z while $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$ are the two shift operators. In contrast to this we now have a Lie algebra in which the diagonal σ_z is a shift operator while the single commuting generator L_2 is non-diagonal and involves two <u>continuous parameters</u> λ_1 and λ_2.

As regards L_4 we have the four eigen vectors:

$$\begin{pmatrix} \pm\Lambda_1 - i\lambda_4 \begin{pmatrix}\lambda_1 - i\lambda_2 \\ \pm\Lambda_1\end{pmatrix} \\ +\Lambda_2 \begin{pmatrix}\lambda_1 - i\lambda_2 \\ \pm\Lambda_1\end{pmatrix} \end{pmatrix}, \begin{pmatrix} \pm\Lambda_1 - i\lambda_4 \begin{pmatrix}\lambda_1 - i\lambda_2 \\ \pm\Lambda_1\end{pmatrix} \\ -\Lambda_2 \begin{pmatrix}\lambda_1 - i\lambda_2 \\ \pm\Lambda_1\end{pmatrix} \end{pmatrix}$$

It is to be noted that the operator which reverses Λ_1 is a product and two matrices $\begin{pmatrix}\alpha_3^3 & 0 \\ 0 & \alpha_3^3\end{pmatrix}$ and α_5^4. α_3^3 reverses the sign of Λ_1 in $\begin{pmatrix}\lambda_1 - i\lambda_2 \\ \pm\Lambda_1\end{pmatrix}$ while α_5^4 reverses the sign of λ_4 and Λ_2 which is equivalent to reversing the sign of Λ_1 only in the term $(\pm\Lambda_1 - i\lambda_4)$.

We now observe some very interesting features in the structure of the Eigenvectors. If a and b are two arbitrary numbers we note that $\begin{pmatrix} a \begin{pmatrix}\lambda_1 - i\lambda_2 \\ \pm\Lambda_1\end{pmatrix} \\ b \begin{pmatrix}\lambda_1 - i\lambda_2 \\ \pm\Lambda_1\end{pmatrix}\end{pmatrix}$ is an eigenvector of the matrix $I \otimes L_3$ while $\begin{pmatrix} \lambda_1 - i\lambda_2 \begin{pmatrix}a \\ b\end{pmatrix} \\ \pm\Lambda_1 \begin{pmatrix}a \\ b\end{pmatrix}\end{pmatrix}$ is an eigenvector of $L_3 \otimes I$. This implies that

$$\begin{pmatrix} \mu_1 - i\mu_2 \begin{pmatrix}\lambda_1 - i\lambda_2 \\ \pm\Lambda_1\end{pmatrix} \\ \pm M_1 \begin{pmatrix}\lambda_1 - i\lambda_2 \\ \pm\Lambda_1\end{pmatrix}\end{pmatrix}$$

are the simultaneous eigenvectors of $I \otimes L_3(\lambda_1,\lambda_2,\lambda_3)$ and $L_3(\mu_1,\mu_2,\mu_3) \otimes I$. It is clear now that if we replace μ_1 by Λ_1 and relabel μ_2 as λ_4 and M_1 as Λ_2 we obtain the simultaneous eigenvectors of $I \otimes L_3$ and L_4 . This implies that the eigen value Λ_1 of L_2 is telescoped into the eigenvector of L_4 , while in the case of $L_3 \otimes I$ and $I \otimes L_3$, the eigen values are not "coupled".

We can now define an operator $(L_3 \otimes I + I \otimes L_3)$ with eigen values $\Lambda_1 + M_1, -\Lambda_1 + M_1, \Lambda_1 - M_1, -\Lambda_1 - M_1$. If we set $\mu_1 = \lambda_1$, $\mu_2 = \lambda_2$ and $M_1 = \Lambda_1$, then the eigen values reduces to $2\Lambda_1$, 0, 0 - $2\Lambda_1$, leading to a trichotomous and a "singlet" eigen value when the matrix I is suitably reduces.

These considerations can be extended for the higher dimensional L-matrices.

The consequences of these mathematical results will be discussed in later contributions.

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A GENERALISATION OF THE L-MATRIX HIERARCHY

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A GENERALISATION OF THE L-MATRIX HIERARCHY

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Recently one of the authors (A.R.)[1] devised a method of building a hierarchy of what he called L-matrices which have the property

$$L^2_{2n+1} = (\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_{2n+1}^2) I \qquad (1)$$

where the matrix L_{2n+1} contains $(2n+1)$ parameters $\lambda_1, \lambda_2, \cdots \lambda_{2n+1}$ and I is a unit matrix of the same dimension as L_{2n+1}. The structure and the interrelations among the eigenvectors of the hierarchy of matrices were studied in detail and it was pointed out that the Dirac Hamiltonian is a member of this hierarchy.

If we call equation (1) as the Clifford condition, Morris[3] has shown in a recent contribution that it is possible to genralize the Clifford condition on the L-matrices by requiring that m-th power of the L matrix is a product of a unit matrix and a number i.e.

$$L^m_{2n+1} = (\lambda_1^m + \lambda_2^m + \cdots + \lambda_{2n+1}^m) I = \Lambda^m_n I \qquad (2)$$

The dimension of the matrix L_{2n+1} is $m^n \times m^n$

We now observe that many considerations of L-matrix theory are applicable to matrices which obey the generalized Clifford condition. We shall, demonstrate this in the case of m=3 by obtaining explicit forms for the matrices, their eigen values, eigenvectors and studying their interrelations. The extension to the case of general m follows directly, of course taking into account the difference in the forms of L-matrices for even and odd m as noted by Morris.

The primitive L matrix obeying (2) involving 3 parameters is a linear combination of three matrices derived by Morris and can be written as

$$L_3 = \begin{pmatrix} \lambda_3 & \lambda_1 + \omega\lambda_2 & 0 \\ 0 & \omega\lambda_3 & \lambda_1 + \omega^2\lambda_3 \\ \lambda_1 + \lambda_2 & 0 & \omega^2\lambda_3 \end{pmatrix} \qquad (3)^{+}$$

Here ω denotes the primitive cube root of unity.

The corresponding matrix for the usual Clifford condition i.e. for m=2 is

$$\begin{pmatrix} \lambda_3 & \lambda_1 - i\lambda_2 \\ \lambda_1 + i\lambda_2 & -\lambda_3 \end{pmatrix} \qquad (4)$$

+) We do not bother to attach additional labels to L-matrices for indicating the generalized Clifford condition since the distinction between the various cases will be obvious for the context.

The higher dimensional L-matrices were obtained for $m=2$ by a prescription known as the σ-operation which consists in replacing any one of the three parameters of the primitive L_3 by the matrix L_{2n-1} and relabelling the other two parameters as λ_{2n} and λ_{2n+1} and attaching unit matrices of the same dimension as L_{2n-1} to them. Thus starting with L_3, we generate L_{2n+1} by the σ-operation.

We now wish to point out that <u>this prescription can be adopted even for L matrices obeying the Clifford condition</u>. For L_3 defined by (3), we can obtain L_{2n+1} by replacing <u>any one</u> of the three parameters in L_3 by L_{2n-1} and relabelling other parameters as λ_{2n} and λ_{2n+1}. In particular we can define

$$L_{2n+1} = \begin{pmatrix} L_{2n-1} & (\lambda_{2n} + \omega\lambda_{2n+1})I & 0 \\ 0 & \omega L_{2n-1} & (\lambda_{2n}+\omega^2\lambda_{2n+1})I \\ (\lambda_{2n}+\lambda_{2n+1})I & 0 & \omega^2 L_{2n-1} \end{pmatrix} \tag{5}$$

The dimension of the matrix L_{2n+1} is obviously $3^n \times 3^n$.

The matrix L_3 given by (3) has three eigenvalues $\Lambda_1(1)$, $\Lambda_1(2)$ and $\Lambda_1(3)$ where

$$\Lambda_1(1) = \Lambda_1 ; \quad \Lambda_1(2) = \Lambda_1 \omega ; \quad \Lambda_1(3) = \Lambda_1 \omega^2 \tag{6}$$

and the eigenvector corresponding to $\Lambda_1(i)$ is

$$\begin{pmatrix} 1 \\ \dfrac{\Lambda_1(i)-\lambda_3}{\lambda_1+\omega\lambda_2} \\ \dfrac{(\Lambda_1(i)-\lambda_3)(\Lambda_1(i)-\omega\lambda_3)}{(\lambda_1+\omega\lambda_2)(\lambda_1+\omega^2\lambda_2)} \end{pmatrix} \tag{7}$$

The third component of the eigenvector (7) can also be written as $(\lambda_1+\lambda_2)/(\Lambda_1(i)-\omega^2\lambda_3)$.

Since we know how L_{2n+1} is generated from L_3, we immediately recognise the structure of eigenvector of L_{2n+1}. If Ω is an arbitrary vector of dimension 3^{n-1}, then the vector of dimension 3^n defined as

$$\begin{pmatrix} \Psi \\ \Omega' \\ \Omega'' \end{pmatrix} = \begin{pmatrix} \Omega \\ \dfrac{\Lambda_n(i)I - L_{2n-1}}{\lambda_{2n}+\omega\lambda_{2n+1}}\,\Omega \\ \dfrac{\lambda_{2n}+\lambda_{2n+1}}{\Lambda_n(i)I-\omega^2 L_{2n-1}}\,\Omega \end{pmatrix} \tag{8}$$

is an eigenvector of L_{2n+1} with eigenvalues $\Lambda_n(i)$ where

$\lambda_1^3 + \lambda_2^3 + \cdots + \lambda_{2n+1}^3 = \Lambda_n^3$ and Λ_n(i) is any one of the three roots of Λ_n^3.

It is to be noted that the matrix L_{2n+1} has only 3 eigenvalues Λ_n ; $\omega\Lambda_n$ and $\omega^2\Lambda_n$ but has 3^n eigenvectors. This degeneracy among the eigenvectors is obvious from the arbitrariness of the 3^{n-1} dimensional vector Ω in (8). The U-matrix or the matrix of eigenvectors can be expressed in an elegant form as

$$U = \left(L_{2n+1}^2 + L\Lambda + \Lambda^2 \right) \tag{9}$$

where

$$\Lambda = \begin{pmatrix} \Lambda_n I & 0 & 0 \\ 0 & \omega\Lambda_n I & 0 \\ 0 & 0 & \omega^2\Lambda_n I \end{pmatrix} \tag{10}$$

For example the U-matrix for L_3 can be written as

$$\begin{bmatrix} \lambda_3^2 + \lambda_3\Lambda_1 + \Lambda_1^2 & (\lambda_1 + \omega\lambda_2)(\omega\Lambda_1 - \omega^2\lambda_3) & (\lambda_1 + \omega\lambda_2)(\lambda_1 + \omega^2\lambda_2) \\ (\lambda_1 + \omega^2\lambda_2)(\lambda_1 + \lambda_2) & \omega^2(\lambda_3^2 + \Lambda_1\lambda_3 + \Lambda_1^2) & (\lambda_1 + \omega^2\lambda_2)(\omega^2\Lambda_1 - \lambda_3) \\ (\lambda_1 + \lambda_2)(\Lambda_1 - \omega\lambda_3) & (\lambda_1 + \lambda_2)(\lambda_1 + \omega\lambda_2) & \omega(\lambda_3^2 + \lambda_3\Lambda_1 + \Lambda_1^2) \end{bmatrix}, \tag{11}$$

The degeneracy among 3^n eigenvectors can be resolved by following the procedure as for $m=2$ [3]. We find that the 3^n eigenvectors of the matrix L_{2n+1} can be obtained as the simultaneous eigenvectors of the n matrices of dimension $3^n \times 3^n$:

$$L_{2n+1}\ ;\ \begin{bmatrix} L_{2n-1} & & \\ & L_{2n-1} & \\ & & L_{2n-1} \end{bmatrix} ;\ \cdots\ ;\ \begin{bmatrix} L_3 & & & & \\ & L_3 & & & \\ & & L_3 & & \\ & & & \ddots & \\ & & & & L_3 \end{bmatrix} \tag{12}$$

corresponding to the sequence of eigenvalues of these matrices.

As in the case of m=2, we can obtain the 'unsaturated matrices' by setting any one of the (2n+1) parameters to be zero. In the case of L_3 when we set $\lambda_3 = 0$, the eigenvectors are

$$\begin{pmatrix} 1 \\ \left(\dfrac{\Lambda_1(i)}{\lambda_1 + \omega\lambda_2}\right) \\ \dfrac{\omega^2 \Lambda_1(i)}{(\lambda_1 + \omega\lambda_2)(\lambda_1 + \omega^2\lambda_2)} \end{pmatrix} \tag{13}$$

If we write $L_3 = \lambda_1 \mathcal{L}_1 + \lambda_2 \mathcal{L}_2 + \lambda_3 \mathcal{L}_3$ (14)

where the matrices $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are given by[8)]

$$\mathcal{L}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} ; \quad \mathcal{L}_2 = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix} ; \quad \mathcal{L}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \tag{15}$$

then we find that $\mathcal{L}_3$ acts as shift operator on the eigenvectors given by (13), i.e. it takes an eigenvector with eigenvalues $\Lambda_1(i)$ to $\omega \Lambda_1(i)$. In the case of the quadratic Clifford condition, the symmetry operations consist of reversing the parameters and the eigenvalues. Here the corresponding operations involve a multiplication by ω.

The extension of these considerations to the case of any general m can be made directly and e.g. the U matrix corresponding to L takes the form

$$U = L^{m-1} + L^{m-2}\Lambda + \cdots + \Lambda^{m-1} \tag{16}$$

where

$$\Lambda = \begin{bmatrix} \Lambda_n(1) I & & & \\ & \Lambda_n(2) I & & \\ & & \ddots & \\ & & & \Lambda_n(m) I \end{bmatrix} \tag{17}$$

and $\Lambda_n^{(1)}, \ldots, \Lambda_n^{(m)}$ are the m-th roots of Λ_n^m and I is unit matrix of dimension m^{n-1}.

However, important differences arise between the quadratic and more general Clifford condition since the anti-commutation operation involves the multiplication by a factor, ω, or by multiples of ω and not just by -1. The similarities and differences will be disucssed in a later contribution.

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L-MATRICES, QUATERNIONS AND PROPAGATORS.

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L-MATRICES, QUATERNIONS AND PROPAGATORS

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We continue our programme of studying the correspondence between the L-matrices and other known systems associated with 'multiple algebras'.*) In particular, we shall relate the L-matrix theory to the concepts of quaternions, their generalizations and to 'propagators'.

An L-matrix of dimension ($2^n \times 2^n$) 'involves' $2n+1$ parameters $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_{2n+1}$ (either pure real or pure imaginary) and satisfies the condition

$$L^2_{2n+1}(\lambda_1, \lambda_2, \ldots, \lambda_{2n+1}) = (\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_{2n+1}^2) I \qquad (1)$$
$$= \Lambda_n^2 I$$

While L_{2n+1} has 2^n independent eigenvectors, it has only two eigenvalues $\pm \Lambda_n$. In a previous contribution we defined a Q-matrix as

$$Q = L_{2n+1} + \lambda I \qquad (2)$$

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*'Multiple Algebras', first invented in the nineteenth century, denote 'algebras requiring more than one term for the specification of the quantity'. For the historical origin of multiple algebras, see R.J. Stephenson, Am. J. Phys. 34, 194 (1966), Alfred M. Bork, Am. J. Phys. 34, 202 (1966).

where λ is an arbitrary parameter. The Q-matrix is nonsingular except for the case $\lambda = \pm \Lambda_n$. It can be expressed as a linear combination of the two singular matrices $L_{2n+1} \pm \Lambda_n I$ as

$$L_{2n+1} + \lambda I = \frac{1}{2\Lambda_n}\Big[(\lambda + \Lambda_n)(L_{2n+1} + \Lambda_n I) - (\lambda - \Lambda_n)(L_{2n+1} - \Lambda_n I)\Big] \quad (3)$$

It may be noted that the eigenvectors of Q are the same as those of L_{2n+1}, while its eigenvalues are $(\pm \Lambda_n + \lambda)$. We also observe that

$$(L_{2n+1} - \lambda I)(L_{2n+1} + \lambda I) = \Lambda_n^2 - \lambda^2 \quad (4)$$

so that

$$\frac{1}{L_{2n+1} - \lambda I} = \frac{L_{2n+1} + \lambda I}{\Lambda_n^2 - \lambda^2} \quad (5)$$

Let us define the transform of the matrix $1/L_{2n+1} - \lambda I$ with respect to the partial set of variables $\lambda_1, \lambda_2, \ldots \lambda_m \ (m \leq 2n+1)$ and λ assuming that they are real, as*

$$Q(x_1, x_2, \ldots x_m; t) = \left(\frac{1}{2\pi}\right)^{m+1} \frac{1}{i} \int \frac{L_{2n+1} + \lambda I}{\Lambda_n^2 - \lambda^2} \times e^{i(\lambda_1 x_1 + \cdots + \lambda_m x_m)} e^{-i\lambda t} \times d\lambda_1 d\lambda_2 \ldots d\lambda_m \, d\lambda \quad (6)$$

---

* The definition of the transform has been made with a suitable choice of numerical factors and coefficients for easy comparison with the known formalism of propagators.

Since the denominator has singularities at $\lambda = \pm\Lambda_n$, the integral will depend on the path of integration chosen for the variable λ. The situation is identical with the evaluation of the propagators corresponding to the retarted, advanced and Feynman Kernels so well known in electrodynamics. The integrals corresponding to the three well known paths of integration are

$$Q_F = -\frac{1}{i}\left(\frac{1}{2\pi}\right)^{m+1}\int \frac{1}{2\Lambda_n}\left\{\frac{L_{2n+1}+\Lambda_n I}{\lambda-\Lambda_n+i\varepsilon} - \frac{L_{2n+1}-\Lambda_n I}{\lambda+\Lambda_n-i\varepsilon}\right\} e^{i\sum_{i=1}^{m}\lambda_i x_i}\, e^{-i\lambda t}\, d\lambda_1 d\lambda_2 \cdots d\lambda_m\, d\lambda \tag{7}$$

$$Q_R = -\frac{1}{i}\left(\frac{1}{2\pi}\right)^{m+1}\int \frac{1}{2\Lambda_n}\left\{\frac{L_{2n+1}+\Lambda_n I}{\lambda+\Lambda_n+i\varepsilon} - \frac{L_{2n+1}-\Lambda_n I}{\lambda-\Lambda_n+i\varepsilon}\right\} e^{i\sum_{i=1}^{m}\lambda_i x_i}\, e^{-i\lambda t}\, d\lambda_1 d\lambda_2 \cdots d\lambda_m\, d\lambda \tag{8}$$

$$Q_A = -\frac{1}{i}\left(\frac{1}{2\pi}\right)^{m+1}\int \frac{1}{2\Lambda_n}\left\{\frac{L_{2n+1}+\Lambda_n I}{\lambda-\Lambda_n-i\varepsilon} - \frac{L_{2n+1}-\Lambda_n I}{\lambda+\Lambda_n-i\varepsilon}\right\} e^{i\sum_{i=1}^{m}\lambda_i x_i}\, e^{-i\lambda t}\, d\lambda_1 d\lambda_2 \cdots d\lambda_m\, d\lambda \tag{9}$$

Integrating with respect to the variable λ we get

$$Q_F = \left(\frac{1}{2\pi}\right)^{m}\int \frac{1}{2\Lambda_n}\left\{\theta(t)\left(L_{2n+1}+\Lambda_n I\right)e^{-i\Lambda_n t} + \theta(-t)\left(L_{2n+1}-\Lambda_n I\right)e^{i\Lambda_n t}\right\} d\lambda_1 d\lambda_2 \cdots d\lambda_m$$

$$\left(\frac{1}{2\pi}\right)^m \int \frac{1}{2\Lambda_n}\,\theta(t)\left\{(L_{2n+1}+\Lambda_n I)\,e^{-i\Lambda_n t} - (L_{2n+1}-\Lambda_n I)\,e^{i\Lambda_n t}\right\} d\lambda_1 d\lambda_2 \cdots d\lambda_m \tag{11}$$

$$\left(\frac{1}{2\pi}\right)^m \int \frac{1}{2\Lambda_n}\,\theta(-t)\left\{(L_{2n+1}+\Lambda_n I)\,e^{-i\Lambda_n t} + (L_{2n+1}-\Lambda_n I)\,e^{i\Lambda_n t}\right\} d\lambda_1 d\lambda_2 \cdots d\lambda_m \tag{12}$$

We then recognize that the parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$ are 'momentum-like', Λ_n 'energy-like' while λ is the free parameter 'off-the-energy shell.' The variables $(x_1, x_2, \ldots, x_m)$ of the transform are 'space-like' and the variable t associated with the parameter λ (or equivalently with the eigenvalues $\pm\Lambda_n$) is 'time-like'. Then $1/(L-\Lambda I)$ can be treated as the propagator in momentum space corresponding to Q_F, the propagator in 'configuration' space.

The Q-matrix for the case n = 1, i.e., corresponding to L_3, is recognized to be a quaternion.

If $x^{\pm}$ are the eigenvectors of L corresponding to the eigenvalues $\pm\Lambda_n$, we can define a wave function as

$$\psi = x^{\pm}\, e^{i\lambda_i x_i \pm i\Lambda_n t} \tag{13}$$

This wave function satisfies the 'equation of motion'

$$\left(i\frac{\partial}{\partial t} - L\right)\psi(x,t) = 0 \tag{14}$$

while the kernel or propagator Q satisfies the inhomogeneous equation

$$\left(i\frac{\partial}{\partial t} - L\right)Q = i\,\delta(t) \tag{16}$$

In 'momentum space', we obtain

$$(\Lambda_n I - L)u = 0\,, \quad (\lambda I - L)Q = i \tag{16}$$

yielding the solutions

$$U^{\pm} = L \pm \Lambda_n t$$
$$Q = \frac{i}{\lambda I - L} \tag{17}$$

where $(L \pm \Lambda_n I)$ are singular matrices and out of the 2^n columns, only half of them are independent. The matrix representing the 2^n independent solutions can be written as $(L + \Lambda I)$ where Λ is the diagonal matrix with half the number $+\Lambda_n$ and the other half $-\Lambda_n$.

We are now encouraged to discuss the distinction between the characteristics of the parameters t and λ on the one hand and x_i and λ_i on the other. This is equivalent to understanding the distinction between the characteristics of a wave function and a kernel of propagator. A wave function is a function of the space and mass-like parameters and also of t . But we refer to it as an amplitude in a space characterized by $x_1, x_2, \ldots, x_n$ at time like t . The mass-like parameters are just constants imbedded in the wave function. If we define a scalar product of the wave function with itself, it represents a distribution in the space $x_1, x_2, \ldots, x_m$ at t and its integral with respect to the space-like variables can be normalized to a scalar for any value of t. Thus in the interpretation of the scalar, t is kept constant. However, the kernel or the propagator is a function not only of the intervals of the space-like parameters, but also of the interval of the time-like t. Hence, it can be transformed not only with respect to the interval of space-like parameters, but also the time-like parameter and this is why an additional parameter λ creeps into the propagator but is absent in the wave function.

The necessity of an additional parameter was realised at the time when the quaternions were first invented by Hamilton in the nineteenth century. The kernel function has singularities at the two values $\pm\Lambda_n$ of the parameters and these are just the eigenvalues of the L-matrix. The momentum-like and mass-like parameters occur in the L-matrix with the same status, but the kernel function is defined as a transform with respect to the partial set of momentum-like parameters and t , the mass-like parameters occurring both in the L-matrix and its transform

The relation between the kernel or the propagator and the wave function was discussed in pedagogic detail in the author's book on 'Elementary Particles and Cosmic Rays,' but at that time there was no indication of the possibility of a generalization.

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A HIERARCHY OF HELICITY OPERATORS IN L-MATRIX THEORY

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A hierarchy of helicity operators in L-matrix theory

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In the theory of L-matrices, we are concerned with the simultaneous eigenvectors of the set of n matrices

$$L^{2n+1}_{2n+1}, L^{2n+1}_{2n-1}, \ldots\ldots, L^{2n+1}_{3}$$

of the same dimension $2^n \times 2^n$ defined as

$$L^{2n+1}_{2n+1} = L_{2n+1}, \quad L^{2n+1}_{2n-1} = \begin{bmatrix} L_{2n-1} & 0 \\ 0 & L_{2n-1} \end{bmatrix}$$

$$\cdots\cdots\cdots$$

$$L^{2n+1}_{2m-1} = \begin{bmatrix} L_{2m-1} & & 0 \\ & \ddots & \\ 0 & & L_{2m-1} \end{bmatrix}$$

$$\cdots\cdots$$

$$L^{2n+1}_{3} = \begin{bmatrix} L_3 & & 0 \\ & L_3 & \\ 0 & & \ddots \; L_3 \end{bmatrix}$$

where

$$L_3 = \begin{bmatrix} \lambda_3 & \lambda_1 - i\lambda_2 \\ \lambda_1 + i\lambda_2 & -\lambda_3 \end{bmatrix}$$

and L_{2m+1} is a matrix involving (2m+1) parameters $\lambda_1, \lambda_2, \ldots$
$\ldots, \lambda_{2m+1}$, obtained from L_{2m-1} by a σ-operation defined as follows.

$$L_{2m+1} = \sigma\left(L_{2m-1}\right) = \begin{bmatrix} \lambda_{2m+1} I & L_{2m-1} - i\lambda_{2m} I \\ L_{2m-1} + i\lambda_{2m} I & -\lambda_{2m+1} I \end{bmatrix}$$

We now recognise that the simultaneous eigenvectors of this set are also the simultaneous eigenvectors of another set of n operators

$$L^{2n+1}_{3(1)}(\lambda_1, \lambda_2, \lambda_3) = \begin{bmatrix} L_{3(1)} & & 0 \\ & L_{3(1)} & \\ 0 & \ddots & L_{3(1)} \end{bmatrix}$$

$$L^{2n+1}_{3(2)}(\lambda_1, \lambda_4, \lambda_5) = \begin{bmatrix} L_{3(2)} & & 0 \\ & L_{3(2)} & \\ 0 & \ddots & L_{3(2)} \end{bmatrix}$$

$$\vdots$$

$$L^{2n+1}_{3(n)}(\lambda_{n-1}, \lambda_{2n}, \lambda_{2n+1}) = \begin{bmatrix} L_{3(n)} & & 0 \\ & L_{3(n)} & \\ 0 & \ddots & L_{3(n)} \end{bmatrix}.$$

where

$$L_{3(m)}(\Lambda_{n-1},\lambda_{2n},\lambda_{2n+1}) = \begin{bmatrix} \lambda_{2m+1} I_{m-1} & (\Lambda_{m-1} - i\lambda_{2m}) I_{m-1} \\ (\Lambda_{m-1} + i\lambda_{2m}) I_{m-1} & -\lambda_{2m+1} I_{m-1} \end{bmatrix}$$

$L_{3(m)}$ is thus a function only of three parameters and is just an "enlarged" L_3 matrix, in which a unit matrix is attached to each of the parameters. For obvious reasons, these can be called the helicity operators of various orders of which the operator

$$\begin{bmatrix} L_3 & & & 0 \\ & L_3 & & \\ & & \ddots & \\ 0 & & & L_3 \end{bmatrix}$$

is the first member.

We have seen earlier that in our notation the Dirac Hamiltonian is an L_5 with $\lambda_4 = 0$. The simultaneous eigen-vectors of L_5 and

$$\begin{bmatrix} L_3 & 0 \\ 0 & L_3 \end{bmatrix}$$

may be recognised to be also the simultaneous eigenvectors of the set.

$$L^{5}_{3(1)}\,(p_x, p_y, p_z) \quad \text{and} \quad L^{5}_{2(2)}\,(p, 0, m)$$

where p is the modulus of the momentum. If we write the eight dimensional Dirac Hamiltonian L_7, with $\lambda_4 = \lambda_5 = \lambda_6 = 0$, then the simultaneous eigenvectors of the set

$$L^{7}_{7}, \quad L^{7}_{5} = \begin{bmatrix} L_5 & 0 \\ 0 & L_5 \end{bmatrix}, \quad L^{7}_{3} = \begin{bmatrix} L_3 & & & 0 \\ & L_3 & & \\ & & L_3 & \\ 0 & & & L_3 \end{bmatrix}$$

are also the simultaneous eigenvectors of the set

$$L^{7}_{3(1)}\,(p_x, p_y, p_z) = \begin{bmatrix} L_3 & & & 0 \\ & L_3 & & \\ & & L_3 & \\ 0 & & & L_3 \end{bmatrix},$$

$$L^{7}_{3(2)}\,(p, 0, 0) = p \begin{bmatrix} 0 & I(1) & 0 & 0 \\ I(1) & 0 & 0 & 0 \\ 0 & 0 & 0 & I(1) \\ 0 & 0 & I(1) & 0 \end{bmatrix}$$

and

$$L^{7}_{3(3)}\,(p, 0, m) = \begin{bmatrix} m\,I(2) & p\,I(2) \\ p\,I(2) & -m\,I(2) \end{bmatrix}$$

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ON THE ALGEBRA OF L-MATRICES *

"The end of method is perspicuity"

by

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ON THE ALGEBRA OF L-MATRICES

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"The end of method is perspicuity"

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There is a singular appropriateness in discussing the algebra of L-matrices in this symposium in which Professor R.H. Good is one of principal participants. For it was a seminar conducted almost twelve years ago, in Madras, on an interesting paper of Professor Good on the gamma matrices which started an investigation that culminated in the present theory of L-matrices which include the gamma matrices within their structure.

