COMPLEX STRUCTURES ON PRODUCT OF CIRCLE BUNDLES OVER COMPACT COMPLEX MANIFOLDS

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THE INSTITUTE OF MATHEMATICAL SCIENCES,
CHENNAI.

A Thesis submitted to the
Board of Studies in Mathematics Sciences

In partial fulfillment of requirements
for the degree of

DOCTOR OF PHILOSOPHY
of
HOMI BHABHA NATIONAL INSTITUTE

Homi Bhabha National Institute

Recommendations of the Viva Voce Board

As members of the Viva Voce Board, we recommend that the dissertation prepared by **Ajay Singh Thakur** titled "Complex Structures on Product of Circle Bundles over Compact Complex Manifolds" may be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copies of the dissertation to HBNI.

DECLARATION

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

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DECLARATION

I, hereby declare that the results presented in the thesis have been carried out by me under the guidance of my supervisor Prof. Parameswaran Sankaran. My collaboration with him in our paper was necessitated by the difficulty and depth of the problem considered. The work is original and has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institute/University.

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Abstract

Let $\bar{L}_i \longrightarrow X_i$ be a holomorphic line bundle over a compact complex manifold for $i = 1, 2$. Let S_i denote the associated principal circle-bundle with respect to some hermitian inner product on $\bar{L}_i$. We construct complex structures on $S = S_1 \times S_2$ which we refer to as *scalar, diagonal, and linear types.* While scalar type structures always exist, diagonal type structures are constructed assuming that $\bar{L}_i$ are equivariant $(\mathbb{C}^*)^{n_i}$-bundles satisfying some additional conditions. The linear type complex structures are constructed assuming X_i are (generalized) flag varieties and $\bar{L}_i$ negative ample line bundles over X_i. When $H^1(X_1; \mathbb{R}) = 0$ and $c_1(\bar{L}_1) \in H^2(X_1; \mathbb{R})$ is non-zero, the compact manifold S does not admit any symplectic structure and hence it is non-Kähler with respect to *any* complex structure.

In the case of diagonal type complex structures on S, we determine their Picard groups and the field of meromorphic function when $X_i = G_i/P_i$ where G_i are simple and P_i maximal parabolic subgroups.

Acknowledgment

I would like to thank my supervisor Professor Parameswaran Sankaran without whom this thesis would not be in the present form. I am extremely grateful to him for his support, patience, understanding and careful attention towards me. On a personal note, I am deeply amazed at the simplicity of his life and this made a great impression on me.

I would like to thank my university teachers Professor Ravi Aithal, Professor Mangala Manohar, Professor R. C. Cowsik, Professor Parvati Shastri and Professor Jyotsna Dani for their encouragement which motivated me to take research in mathematics seriously.

I had many good opportunities to discuss and learn mathematics with several of my colleagues. I am specially grateful to Umesh, Pooja, Preena, Sarbeswar, Mahender, Swagata and Prateep. I am also grateful to Professor D. S. Nagaraj for his encouragement during the course of learning Algebraic Geometry.

I would like to thank Umesh, Sundar, Krishnan, Soumya, Prem, Pooja, Alok, Preena, Sarbeswar, Issan, Srikanth, Swagata, Shilpa, Sudhir, Niraj, Krishnakumar and Ramchandra for making my stay at Matscience memorable and enjoyable. I am specially grateful to Sandipan and Vijaykumar with whom I shared many good memories.

I would like to thank the office staff of the Institute of Mathematical Sciences for being very helpful and providing an excellent environment for research. I sincerely thank IMSc Library staff Raina and Dinesh for their kind help and for keeping the rules a bit informal.

I am blessed to have unconditional love and support from my father and mother. They will always be an inspiration to me throughout my life. I would like to dedicate this thesis to them.

Contents

List of Publication(s)

Parameswaran Sankaran and Ajay Singh Thakur. Complex structures on products of circle bundles over complex manifolds, C. R. Math. Acad. Sci. Paris, Volume 349, Issues 7-8, April 2011, Pages 437-439.

Chapter 1

Introduction

The main aim of the thesis is to construct and study the complex structures on the product of circle bundles, where these circle bundles are associated to holomorphic line bundle over compact complex manifold X_i, $i = 1, 2$. In the special case when the X_i are projective spaces and the circle bundles are associated to the tautological line bundle over X_i, we get complex structures on the product of odd dimensional spheres. The complex structures on $\mathbb{S}^{2m-1} \times \mathbb{S}^{2n-1}$, for positive integer n and m, were first studied by Riemann for the case $m = n = 1$, H. Hopf for the case $m = 1$ and for the rest of the case by Calabi-Eckmann [5]. Later, Loeb-Nicolau [19] constructed and studied a more general family of complex structures on $\mathbb{S}^{2m-1} \times \mathbb{S}^{2n-1}$, for positive integer m, n . The class of manifolds we consider here is obtained by generalizing the construction of Loeb-Nicolau [19].

The general case of construction of complex structures on the product of sphere bundles over compact complex manifolds can be reduced to the case of line bundle. This is obtained by identifying the sphere bundles to circle bundles associated to canonical line bundles over the projective space bundles associated to the vector bundles. See the Section 3.4. Thus complex structures on product of sphere bundles are obtained as that of complex structures of product of circle bundles.

1.1 Hopf and Calabi-Eckmann Manifolds and their Generalizations

Compact Riemann surfaces are Kähler manifolds and are in fact projective varieties. H. Hopf [13] gave the first examples of compact complex manifolds which are non-Kähler by showing that $\mathbb{S}^1 \times \mathbb{S}^{2n-1}$ admits a complex structure for any positive integer n. Complex structures on $\mathbb{S}^1 \times \mathbb{S}^{2n-1}$ are obtained by identifying it with the quotient of $\mathbb{C}^n \setminus \{0\}$ under a free and properly discontinuous holomorphic action of $\mathbb{Z}$. Here the $\mathbb{Z}$-action on $\mathbb{C}^n \setminus \{0\}$ is generated by the automorphism

$$(z_1, z_2, \dots, z_n) \to (\exp(2\pi\sqrt{-1}\tau_1)z_1, \exp(2\pi\sqrt{-1}\tau_2)z_2, \dots, \exp(2\pi\sqrt{-1}\tau_n)z_n)$$

for some fixed constants $\tau_1, \tau_2, \dots, \tau_n$ such that $\mathrm{Im}(\tau_i) > 0$, for $i = 1, 2, \dots, n$. As their second Betti number vanishes, these manifolds are non-Kähler. Complex manifolds thus obtained are called *Hopf manifolds*. In the case when $\tau := \tau_1 = \tau_2 = \cdots = \tau_n$, the corresponding Hopf manifold is the total space of a holomorphic principal bundle over a projective space $\mathbb{P}^{n-1}$ with fibre the elliptic curve $\mathbb{C}^*/\mathbb{Z}$, where the $\mathbb{Z}$-action on $\mathbb{C}^*$ is generated by

$$z \to \exp(2\pi\sqrt{-1}\tau)z \ ; \text{for } z \in \mathbb{C}^*.$$

Haefliger [10] generalized Hopf's construction by considering a free and properly discontinuous $\mathbb{Z}$-action on $\mathbb{C}^n \setminus \{0\}$, now generated by a more general holomorphic automorphism f of $\mathbb{C}^n$ fixing 0 and such that eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the derivative $f'(0)$ at 0 are inside the unit circle. See also [8].

Calabi and Eckmann [5] showed that the product of any two odd dimensional spheres admit complex structures and thus obtained a new class of *simply connected* non-Kähler compact complex manifolds. To obtain complex structures on $\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}$, for $n, m > 0$, they explicitly constructed complex analytic charts such that the following differential fibre bundle

$$\mathbb{S}^1 \times \mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1} \to \mathbb{CP}^n \times \mathbb{CP}^m$$

becomes a holomorphic bundle with fibre an elliptic curve.

Loeb and Nicolau [19], inspired by Haefliger's paper [10], constructed a much larger class of complex structures on the product of odd dimensional spheres. To achieve this they considered certain proper holomorphic $\mathbb{C}$-actions on $\mathbb{C}^N$. These actions arise as a one-parameter family of biholomorphism associated to a vector field ξ on $\mathbb{C}^N$. The vector field considered here is of the form $\xi = \xi_0 + \xi_1 + \xi_2$. The diagonal part $\xi_0 = \sum_{1=1}^{N} \lambda_i z_i \partial/\partial z_i$ is required to satisfy the so called *weak hyperbolicity condition of type* (m, n) for $N = m + n$. The linear part $\xi_0 + \xi_1 = \sum_{i,j} a_{ij} z_i \partial/\partial z_j$ is such that the matrix (a_{ij}) is in an upper triangular form. The non-linear part ξ_2 is a sum of *resonant monomial* vector fields. This $\mathbb{C}$-action induces a one dimensional foliation on $(\mathbb{C}^m \setminus \{0\}) \times (\mathbb{C}^n \setminus \{0\})$. They showed that there is an embedding of $\mathbb{S}^{2m-1} \times \mathbb{S}^{2n-1}$ in $(\mathbb{C}^m \setminus \{0\}) \times (\mathbb{C}^n \setminus \{0\})$ transverse to each leaf and hence inducing a complex structure on this manifolds. This new class of manifolds contains as special cases the elliptic curves, the Hopf manifolds and the

Calabi-Eckmann manifolds. They studied the Dolbeault cohomology groups and the Picard group of these manifolds, and, moreover, they were also able to obtain a *versal deformation* of these of manifolds.

Recently, there have been many generalizations of Calabi-Eckmann manifolds leading to new classes of compact complex non-Kähler manifolds by López de Medrano and Verjovsky [20], Meersseman [21], Meersseman and Verjovsky [22] and Bosio [4]. See also Ramani-Sankaran [27] and Sankaran [29].

López de Medrano and Verjovsky [20] considered the one dimensional foliation on $\mathbb{C}^N$ associated to a diagonal vector field $\xi_0 = \sum_{1=1}^{N} \lambda_i z_i \partial/\partial z_i$, where the sequence of complex numbers $(\lambda_1, \lambda_2, \ldots, \lambda_N)$ satisfy following condition: For no pair of indices i, j does the segment $[\lambda_i, \lambda_j]$ contains 0. Let $\mathcal{S}$ be the union of Siegel leaves, that is, leaves which do not contain the origin in their closures. Then $\mathcal{S}$ is either empty or is an open dense subset of $\mathbb{C}^N$. Then the action of $\mathbb{C}$ given by the diagonal vector field ξ_0 commutes with the scalar multiplication of $\mathbb{C}^*$ on $\mathbb{C}^N$. The quotient space $N := \mathcal{S}/(\mathbb{C} \times \mathbb{C}^*)$ is a compact complex manifold and is non-symplectic. As a special case, they obtained the complex manifolds considered by Loeb and Nicolau [19] corresponding to diagonal vector field.

Meersseman [21] generalized the construction of López de Medrano and Verjovsky [20] to obtained a large class of compact complex manifolds. In this case Meersseman considered the foliation obtained by an action of the $\mathbb{C}^m$ on $\mathbb{C}^n$ such that $n > 2m$. This $\mathbb{C}^m$ action is given by a system of m diagonal vector fields satisfying certain condition. In the case $m = 1$ this condition is similar to that of López de Medrano and Verjovsky [20]. The quotient $N := \mathcal{S}/(\mathbb{C}^m \times \mathbb{C}^*)$ of the union $\mathcal{S}$ of Siegel leaves under the action of $\mathbb{C}^m \times \mathbb{C}^*$ is a compact complex manifolds. Here the $\mathbb{C}^*$-action is by scalar multiplication. In particular, it gives many examples of complex structures on connected sums of products of spheres. If $n = 2m+1$ then N is a complex torus. In other cases, N in non-symplectic and non-Kähler.