The fountainhead of the theory of elementary particles is the Dirac equation which rests on four Dirac matrices and which are of dimension 4 x 4 but obey the same anti-communication relations as the 2 x 2 Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1)$$

Four mutually anti-commuting matrices were needed since the Dirac Hamiltonian, was postulated to be a linear combination of the four matrices with the four quantities p_x, p_y, p_z and m as their coefficients, respectively. We can obtain anticommuting matrices of higher dimensions by defining the left and right direct products of Pauli matrices with a unit matrix of arbitrary dimension as

$$I \otimes \sigma_x = S_x = \begin{pmatrix} \sigma_x & & & \\ & \sigma_x & & \\ & & \ddots & \\ & & & \sigma_x \end{pmatrix} \tag{2a}$$

$$I \otimes \sigma_y = S_y = \begin{pmatrix} \sigma_y & & \\ & \ddots & \\ & & \sigma_y \end{pmatrix} \tag{2b}$$

$$I \otimes \sigma_z = S_z = \begin{pmatrix} \sigma_z & & \\ & \ddots & \\ & & \sigma_z \end{pmatrix} \tag{2c}$$

$$\sigma_x \otimes I = \rho_x = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{3a}$$

$$\sigma_y \otimes I = \rho_y = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix} \tag{3b}$$

$$\sigma_z \otimes I = \rho_z = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{3c}$$

For obvious reasons we shall call S_x, S_y and S_z as Pauli matrices enlarged by repetition and ρ_x, ρ_y and ρ_z Pauli matrices enlarged by dilation. Though the dimension is increased by the direct-product operation, we obtain only sets of three anticommuting matrices and not four as required.

In the case when I is of dimension two, Dirac noticed that the set of four matrices

$$\alpha_x = \rho_x S_x \;, \quad \alpha_y = \rho_x S_y \;, \quad \alpha_z = \rho_x S_z \qquad (4)$$

will satisfy the requirements for his relativistic wave equation. The entire algebra of Dirac matrices was built out of the set of four defined above, their products, sums and differences.

However, from an algebraic point of view the procedure of constructing the Dirac matrices from the Pauli matrices is only part of a general method of constructing higher dimensional anti-commuting matrices from a primitive set of three (2 x 2) Pauli matrices. During the past year, a systematic study of this method has been made by the author which can be summarized as an algebra of L-matrices in the following manner.

We are aware that the only (2 x 2) matrix which commutes with the three Pauli matrices is a multiple of the unit matrix. However, it is a remarkable fact that if we build 'dilated' Pauli matrices ρ_x, ρ_y, ρ_z , any matrix of the form

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \qquad (5)$$

where A is an arbitrary matrix of the same dimension as the unit matrix I in ρ_x, ρ_y and ρ_z commutes with all the three dilated Pauli matrices. Thus, it is this commuting matrix that corresponds to the multiple of unit matrix in two dimensions.

As in the case of Pauli matrices we can form a 'helicity matrix' through a linear combination of the three anticommuting 'dilated' Pauli matrices

$$\rho\cdot\lambda = \lambda_1\rho_1 + \lambda_2\rho_2 + \lambda_3\rho_3 \qquad (6)$$

From now on we use the numerical suffix 1,2 and 3 instead of x,y and z for the enlarged Pauli matrices with the explicit understanding that any one of the set ρ_1, ρ_2, ρ_3 can denote ρ_x and another ρ_y and the other ρ_z. This convention will be adopted in the ensuing discussion to simplify the notation and procedure.

For reasons which will be apparent presently we shall call the $\rho\cdot\lambda$ helicity matrix of mth order if ρ_1, ρ_2 and ρ_3 are of the dimension $2^m \times 2^m$ and denote them by $\rho_1^{(m)}, \rho_2^{(m)}, \rho_3^{(m)}$.

We can now form a set of five anti commuting matrices of dimension $2^n \times 2^n$

$$\rho_1^{(n)}, \rho_2^{(n)}, \rho_3^{(n)}, \rho_3^{(n)}\rho_1^{(n)}(n-1), \rho_3^{(n)}\rho_2^{(n)}(n-1) \qquad (7)$$

where

$$\rho_i^{n}(n-1) = \begin{pmatrix} \rho_i^{(n-1)} & 0 \\ 0 & \rho_i^{(n-1)} \end{pmatrix} \tag{8}$$

Considering L-matrices of (n - 1)th order we note that they commute with any one of the three matrices of order $2^{n-1} \times 2^{n-1}$

$$\begin{pmatrix} \rho_i^{(m-2)} & \\ & \rho_i^{(m-2)} \end{pmatrix} \tag{9}$$

and, therefore, the product of any one of the Pauli matrices of order (n-1) with the above anticommutes with the other two. Thus, we arrive at seven anticommuting matrices of dimensions $2^n \times 2^n$

$$\rho_i^{(m)} \qquad (i = 1, 2)$$

$$\rho_3^{(m)} \rho_i^{(m)}(m-1) \qquad (i = 1, 2)$$

$$\rho_3^{(m)} \rho_3^{(m)}(m-1)\, \rho_i^{(m)}(m-2) \qquad (i = 1, 2)$$

and

$$\rho_3^{(m)} \rho_3^{(m)}(m-1)\, \rho_3^{(m)}(m-2) \tag{10}$$

where

$$\rho_i^{(m)}(m-2) = \begin{pmatrix} \rho_i^{(m-2)} & & & \\ & \rho_i^{(m-2)} & & \\ & & \rho_i^{(m-2)} & \\ & & & \rho_i^{(m-2)} \end{pmatrix} \tag{11}$$

These are Pauli matrices enlarged both by dilation and repetition.

This procedure can be continued until we arrive at the $(2n + 1)$ anticommuting matrices of dimension 2^n which can be conveniently arranged as $(n - 1)$ sets of two matrices and one set of three matrices

$$\rho_i^{(m)} \qquad (i = 1,2)$$

$$\rho_3^{(m)} \; \rho_i^{(m)}(m-1) \qquad (i = 1,2)$$

$$\rho_3^{(m)} \; \rho_3^{(m)}(m-1) \; \rho_i^{(m)}(m-2) \qquad (i = 1,2)$$

$$\rho_3^{(m)} \; \rho_3^{(m)}(m-1) \; \rho_3^{(m)}(m-2) \; \rho_i^{(m)}(m-3) \qquad (i = 1,2)$$

$$\vdots$$

$$\rho_3^{(m)} \; \rho_3^{(m)}(m-1) \; --- \; \rho_3^{(m)}(2) \; \rho_i^{(m)}(1) \qquad (i = 1,2)$$

and

$$\rho_3^{(m)} \; \rho_3^{(m)}(m-1) \; --- \; \rho_3^{(m)}(2) \; \rho_3^{(m)}(1)$$

where

$$\rho_i^{(m)}(n-r) = \begin{pmatrix} \rho_i^{(n-r)} & & \\ & \ddots & \\ & & \rho_i^{(n-r)} \end{pmatrix} \tag{12}$$

where $\rho_i^{(n-r)}$ is repeated 2^r times.

This step by step procedure of obtaining $(2n + 1)$ anti-commuting matrices can be expressed through a single prescription called the σ-operation by the author which can be described as follows:

Taking the primitive L-matrix of dimension 2×2

$$L_3 = \begin{pmatrix} \lambda_3 & \lambda_1 - i\lambda_2 \\ \lambda_1 + i\lambda_2 & -\lambda_3 \end{pmatrix} \tag{13}$$

which is a linear combination of the three Pauli matrices with λ_1, λ_2 and λ_3 as their respective coefficients, we obtain a matrix of dimension $(2^n \times 2^n)$ involving $(2n + 1)$ parameters by adopting the following procedure.

Replace any one of the three parameters λ_1, λ_2 and λ_3 by a matrix L_{2n-1} of dimension $(2^{n-1} \times 2^{n-1})$ and relabel the other two parameters as λ_{2n} and λ_{2n+1} attaching unit matrices of dimension $(2^{n-1} \times 2^{n-1})$ to them. That is, we can define L_{2n+1} as

$$L_{2n+1} = \sigma(L_{2n-1}) = \begin{pmatrix} \lambda_{2n+1} I & L_{2n-1} - i\lambda_{2n} I \\ L_{2n-1} + i\lambda_{2n} I & -\lambda_{2n+1} I \end{pmatrix} \tag{14}$$

or,

$$= \begin{pmatrix} \lambda_{2n+1} I & \lambda_{2n} I - i L_{2n-1} \\ \lambda_{2n} I + i L_{2n-1} & -\lambda_{2n+1} I \end{pmatrix} \tag{15}$$

or

$$= \begin{pmatrix} L_{2n+1} & (\lambda_{2n} - i\lambda_{2n+1})I \\ (\lambda_{2n} + i\lambda_{2n+1})I & -L_{2n+1} \end{pmatrix} \tag{16}$$

This amounts to writing as

$$L_{2n+1} = \lambda_{2n+1}\,\rho_3^{(n)} + \lambda_{2n}\,\rho_3^{(n)} + \rho_1^{(n)} \begin{pmatrix} L_{2n-1} & 0 \\ 0 & L_{2n-1} \end{pmatrix} \tag{17}$$

The first two are enlarged Pauli matrices, while the third is just the product of the third enlarged Pauli matrix and a matrix which commutes with all the three enlarged Pauli matrices. The procedure for L_{2n-1} is identical with the step by step procedure described in the definition of the σ-operation, we can replace any one of the parameters by L_{2n-1}. This feature is best expressed through the use of numeral suffixes for the enlarged Pauli matrices. Writing L_{2m+1} as

$$L_{2m+1} = \sum_{i=1}^{2n+1} \lambda_i \, \mathcal{L}_i^{2m+1} \tag{18}$$

we immediately notice that the L-matrices can be expressed as products of ρ-matrices of various orders as given in equation (10). The algebra of the L-matrices follows from the algebra of the ρ-matrices as ex-emplified in equation (12). So we get the product of (2m + 1) L-matrices as

$$\alpha_1^{(2m+1)}\alpha_2^{(2m+1)}\cdots\alpha_{2m+1}^{(2m+1)} = i^m I \tag{19}$$

L_{2n+1} is a function of 2n + 1 parameters λ_1, λ_2 ~~and~~ to λ_{2n+1} and

$$L^2 = (\lambda_1^2 + \cdots + \lambda_{2n+1}^2) I = \Lambda_n^2 I \tag{20}$$

The Dirac Hamiltonian was identified to be L_5 with $\lambda_1 = p_x, \lambda_2 = p_y$, $\lambda_3 = p_z, \lambda_4 = 0, \lambda_5 = m$ and $\Lambda_2 = E$

$$L_5^2 = E^2 I = (p_x^2 + p_y^2 + p_z^2) I + m^2 I \tag{21}$$

where p_x, p_y, p_z. are the three components of momenta, m the mass, and E the energy.

To establish closer contact with relativistic transformation we define

$$v_1 = \frac{\lambda_1}{\Lambda};\ v_2 = \frac{\lambda_2}{\Lambda};\ \cdots;\ v_{2n} = \frac{\lambda_{2n}}{\Lambda};\ \lambda = \lambda_{2n+1} \tag{22}$$

The parameters $\lambda_1, \lambda_2, \cdots, \lambda_{2n}$ can then be expressed in terms of $\lambda, v_1, \cdots, v_{2n}$ as

$$\lambda_1 = \frac{\lambda v_1}{\sqrt{1-v^{2}}};\ \cdots;\ \lambda_{2n} = \frac{\lambda v_{2n}}{\sqrt{1-v^{2}}} \tag{23}$$

with

$$v^2 = \frac{\Lambda^2 - \lambda^2}{\Lambda^2} + v_1^2 + \cdots + v_{2n}^2. \tag{24}$$

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THE INSTITUTE OF MATHEMATICAL SCIENCES
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A HIERARCHY OF IDEMPOTENT MATRICES

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A HIERARCHY OF IDEMPOTENT MATRICES

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We notice that any (3 x 3) antisymmetric matrix

$$\begin{bmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{bmatrix} \tag{1}$$

where $\lambda_1, \lambda_2, \lambda_3$ are pure real or pure imaginary parameters has the very interesting property

$$A^3 = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\, A \tag{2}$$

The determinant of A is zero. hence one of the three eigenvalues of A is zero, the other two being given by $\pm \Lambda_1$ where

$$\Lambda_1 = +\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \tag{3}$$

The eigenvectors corresponding to the eigenvalues $\pm$ and 0 are:

$$u_+ = \begin{pmatrix} i\lambda_1\lambda_3 + \Lambda_1\lambda_2 \\ i\lambda_2\lambda_3 - \lambda_1\Lambda_1 \\ -i(\lambda_1^2+\lambda_2^2) \end{pmatrix}, \quad u_- = \begin{pmatrix} i\lambda_1\lambda_3 - \Lambda_1\lambda_2 \\ i\lambda_2\lambda_3 + \lambda_1\Lambda_1 \\ -i(\lambda_1^2+\lambda_2^2) \end{pmatrix}; \quad u_0 = i\lambda_3 \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \tag{4}$$

The matrix A^2 is non-diagonal. Though it is a 3 x 3 matrix it has only two eigenvalues $\pm\Lambda_1$ and 0. There are two independent eigenvectors which can be obtained as linear combination of the two eigenvectors of A corresponding to the eigenvalue $\pm\Lambda_1$. It is to be noted that while an eigenvector A is an eigenvector of A^2 the converse is not true since an eigenvector of A^2 corresponding to Λ^2 may be a linear combination of the eigenvector of A with different eigenvalues of $\pm\Lambda$ and hence is not an eigenvector of A.

Writing the equation (2) as

$$A_3^2 A_3 = \Lambda_1^2 A_3 \tag{5}$$

we immediately recognise that the three columns of A are just the eigenvectors of A^2 corresponding to the eigenvalue Λ_1^2. Out of these columns only two are independent since A is singular. This is as it should be since these eigenvectors correspond only to the doubly degenerate eigenvalue Λ_1^2.

In the theory of L matrices, we started with the primitive(2 x 2) L-matrix L_3 such that

$$L_3^2 = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\, I \tag{6}$$

To obtain the matrix L_{2n+1} of dimension 2^n, having (2n+1) parameters we replace any one of the three parameters in (6) by L_{2n-1} and the other two by (λ_{2n+1}) I and $\lambda_{2n} I$ where I is a unit matrix of the same dimension as L_{2n-1}.

In the case of the A matrix we can adopt a similar procedure if we recognise that the numerical coefficient of A on the right hand side of eq.(5) is quadratic in λ_1 , λ_2 and λ_3. In the A matrix we can replace any one of the parameters by L_{2n-1} of dimension 2^{n-1} and the other two parameters by $\lambda_{2n} I$ and $\lambda_{2n+1} I$. We thus obtain :

$$A_{2n+1} = \begin{pmatrix} 0 & L_{2n-1} & \lambda_{2n+1} I \\ -L_{2n-1} & 0 & -\lambda_{2n} I \\ -\lambda_{2n+1} I & \lambda_{2n} I & 0 \end{pmatrix} \tag{7a}$$

or

$$A_{2n+1} = \begin{pmatrix} 0 & \lambda_{2n+1} I & L_{2n-1} \\ -\lambda_{2n+1} I & 0 & -\lambda_{2n} \\ -L_{2n-1} & \lambda_{2n} I & 0 \end{pmatrix} \tag{7b}$$

or

$$A_{2n+1} = \begin{pmatrix} 0 & \lambda_{2n+1} I & \lambda_{2n} I \\ -\lambda_{2n+1} I & 0 & -L_{2n-1} \\ -\lambda_{2n} I & L_{2n-1} & 0 \end{pmatrix} \tag{7c}$$

It is to be emphasised that higher dimensional A matrices are obtained by imbedding an L matrix and not the A matrix.

The eigenvalues of A_{2n+1} are $\pm \Lambda_n$ and 0 where

$$\Lambda_n = \pm \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \cdots + \lambda_{2n+1}^2} = \sqrt{\Lambda_{n-1}^2 + \lambda_{2n}^2 + \lambda_{2n+1}^2} \quad (8)$$

Possible Physical Interpretations:

In the case of the L matrices we were able to identify the helicity matrix to be L_3 with $\lambda_1 = p_x$; $\lambda_2 = p_y$; $\lambda_3 = p_z$ and the Dirac Hamiltonian with L_5 with the realisations:

$$\lambda_1 = p_x \;;\; \lambda_2 = p_y \;;\; \lambda_3 = p_z \;;\; \lambda_4 = 0 \;;\; \lambda_5 = m \quad (9)$$

The question now arises whether a similar interpretation can be attempted in the case of the A matrices which involve one set of trichotomous eigenvalues and the rest are dichotomous sets.

In the case of A_3, there seems to be a direct connection with the polarization states of the photon. Since the operator A^2 yields only the square of energy as it meaningful to speculate that the particle and the antiparticle of the photon are the same?

In the case of A_5, if we choose L_3 to be the helicity matrix with $\lambda_1 = p_x$, $\lambda_2 = p_y$, $\lambda_3 = p_z$ and if set $\lambda_4 = 0$; $\lambda_5 = m$ is the matrix connected with the relativistic eigenstates of an elementary particle? In this case it will still represent a particle of spin $1/2$ if L_3 is chosen to be the helicity matrix.

If A^2 is to represent the square of energy are we dealing only with particles and not antiparticles and does the triplet of eigenvalues corresponding to A^3 have any connection with a fundamental triplet of particles?

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THE INSTITUTE OF MATHEMATICAL SCIENCES
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L-MATRICES AND PROPAGATORS WITH IMAGINARY PARAMETERS

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L-MATRICES AND PROPAGATORS WITH IMAGINARY PARAMETERS

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Introduction:

The study of L-matrices leads us naturally to the definition of the matrices

$$Q = L - \lambda I \tag{1}$$

and

$$R = \frac{1}{L - \lambda I} \tag{2}$$

where Q is a 'quaternion-like' object and its reciprocal R is the resolvent which can be interpreted as the propagator associated with L under 'suitable' circumstances.

The 'suitable' circumstances relate to the interpretation of the $2n+1$ parameters $\lambda_1, \lambda_2, \lambda_3, \ldots$ and λ_{2n+1} which are imbedded in L such that

$$\begin{aligned} L^2 &= (\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_{2n+1}^2) I \\ &= \Lambda_n^2 I \end{aligned} \tag{3}$$

where $\pm \Lambda_\eta$ are the two eigenvalues of the $2^n \times 2^n$ dimensional matrix and I is a unit matrix of the same dimension as that of L. It was emphasised by the author that these parameters are either pure real or pure imaginary which implies that the eigenvalue also is pure real or pure imaginary.

The distinction between real and imaginary parameters and the corresponding eigenvalues becomes significant when we are interested in the existence of a Fourier transform of L with respect to the parameters. To understand the consequences which follow from the introduction of imaginary quantities, we recall a very simple feature in the integration of an exponential function $e^{a\tau}$ over an infinite domain in τ, from $-\infty$ to ∞.

Case i) If a is real and negative, the integral is finite even when the upper limit is ∞. The lower limit should not extend to $-\infty$. Without loss of generality we can set the lower limit to be zero. The integral is

$$\int_0^\infty e^{a\tau} d\tau = -\frac{1}{a} \tag{4}$$

Case ii) a is real and positive. Now the lower limit can be but the upper limit should not extend to ∞. We have for the integral

$$\int_{-\infty}^0 e^{a\tau} d\tau = \frac{1}{a} \tag{5}$$

Case iii) a is pure imaginary. The integration can be performed over the entire domain. If $a = i\beta$, the integral

$$\int_{-\infty}^{\infty} e^{i\beta\tau} \, d\tau = 2\pi i \, \delta(\beta) \tag{6}$$

where δ is the Dirac-delta function.

Case iv) a is complex. Set a = $\beta + i\gamma$. The domain of integration will depend upon whether β is positive or negative, as given in (4) and (5).

II. Fourier-Transforms of the Resolvent

We now define the Fourier transform of the resolvent with respect to a partial set of variables $\lambda_1, \ldots \lambda_p$. For reasons of convenience, we relabel the other parameters $\lambda_{p+1}, \ldots, \lambda_{2n+1}$ as $m_1, \ldots, m_{2n-p+1}$. Let us define

$$p^2 = \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_p^2 \tag{7}$$

$$M^2 = m_1^2 + \ldots + m_{2n-p+1}^2 \tag{8}$$

Thus $\Lambda_n^2 = P^2 + M^2$. We call the parameters $\lambda_1, \ldots, \lambda_p$ 'momentum like', since the Fourier transformation is defined with respect to them, the parameters $m_1, \ldots, m_{2n-p+1}$ 'mass like' since they are kept constant in the Fourier transformation. Λ_n is 'energy like' since it is eigenvalue. The variables $x_1, \ldots, x_p$ of the transform associated with $\lambda_1, \ldots, \lambda_p$ are called 'space like' while the variable t associated with free parameter λ is called time like.

To facilitate the discussion on various types of particles later in this paper, we now introduce 'velocity like' parameters

$$v_1 = \frac{\lambda_1}{\Lambda_n} \; ; \quad v_2 = \frac{\lambda_2}{\Lambda_n} \; ; \quad \ldots \; ; \quad v_p = \frac{\lambda_p}{\Lambda_n} \qquad (9)$$

We can express the parameters $\lambda_1, \ldots, \lambda_p$ in terms of $v_1, \ldots, v_p$ and M as

$$\lambda_1 = \frac{M v_1}{\sqrt{1-v^2}} \; ; \quad \ldots \; ; \quad \lambda_p = \frac{M v_p}{\sqrt{1-v^2}}$$

with

$$v^2 = \frac{\Lambda_n^2 - M^2}{\Lambda_n^2} = v_1^2 + \cdots + v_p^2 \qquad (10)$$

(11)

We shall now classify 'particles' as follows making the postulate that the integration over the parameters is from $-\infty$ to ∞.

Case i) $p^2 > 0$, $M^2 > 0$ and hence $\Lambda_n^2 > 0$. The denominator has singularities at $\pm\Lambda_n$ on the real axis. The integral will depend upon the path of integration chosen suitably. The situation is identical to the definition of propagators corresponding to the advanced, retarded and Feynman kernels so well known in Quantum Electrodynamics. The integrals corresponding to the well known paths of integration are:

$$K_F = \int_{-\infty}^{+\infty}\!\!\int \Big[d\lambda_1 \cdots d\lambda_p \, d\lambda \, \frac{1}{2\Lambda_n} \left(\frac{L_{2n+1} + \Lambda_n I}{\lambda - \Lambda_n + i\varepsilon} - \frac{L_{2n+1} + \Lambda_n I}{\lambda + \Lambda_n - i\varepsilon} \right) e^{i(\lambda_1 x_1 + \cdots + \lambda_p x_p) - i\lambda t} \Big] \quad (12)$$

$$K_R = \int_{-\infty}^{+\infty} d\lambda_1 \cdots d\lambda_p \, d\lambda \, \frac{1}{2\Lambda_n} \left(\frac{L_{2n+1} + \Lambda_n I}{\lambda - \Lambda_n + i\varepsilon} - \frac{L_{2n+1} - \Lambda_n I}{\lambda + \Lambda_n + i\varepsilon} \right) e^{i(\lambda_1 x_1 + \cdots + \lambda_p x_p) - i\lambda t} \quad (13)$$

$$K_A = \int_{-\infty}^{\infty} d\lambda_1 \cdots d\lambda_p \, d\lambda \, \frac{1}{2\Lambda_n} \left(\frac{L_{2n+1} + \Lambda_n I}{\lambda - \Lambda_n - i\varepsilon} - \frac{L_{2n+1} - \Lambda_n I}{\lambda + \Lambda_n - i\varepsilon} \right) \quad (14)$$

Integrating with respect to variable λ we obtain

$$Q_F = \int_{-\infty}^{+\infty} \frac{1}{2\Lambda_n}\Big[\Theta(t)(L_{2n+1} + \Lambda_n I)e^{-i\Lambda_n t} + \Theta(-t)(L_{2n+1} - \Lambda_n I)e^{i\Lambda_n t}\Big] d\lambda_1 \cdots d\lambda_p \tag{15}$$

$$Q_R = \int_{-\infty}^{\infty} \frac{1}{2\Lambda_n}\Theta(t)\Big[(L_{2n+1} + \Lambda_n I)e^{-i\Lambda_n t} - (L_{2n+1} - \Lambda_n I)e^{i\Lambda_n t}\Big] d\lambda_1 \cdots d\lambda_p \tag{16}$$

$$Q_A = \int_{-\infty}^{\infty} \frac{1}{2\Lambda_n}\Theta(-t)\Big[(L_{2n+1} + \Lambda_n I)e^{-i\Lambda_n t} + (L_{2n+1} - \Lambda_n I)e^{i\Lambda_n t}\Big] d\lambda_1 \cdots d\lambda_p \tag{17}$$

Case ii) Assume $M^2 < 0$. As $\Lambda_n^2 = p^2 + M^2$, Λ_n is real when $p^2 \geqslant M^2$, while Λ_n will be imaginary when $p^2 < M^2$. Thus the integral splits into two parts: corresponding to $p^2 \geqslant M^2$ and $p^2 < M^2$. For the region, $p^2 \geqslant M^2$, the above expressions for advanced, retarded and Feynman kernels will still hold good. However, when $p^2 < M^2$ is pure imaginary the exponential $e^{i\Lambda_n t}$ involves a real exponent and hence corresponds to a rapidly decreasing or increasing function of t.

All these features can be imbedded in the above kernels provided we require Λ_m to be pure positive real or pure negative imaginary, in the case of imaginary mass and use the same expressions for the kernels.

Case iii) Let us assume $p^2 < 0$. Now the integration with respect to space like variables has to be split up into two halves. This division looks artificial and hence there is a difficulty in understanding its significance. However, in the case of a 'radial' distributions we can integrate in space over the semi infinite range 0 to ∞. The physical meaning will be explained presently.

From the above discussion, it is clear that it is possible to have imaginary values for a partial set of parameters in the L-matrix provided the domains of space like and time like variables are divided dichotomously into positive and negative domains.*

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*In such case it is called Fourier-Carleman transformation familiar in the Theory of Fourier Transforms.

III. Physical Interpretation

To discuss the relevance of the above considerations to physical problems we now set the parameters in L_5 as follows:

$$\lambda_1 = p_x ; \quad \lambda_2 = p_y ; \quad \lambda_3 = p_z ; \quad \lambda_4 = m_1 = 0 ; \quad \lambda_5 = m_2 = M \tag{18}$$

With this choice we can classify particles according to the following prescription.

a) Ordinary Free Particles:

$$M^2 \geq 0 ; \quad p^2 \geq 0 ; \quad E^2 \geq 0$$

In this case, there exists a rest system ($p=0$) for the particle when $M^2 > 0$. However, when $M = 0$, there is no rest system as is in the case for photons or neutrons.

b) If M = imaginary, $E^2 < p^2$. Two classes can now be distinguished:

i) $E^2 > 0$ i.e. E is real and

ii) $E^2 < 0$ E is imaginary

Tanaka has considered this question in great detail by assuming the mass to be imaginary in the Dirac equation and postulating 'superlight particles' with the velocities greater than of light. We see from equations (12), (13) and (14) that the expression for the propagators in configuration space with imaginary energies involves an exponential decaying function of time. However, the integrals for these functions exist for time integration from 0 to ∞. If we consider the wave functions of a particle corresponding to this, it decays almost instantaneously i.e. it is 'evanescent'. However, such particles will make a contribution to the momentum transform of the resolvent. This means that though we cannot have incident or emergent particles with imaginary energy, they can still contribute to the energy denominators. These particles have imaginary velocities. Sudarshan et al have assumed that particles of imaginary mass must have real energy but the propagators formalism necessitates the inclusion of evanescent particles of imaginary energy and velocity when considering particles of imaginary mass.

c) $p^2 < 0;\ M^2 > 0$ with $-p^2 < M^2$ This case corresponds to bound particles. If we wish to make a space integration from 0 to ∞.

Hence bound particles have as much extension in space in relation to the universe as evanescent particles have existence in time in relation to eternity. Further just as bound particles with imaginary momentum are expected to play a role in the scattering of real particles, the evanescent particles should play a similar role in the interactions of particles with imaginary mass. It is expected that these considerations will be important in postulating the possible existence of faster than light particles and examining the consistency of such a postulate with the Lorentz group.

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THE INSTITUTE OF MATHEMATICAL SCIENCES
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GENERALIZED HELICITY MATRICES

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GENERALISED HELICITY MATRICES

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In the theory of L-matrices developed by the author, a method was described for generating a matrix L_{2n+1} of dimension $2^n \times 2^n$ involving $(2n+1)$ pure real or imaginary parameters $\lambda_1, \lambda_2, \ldots, \lambda_{2n+1}$ such that

$$L_{2n+1}^2 = (\lambda_1^2 + \lambda_2^2 + \ldots\ldots + \lambda_{2n+1}^2) I = \Lambda_n^2 I, \qquad (1)$$

where I is a unit matrix of the same dimension as L_{2n+1}. The matrix L_{2n+1} can be written as a linear combination of $(2n+1)$ mutually ancitommuting $\mathcal{L}$-matrices as

$$L_{2n+1} = \sum_{i}^{2n+1} \lambda_i \mathcal{L}_i^{2n+1}, \qquad (2)$$

with

$$(\mathcal{L}_i)^2 = 1, \; (i = 1,2,\ldots,n), \qquad (3)$$

the superscript $(2n+1)$ on $\mathcal{L}$ denoting that the $\mathcal{L}$ matrices "belong" to L_{2n+1} and are of dimension $2^n \times 2^n$. The set of these $(2n+1)$ mutually anticommuting $\mathcal{L}_i^{2n+1}$, $i = 1,2,\ldots (2n+1)$, satisfy the product property

$$\mathcal{L}_1^{2n+1} \mathcal{L}_2^{2n+1} \ldots\ldots \mathcal{L}_{2n+1}^{2n+1} = i^2 I. \tag{4}$$

This implies that only 2n out of the (2n+1) matrices are independent.

It was also pointed out that if A is any non-singular matrix, then

$$L' = A L A^{-1} \tag{5}$$

is also an L-matrix i.e. it satisfies (1). This in turn implies that

$$\mathcal{L}' = A \mathcal{L} A^{-1} \tag{6}$$

is also an $\mathcal{L}$-matrix i.e. square of each $\mathcal{L}_i'$ is unity; the $\mathcal{L}_i'$ anticommute with one another and satisfy the product relation (4). We shall now demonstrate that from the set $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{2n+1}$ we can obtain n sets of "generalized Pauli matrices", each set $\{H^i\}$ consisting of three mutually anticommuting matrices

$$\{H^i\} = (H_1^i, H_2^i, H_3^i)$$
$$i = 1,2,\ldots,n \tag{7}$$

Not only is the square of each matrix equal to unity but their product is a multiple of unity

$$H_1^i \, H_2^i \, H_3^i = i^3 \, I \tag{8}$$

i.e. only two out of the set of three are independent and since there are n sets, we have $2n$ independent anticommuting matrices in conformity with the property of $\mathcal{L}$-matrices.

We now define

$$H_1^n = \mathcal{L}_{2n+1} \, \mathcal{L}_{2n} ,$$
$$H_2^n = \mathcal{L}_{2n+1} ,$$
$$H_3^n = \mathcal{L}_{2n} . \tag{9}$$

The set $\{ H_1^n, H_2^n, H_3^n \}$ has the required properties of "generalized Paulit matrices". Let us write

$$\mathcal{L}_{2n-1} = H_1^n \, \mathcal{L}_{2n-1}(1),$$
$$\mathcal{L}_{2n-2} = H_1^n \, \mathcal{L}_{2n-2}(1),$$
$$\cdots$$
$$\cdots$$
$$\mathcal{L}_1 = H_1^n \, \mathcal{L}_1(1) \tag{10}$$

Then $\mathcal{L}_{2n-1}(1)$, $\mathcal{L}_{2n-2}(1)$, ... $\mathcal{L}_1(1)$ form a set of $(2n-1)$ mutually anticommuting matrices with the product property,

$$\mathcal{L}_{2n-1}(1)\ \mathcal{L}_{2n-2}(1)\ \ldots\ \mathcal{L}_1(1) = i^n\ I \tag{11}$$

i.e. (2n-2) of these $\mathcal{L}(1)$ matrices are independent. These $\mathcal{L}(1)$ matrices are not $\mathcal{L}$ matrices in the strict sense though their dimension is the same as that of the $\mathcal{L}$ matrices. There are only (2n-2) independent $\mathcal{L}(1)$ matrices in contrast with the 2n independent $\mathcal{L}$ matrices.

We nest proceed to define

$$H_1^{n-1} = \mathcal{L}_{2n-1}(1)\ \mathcal{L}_{2n-2}(1),$$

$$H_2^{n-1} = \mathcal{L}_{2n-1}(1),$$

$$H_3^{n-1} = \mathcal{L}_{2n-2}(1). \tag{12}$$

The set $\left\{ H_1^{n-1},\ H_2^{n-1},\ H_3^{n-1} \right\}$ again has the properties of generalized Pauli γ^{+} matrices. We now factor out H_3^{n-1} from $\mathcal{L}_{2n-3}(1)$, $\mathcal{L}_{2n-4}(1), \ldots$ and $\mathcal{L}_1(1)$ and write

$$\mathcal{L}_{2n-3}(1) = H_1^{n-1}\ \mathcal{L}_{2n-3}(2)$$

$$\mathcal{L}_{2n-4}(1) = H_1^{n-1}\ \mathcal{L}_{2n-4}(2)$$

$$\mathcal{L}_1(1) = H_1^{n-1}\ \mathcal{L}_1(2). \tag{13}$$

Then $\mathcal{L}_{2n-3}(2)$, $\mathcal{L}_{2n-4}(2), \ldots \mathcal{L}_1(2)$ form a set of (2n-3) mutually anticommuting matrices with the product property.

$$\mathcal{L}_{2n-3}(2)\,\mathcal{L}_{2n-4}(2)\ldots\ \mathcal{L}_1(2) = i^n\, I.$$

i.e. only (2n-4) of these $\mathcal{L}(2)$ matrices are independent. This procedure can be iterated till we arrive at the set

$$\begin{aligned} H^1_1 &= \mathcal{L}_1(n-1)\ \mathcal{L}_2(n-1), \\ H^1_2 &= \mathcal{L}_2(n-1), \\ H^1_3 &= \mathcal{L}_1(n-1). \end{aligned} \tag{14}$$

Any member of the set $\{H^i\}$ commutes with any member of another set $\{H^j\}$, $i \neq j$. If we now define <u>the helicity matrix</u> H^i <u>as the linear combination of the members of the set</u> H^i and choose in particular

$$\begin{aligned} H^n &= \lambda_{2n+1}\, H^n_3 + \lambda_{2n}\, H^n_2 + \Lambda_{n-1}\, H^n_1, \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ H^i &= \lambda_{2i+1}\, H^i_3 + \lambda_{2i}\, H^i_2 + \Lambda_{i-1}\, H^i_1, \\ H^1 &= \lambda_3\, H^1_3 + \lambda_2\, H^1_2 + \lambda_1\, H^1_1, \end{aligned} \tag{15}$$

it follows that

$$(H^n)^2 = \Lambda_n^2 I = (\Lambda_{n-1}^2 + \lambda_{2n}^2 + \lambda_{2n+1}^2) I$$

$$(H^i)^2 = \Lambda_i^2 I = (\Lambda_{i-1}^2 + \lambda_{2i}^2 + \lambda_{2i+1}^2) I,$$

$$(H^1)^2 = \Lambda_1^2 I = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) I \qquad (16)$$

Since H^i commutes with H^j for $j \neq i$, we find it is possible to obtain eigenvectors of L_{2n+1} which are simultaneous eigenvectors of the " helicity " eigen values $\Lambda_n, \Lambda_{n-1}, \ldots \Lambda_1$ respectively.

If we obtain the particular representation of L_{2n+1} by the iteration procedure known as the σ-operation defined previously as

$$L_{2n+1} = \sigma(L_{2n-1}) = \begin{pmatrix} \lambda_{2n+1} I & L_{2n-1} - i \lambda_{2n} I \\ L_{2n-1} + i \lambda_{2n} I & -\lambda_{2n+1} I \end{pmatrix}$$

with

$$L_3 = \begin{pmatrix} \lambda_3 & \lambda_1 - i \lambda_2 \\ \lambda_1 + i\lambda_2 & -\lambda_3 \end{pmatrix} = \sigma.\lambda ,$$

where σ's are the three Pauli matrices then the set $\{ H^i \}$ can be identified to be

$$H_k^i = \begin{pmatrix} \sigma_k \otimes I & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \sigma_k \otimes I \end{pmatrix} \quad , \quad k = 1,2,3,$$

where $\sigma_k \otimes I$, the direct product of σ_k and the unit matrix I of dimension $2^{i-1} \times 2^{i-1}$ is repeated $2^{(n-i)}$ times along the diagonal.

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HELICITY MATRICES FOR GENERALISED CLIFFORD ALGEBRA

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HELICITY MATRICES FOR GENERALISED CLIFFORD ALGEBRA

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In a previous contribution[1)] to this journal, one of the authors (A.R) has shown that to a set of (2n+1) mutually anticommuting matrices of dimension $2^n \times 2^n$, α_i (i = 1,2,..., 2n+1) satisfying the Clifford conditions

$$\alpha_i \alpha_j = -\alpha_j \alpha_i , \quad (i \neq j)$$
$$(\alpha_i)^2 = I \tag{1}$$

there correspond n sets of three matrices $\{H_i^{\mu}\}$ $i = 1,2,3.$ $\mu = 1,2,\ldots,n$ with the following properties:

(1) each set obeys the Clifford conditions.

(2) Any member of one set commutes with any member of any other set.

For obvious reasons the triplets of these matrices are called the generalised Pauli matrices. A linear combination of this triplet of matrices is called the generalised helicity matrix.

Our object now is to prove that such helicity matrices can be defined corresponding to $(2n+1)$ matrices $\mathcal{L}_i$ of dimension $m^n \times m^n$ which obey the generalised Clifford conditions prescribed by A.O.Morris[2)]

$$\mathcal{L}_i \mathcal{L}_j = \omega \mathcal{L}_j \mathcal{L}_i ,$$

$$\text{and } (\mathcal{L}_i)^m = I , \tag{2}$$

where ω is the primitive m-th root of unity. We notice that for these relations there is a symmetry principle that the Clifford conditions are unaltered if any $\mathcal{L}$ is replaced by $\omega \mathcal{L}$.

If we now define

$$H_1^n = \mathcal{L}_1 ; \; H_2^n = \mathcal{L}_2 ; \quad H_3^n = \zeta (H_1^n)^{m-1} H_2^n \tag{3}$$

where

$$\zeta = 1 \text{ for } m \text{ odd}$$

$$\zeta = \omega^{1/2} \text{ for } m \text{ even.}$$

The set $\{ H_i^n \}$ $(i = 1,2,3)$ satisfies the required Clifford conditions; though H_3^n is not a member of the set $\mathcal{L}_i$. It is to be noted that we have to attach the factor $\omega^{1/2}$ to the member of the triplet for the case of even m to satisfy the second Clifford condition $(H_3^n)^m = I$. In a completely analogous way to the construction of the helicity matrices obeying the quadratic Clifford conditions[1] we define

$$\mathcal{L}_i = (H_3^n)^{m-1} \mathcal{L}_i(1) \qquad (i = 3,\ldots, 2n+1) \qquad (4)$$

i.e. we factor out $(H_3^n)^{m-1}$ for each of the remaining matrices of the set $\mathcal{L}_i$. Then $\mathcal{L}_3(1),\ldots, \mathcal{L}_{2n+1}(1)$ form a set of $(2n-1)$ matrices which obey the generalised Clifford conditions.

We next proceed to define,

$$\begin{aligned} H_1^{n-1} &= \mathcal{L}_3(1) \\ H_2^{n-1} &= \mathcal{L}_4(1) \\ H_3^{n-1} &= \zeta\, (H_1^{n-1})^{m-1} (H_2^{n-1}). \end{aligned} \qquad (5)$$

This set $\left\{ H_1^{n-1}, H_2^{n-1}, H_3^{n-1} \right\}$ again satisfies the generalised Clifford conditions.

We now factor out (H_3^{n-1}) from the set of $2n-1$ matrices $\mathcal{L}_i(1)$ $(i = 3,\ldots, 2n+1)$ and write

$$\mathcal{L}_i(1) = (H_3^{n-1})^{m-1} \mathcal{L}_i(2) \qquad (i = 5,\ldots,2n+1) \qquad (6)$$

Then $\mathcal{L}_5(2)$, $\mathcal{L}_6(2)$, $\mathcal{L}_7(2)$,...., $\mathcal{L}_{2n+1}(2)$ form a set of $2n-3$ matrices which again satisfy the generalised Clifford conditions. This procedure can be iterated till we arrive at

$$
\begin{aligned}
H_1^1 &= \mathcal{L}_{2n-1}\,(n-1) \\
H_2^1 &= \mathcal{L}_{2n}\,(n-1) \\
H_3^1 &= \zeta\,(H_1^1)^{m-1}\,H_2^1\,.
\end{aligned}
\tag{7}
$$

Any member of the set $\{H^i\}$ commutes with any member of the set $\{H^j\}$, $i \neq j$. If we now define the helicity matrix as the linear combination of the members of the set $\{H^i\}$ and choose in particular

$$
\begin{aligned}
H^n &= \lambda_{2n+1}\,H_3^n + \lambda_{2n}\,H_2^n + \Lambda_{n-1}\,H_1^n\,, \\
&\;\;\vdots \\
H^i &= \lambda_{2i+1}\,H_3^i + \lambda_{2i}\,H_2^i + \Lambda_{i-1}\,H_1^i\,, \\
&\;\;\vdots \\
H^1 &= \lambda_3\,H_3^1 + \lambda_2\,H_2^1 + \lambda_1\,H_1^1\,,
\end{aligned}
\tag{8}
$$

where we have used the notation of reference (3) with

$$
\Lambda_n = (\lambda_1^m + \lambda_2^m + \ldots + \lambda_{2n+1}^m)^{1/m}
\tag{9}
$$

and the m roots of $\left(\lambda_1^m + \lambda_2^m + \ldots + \lambda_{2n+1}^m\right)$ are, Λ_n, $\Lambda_n \omega, \ldots, \Lambda_n \omega^{m-1}$. It follows that

$$\begin{aligned}
(H_n)^m &= \Lambda_n^m I = (\Lambda_{n-1}^m + \lambda_{2n}^m + \lambda_{2n+1}^m)\, I, \\
&\vdots \\
(H^i)^m &= \Lambda_i^m I = (\Lambda_{i-1}^m + \lambda_{2i}^m + \lambda_{2i+1}^m)\, I, \\
&\vdots \\
(H^1)^m &= \Lambda_1^m I = (\lambda_1^m + \lambda_2^m + \lambda_3^m)\, I.
\end{aligned} \tag{10}$$

Since $\left\{H^i\right\}$ commutes with $\left\{H^j\right\}$ for $i \neq j$, we find that it is possible to obtain the eigenvectors of

$$L_{2n+1} = \sum_{i=1}^{2n+1} \lambda_i \mathcal{L}_i\,, \tag{11}$$

which are the simultaneous eigenvectors of the "helicity" matrices $H^1, H^2, \ldots, H^n$.

It is a remarkable fact that the 2m-th root of unity enters in the definition of H_3^i for m even, although we are dealing with the m-th root of the unit matrix. For the spacial case of $m = 2$ and $n = 1$, we have

$$\omega = -1,\ \omega^{1/2} = i. \tag{12}$$

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APPENDIX

We prove the following Theorem:

If A and B are two matrices satisfying the generalised Clifford conditions:

$$AB = \omega BA \tag{A 1}$$

$$\text{and} \quad A^n = B^n = I$$

where ω is the primitive n-th root of unity then

$$(\lambda A + \mu B)^n = \lambda^n + \mu^n.$$

PROOF: Condition (A 2) implies that A and B are non-singular and so the matrices A^{-1} and B^{-1} exist. Now

$$(\lambda A + \mu B)^n = \mu^n B^n \Big\{ 1 + C_1 (B^{-1} A) + C_2 (B^{-1} A)^2 + \dots C_{n-1} (B^{-1} A)^{n-1} + C_n (B^{-1} A)^n \Big\}.$$

where

$$C_1 = \frac{\lambda}{\mu} \sum_0^{n-1} \omega^i$$

$$C_2 = \frac{\lambda^2}{\mu^2} \sum_{0}^{n-1} \omega^{i_1} \omega^{i_2} \; (i_1 \neq i_2)$$

$$\vdots$$

$$C_{n-1} = \frac{\lambda^{n-1}}{\mu^{n-1}} \sum_{0}^{n-1} \omega^{i_1} \ldots \omega^{i_{n-2}} \; (i_1 \neq i_2 \ldots \neq i_{n-2})$$

and $$C_n = \prod_{i_1=0}^{n-1} \omega^{i_1} .$$

Since ω is the n-th root of unity $C_1, \ldots C_{n-1}$ vanish and $C_n = 1$.

Hence we have

$$(\lambda A + \mu B)^n = \lambda^n A^n + \mu^n B^n$$

$$= \lambda^n + \mu^n$$

using condition (A.2).

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L-MATRICES AND THE FUNDAMENTAL THEOREM OF SPINOR THEORY*

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During the past two years one of the authors (A.R.) has developed a theory of L-matrices which amounts to a generalization of the Pauli and Dirac matrices to higher dimensions[1]. It was shown that there were two methods of generating higher dimensional matrices with the requisite properties of a Clifford Algebra starting with a primitive set of Pauli matrices. The first method is traced to the famous derivation of the γ matrices by Dirac[2] in 1928. The second method due to one of the authors (A.R.) is known as the 'σ-operation'. The two methods have been shown to be equivalent by demonstrating that in the first method we compose elements of Clifford Algebra to arrive at suitable linear combinations while in the second method we obtain a suitable matrix and then decompose it into matrices which are elements of a Clifford Algebra.

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Examining the literature we find that a third and distinct method has been described by Rasevskii[3] using the fundamental theorem in spinor analysis. To understand the method of Rasevskii we shall first summarise the two methods of generalising Pauli matrices.

Method 1: We can obtain sets of three anticommuting matrices of higher dimensions by defining their left and right direct products of Pauli matrices with a unit matrix of arbitrary dimension:

$$I \otimes \sigma_x = S_x ; \quad I \otimes \sigma_y = S_y ; \quad I \otimes \sigma_z = S_z$$

$$\sigma_x \otimes I = \rho_x ; \quad \sigma_y \otimes I = \rho_y ; \quad \sigma_z \otimes I = \rho_z \tag{1}$$

Taking I to be a 2 x 2 unit matrix, we note that any member of the set S_x, S_y, S_z commutes with any member of the set ρ_x, ρ_y, ρ_z. Therefore the set of five matrices

$$\alpha_x = \rho_x S_x ; \quad \alpha_y = \rho_x S_y ; \quad \alpha_z = \rho_x S_z ; \quad \rho_y ; \quad \rho_z \tag{2}$$

anticommute with one another.

Dirac chose only four of them since he was concerned only with a linear combination of matrices with only four coefficients: p_x, p_y, p_z and m where p_x, p_y and p_z are the components of a three-momentum vector and m the mass. The same procedure has been used by one of the authors (A.R.) to generate (2n+1) mutually anticommuting matrices of dimension $2^n \times 2^n$. If we denote these anticommuting matrices by

$$\mathcal{L}_1^{2n+1};\ \mathcal{L}_2^{2n+1};\ \ldots;\ \mathcal{L}_{2n+1}^{2n+1}$$

they possess the product property:

$$\mathcal{L}_1^{2n+1}\,\mathcal{L}_2^{2n+1}\ \ldots\ \mathcal{L}_{2n+1}^{2n+1} = i^n I \tag{3}$$

and if we define L_{2n+1} as

$$L_{2n+1} = \sum_{i=1}^{2n+1} \lambda_i \mathcal{L}_i^{2n+1} \tag{4}$$

where the $\lambda's$ are scalar numbers, from the anticommuting property of the above (2n+1) matrices, we have

$$L_{2n+1}^2 = (\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_{2n+1}^2) I \tag{5}$$

<u>Method II</u>: In this method we first obtain the matrix L_{2n+1} and then express it as a linear combination of (2n+1) mutually anticommuting matrices. We start with the primitive matrix

$$L_3 = \begin{pmatrix} \lambda_3 & \lambda_1 - i\lambda_2 \\ \lambda_1 + i\lambda_2 & -\lambda_3 \end{pmatrix} \tag{6}$$

L_{2n+1} can be obtained from L_3 by a 'σ-operation' which is defined as follows:

Replace any one of the parameters i.e. λ's in (5) by L_{2n-1} and relabel the other two as λ_{2n} and λ_{2n+1} and attach a unit matrix of dimension $2^{n-1} \times 2^{n-1}$. For example,

$$L_{2n+1} = \begin{pmatrix} \lambda_{2n+1} I & L_{2n-1} - i\lambda_{2n} I \\ L_{2n-1} + i\lambda_{2n} I & -\lambda_{2n+1} I \end{pmatrix} \tag{7}$$

Since we have assumed the form of L_3, the above recurrence relation defines L_{2n+1}. From the form of L_{2n+1} it is clear that it can be expressed as the linear combination of (2n+1) mutually anticommuting $2^n x 2^n$ matrices.

<u>Method III</u>: This method described in great detail by Rasevskii makes use of the fundamental theorem of spinor analysis. To facilitate the understanding of this method in comparison with the previously discussed Dirac procedure and the σ-operation we shall describe it in a language as close as possible to the theory of L-matrices.

The Clifford algebra C_m consists of the p-fold products (p = 0,1,....,m) of mutually anticommuting elements e_i $(i = 1, 2, \ldots m)$:

$$1,\ e_i,\ \underset{(i<j)}{e_i e_j},\ \underset{(i<j<k)}{e_i e_j e_k},\ \ldots,\ e_1 e_2 \ldots e_m \tag{8}$$

If m = 2n, in the language of L-matrices, the basic elements correspond to

$$\mathcal{L}_i^{2n+1} \qquad (i = 1, 2, \ldots, 2n)$$

the (2n+1)th element being obtained from the product relation

$$\mathcal{L}_1^{2n+1}\,\mathcal{L}_2^{2n+1}\,\cdots\,\mathcal{L}_{2n+1}^{2n+1} = i^n I$$

An aggregate A of C_m is written as a linear combination of these 2^n elements:

$$A = a_0 + a_i e_i + a_{ij} \underset{(i<j)}{e_i e_j} + \cdots + a_{12\ldots m} e_1 e_2 \cdots e_m \tag{9}$$

$$= A_0 + A_1 \tag{10}$$

where A_0 consists of the even products only and A_1 those of odd products only.

Denoting the representation matrices of the 2n basic elements of C_{2m} as $\hat{E}$'s, the first m basic elements are obtained as being isomorphic to the mapping.

$$A \xrightarrow{\hat{E}_i} A e_i \qquad (i = 1, \ldots, m) \tag{11}$$

so that $$\hat{E}_i \hat{E}_j = -\hat{E}_j \hat{E}_i \;;\; i \neq j \;;\; i, j = (1, \ldots, m) \tag{12}$$

and $$\hat{E}_i^2 = I \tag{13}$$

The next m elements $\hat{E}_{m+i}$ (i = 1,...,m) are obtained as being isomorphic to the mapping

$$A \xrightarrow{\hat{E}_{m+i}} i e_i (A_0 - A_1) \tag{14}$$

The factor $i = \sqrt{-1}$ is introduced so that

$$(\hat{E}_{m+i})^2 = I \qquad ; i = 1, 2, \ldots, m \tag{15}$$

It is easily checked that the 2m basic elements so obtained are mutually anticommuting. All the elements so obtained by taking p-fold products (p = 0,1,...,2m) of $\hat{E}_i$ (i =1,...,2m) give representation of the Clifford Algebra .

To fix our ideas, we demonstrate the above method for the case 2m = 4 i.e. we obtain the representation of the Clifford Algebra C_4, considering C_2 as the vector space and transformations of C_2 such that it maps onto itself.

The Clifford Algebra C_2 generated by two basic elements e_1, e_2 has four elements

$$1, e_1, e_2, e_{12} \tag{16}$$

An aggregate of C_2 is

$$A = a_0 \cdot 1 + a_1 e_1 + a_2 e_2 + a_{12} e_{12} \tag{17}$$

where $e_{12} = e_1 e_2$. The algebra C_2 maps onto itself when we multiply by $1, e_1, e_2$ or e_{12}. But we consider the mappings obtained by e_1, e_2 only.

Let $\hat{E}_1$ be defined by

$$A \xrightarrow{\hat{E}_1} A e_1 \tag{18}$$

The $\hat{E}_1$ mapping takes (a_0, a_1, a_2, a_{12}) to $(a_1, a_0, -a_2, -a_{12})$ and in a matrix form of

$$\hat{E}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{19}$$

$\hat{E}_2$ is defined by the mapping

$$A \xrightarrow{\hat{E}_2} A e_2 \tag{20}$$

such that $(a_0, a_1, a_2, a_{12}) \xrightarrow{\hat{E}_2} (a_2, a_{12}, a_0, a_1)$

and $\hat{E}_2$ has thus the form

$$\hat{E}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{21}$$

$\hat{E}_3$ is obtained through the mapping

$$(a_0 1 + a_1 e_1 + a_2 e_2 + a_{12} e_{12}) \xrightarrow{\hat{E}_3} e_1 (ia_0 + ia_{12} - ia_1 - ia_2) \qquad (22)$$

$$\text{i.e. } (a_0, a_1, a_2, a_{12}) \xrightarrow{\hat{E}_3} (-ia_1, ia_0, ia_{12}, -ia_2)$$

Therefore

$$\hat{E}_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \qquad (23)$$

Similarly $\hat{E}_4$ is given by the mapping

$$(a_0 1 + a_1 e_1 + a_2 e_2 + a_{12} e_{12}) \xrightarrow{\hat{E}_4} e_2 (ia_0 + ia_{12} - ia_1 e_1 - ia_2 e_2) \qquad (24)$$

$$\text{i.e. } (a_0, a_1, a_2, a_{12}) \xrightarrow{\hat{E}_4} (-ia_2, -ia_{12}, ia_0, ia_1)$$

Therefore,

$$\hat{E}_4 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \qquad (25)$$

The procedure described above yields a representation of C_{2n} from C_n i.e. starting from C_2 we get $C_4, C_8 \ldots$ etc. This is equivalent to generating L_{2n+1} from L_{n+1} i.e. starting from L_3 we get L_5, L_9, L_{17}, L_{13}, ... etc. But to obtain L_7, L_{11}, L_{13} etc., we have to consider $C_6, C_{10}, C_{14}, \ldots$, etc. i.e. we should start with C_3, C_5, C_7 etc.

Even for these algebras Raseviskii has shown that the above procedure is applicable by writing formally

$$C_3 = C_2 \oplus C_2 \qquad \text{([illegible])}$$

and in general

$$C_{2m+1} = C_{2m} \oplus C_{2m} \tag{26}$$

The algebra C_{2m+1} has twice as many elements as C_{2m}.

However, it looks as if by taking n elements out of (2n-1) anticommuting basic-elements of L_{2n-1} and generating 2^n elements by the p-fold products (p=1,...n) of the n elements we can obtain the representation of the basic elements of L_{2n+1}. Hence the entire hierarchy $L_3, L_5, L_7 \ldots$ etc. can be easily generated, using Rasevskii's method. The validity of the above suggestion has to be examined.

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THE INSTITUTE OF MATHEMATICAL SCIENCES
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ON THE REPRESENTATIONS OF GENERALIZED CLIFFORD ALGEBRAS

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It has been shown by one of the authors (A.R.)[1] that there are (2n+1) anticommuting matrices of dimension $2^n \times 2^n$ satisfying the two Clifford conditions

$$\text{I:} \quad \alpha_i \alpha_j = -\alpha_j \alpha_i$$

$$\text{II:} \quad \alpha_i^2 = I \qquad (i = 1, \ldots, 2n+1) \qquad (1)$$

If we form p-fold products (p = 0,1,....2n) of the α_i's, we obtain an aggregate of 2^{2n} matrices constituting the elements of the Clifford algebra C_{2n}. Out of these 2^{2n} elements, only the (2n+1) base elements α_i satisfy both the Clifford conditions I and II while the other elements obey only the second Clifford condition of (1). In conformity with the mathematical literature we denote the 2^{2n} elements of C_{2n} by

$$1; \; e_i; \; \underset{(i<j)}{e_i e_j}; \; \underset{(i<j<k)}{e_i e_j e_k}; \; \ldots ; \; e_1 e_2 \ldots e_{2n} \qquad (2)$$

It has been pointed out[2] that there are three methods of generating the (2n+1) base elements which can be represented as matrices of dimension $2^n \times 2^n$ of C_{2n}; the first being traced to the primary derivation of the α -matrices by Dirac, the second one known as the ' σ ' operation, formulated in detail in reference 1) and the third one is due to Rasevskii[3]. While the first two methods which have been shown to be equivalent[2], generate the 2n independent base matrices of C_{2n} of dimension $2^n \times 2^n$ from the (2n-2) independent base matrices of C_{2n-2} of dimension $2^{n-1} \times 2^{n-1}$, in the third method of Rasevskii[3] the 2n independent base elements of C_{2n} are generated as a mapping on vectors constructed out of the complete set of 2^n elements of C_n.

Recently A.O.Morris[4] has obtained explicit representations of the Generalized Clifford Algebra (G.C.A.)

$$e_i e_j = \omega e_j e_i \quad ; i < j ; \; i = j = 1, \ldots, 2n$$

$$e_i^m = I$$

and (3)

where ω is the m th primitive root of unity. It is to be noted that although the dimension of the base matrices is $m^n \times m^n$, yet there are only (2n+1) base matrices as in the case of ordinary Clifford algebra, the last one obtained as the product of the first 2n base matrices. The complete set of m^{2n}

elements of C_n^m is constructed as the products

$$e_1^{k_1} e_2^{k_2} \cdots e_{2n}^{k_{2n}}$$

$$0 \leq k_1, k_2, \ldots, k_{2n} \leq m-1 \tag{4}$$

The usual Clifford algebra, Eq.(1) is obtained as a special case when m = 2.

The most striking feature of the **G.C.A.** is that it has properties closely analogous to the usual Clifford algebra with suitable modifications to incorporate that ω is the m^{th} root of unity. The representation of the G.C.A. has been earlier obtained using the first two methods[5].

We now use the method of Rasevskii to get the representation of the G.C.A. C_{2n}^m. We can write an arbitrary element A of C_{2n}^m as

$$A = a_0 1 + a_{1 i_1, 2 i_2 \ldots (2n) i_{2n}} \, e_1^{i_1} e_2^{i_2} \cdots e_{2n}^{i_{2n}}$$

$$0 \leq i_1, i_2, \ldots, i_{2n} \leq m-1 \tag{5}$$

where we have used the summation convention of repeated indices.