Ramani-Sankaran [27] introduced the notion of *generalized Hopf manifolds* which are connected compact complex homogeneous manifolds and which fibres over a projective variety G/P, where G is a simple complex Lie group and P is a maximal parabolic subgroup of G, with fibre and structure group a one dimensional complex torus. For this they consider a free and properly discontinuous $\mathbb{Z}$-action on total space E of the principal $\mathbb{C}^*$-bundle associated to the negative ample line bundle over the projective variety G/P, which generates the Picard group of G/P. The $\mathbb{Z}$-action, via the structure group action of $\mathbb{C}^*$, on E is generated by the bundle isomorphism

$$e \mapsto \exp(2\pi\sqrt{-1}\tau).e$$

for $e \in E$ and τ is a fixed complex number with $\mathrm{Im}(\tau) > 0$. The resulting space $W := E/\mathbb{Z}$ is a non-Kähler homogeneous compact complex manifold which is diffeomorphic to $\mathbb{S}^1 \times S(E)$ where $S(E)$ is a total space of the circle bundle associated to a $U(1)$-invariant hermitian metric on the line bundle E. For example, when $G = SL(n, \mathbb{C})$ and $\mathbb{P}^{n-1} = SL(n, \mathbb{C})/P$ for a maximal parabolic subgroup P of $SL(n, \mathbb{C})$, the total space of the principal $\mathbb{C}^*$-bundle associated to the tautological line bundle over the projective space $\mathbb{P}^{n-1} = SL(n, \mathbb{C})/P$ is $\mathbb{C}^n \backslash \{0\}$ and the quotient $\mathbb{C}^n \backslash \{0\}/\mathbb{Z}$ is a Hopf manifold. They showed that the *algebraic dimension* of W (i.e. transcendental degree for the field of meromorphic functions on W) is equal to $\dim_{\mathbb{C}}(W) - 1$. Now suppose that we have two such pair $P_i \subset G_i$ where G_i,P_i for $i = 1, 2$ are as above and that L_i is the total space of the principal $\mathbb{C}^*$-bundle associated to the negative ample line bundle on G_i/P_i, which generates the Picard group of G_i/P_i. The bundle $L_1 \times L_2$ is a principal $\mathbb{C}^* \times \mathbb{C}^*$ bundle over $G_1/P_1 \times G_2/P_2$. For $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$ take a $\mathbb{C}$-action, via the structure group action of $\mathbb{C}^* \times \mathbb{C}^*$, on $L_1 \times L_2$ as follow

$$(e_1, e_2) \mapsto (\exp(2\pi\sqrt{-1}\tau z).e_1, \exp(2\pi\sqrt{-1}z).e_2)$$

for $z \in \mathbb{C}$, $e_1 \in L_1$ and $e_2 \in L_2$. The quotient space $L_1 \times L_2/\mathbb{C}$ is the total space of a principal bundle over $G_1/P_1 \times G_2/P_2$ with an elliptic curve as the fibre and the structure group. Identifying $S(L_1) \times S(L_2)$ with the quotient space $L_1 \times L_2/\mathbb{C}$, we obtain a class of homogeneous, non-Kähler, compact complex manifold. Sankaran [29] called them *generalized Calabi-Eckmann manifolds.*

1.2 Standard Action of a Torus

We extend the above construction in a more general setting. Let X_1, X_2 be connected compact complex manifolds and let L_i be a principal $\mathbb{C}^*$-bundle on X_i, for $i = 1, 2$. The bundle $L := L_1 \times L_2$ is a principal $\mathbb{C}^* \times \mathbb{C}^*$ bundle over $X := X_1 \times X_2$. For $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$, we get a $\mathbb{C}$-action on L as in the construction of generalized Calabi-Eckmann manifolds and we have a diffeomorphism of $S(L) := S(L_1) \times S(L_2)$ with the quotient manifold $L/\mathbb{C}$. The complex structure thus obtained on $S(L)$, will be called *scalar type.* Following Loeb-Nicolau [19], we consider more general $\mathbb{C}$-actions on L to obtain a larger class of complex structures on $S(L)$. In this case the basic construction involves the notion of *standard action* by the torus $(\mathbb{C}^*)^{n_i}$ on the total space $\bar{L}_i$ of the line bundle associated to the principal $\mathbb{C}^*$-bundle L_i over a complex manifold X_i, for $i = 1, 2$. Fixing a hermitian metric on the line bundle $\bar{L}_i$ which is invariant under the maximal compact subgroup ($\cong (\mathbb{S}^1)^{n_i}$), we denote the corresponding circle bundle again by $S(L_i)$.

Let $E \longrightarrow B$ be a $T = (\mathbb{C}^*)^n$-equivariant principal $\mathbb{C}^*$-bundle over a complex manifold B. The associated line bundle $\bar{E} \longrightarrow B$ is again T-equivariant. Identifying B with the zero cross section of the line bundle $\bar{E}$, we have $E = \bar{E} \setminus B$. Fix a hermitian metric on the line bundle $\bar{E}$ which is invariant under the maximal compact subgroup ($\cong (\mathbb{S}^1)^n$). We shall denote by $\epsilon_j : \mathbb{C}^* \subset (\mathbb{C}^*)^n$ the inclusion of the jth factor and write $t\epsilon_j$ to denote $\epsilon_j(t)$ for $1 \leq j \leq n$. Thus any $(t_1, \cdots, t_n) \in T$ equals $\prod_{1 \leq j \leq n} t_j \epsilon_j$. Let d be a positive integer.

Definition 1.2.1. *We say that the T-action on E is d-*standard *(or more briefly* standard*) if the following conditions hold:*
(i) the restricted action of the diagonal subgroup $\Delta \subset T$ on E is via the d-fold covering projection $\Delta \longrightarrow \mathbb{C}^*$ onto the structure group $\mathbb{C}^*$ of $E \longrightarrow B$. (Thus if $d = 1$, the action of Δ coincides with that of the structure group of E.)
(ii) For any $0 \neq v \in E$ and $1 \leq j \leq n$ let $\nu_{v,j} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be defined as $t \mapsto ||t\epsilon_j.v||$. Then $\nu'_{v,j}(t) > 0$ for all t unless $\mathbb{R}_+\epsilon_j$ is contained in the isotropy at v.

When $L_i \longrightarrow X_i$, $i = 1, 2$, admit standard actions of torus $(\mathbb{C}^*)^{n_i}$, any choice of a sequence of complex numbers $\lambda = (\lambda_1, \ldots, \lambda_N)$, $N = n_1 + n_2$, gives a $\mathbb{C}$-action on L, via the embedding $\alpha_\lambda : \mathbb{C} \to (\mathbb{C}^*)^{n_1} \times (\mathbb{C}^*)^{n_2}$ which is defined as follow:

$$z \to (\exp(\lambda_1 z), \exp(\lambda_2 z), \ldots, \exp(\lambda_N z)).$$

Definition 1.2.2. *We say the above $\mathbb{C}$-action on $L := L_1 \times L_2$ is admissible, if $\lambda_k \neq 0$ for $1 \leq k \leq N$ and λ satisfies the* weak hyperbolicity condition of type (n_1, n_2) *(in the sense of Loeb-Nicolau [19, p. 788]), i.e.,*

$$0 \leq \arg(\lambda_i) < \arg(\lambda_j) < \pi, \;\; 1 \leq i \leq n_1 < j \leq N.$$

It can be shown that any admissible $\mathbb{C}$-action on L is free and each orbit is closed and properly embedded in L.

1.3 The Main Results

We shall continue with the notations of the previous section.

Theorem 1.3.1 (Theorem 3.3.3). *Let L_i be a principal $\mathbb{C}^*$-bundle over a compact complex manifold X_i with a standard action of torus T_i, for $i = 1, 2$. Suppose that $\alpha_\lambda : \mathbb{C} \longrightarrow T = T_1 \times T_2$ defines an admissible action of $\mathbb{C}$ on $L = L_1 \times L_2$. Then $L/\mathbb{C}$ is a (Hausdorff) complex analytic manifold and the quotient map $L \longrightarrow L/\mathbb{C}$ is the projection of a holomorphic principal $\mathbb{C}$-bundle. Furthermore, $L/\mathbb{C}$ is diffeomorphic to $S(L)$.* $\square$

The complex structure thus obtained on $S(L)$ is called *diagonal type.* The scalar type structure arises as a special case of the diagonal type where $(\mathbb{C}^*)^{n_i} = \mathbb{C}^*$ is the structure group of L_i, $i = 1, 2$. In the case of scalar type complex structure the differentiable $\mathbb{S}^1 \times \mathbb{S}^1$-bundle with projection $S(L) \longrightarrow X$ is a holomorphic principal bundle with fibre and structure group an elliptic curve.

By the above construction a new class of non-symplectic, non-Kähler compact complex manifolds are obtained:

Theorem 1.3.2 (Theorem 3.3.9). *Suppose that $H^1(X_1; \mathbb{R}) = 0$ and that the Chern class $c_1(\bar{L}_1) \in H^2(X_1; \mathbb{R})$ is non-zero. Then $S(L)$ is not symplectic and hence non-Kähler with respect to* any *complex structure.* $\square$

Remark 1.3.3. The above construction of complex manifold $S(L)$ is even valid in the more general setting when the complex manifolds $X_i, i = 1, 2$, are non-compact. In this thesis, the computation of the Picard group and the algebraic dimension of $S(L)$ are done in the case when X_i are certain compact manifolds. And hence we shall restrict ourselves to the case where $X_i, i = 1, 2$, are compact complex manifolds.

The action of the structure group ($\cong \mathbb{C}^*$) of a principal $\mathbb{C}^*$-bundle is a standard action in a natural way. We construct a standard action of a complex torus $\widetilde{T} \simeq \mathbb{C}^{l+1}$ on a line bundle over a class of homogeneous complex manifolds, namely the generalized flag variety which are of the form G/P, where G is a simply connected semi simple linear algebraic group over $\mathbb{C}$ of rank l and P a parabolic subgroup of G.

The construction of a linear type complex structure is carried out under the assumption that X_i is a generalized flag variety G_i/P_i, $i = 1, 2$, where G_i is a simply connected semi simple linear algebraic group over $\mathbb{C}$ and P_i a parabolic subgroup and L_i is a principal $\mathbb{C}^*$-bundle associated to a negative ample line bundle $\bar{L}_i$ over X_i. In this case L_i is acted on by the reductive group $\widetilde{G}_i = G_i \times \mathbb{C}^*$ in such a manner that the action of a maximal torus $\widetilde{T}_i \subset \widetilde{G}_i$ on L_i is standard. Fix a Borel subgroup $\widetilde{B}_i \supset \widetilde{T}_i$ and choose an element $\lambda \in Lie(\widetilde{B})$ where $\widetilde{B} = \widetilde{B}_1 \times \widetilde{B}_2 \subset \widetilde{G}_1 \times \widetilde{G}_2 =: \widetilde{G}$. Writing the Jordan decomposition $\lambda = \lambda_s + \lambda_u$ where λ_s belongs to the the Lie algebra of

$\widetilde{T} := \widetilde{T}_1 \times \widetilde{T}_2$, we assume that λ_s satisfies the weak hyperbolicity condition. For each such λ we have a $\mathbb{C}$-action on $L = L_1 \times L_2$, induced by the embedding $\mathbb{C} \to \widetilde{G}$ defined as follow,

$$z \to \exp(\lambda z) \text{ ; for } z \in \mathbb{C}.$$

We say that this $\mathbb{C}$-action is admissible if λ_s satisfies the weak hyperbolicity condition. As in the diagonal case we show that for an admissible $\mathbb{C}$-action, $S(L)$ is diffeomorphic to the quotient space $L/\mathbb{C}$. Thus we obtain a complex structure on $S(L)$ of *linear type* and we denote it by $S_\lambda(L)$. When $\lambda = \lambda_s$, we get back the diagonal type complex structure.

This description of complex structures on $S(L)$ helps to compute the cohomology $H^q(S_\lambda(L); \mathcal{O})$. In the case when the X_i are generalized flag manifolds, using the Künneth formula due to A. Cassa [7] we show that $H^q(S_\lambda(L); \mathcal{O})$ vanishes for most values of q. Namely, we prove:

Theorem 1.3.4 (Theorem 5.0.7)**.** *Suppose that $L = L_1 \times L_2$ where the L_i is principal $\mathbb{C}^*$-bundle associated to a negative ample line bundle over generalized flag variety $X_i = G_i/P_i$, for $i = 1, 2$. Suppose that $1 \leq r_1 \leq r_2$ where $r_i = \dim X_i$. Let $S_\lambda(L)$ be the complex manifold associated to λ such that the semi-simple part λ_s of λ satisfies the weak hyperbolicity condition. Then $H^q(S_\lambda(L); \mathcal{O}) = 0$ provided $q \notin \{0, 1, r_1, r_1 + 1, r_2, r_2 + 1, r_1 + r_2, r_1 + r_2 + 1\}$.* □

Theorem 1.3.5 (Theorem 5.1.2)**.** *If P_i are maximal and L_i are negative ample generators of $Pic(X_i) \cong \mathbb{Z}$, then $Pic(S_\lambda(L)) \cong Pic^0(S_\lambda(L)) \cong \mathbb{C}$.* □

We have the following result concerning the field of meromorphic functions on $S_\lambda(L)$ with diagonal type complex structure.

Theorem 1.3.6 (Theorem 5.2.1)**.** *Let L_i be the negative ample generator of $Pic(G_i/P_i) \cong \mathbb{Z}$ where P_i is a maximal parabolic subgroup of $G_i, i = 1, 2$. Assume that $S_\lambda(L)$ is of diagonal type. Then the field $\kappa(S_\lambda(L))$ of meromorphic functions of $S_\lambda(L)$ is purely transcendental over $\mathbb{C}$. The transcendence degree of $\kappa(S_\lambda(L))$ is less than* $\dim S_\lambda(L)$. □

Construction of linear type complex structure, applications to Picard groups and the field of meromorphic functions when $X_i = G_i/P_i$ involve some elementary concepts from representation theory of complex Lie groups. We shall describe these preliminaries in Chapter 2.

Chapter 2

Preliminaries

In this chapter, we recall certain well known definition and results which will be used in the thesis. As described in the introduction, the aim of the thesis is to construct a family of complex structure on the product of circle bundles over compact complex manifolds $X_i, i = 1, 2$. In the case when complex manifolds X_i are (generalized) flag manifolds, we shall study the cohomology groups and the field of meromorphic functions of the constructed manifolds.

In first section of this chapter we state the definition and describe some basic properties of flag manifolds. To study the cohomology groups we shall require the Künneth formula for analytic sheaves. The second section of this chapter deals with the Künneth formula. The third section deals with the description of Loeb and Nicolau's construction of complex structures on the product of odd dimensional spheres $\mathbb{S}^m \times \mathbb{S}^n$ for $m, n > 0$ [19].

2.1 Flag Manifolds

Let G be a connected complex Lie group. A homogeneous complex manifold $X = G/P$ is called a *flag manifold* if P is a parabolic subgroup of G. Since a parabolic subgroup contains radical of the group G, we can assume, without loss of generality that a flag manifold is of the form G/P, where G is a semisimple complex Lie group and P is a parabolic subgroup of G. For detailed expositions on complex homogeneous varieties we refer to [3] and [14].

We now fix some basic notations about complex Lie group.