We now divide the m^{2n} elements into m sets as

$$A = A_0 + A_1 + \cdots + A_{m-1} \tag{6}$$

where A_i contains terms of degree $i \bmod m$, each having m^{2n-1} elements.

We now construct after Raseveskii, the representation of the 2n base elements of G.C.A. C^{m}_{2n} obeying

$$e_i e_j = \omega e_j e_i \; ; \; i < j \; ; \; i, j = 1, \cdots, 2n$$
$$e_i^m = I$$

The first n elements of C^{m}_{2n} are obtained as the manning

$$A \xrightarrow{\hat{E}_i} A e_i \tag{7}$$

The other n elements are obtained as

$$A \xrightarrow{\hat{E}_{n+i}} \zeta\, e_i \left[\omega^{m-1} A_0 + \omega^{m-2} A_1 + \cdots + 1 \cdot A_{m-1} \right] \tag{8}$$

where

$$\zeta = 1 \quad \text{for } m \text{ odd}$$
$$= \omega^{1/2} \quad \text{for } m \text{ even}.$$

The affinors $\hat{E}_1, \cdots, \hat{E}_{2n}$ can be shown to furnish a representation of the generalized Clifford Algebra C^{m}_{2n}.

Case 1: For the first n elements, $E_1, E_2, \cdots, E_n$ the proof is obvious since

$$A \xrightarrow{\hat{E}_i \hat{E}_j} A e_i e_j \tag{9}$$

But $$A \xrightarrow{\hat{E}_j \hat{E}_i} A\, e_j e_i = \omega^{m-1} A\, e_i e_j$$

$$\therefore\ \hat{E}_i \hat{E}_j = \omega\, \hat{E}_j \hat{E}_i \tag{10}$$

and $$A \xrightarrow{\hat{E}_i \ldots \hat{E}_i \ (m \text{ times})} A \underbrace{e_i e_i \ldots e_i}_{(m \text{ times})} = A \tag{11}$$

Therefore by eq. (9), (10), (11) it follows that $\hat{E}_i$ $(i = 1, \ldots, n)$ obey the algebra C^m_{2n}.

Case 2.

To prove that $\hat{E}_{n+i}$ also obey the algebra C^m_{2n} we proceed as follows:

$$A \xrightarrow{\hat{E}_{n+i}} \zeta e_i \left[\omega^{m-1} A_0 + \omega^{m-2} A_1 + \ldots + 1 \cdot A_{m-1} \right] \tag{12}$$

Notice that the degree of the terms has been increased by one. Hence

$$A \xrightarrow{\hat{E}_{n+j} \hat{E}_{m+i}} \zeta e_j e_i \left[\omega^{2m-3} A_0 + \omega^{2m-5} A_1 + \cdots + \omega^{m-1} A_{m-1} \right] \tag{13}$$

lly $$A \xrightarrow{\hat{E}_{n+i} \hat{E}_{n+j}} \zeta e_i e_j \left[\omega^{2m-3} A_0 + \ldots + \omega^{m-1} A_{m-1} \right] \tag{14}$$

Thus, $$\hat{E}_{n+i} \hat{E}_{n+j} = \omega\, \hat{E}_{n+j} \hat{E}_{n+i}$$

It is then not hard to prove that

$$(\hat{E}_{n+i})^m = I$$

<u>Case 3</u>: We shall prove now, that $\hat{E}_i$ and $\hat{E}_{n+j}$ obey the algebra

$$A \xrightarrow{\hat{E}_i \hat{E}_{n+j}} \zeta_{\ell j} \left[\omega^{m-1} A_0 + \omega^{m-2} A_1 + \cdots + A_{m-1} \right] e_i \tag{15}$$

and

$$A \xrightarrow{\hat{E}_{n+j} \hat{E}_i} \zeta_{\ell j} \left[\omega^{m-2} A_0 + \omega^{m-3} A_1 + \cdots + \omega^{m-1} A_{m-1} \right] e_i \tag{16}$$

From (15) and (16) it is clear that

$$\hat{E}_i \hat{E}_{n+j} = \omega \hat{E}_{n+j} \hat{E}_i \tag{17}$$

Thus Eq. (7) and (8) yield a representation of the 2n basic elements of C_{2n}^m.

We shall demonstrate the above procedure for the case $m = 3; 2n = 4$. i.e. to obtain the representation of from

We start with two basic elements e_1, e_2 of C_2^3 obeying the G.C.A. The complete set of 9 elements of C_2^3 is given by

$$1, e_1, e_2, e_1^2, e_1 e_2, e_2^2, e_1^2 e_2, e_1 e_2^2, e_1^2 e_2^2$$

An arbitrary element of C_2^3 is therefore written as

$$A = a_0 1 + a_1 e_1 + a_2 e_2 + a_{11} e_1^2 + a_{12} e_1 e_2 + a_{22} e_2^2 + a_{1^2 2}\, e_1^2 e_2 + a_{12^2}\, e_1 e_2^2 + a_{1^2 2^2}\, e_1^2 e_2^2 \qquad (18)$$

Now $$A = A_0 + A_1 + A_2$$

where
$$A_0 = a_0 + a_{1^2 2}\, e_1^2 e_2 + a_{12^2}\, e_1 e_2^2$$
$$A_1 = a_1 e_1 + a_2 e_2 + a_{1^2 2^2}\, e_1^2 e_2^2$$
$$A_2 = a_{11} e_1^2 + a_{12} e_1 e_2 + a_{22} e_2^2$$

The mapping $A \xrightarrow{\hat{E}_1} A e_1$ yields the matrix: (19)

$$\hat{E}_1 = \begin{bmatrix} 0&0&0 & 1&0&0 & 0&0&0 \\ 1&0&0 & 0&0&0 & 0&0&0 \\ 0&0&0 & 0&0&0 & \omega^2&0&0 \\ 0&1&0 & & & & 0&0&0 \\ 0&0&\omega^2 & &0& & 0&0&0 \\ 0&0&0 & & & & 0&0&\omega \\ & & & 0&\omega^2&0 & 0&0&0 \\ &0& & 0&0&\omega & 0&0&0 \\ & & & 0&0&0 & 0&\omega&0 \end{bmatrix} \qquad (20)$$

The mapping $A \xrightarrow{\hat{E}_2} Ae_2$ yields the matrix:

$$\hat{E}_2 = \begin{bmatrix} 000 & 001 & 000 \\ 000 & 000 & 010 \\ 100 & 000 & 000 \\ 000 & & 001 \\ 010 & 0 & 000 \\ 001 & & 000 \\ & 100 & 000 \\ 0 & 010 & 000 \\ & 000 & 100 \end{bmatrix} \tag{21}$$

The mapping $A \xrightarrow{\hat{E}_3} \zeta e_1 \left[\omega^2 A_0 + \omega A_1 + 1 \cdot A_2 \right]$ gives the matrix (Here ζ = 1 since m = 3)

$$\hat{E}_3 = \begin{bmatrix} 000 & 100 & 000 \\ \omega^2 00 & 000 & 000 \\ 000 & 000 & \omega^2 00 \\ 0\omega 0 & & 000 \\ 00\omega & 0 & 000 \\ 000 & & 00\omega \\ & 010 & 000 \\ 0 & 001 & 000 \\ & 000 & 0\omega^2 0 \end{bmatrix} \tag{22}$$

The mapping $A \xrightarrow{\hat{E}_4} \zeta e_2 \left[\omega^2 A_0 + \omega A_1 + 1 \cdot A_0 \right]$ gives the last matrix

$$\hat{E}_4 = \begin{bmatrix} 000 & 001 & 000 \\ 000 & 000 & 0\omega 0 \\ 00\omega^2 & 000 & 000 \\ 000 & & 00\omega^2 \\ 010 & 0 & 000 \\ 00\omega & & 000 \\ & \omega 00 & 000 \\ 0 & 0\omega^2 0 & 000 \\ & 000 & 100 \end{bmatrix} \tag{23}$$

It is important to note that in contrast to the other two methods, the methods due to Rasevskii does not require the explicit form of the matrices of C_n^m to construct the representation of C_{2n}^m.

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THE GENERALISED CLIFFORD ALGEBRA AND THE UNITARY GROUP

by

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THE GENERALISED CLIFFORD ALGEBRA AND THE UNITARY GROUP

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During the past two years following the first formulation of L-matrix theory[1] the Matscience group has been concerned with the generalised Clifford algebra of matrices which are the m-th roots of unity. The generalised algebra was discovered by Yamazaki[2] in 1964 and the matrix representations in the lowest dimension were first given by Morris in 1967[3]. We shall now present some new results on the subject and point out a surprising and unexpected connection with the generators of the special unitary group.

It has been established that there are (2n+1) matrices $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{2n+1}$ of dimension $m^n \times m^n$ obeying the two generalised Clifford conditions:

$$\text{I} \qquad \mathcal{L}_i \mathcal{L}_j = \omega \mathcal{L}_j \mathcal{L}_i ; \; i<j ; \; i,j = 1,2,\ldots,2n+1 \qquad (1)$$

$$\text{II} \qquad \mathcal{L}_i^m = I$$

where ω is the primitive m-th root of unity. The (2n+1) matrices obey the product rule:

$$\alpha_1^{m-1}\alpha_{n+1}\alpha_2^{m-1}\alpha_{n+2}\cdots\alpha_{2n}\alpha_{2n+1}^{m-1} = \omega^{m-1} I \tag{2}$$

It follows therefore that in the lowest dimension (n = 1) there are only three matrices satisfying the two Clifford conditions. The case m = 2 corresponds to the usual Clifford algebra.

The representations of the three matrices P, Q and R of dimension m x m satisfying the two Clifford conditions have been given by Morris[3] as

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & & 1 \\ 1 & 0 & 0 & \cdots & & 0 \end{bmatrix} ; \quad Q = \begin{bmatrix} 0 & \omega & 0 & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & 0 & \cdots & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & & \omega^{m-1} \\ 1 & 0 & 0 & \cdots & & 0 \end{bmatrix} \tag{3}$$

The third matrix R is connected to P, Q by the relation

$$R = \zeta P^{m-1} Q \tag{4}$$

where

$$\gamma = 1 \text{ for } m \text{ odd}$$
$$= \omega^{1/2} \text{ for } m \text{ even}$$

The method of obtaining 2n+1 matrices of dimension $m^n \times m^n$ from these two matrices has been described in a previous paper[4]. This is through the generalisation of σ-operation defined in the first formulation of the L-matrix theory[1].

Through the work of Rasevskii[5] on the ordinary Clifford algebra, we realised that even for the generalised case if we form all possible linearly independent products from the basic 2n elements satisfying the two Clifford conditions[6],

$$\gamma_1^{k_1} \gamma_2^{k_2} \ldots \gamma_{2n}^{k_{2n}}$$

with

$$0 \leq k_1, k_2, \ldots, k_{2n} \leq m-1 \tag{5}$$

we obtain m^{2n} matrices constituting the algebra . While all these elements satisfy the Clifford condition II only the 2n base elements satisfy both the Clifford conditions. We obtain the following rule:

The total number of matrix roots of unity of dimension $m^n \times m^n$ is equal to the order of the root raised to the power of the number of independent base elements: i.e. m^{2n}.

It is important to note that the algebra C^m_{2n} has a symmetry principle inherent in it; it is unaffected if we multiply the matrices by an m-th root of unity.

We now consider the case m = 3; n = 1, and now   is a primitive cube root of unity. If we form all possible products of P and Q we obtain eight matrices

$$P, Q, P^2Q^2; \quad P^2, Q^2, PQ; \quad P^2Q, PQ^2$$

or equivalently

$$P, P^2; Q, Q^2; R, R^2; PQ, QR \tag{6}$$

With the unit matrix they form the elements of the generalised Clifford algebra C^3_2.

We now list below explicitly these matrices

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad ; \quad P^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix} \quad ; \quad Q^2 = \begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad ; \quad R^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

$$PQ = \begin{pmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix} \quad ; \quad QR = \omega P^2 Q^2 = \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix} \tag{7}$$

The set of eight matrices consists of four sets of two commuting matrices in which one set, P^2Q and PQ^2 is diagonal. Each set of commuting matrices consists of a matrix and its square.

We now observe the remarkable fact that the two commuting generators of SU_3, λ_3 and λ_8 in the Gell-Mann notation[7], can be expressed as a linear combination of just the diagonal matrices of the above set.

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{\omega P^2 Q - PQ^2}{\omega(1-\omega)}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = -\frac{1}{\sqrt{3}} \left(PQ^2 + \omega P^2 Q \right) \tag{8}$$

For the sake of completeness we express the other six matrices in terms of the base matrices P and Q.

$$\lambda_1 = \frac{1}{3}\left\{ (P^2 + \omega Q^2 + PQ) + (P + Q + \omega P^2 Q^2) \right\}$$

$$\lambda_2 = \frac{i}{3}\left\{ (P^2 + \omega Q^2 + PQ) - (P + Q + \omega^2 P^2 Q^2) \right\}$$

$$\lambda_4 = \frac{1}{3}\left\{ (P^2 + Q^2 + \omega PQ) + (P + Q + \omega P^2 Q^2) \right\}$$

$$\lambda_5 = \frac{i}{3}\left\{ (P + Q + \omega P^2 Q^2) - (P^2 + Q^2 + \omega PQ) \right\}$$

$$\lambda_6 = \frac{1}{3}\left\{ (P + \omega Q + P^2 Q^2) + (P^2 + \omega^2 Q^2 + \omega^2 PQ) \right\}$$

$$\lambda_7 = \frac{i}{3}\left\{ (P^2 + \omega^2 Q^2 + \omega^2 PQ) - (P + \omega Q + P^2 Q^2) \right\} \tag{9}$$

In the last few years, attempts have been made[8] to classify the various elementary particles by assuming that they are composed of three fundamental objects called quarks (i.e. a triplet) which are simultaneous eigenstates of λ_3 and λ_8 i.e. the operators corresponding to the third component of isotopic spin I_Z and hypercharge Y. Since I_Z and Y can be expressed in terms of P^2Q and PQ^2 only, we wish to suggest that the states of the triplets can be considered as the simultaneous eigenstates of P^2Q and PQ^2.

Relabelling for convenience of notation, P^2Q and ωPQ^2 as M_1 and M_2 and their eigenvalues as μ_1 and μ_2 we give below a table of the quantum numbers of the quark triplet and μ_1 and μ_2.

	I_Z	Y	μ_1	μ_2
A	$1/2$	$1/3$	1	1
B	$-1/2$	$1/3$	ω	ω^2
C	0	$-2/3$	ω^2	ω

Further we have

$$Q(\text{Charge}) = \frac{1}{3}(\mu_1 + \mu_2) \tag{10}$$

$$I_z = \frac{1}{2}\left(\frac{\mu_1 - \omega\mu_2}{1-\omega}\right) \tag{11}$$

$$Y = -\frac{1}{3}\left(\omega\mu_1 + \omega^2\mu_2\right) \tag{12}$$

Since μ_2 is the square of μ_1 'conservation' of μ_1 does not imply 'conservation of μ_2 and so we have to postulate separate conservation laws for μ_1 and μ_2 in just the same way as we postulate separate conservation laws for Q and Y or equivalently Q and I_z or I_z and Y.

The above considerations can be carried over to obtain the generators of SU(m) from the two base elements P and Q, the (m-1) commuting matrices in this case being

$$P^{k_1} Q^{k_2}$$

with

$$k_1 + k_2 = m$$

In such a case there are (m+1) sets of (m-1) commuting matrices, the unit matrix commuting with all of them. Members in a given set can be expressed as powers of one of its elements say B i.e. any set can be written as $B, B^2, \ldots\ldots, B^{m-1}$. If we add the unit element we get a cyclic group of order m.

We point out that if $\mu_1, \mu_2, \ldots, \mu_n$ are 'quantum numbers' where μ_1 can take any one of the n values of the n-th root of unity and

$$\mu_2 = \mu_1^{\,2} \; ; \; \mu_3 = \mu_1^{\,3} \; ; \; \ldots \; ; \; \mu_{n-1} = \mu_1^{\,n-1} \qquad (12)$$

then we can ascribe to any multiplet of SU(n), n 'scalar' quantum numbers, $s_1, s_2, \ldots, s_n$ which can be expressed in terms of μ_i as follows:

$$s_1 = \frac{1}{n}\left\{ \mu_1 + \mu_2 + \cdots + \mu_{n-1} \right\} = \mu_1 + \mu_1^2 + \cdots + \mu_1^{n-1}$$

$$s_2 = \frac{1}{n}\left\{ (\omega\mu_1) + (\omega\mu_1)^2 + \cdots + (\omega\mu_1)^{n-1} \right\}$$

$$\vdots$$

$$s_r = \frac{1}{n}\left\{ (\omega^{r-1}\mu_1) + (\omega^{r-1}\mu_1)^2 + \cdots + (\mu_1\omega^{r-1})^{n-1} \right\}$$

$$\vdots$$

$$s_n = \frac{1}{n}\left\{ (\omega^{n-1}\mu_1) + (\omega^{n-1}\mu_1)^2 + \cdots + (\omega^{n-1}\mu_1)^{n-1} \right\}$$

(13)

These scalar quantum numbers obey the relation:

$$s_1 + s_2 + s_3 + \cdots + s_n = 0 \qquad (14)$$

We can also define vector uantum numbers

$$v_j = (s_j - s_{j+1})/2 \tag{15}$$

and hence we have

$$v_1 + v_2 + \dots + v_n = 0 \tag{16}$$

assuming for $j = n$ ~~+~~ 1, S_{j+1} is S_1 itself.

It is enough to establish the above relations for the basic multiplets of SU(n). Since the s_i and v_i are linear eq.(15) is propagated for the higher representations also i.e. if

$$\begin{aligned} S_i &= \sum_k s_i^{(k)} \\ M_i &= \sum_k \mu_i^{(k)} \\ V_i &= \sum_k v_i^{(k)} \end{aligned} \tag{17}$$

where the superscript denotes the 'quark' components of the 'composite particles' corresponding to the higher representation. The relation between S_i and M_i can be expressed through (13) by just replacing s_i by S_i and μ_i by M_i. The [illegible] generalised Gell-Mann-Nishijima relation can be written similar to (15).

$$V_j = (S_j - S_{j+1})/2 \tag{18}$$

In the particular case of SU_3 we can set

$$\begin{aligned} S_3 &= Q \\ S_2 &= -Y \\ S_1 &= -Q + Y \end{aligned}$$

where Q and Y are charge and hypercharge respectively and

$V_3 = I_Z$, Z component of Isospin

$V_2 = V_Z$, " spin

$V_1 = U_Z$, " spin

so that

$$I_Z = Q/2 - (Y-Q)/2$$

$$Q = I + Y/2$$

The table for the values of $s_1, s_2, \ldots, s_n$ for the basic multiplet SU(n) quark $A_1, A_2, \ldots, A_n$ is

	s_1	s_2	s_3	$\ldots$	s_n
A_1	$\frac{n-1}{n}$	$-\frac{1}{n}$	$-\frac{1}{n}$		$-\frac{1}{n}$
A_2	$-\frac{1}{n}$	$\frac{n-1}{n}$	$-\frac{1}{n}$	$\ldots$	$-\frac{1}{n}$
$\vdots$	$\vdots$	$\vdots$	$\frac{n-1}{n}$		$\vdots$
A_n	$-\frac{1}{n}$	$-\frac{1}{n}$	$-\frac{1}{n}$		$\frac{n-1}{n}$

Similarly the table for v_n's for the basic representation is given as:

	v_1	v_2	$\ldots$	v_n
A_1	$\frac{1}{2}$	0		$-\frac{1}{2}$
A_2	$-\frac{1}{2}$	$\frac{1}{2}$		0
A_3	0	$-\frac{1}{2}$	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$
A_n	0	0		$\frac{1}{2}$

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THE INSTITUTE OF MATHEMATICAL SCIENCES
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IDEMPOTENT MATRICES FROM A GENERALIZED CLIFFORD ALGEBRA

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In an earlier contribution[1)] we pointed out that linear combinations of matrices which satisfy the condition

$$\mathcal{L}^m = I \tag{1}$$

can be so defined as to obtain the elements of a Lie algebra and in particular SU(m) in their self-representation. The collection of $\mathcal{L}$-matrices satisfying eqn.(1) form the elements of a generalized Clifford algebra. The connection between the Lie and the Clifford algebras seemed at that time extremely surprising, but we shall now show that it is a natural consequence of the properties of the matrix $\mathcal{L}$. We can choose a linear combination of an $\mathcal{L}$ matrix and its powers to define a matrix A such that

$$\begin{aligned} A^m &= A \text{ if } m \text{ is odd} \\ &= A^2 \text{ if } m \text{ is even} \end{aligned} \tag{2}$$

Since the elements of the self representation of SU(m) can be represented by A-matrices, it becomes clear why there exists a connection between the elements of the Lie and generalized Clifford algebras.

Let us define the matrix A as

$$A = \frac{1}{m}\sum_{k=1}^{m-1} (1-\omega^k)\,\chi^k \tag{3}$$

where ω is the primitive m^{th} root of unity.

It can be verified that A has the strikingly interesting property

$$A^3 = A, \quad A^2 \neq I \tag{4}$$

for __any__ n. It follows immediately that

$$\begin{aligned} A^m &= A \text{ when m is odd} \\ &= A^2 \text{ when m is even} \end{aligned} \tag{5}$$

A^2 can be expressed elegantly as a linear combination of and its powers as

$$A^2 = \frac{1}{m}\sum_{k=0}^{m-1} (1+\omega^k)\,\chi^k \tag{6}$$

The linear combination (3) is suggested by the generalized Gell-Mann-Nishijima relation we have recently obtained[1] between the vector quantum numbers and the scalar quantum numbers of SU(m). The vector quantum number is just the eigenvalue of the matrix A while the scalar quantum numbers are linear combination of the roots of unity.

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THE INSTITUTE OF MATHEMATICAL SCIENCES
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KEMMER ALGEBRA FROM GENERALISED CLIFFORD ELEMENTS

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KEMMER ALGEBRA FROM GENERALISED CLIFFORD ELEMENTS

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1. Introduction

Our earlier studies[1,2,3] on the generalised Clifford algebra (G.C.A.) formulated by K. Yamazaki[4] led us to a surprising connection between the generalised Clifford algebra and the unitary groups which describe the internal symmetry of elementary particles. We shall now show that it is possible to obtain the matrices of the Duffin-Kemmer-Petiau[5] (D.K.P.) algebra which enter the space-time description of particles having spin zero or one through a wave equation, known in literature as the D.K.P. equation. Such a derivation of D.K.P. algebra from the generalised Clifford algebra leads us automatically to a method of constructing the elements of the algebra of orthogonal groups also.

2. The β-matrices of Duffin-Kemmer-Petiau

The generalised Clifford algebra C_n^m has n basic elements e_i's such that

$$e_i^m = 1 \; ; \quad e_i e_j = \omega e_j e_i \quad (i < j; \; i, j = 1, \ldots, n) \qquad (2.1)$$

where ω is the primitive m-th root of unity.

In the following we shall be making use of the C_2^{r+1} algebra for derivating the elements of K(r) whose generators β_i's (i=1,...,r) obey

$$\beta_l \beta_m \beta_n + \beta_n \beta_m \beta_l = \delta_{mn} \beta_l + \delta_{ml} \beta_n \quad (2.2)$$

From the above we are led to the relations

$$\beta_l^3 = \beta_l \quad ;$$

$$\beta_l \beta_m^2 + \beta_m^2 \beta_l = \beta_l \qquad (l \neq m) \text{ etc.}$$

(2.3)

It will be important for us to note that the algebra K(r) has always an irreducible representation of dimension r+1. To obtain the basic r generators of K(r), consider the elements of C_2^{r+1}. Let us now define entities, E_{ij}'s which are linear combinations, of the elements of C_2^{r+1} as,

$$E_{ij} = \frac{1}{r+1} \sum_{\nu=0}^{r} \omega^{\nu(i-1)} e_1^{\nu} e_2^{(j-i)}$$

$$(i, j = 1, \ldots, r+1)$$

(2.4)

These entities possess an important property as shown by A.O.Morris[7]

$$E_{ij} E_{kl} = E_{il} \delta_{jk}$$

(2.5)

Let us now pick out of the E_{ij}'s, r elements

$$E_{i,r+1} \qquad (i=1,\ldots,r)$$

and define β_i's such that

$$\beta_i = \left(E_{i,r+1} + E_{r+1,i} \right) \quad (i=1,\ldots,r) \tag{2.8}$$

We assert that the β_i's constitute the r basic elements of the algebra K(r). This would be true if we can show that β_i's obey the fundamental relations (2.2) of this algebra. Using (2.5) and (2.6) it is easily checked that

$$\beta_i \beta_k \beta_l + \beta_l \beta_k \beta_i = \delta_{kl} \beta_i + \delta_{ki} \beta_l$$

and hence $\beta_i^3 = \beta_i$ etc. The irreducible representation of is of dimension (r+1) and hence we have obtained one representation of the Kemmer algebra K(r) of (r+1) dimension, since β_i's constructed from the elements of C_2^{r+1} are of (r+1) dimension. In particular for r=4, we obtain the 5 dimensional representation of the Kemmer matrices relating to spin zero particles.

3. Lie algebra of orthogonal groups

It is well known that one can construct from the elements of the ordinary Clifford algebra C_2^n (of n basic elements) the Lie algebra of the rotation group in (n+1) dimensions (for example see H. Boerner[8]). In this paper we are interested in constructing the elements of the algebra of proper orthogonal group from the generalised Clifford elements $C^{\gamma+1}_{2}$. To achieve this, let us define J_{mn}'s as

$$J_{mn} = \left(\beta_m \beta_n - \beta_n \beta_m \right) \quad (m \neq n) \qquad (3.1)$$

where β_i's are the r basic elements of the K(r) as defined in (2.6). These give J_{mn}'s which are $\binom{\gamma}{2}$ in number. Also define r quantities J_{m0}, (m=1,2,..,r) given by

$$J_{m0} = \left(\beta_m \beta_0 - \beta_0 \beta_m \right) \qquad (3.2)$$

with

$$\beta_0 = E_{\gamma+1,\gamma+1} \qquad (3.3)$$

$E_{\gamma+1,\gamma+1}$ being defined as in (2.4).

Hence the total number of J_{mn}'s constructed as above will be

$$\binom{\gamma}{2} + \gamma = \gamma(\gamma+1)/2 \qquad (3.4)$$

It is easy check, by substituting E_{ij}'s for β_i's in (3.1) and (3.2) that J_{mn}'s reduce to $(E_{mn} - E_{nm})$. With little effort one can also derive the commutator relationship of the J_{mn}'s given by

$$[J_{mn}, J_{m'n'}] = [\delta_{nm'} J_{mn'} + \delta_{mn'} J_{nm'} - \delta_{mm'} J_{nn'} - \delta_{nn'} J_{mm'}] \qquad (3.5)$$

Hence we conclude that the J_{mn}'s which are $r(r+1)/2$ in number satisfying (3.5) are the generators of the Lie algebra of the orthogonal group in (r+1) dimension.

Since we know the explicit representation of the basic elements e_1 and e_2 of C_2^{r+1}, we are now in possession of (r+1) dimensional representation of K(r) and that of the rotation group in (r+1) dimensions.

It has been pointed out by C. Ryan and E.C.G. Sudarshan[6)] that the representation of the parafermi rings of order p=2 coincide with the irreducible representation of D.K.P. algebra. Elements of the parafermi rings also provide a method of constructing the algebra of orthogonal groups. It will be the future programme of the authors to arrive at the algebra of parafermi rings of any order starting from the generalised Clifford elements.

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THE INSTITUTE OF MATHEMATICAL SCIENCES
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The Pageant of Modern Physics -

Planck to Gell-Mann*

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The Pageant of Modern Physics

Planck to Gell-Mann

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The dawn of modern physics burst on the physical world at the beginning of this century with the discovery by Max Planck of the quantum (particle) nature of light. This received spectacular confirmation in the successful interpretation of atomic spectra by Niels Bohr, through his atom model in which the radiant energy was ascribed to transitions between the discrete energy levels of the electrons.

The theory of these physical phenomena is known as relativistic quantum mechanics, which takes into account the constancy of the velocity of light, the dualism of particles and waves and the discrete nature of the energy levels of atomic and nuclear matter. The growth of relativistic quantum mechanics has been characterised by six major developments.

(i) the formulation of theory of Relativity by Einstein (1905), (1905),

(ii) the invention of non-relativistic quantum mechanics (1925-30) by de Broglie, Schrodinger and Heisenberg.

(iii) the introduction of concept of intrinsic spin by Pauli (1926),

(iv) the fusion of relativity and quantum mechanics by Dirac (1928),

(v) the propagator formalism of Feynman (1949),

(vi) the inclusion of internal quantum numbers in the description of elementary particles by Gell-Mann (1954-68).

These distinct discoveries are well separated in time and are the independent products of the inventive genius of their creators. Since all these theories are mathematical creations to explain various aspects of the same physical phenomena, the collisions of elementary particles, we are tempted to pose the following question:

Is there a common thread which runs through this fabric of modern physics? Can we discern a pattern in this pageant of discoveries?

Very recent investigations at Matscience indicate that there exists a mathematical structure which can serve as a common basis for these six discoveries. This claim, which may seem almost utopian, has a reasonable basis since the principal features of this structure were already known to mathematicians for over a century.

We shall briefly describe these major developments in relativistic quantum mechanics and express their essential features through symbolic equations. These equations are recognised to be different facets of the same mathematical structure.

1. The Theory of Relativity

In contrast with later developments, the theory of relativity stands apart as the creation of a single mind, that of Einstein and consists of the transformation laws of dynamical variables relating to coordinate systems moving relative to one another with uniform velocity. In essence this transformation expresses the invariance of the length of a four vector leading to the quadratic relation:

$$\boxed{E^2 = p^2 + m^2} \qquad \text{I}$$

where E, p and m are the energy, linear momentum and the rest mass of the particle respectively. The ramifications of this fundamental relation comprise the theory of relativity.

2. Quantum Mechanics

Quantum mechanics is based on the principle of complementarity which states that a description of the microscopic world is possible either in terms of space variables or momentum variables but not in terms of both simultaneously. Mathematically this implies that the wave function in momentum space $\phi(p)$ is obtained through a Fourier transformation of the wave function in coordinate space $\psi(x)x$). Symbolically we can write:

$$\boxed{\phi(p) = \int \psi(x) e^{ipx} dx} \qquad \text{II}$$

From this fundamental relation flows the de Broglie principle of dualism of particles and waves, the uncertainty principle of Heisenberg and the all-pervasive Schrodinger equation. The discrete nature of the energy levels of an electron moving round the nucleus in an atom is understood as the manifestation of the discrete nature of the eigenvalues in the solutions of the Schrodinger equation.

4. Intrinsic Spin

In such a description of matter it was found necessary to attribute an intrinsic and indestructible spin to the elementary particles. This was achieved through the use of the famous Pauli matrices:

$$\boxed{\sigma_x , \quad \sigma_y , \quad \sigma_z} \qquad \text{III}$$

which have the necessary characteristics of the three components of angular momentum.

4. The Dirac Equation

The confluence of relativity and quantum mechanics was achieved by Dirac in 1928 when he just wrote down his relativistic equation for electron. This equation involves the generalisation of Pauli matrices and can be symbolically represented by

$$\boxed{H = \alpha \cdot p + \beta m} \qquad \text{IV}$$

where the α and β are the Dirac matrices. This generalization of Pauli matrices is a mathematical operation which achieved the simultaneous inclusion of relativity and spin into the wave equation. One of the profound results of this fusion is the emergence of the concept of antiparticle as a necessary mathematical consequence of the quadratic relation between energy and momentum.

5. Feynman Formalism

The triumphant career of Dirac's theory for two decades was crowned with the elegant formulation by Feynman of the propagator formalism which enabled the study of any fundamental process 'at one stroke' where 'many' were needed in the original form of Dirac theory. The essence of this formalism is symbolically expressed through the two-point propagator function:

$$\boxed{K(x_2, x_1) = \sum_i \psi_i(x_2)\, \psi_i^*(x_1)} \qquad \text{V}$$

6. Internal Symmetry

In the early fifties, relativistic quantum mechanics met with an unprecedented challenge with the proliferation of new types of particles in high energy accelerators. It was possible to understand the phenomena relating to these strange particles only after the introduction of 'internal' quantum numbers like isotopic spin and strangeness or hypercharge. The new era of phenomenological physics was ushered in by the Gell-Mann-Nishijima relation connecting the internal symmetry quantum numbers:

$$\boxed{Q = I_z + \frac{Y}{2}} \qquad \text{VI}$$

Investigators in the last decade accepted without question the opinion that the problems of internal symmetry should be kept distinct from those relating to the dynamics of the system. Sporadic attempts to admit a unified approach met with little success.

7. A Common Thread?

We now wish to point out that there exists a mathematical structure which seem to provide a common basis for all these six developments. This mathematical structure can be traced to the Pauli matrices and their varied generalizations. In our investigation we are concerned with two classes of matrices L and A which obey the relations:

$$\boxed{L^n = I, \; A^3 = A} \qquad \text{VII}$$

When we consider the case $n = 2$, we are led to the Pauli and Dirac matrices and a quadratic relation between the eigenvalues and parameters. We are able to relate the Fourier transforms of L-matrices to the propagator formalism of Feynman. The manner in which we take the Fourier transform is an expression of the principle of complementarity in quantum mechanics.

If we consider the case of $n = 3$, we are led to the relations between the internal quantum numbers which dominates the present Gell-Mannic era of elementary particle physics.

The study of these two classes of matrices has been made possible through the classic contributions of Galois, Clifford Lie and more recently of Yamazaki. We have reached a stage when we have a right to hope that the hitherto unobserved connection between the Lie and the Clifford algebras will lead to new results and a deeper understanding of natural phenomena.

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UNITARY GENERALIZATION OF PAULI MATRICES*

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Recently it was shown that suitable linear combinations of α-matrices which satisfy the condition

$$\alpha_i^n = I; \quad \alpha_i \alpha_j = \omega \alpha_j \alpha_i \quad (i<j)$$

can be so defined as to obtain the elements of a Lie algebra and in particular those of SU(n) in their self-representation[1]. This connection seems unexpected and surprising but we shall now derive a unitary generalization of Pauli matrices in a manner as to make this connection quite perspicuous.*

We start with observation that a partial unit matrix i.e. a diagonal matrix which has unity at $(k,k), (\ell,\ell), \quad , \ldots, (m,m)$ and zero elsewhere i.e. has the following interesting property. When multiplied on its right by a matrix B, it annihilates the r^{th} row of B if there is a zero at (r,r) in the partial unit matrix and it annihilates the r^{th} column of B when it is multiplied by B on its left. Hence the following result which may

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* It is interesting to compare this with the original derivation of Gell-Mann[2] and subsequent treatments like that of Schiff.[3]

seem trivial emerges as a natural consequence. If a partial unit matrix has unity at (k,k), (ℓ, ℓ), (m,m)... and zero elsewhere then it behaves like unit matrix regarding matrices which have zeros along k^{th} column and k^{th} row, row and column. Thus if we have matrices which obey (1) then can dilate them into higher dimensional matrices by adding columns and rows with zero such that the new matrices A obey the property

$$A^3 = A \tag{1}$$

i.e. by position (j,k) is meant the intersection of the j^{th} row and k^{th} column. We shall now apply these considerations to Pauli matrices.

The Pauli matrices σ_x, σ_y and σ_z not only obey anti-commutation relations characteristic of the Clifford algebra but also commutation relations among themselves which yield the SU(2) algebra. Till now we were concerned with the generalizations of Pauli matrices, so as to preserve the anticommutation relations. We shall now generalize the Pauli matrices in such a way that their commutations yield the algebra of SU(n). We shall term this as unitary generalizations of Pauli matrices. <u>A generalized Gell-Mann-Nishijima relation will emerge as a natural consequence of this derivation</u>[4].

Defining as usual

$$\sigma_+ = \frac{\sigma_x + i\sigma_y}{2} \; ; \; \sigma_- = \frac{\sigma_x - i\sigma_y}{2} \tag{2}$$

we immediately find that σ_+, σ_- and σ_z obey the commutation relations of the SU(2) algebra.

$$[\sigma_z, \sigma_+]_- = \sigma_+ \; ; \; [\sigma_z, \sigma_-] = -\sigma_- \; ; \; [\sigma_+, \sigma_-] = \sigma_z . \tag{3}$$

Let us now define $(n \times n)$ diagonal matrices, such that the only two nonvanishing elements equal to ± 1 occur along the diagonal at (k,k) and (ℓ,ℓ). The rest of the elements along the diagonal and off diagonal are zero. We shall denote such a generalized σ_z matrix by $\Sigma_z(k,\ell)$. It is clear that we have $n(n-1)$ such matrices out of which $n(n-1)/2$ matrices are just the negative of the other $n(n-1)/2$.

We can now construct non-diagonal $(n \times n)$ matrices which are the generalized σ_x and σ_y matrices i.e. $\Sigma_x(k,\ell)$ and $\Sigma_y(k,\ell)$. $\Sigma_x(k,\ell)$ has only two non-vanishing elements equal to 1 occurring at the position (k,ℓ) and (ℓ,k). Similarly $\Sigma_y(k,\ell)$ has $-i$ at (k,ℓ) and i at (ℓ,k). Obviously we have a total of $n(n-1)$ nondiagonal $\Sigma_x(k,\ell)$ and $\Sigma_y(k,\ell)$ matrices.

This implies that there are $n(n-1)/2$ <u>generalized σ_+ matrices</u> denoted by $\Sigma_+(k,\ell)$ and $n(n-1)/2$ <u>generalized σ_- -matrices</u> denoted by $\Sigma_-(k,\ell)$. $\Sigma_+(k,\ell)$ will have only one non-vanishing element equal to 1 at (k,ℓ) while $\Sigma_-(k,\ell)$ will have one at (ℓ,k). Just like the σ_+ and σ_-, $\Sigma_+(k,\ell)$ and $\Sigma_-(k,\ell)$ matrices will act as shift operators on the eigenvectors of $\Sigma_z(k,\ell)$.

We observe that the $\Sigma(k,\ell)$ matrices obey the idempotent property

$$\chi^3 = I$$

in contrast to

$$\chi^2 = I$$

To build the (n^2-1) generators of SU(n) from these matrices we note that the $n(n-1)$ non-diagonal Σ matrices are part of this set. The other $n-1$ matrices have to be obtained from the $n(n-1)/2$ diagonal Σ_z matrices. For reasons, which will be apparent presently we express these Σ_z matrices as the differences of a set of n matrices $S_1, S_2, \ldots, S_n$ as follows. We write

$$\Sigma_z(k,\ell) = S_k - S_\ell \tag{4}$$

where S_j is a diagonal matrix with $(n-1)/n$ occurring at the position (j,j) and the rest of the $(n-1)$ elements along the diagonal are equal to $-\frac{1}{n}$, i.e.

$$S_j = \begin{pmatrix} -\frac{1}{n} & & & & & & & 0 \\ & -\frac{1}{n} & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \frac{n-1}{n} & & \\ & 0 & & & & & \cdot & \\ & & & & & & & \cdot \\ & & & & & & & \cdot \\ & & & & & & & -\frac{1}{n} \end{pmatrix} \qquad (5)$$

If we now add to the collection of n(n-1) non-diagonal Σ_x and Σ_y matrices, one of the Σ_z matrices and (n-2) of the S-matrices, then this collection of (n^2-1) matrices turn out to be generators of SU(n). We are now confronted with the unexpected realisation that equation (4) between Σ_z and S_j is only a <u>generalized Gell-Mann-Nishijima relation (G-N relation)</u>. An interesting feature of the G-N relation is that it establishes a connection between two scalar and one vector numbers or in case of SU(2) and between one vector and (n-1) scalar numbers in the case of SU(n). In order to express (4) in the similar form we use the condition that

$$\sum_{j=1}^{n} S_j = 0 \qquad (6)$$

If we write

$$T_z = \frac{1}{2} \Sigma_z(k,l) \tag{7}$$

then we have

$$S_k = T_z - \sum_{i \neq k \neq l} S_i \tag{8}$$

These remarks will become apparent when we consider the case SU(3) and then make the generalization to SU(n).

Following the above procedure we obtain the following 3 x 3 matrices.

<u>Diagonal matrices</u>

$$2I_z = \Sigma_z(1,2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad 2U_z = \Sigma_z(2,3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$2V_z = \Sigma_z(3,1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

<u>Non-diagonal</u> Σ_x <u>and</u> Σ_y

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad : \Sigma_x$$

$$\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad : \Sigma_y$$

The three diagonal Σ matrices can be recognized to be the $2\,I_z$, $2\,U_z$ and $2\,V_z$ while the non-diagonal Σ_x and Σ_y matrices are the non-diagonal λ matrices of Gell-Mann.[3)]

$$S_1 = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}; \quad S_2 = \begin{pmatrix} -1/3 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}; \quad S_3 = \begin{pmatrix} -1/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}$$

We immediately find that

$$\Sigma_z(1,2) = S_1 - S_2$$
$$\Sigma_z(2,3) = S_2 - S_3$$
$$\Sigma_z(3,1) = S_3 - S_1 \qquad (9)$$

Let us consider the quark model, now all too familiar in elementary particle physics, consisting of triplet, A,B,C having charges $Q = 2/3, -1/3, -1/3$. If we denote the eigenvalues of S_1, S_2, S_3 as s_1, s_2 and s_3 respectively we can recognize them to be identical to the quantum numbers Q,Y-Q and -Y i.e.

$$s_1 = Q; \quad s_2 = Y-Q; \quad s_3 = -Y \qquad (10)$$

The eigenvalues of I_z is $\frac{1}{2}$, $-\frac{1}{2}$ and 0, we have because of equation (9)

$$I_z = \frac{s_1 - s_2}{2} \qquad (11a)$$

Similarly

$$U_z = \frac{s_2 - s_3}{2} \; ; \quad V_z = \frac{s_3 - s_1}{2} \tag{11b}$$

These can be recognised to be the coupled generalized Gell-Mann-Nishijima relations between the vector quantum numbers I_z, U_z, V_z and the scalar quantum numbers s_1, s_2 and s_3. We now observe that along with the generalized Σ_x, Σ_y matrices, if we take I_z and $-\frac{1}{\sqrt{3}} S_3$ we have the 8 generators of SU(3), given by the matrices of Gell-Mann.

Finally we note that the scalar vector quantum numbers satisfy the relation,

$$s_1 + s_2 + s_3 = 0$$

$$I_z + U_z + V_z = 0 \tag{12}$$

Using this we can rewrite (11a) as

$$s_1 = I_z - \frac{s_3}{2} \tag{13}$$

which is the conventional G-N-relation.

We can easily generalize the above considerations to the case of SU(n). Essentially as in SU(3) we introduce n scalar quantum numbers $s_1, s_2, \ldots, s_n$ for the SU(n) multiplet $A_1 A_2, \ldots, A_n$.

These are given below in the table

Scalar quantum numbers of SU(n) multiplet

	s_1	s_2	. . .	s_n
A_1	$\frac{n-1}{n}$	$-\frac{1}{n}$	. . .	$-\frac{1}{n}$
A_2	$-\frac{1}{n}$	$\frac{n-1}{n}$	. . .	$-\frac{1}{n}$
.	.	.	. . .	.
.	.	.	. . .	.
.	.	.	. . .	.
A_n	$-\frac{1}{n}$	$-\frac{1}{n}$	. . .	$\frac{n-1}{n}$

We can define the vector quantum numbers V_{jk} for the SU(n) multiplet as

$$V_{jk} = \frac{s_j - s_k}{2} \tag{14}$$

As argued earlier, we need take only one vector quantum number and n-2 scalar numbers to arrive at the generators of SU(n). As in SU(n) the scalar and vector quantum numbers satisfy the relation

$$\sum_{i=0}^{n} s_i = 0 \; ; \qquad \sum_{j,k} V_{j,k} = 0 \tag{15}$$

We shall now establish a hitherto unobserved and also surprising connection by expressing these scalar quantum numbers in terms of the eigenvalues of the (n-1) commuting matrices in the generalized Clifford algebra of the order $n^{4)}$. Let $\mu_1, \mu_2, \ldots, \mu_{n-1}$ be the eigenvalues of the (n-1) commuting matrices in the generalized

Clifford algebra of order n where μ_i can take any one of the n values of n^{th} root of unity and

$$\mu_2 = \mu_1^2 ; \mu_3 = \mu_1^3 ; \cdots ; \mu_{n-1} = \mu_1^{n-1} \tag{16}$$

then the scalar quantum numbers $s_1, s_2, \ldots, s_n$ can be expressed in terms of μ_i s as follows

$$\begin{aligned} s_1 &= \frac{1}{n}\{ \mu_1 + \cdots + \mu_{n-1} \} \\ &= \mu_1 + \mu_1^2 + \cdots + \mu_1^{n-1} \\ s_2 &= \frac{1}{n}\{ \omega^{n-1}\mu_1 + \cdots + \omega^{n-1}\mu_1^{n-1} \} \\ &\vdots \\ s_n &= \frac{1}{n}\{ (\omega\mu_1) + \cdots + (\omega\mu_1)^{n-1} \} \end{aligned}$$

We wish to emphasize that the derivation of the generalized G.N. relation is not just of academic or formal interest. These have many speculative attempts towards generalizing the G-N relation of SU(3) to other groups like SU(4) etc. For example Amati et al[5) have considered the relations of the type

$$Q = I_z + \frac{Y}{2} + \frac{2}{3} \pm \frac{N}{4} \tag{18}$$

which, in the light of our discussion, is without any logical foundation. All the scalar quantum numbers occuring in the G-N relation should have a equal status.

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GENERALISED CLIFFORD BASIS AND INFINITESIMAL GENERATORS OF UNITARY GROUPS

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It is suggested in some recent papers[1,2] that there exists an intimate connection between the irreducible representation of a generalised Clifford algebra $C_m^{(n)}$ and the infinitesimal generators of $U(n)$ and $SU(n)$ groups and their direct products. In this paper we bring out explicitly this aspect which is not very well recognised in Mathematical as well as Theoretical Physics literature especially particle physics.

It is very well known that a certain algebra $C_m^{(n)}$ called generalised Clifford algebra, may be defined as polynomial algebra generated over the field K by a set $\{ e_1, e_2, \ldots, e_m \}$ of elements subject to the generalised Clifford conditions,

$$e_i^n = 1; \quad e_i\, e_j = \omega\, e_j\, e_i \quad (i < j;\ i,j = 1,2,\ldots,m) \qquad (1)$$

if n is odd, K is a field which contains ω and if n is even, K is a field which contains ξ, where ω is a primitive n-th root of unity and ξ a primitive $2n$-th root of unity such that $\xi^2 = \omega$, where $n \geqslant 2$. When n is 2, (1) reduces to the well known Clifford algebra.

In this paper the following theorem will be proved which establishes the relation between the irreducible representation of $C_2^{(n)}$ and the unitary basis of $U(n)$ and $SU(n)$. Then it will be generalised to the case of $C_{2m}^{(n)}$ which will be used to obtain the basis for the groups $U(n) \otimes U(n) \ldots \otimes U(n)$ and $SU(n) \otimes SU(n) \ldots \otimes SU(n)$ where each product contains m factors.

Now $C_2^{(n)}$ is the K-algebra with K-basis, $\left\{ e_1^i \, e_2^j \,\middle|\, i,j = 1,2,\ldots,n \right\}$. Define

$$E_{ij} = 1/n \sum_{p=0}^{n-1} \omega^{p(i-1)} \, e_1^p \, e_2^{j-i} \tag{2}$$

Set $\quad {}^{(1)}F_j^{(i)} = (E_{ij} + E_{ji}), \quad {}^{(2)}F_j^{(i)} = -\, i\,(F_{ij} - F_{ji}) \quad (3)$

and $\quad {}^{(3)}F_j^{(i)} = (E_{ii} - E_{jj})$ for $i < j$

and

$${}^{(3)}G_i = \frac{1}{i} \left\{ {}^{(3)}F_2^{(1)} + 2 \times {}^{(3)}F_3^{(2)} + \ldots + i \times {}^{(3)}F_{(i+1)}^{(i)} \right\}$$

Now we state the main

<u>Theorem</u>: Given $C_2^{(n)}$ over the K-algebra then

(1) E_{ij} are the infinitesimal basis of $U(n)$ in terms of the raising and lowering operations,

(2) ${}^{(1)}F_j^{(i)}$, ${}^{(2)}F_j^{(i)}$ and ${}^{(3)}F_j^{(i)}$ for $i < j$ and $I = \Sigma E_{ii}$ are the infinitesimal basis of U8n) in terms of the Pauli spin operators bounded by zeros, and

(3) ${}^{(1)}F_j^{(i)}$, ${}^{(2)}F^i$ and ${}^{(3)}G_j$ from the infinitesimal generators of $SU(n)$ in terms of the Pauli spin operators bounded by zeros.

PROOF. The proof of the above theorem will be immediate if we establish that E_{ij} are linearly independent and satisfy the generating property

$$E_{ij} \; E_{k\ell} = E_{i\ell} \; \delta_{jk} \tag{5}$$

Then as E_{ij} are n^2 in number we immediately note that an isomorphism of E_{ij} with the n-th order matrices I_{ij} having unity in i-th row and j-th column and zeros in other places, exists as the later form a linarly independent basis and satisfy (5). It is very well known that the raising and lowering operations I_{ij} form the infinitesimal basis of $U(n)$.

Relation (5) can be established very easily by considering the product $E_{ij}\, E_{k\ell}$ as was done in detail by Morris[3]. Then we obtain

$$E_{ij}\, E_{k} = \frac{1}{n^2} \sum_{p=0}^{n-1} \left(\omega^{p(i-1)}\, e_1^{p} \right) \left(e_2^{j-i} \right) \left(\sum_{q=0}^{n-1} \omega^{q(k-1)}\, e_1^{q} \right) e_2^{-k}$$

$$= \frac{1}{n^2} \left(\sum_{p=0}^{n-1} \sum_{q=0}^{n-1} \omega^{(p+q)(i-1)+q(k-j)}\, e_1^{p+q}\, e_2^{j+\ell-i-k} \right) \tag{6}$$

Now the coefficient of $e_1^{p+q}\, e_2^{j+\ell-i-k}$ in (6) is $n\,\omega^{r(i-1)}$ if $k = j$ and zero otherwise which leads to relation (5) as required.

To prove the independence of E_{ij} we proceed as Morris did and consider sets

$$S_r = \left\{ E_{ij} \,/\, j-1 \equiv r \pmod{n},\ r = 0, \ldots, n-1 \right\}. \tag{7}$$

Then $S_r \cap S_t = \emptyset$, $(r \neq t)$ and set $S = \bigcup_{r=0}^{n-1} S_r$. Since ω is a primitive n-th root of unity

$$\det\left[\omega^{(i-1)}\ \omega^{(j-1)} \right] = \pi\, (\omega^{i} - \omega^{j}) \neq 0. \tag{8}$$

Thus each set S_r, $(r = 0,\ldots,n-1)$ is linearly independent over K and hence S is a linearly independent set. The second part of the theorem is immediate when we note that in the matrix representation of E_{ij} by I_{ij} which follows by the isomorphism between their generating relations (5), the matrix representation of ${}^{(\alpha)}F^{i}_{j}$ is given by

$${}^{(1)}F^{(i)}_{j} \longrightarrow {}^{(1)}J^{(i)}_{j} = I_{ij} + I_{ji} \;;\quad {}^{(2)}F^{(i)}_{j} \longrightarrow {}^{(2)}J^{(i)}_{i} = i(I_{ij} - I_{ji});$$

$$\text{and}\quad {}^{(3)}F^{(i)}_{j} \longrightarrow {}^{(3)}J^{(i)}_{j} = (I_{ii} - I_{ji}) \tag{9}$$

for $i < j$ which for a pair of values of i and j correspond to a set of Pauli σ-matrices σ^1, σ^2 and σ^3 bounded by zero elements. In fact we can verify directly that

$$\left[{}^{(\alpha)}F^{(i)}_{j},\ {}^{(\beta)}F^{(i)}_{j}\right]_{+} = 2\,\delta_{\alpha\beta}\, I^{i}_{j} \quad\text{and}\quad \left[{}^{(\alpha)}F^{(i)}_{j},\ {}^{(\beta)}F^{(i)}_{j}\right] = \epsilon_{\alpha\beta r}\,{}^{(r)}F^{(i)}_{j} \tag{10}$$

where I^{i}_{j} is a matrix having unit in i-th and j-th diagonal positions and zeros elsewhere and $\alpha, \beta, r = 1,2,3$. As ${}^{(3)}J^{(i)}_{(j)}$ are not linearly independent, such an independent basis that is used in SU(n) theory is defined by $^{(3)}G^{(i)}_{(j)}$ with $^{(1)}F^{(i)}_{j}$ and $^{(2)}F^{(i)}_{j}$ where

$$^{(3)}G_i \longrightarrow {}^{(3)}K_i = \frac{1}{i}\left\{ {}^{(3)}J^{(1)}_{(2)} + 2\times {}^{(3)}J^{(2)}_{3} + \ldots + i\ {}^{(3)}J^{(i)}_{i+1} \right\} \tag{11}$$

This establishes the theorem completely.

In view of the fact that

$$G^{n}_{2m} \equiv M_n(K) \otimes_k \ldots \otimes_k M_n(K)$$

as is established by Morris[3], the above theorem immediately can be extended directly to obtain the basis of the direct products,

$$U(n) \otimes \ldots \times U(n) \left[SU(n) \otimes \ldots \otimes SU(n) \right]$$

of U(n) [SU(n)] with itself m-times.

The study of the importance of this connection between the generalised Clifford algebras and the unitary groups will be the subject matter of some more papers.

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REPRESENTATION OF PARA-FERMI RINGS AND GENERALISED CLIFFORD ALGEBRA

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REPRESENTATION OF PARA-FERMI RINGS AND GENERALISED CLIFFORD ALGEBRA

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1. INTRODUCTION

Recently a programme was initiated to arrive at the representations of different types of algebras from the elements of the generalised Clifford algebra. As a first step[1] the generators of the Kemmer algebra K(n) of n elements were synthesised from the elements of C_2^{n+1}, the generalised Clifford algebra whose two generating elements are the (n+1) roots of the unit matrix.

It is well known that Kemmer algebra corresponds to para-fermi statistics of order p=2. We are now encouraged to carry the programme further to obtain representations of para-fermi rings of any order p, relating to any number of operators v. We therefore outline the relations defining the operators occuring in para-fermi rings. We extend the results of reference (1) to obtain the next higher representation of K(n). We then obtain the representations for any number of these operators, for any order p of the statistics and deduce the dimensions of the representations.

2. OPERATORS OF PARA-FERMI THEORY

The theory of generalised statistics, including the Bose and Fermi statistics as special cases has been studied by a number of authors[(2),(3),(4)]. We here attempt to give an explicit representation of the para-fermi operators for any order p of the para-fermi statistics.

Let a_α $(\alpha = 1,2,\ldots,v)$ and their adjoints a^+_α be the operators of the para-fermi rings satisfying commutation relations

$$\left[a_\lambda, \tfrac{1}{2}\left[a^+_\mu, a_\nu\right]_-\right]_- = \delta_{\lambda\mu} a_\nu$$

$$(\lambda, \mu, \nu = 1,\ldots,v) \tag{2.1}$$

$$\left[a_\lambda, \tfrac{1}{2}\left[a_\mu, a_\nu\right]_-\right]_- = 0 \tag{2.2}$$

and if the order of the para-fermi statistics is p,

$$(a_\alpha)^{p+1} = 0 \; ; \quad a^j_\alpha \neq 0 \qquad \text{for} \quad j \leq p \tag{2.3}$$

The operators a_α and a^+_α are identified in physics as the creation and annihilation operators. Green[(2)] had noticed that

$$a_\alpha = \sum_{r=1}^{p} b^{(r)}_\alpha \tag{2.4}$$

will yield the para-fermi ring as defined in (2.1 to 2.3) if $b^{(r)}_\alpha$ are commuting Fermi-Dirac operators. This gives a reducible representation of dimension 2^{pv}.

Let us define $2v$ hermitian operators[(3,4)]

$$\beta_{2\alpha-1} = \tfrac{1}{2}(a_\alpha + a^+_\alpha) \; ; \quad \beta_{2\alpha} = \tfrac{1}{2}(a_\alpha - a^+_\alpha)$$

$$(\alpha = 1,\ldots,v) \tag{2.5}$$

obeying the commutation relations:

$$\left[\beta_\lambda,\left[\beta_\mu,\beta_\nu\right]_-\right]_- = \delta_{\lambda\mu}\beta_\nu - \delta_{\lambda\nu}\beta_\mu \qquad (2.6)$$

The condition (2.3) is equivalent to

$$\left(\beta_{2\alpha-1} - i\,\beta_{2\alpha}\right)^{p+1} = 0 \;:\; \left(\beta_{2\alpha-1} - i\,\beta_{2\alpha}\right)^{j} \neq 0$$

$$\text{for } j \leq p. \qquad (2.7)$$

It is known(3) that one can generate the algebra of the rotation group in (2v+1) dimensions, i.e. $O(2v+1)$ from the β's.

For the case p=2 these β's are the Kemmer elements of K(2v) and representations of the lowest dimension for K(2v) using Clifford elements have been constructed earlier(1).

We will now describe a method by which we obtain the next higher representation of the Kemmer algebra. The lowest representation of the β's is of dimension (2v+1). We can show that the next representation is of dimension

$$N = 2v + {}^{2v}C_2 \qquad (2.8)$$

To obtain this, we take all commutators $[\beta_m, \beta_n]_- = J_{mn}$ of the generating elements, ${}^{2v}C_2$ in number. If we add the 2v generators, $\beta_m = J_{om}$ to the above we get a closed set under commutation. Let us take an aggregate A of the resulting set, say,

$$A = \sum a_{mn} J_{mn} \;; \quad (m \neq n,\ m,n = 0,1,\ldots,2v) \qquad (2.9)$$

Let us now define mappings $\hat{E}_i$'s such that

$$A \xrightarrow{\hat{E}_i} A' = \left[A, J_{oi}\right]_- ; \; (i = 1, \ldots, 2v) \tag{2.10}$$

It is verified that the $\hat{E}_i$'s obey

$$\left[\hat{E}_\lambda, \left[\hat{E}_\mu, \hat{E}_\nu\right]_-\right]_- = \delta_{\mu\lambda}\hat{E}_\nu - \delta_{\lambda\nu}\hat{E}_\mu$$

and $\hat{E}^3 = \hat{E}$ for all λ, μ and ν. (2.11)

Thus we have the representation of the generators of the Kemmer algebra K(2v) which is of dimension $2v + {}^{2v}C_2$. It is to be noted that we need not know the actual matrix representation of the β's themselves to obtain $\hat{E}_i$ matrices.