Fix a maximal torus T and a Borel subgroup B, containing T, of a complex semisimple Lie group G. Let P be a parabolic subgroup containing B. Let

$\chi(T) := Hom(T, \mathbb{C}^*)$ be the group of character of T. Let $R \subset \chi(T)$ denote the root system of the pair (G, T) with $\mathfrak{g}_\mu$ the root space corresponding to a root $\mu \in R$. Let R^+ be the set of positive roots corresponding to the Borel subgroup B. Let $\Phi = \{\mu_1, \mu_2, \ldots, \mu_l\}$ be the set of simple roots. Corresponding to Φ, let $S = \{\varpi_1, \varpi_2, \ldots, \varpi_l\}$ be the set of fundamental weights. Let Λ be the weight lattice. When G is simply connected we have $\Lambda = \chi(T)$. Let $\Lambda^+ \subset \Lambda$ denote the set of dominant weights.

The set of parabolic subgroups are in one to one correspondence with the power set of Φ. For a weight $\varpi = \sum c_i \varpi_i$, consider the subset $J := \{\mu_i \mid c_i \neq 0\}$ of Φ. We denote P_ϖ for the parabolic subgroup which corresponds to 'omitting' the subset $\Phi \setminus J$ of the simple roots. Thus $P_\varpi = B$ if ϖ is regular.

For a dominant weight ϖ, we denote the corresponding finite dimensional irreducible G-module with the highest weight ϖ by $V(\varpi)$. Furthermore, for a dominant weight ϖ, the dual G-module $V(\varpi)^*$ is again an irreducible G-module with the highest weight $-w_0(\varpi)$, where w_0 is the element of largest length in the Wely group W of G with respect to T.

Example 2.1.1. For the semisimple complex Lie group $SL(n, \mathbb{C})$, the subgroup of upper triangular matrices is a Borel subgroup B. All maximal parabolic subgroups of $SL(n, \mathbb{C})$ containing B are of the form:

$$P_i := \{g = (g_{lm}) \in SL(n, \mathbb{C}) \mid g_{l,m} = 0 \text{ for } i + 1 \leq l \leq n, 1 \leq m \leq i\}$$

where $1 \leq i \leq n - 1$.

Example 2.1.2. For a parabolic subgroup P of the semisimple complex Lie group $SL(n, \mathbb{C})$, containing the Borel subgroup B of upper triangular matrices, there exist a subset $\{k_1, k_2, \ldots, k_r\}$ of $\{1, 2, \ldots, n - 1\}$ such that:

$$P = P_{k_1} \cap P_{k_2} \cap \ldots \cap P_{k_r}$$

where $P_{k_1}, P_{k_2}, \ldots, P_{k_r}$ are as defined in the Example 2.1.1.

Example 2.1.3. Let V be a complex vector space of dimension n. A flag F in V is :

$$F : 0 \subset V_{k_1} \subset V_{k_2} \subset \cdots \subset V_{k_r} \subset V$$

where V_{k_i} is of the dimension k_i. Let $\mathcal{F}$ be the set of all such flags in V. The group $SL(n, \mathbb{C})$ acts transitively on $\mathcal{F}$. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for the vector space V and let W_{k_i} be the subspace generated by $\{e_1, e_2, \ldots, e_{k_i}\}$. Then the isotropy subgroup of the flag $F : 0 \subset W_{k_1} \subset W_{k_2} \subset \cdots \subset W_{k_r} \subset V$ is a parabolic subgroup P. Thus $\mathcal{F}$ acquires the structure of a flag manifold $SL(n, k)/P$. In particular, when $r = 1$, we get the Grassmann manifold $\mathbb{G}_{n.k}$.

2.1.1 Line Bundles over a Flag Manifold

Let G be a simply connected semisimple complex Lie group. Let T be a maximal torus in G and let B be a Borel subgroup containing T. Any character $\varpi \in \chi(T)$ extends to a character $\varpi : B \to \mathbb{C}^*$ obtained by composing the natural map $B \to T$ (as $B = T.B_u$, B_u is unipotent part of B) with $\varpi : T \to \mathbb{C}^*$.

The character $\varpi : B \to \mathbb{C}^*$ can be further extended to give a character $\varpi : P_\varpi \longrightarrow \mathbb{C}^*$. For the character $\varpi : P_\varpi \longrightarrow \mathbb{C}^*$, one has a G-equivalent line bundle, whose total space is

$$G \times_{P_\varpi} \mathbb{C} := G \times \mathbb{C}/\sim, \text{ where } (gb, x) \sim (g, \varpi(b)x)$$

for $g \in G, x \in \mathbb{C}, b \in P_\varpi$. We denote this line bundle on G/P_ϖ by $\bar{L}_{-\varpi}$. These are all the line bundles over G/P_ϖ.

Theorem 2.1.4. *For a dominant weight ϖ,*

$$H^0(G/P_\varpi, \bar{L}_\varpi)^* \cong V(\varpi) \text{ and } H^0(G/P_\varpi, \bar{L}_\varpi) \cong V(-w_0(\varpi)). \quad \square$$

For a dominant weight ϖ, let v_ϖ be "the" highest weight vector in $V(\varpi)$. Let P be the subgroup of G which stabilizes one dimensional vector space $\mathbb{C}v_\varpi$. The subgroup P is a parabolic subgroup as the Borel subgroup B is contained in P and moreover $P = P_\varpi$. Every parabolic subgroup arises in this manner. This gives an algebraic embedding of $G/P_\varpi \hookrightarrow \mathbb{P}(V(\varpi))$. The line bundle $\bar{L}^*_\varpi = \bar{L}_{-\varpi}$ over G/P_ϖ is the pull back of the tautological bundle over $\mathbb{P}(V(\varpi))$.

2.1.2 Cone over a Flag Manifold

Let $Y \subset \mathbb{P}^n$ be a closed holomorphic submanifold. By the Theorem of Chow, Y is an algebraic projective variety. Let $I(Y)$ be the ideal consisting of all homogeneous polynomials in $n+1$ variables which vanishes on Y. Forgetting the homogeneous structure of $I(Y)$, let $\hat{Y} \subset \mathbb{C}^{n+1}$ be the affine subvariety corresponding to the ideal $I(Y)$. We call the affine variety $\hat{Y}$ as *the cone over* Y. We call $0 \in \hat{Y}$ the *vertex* of the cone $\hat{Y}$.

Let G be a semisimple complex Lie group. Fix a maximal torus T and a Borel subgroup B of G containing T. Let $\varpi \in \chi(T)$ be a dominant weight. Let P_ϖ be the parabolic subgroup associate to ϖ. Let $\bar{L}_\varpi$ be the line bundle associated to the weight ϖ. We have an embedding of the flag manifold G/P_ϖ into projective space $\mathbb{P}(V(\varpi))$ determined by the line bundle $\bar{L}_\varpi$. For this embedding the line bundle $\bar{L}_{-\varpi}$ over G/P_ϖ is the pullback of the tautological bundle over $\mathbb{P}(V(\varpi))$. We shall denote the cone over G/P_ϖ by $\hat{L}_{-\varpi}$. Then $\hat{L}_{-\varpi}$ is the affine subvariety of the affine space $V(\varpi)$.

The homogeneous coordinate ring of the projective variety G/P_ϖ for the projective embedding $G/P_\varpi \hookrightarrow \mathbb{P}(V(\varpi))$ is given by:

$$R = \oplus_{r\geq 0} H^0(G/P_\varpi, \bar{L}_{r\varpi}) = \oplus_{r\geq 0} V(r\varpi)^*.$$

Moreover, the cone $\hat{L}_{-\varpi}$ over G/P_ϖ is an affine variety with the coordinate ring R.

Let X be an analytic variety. We say X is Cohen-Macaulay (resp. normal) at a point $p \in X$, if the local ring $\mathcal{O}_{X,p}$ at the point p is Cohen-Macaulay i.e., depth $\mathcal{O}_{X,p} = \dim \mathcal{O}_{X,p}$ (resp. $\mathcal{O}_{X,p}$ is normal i.e., it is integrally closed domain). The analytic variety X is called Cohen-Macaulay (resp. normal) if it is Cohen-Macaulay (resp. normal) at all its points. Similarly, an algebraic variety is Cohen-Macaulay (resp. normal), if the local ring at all its point is Cohen-Macaulay (resp. normal).

An algebraic variety Y has a unique structure of an analytic variety. We shall show that the cone $\hat{L}_{-\varpi}$ over G/P_ϖ for the projective embedding $G/P_\varpi \hookrightarrow \mathbb{P}(V(\varpi))$ is Cohen-Macaulay and normal as an analytic variety.

For a point $p \in Y$ we denote the algebraic local ring by $\mathcal{O}^{alg}_{Y,p}$ and analytic local ring by $\mathcal{O}^{h}_{Y,p}$.

Lemma 2.1.5. *(1) $\mathcal{O}^{alg}_{Y,p}$ is Cohen-Macaulay if and only if $\mathcal{O}^{h}_{Y,p}$ is Cohen-Macaulay.*

(2) $\mathcal{O}^{alg}_{Y,p}$ is normal if and only if $\mathcal{O}^{h}_{Y,p}$ normal.

Proof. Since the completion of the local rings $\mathcal{O}^{alg}_{Y,p}$ and $\mathcal{O}^{h}_{Y,p}$ are the same, the first statement follows because of the fact that a local ring $(A, \mathfrak{m})$ is Cohen-Macaulay if and only if its completion $(\hat{A}, \hat{\mathfrak{m}})$ is Cohen-Macaulay. Second statement is a theorem by Zariski [33, p.320,Theorem 32]. □

The Lemma 2.1.5 implies that an algebraic variety Y is Cohen-Macaulay (resp. normal) if and only if it is Cohen-Macaulay (resp. normal) as an analytic variety.

Theorem 2.1.6. *[26] Let $\bar{L}_\varpi$ be an ample line bundle over G/P_ϖ. The projective variety G/P_ϖ is arithmetically Cohen-Macaulay with respect to $\bar{L}_\varpi$ i.e., the cone $\hat{L}_{-\varpi}$, is a Cohen-Macaulay affine algebraic variety.* □

Theorem 2.1.7. *[25] Let $\bar{L}_\varpi$ be an ample line bundle over G/P_ϖ. The projective variety G/P_ϖ is arithmetically normal with respect to $\bar{L}_\varpi$ i.e., the cone $\hat{L}_{-\varpi}$ is a normal affine algebraic variety.* □

The above results hold also for Schubert varieties in flag manifolds and also over any algebraically closed field of arbitrary characteristics.

From the Theorems 2.1.6, 2.1.7 and Lemma 2.1.5 it follows that the affine analytic variety $\hat{L}_{-\varpi}$ is Cohen-Macaulay and normal.

Let X be a smooth connected projective variety of dimension q and let $\bar{L}$ be a negative ample line bundle over X. Let X be arithmetically Cohen-Macaulay for the projective embedding determined by the ample line bundle $\bar{L}^*$. This means that the cone $\hat{L}$ is a Cohen-Macaulay affine analytic space. We denote the vertex of the cone $\hat{L}$ by a. We identify the space X with the image of the zero cross section of the line bundle $\bar{L} \longrightarrow X$. Let L be the total space of the holomorphic principal $\mathbb{C}^*$-bundle over X corresponding to line bundle $\bar{L}$. We have the following identification:

$$L = \bar{L} \setminus \{X\} = \hat{L} \setminus \{a\}.$$

We have the following computation of the cohomology groups $H^i(L, \mathcal{O}_L)$.

Proposition 2.1.8. *Let $\bar{L}$ be a negative ample line bundle over a smooth connected projective variety X of dimension q. Suppose X is arithmetically Cohen-Macaulay for the projective embedding determined by the ample line bundle $\bar{L}^*$. Then,*

$$H^i(L, \mathcal{O}_L) = 0 \textit{ for all } i \neq 0, q, \textit{ where } q = \dim X.$$

Moreover, $H^0(L, \mathcal{O}_L) \cong H^0(\hat{L}, \mathcal{O}_{\hat{L}})$.

Proof. The local ring at the vertex $a \in \hat{L}$ is Cohen-Macaulay ring. This means that the depth of the local ring $\mathcal{O}_{\hat{L},a}$ at the vertex a is $q+1$, i.e., depth $\mathcal{O}_{\hat{L},a} = q+1$. In this case, by [2, Corollary 3.9], the restriction maps

$$H^i(\hat{L}, \mathcal{O}_{\hat{L}}) \longrightarrow H^i(L, \mathcal{O}_L)$$

is isomorphism for $i < q$. Now since $\hat{L}$ is an affine analytic space and hence a Stein space, Cartan's Theorem B [6] implies that the cohomology groups $H^i(\hat{L}, \mathcal{O}_{\hat{L}})$ vanish for $i \neq 0$. This implies that $H^i(L, \mathcal{O}_L) = 0, 0 < i < q$.

Vanishing of the cohomology group $H^{q+1}(L, \mathcal{O}_L)$ follows by [9, Theorem 3.4] because L is non-compact and connected. All other higher cohomology groups $H^i(L, \mathcal{O}_L), i > q+1$, vanishes as L is a complex manifold of dimension $q+1$. ∎

2.2 Künneth Formula for Analytic Sheaves

The main aim of this section is to state the Künneth formula for the analytic coherent sheaves over complex manifolds. The Künneth formula will enable us to compute the cohomology groups of the complex manifolds that will be constructed in the later part of the thesis.

In the case of algebraic category we have the following Künneth formula due to J. H.Sampson and G.Washnitzer [28]. Let X and Y be algebraic varieties over $\mathbb{C}$. Let $\mathcal{F}$ and $\mathcal{G}$ be algebraic coherent sheaf on X and Y respectively. Then:

$$H^k(X \times Y, p_X^* \mathcal{F} \otimes_{\mathcal{O}} p_Y^* \mathcal{G}) \cong \bigoplus_{i+j=k} H^i(X, \mathcal{F}) \otimes_{\mathbb{C}} H^j(Y, \mathcal{G})$$

where $p_X : X \times Y \longrightarrow X$ and $p_Y : X \times Y \longrightarrow Y$ are the canonical projections and $\mathcal{O}$ is the structure sheaf on $X \times Y$. A similar kind of formula holds in the analytic case in the situation that we are in.