3. REPRESENTATIONS FOR ANY ORDER

This section deals with a method of obtaining representations of the β's given by (2.5) and (2.6) for any order p of the para-fermi ring. To be specific we define $\beta_\alpha^{(p)}$, (α running from 1 to 2v) as the generators of the ring belonging to the order p of the statistics. Let $\beta_\alpha^{(p)}$ be constructed as

$$\beta_\alpha^{(p)} = \frac{\gamma_\alpha}{2} \otimes 1 + 1 \otimes \beta_\alpha^{(p-1)}; \; (p = 3, 4 \text{ etc.}) \tag{3.1}$$

where γ_α's are the elements of the Clifford algebra, C_{2v}^2, where the square of each of the 2v generators is the identity, the generators obeying the anticommutation relations. Starting from Pauli matrices generators of C_{2v}^2 can be obtained by the σ-operation detailed by one of the authors[5] (A.R.). The dimension of $\gamma_\alpha \in C_{2v}^2$

is 2. ν If we start with p=3, we have

$$\beta_\alpha^{(3)} = \frac{\gamma_\alpha}{2} \otimes 1 + 1 \otimes \beta_\alpha^{(2)} \tag{3.2}$$

$\beta_\alpha^{(2)} \varepsilon K(2v)$ which for the basic representation has the dimension $(2v+1)$. It can be seen that $\beta_\alpha^{(3)}$ obeys the triple commutation relation (2.5). Equations (2.7) for any α is also seen to be satisfied noting that $(\gamma_\mu - i\,\gamma_{\mu-1})^2 = 0$ and $(\beta_\mu^{(2)} - i\,\beta_{\mu-1}^{(2)})^j = 0$ for j = 3 only; for j < 3 it is non-zero. Similarly defining

$$\beta_\alpha^{(4)} = \frac{\gamma_\alpha}{2} \otimes 1 + 1 \otimes \beta_\alpha^{(3)} \tag{3.3}$$

it is verified that all the relations for para-fermi statistics for order p=4. are obeyed. In general (3.1) is found to be valid for all p. Starting with the $(2v+1) \times (2v+1)$ dimensional representation of $\beta_\alpha^{(2)}$, the matrices $\beta_\alpha^{(p)}$ has the dimension

$$N \times N = \left[2^{v(p-2)} \times (2v+1)\right] \times \left[2^{v(p-2)} \times (2v+1)\right] ; \quad (p = 3,4,\ldots). \tag{3.4}$$

Starting with Kemmer matrices a representation of lower dimension can be obtained by compounding the β's in the following way also;-

$$\beta_\alpha^{(2m)} = \beta_\alpha^{m} \otimes 1 + 1 \otimes \beta_\alpha^{(m)} \tag{3.5a}$$

and

$$\beta_\alpha^{(2m+1)} = \frac{\gamma_\alpha}{2} \otimes 1 + 1 \otimes \beta_\alpha^{(2m)} \quad (m=1,2,..). \tag{3.5b}$$

$\beta_\alpha^{(p)}$'s defined by (3.5) satisfy the equations (2.6) and (2.7).

This representation of the para-fermi ring of order p is of dimensions

$$N \times N = \left[(2v+1)^{2n-1} \times 2^{(p-2^n)v}\right] \times \left[(2v+1)^{2n-1} \times 2^{(p-2^n)v}\right] \quad (3.6)$$

where n is the maximum power of 2 such that 2^n is less than p. If we use for $\beta_\alpha^{(2)}$ higher representations(6), N will naturally be altered. However, it is to be noted that if we had begun with $\beta_\alpha^{(2)}$ given by $\frac{\gamma_\alpha}{2} \otimes 1 + 1 \otimes \frac{\gamma_\alpha}{2}$, and proceeded further according to the equation (3.1) we would have obtained 2^{pv} dimensional representation for $\beta_\alpha^{(p)}$ which is the same as that of Green(2). The representations arrived at in this section are also not irreducible.

In the above, representations of para-fermi rings are obtained by adding γ_μ's of suitable dimension to the para-fermi operators of smaller order. On referring to the literature of the theory of relativistic wave equations(7,8,9) there seems to be grounds for hoping that higher values of p may be related to higher spin. Hence successive additions of half spin fields may be thought of as a method of obtaining representations for higher p corresponding to those given here.

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THE INSTITUTE OF MATHEMATICAL SCIENCES
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ALGEBRAS DERIVED FROM POLYNOMIAL CONDITIONS

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Introduction

The essential feature of L-matrix theory introduced by one of the authors (A.R.)[1] consists in realising that the multiple solutions of a polynomial matrix equation form an algebra under suitable conditions. This polynomial condition must be more restrictive than the characteristic equation which any matrix satisfies according to the Cayley-Hamilton theorem. The simplest and perhaps the most important to physics in the class of matrix solutions which satisfy $A^2 = 1$. The dimension of the matrix representation is related to the number of elements and they possess the anticommuting property. The next polynomial condition which was investigated was $A^n = 1$ yielding the generalised Clifford algebra. We now investigate other polynomials which yield interesting algebras[2,3] with $A^2 = 0$; $A^3 = A$; $A^4 = A^2$; $A^s = A^{s-2}$ along with the condition that linear combinations of these also obey corresponding equations. More general conditions such as $A^l = A^m$ $(l > m)$ are also studied.

Section 2.

(i) We deal herein with types of matrices obeying the conditions

$$A_r^2 = 0\ ;\ r = 1,\dots,n \text{ and } \Big(\sum_{r=1}^{n \text{ or less}} A_r\Big)^2 = 0 \qquad (2.1)$$

It is verified that the matrices A_r's anticommute among themselves and the representation for such matrices will be

$$A_r = \alpha_r \prod_{q=1}^{r-1} \beta_q \quad \text{where } \alpha_r = 1\times 1\times\cdots 1\times \underset{(r^{th}\text{ place})}{\alpha} \times 1\times\cdots\times 1 \quad (n \text{ terms})$$

$$\text{and } \beta_q = 1\times 1\times\cdots 1\times \underset{(q^{th}\text{ place})}{\beta} \times 1\times\cdots\times 1 \quad (\quad " \quad) \tag{2.2}$$

Also

$$\tfrac{1}{2}[\alpha, \beta] = -\alpha \; ; \; \beta = [\alpha, \alpha^+]_-$$
$$\alpha^2 = \alpha^{+2} = 0 \; ; \; \alpha = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{2.3}$$

This structure is nothing new but that given by Jordan and Wigner[4] for the representation of ordinary fermi operators.

ii) Exploiting similar methods we can obtain other types of relations like

$$A_r^2 = 0 ; \; B_r^2 = 0 \quad \text{and} \quad (A_r + B_r)^2 = 1 \tag{2.4}$$

This can be satisfied if A_r's are those given above and B_r's are their respective adjoints: $A_r^+ = B_r$ Also it is seen that

$$\left(\sum A_r + \sum B_r\right)^2 = q \tag{2.5}$$

where q is the number of adjoint pairs that occur inside the bracket in equation (2.5)

iii) Matrices satisfying $\beta_1^2 = 1$

$\beta_2^2 = 1$ and $(\beta_1 + \beta_2)^2 = 0$ can be represented

(2.6)

by the construction

$$\beta_1 = \frac{A + A^+}{2} \quad ; \quad \beta_2 = i\,\frac{A - A^+}{2} \tag{2.7}$$

where A and A^+ are the same as above.

<u>Section 3.</u>

We now look for the matrices satisfying the relationships

$$\beta_r^3 = \beta_r \quad \text{and} \quad \left(\sum \lambda_r \beta_r\right)^2 = \left(\sum \lambda_r^2\right)\left(\sum \lambda_r \beta_r\right) \tag{3.1}$$

The solutions are the well-known Kemmer matrices satisfying the commutation relations

$$[\beta_\mu, [\beta_\nu, \beta_\lambda]_-]_- = \delta_{\mu\nu}\beta_\lambda - \delta_{\mu\lambda}\beta_\nu \tag{3.2}$$

If one goes further and prescribes relations

$$\alpha_r^4 = \alpha_r \quad (r = 1, \ldots, n) \quad \text{and} \quad \left(\sum_r \alpha_r\right)^4 = \left(\sum \alpha_r\right)^2 \tag{3.3}$$

The following types of matrices satisfy the requirements completely. They are

$$\alpha_r = [\gamma_r \otimes 1 + i\gamma \otimes \beta_r] \tag{3.4}$$

where γ_r are the generalised Riac matrices and β_r's are the well known Kemmer matrices and γ is the product of all the Dirac matrices. These are the elements of the Chandra ring[5].
We can also give the following construction satisfying equation (3.3):

$\bar{\alpha}_r = (\eta \otimes 1 + i\beta_r \otimes \beta_r)$ if number of β elements is odd,
such that

$$\bar{\alpha}_r^{\,4} = \bar{\alpha}_r^{\,2} \tag{3.5}$$

and $\eta = \prod_{r=1}^{n} \eta_r$ where $\eta_r = 2\beta_r^2 - 1$ and β_r's are the usual Kemmer matrices. α and $\bar{\alpha}$ satisfy the condition that linear combination of them also behave accordingly.

We now ask the more general question: What are the conditions to be satisfied by the elements A_r's such that

$$A_r^{\,l} = A_r^{\,l-2} \quad \text{and} \quad \left(\sum \lambda_r A_r\right)^l = \sum \lambda_r^2 \left(\sum \lambda_r A_r\right)^{l-2} \tag{4.1}$$

We recognise that

$$\left(\sum \lambda_r A_r\right)^l = \sum_{\substack{\text{all} \\ \text{permutations}}} \left(\lambda_{\mu_1} A_{\mu_1} \cdots \lambda_{\mu_l} A_{\mu_l}\right) \tag{4.2}$$

Hence if we prescribe the condition

$$\sum_{\substack{\text{all}\\ \text{perm.}}} \left[A_{\mu_1} \cdots A_{\mu_{l-2}} \left(\delta_{\mu_{l-1}\mu_l} - A_{\mu_{l-1}} A_{\mu_l} \right) \right] = 0 \qquad (4.3)$$

the matrices A_γ's satisfy the equation (4.1). It is relevant to point out that such conditions for matrices for relativistic higher spin equations were derived by Umezawa and Visconti[6]. However, it has not been explicitly mentioned in literature that they satisfy the second part of the equation (4.1).

In a similar way, we can take matrices satisfying relations

$$A_\gamma^{\,l} = A_\gamma \quad \text{and} \quad \left(\sum \lambda_\gamma A_\gamma \right)^l = \left(\sum \lambda_\gamma^{l-1} \right) \left(\sum \lambda_\gamma A_\gamma \right) \qquad (4.4)$$

and arrive at the necessary condition as

$$\sum_{\substack{\text{all}\\ \text{perm}}} A_{\mu_1} \left(\delta_{\mu_2 \cdots \mu_l} - A_{\mu_2} \cdots A_{\mu_l} \right) = 0 \qquad (4.5)$$

The generalization of the above relationships for the polynomial condition of the form $A^l = A^{l-s}$ $(s = 1, \ldots, l-1)$ can be easily obtained. For $l = s$; $A^l = I$ and the condition is

$$\sum_{\substack{\text{all}\\ \text{perm}}} \left(\delta_{\mu_1 \cdots \mu_l} - A_{\mu_1} \cdots A_{\mu_l} \right) = 0 \qquad (4.6)$$

For the general case $A^l = A^{l-s}$ the necessary requirement is

$$\sum_{\substack{\text{all}\\ \text{perm.}}} A_{\mu_1} \cdots A_{\mu_{l-s}} \left(\delta_{\mu_{l-s+1} \cdots \mu_l} - A_{\mu_{l-s+1}} \cdots A_{\mu_l} \right) = 0. \qquad (4.7)$$

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THE INSTITUTE OF MATHEMATICAL SCIENCES
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ON THE COMPOSITION OF GENERALIZED HELICITY MATRICES

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ON THE COMPOSITION OF GENERALIZED HELICITY MATRICES

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In developing the theory of L-matrices the concept of helicity was generalized[1] to mean the eigenvalue of a linear combination of three <u>enlarged</u> Pauli matrices. Starting from a primitive set of the three Pauli matrices, n sets of enlarged Pauli matrices

$$H_{\nu}^{\mu} \qquad \begin{pmatrix} \mu = 1,2,\ldots,n \\ \nu = 1,2,3 \end{pmatrix} \tag{1}$$

of dimension $2^n \times 2^n$ can be obtained with the following properties:

i) each set consists of three anticommuting matrices.

ii) the members of one set commute with the members of another.

Initially, a method known as the σ-operation[2] was described following the original derivation of Dirac to obtain 2n+1 anticommuting matrices of dimension $2^n \times 2^n$ from the basic Pauli matrices. Secondly, n helicity operators each of which is a linear combination of three enlarged matrices, were obtained[1] with the above mentioned properties i and ii).

We now show that the σ-operation defined previously can be generalized further into a 'tenon and mortice'* method

* This is a simple technique used in a carpentry where the edge of one piece (tenon) is put into a cavity (mortice) of another.

of fastening helicity matrices. Such a method demonstrates the power and scope of the L-matrix approach in a strikingly direct manner. For this we proceed as follows:

First we recall that there are two methods of enlarging the Pauli matrices to higher dimensions.

(i) by dilatation

$$\sigma_x \otimes I = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \; ; \; \sigma_y \otimes I = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix} \; ; \; \sigma_z \otimes I = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

(ii) by repetition

$$I \otimes \sigma_k = \begin{pmatrix} \sigma_k & & \\ & \ddots & \\ & & \sigma_k \end{pmatrix} \qquad (k = 1,2,3) \tag{2}$$

where the notation of the previous paper[1)] is used whereby the suffix 1 stands for x or y or z, 2 for any of the remaining two and 3 for the last one. I is the identity matrix of arbitrary dimension called the unit matrix of dilatation in (i).

Let $H_k(i|j)$ denote the matrix of dimension $2^{i+j+1} \times 2^{i+j+1}$, such that

$$H_k(i|j) = I_i \times \sigma_k \times I_j \qquad (k = 1,2,3) \tag{3}$$

The suffix i denotes the number of times σ_k is repeated in the H matrix and $2^i \times 2^i$ is the dimension of the unit matrix I_i and $2^j \times 2^j$ the dimension of the matrix I_j.

In the case of $2^n \times 2^n$ dimensional [illegible] n [illegible]
are

$$H_k(o|n-1), \ldots, H_k(n-1|o) \tag{4}$$

We define a linear combination of the three matrices $H_k(i|j)$, $(k = 1,2,3)$ as a helicity matrix

$$H(i|j) = \lambda_1 H_1(i|j) + \lambda_2 H_2(i|j) + \lambda_3 H_3(i|j) \tag{5}$$

We 'fasten' the helicity matrices by the <u>'tenon and mortice'</u> method to obtain L_{2n+1} which is a linear combination of $(2n+1)$ anticommuting matrices. We choose <u>any</u> set $H(i|j)$ and call it for reasons which will be obvious presently as H^n and define

$$H_1^n \doteq \mathcal{L}_{2n+1}^{2n+1}$$

$$H_2^n = \mathcal{L}_{2n}^{2n+1} \tag{6}$$

Now we fasten H_3^n, the third member of H^n, by multiplication, to three members of any set H^{n-1} other than H^n. On multiplication by H_3^n we obtain three matrices $H_3^n H_1^{n-1}$, $H_3^n H_2^{n-1}$ and $H_3^n H_3^{n-1}$. We now define

$$H_3^n H_1^{n-1} = \mathcal{L}_{2n-1}^{2n+1}$$

$$H_3^n H_2^{n-1} = \mathcal{L}_{2n-2}^{2n+1} \tag{7}$$

Now fastening the third matrix $H_3^n\, H_3^{n-1}$ to H^{n-2} we define

$$H_3^n\, H_3^{n-1}\, H_1^{n-2} = \mathcal{L}^{2n+1}_{2n-3}$$

$$H_3^n\, H_3^{n-1}\, H_2^{n-2} = \mathcal{L}^{2n+1}_{2n-4} \tag{8}$$

Continuing this process of fastening the third remaining matrix to the next set we reach

$$H_3^n\, H_3^{n-1} \quad \cdots \quad H_3^2\, H_1^1 = \mathcal{L}^{2n+1}_3$$

$$H_3^n\, H_3^{n-1} \quad \cdots \quad H_3^2\, H_2^1 = \mathcal{L}^{2n+1}_2$$

and

$$H_3^n\, H_3^{n-1} \quad \cdots \quad H_3^2\, H_3^1 = \mathcal{L}^{2n+1}_1 \tag{9}$$

Thus we obtain (2n+1) anticommuting matrices and define L_{2n+1} as the linear combination of the $\mathcal{L}^{2n+1}_i$ $(i = 1,\ldots,2n+1)$ matrices.

We recognize that the above procedure is equivalent to the following method. Writing

$$H^n = \lambda_1 H_1^n + \lambda_2 H_2^n + \lambda_3 H_3^n$$

we replace λ_3 by H^{n-1} and relabel the other two parameters so that we get a matrix with five anticommuting elements. Again starting with H^n we can replace λ_3 by the five parameter matrix constructed as before which has no common matrices with H^n to get a linear combination of seven anticommuting matrices. The procedure is repeated till we get a linear combination of 2n+1 anticommuting matrices which is defined as L_{2n+1}.

It is observed that out of the (2n+1) matrices two are members of a helicity matrix. There are n-2 sets of two matrices which are m fold products of the members of helicity matrices (m = 2,...,n-1). There is only one set with three elements which n-fold products of the members of the helicity matrices. We have thus demonstrated the irrelevance of the choice of the helicity matrices and relevance of the sequence in which they are chosen to obtain (2n+1) anticommuting matrices.*

Earlier[3], the anticommuting matrices were obtained from helicity matrices but the irrelevance of the choice of helicity matrices were not realized at that time.

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* It was a surprise to find that the (2n+1) higher dimensional anticommuting matrices were derived by Kestelman[4] as early as 1961 by a recurrence method which can be shown to be identical with the -operation. However, the concepts of generalized helicity matrices their eigenvalues and eigenvectors did not engage his attention.

If we now attach parameters λ_i to these anticommuting matrices $\mathcal{L}_i^{2n+1}$, we perceive a 'shell structure' in which there are three matrices in the first shell, two matrices in each of the other (n-1) shells. Thus we have a telescoping of eigenvalues defined as

$$\Lambda_i^2 = \Lambda_{i-1}^2 + \lambda_{2i}^2 + \lambda_{2i+1}^2$$

where
$$\Lambda_{i-1}^2 = \lambda_1^2 + \dots + \lambda_{2i-1}^2$$

the eigenvalue Λ_j being the characteristic of the j^{th} shell.

We can also visualise the 2n+1 dimensional space of the parameters as being decomposed into a sequence of a space of three dimensions each. The eigenvalue of a helicity operator corresponds to the length in such a space with the important additional feature that it takes dichotomous values positive and negative. The length in one space is then imbedded into the other space as one of the parameters.

It is to be noted that Λ_i $(i = 1, 2, \dots, n)$ are the eigenvalues of the helicity operators

$$H^n, H^{n-1}, \dots, H^1$$

and also of the n-operators

$$L_{2n+1}, \dots, I \otimes L_{2i+1}, \dots, I \otimes L_3 .$$

While the simultaneous eigenvector of the complete set n helicity operators is also the simultaneous eigenvectors of the L-operators it does not imply that the eigenvector of L_{2n+1} is an eigenvector of H_n. This is obvious from the form of the eigenvector of L_{2n+1} given in the original formulation of L-matrix theory.

The tenon and mortice method we have described is only a generalization of the σ-operation formulated earlier[2). This generalization can be equally applied to the helicity matrices[5) of the generalized Clifford algebra[6) defined by the ω-commutation relation

$$AB = \omega BA$$

$$A^n = B^n = I$$

$$\omega^n = 1$$

where ω is the primitive n^{th} root of unity. This is possible since as in the case of Pauli matrices there are only three matrices in the lowest dimension obey is the ω-commutation relations.

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APPENDIX

(To the paper 'On the composition of generalised Helicity matrices')

There is a particular and interesting case when the eigenvalues Λ_i $(i = 1,\dots,n)$ are set equal to zero. This implies that

$$\lambda_{2i}^2 + \lambda_{2i+1}^2 = 0.$$

The linear combinations of the two members of a helicity matrix with coefficients λ_{2i} and λ_{2i+1} can be interpreted as the annihilation and creation operators as recognised earlier by the author and his collaborators(7) and also by Raghavacharyulu(8).

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SHOULD WE REVISE OUR NOTIONS ABOUT SPIN AND PARITY IN RELATIVISTIC QUANTUM THEORY

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SHOULD WE REVISE OUR NOTIONS ABOUT SPIN AND PARITY IN RELATIVISTIC QUANTUM THEORY?

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We shall show by purely mathematical arguments that there is a necessity to re-examine the conventional concepts of spin and parity in relativistic quantum mechanics. These arguments are based on the theory of L-matrices developed recently by the author in a series of contributions wherein it was demonstrated that the Dirac-Hamiltonian is just a member of a hierachy of matrices with well defined algebraic properties.

This paper is divided into two sections. The first deals with the proof of the existence of a hierachy of generalized 'helicity operators' and 'helicity eigenvalues'. The second deals with the fundamental problem of imbedding spin into a relativistic Hamiltonian.

1. The Helicity Hierachy

In the theory of L-matrices, we are concerned with the simultaneous eigenvectors of the set of n matrices

$$L^{2n+1}_{2n+1},\ L^{2n+1}_{2n-1},\ \ldots,\ L^{2n+1}_{3}$$

of the same dimension $2^n \times 2^n$ defined as

$$L^{2n+1}_{2n+1} = L_{2n+1}\,;\quad L^{2n+1}_{2n-1} = \begin{bmatrix} L_{2n-1} & 0 \\ 0 & L_{2n-1} \end{bmatrix},$$

$$L^{2n+1}_{2m-1} = \begin{bmatrix} L_{2m-1} & & & \\ & L_{2m-1} & & \\ & & \ddots & \\ & & & L_{2m-1} \end{bmatrix}, \ldots,$$

$$\ldots,\quad L^{2n+1}_{3} = \begin{bmatrix} L_3 & & & \\ & L_3 & & \\ & & \ddots & \\ & & & L_3 \end{bmatrix} \qquad (1)$$

where

$$L_3 = \begin{bmatrix} \lambda_3 & \lambda_1 - i\lambda_2 \\ \lambda_1 + i\lambda_2 & -\lambda_3 \end{bmatrix}$$

and L_{2m+1} is a matrix involving (2m+1) parameters $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{2m+1}$, obtained from L_{2m-1} by a σ-operation defined as follows.

$$L_{2m+1} = \sigma(L_{2m-1}) = \begin{bmatrix} \lambda_{2m+1} I & L_{2m-1} - i\lambda_{2m} I \\ L_{2m-1} + i\lambda_{2m} I & -\lambda_{2m+1} I \end{bmatrix}$$

We ~~now recognize that the simultaneous eigenvectors of this~~ set are also the simultaneous eigenvectors of another set of n operators

$$L^{2n+1}_{3(1)} = \begin{bmatrix} L_{3(1)} & & & \\ & L_{3(1)} & & \\ & & \ddots & \\ & & & L_{3(1)} \end{bmatrix}$$

$$L^{2n+1}_{3(2)}(\Delta_1, \lambda_4, \lambda_5) = \begin{bmatrix} L_{3(2)} & & & \\ & L_{3(2)} & & \\ & & \ddots & \\ & & & L_{3(2)} \end{bmatrix}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$L^{2n+1}_{3(n)}(\Delta_{n-1}, \lambda_{2n}, \lambda_{2n+1}) = \begin{bmatrix} L_{3(n)} & 0 \\ 0 & L_{3(n)} \end{bmatrix}$$

where

$$L_{3(m)}(\Delta_{m-1}, \lambda_{2m}, \lambda_{2m+1}) = \begin{bmatrix} \lambda_{2m+1} I_{m-1} & (\Delta_{m-1} - i\lambda_{2m}) I_{m-1} \\ (\Delta_{m-1} + i\lambda_{2m}) I_{m-1} & -\lambda_{2m+1} I_{m-1} \end{bmatrix}$$

$L_{3(m)}$ is thus a function only of three parameters and is just an 'enlarged' L_3 matrix, in which a unit matrix is attached to each of the parameters. For obvious reasons, these can be called the helicity operators of various orders of which the operator

$$\begin{bmatrix} L_3 & & & \\ & L_3 & & \\ & & \ddots & \\ & & & L_3 \end{bmatrix}$$

is the first member.

We have seen earlier that in our notation the Dirac Hamiltonian is an L_5 with $\lambda_4 = 0$. The simultaneous eigenvectors of L_5 and

$$\begin{bmatrix} L_3 & 0 \\ 0 & L_3 \end{bmatrix}$$

may be recognized to be also the simultaneous eigenvectors of the set

$$L^5_{3(1)}(p_x, p_y, p_z) \text{ and } L^5_{2(2)}(p, 0, m)$$

where $|\vec{p}|$ is the modulus of the momentum. If we write the eight dimensional Dirac Hamiltonian L_7 , with $\lambda_4 = \lambda_5 = \lambda_6 = 0$ then the simultaneous eigenvectors of the set

$$L^7_7, \quad L^7_5 = \begin{bmatrix} L_5 & 0 \\ 0 & L_5 \end{bmatrix}, \quad L^7_3 = \begin{bmatrix} L_3 & & & \\ & L_3 & & \\ & & L_3 & \\ & & & L_3 \end{bmatrix}$$

are also the simultaneous eigenvectors of the set

$$L^7_{3(1)}(p_x, p_y, p_z) = \begin{bmatrix} L_3 & & & \\ & L_3 & & \\ & & L_3 & \\ & & & L_3 \end{bmatrix}$$

$$L^7_{3(2)}(p, 0, 0) = p \begin{bmatrix} 0 & I(1) & 0 & 0 \\ I(1) & 0 & 0 & 0 \\ 0 & 0 & 0 & I(1) \\ 0 & 0 & I(1) & 0 \end{bmatrix}$$

and

$$L^7_{3(3)}(p, 0, m) = \begin{bmatrix} m\,I(2) & p\,I(2) \\ p\,I(2) & -m\,I(2) \end{bmatrix}$$

2. Some Fundamental Questions

The identification of simultaneous eigenvector of the hierarchy of helicity operators clearly reveals the manner in which the parameters and eigenvalues are imbedded in the eigenvectors of the L-matrices. This facilitates the study of transformations which reverse the signs of the parameters and eigenvalues. Writing

$$L_{2n+1} = \sum_{i=1}^{2n+1} \lambda_i \mathcal{L}_i^{2n+1} \tag{1}$$

and defining

$$\mathcal{L}_i^{2n+1} L_{2n+1} \left(\mathcal{L}_i^{2n+1}\right)^{-1} = L'_{2n+1} \tag{2}$$

we note that L' can be obtained from L by reversing the signs of all λ's except that of λ_i. If u and u' are the eigenvectors of L and L',

$$L_{2n+1} u = \Delta_n u \tag{3}$$

and

$$L'_{2n+1} u' = \Delta_n u' \tag{4}$$

where

$$u = \begin{pmatrix} (L_{2n-1} - i \lambda_{2n} I)\omega \\ (\Delta_n \mp \lambda_{2n+1}) I \omega \end{pmatrix} \tag{5}$$

ω being an arbitrary vector of half the dimension of u. u' is obtained from u by reversing the signs of all the parameters except λ_i in the eigenvector u <u>but leaving</u> ω <u>unaffected</u>. Therefore, even if we choose ω to be the eigenvector of L_{2n-1} and imbed the parameters λ_1, λ_2 $\lambda_3, \lambda_4, \lambda_5, \ldots, \lambda_{2n-1}$ in it, these parameters, and Λ_{n-1} in ω are unaffected by the operator $\mathcal{L}_i^{2n+1}$. To fix our ideas, we apply these considerations to the Dirac equation. The Dirac Hamiltonian is given by L_5 with

$$\lambda_1 = p_x \; ; \; \lambda_2 = p_y \; ; \; \lambda_3 = p_z \; ; \; \lambda_4 = 0 \; ; \; \lambda_5 = m$$

In the eigenvectors of L_5 occur two eigenvalues <u>explicitly</u> – one of L_5 with the above identification of parameters and the other of

$$L_3^5 = \begin{pmatrix} \sigma \cdot p & 0 \\ 0 & \sigma \cdot p \end{pmatrix}$$

or, equivalently of the two helicity operators $L^5_{3(1)}$ and $L^5_{3(2)}$. <u>The value of spin as such does not occur explicitly in the eigenvector and has to be obtained from a knowledge of the helicity and the direction of momentum</u>. This feature which is revealed only by the recognition of the Dirac spinors as the eigenvectors of helicity operators is far from trivial; on the contrary it raises a deep and fundamental question:

Should we re-examine the conventional concepts of spin and parity in relativistic quantum mechanics?

The eigenvector of L_5 (with $\lambda_4 = 0$) is given by

$$u = \begin{pmatrix} L_3\, \omega \\ \Lambda_2 - m\, \omega \end{pmatrix}$$

where ω is an ~~arbitrary~~ vector with two components. We can reverse ~~the~~ signs of p_x, p_y, p_z in L_3, leaving ω untouched by operating $\mathcal{L}^5_5$ which is the $\mathcal{L}$ matrix attached to the parameter m. If ω is an eigenvector of L^5_3

,

$$\omega = \begin{pmatrix} p_x - i p_y \\ \Lambda_1 - p_z \end{pmatrix}$$

the operator $\mathcal{L}$ will still leave the parameters p_x, p_y, p_z in ω unchanged.