Before we state the formula in the analytic case we shall describe the basic ingredients involved in it, which includes the notion of completed tensor product of analytic Fréchet-nuclear coherent sheaves. For more details and examples we refer to [24].

A subset A of a linear space E over $\mathbb{C}$ is *absolutely convex* if $\alpha x + \beta y \in A$, whenever $x, y \in A$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| \leq 1$. If for each element $x \in E$ there is a positive number ϱ with $x \in \varrho A$ then A is called *absorbing.* Finally we shall call an absorbing absolutely convex subset A *central* if $x \in A$ whenever $\alpha x \in A$ for all $\alpha \in \mathbb{C}$ with $|\alpha| < 1$.

A *semi-norm* on a linear space E is a real valued function p with the following properties

(1) $p(x + y) \leq p(x) + p(y)$ for $x, y \in E$

(2) $p(\alpha x) = |\alpha| p(x)$ for $\alpha \in K$ and $x \in E$.

In each linear space E there is a one to one relation between semi-norms and central subsets. For each central subset A, the equation

$$p_A(x) = \inf\{\varrho > 0 \mid x \in \varrho A\} \text{ for } x \in E$$

determines a semi-norm p_A for which

$$A = \{x \in E \mid p_A(x) \leq A\}.$$

Conversely each semi-norm p can be obtained in this way from the central subset

$$A = \{x \in E \mid p(x) \leq 1\}.$$

Let $\wp(E)$ be a system of semi-norms p on a linear space E with the following properties:

(1) For any finitely many semi-norms $p_1, p_2, \ldots, p_n \in \wp(E)$ there is a semi-norm $p \in \wp(E)$ with $p_r(x) \leq p(x)$ for $x \in E$ and $r = 1, 2, \ldots, n$.

(2) For each element $x_0 \neq 0$ of E there is a semi-norm $p_0 \in \wp(E)$ with $p_0(x) > 0$.

The system $\wp(E)$ determine a topology by demanding that the all semi-norms $p \in \wp(E)$ be continuous. The linear space E which is made into a topological Hausdorff space in this way is called a *locally convex topological space*. The sets

$$U_\epsilon = \{x \in E \mid p(x) \leq \epsilon\} \text{ for } p \in \wp(E) \text{ and } \epsilon > 0$$

then form a fundamental system $\mathcal{U}_{\mathfrak{F}}(E)$ of central zero neighborhoods. The set $\mathfrak{U}(E)$ is the collection of all central subsets U for which there is a set $U_0 \in \mathcal{U}_{\mathfrak{F}}(E)$ with $U_0 \subset U$.

For a central subset U of a linear space E, if the semi-norm p_U,

$$p_U(x) = \inf\{\varrho > 0 \mid x \in \varrho U\} \text{ for } x \in E$$

is such that $p_U(x) > 0$ for $x \neq 0$, then the linear space E is a normed linear space with the norm p_U. Given a central subset $U \in \mathfrak{U}(E)$ in a locally convex space E, the quotient space $E(U) := E/p_U^{-1}(0)$ is a normed space and the norm is given by:

$$x \mapsto ||x|| := p_U(x).$$

Moreover, any normed linear space E with the norm p, is obtained in this way with $p = p_U$, where

$$U = \{x \in E \mid p(x) \leq 1\}$$

is a central closed subset. In this case U is the closed unit ball in the normed space E.

The *topological dual* of a locally convex space E is the linear subspace E' of the algebraic dual E^* which consists of all continuous linear forms. For every subset A of a locally convex space E we denote by A^0 the polar

$$A^0 = \{a \in E' \ : \ |\langle x, a\rangle| \leq 1 \text{ for } x \in A\}.$$

Definition 2.2.1. Let E and F be locally convex normed spaces, with closed unit balls $U \subset E$ and $V \subset F$. A linear operator $T : E \longrightarrow F$ is called *nuclear* if there are continuous linear forms $a_n \in E'$ and elements $y_n \in F$ with

$$\sum p_{U^0}(a_n) p_V(y_n) < +\infty$$

such that the T has the form

$$Tx = \sum_{i=1}^{\infty} \langle x, a_n \rangle y_n, \text{ for } x \in E$$

Definition 2.2.2.

1. A locally convex space E is called a *nuclear space* if for each $U \in \mathfrak{U}(E)$ there exists $V \in \mathfrak{U}(E)$ with $V \subset \varrho U$ for some positive number ϱ, such that the canonical mapping from $E(V)$ onto $E(U)$ of normed spaces is nuclear.

2. A *Fréchet space* is a locally convex topological space which is metrizable and complete.

A closed subspace of a Fréchet space (resp. nuclear space) is a Fréchet space (resp. nuclear space). A Hausdorff quotient, i.e. quotient of a topological vector space by its closed subspace, for a Fréchet space (resp. nuclear space) is again a Fréchet space (resp. nuclear space) with the usual quotient topology.

For a separable complex analytic space X and a coherent analytic sheaf $\mathcal{F}$ on X, the vector space $\Gamma(X, \mathcal{F})$, of global analytic sections of $\mathcal{F}$, has a natural structure of topological vector space with respect to which $\Gamma(X, \mathcal{F})$ is a Fréchet-nuclear space. For $\mathcal{F} = \mathcal{O}$, this topology on $\Gamma(X, \mathcal{F})$ is the same as the topology of uniform convergence of analytic functions on compact subsets of X.

2.2.1 Analytic Tensor Product

Let E and F be two linear locally convex topological space. On the algebraic tensor product $E \otimes F$ we shall construct two locally convex topologies using two system of semi-norms on $E \otimes F$. In the case when E and F are nuclear spaces, these two topologies turns out to be identical. In this case, the completion of this topology on $E \otimes F$ will be called the completed analytical tensor product of E and F.

For arbitrary central subsets $U \in \mathfrak{U}(E)$ and $V \in \mathfrak{U}(F)$ we associate a semi-norm $\pi_{(U,V)}$ by:

$$\pi_{(U,V)}(z) = \inf \left\{ \sum_{r=1}^{n} p_U(x_r) p_V(y_r) \right\}.$$

Here the infimum is taken over all possible representations of the element z in the form, $z = \sum_{r=1}^{n} x_r \otimes y_r$, with $x_r \in E$ and $y_r \in F$.

Similarly for $U \in \mathfrak{U}(E)$ and $V \in \mathfrak{U}(F)$ we associate a semi-norm $\epsilon_{(U,V)}$ as follow:

$$\epsilon_{(U,V)}(z) = \sup \left\{ \left| \sum_{r=1}^{n} \langle x_r, a\rangle \langle y_r, b\rangle \right| \;:\; a \in U^0, b \in V^0 \right\}.$$

The above expression of $\epsilon_{(U,V)}(z)$ is independent of the representation of the element $z = \sum_{r=1}^{n} x_r \otimes y_r$.

Using these two system of semi-norms we obtain two locally convex topologies on the algebraic tensor product $E \otimes F$, called π-topology, denoted by $E \otimes_\pi F$, and ϵ-topology, denoted by $E \otimes_\epsilon F$. The π-topology is finer than ϵ-topology. We denote the respective completion by $E\hat{\otimes}_\pi F$ and $E\hat{\otimes}_\epsilon F$.

Theorem 2.2.3. *If E and F are nuclear space, then $E \otimes_\pi F = E \otimes_\epsilon F$. Moreover, $E \otimes_\pi F = E \otimes_\epsilon F$ is nuclear space.* □

Under the hypothesis of the Theorem 2.2.3, we shall denote the completed analytic tensor product $E\hat{\otimes}_\pi F = E\hat{\otimes}_\epsilon F$ by $E\hat{\otimes}F$.

Example 2.2.4. If U is an open subset of $\mathbb{C}^n$ and V is an open subset of $\mathbb{C}^m$, then the set of holomorphic functions, $\Gamma(U \times V, \mathcal{O}_{U\times V}) \cong \Gamma(U, \mathcal{O}_U)\hat{\otimes}\Gamma(V.\mathcal{O}_V)$.

2.2.2 The Künneth Formula

Definition 2.2.5.

1. A coherent analytic sheaf $\mathcal{F}$ on a complex analytic space X is a *Fréchet sheaf* if $\mathcal{F}(U)$ is a Fréchet space for any open set $U \subset X$ and if the restriction homomorphism $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is continuous for any open set $V \subset U$.

2. A Fréchet sheaf $\mathcal{F}$ is said to be *nuclear* if $\mathcal{F}(U)$ is a nuclear space for any open set U in X.

3. A Fréchet sheaf $\mathcal{F}$ is called *normal* if there exists an open cover for X which is a Leray cover for $\mathcal{F}$ i.e there X can be covered with open sets U_i such that $H^k(U_i, \mathcal{F}) = 0$, for $k > 0$.

Lemma 2.2.6. *[7, p. 927]. For a complex manifold X, any coherent analytic sheaf is Fréchet-nuclear and normal.* □

On a complex space X with countable topology, one has the notion of completed tensor product $\mathcal{F}\hat{\otimes}\mathcal{G}$ of coherent analytic Fréchet nuclear sheaves $\mathcal{F}$ and $\mathcal{G}$. By definition $\mathcal{F}\hat{\otimes}\mathcal{G}(U) := \mathcal{F}(U)\hat{\otimes}\mathcal{G}(U)$, for U open in X.

Example 2.2.7. The structure sheaf of complex manifold is Fréchet nuclear and from the Example 2.2.4, it follows that $\mathcal{O}_{X\times Y} = pr_X^*\mathcal{O}_X\hat{\otimes}pr_Y^*\mathcal{O}_Y$, where pr_X denotes the projection $X \times Y \longrightarrow X$.

Let now X be a complex space with countable topology. Let $\mathcal{F}$ be a Fréchet-nuclear coherent sheaf on X. Let $\mathcal{U}$ be a countable Stein open covering of X. We have a canonical isomorphism $H^*(\mathcal{U},\mathcal{F}) \cong H^*(X,\mathcal{F})$. If we put on $\Gamma(U,\mathcal{F})$, for an arbitrary open set U of X, the usual Fréchet-nuclear topology, then $C^*(\mathcal{U},\mathcal{F})$ becomes a Fréchet nuclear space and so by passing to cohomology, we get the quotient topology on $H^*(X,\mathcal{F})$, which is generally not separated. The cohomology group $H^*(X,\mathcal{F})$ is Fréchet-nuclear locally convex space if the quotient topology obtained on $H^*(X,\mathcal{F})$ is separated.

We now state the Künneth formula in case when the cohomology groups $H^*(X,\mathcal{F})$ and $H^*(Y,\mathcal{G})$ are separated. For more general treatment see [7].

Theorem 2.2.8. *[7, Teorema 3] Let $\mathcal{F}$ and $\mathcal{G}$ be Fréchet-nuclear and normal coherent analytic sheaves on complex space X and Y. Assume that X and Y are second countable and that the cohomology groups $H^i(X,\mathcal{F})$ and $H^i(Y,\mathcal{G}), \forall\ i \geq 0$, are separated. Then for every non-negative integer k there exists a topological isomorphism*

$$H^k(X \times Y, \mathcal{F}\hat{\otimes}\mathcal{G}) \cong \bigoplus_{i+j=k} H^i(X,\mathcal{F})\hat{\otimes}H^j(Y,\mathcal{G})$$

of topological vector space. □

2.3 Complex Structures on $\mathbb{S}^{2n_1-1} \times \mathbb{S}^{2n_2-1}$

Let $S(L_i)$ be the $\mathbb{S}^1$-bundle over a complex manifold X_i associated to a holomorphic principal $\mathbb{C}^*$- bundle $L_i \longrightarrow X_i$ for $i = 1, 2$. We shall be concerned with complex structures on $S(L_1) \times S(L_2)$. We first consider case where $X_i := \mathbb{P}^{n_i-1}$, the projective spaces, for positive integer $n_i, i = 1, 2$, and $L_i := \mathbb{C}^{n_i} \setminus \{0\}$ the holomorphic tautological principal $\mathbb{C}^*$-bundle over $X_i, i = 1, 2$. For a fixed hermitian inner product on the vector space $\mathbb{C}^{n_i}$, the unit sphere $\mathbb{S}^{2n_i-1} \subset \mathbb{C}^{n_i} \setminus \{0\}$ is the total space of the $\mathbb{S}^1$-bundle $S(L_i)$.

Complex structures on the product of odd dimensional spheres $\mathbb{S}^{2n_1-1} \times \mathbb{S}^{2n_2-1}$ were first studied by Riemann for the case $n_1 = n_2 = 1$, by H. Hopf for the case $n_1 = 1, n_2 \geq 1$ and for the case $n_i > 1, i = 1, 2$ by Calabi-Eckmann [5]. Later, Loeb-Nicolau [19] constructed and studied a more general family of complex structure on $\mathbb{S}^{2n_1-1} \times \mathbb{S}^{2n_2-1}$, for positive integers n_1 and n_2. In this thesis we shall generalize the construction of Loeb and Nicolau to obtain a family of complex structures on the more general space $S(L_1) \times S(L_2)$.