If we wish to reverse the parameters p_x, p_y, p_z keeping 'the direction of spin' unchanged such an operation is identical with reversing both $\vec{p}$ and the eigenvalue p occurring in ω. This amounts to multiplying ω by (-1) and this does not alter the eigenvalue of L_3. Therefore it is not possible to reverse $\vec{p}$ and helicity in the eigenvector of L_3. This difficulty arises because of the <u>saturated character of</u> L_3 in the sense defined in an earlier contribution.

A way out of this difficulty is to consider a case in which L_3, L_5 and L_7 are unsaturated, the parameters being identified as

$$\lambda_1 = p_x ; \quad \lambda_2 = p_y ; \quad \lambda_3 = p_z ; \quad \lambda_4 = 0 ; \quad \lambda_7 = m ; \quad \lambda_5 = \lambda_6 = 0 \qquad (17)$$

For such unsaturated matrices* the eigenvalue are defined as

$$\begin{aligned} \Lambda_1 &= (p_x^2 + p_y^2)^{1/2} \\ \Lambda_2 &= (p_x^2 + p_y^2 + p_z^2)^{1/2} \\ \Lambda_3 &= (p_x^2 + p_y^2 + p_z^2 + m^2)^{1/2} \end{aligned} \qquad (18)$$

These ~~are also the eigenvalues of~~ the helicity operators

$$L^7_{3(1)}(p_x, p_y, 0) \; ; \; L^7_{3(2)}(\Lambda_1, 0, p_z) \; ; \; L^7_{3(3)}(\Lambda_2, 0, m) \qquad (19)$$

The eigenvector v of L_7 can be written as

$$v = \begin{pmatrix} L_5 & \omega \\ \Lambda_3 - m & \omega \end{pmatrix} \qquad (20)$$

where ω is an arbitrary four component vector.

The matrix

$$\mathcal{L}^7_7 = \begin{pmatrix} I(4) & \\ & -I(4) \end{pmatrix} \qquad (21)$$

*This case has obviously to be distinguished from an L_7 with $\lambda_4 = \lambda_5 = \lambda_6 = 0$ which has been discussed before.

where $I(4)$ is the 4 x 4 unit matrix, will reverse the sign of Λ_3 and m or p_x, p_y, p_z in L_7 leaving ω untouched.

If ω is an eigenvector of L_5 then

$$\omega = \begin{pmatrix} L_3 & u \\ \Delta_2 - p_z & u \end{pmatrix}$$

where u is an arbitrary vector with 2 components, the matrix

$$\mathcal{L}^5_5 = \begin{pmatrix} I(2) & \\ & -I(2) \end{pmatrix}$$

will reverse p_z and Λ_2 leaving u untouched.

If u is an eigenvector of L_3 then

$$u = \begin{pmatrix} p_x - i p_y \\ \Delta_1 \end{pmatrix}$$

the matrix

$$\mathcal{L}^3_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

will reverse p_x and p_y in u.

Thus we find that if u, v and w are eigenvectors of L_7, L_5 and L_3 the matrix product with $\lambda_3 = \lambda_4 = \lambda_6 = 0$

$$\mathcal{L}_7^7 \, \mathcal{L}_5^7 \, \mathcal{L}_3^7 \tag{26}$$

will reverse $\lambda_1, \lambda_2, \lambda_5$ and Λ_2 leaving Λ_1 and Λ_3 untouched. This is just what we require the parity operator to perform if we set

$$\lambda_1 = p_x; \quad \lambda_2 = p_y; \quad \lambda_3 = 0; \quad \Lambda_1 = (p_x^2 + p_y^2)^{1/2}$$

$$\lambda_4 = 0; \quad \lambda_5 = p_z; \quad \lambda_6 = 0; \quad \Lambda_2 = p$$

$$\lambda_7 = m; \quad \Lambda_3 = E = (p_x^2 + p_y^2 + p_z^2 + m^2)^{1/2}$$

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SYMMETRIES ASSOCIATED WITH THE ROOTS OF THE UNIT MATRIX

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SYMMETRIES ASSOCIATED WITH THE ROOTS OF THE UNIT MATRIX.

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From the study of the algebra of matrices[1,2] which are the nth-roots of the unit matrix, it was clear that we could start with two $n \times n$ dimensional base matrices A and B defined for the case n odd as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & 1 \\ 1 & 0 & 0 & 0 & . & . & . & 0 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 0 & \omega & 0 & 0 & . & . & . & 0 \\ 0 & 0 & \omega^2 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & \omega^{n-1} \\ 1 & 0 & 0 & 0 & . & . & . & 0 \end{bmatrix}$$

where ω is the primitive nth-root of unity and form all possible products

$$A^k B^{\ell}, \; (k,l = 0,1,\ldots,n-1)$$

representing the n^2-1 non-trivial roots of the unit matrix, the trivial being the unit matrix itself. Among these, we can choose the matrix $C = A^{n-1} B$ such that A,B and C obey the commutation relations

$$\begin{aligned} AB &= \omega BA \\ AC &= \omega CA \\ BC &= \omega CB \end{aligned}$$

For the case when n is even we define B with an extra factor $\omega^{1/2}$ and C as $\omega^{1/2} A^{n-1} B$.

We now point out an extraordinarily interesting property of this collection of n^2 roots of the unit matrix.

Any matrix R in this collection operating on the eigenvector of any other matrix S corresponding to an eigenvalue η yields another eigenvector of S corresponding to another eigenvalye η' where η and η' are roots of unity, provided R is not expressible as S^k, $k = 1,2,\ldots,n-1$.

This property becomes obvious if we realize that since R and S are the products of the powers of the matrices A and B they obey the relation

$$RS = \epsilon\, SR$$

where ϵ is a root of unity. We can rewrite the above equation as

$$RSR^{-1} = \epsilon S.$$

If u is an eigenvector of S corresponding to the eigenvalue η then

$$u' = Ru$$

is an eigenvector of $RSR^{-1} = \epsilon S$ corresponding to the <u>same</u> eigenvalue η. Therefore Ru is an eigenvector of S corresponding to the eigenvalue $\frac{\eta}{\epsilon} = \eta'$.

It is very interesting to observe that from this Clifford algebraic point of view any one of the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ acts as a shift operator on the eigenvectors of any other operator, while from the Lie algebraic point of view it is customary to look at $\sigma_{\pm} = \frac{\sigma_x + i\sigma_y}{2}$ as the shift operators on the eigenvectors of the diagonal matrix σ_z.

Once this property is realized, we can generate the matrix A from B or B from A and therefore the set of n^2 matrices can be generated from any one root of the unit matrix.

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ON GENERALISED IDEMPOTENT MATRICES

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ON GENERALISED IDEMPOTENT MATRICES

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1. Introduction:

Study of matrices representing the elements of the generalised Clifford algebra and σ-operations[1)] on them have been extensively made in a series of contributions to this journal. This work collectively called the L-matrix theory related to matrices which satisfied the generalised Clifford condition $A^n = 1$, the case n=2 representing the simplest but perhaps the most significant to elementary particle physics. Equally important is the study of matrices of the type $A^2 = A$, or $A^n = A$ or in general $A^n = A^p$ $p<n$. In this note we study such matrices and establish a theorem that such matrices can be decomposed into products of other types of idempotent matrices.

In section 2 we first define generalised idempotency and give a method of constructing matrices obeying conditions like $A^n = A^p$, $p < n$; we also examine how linear sums of such matrices carry over the property of idempotency. In section 3, we describe a method of extending the dimensions of such matrices preserving their generalised idempotent property. In section 4, we give a theorem relating a singular matrix having generalised idempotent property to a product of idempotent matrices i.e. matrices whose square is the matrix itself.

2. Generalised idempotent matrices:

Before taking up the general method of constructing matrices $A^n = A$, we consider the simplest case $A^3 = A$, which can easily be obtained from matrices satisfying the relation $A^3 = 1$. Let

$$\tilde{A} = \frac{A - A^{+}}{\sqrt[3]{3}} \qquad (2.1)$$

It is easy to check that $\tilde{A}^3 = \tilde{A}$ which is nothing but the anti-symmetric matrices of the type

$$\tilde{A} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \omega & -1 \\ -\omega^2 & 0 & \omega^2 \\ 1 & -\omega & 0 \end{pmatrix} \quad ; \quad \omega^3 = 1 \qquad (2.2)$$

which were studied earlier[2].

In passing we may remark that the matrices representing the generalised Clifford algebraic elements are special cases of circulants[3] which play an important role in many physical situations. The circulants are NxN matrices of the form

$$C = \begin{pmatrix} C_0 & C_1 & \cdots & C_{N-1} \\ C_{N-1} & C_0 & \cdots & C_{N-2} \\ & & \vdots & \\ C_1 & C_2 & \cdots & C_0 \end{pmatrix} \quad \text{for a given } N \qquad (2.3)$$

The eigenvectors and eigensolutions of these are obtained from the scalar equation $r^N = 1$, and the eigenvalues of the matrix are y_i's : $y_i = c_0 + c_1 r_i + \cdots + c_\ell r_i^\ell + \cdots c_{N-1} r_i^{N-1}$ where r_i is one of the N roots of unity. Hence if only one C_ℓ is non-zero and is equal to unity and all the other C_ℓ's are zero, it is easy to observe that $C^N = 1$, since all the eigenvalues obey

$$y_i^N = 1 \tag{2.4}$$

To find representations of polynomial conditions on n x n matrices we turn to the well-known matrix called the companion matrix[3].

$$M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \tag{2.5}$$

This has the characteristic equation

$$\lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = 0 \tag{2.6}$$

and by the Cayley Hamilton theorem, the matrix M satisfies the equation

$$M^n + a_1 M^{n-1} + \cdots + a_n M^0 = 0 \tag{2.7}$$

If all the a's i.e., $a_1, a_2, \dots, a_n$ are zero $M^n = 0$; if all the a's except a_n are zero and if $a_n = (-1)$ we call the matrix M as A satisfying $A^n = 1$. If $a_{n-1} = (-1)$ and all the rest zero, the $M = \tilde{A}$ matrix obeys the condition $\tilde{A}^n = \tilde{A}$. We will call this as the <u>generalised idempotency condition</u>. By choosing the elements properly we can arrive at a matrix obeying any given polynomial condition. As pointed out one can construct matrices obeying conditions like say $M^n = CM+1$ and the corresponding matrix will be given by

$$M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \qquad \text{such that } M^n = CM+1$$

For the generalised idempotency we can easily obtain the (n x n) matrix

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{2.8}$$

which obeys the equation $\tilde{A}^n = \tilde{A}$; there can be other matrices of the type $\tilde{B}$ such that for n odd

$$\tilde{B} = \begin{bmatrix} 0 & \omega & 0 & - - - & 0 & 0 \\ 0 & 0 & \omega^2 & - - - & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & - - - & 0 & \omega^{n-1} \\ 0 & 1 & 0 & - - - & 0 & 0 \end{bmatrix} ; \quad \tilde{B}^n = \frac{1}{\omega} \tilde{B} \tag{2.9}$$

For n even

$$\tilde{B} = \begin{bmatrix} 0 & \xi & 0 & - - - & 0 & 0 \\ 0 & 0 & \xi^3 & - - - & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & - - - & 0 & \xi^{2n-3} \\ 0 & \xi^{2n-1} & 0 & - - - & 0 & 0 \end{bmatrix} \tag{2.10}$$

where $\xi^2 = \omega$, the n^{th} root of unity.

It can be easily checked that $\tilde{B}^n = \frac{1}{\xi} \tilde{B}$ (2.11)

It is also easy to see that a linear addition of matrices of type $\tilde{A}$ and type $\tilde{B}$, will also obey the generalised idempotency condition similar to equations (2.11) with an appropriate multiplying factor. For example if we take the linear sum

$$D = \lambda_1 \tilde{A} + \lambda_2 \tilde{B} \tag{2.12}$$

where λ_1 and λ_2 are parameters and A and B are (n x n) matrices such that $\tilde{A}^n = \tilde{A}$ and $\tilde{B}^n = \frac{1}{\omega} \tilde{B}$ for n odd. Then we deduce that

$$\tilde{A}\tilde{B} = \omega \, \tilde{B}\tilde{A} \tag{2.13}$$

where ω is the n^{th} root of unity and ℓ is a matrix of dimension (n x n) given by

$$\ell = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{2.14}$$

Similarly

$$\tilde{B}\tilde{A} = \omega^{n-1} \Delta \, \tilde{A}\tilde{B} \tag{2.15}$$

where Δ is a (n x n) matrix given by

$$\Delta = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \omega^{n-1} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{2.16}$$

We also deduce that, for n odd,

$$\left(\lambda_1 \tilde{A} + \lambda_2 \tilde{B}\right)^n = (\lambda_1 + \lambda_2)(\lambda_1 + \omega^2 \lambda_2) \cdots (\lambda_1 + \omega^{n-1}\lambda_2) \left(\lambda_1 \tilde{A} + \lambda_2 \tilde{B}\right) \tag{2.17}$$

Similarly for n even $\tilde{B}$ matrix is of the type given by (2.10) and the linear sum obeys the rule

$$\left(\lambda_1 \tilde{A} + \lambda_2 \tilde{B}\right)^n = (\lambda_1 + \lambda_2 \xi^{2n-1}) \cdots (\lambda_1 + \lambda_2 \xi^3)\left(\lambda_1 \tilde{A} + \lambda_2 \tilde{B}\right) \tag{2.18}$$

3. Increasing the dimensions of the generalised idempotent matrices.

In a previous contribution the authors[2] extended the dimension of a 3 x 3 matrix which has the property $\tilde{A}^3 = \tilde{A}$ by a σ-operation i.e. by replacing the elements of a matrix by L matrices at suitable places. This idea can be easily extended to matrices which obey

$$\tilde{A}^n = \tilde{A}, \quad \text{or} \quad \tilde{B}^n = \tilde{B} \tag{3.1}$$

conditions and thereby introducing parametrisation in the same sense as was done for the L matrices[1]. Moreover the generalisation of this scheme is such that the condition (3.1) is preserved after addition of these matrices.

Let us take matrices $\tilde{\mathcal{A}}_i$ defined by

$$\tilde{\mathcal{A}}_i = \begin{bmatrix} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathcal{L}_i & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 & \mathcal{L}_i \\ 0 & \mathcal{L}_i & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{3.2}$$

$\tilde{\mathcal{A}}_i$ matrices are obtained by enlargement from $\tilde{A}^n = \tilde{A}$ matrices by replacing matrices $\mathcal{L}_i$ for unity in all the rows except the first row. $\mathcal{L}_i$ are matrices which obey the $\mathcal{L}_i^{n-1} = I$ condition $\mathcal{L}_i^{n-1} \neq I$. They are of minimum dimension (n-1) x (n-1). Hence the dimension of $\tilde{\mathcal{A}}_i$ next higher to n x n is [n x (n-1)] x [n x (n-1)]. The dimensions of $\tilde{\mathcal{E}}_i$ are determined by the number of parameters we want to use in the linear sum of $\tilde{\mathcal{E}}_i$-matrices. This is also the usual σ-operation and has been elegantly formalised for matrices representing generalised Clifford algebra by Morris[4]. According to him if P and Q are (linearly independent) (n-1) x (n-1) square the matrices, such that

$$P^{n-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} = I \text{ and } Q^{n-1} = \begin{bmatrix} 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & \omega^{n-1} \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} = I \quad \text{for } n-1 \text{ odd.} \tag{3.3}$$

then if we are in need of constructing r elements such that $\mathcal{L}_i^{n-1} = I$ and $\mathcal{L}_i \mathcal{L}_{i+k} = \omega \mathcal{L}_{i+k} \mathcal{L}_i$ we write

$$\mathcal{L}_i = \underbrace{P^{n-2}Q \otimes P^{n-2}Q \otimes \cdots}_{(i-1)\text{ terms}} \otimes P \otimes \underbrace{I \otimes \cdots \otimes I}_{(r-i)\text{ terms}} \tag{3.4}$$

The dimensions of $\mathcal{L}_i$ is $(n-1)^r$. Hence the dimension of is $(n-1)^r \times n$. $\mathcal{L}_i$'s could also be constructed by the use of Q instead of the P's. If (n-1) is even the construction goes through with suitable modifications[4]. The linear addition of $\tilde{\mathcal{E}}_i$'s given by $\hat{A} = \sum_{i=1}^{r} \lambda_i \tilde{\mathcal{E}}_i$ will be given by

$$\hat{A} = \begin{bmatrix} 0 & \Sigma \lambda_i I & 0 & - & - & - & 0 \\ 0 & 0 & L & - & - & - & 0 \\ & & & & & \vdots & \\ 0 & 0 & 0 & - & - & - & L \\ 0 & L & 0 & - & - & - & 0 \end{bmatrix} \tag{3.5}$$

Each element of the matrix $\hat{A}$ given below is of the same dimensions as that of the L matrix occurring in it. These L matrices being a linear addition of generalised Clifford elements obeys the condition

$$L^{n-1} = (\lambda_1^{n-1} + \cdots + \lambda_r^{n-1}) I \tag{3.6}$$

We then arrive at the result that $\hat{A} = \sum_{i=1}^{r} \lambda_i \, c\tilde{e}_i$ obeys the condition

$$\hat{A}^n = \left(\sum_{i=1}^{r} \lambda_i^{n-1} \right) \hat{A} \tag{3.7}$$

One can easily extend the same procedure to the $\tilde{B}$ type matrices and obtain the following result. If $\tilde{B}$ is properly enlarged in its dimensions such that for n odd

$$\hat{B} = \begin{bmatrix} 0 & \Sigma \lambda_i \omega & 0 & - & - & - & 0 \\ 0 & 0 & \omega^2 L & - & - & - & 0 \\ & & & & & \vdots & \\ 0 & 0 & 0 & - & - & - & \omega^{n-1} L \\ 0 & L & 0 & - & - & - & 0 \end{bmatrix} \tag{3.8}$$

where $L^{n-1} = \left(\sum_{i=1}^{n} \lambda_i^{n-1}\right) I$, then

$$\hat{B}^n = \frac{1}{\omega}\left(\sum_{i=1}^{n} \lambda_i^{n-1}\right) \hat{B} \tag{3.9}$$

For n even,

$$\hat{B} = \begin{bmatrix} 0 & \left(\sum_{i=1}^{n}\lambda_i\right)\xi & 0 & \cdots & 0 \\ 0 & 0 & \xi^3 L & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & \xi^{2n-3} L \\ 0 & \xi^{2n-1} L & 0 & \cdots & 0 \end{bmatrix} \tag{3.10}$$

and

$$\hat{B}^n = \frac{i}{\xi}\left(\sum_{i=1}^{n} \lambda_i^{n-1}\right) \hat{B} \tag{3.11}$$

Similarly we can construct matrices of types $\approx\!\!A$ and $\approx\!\!B$ which **obey conditions $\tilde{\tilde{A}}^n = \tilde{\tilde{A}}^{n-2}$, and $\tilde{\tilde{B}}^n$ = constant x $\tilde{\tilde{B}}^{n-2}$ by** making use of the companion matrix of equation (2.5). Such matrices $\tilde{\tilde{A}}$, are obtained from the usual generalised Clifford matrices by altering the last row as below

$$\tilde{\tilde{A}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 0 \end{bmatrix} \tag{3.12}$$

This automatically leads to

$$\tilde{\tilde{A}}^{n} = \tilde{\tilde{A}}^{2} \qquad (3.13)$$

Similarly $\tilde{\tilde{B}}$ matrices are given by

$$\tilde{\tilde{B}} = \begin{bmatrix} 0 & \omega & 0 & - - - & 0 \\ 0 & 0 & \omega^2 & - - - & 0 \\ & & & \vdots & \\ 0 & 0 & 1 & - - - & 0 \end{bmatrix} \qquad \text{for n odd} \qquad (3.14)$$

and

$$\tilde{\tilde{B}} = \begin{bmatrix} 0 & \xi & 0 & - - - & 0 \\ 0 & 0 & \xi^3 & - - - & 0 \\ & & & \vdots & \\ 0 & 0 & \xi^{2n-1} & - - - & 0 \end{bmatrix} \qquad \text{for n even} \qquad (3.15)$$

where $\xi^2 = \omega$ is the n^{th} root of unity. $\tilde{\tilde{B}}^{n} = \text{Const.}\ \tilde{\tilde{B}}^{2}$ for n odd and for n even.

To extend the dimensions of these matrices and to put paramatrisation in their linear sum, we adopt the now familiar method of constructing $\tilde{\tilde{\mathcal{Q}}}_{i}$'s, such that

$$\tilde{\tilde{\mathcal{Q}}}_{i} = \begin{bmatrix} 0 & I & 0 & - - - & 0 \\ 0 & 0 & \mathcal{L}_i & - - - & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & - - - & \mathcal{L}_i \\ 0 & 0 & \mathcal{L}_i & - - - & 0 \end{bmatrix} \qquad (3.16)$$

where $\mathcal{L}_i$ are matrices obeying $\mathcal{L}_i^{n-2} = I$; such matrices for any given number of them are obtained from generalised Clifford elements as shown by Morris[4)] and as detailed above. Now it is easy to see that we can form matrices $\hat{\hat{A}}$ by linear addition with parameters

$$\hat{\hat{A}} = \sum_{i=1}^{n} \lambda_i \, e \tilde{\hat{\mathcal{L}}}_i = \begin{bmatrix} 0 & (\sum_{i=1}^{n}\lambda_i)I & 0 & \cdots & 0 \\ 0 & 0 & L & \cdots & 0 \\ & & & & \vdots \\ 0 & 0 & 0 & \cdots & L \\ 0 & 0 & L & \cdots & 0 \end{bmatrix}$$

and L's obey the condition (3.17)

$$L^{n-2} = \left(\sum_{i=1}^{n} \lambda_i^{n-2} \right) I \tag{3.18}$$

We can obtain then

$$(\hat{\hat{A}})^n = \left(\sum_{i=1}^{n} \lambda_i^{n-2} \right) \hat{\hat{A}}^2 \tag{3.19}$$

Thus any polynomial condition like $A^n = A^p$ $p < n$ can be obtained from the companion matrix A and its dimension can be extended by inserting in the place of unity L matrices having the property L^{n-p} = constant in all rows of A except the first row. A similar construction works equally well for the B type matrices.

4. Decomposition of generalised idempotent matrices.

In this brief section we give an interesting theorem relating a generalised idempotent matrix such as those which obey $\tilde{A}^n = \tilde{A}$, or in general $A^n = A^p$ to a product of regular idempotent matrices which obey the condition that the square of each matrix equals the original matrix.

Theorem: The n x n matrix $\tilde{A}$, which is given by

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & - & - & - & 0 & 0 \\ 0 & 0 & 1 & - & - & - & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & - & - & - & 0 & 1 \\ 0 & 1 & 0 & - & - & - & 0 & 0 \end{bmatrix} \qquad (4.1)$$

is a singular matrix obeying the condition $\tilde{A}^n = \tilde{A}$. This can be decomposed as a product of n idempotent matrices $\tilde{\ell}^{(1)}$ obeying $\left(\tilde{\ell}_r^{(1)}\right)^2 = \tilde{\ell}_r^{(1)}$ where $\tilde{\ell}_r^{(1)} = A^r \ell^{(1)} (A^r)^+$;

and A is a matrix representing the generalised Clifford elements obeying $A^n = I$, i.e.

$$\tilde{A} = \prod_{r=0}^{n-1} \tilde{\ell}_r^{(1)} = \tilde{\ell}_0^{(1)} \tilde{\ell}_1^{(1)} - - - \tilde{\ell}_{n-1}^{(1)} \qquad (4.2)$$

The matrix $\ell^{(1)}$ is given by

$$\ell^{(1)} = \begin{bmatrix} 1 & 0 & 0 & - & - & - & 0 & 0 \\ 0 & 1 & 0 & - & - & - & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & - & - & - & 1 & 0 \\ 1 & 0 & 0 & - & - & - & 0 & 0 \end{bmatrix} \qquad (4.3)$$

The proof is obvious if one remembers the fact that

$$\ell^{(1)} A = \tilde{A} \quad \text{and} \quad \left(\ell^{(1)}\right)^2 = \ell^{(1)} \quad \text{itself.}$$

A similar theorem can be stated for the B type of matrices, say for n odd

$$\tilde{B}^{n} = \omega^{n-1} \tilde{B} \qquad (4.4)$$

$\omega^{n-1} \tilde{B}$ can be decomposed into product idempotent of matrices

$$\tilde{\delta}^{(1)}_{r} = B^{r} \delta_{1}^{(1)} (B^{r})^{+} \qquad \text{where} \qquad \delta^{(1)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \omega^{n-1} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

is a (n x n) matrix. It can be shown that

$$\omega^{n-1} \tilde{B} = \prod_{r=0}^{n-1} \tilde{\delta}^{(1)}_{r} \qquad (4.5)$$

This theorem can be easily extended to singular matrices obeying the condition $A^n = A^p$ suitably changing ℓ.

For p = 2; $$\ell^{(2)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad (4.6)$$

$$\tilde{A} = \prod_{r=0}^{n-1} \tilde{\ell}^{(2)}_{r} \qquad (4.7)$$

where $A^n = 1$. In general for any p, the corresponding $\mathcal{L}^{(p)}$ has unity along its diagonal ones and zero everywhere for the first n-1 rows. The last row contains zero everywhere except for the position corresponding to the p^{th} column. The generalisation for the B type matrices is immediate: (a factor) x B = $\prod_{r=0}^{n-1} \tilde{\mathcal{S}}^{(p)}$ where $\tilde{\mathcal{S}}^{(p)} = B^r \mathcal{S}^{(p)} (B^r)^+$ and $B^n = I$; and $\mathcal{S}^{(p)}$ contains unity along its diagonal only upto n-1 the row. Corresponding to the p^{th} column in the last one we have a term ω^{n-p} for odd n. The extension to even values of n is also obvious. The existence of such relations for singular matrices was indicated earlier[5].

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THE INSTITETE OF MATUEMATICAL SCIENCES
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MATHEMATICAL LOGIC AS A GUIDE TO PHYSICAL THOUGHT*

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MATHEMATICAL LOGIC AS A GUIDE TO PHYSICAL THROUGHT

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"Mathematical science is an indivisible whole, an organism whose vitality depends upon its connections between its parts. Advancement in Mathematics is made by simplification of methods, the disappearance of old procedures which have lost their usefulness and the unification of fields until then foreign."

-- Hibert (1900)

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Today theoretical physics occupies a privileged position among the fundamental sciences. It is closely related to experimental physics and draws its strength from the extensive use of mathematical methods. In the understanding of the universe around us, the efforts of the investigator are subject to two types of rigorous test:

(i) Experimental observation, whereby the predictions are to be verified in the laboratory and the observatory.

(ii) The theory must equally satisfy the demands of logic and mathematical consistency.

This has been the story of the growth of modern physics and of the most recent developments after the discovery, in high energy physics laboratories, of new elementary particles as the constituents of the matter.

This collection of particles was called the sub-nuclear zoo by Oppenheimer and the energies of experimenters all over the world were directed towards producing more accurate data about these particles. This only increased the complexity of the problem and it was the systematic attempt to understand the plan behind this maze through mathematical methods that ushered in the Gell-Mannic era of modern physics. It started with the postulate of the famous Gell-Mann--Nishijima relation in 1954 and the next ten years were devoted to understanding the mathematical significance of this relation. Again it was given to Gell-Mann to postulate the principle of unitary symmetry by which he was able to classify the known elementary particles and predict the existence of the Omega mimus which was discovered at Brookhaven a few years later. The triumph of Gell-Mann is an outstanding instance of the triumph of mathematical logic as a guide to physical thought.

Investigations at Matscience during the past few years encourage us to believe in the existence of a mathematical structure which not only includes unitary symmetry but has in it imbedded the fundamental principles of quantum mechanics as developed through the entire period 1900-1970 from Planck to Gell-Mann.

We shall first outline the famous developments in modern physics and then indicate how a common mathematical scheme unites these independent discoveries.

The dawn of modern physics burst on the physical world at tho begining of this century with the discovery by Max Planck of the quantum (particle) nature of light. This received spectacular confirmation in the successful interpretation of atomic spectra by Niels Bohr, through his atom model in which the radiant energy was ascribed to transitions between the discrete energy levels of the electrons.

The theory of these physical phenomona is known as <u>relativistic quantum mechanics</u>, which takes into account the constancy of the velocity of light, the dualism of particles and waves and the discrete nature of the energy levels of atomic and nuclear matter. The growth of relativistic quantum mechanics has been characterised by six major developments.

(i) the formulation of theory of Relativity by Einstein (1905), (1905),

(ii) the invention of non-reletivistic quantum mechanics (1925-30) by de Broglie, Schrodinger and Heisenberg.

(iii) the introduction of concept of intrinsic spin by Pauli (1926),

(iv) the fusion of relativity and quantum mechanics by Dirac (1928),

(v) the propagator formalism of Feynman (1949),

(vi) the inclusion of internal quantum numbers in the description of elementary particles by Gell-Mann (1954-68).

These distinct discoveries are well separated in time and are the independent products of the inventive genius of their creators. Since all these theories are mathematical creations to explain various aspects of the same physical phenomena, the collisions of elementary particles, we are tempted to pose the following question:

Is there a common thread which runs through this fabric of modern physics? Can we discern a pattern in this pageant of discoveries?

Very recent investigations at Matscience indicate that there exists a mathematical structure which can serve as a common basis for these six discoveries. This claim, which may seem almost utopian, has a reasonable basis since the principal features of this structure were already known to mathematicians for over a century.

We shall briefly describe these major developments in relativistic quantum mechanics and express their essential features through symbolic equations. These equations are recognised to be different facets of the same mathematical structure.