2.3.1 Classical Examples

Elliptic Curves:

Complex structures on the compact torus $\mathbb{S}^1 \times \mathbb{S}^1$ are obtained by identifying it with the quotient space $\mathbb{C}^*/\mathbb{Z}$, where the $\mathbb{Z}$-action on $\mathbb{C}^*$ is generated by the automorphism: $z \mapsto \exp(2\pi\sqrt{-1}\tau)z$, for $\tau \in \mathbb{C}^*$ such that $\mathrm{Im}(\tau) > 0$. The $\mathbb{Z}$-action on $\mathbb{C}^*$ is properly discontinuous and free. Thus we obtain a complex structure on $\mathbb{C}^*/\mathbb{Z}$ and hence complex structure on $\mathbb{S}^1 \times \mathbb{S}^1$. The compact torus $\mathbb{S}^1 \times \mathbb{S}^1$, endowed with the above complex structure, and denoted by $\mathbb{E}_\tau$, is the elliptic curve. Moreover, elliptic curves are algebraic manifolds.

Hopf Manifolds

Hopf manifold is a compact complex manifolds which is the quotient $\mathbb{C}^n \setminus \{0\}/\mathbb{Z}$, for $n \geq 2$, where the $\mathbb{Z}$ action is generated by an automorphism g, obtained as follow: Fix complex numbers $\tau_j \in \mathbb{C}, 1 \leq j \leq n$, where $\mathrm{Im}(\tau_j) > 0$ for all j. Then for $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \setminus \{0\}$,

$$g(z) = (\exp(2\pi\sqrt{-1}\tau_1)z_1, \exp(2\pi\sqrt{-1}\tau_2)z_2, \ldots, \exp(2\pi\sqrt{-1}\tau_n)z_n).$$

As a differential manifold, the Hopf manifold is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{2n-1}$. See [16, p.49]. Complex manifolds considered by Hopf [13] are those when $\tau_j = \tau$ for all j, where τ is a fixed complex number with $\mathrm{Im}(\tau) > 0$.

Calabi-Eckmann Manifolds

Calabi and Eckmann [5] constructed complex structures on the product of odd dimensional spheres, $\mathbb{S}^{2n_1-1} \times \mathbb{S}^{2n_2-1}, n_1, n_2 \geq 2$. The complex structures were obtained by explicitly constructing holomorphic chart. The complex manifolds thus obtained are first examples of compact, simply connected, non-Kähler manifolds. The Calabi-Eckmann manifolds are total spaces of a holomorphic principal bundles over $\mathbb{P}^{n_1-1} \times \mathbb{P}^{n_2-1}$ with fibre an elliptic curve.

2.3.2 Loeb and Nicolau's Construction

A larger class of complex structures were obtained by Loeb and Nicolau [19] on the product of odd spheres $\mathbb{S}^{2n_1-1} \times \mathbb{S}^{2n_2-1}$ than those considered by Calabi-Eckmann [5]. Their construction was greatly inspired by Haefliger's paper [10]. We start this section by discussing the results of the Haefliger's paper [10] and then we shall describe Loeb and Nicola's construction.

A.Haefliger, in [10], described the versal deformation of transversely holomorphic foliations on the sphere $\mathbb{S}^{2N-1}$ induced by the holomorphic flows associated to certain vector fields on $\mathbb{C}^N$. Let ξ be a vector field such that $\xi(0) = 0$. Then

$$\xi(z) = \sum a_{ij} z_i \frac{\partial}{\partial z_j} + \sum z_i z_j f_{ijk}(z) \frac{\partial}{\partial z_k},$$

where $z = (z_1, z_2, \ldots, z_N)$ and $f_{ijk}(z), 1 \leq i, j, k \leq N$, are holomorphic functions on $\mathbb{C}^N$. We say ξ is in the *Poincaré domain* if the convex hull in $\mathbb{C}$ of the set of eigenvalues of the matrix (a_{ij}), associated to the linear part $\sum a_{ij} z_i \frac{\partial}{\partial z_j}$ of ξ, does not contain 0. The orbit of the holomorphic flow associated with a vector field ξ, which is in the Poincaré domain, will induce a one dimensional holomorphic foliation $\mathcal{F}_\xi$ on $U \setminus \{0\}$, where U is a small enough neighborhood of the origin. Haefliger showed that there exists an embedding $\mathbb{S}^{2N-1} \to U \setminus \{0\}$ transverse to leaves of the foliation $\mathcal{F}_\xi$. Thus the restriction of the foliation $\mathcal{F}_\xi$ induces a *transversely holomorphic* foliation $\mathcal{F}_\xi^0$ on the sphere $\mathbb{S}^{2N-1}$. Recall that a foliation $\mathcal{F}$ of codimension k on a differentiable manifolds M is defined by a family (U_i, ϕ_i) where $\{U_i\}$ is an open covering of M and $\phi_i : U_i \longrightarrow \mathbb{R}^k$ are smooth submersions such that there exist cocycle $\{g_{ij}\}$ of local transformation of $\mathbb{R}^k$ such that $\phi_i = g_{ij} \circ \phi_j$. The foliation $\mathcal{F}$ on M is said to be transversely holomorphic foliation of (complex) codimension q if submersions ϕ_i takes value in $\mathbb{C}^q = \mathbb{R}^k$ and the cocycle $\{g_{ij}\}$ are local holomorphic transformation of $\mathbb{C}^q$.

Let $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ be a sequence of non-zero complex numbers. A λ-*resonant* monomial vector field in $\mathbb{C}^N$ is a vector field of the form

$$a.z^{\mathbf{m}} \frac{\partial}{\partial z_k} = a.z_1^{m_1} z_2^{m_2} \cdots z_N^{m_N} \frac{\partial}{\partial z_k}, \qquad a \in \mathbb{C},$$

such that

$$\lambda_k = (\mathbf{m}, \lambda) := m_1\lambda_1 + m_2\lambda_2 + \cdots + m_N\lambda_N.$$

Let $\mathfrak{g}_\lambda$ be the set of vector fields commuting with the diagonal vector field $\xi_0 = \sum_{i=1}^N \lambda_i z_i \frac{\partial}{\partial z_i}$. The set $\mathfrak{g}_\lambda$ is a Lie subalgebra of the Lie algebra of holomorphic vector fields on $\mathbb{C}^N$. If the convex hull of $\lambda_1, \lambda_2, \ldots, \lambda_N$ does not contain $0 \in \mathbb{C}$, then $\mathfrak{g}_\lambda$ is of finite dimension. The set of λ-resonant monomial vector fields forms a basis for the subalgebra $\mathfrak{g}_\lambda$. Elements of $\mathfrak{g}_\lambda$ are called λ-resonant vector fields.

Given a vector field η in the Poincaré domain, let $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ be the set of eigenvalues of linear part of η. The theorem of Poincaré-Dulac [1, p.190] states that there is a biholomorphic map h defined in a neighborhood of the origin such that $h_*(\eta) := \xi$ is a λ-resonant vector field in *normal form*, i.e, ξ can be written as a sum

$$\xi = \xi_0 + \xi_1 + \xi_2 = \xi_0 + \sum_{(\mathbf{m}, \lambda) = \lambda_k} a_{\mathbf{m}}^s z^{\mathbf{m}} \frac{\partial}{\partial z_k} \tag{1}$$

where ξ_0 is the diagonal vector field $\sum_{i=1}^N \lambda_i z_i \frac{\partial}{\partial z_i}$, $\xi_0+\xi_1$ is the linear part of ξ in lower Jordan form and ξ_2 is the sum of λ-resonant monomial vector fields. By a further diagonal coordinate change by a diagonal matrix with diagonal entries $\epsilon^{-1}, \epsilon^{-2}, \ldots, \epsilon^{-N}$ for small enough ϵ, one can make the coefficients $a^s_{\mathbf{m}}$ of the non-diagonal part $\xi_1 + \xi_2$ as close to the zero as possible. In this case we shall say that non-diagonal part of ξ is close to the zero vector field.

Lemma 2.3.1. *Let $\xi_0 = \sum_{i=1}^N \lambda_i z_i \frac{\partial}{\partial z_i}$ be a diagonal vector field on $\mathbb{C}^N$ such that convex hull of $\lambda_1, \lambda_2, \ldots, \lambda_N$ does not contain 0. Then the leaves of the foliation $\mathcal{F}_{\xi_0}$ on $\mathbb{C}^0 \setminus \{0\}$ are transverse to the unit sphere $\mathbb{S}^{2N-1}$.*

Proof. Let $v = \sum_{i=1}^N p_i \frac{\partial}{\partial z_i}$. The real vector v is tangent to $\mathbb{S}^{2N-1}$ at a point $z = (z_1, z_2, \ldots, z_N)$ if and only if

$$\sum_{i=1}^N \mathrm{Re}(\bar{z}_i p_i) = 0.$$

Hence $\mathcal{F}_{\xi_0}$ is not transversal to $\mathbb{S}^{2N-1}$ at a point $z = (z_1, z_2, \ldots, z_N)$ if and only if for all non-zero $a \in \mathbb{C}^*$

$$\sum_{i=1}^N \mathrm{Re}(a\lambda_i)|z_i|^2 = 0.$$

This is possible only if $|z_1|^2\lambda_1 + |z_2|^2\lambda_2 + \cdots + |z_N|^2\lambda_N = 0$. Since $\sum |z_i|^2 = 1$ as $z \in \mathbb{S}^{2N-1}$, this implies that 0 belongs to the convex hull of $\lambda_1, \lambda_2, \ldots, \lambda_N$. Hence we get a contradiction. □

Now let ξ be a vector field on $\mathbb{C}^N$, which is in the Poincaré domain. In the view of the Poincaré -Dulac theorem, without lost of generality, we assume that ξ is a λ-resonant vector field in normal form (1) and that the non-diagonal part if ξ is sufficiently close to the zero vector field. The orbit of the flow associated to the vector field ξ induces a foliation $\mathcal{F}_\xi$ on $\mathbb{C}^N \setminus \{0\}$. Using the Lemma 2.3.1 it can be shown that the leaves of the foliation $\mathcal{F}_\xi$ are transverse to the unit sphere $\mathbb{S}^{2N-1}$ and hence induces a transversely holomorphic foliation $\mathcal{F}^0_\xi$ of dimension one on $\mathbb{S}^{2N-1}$. Haefliger obtained a versal deformation of the transversely holomorphic foliation $\mathcal{F}^0_\xi$ on $\mathbb{S}^{2N-1}$.

Theorem 2.3.2. *[10, p.243] Let ξ be a vector field in the Poincaré domain. Assume that ξ is λ-resonant vector field in normal form (1) and the non-diagonal part of ξ is close to zero vector field. Let S be a small enough neighborhood of 0 in a vector subspace of $\mathfrak{g}_\lambda$ complementary to the vector subspace generated by $[\xi, \mathfrak{g}_\lambda]$ and ξ.*

The family $\mathcal{F}^0_{\xi+s}$ of the transversely holomorphic foliations on $\mathbb{S}^{2N-1}$ obtained by intersecting the orbits of the flows generated by $\xi + s$ is a versal deformation of $\mathcal{F}^0_\xi$ parametrized by $s \in S$.

The above theorem is proved by showing that the Kodaira-Spencer map $\rho : T_0S \longrightarrow H^1(\mathbb{S}^{2N-1}, \theta^{tr}_{\mathcal{F}^0_\xi})$ is an isomorphism. Here $\theta^{tr}_{\mathcal{F}^0_\xi}$ is the sheaf of germs of transversely holomorphic vector fields for the foliation $\mathcal{F}^0_\xi$ on $\mathbb{S}^{2N-1}$.

Loeb and Nicolau obtained a transversely holomorphic foliation on $\mathbb{S}^{2n_1-1} \times \mathbb{S}^{2n_2-1}$ as in the same way by a mean of a vector field on $\mathbb{C}^N, N = n_1 + n_2$, satisfying certain hypotheses. The transversely holomorphic foliations on $S^{n_1,n_2} := \mathbb{S}^{2n_1-1} \times \mathbb{S}^{2n_2-1}$ turns out to be of zero dimensional and hence endows a complex structure on it. We now describe the construction.

A vector field ξ on $\mathbb{C}^N$ in Poincaré domain is called *weakly hyperbolic of type* (n_1, n_2) if the the set of eigenvalues, $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ of the linear part of ξ fulfills the following condition:

$$0 \leq \arg \lambda_i < \arg \lambda_j < \pi \text{ for } 1 \leq i \leq n_1 \text{ and } n_1 < j \leq n_1 + n_2 = N.$$

Upto to a change in the coordinate order and multiplication by a non-zero constant we can assume that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ fulfill the following *normalization* conditions

$$0 \leq \arg \lambda_1 \leq \arg \lambda_2 \leq \cdots \leq \arg \lambda_N < \pi$$

and

$$|\lambda_k| \leq |\lambda_{k+1}| \text{ if } \arg \lambda_k = \arg \lambda_{k+1}.$$

Theorem 2.3.3. *[19, Theorem 1] Let ξ be a holomorphic vector field, which in the Poincaré domain, defined in a neighborhood of origin in $\mathbb{C}^N$. Assume that ξ is weakly hyperbolic of type (n_1, n_2) and fulfilling the normalization conditions. Then there are two complex submanifolds Y_1 and Y_2 of dimension n_1, n_2 respectively, contained in a neighborhood U of the origin and which are saturated by the foliation $\mathcal{F}_\xi|U$. Moreover there is an embedding τ of $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ in $\mathbb{C}^N$ transverse to $\mathcal{F}_\xi|U$ and meets each leaf of $\mathcal{F}_\xi|U \setminus (Y_1 \cup Y_2)$ at exactly one point.* $\square$

Using the Poincaré-Dulac theorem we can assume that the vector field ξ is λ-resonant in the normal form (1) and furthermore, the non-diagonal part of ξ is sufficiently close to zero vector field. Under these assumptions on ξ, in the above theorem one can take $U = \mathbb{C}^N$, $Y_1 = \mathbb{C}^{n_1} \times \{0\}$, $Y_2 = \{0\} \times \mathbb{C}^{n_2}$ and τ to be the canonical embedding of S^{n_1,n_2} in $\mathbb{C}^N$.

Corollary 2.3.4. *[19, Corollary 2] Let ξ be a vector field on $\mathbb{C}^N$ in Poincaré domain satisfying the weak hyperbolicity condition and normalization conditions. Assume that ξ is a λ-resonant vector field in the normal form (1) and the non-diagonal part of ξ is sufficiently close to the zero vector field. Then the canonical embedding $i : S^{n_1,n_2} \longrightarrow \mathbb{C}^N$ induces a complex structure on S^{n_1,n_2}. This complex manifold, denoted by $S_\xi^{n_1,n_2}$, is naturally identified with the leaf space of $\mathcal{F}_\xi$. Moreover the vector field ξ defines on $(\mathbb{C}^{n_1} \setminus \{0\}) \times (\mathbb{C}^{n_2} \setminus \{0\})$ a structure of holomorphic principal $\mathbb{C}$- bundle over $S_\xi^{n_1,n_2}$.* □

The above corollary provides a general method of constructing complex structures on a product of odd dimensional spheres. By this method, the classical examples, namely elliptic curves, Hopf manifolds and Calabi-Eckmann manifolds, can also be obtained as particular cases. We describe this as follow.

Elliptic curve: Let $n_1 = n_2 = 1$ and take τ such that $\mathrm{Im}(\tau) > 0$. Consider the diagonal vector field $\xi = z_1\frac{\partial}{\partial z_1} + z_2\tau\frac{\partial}{\partial z_2}$ on $\mathbb{C}^2$. The $\mathbb{C}$-action on $\mathbb{C}^* \times \mathbb{C}^*$ induced by ξ is given by:

$$(t, z_1, z_2) \mapsto (\exp(t)z_1, \exp(\tau t)z_2).$$

Each orbit of this $\mathbb{C}$-action intersect $\{1\} \times \mathbb{C}^*$ and in this case, for $(1, z)$ and $(1, w)$ of $\{1\} \times \mathbb{C}^*$, $t.(1, z) = (1, w)$ only if $t = 2\pi\sqrt{-1}k$, for some $k \in \mathbb{Z}$. Hence the orbit space $(\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{C}$ is then identified to the quotient of $\{1\} \times \mathbb{C}^* \cong \mathbb{C}^*$ by the discrete subgroup of $\mathbb{C}^*$ generated by $\exp(2\pi\sqrt{-1}\tau)$. Any elliptic curve is obtained in this way.

Hopf manifolds: Hopf obtained a family of complex structures on $\mathbb{S}^1 \times \mathbb{S}^{2n-1}, n > 1$. These complex manifolds can again be obtained by the method of Loeb and Nicolau. For this case we take $n_1 = 1$ and $n_2 = n$. Let $\tau_1, \tau_2, \ldots, \tau_n$ be non-zero complex numbers such that $\mathrm{Im}(\tau_i) > 0$, for $1 \leq i \leq n$. Consider the diagonal vector field

$$\xi = z\frac{\partial}{\partial z} + w_1\tau_1\frac{\partial}{\partial w_1} + w_2\tau_2\frac{\partial}{\partial w_2} + \cdots + w_n\tau_n\frac{\partial}{\partial w_n}$$

on $\mathbb{C}^{n+1}$. The $\mathbb{C}$-action induced by ξ on $\mathbb{C}^* \times \mathbb{C}^n \setminus \{0\}$ is given by

$$(t, z_1, w_1, w_2, \ldots, w_n) \mapsto (\exp(t)z_1, \exp(\tau_1 t)w_1, \exp(\tau_2 t)w_2, \ldots, \exp(\tau_n t)w_n)$$

Each orbit of the $\mathbb{C}$-action on $\mathbb{C}^* \times \mathbb{C}^n \setminus \{0\}$ intersect $\{1\} \times \mathbb{C}^n \setminus \{0\}$. And as in the case of elliptic curve, the orbit space $(\mathbb{C}^* \times \mathbb{C}^n \setminus \{0\})/\mathbb{C}$ can be identified to quotient of $\{1\} \times \mathbb{C}^n \setminus \{0\} \cong \mathbb{C}^n \setminus \{0\}$ by the action of group $\mathbb{Z}$ generated by the automorphism

$$(w_1, w_2, \ldots, w_n) \mapsto (\exp(2\pi\sqrt{-1}\tau_1)w_1, \ldots, \exp(2\pi\sqrt{-1}\tau_n)w_n).$$

Thus we obtain a Hopf manifold.

Calabi-Eckmann manifolds: Calabi-Eckmann [5] constructed a family complex structures on $\mathbb{S}^{2n_1-1} \times \mathbb{S}^{2n_2-1}, n_1, n_2 > 1$. These manifolds are the total spaces of a principal bundles over $\mathbb{P}^{n_1-1} \times \mathbb{P}^{n_2-1}$ with fibre an elliptic curve. They can be viewed as the orbit space of the $\mathbb{C}$-action on $(\mathbb{C}^{n_1} \setminus \{0\}) \times (\mathbb{C}^{n_2} \setminus \{0\})$, induced by the vector field

$$\xi = \sum_{i=1}^{n_1} z_i \frac{\partial}{\partial z_i} + \tau(\sum_{j=1}^{n_2} w_j \frac{\partial}{\partial w_j})$$

for non-zero τ such that $\mathrm{Im}(\tau) > 0$.

Loeb and Nicolau studied the Dolbeault's cohomology and deformation of the complex manifolds $S_\xi^{n_1,n_2}$, where the vector field ξ is as in the Theorem 2.3.3. A versal deformation were obtained much the same way as by Haefliger [10]. Using the cohomology computation, they gave the description of the holomorphic principal $\mathbb{C}$-bundle and the holomorphic line bundles over $S_\xi^{n_1,n_2}$.

The open subset $\mathbb{C}^{n_i} \setminus \{0\}$ of $\mathbb{C}^{n_i}$, for $i = 1, 2$, is the total space of the tautological principal $\mathbb{C}^*$-bundle over the projective space $\mathbb{P}^{n_1-1}$. For a fixed hermitian inner product on $\mathbb{C}^{n_i}$, the unit sphere $\mathbb{S}^{2n_i-1}$ is the total space of the corresponding circle bundle over $\mathbb{P}^{n_i-1}$. Now, by the assertion of the Corollary 2.3.4, complex structures on $\mathbb{S}^{2n_1-1} \times \mathbb{S}^{2n_2-1}$ are obtained by identifying it to the quotient of $(\mathbb{C}^{n_1} \setminus \{0\}) \times (\mathbb{C}^{n_2} \setminus \{0\})$ by a $\mathbb{C}$-action. This $\mathbb{C}$-action is a flow associated to a vector field in the Poincaré domain, on the product $(\mathbb{C}^{n_1} \setminus \{0\}) \times (\mathbb{C}^{n_2} \setminus \{0\})$.

We extend this idea of Loeb and Nicolau to obtain a family of complex structures on the product $S(L_1) \times S(L_2)$, where $S(L_i)$ is the circle bundle associated to a holomorphic principal $\mathbb{C}^*$-bundle $L_i \longrightarrow X_i$ over complex manifold X_i for $i = 1, 2$. For this we consider an *admissible* $\mathbb{C}$-action (see Definition 1.2.2) on the product $L_1 \times L_2$. We show that each orbit of an admissible $\mathbb{C}$-action intersects $S(L_1) \times S(L_2)$ transversely and at exactly one point. Thus by identifying $S(L_1) \times S(L_2)$ with the quotient space $(L_1 \times L_2)/\mathbb{C}$, we get a complex structure on $S(L_1) \times S(L_2)$ in a natural way. Moreover, $L_1 \times L_2$ is viewed as holomorphic principal $\mathbb{C}$-bundle over the obtained complex manifold $S(L_1) \times S(L_2)$. The complex structures obtained on $S(L_1) \times S(L_2)$ will correspond to complex structures on S^{n_1,n_2} obtained by Loeb and Nicolau [19] by considering only linear vector fields.

Chapter 3

Basic Construction

Let X_1, X_2 be any two compact complex manifolds and let $p_1 : L_1 \longrightarrow X_1$ and $p_2 : L_2 \longrightarrow X_2$ be holomorphic principal $\mathbb{C}^*$-bundles over X_1 and X_2 respectively. Denote by $p : L_1 \times L_2 =: L \longrightarrow X := X_1 \times X_2$ the product $\mathbb{C}^* \times \mathbb{C}^*$-bundle. We shall denote by $\bar{L}_i$ the line bundle associated to L_i and identify X_i with the zero cross-section in L_i so that $L_i = \bar{L}_i \setminus X_i$. We put a hermitian metric on L_i and denote by $S(L_i) \subset L_i$ the unit sphere bundle with fibre and structure group $\mathbb{S}^1$. We shall denote by $S(L)$ the compact torus $\mathbb{S}^1 \times \mathbb{S}^1$-bundle $S(L_1) \times S(L_2) \longrightarrow X$. Our aim is to study complex structures on $S(L) := S(L_1) \times S(L_2)$.

Inspired by Loeb and Nicolau's construction [19], we shall obtain complex structures on $S(L)$ by identifying it with the orbit space $L/\mathbb{C}$, where the $\mathbb{C}$-action on L is the flow associated to certain holomorphic vector fields.

In this section we consider holomorphic $\mathbb{C}$-actions on $L_1 \times L_2$ which lead to a complex structure on $S(L)$ of *scalar* and *diagonal* types. The scalar type complex structures always exist. The construction of diagonal type complex structures involves the notion of standard action by a torus $(\mathbb{C}^*)^{n_i}$, $n_i > 0, i = 1, 2$, on a principal $\mathbb{C}^*$-bundle L_i over a complex manifold X_i. See Definition 1.2.1. Such standard actions always exists in the case when $X_i, i = 1, 2$ are flag manifolds. In the case when X_i are flag manifolds, we shall use the setup to construct *linear type* complex structures on $S(L)$ (see Chapter 4).

3.1 Examples of Standard Action of a Torus

Let $E \longrightarrow B$ be a holomorphic principal $\mathbb{C}^*$-bundle over a complex manifold B. To the principal bundle we associate a line bundle $\bar{E} \longrightarrow B$, where the total space $\bar{E} := E \times_{\mathbb{C}^*} \mathbb{C}$ is constructed from $E \times \mathbb{C}$ by the identification :

$$(e, z) \sim (et^{-1}, tz) \text{ for } (e, z) \in E \times \mathbb{C} \text{ and } t \in \mathbb{C}^*.$$

Identifying B with the image of the zero cross section of the line bundle $\bar{E}$, we have $E = \bar{E} \setminus B$.

Let T be a complex torus group. We identify T with $(\mathbb{C}^*)^n$ by choosing an isomorphism $T \cong (\mathbb{C}^*)^n$ for some n. We shall denote by $\epsilon_j : \mathbb{C}^* \subset (\mathbb{C}^*)^n$ the inclusion of the jth factor and write $t\epsilon_j$ to denote $\epsilon_j(t)$ for $1 \leq j \leq n$. Thus any $(t_1, \cdots, t_n) \in T$ equals $\prod_{1 \leq j \leq n} t_j \epsilon_j$. One has the polar decomposition $T = (\mathbb{S}^1)^n \times \mathbb{R}^n_+$.

Suppose that the total space $\bar{E}$ and the base space B of the associated line bundle are acted upon holomorphically by the torus $T(\cong (\mathbb{C}^*)^n)$ such the line bundle $\bar{E} \longrightarrow B$ is T equivariant. Fix a hermitian metric on the line bundle $\bar{E}$ which is invariant under the maximal compact subgroup $(\cong (\mathbb{S}^1)^n)$ of T. Let d be a positive integer. Recall the definition of standard d-action of T on E from Definition 1.2.1. In this section we give examples of such actions.

Note that condition (i) in the Definition 1.2.1 implies that the Δ-orbit of any $e \in E$ is just the fibre of the bundle $E \longrightarrow B$ containing e. The exact value of d will not be of much significance for us. However, it will be too restrictive to assume $d = 1$. See Chapter 4.

We shall see below some examples of a standard action of a torus on a holomorphic principal $\mathbb{C}^*$-bundle. For a line bundle γ over a complex manifold B, we shall denote the total space of γ by $\bar{E}(\gamma)$. The total space of the corresponding holomorphic principal $\mathbb{C}^*$-bundle is denoted by $E(\gamma)$.

Example 3.1.1. For any principal $\mathbb{C}^*$-bundle $E \longrightarrow B$, the corresponding line bundle $\bar{E} \longrightarrow B$ is $\mathbb{C}^*$-equivariant, where the $\mathbb{C}^*$-action on $\bar{E}$ is that of the structure group. For a hermitian metric on the line bundle $\bar{E} \longrightarrow B$, we have : $||(t.e)|| = |t|||e||$, for $t \in \mathbb{C}^*, e \in \bar{E}$. It readily follows that this action gives a d-standard action on the principal $\mathbb{C}^*$-bundle $E \longrightarrow B$, where $d = 1$ in this case.