1. The Theory of Relativity

In contrast with later developments, the theory of relativity stands apart as the creation of a single mind, that of Einstein and consists of the transformation laws of dynamical variables relating to coordinate systems moving relative to one another with uniform velocity. In essence this transformation expresses the invariance of the length of a four vector leading to the quadratic relation:

$$\boxed{E^2 = p^2 + m^2} \qquad \text{I}$$

where E, p and m are the energy, linear momentum and the rest mass of the particle respectively. The ramifications of this fundamental relation comprise the theory of relativity.

2. Quantum Mechanics

Quantum mechanics is based on the principle of complementarity which states that a description of the microscopic world is possible either in terms of space variables or momentum variables but not in terms of both simultaneously. Mathematically this implies that the wave function in momentum space $\phi(p)$ is obtained through a Fourier transformation of the wave function in coordinate space $\psi(x)x$). Symbolically we can write:

$$\boxed{\phi(p) = \int \psi(x)e^{ipx}dx} \qquad \text{II}$$

From this fundamental relation flows the de Broglie principle of dualism of particles and waves, the uncertainty principle of Heisenberg and the all-pervasive Schrodinger equation. The discrete nature of the energy levels of an electron moving round the nucleus in an atom is understood as the manifestation of the discrete nature of the eigenvalues in the solutions of the Schrodinger equation.

4. Intrinsic Spin

In such a description of matter it was found necessary to attribute an intrinsic and indestructible spin to the elementary particles. This was achieved through the use of the famous Pauli matrices:

$$\boxed{\sigma_x, \sigma_y, \sigma_z} \qquad \text{III}$$

which have the necessary characteristics of the three components of angular momentum.

4. The Dirac Equation

The confluence of relativity and quantum mechanics was achieved by Dirac in 1928 when he just wrote down his relativistic equation for electron. This equation involves the generalisation of Pauli matrices and can be symbolically represented by

$$\boxed{H = \alpha \cdot p + \beta m} \qquad \text{IV}$$

where the α and β are the Dirac matrices. This generalization of Pauli matrices is a mathematical operation which achieved the simultaneous inclusion of relativity and spin into the wave equation. One of the profound results of this fusion is the emergence of the concept of antiparticle as a necessary mathematical consequence of the quadratic relation between energy and momentum.

5. Feynman Formalism

The triumphant career of Dirac's theory for two decades was crowned with the elegant formulation by Feynman of the propagator formalism which enabled the study of any fundamental process 'at one stroke' where 'many' were needed in the original form of Dirac theory. The essence of this formalism is symbolically expressed through the two-point propagator function:

$$\boxed{K(x_2, x_1) = \sum_i \psi_i(x_2)\, \psi_i^*(x_1)} \qquad \text{V}$$

6. Internal Symmetry

In the early fifties, relativistic quantum mechanics met with an unprecedented challenge with the proliferation of new types of particles in high energy accelerators. It was possible to understand the phenomena relating to these strange particles only after the introduction of 'internal' quantum numbers like isotopic spin and strangeness or hypercharge. The new era of phenomenological physics was ushered in by the Gell-Mann-Nishijima relation connecting the internal symmetry quantum numbers:

$$\boxed{Q = I_Z + \frac{Y}{2}} \qquad \text{VI}$$

Investigators in the last decade accepted without question the opinion that the problems of internal symmetry should be kept distinct from those relating to the dynamics of the system. Sporadic attempts to adopt a unified approach met with little success.

7. A Common Thread?

We now wish to point out that there exists a mathematical structure which seem to provide a common basis for all these six developments. This mathematical structure can be traced to the Pauli matrices and their varied generalizations. In our investigation we are concerned with two classes of matrices L and Λ which obey the relations:

$$\boxed{L^n = I, \quad \Lambda^3 = \Lambda} \qquad \text{VII}$$

When we consider the case $n = 2$, we are led to the Pauli and Dirac matrices and a quadratic relation between the eigenvalues and parameters. We are able to relate the Fourier transforms of L-matrices to the propagator formalism of Feynman. The manner in which we take the Fourier transform is an expression of the principle of complementarity in quantum mechanics.

If we consider the case of $n = 3$, we are led to the relations between the internal quantum numbers which dominates the present Gell-Mannic era of elementary particle physics.

The study of these two classes of matrices has been made possible through the classic contributions of Galois, Clifford Lie and more recently of Yamazaki. We have reached a stage when we have a right to hope that the hitherto unobserved connection between the Lie and the Clifford algebras will lead to new results and a deeper understanding of natural phenomena.

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NEW GENERALIZATION OF PAULI MATRICES

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NEW GENERALISATIONS OF PAULI MATRICES

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It is a puzzling fact of scientific history that seven major developments in the triumphant career of quantum mechanics have been the result of discoveries which stand independent of one another. The relativistic quadratic relation, the principle of complementarity, the Pauli matrices, the Dirac algebra, the Feynman formalism, the concept of isotopic spin and the Gell-Mann-Nishijima relation were formulated by their creators to meet a series of challenges well separated in time during the past six decades. It looks almost incredible that it is now possible to assert that a common thread runs through them in the form of mathematical generalisations of the Pauli matrices. We shall now summarize how these seven developments imply some particular aspect of the generalisations.

The starting point of investigating such a common thread is the realisation that the Pauli matrices satisfy two distinct Clifford conditions:

i) $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$

ii) $\sigma_x \sigma_y = -\sigma_y \sigma_x$ and cyclically which are capable of independent generalisations.

We study the following modes of generalisations of the above conditions:

(I) Construction of matrices of higher dimensions which satisfy either of the conditions or both.

(II) Study of matrices A which obey the relation $A^n = I$.

(III) Investigation of matrices A such that $A^n = A$.

(IV) Study of the algebra of such matrices with the addition of identity matrix to the collection.

(V) Definition of Fourier transform of linear combinations of such matrices.

We shall show that (I) yields us the relativistic quadratic relation and the famous matrices of Dirac, (II) gives us a generalisation of commutation relations involving the higher roots of unity, (III) generates the algebra of unitary matrices and leads to a generalised Gell-Mann-Nishijima relation, (IV) deals with the propagator formalism and finally (V) implies the principle of complementarity and a new interpretation of mass in the relativistic quadratic relation involving momentum and energy.

(I) Clifford algebraic generalisation of the Pauli matrices.

In order to build higher dimensional matrices which obey either of the Clifford conditions or both, we start with a linear combination of the Pauli matrices:

$$L_3 = \sigma\cdot\lambda = \begin{pmatrix} \lambda_3 & \lambda_1 - i\lambda_2 \\ \lambda_1 + i\lambda_2 & -\lambda_3 \end{pmatrix}$$

satisfying

$$L_3^2 = (\lambda_1^2+\lambda_2^2+\lambda_3^2)I$$

where λ's are pure real or imaginary parameters. To obtain higher dimensional matrices L_{2n+1} we define a "σ-operation" as follows: We replace any one of the three parameters in L_3 by L_{2n-1} itself and relabel the other two as $\lambda_{2n}I$ and $\lambda_{2n+1}I$, where I is of the same dimension as L_{2n-1}, e.g.

$$L_{2n+1} = \sigma(L_{2n-1})$$

$$= \begin{pmatrix} \lambda_{2n+1}I & L_{2n-1}-i\lambda_{2n}I \\ L_{2n-1}+i\lambda_{2n}I & -\lambda_{2n+1}I \end{pmatrix}$$

It can be seen that the dimension of L_{2n+1} is $2^n \times 2^n$ and $(2n+1)$ parameters are imbedded in it. If we now write L_{2n+1} as

$$L_{2n+1} = \sum_{i=1}^{2n+1} \lambda_i \chi_i^{2n+1} \quad ; \; L_{2n+1}^2 = \Lambda_n^2 I$$

where $$\Lambda_n^2 = (\lambda_1^2 + \cdots + \lambda_{2n+1}^2)I$$

We immediately recognize χ_i^{2n+1} $(i=1,\ldots,2n+1)$ as the set of $(2n+1)$ anticommuting matrices of dimension $2^n \times 2^n$. By taking p-fold products $(p = 0,1,2,\ldots,2n)$, we obtain the entire Clifford algebra C^2_{2n}. The well-known Dirac matrices can be identified with the particular case of L_5 and if we choose $\lambda_1 = p_x; \lambda_2 = p_y; \lambda_3 = p_z; \lambda_4 = 0; \lambda_5 = m$ and $\Lambda_2 = E$ we obtain the relativistic relation $E^2 = p^2 + m^2$.

II. Generalised Clifford algebra.

We shall consider the representation of matrices, A,B obeying two distinct generalised Clifford conditions:

$$\text{i) } A^m = B^m = I$$

$$\text{ii) } AB = \omega BA.$$

where ω is the m-th root of unity. These relations are characterising the generalised Clifford algebra first formulated by Yamazaki. The above relations imply that

$$(\lambda A + \mu B)^m = \lambda^m + \mu^m$$

If we obtain the representations in lowest dimension, then there are only three independent matrices which obey both the generalised Clifford conditions as is true also for the usual Clifford conditions. In the case of m=3, we have

$$\mathcal{L}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \quad \mathcal{L}_2 = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix}; \quad \mathcal{L}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

The higher dimensional representations for any m can be generated by means of the σ-operation on these three matrices.

Thus we can obtain all the $2n+1$ elements which obey both the generalised Clifford conditions for any m. In order to obtain the remaining elements of C^{m}_{2n} which obey only the first condition, we consider all elements of the form

$$\mathcal{L}_1^{k_1} \mathcal{L}_2^{k_2} \ldots \mathcal{L}_{2n}^{k_{2n}}$$

with

$$0 \leq k_1, \ldots, k_{2n} \leq m-1$$

An interesting aspect of the generalised Clifford algebra is that it has properties analogous to the usual Clifford algebra with the necessary modifications to incorporate ω, the primitive m-th root of unity. The most striking feature lies in the fact that when we consider matrices $A^m=1$ for even m, the 2m-th root of unity enter in the definition of such matrices. For the special case of m=2 we have the Pauli matrices and contains 'i' <u>which is the fourth root of unity</u>.

III. <u>The unitary matrices and generalised Gell-Mann-Nishijima relations.</u>

There exists a rather unexpected and surprising connection between the elements of generalised Clifford algebra and the elements of Lie algebra of the special unitary group. In particular we have found that a suitable linear combination of the L-matrices obeying

$$\mathcal{L}_i^m = I \; ; \; \mathcal{L}_i \mathcal{L}_j = \omega \mathcal{L}_j \mathcal{L}_i \quad (i<j)$$

can be used to obtain the elements of SU(m) algebra in the self representation. We can make the connection quite perspicuous by a unitary generalisation of Pauli matrices which satisfy also the SU_2 algebra. A generalised Gell-Mann-Nishijima relations will emerge as a natural consequence of this derivation. We define (m x m) generalised σ_Z matrices, $\Sigma_Z(k,\ell)$ such that the only two non-vanishing elements equal to ± 1 occur along the diagonal at the (k,k) and (ℓ,ℓ)th position. The rest of the elements along diagonal and off diagonal are zero. In a similar way we can construct generalised σ_x and σ_y matrices, $\Sigma_x(k,\ell)$ and $\Sigma_y(k,\ell)$. To build the m^2-1 generators of SU(m) from these Σ-matrices, we note that the m(m-1) non-diagonal Σ_x and Σ_y are part of this set. To obtain the other m-1 we express the Σ_Z matrices as the differences of m matrices $S_1, S_2, \ldots, S_m$ such that

$$\Sigma_Z(k,\ell) = S_k - S_\ell$$

where S_j is a diagonal matrix of dimension (m x m) with $m-1/m$ occurring at the (j,j)th position and the rest of the (m-1) elements along the diagonal are equal to 0. The collection of m(n-1) non-diagonal Σ_x and Σ_y, one of the Σ_Z matrices and (m-2) of the S matrices comprise the m^2-1 generators of SU(m). We are now confronted with the unexpected realisation that the relation between $\Sigma_Z(k,\ell)$, S_k and S_ℓ is the generalised Gell-Mann-Nishijima relation between the vector and scalar quantum numbers. The most important result is that the scalar quantum numbers can be expressed in terms of the eigenvalues of the (m-1) commuting matrices in the generalised Clifford algebra or in other words they can be identified with the linear combination of the roots of unity. The vector quantum numbers are the eigenvalues of matrices A such that

$$A^n = A, \; n \quad \text{odd}$$
$$= A^2, \; n \quad \text{even}$$

The striking feature is that it is possible to express A as linear combination of an element from generalised Clifford algebra and its powers as

$$A = \frac{1}{n} \sum_{k=1}^{m-1} (1-\omega^k)\mathcal{L}^k$$

where

$$\mathcal{L}^m = I$$

IV. <u>Feynman propagator through the L-matrix formulation</u>.

If we define the matrix

$$Q = L_{2n+1} + \lambda I$$

where λ is an arbitrary parameter, it is interesting to note that Q is singular for $\lambda = \pm\Delta_n$ where Δ_n is the eigenvalues of L_{2n+1}. The non-singular Q can be expressed as the linear combination of singular matrices as

$$L_{2n+1} + I = \frac{1}{2\Lambda_n}\left\{\begin{array}{l}(\lambda+\Lambda_n)\,(L_{2n+1}+\lambda I)\\ -(\lambda-\Lambda_n)\,(L_{2n+1}-\lambda I)\end{array}\right\}$$

Since

$$L_{2n+1} + \lambda I = \frac{\Lambda_n^2-\lambda^2}{L_{2n+1}-\lambda I} \equiv (\Delta_n^2-\lambda^2)R \;;\; R = \text{resolvent.}$$

where the r.h.s. can be recognised to be the Feynman propagator in momentum space, we are led naturally to the fact that the best way to understand the equivalence of $n!$ diagrams of field theory to a single Feynman diagram <u>through an intermediate stage of 2^n diagrams</u>. ($L_{2n+1} + \lambda I$ can be identified to be a generalised quaternion).

V. <u>A new interpretation for mass.</u>

We define the Fourier transform of the resolvent $R = \frac{1}{L-\lambda I}$ with respect to a <u>partial set</u> of variables $\lambda_1, \dots \lambda_p$ out of $(2n+1)$ λ's. For reasons of convenience, we relabel the other parameters $\lambda_{p+1}, \dots, \lambda_{2n+1}$ as $m_1, \dots, m_{2n-p+1}$. Let us define

$$p^2 = \lambda_1^2 + \cdots + \lambda_p^2$$

$$M^2 = m_1^2 + \dots + m_{2n-p+1}^2$$

Thus $\Lambda_n^2 = p^2 + M^2$. We call the parameters $m_1, \dots, m_{2n-p+1}$ 'momentum like', since the Fourier transformation is defined with respect to them, the parameters $m_1, \dots, m_{2n-p+1}$ are 'mass like' since they are kept constant in the Fourier transformation, Λ_n is 'energy like' since it is eigenvalue. The variables $x_1, \dots, x_p$ of the transform associated with $\lambda_1, \dots, \lambda_p$ are called 'space like', while the variable t associated with free parameter is called time like.

We now introduce 'velocity like' parameters

$$v_1 = \frac{\lambda_1}{\Lambda_1} \; ; \quad \dots \quad ; \quad v_p = \frac{\lambda_p}{\Lambda_n}$$

We can express the parameters $\lambda_1, \dots, \lambda_p$ in terms of $v_1, \dots, v_p$ and M as

$$\lambda_1 = \frac{M v_1}{\sqrt{1-v^2}} \; ; \quad \dots \quad ; \quad \lambda_p = \frac{M v_p}{\sqrt{1-v^2}}$$

with

$$v^2 = \frac{\Lambda_n^2 - M^2}{\Lambda_n^2} = v_1^2 + \dots + v_p^2$$

In the formulation of L-matrix theory, the connection between the characteristics of the parameters t and x on the one hand and on the other becomes very perspicuous. A wave function is a function of the space and mass like parameters and also of t. It is an amplitude in a space <u>at</u> time t. The mass like parameters are just constants imbedded in the wave function. If we define a scalar product of the wave function with itself, it represents a distribution in space at t and its integral w.r.t. space-like variables can be normalized to a scalar for <u>any</u> value of t. Thus, t is kept fixed. However, the propagator is a function of not only of the intervals of the space-like parameters, but also of the interval in the time-like t. Hence, it can be transformed not only w.r.t. space-like parameter but also time-like parameters and thus an additional parameter (off mass-shell parameter) creeps into the propagator, but is absent in the wave function. The propagator $Q = L_{2n+1} + \lambda I$ can be recognized to be the quaternion first invented by Hamilton. The momentum-like and mass-like parameters occur with the same status in the L-matrix theory, but the kernel function is defined as a transform w.r.t. the partial set of momentum like parameters and t, the mass-like parameters occurring both in L and its transforms. In this formulation the mass like parameters acquire a new interpretation in that <u>they are the variables which are kept fixed in taking the fourier transform of the resolvent</u>.

THE WEAK INTERACTION HAMILTONIAN IN L-MATRIX THEORY

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1. INTRODUCTION

During the past three years the author and his collaborators have made a systematic study of matrices A and B which satisfy the Clifford conditions

$$A^2 = B^2 = I \qquad (1)$$

$$AB = -BA\ .$$

It was demonstrated that there is a "shell structure" in the anticommuting matrices and a consequent telescoping of eigenvalues of an L-matrix which is a linear combination of anticommuting matrices. We now observe that the form of the weak interaction Hamiltonian in elementary particle physics is an interesting manifestation of the shell structure.

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2. SHELL STRUCTURE OF AN L-MATRIX

Starting with the three fundamental mutually anti-commuting (Pauli) matrices σ_1, σ_2, σ_3 of dimension 2×2 we can form n sets of three matrices each of dimension $2^n \times 2^n$ by forming direct products of the σ matrices with p unit matrices of the same dimension to the right and n-1-p to the left ($p = 0, 1, \ldots, n-1$). We denote <u>any one</u> set by $\{H^1\}$, any other by $\{H^2\}$ and the last remaining as $\{H^n\}$

$$\{H^1\} \equiv H_1^1,\ H_2^1,\ H_3^1\ ,$$

$$\{H^2\} \equiv H_1^2,\ H_1^2,\ H_3^2\ .$$

$$\{H^n\} \equiv H_1^n,\ H_2^n,\ H_3^n\ .$$

Members of any set anticommute with one another and commute with the members of any other set.

We can form 2n+1 mutually anticommuting matrices in the following way.

$$\mathcal{L}_{2n},\ \mathcal{L}_{2n+1} \equiv H_2^n,\ H_3^n\ .$$

$$\mathcal{L}_{2n-2},\ \mathcal{L}_{2n-1} \equiv H_1^n(H_2^{n-1},\ H_3^{n-1})$$

$$\cdots\cdots\cdots$$

$$\mathcal{L}_4,\ \mathcal{L}_5 \equiv H_1^n\ H_1^{n-1}\ \ldots\ H_1^3(H_2^2,\ H_3^2)$$

$$\mathcal{L}_1,\ \mathcal{L}_2,\ \mathcal{L}_3 \equiv H_1^n\ H_1^{n-1}\ \ldots\ H_1^2(H_1^1,\ H_2^1,\ H_3^1)\ .$$

We recognize a shell structure in $\mathcal{L}$ matrices with $\{H^1\}$ in the first shell, $\{H^2\}$ in the second shell and $\{H^n\}$ in the outermost n-th shell.

We can form n commuting matrices as follows:

$$\mathcal{L}_{2n+1} = \sum_{i=1}^{2n+1} \lambda_i \mathcal{L}_i ,$$

$$\mathcal{L}^*_{2n-1} = \sum_{i=1}^{2n-1} \lambda_i H^n_1 \mathcal{L}_i ,$$

$$\vdots$$

$$\mathcal{L}^*_3 = \sum_{i=1}^{3} \lambda_i H^n_1 H^{n-1}_1 H^2_1 H^1_i .$$

The eigenvalues of L_{2n+1}, L^*_{2n-1}, ..., L^*_3 are $\pm \Lambda_n$, $\pm \Lambda_{n-1}$, ..., $\pm \Lambda_3$, respectively, where

$$\Lambda^2_n = \Lambda^2_{n-1} + \lambda^2_{2n} + \lambda^2_{2n+1} .$$

This implies a telescoping of eigenvalues corresponding to the "shells". We can specify the simultaneous eigenvectors of the n commuting operators by specifying the signs of the eigenvalues.

On the other hand, we can form n commuting "helicity" matrices, with only three anticommuting matrices in each helicity matrix.

$$\{H(\lambda_1, \lambda_2, \lambda_3)\} \equiv H_1^1 \lambda_1 + H_2^1 \lambda_2 + H_3^1 \lambda_3,$$

$$\{H(\Lambda_1, \lambda_4, \lambda_5)\} \equiv H_1^2 \Lambda_1 + H_2^2 \lambda_4 + H_3^2 \lambda_5,$$

$$\vdots$$

$$\{H(\Lambda_{n-1}, \lambda_{2n}, \lambda_{2n+1})\} \equiv H_1^n \Lambda_{n-1} + H_2^n \lambda_{2n} + H_2^n \lambda_{2n+1}.$$

These operators have eigenvalues $\pm \Lambda_1, \pm \Lambda_2, \ldots, \pm \Lambda_n$, respectively.

Thus there is a duality in setting up the eigenstates corresponding to the eigenvalues $\pm \Lambda_1, \pm \Lambda_2, \ldots, \pm \Lambda_n$.

However, we notice an ambiguity in the case of the helicity matrices which is not present in the L matrices as regards the eigenvector corresponding to the eigenvalues. For any helicity matrix, say

$$\{H(\Lambda_K, \lambda_{2K1}, \lambda_{2K+1})\} ,$$

we can replace Λ_K by $-\Lambda_K$, without altering its eigenvalue $\pm \Lambda_{K+1}$. This replacement has no relation whatsoever to the choice of the eigenvalues $\pm \Lambda_K$ for $H(\Lambda_{K-1}, \lambda_{2K-3}, \lambda_{2K-3})$. This feature is not present if we require the simultaneous vectors of the n commuting L-matrices since the eigenvalues do not occur inside the matrices.

If we take "<u>unsaturated</u>" L matrices, i.e., if we assume $\lambda_{2K} = 0$, $K = 1, \ldots, n$, then we note that reversing the sign of Λ_k in $H(\Lambda_K, 0, \lambda_{2K+1})$ without requiring a change of sign of the eigenvalue of $H(\Lambda_{K-1}, 0, L_{2K-1})$ is equivalent to reversing the sign of λ_{2K+1}. This is done by operating the product

$$H_2^L \; H_3^K$$

on the eigenvector and in the case of 4×4 matrices this is just the gamma matrix γ_5 which reverses the sign of m.

If we require the eigenstate corresponding to a particular choice of signs for the eigenvalues of Λ_1 and Λ_2 in the helicity formalism we can use the eigenvector u or γ_5 u where u is the simultaneous eigenvector of L_5 and L_3^*. Thus $(1 + \gamma_5)u$ is a valid choice and this is the basis of the theory of weak interactions.

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The L-matrix theory has been described in a series of contributions to the Journal of Mathematical Analysis and Applications (Academic Press) during the period 1967-1970.

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THE INSTITUTE OF MATHEMATICAL SCIENCES
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ON THE SHELL-STRUCTURE OF AN L-MATRIX

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In a series of contributions to this journal, methods were studied for generating $2n+1$ mutually anticommuting matrices of dimension $2^n \times 2^n$, the square of each being the unit matrix, starting from three mutually anticommuting Pauli matrices of dimensions 2×2.[1] The L-matrix was defined as the linear combination of these $2n+1$ matrices with the parameters $\lambda_1, \lambda_2, \dots, \lambda_{2n+1}$ as the coefficients of these matrices. The L-matrix has only two eigenvalues, $\pm\Lambda_n$ where

$$\Lambda_n = +\left(\lambda_1^2 + \dots + \lambda_{2n+1}^2\right)^{1/2}$$

The very method of derivation of the $(2n+1)$ anticommuting matrices yielded $n-1$ matrices

$$L^*_{2n-1},\ L^*_{2n-3},\ \dots,\ L^*_3$$

which commute with one another and with L_{2n+1}. L^*_{2k+1} is a linear combination of $2k+1$ mutually anticommuting matrices with $2k+1$ parameters $\lambda_1, \lambda_2, \dots, \lambda_{2k+1}$ as the coefficients. L^*_{2k+1} has two eigenvalues $\pm\Lambda_k$ where

$$\Lambda_k = +\left(\lambda_1^2 + \dots + \lambda_{2k+1}^2\right)^{1/2}$$

This implies a <u>shell structure</u> in the eigenvalues with the parameters λ_{2n}, λ_{2n+1}, in the outermost n^{th} shell, λ_{2n-1}, λ_{2n-2} in the (n-1)-th shell and finally λ_1, λ_2, λ_3 in the innermost or the first shell.[2)]

We shall now deal with the <u>inverse</u> problem of starting with (2n+1) anticommuting matrices, identifying the commuting matrices and discerning the shell-structure.

Given a set $\{\mathcal{L}\}_{2n+1}$ of (2n+1) anticommuting matrices the square of each being the identity, and a set of 2n+1 parameters $\lambda_1, \lambda_2, \ldots, \lambda_{2n+1}$ we can form an L-matrix which is a linear combination of the $\mathcal{L}$'s with λ's as coefficients. In doing so we are at liberty to choose λ_{2n+1} as the coefficient of any matrix in the set $\{\mathcal{L}\}_{2n+1}$ and λ_{2n} as that of any other and so on and λ_1 as the coefficient of the last remaining matrix. Without loss of generality we can denote the matrix for which λ_k is chosen as the coefficient to be $\mathcal{L}_k$. We now define, for reasons which will be apparent presently,

$$H_3^n = \mathcal{L}_{2n+1}$$

$$H_2^n = \mathcal{L}_{2n}$$

and

$$H_1^n = -i\, H_2^n H_3^n$$

This implies that

$$(H_1^n)^2 = (H_2^n)^2 = (H_3^n)^2 = I$$

We note that H_1^n anticommutes with H_2^n and H_3^n but commutes with all α_k $(k = 1, 2, \ldots, 2n-1)$.

We now factor out H_1^n from the remaining (2n-1) matrices α_k $(k = 1, 2, \ldots, 2n-1)$. This is done by premultiplying the α_k $(k = 1, 2, \ldots, 2n-1)$ by H_1^n since H_1^n is its own inverse. We then write

$$L^*_{2n-1} = \sum_{i=1}^{2n-1} \lambda_i H_1^n \alpha_i$$

and note that L^*_{2n-1} commutes with L_{2n+1}. We then proceed to define

$$H_3^{n-1} = H_1^n \alpha_{2n-1}$$

$$H_2^{n-1} = H_1^n \alpha_{2n-2}$$

$$H_3^{n-1} = i H_2^{n-1} H_3^{n-1}$$

and

$$L^*_{2n-3} = \sum_{i=1}^{2n-3} H_1^n H_1^{n-1} \alpha_i$$

Continuing the above process we get finally

$$H_1^1 = H_1^n H_1^{n-1} \cdots H_1^2 \alpha_1$$

$$H_2^1 = H_1^n H_1^{n-1} \cdots H_1^2 \alpha_2$$

$$H_3^1 = H_1^n H_1^{n-1} \cdots H_1^2 \alpha_3$$

and
$$L_3^* = \sum_{i=1}^{3} \lambda_i H_1^n H_1^{n-1} \cdots H_1^2 \alpha_i$$

The eigenvalues of $L_{2n+1}, L^*_{2n-1}, \cdots, L^*_3$ are $\pm \Lambda_n, \pm \Lambda_{n-1}, \cdots, \pm \Lambda_1$ respectively where
$$\Lambda_i^2 = \Lambda_{i-1}^2 + \lambda_{2i}^2 + \lambda_{2i+1}^2 \quad (i=1,\cdots,n)$$

The $H's$ which have been defined are recognised to be the components of helicity matrices, in the sense of the earlier contributions[3)], the helicity matrix $\{H^i\}$ being a linear combination of H_1^i, H_2^i, H_3^i defined as
$$H^i(\Lambda_{i-1}, \lambda_{2i}, \lambda_{2i+1}) = \Lambda_{i-1} H_1^i + \lambda_{2i} H_2^i + \lambda_{2i+1} H_3^i$$
with
$$\Lambda_i^2 = \Lambda_{i-1}^2 + \lambda_{2i}^2 + \lambda_{2i+1}^2 \quad (i=1,2,\cdots,n).$$

Any component of the helicity matrix $\{H^i\}$ commutes with any component of any other helicity matrix $\{H^j\}$, $i \neq j$. There are n such commuting helicity matrices but members of one set anticommute with one another. The eigenvalues of $\{H^i\}$ are $\pm\Lambda_i$.

In the earlier papers the components of the helicity matrices were obtained as the right and left direct products of the Pauli matrices with 2 x 2 unit matrices. The considerations here are quite general and this is as it should be since it is obvious that starting with the helicity matrices obtained as direct products of Pauli matrices and the unit matrices, we can make similarity transformations to obtain another sequence of helicity matrices.

In the case of the Dirac equation[1] we set $\lambda_1 = p_x$, $\lambda_2 = p_y$, $\lambda_3 = p_z$, $\lambda_4 = 0$ and $\lambda_5 = m$ such that m is in the second shell whereas p_x, p_y, p_z are in the first shell. It will be interesting to investigate the case when m is assigned to the first shell and one ortwo of the components of momenta are in the second shell. It is hoped the recognition of the shell structure will throw light on the problem of telescoping one eigenvalue into the other like helicity being telescoped into energy as in the Dirac equation.

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