Example 3.1.2. Consider a complex projective space $\mathbb{P}^{n-1}$. Let $\gamma_{n,1}$ be the tautological line bundle over $\mathbb{P}^{n-1}$. The total space $\bar{E}$ of the bundle $\gamma_{n,1}$ is the subset of $\mathbb{P}^{n-1} \times \mathbb{C}^n$ define by:

$$\bar{E} := \{(V, v) \mid v \in V\}.$$

The natural action of the torus T:= $(\mathbb{C}^*)^n$ on $\mathbb{C}^n$ given by

$$t.(z_1, z_2, \ldots, z_n) = (t_1 z_1, t_2 z_2, \ldots, t_n z_n), \text{ for } t = (t_1, t_2, \ldots, t_n) \in (\mathbb{C}^*)^n$$

can be extended to give an action of $(\mathbb{C}^*)^n$ on $\mathbb{P}^{n-1} \times \mathbb{C}^n$ given by

$$(t_1, t_2, \ldots, t_2).([z_1 : z_2 : \ldots : z_n], (z_1, z_2, \ldots, z_n)) =$$

$$([t_1 z_1 : t_2 z_2 : \ldots : t_n z_n], (t_1 z_1, t_2 z_2, \ldots, t_n z_n)).$$

This action can be restricted on $\bar{E}$ to give an action of T on the line bundle $\bar{E} \longrightarrow \mathbb{P}^{n-1}$. The corresponding holomorphic principal $\mathbb{C}^*$-bundle is $\mathbb{C}^n \setminus \{0\} := E \longrightarrow \mathbb{P}^{n-1}$. The standard hermitian metric on $\mathbb{C}^n$,

$$||(z_1, z_2, \ldots, z_n)|| = \sqrt{\sum_i |z_i|^2},$$

yields a hermitian metric on the line bundle $\bar{E}$. The induced T action on the holomorphic principal $\mathbb{C}^*$-bundle, $\mathbb{C}^n \setminus \{0\} \longrightarrow \mathbb{P}^{n-1}$, is d-standard with $d = 1$.

Example 3.1.3. Consider a collection of n monomials $P_1, P_2, \ldots, P_N$, all of fixed degree $d > 0$, in variable $t_1, t_2, \ldots, t_n$ for some positive integer n. Suppose that for each variable t_i, there exists j such that P_j is dependent on t_i, i.e, positive exponent of t_i occurs in the monomial P_j. As in the Example 3.1.2 we consider a more general action of a torus group $T := (\mathbb{C}^*)^n$ on $\mathbb{C}^N$ define as :

$$t.(z_1, z_2, \ldots, z_N) = (P_1(t)z_1, P_2(t)z_2, \ldots, P_N(t)z_N), \text{ for } t := (t_1, t_2, \ldots, t_n).$$

This torus T action on $\mathbb{C}^N$ can be extended to give a T action on the line bundle $\bar{E}(\gamma_{N,1}) \longrightarrow \mathbb{P}^{N-1}$. Under our hypothesis on the polynomials $P_1, P_2, \ldots, P_N$, it is clear that the induced torus T action on the principal $\mathbb{C}^*$-bundle $\mathbb{C}^N \setminus \{0\}$ is d-standard. Here we take the hermitian metric on the line bundle $\gamma_{N,1}$ as that in the previous Example 3.1.2.

Next we shall give some examples of standard action of a torus group on certain holomorphic principal $\mathbb{C}^*$-bundle E over a flag manifold $SL(n, \mathbb{C})/P$ for a parabolic subgroup P. The bundle E over $SL(n, \mathbb{C})/P$ is assumed to be such that there is a T-equivariant embedding of $SL(n, \mathbb{C})/P$ into a projective space $\mathbb{P}^{N-1}$, for some large positive number N, such that the tautological principal $\mathbb{C}^*$-bundle $\mathbb{C}^N \setminus \{0\} \longrightarrow \mathbb{P}^{N-1}$ restricts to E on $SL(n, \mathbb{C})/P$. Here $T = (\mathbb{C}^*)^n$. The T action on the principal $\mathbb{C}^*$-bundle $\mathbb{C}^N \setminus \{0\} \longrightarrow \mathbb{P}^{N-1}$ turns out to be d-standard and thus we shall obtain a d-standard action on the principal $\mathbb{C}^*$-bundle $E \longrightarrow SL(n, \mathbb{C})/P$, for some d.

Example 3.1.4. Let $\gamma_{n,k}$ be the k-plane bundle over the Grassmannian $\mathbb{G}_{n,k}$ of k-dimensional subspace of $\mathbb{C}^n$. The total space of $\gamma_{n,k}$ is the subset of $\mathbb{G}_{n,k} \times \mathbb{C}^n$ define as :

$$\{(V, v) \mid v \in V\}.$$

The k^{th}-exterior bundle $\wedge^k \gamma_{n,k}$, is the "tautological" line bundle over $\mathbb{G}_{n,k}$ with the total space:

$$\bar{E}(\wedge^k \gamma_{n,k}) := \{(V, v_1 \wedge v_2 \wedge \ldots \wedge v_k) \mid V \subset \mathbb{C}^n \text{of dim } k \text{ and } v_i \in V, 1 \leq i \leq k\}.$$

The natural action of the torus $(\mathbb{C}^*)^n$ on $\mathbb{C}^n$ given by

$$t.(z_1, z_2, \ldots, z_n) = (t_1 z_1, t_2 z_2, \ldots, t_n z_n), \text{ for } t = (t_1, t_2, \ldots, t_n) \in (\mathbb{C}^*)^n$$

can be extended to give an action of $(\mathbb{C}^*)^n$ on $\bar{E}(\wedge^k \gamma_{n,k})$,

$$t.(V, v_1 \wedge v_2 \wedge \ldots \wedge v_k) \mapsto (t.V, t.v_1 \wedge t.v_2 \wedge \ldots \wedge t.v_k),$$

where $t.V$ is the image of V under the action of t on $\mathbb{C}^n$. We have the following embedding of the line bundles for $N = \binom{n}{k}$:

$$\begin{array}{ccc} \bar{E}(\wedge^k \gamma_{n,k}) & \hookrightarrow & \bar{E}(\gamma_{N,1}) \\ \downarrow & & \downarrow \\ \mathbb{G}_{n,k} & \hookrightarrow & \mathbb{P}(\wedge^k \mathbb{C}^n) \end{array}$$

The embedding $\mathbb{G}_{n,k} \hookrightarrow \mathbb{P}(\wedge^k \mathbb{C}^n)$ is the Plücker embedding given by:

$$V \mapsto \mathbb{C}.v_1 \wedge v_2 \wedge \ldots \wedge v_k, \text{ the line spanned by } v_1 \wedge v_2 \wedge \ldots \wedge v_k,$$

where $\{v_1, v_2, \ldots, v_k\}$ is a set of basis for the subspace V and the embedding is independent of the choice of the basis. The corresponding embedding of the principal $\mathbb{C}^*$-bundle $E(\wedge^k \gamma_{n,k}) \hookrightarrow E(\gamma_{N,1}) = \wedge^k \mathbb{C}^n \setminus \{0\}$ is given by:

$$(V, v_1 \wedge v_2 \wedge \ldots \wedge v_k) \mapsto v_1 \wedge v_2 \wedge \ldots \wedge v_k$$

where $\{v_1, v_2, \ldots, v_k\}$ is a set of basis for the subspace V.

We denote by $I(n, k)$ the collection

$$\{(i_1, i_2, \ldots, i_k) \mid 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}.$$

For the standard basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{C}^n$, the set

$$\{e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mid I = (i_1, i_2, \ldots i_k) \in I(n, k)\}$$

is a basis for the vector space $\wedge^k\mathbb{C}^n$. The action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^n$ induces an action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^N \cong \wedge^k\mathbb{C}^n, N = \binom{n}{k}$, in a natural way. Namely, on a basis vector e_I,

$$t.e_I = t.(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) = t.e_{i_1} \wedge t.e_{i_2} \wedge \cdots \wedge t.e_{i_k} = P_I(t).e_I$$

where $P_I := t_{i_1}t_{i_2}\dots t_{i_k}$, for $(i_1, i_2, \dots, i_k) \in I$, is monomials in $t_1, t_2, \cdots, t_n$. More generally,

$$t.(\sum_{I\in I(n,k)} z_I e_I) = \sum_{I\in I(n,k)} P_I(t) z_I e_I, \text{ for } t = (t_1, t_2, \dots, t_n) \in \mathbb{C}^n.$$

It is clear that the embedding $E(\wedge^k\gamma_{n,k}) \hookrightarrow E(\gamma_{N,1}) = \wedge^k\mathbb{C}^n \setminus \{0\}$ is T-equivariant. Using the monomials $P_1, P_2, \dots, P_N$, as in the Example 3.1.3, the k-standard action of $(\mathbb{C}^*)^n$ obtained on the principal $\mathbb{C}^*$-bundle $E(\gamma_{N,1}) = \wedge^k\mathbb{C}^n \setminus \{0\} \longrightarrow \mathbb{P}(\wedge^k\mathbb{C}^n)$ restricts to give a k-standard action of the torus $(\mathbb{C}^*)^n$ on the principal $\mathbb{C}^*$-bundle $E(\wedge^k\gamma_{n,k}) \longrightarrow \mathbb{G}_{n,k}$.

Remark 3.1.5. In the above example, the vector space $\wedge^k(\mathbb{C}^n)$ can be decomposed into sum of one dimensional weight spaces corresponding to the action of torus $(\mathbb{C}^*)^n$ on $\mathbb{C}^N$. Each basis element $e_I, I \in I(n,k)$, generates the weight space of with character $P_I : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}^*$.

Example 3.1.6. Any negative ample line bundle over a projective space $\mathbb{P}^{n-1}$ is of the form $\gamma_{n,1}^{\otimes d}$ for some positive d. Let $P_1, P_2, \dots, P_N$ be all the monomials of degree d in n variables, where $N := \binom{d+n-1}{d}$. For the line bundle $\gamma_{n,1}^{\otimes d}$ we have the *d-tuple* embedding $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{N-1} := \mathbb{P}(sym^d(\mathbb{C}^n))$ given as

$$[z_1 : z_2 : \dots : z_n] \mapsto [P_1 : P_2 : \cdots : P_N].$$

For the d-tuple embedding, the line bundle $\gamma_{n,1}^{\otimes d}$ over $\mathbb{P}^{n-1}$ is the pullback of the tautological bundle $\gamma_{N,1}$ over $\mathbb{P}^{N-1}$ and we have the following diagram for the holomorphic principal $\mathbb{C}^*$-bundles,

$$\begin{array}{ccc} E(\gamma_{n,1}^{\otimes d}) & \hookrightarrow & E(\gamma_{N,1}) \\ \downarrow & & \downarrow \\ \mathbb{P}^{n-1} & \hookrightarrow & \mathbb{P}(sym^d\mathbb{C}^n) \end{array}$$

The natural action of the torus $T := (\mathbb{C}^*)^n$ on $\mathbb{C}^n$ extends to give an action of T on $E(\gamma_{n,1}^{\otimes d})$ and $E(\gamma_{N,1}) = sym^d(\mathbb{C}^n) \setminus \{0\}$. Moreover the embedding $E(\gamma_{n,1}^{\otimes d}) \hookrightarrow E(\gamma_{N,1})$ is T-equivariant. Using the monomials $P_1, P_2, \dots, P_N$, as in the Example 3.1.3, we can construct a d-standard action of the torus $(\mathbb{C}^*)^n$ on the holomorphic principal $\mathbb{C}^*$-bundle $E(\gamma_{N,1}) \longrightarrow \mathbb{P}^{N-1}$. This $(\mathbb{C}^*)^n$-action on $E(\gamma_{N,1})$ restricts to give a d-standard action on the principal $\mathbb{C}^*$-bundle $E(\gamma_{n,1}^{\otimes d}) \longrightarrow \mathbb{P}^{n-1}$.

Remark 3.1.7. For any sequence $J := \{i_1 \leq i_2 \leq \ldots \leq i_d\}, 1 \leq i_r \leq n$ the standard basis element $e_J = e_{i_1} e_{i_2} \cdots e_{i_d}$ of $sym^d(\mathbb{C}^n)$ generates the weight space with character $P_J : T \longrightarrow \mathbb{C}^*; (t_1, t_2, \ldots t_n) \mapsto t_{i_1} t_{i_2} \ldots t_{i_d}$. The set of all monomials $\{P_J\}$ of degree d in n variables $t_1, t_2, \ldots, t_n$ are T-weights of $sym^d(\mathbb{C}^n)$. (cf. 3.1.5).

Example 3.1.8. More generally, let $\mathcal{F}$ be the set of flags, in a vector space V, of the form

$$F : 0 \subset V_{k_1} \subset V_{k_2} \subset \cdots \subset V_{k_r} = V,$$

where V_{k_i} is of the dimension k_i (see Example 2.1.3). The set $\mathcal{F}$ acquires the structure of a flag manifold $SL(n, \mathbb{C})/P$ for a parabolic subgroup P. Conversely, for any parabolic subgroup P of $SL(n, \mathbb{C})$, the flag manifold $SL(n, \mathbb{C})/P$ can be identified to a set of flags. We have a projection map of the flag manifold $SL(n, \mathbb{C})/P$ onto the Grassmannian $\mathbb{G}_{n,k_i}$

$$\begin{array}{rccc} \sigma_i : & SL(n,\mathbb{C})/P & \to & \mathbb{G}_{n,k_i} \\ & F & \mapsto & V_{k_i} \end{array}$$

Any negative ample line bundle over a Grassmannian $\mathbb{G}_{n,k_i}$ is of the form $(\wedge^{k_i} \gamma_{n,k_i})^{\otimes d_i}$, for some positive integer d_i. A negative ample line bundle γ over $SL(n, \mathbb{C})/P$ is of the form $\otimes_i (\sigma_i^*((\wedge^{k_i} \gamma_{n,k_i})^{\otimes d_i}))$. For this line bundle we have an embedding of the flag manifold $SL(n, \mathbb{C})/P$ into a projective space $\mathbb{P}^{N-1} := \mathbb{P}(\otimes_i (sym^{d_i}(\wedge^{k_i} \mathbb{C}^n)))$ such that the line bundle γ over $SL(n, \mathbb{C})/P$ is the pullback of the tautological bundle over $\mathbb{P}^{N-1}$. We have,

$$\begin{array}{ccc} E(\gamma) & \hookrightarrow & E(\gamma_{N,1}) \\ \downarrow & & \downarrow \\ SL(n,\mathbb{C})/P & \hookrightarrow & \mathbb{P}(\otimes_i(sym^{d_i}(\wedge^{k_i}\mathbb{C}^n))) \end{array}$$

Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for the vector space $\mathbb{C}^n$. Let $\{f_1, f_2, \ldots, f_N\}$ be the corresponding basis for the vector space $W := \otimes_i sym^{d_i}(\wedge^{k_i} \mathbb{C}^n))$. Now the natural action of $T := (\mathbb{C}^*)^n$ on $\mathbb{C}^n$ yields an action of torus $(\mathbb{C}^*)^n$ on W. Under this action, each basis element f_i of W generates a weight space with character $P_i : T \longrightarrow \mathbb{C}^*, 1 \leq i \leq N$. The set of characters $P_1, P_2, \ldots, P_N$ are monomials of degree $d = \sum_i k_i d_i$ in n variables $t_1, t_2, \ldots t_n$. As in the Example 3.1.3, we obtain a d-standard action of the torus $(\mathbb{C}^*)^n$ on the principal $\mathbb{C}^*$-bundle $E(\gamma_{N,1}) := \mathbb{C}^N \setminus \{0\} \longrightarrow \mathbb{P}(W)$. This action restricts to give a d-standard action of the torus $(\mathbb{C}^*)^n$ on the principal $\mathbb{C}^*$-bundle $E(\gamma) \longrightarrow SL(n, \mathbb{C})/P$.

In Chapter 4, we shall construct a d-standard action of a torus on a negative ample line bundle over a more general flag manifold G/P where G is a semi-simple algebraic group and P is a parabolic subgroup of G. We use the similar argument as in above examples to obtain the standard action.

Lemma 3.1.9. *Suppose that $E \longrightarrow B$ be a principal $\mathbb{C}^*$-bundle with a d-standard action of $T(\cong (\mathbb{C}^*)^n)$. Then:*

(i) One has $||t\epsilon_j.v|| \leq ||v||$ for $0 < |t| < 1$ where equality holds if and only if $\mathbb{R}^+\epsilon_j$ is contained in the isotropy at v.

(ii) For any $t = (t_1, \ldots, t_n) \in T$, one has

$$|t_{k_0}|^d.||v|| \leq ||t.v|| \leq |t_{j_0}|^d.||v||, \forall t \in T, \forall\ v \in E,$$

where $j_0 \leq n$ (resp. k_0) is such that $|t_{j_0}| \geq |t_j|$ (resp. $|t_{k_0}| \leq |t_j|$) for all $1 \leq j \leq n$. Also $||t.v|| = |t_{j_0}|^d.||v||$ if and only if $|t_j| = |t_{j_0}|$ for all j such that $(t_j/t_{j_0})\epsilon_j.v \neq v$ and $||t.v|| = |t_{k_0}|^d.||v||$ if and only if $|t_j| = |t_{k_0}|$ for all j such that $(t_j/t_{k_0})\epsilon_j.v \neq v$.

Proof. (i) Suppose that $\mathbb{R}_+\epsilon_j$ is not contained in the isotropy at v. Since the compact subgroup $(\mathbb{S}^1)^n \subset T$ preserves the norm, we may assume that $t \in \mathbb{R}_+$. In view of 1.2.1(ii), $\nu_{v,j}$ is strictly increasing. Hence $||t\epsilon_j.v|| < ||v||$ for $0 < t < 1$.

(ii) Write $s = (s_1, \ldots, s_{n_1})$ where $s_j = t_j/t_{j_0}\ \forall j$. Denoting the diagonal imbedding $\mathbb{C}^* \longrightarrow T$ by δ, we have $t = \delta(t_{j_0})s$. Now $\delta(t_{j_0}).v = t_{j_0}^d v$ in view of 1.2.1 (i).

By repeated application of (i) above, we see that $||t.v|| = ||s(\delta(t_{j_0})v)|| = ||s.t_{j_0}^d v|| \leq |t_{j_0}|^d.||v||$ where the inequality is strict unless $|t_j| = |t_{j_0}|$ for all j such that $s_j\epsilon_j.v \neq v$. A similar proof establishes the inequality $||t.v|| \geq |t_{k_0}|^d.||v||$ as well as the condition for equality to hold. □

3.2 Admissible $\mathbb{C}$-action

Let $L_i \longrightarrow X_i, i = 1, 2$, be a holomorphic principal $\mathbb{C}^*$-bundle such that the corresponding line bundle $\bar{L}_i \longrightarrow X_i, i = 1, 2$, is a $(\mathbb{C}^*)^{n_i} =: T_i$-equivariant line bundle. Let the action of T_i on $L_i \longrightarrow X_i$ be d_i-standard. We denote the torus $T_1 \times T_2$ by T. We get a principal $\mathbb{C}^* \times \mathbb{C}^*$-bundle $L := L_1 \times L_2 \longrightarrow X_1 \times X_2 =: X$, with the torus $T := T_1 \times T_2$ action on $L = L_1 \times L_2$. In this section we consider holomorphic $\mathbb{C}$-actions on L which lead to complex structure on $S(L) := S(L_1) \times S(L_2)$ of *scalar* and *diagonal* types. Whereas scalar type complex structures always exist, in order to obtain diagonal type complex structure we need additional hypotheses.

Let $\lambda \in Lie(T) = \mathbb{C}^N, N := n_1 + n_2$. There exists a unique Lie group homomorphism $\alpha_\lambda : \mathbb{C} \longrightarrow T$ defined as

$$z \mapsto \exp(z\lambda).$$

When λ is clear from the context, we write α to mean α_λ. We denote by $\alpha_{\lambda,i}$ (or more briefly α_i) the composition $\mathbb{C} \longrightarrow T \xrightarrow{pr_i} T_i, i = 1, 2$.

We recall the definition of weak hyperbolicity [19]. Let $\lambda = (\lambda_1, \ldots, \lambda_N)$, $N = n_1 + n_2$. One says that λ satisfies the *weak hyperbolicity condition of type* (n_1, n_2)(in the sense of Loeb-Nicolau [19, p. 788]) if

$$0 \leq \arg(\lambda_i) < \arg(\lambda_j) < \pi, \ 1 \leq i \leq n_1 < j \leq N. \tag{2}$$

If $\lambda_j = \lambda_1 \ \forall j \leq n_1, \lambda_j = \lambda_N \ \forall j > n_1$, we say that λ is of scalar type.

We denote by C_i the cone $\{\sum r_j\lambda_j \in \mathbb{C} \mid r_j \geq 0, n_{i-1}+1 \leq j \leq n_i\}$ where $n_0 = 0$. We shall denote $C_i \setminus \{0\}$ by C_i° and referred to it as the *deleted cone*. Weak hyperbolicity is equivalent to the requirement that the cones are disjoint and are contained in the half-space $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} \cup \mathbb{R}_{\geq 0}$.

The weak hyperbolicity condition implies that $(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ belongs to the Poincaré domain [1], (that is, 0 is not in the convex hull of $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$) and that α_λ is a proper holomorphic imbedding. Thus $\mathbb{C}_\lambda \cong \mathbb{C}$. When there is no risk of confusion, we merely write $\mathbb{C}$ to mean $\mathbb{C}_\lambda$.

We get a $\mathbb{C}$-action on L, via the embedding $\alpha_\lambda : \mathbb{C} \to T_1 \times T_2$. Recall the notions of an admissible $\mathbb{C}$-action from Definition 1.2.2.

Example 3.2.1. *Let $T_i = \mathbb{C}^*$ be the structure group of $L_i \longrightarrow X_i$ so that the T_i-action on L_i is standard (refer eg.3.1.1). If $\tau \in \mathbb{C}^*$ is such that $0 < \arg(\tau) < \pi$, then the imbedding $\alpha(z) = (\exp(z), \exp(\tau z)) \in \mathbb{C}^* \times \mathbb{C}^*$ is admissible.*

Proposition 3.2.2. *Any admissible $\mathbb{C}$-action of diagonal type on $L_1 \times L_2$ is free.*

Proof. . Suppose that $z \in \mathbb{C}, z \neq 0, (p_1, p_2) \in L$. Let $z.(p_1, p_2) = (q_1, q_2)$. It is readily seen that one of the deleted cones zC_1°, zC_2° lies entirely in the left-half space $\mathcal{R}_- := \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$ or the right-half space $\mathcal{R}_+ := \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. Consider the case $zC_1^\circ \subset \mathcal{R}_-$. Then $|\exp(z\lambda_j)| < 1$ for all $j \leq n_1$. We claim that there is some j such that $\exp(z\lambda_j)\epsilon_j.p_1 \neq p_1$, for, otherwise, the action of T_1-action, restricted to the orbit through p_1 factors through the compact group $T_1/\langle\exp(z\lambda_j)\epsilon_j, 1 \leq j \leq n_1\rangle \cong (\mathbb{S}^1)^{2n_1}$. This implies that the T_1-orbit of p_1 is compact, contradicting 1.2.1 (i), since by the definition of standard action of the T_1, orbit through p_1 contains the fibre $\mathbb{C}^*$ of the principal $\mathbb{C}^*$-bundle $L_1 \longrightarrow X_1$. Now it follows from Lemma 3.1.9 that $||q_1|| = ||(\prod_{1\leq j\leq n_1} \exp(z\lambda_j)\epsilon_j).p_1|| < ||p_1||$. Thus $q_1 \neq p_1$ in this case. Similarly, we see that $(p_1, p_2) \neq (q_1, q_2)$ in the other cases also, showing that the $\mathbb{C}$-action on L is free. $\square$

Lemma 3.2.3. *The orbits of an admissible $\mathbb{C}$-action on L are closed and properly imbedded in L.*

Proof. Let $p = (p_1, p_2) \in L$. Let (z_n) be any sequence of complex numbers such that $|z_n| \to \infty$. We shall show that $z_n.p$ has no limit points in L. Without loss of generality, we may assume that the z_n are such that $z_n/|z_n|$ have a limit point $z_0 \in \mathbb{S}^1$. By the weak hyperbolicity condition (2), one of the deleted cones $z_0 C_i^\circ$ is contained in one of the sectors $\mathcal{S}_+(\theta) := \{w \in \mathbb{C} \mid -\theta < \arg(w) < \theta\} \subset \mathcal{R}_+$ or $\mathcal{S}_-(\theta) = -\mathcal{S}_+(\theta) \subset \mathcal{R}_-$ for some θ, $0 < \theta < \pi/2$. Say $z_0 C_i^0 \subset \mathcal{S}_-(\theta)$. Then $z_n C_i^\circ \subset \mathcal{S}_-(\theta)$ for all n sufficiently large. It follows that $|\exp(z_n \lambda_j)| \to 0$ as $n \to \infty$ for $n_{i-1} < j \leq n_{i-1} + n_i$ (where $n_0 = 0$). By Lemma 3.1.9 we conclude that the sequence $(\alpha_i(z_n)(p_i))$ does not have a limit in L_i. □

3.3 Complex Structures on $S(L_1) \times S(L_2)$

We continue with the notation from previous section.

Definition 3.3.1. *Given standard $T_i = (\mathbb{C}^*)^{n_i}$-actions on the L_i, $i = 1, 2$, we obtain holomorphic vector fields $v_1, \ldots, v_N$ on $L = L_1 \times L_2$ as follows. Let $p = (p_1, p_2) \in L$. Suppose that $1 \leq j \leq n_1$. The holomorphic map $\mu_{p_1} : T_1 \longrightarrow L_1$, $s \mapsto s.p_1$, induces $d\mu_{p_1} : Lie(T_1) = \mathbb{C}^{n_1} \longrightarrow \mathcal{T}_{p_1} L_1$. Set $v_j(p) := (d\mu_p(e_j), 0) \in \mathcal{T}_{p_1} L_1 \times \mathcal{T}_{p_2} L_2 = \mathcal{T}_p L$. The vector fields $v_j, n_1 < j \leq N$, are defined similarly. We refer to $v_j, 1 \leq j \leq N$ as the* standard vector fields *on L.*

Remark 3.3.2. Let $1 \leq j \leq n_1$. Consider the differential $d\nu_1 : \mathcal{T}_p L \longrightarrow \mathbb{R}$ of the norm map $\nu_1 : L \longrightarrow \mathbb{R}_+$ defined as $q = (q_1, q_2) \mapsto ||q_1||$. It is readily verified that, if $\mathbb{R}_+ \epsilon_j$ is not contained in the isotropy at p_1, then by standardness of the action, $d\nu_1(v_j(p)) = v_j(p)(\nu_1) = \nu'_{j,p_1}(1) > 0$. (Here ν_{j,p_1} is as in the Definition 1.2.1(ii) of standard action.) On the other hand, since $\nu_1(s.p) = \nu_1(p)$ for all $s \in (\mathbb{S}^1)^{n_1} = \exp(\sqrt{-1}\mathbb{R}^{n_1}) \subset T_1$ we obtain that $d\nu_1(\sqrt{-1} v_j(p)) = 0$. Thus, for any $z \in \mathbb{C}$, we obtain that $d\nu_1(z v_j(p)) = \mathrm{Re}(z) \nu'_{j,p_1}(1)$. An entirely analogous statement holds when $n_1 < j \leq N$.

Assume that $\lambda \in \mathbb{C}^N$ yields an admissible imbedding $\alpha : \mathbb{C} \longrightarrow T$, $\alpha(z) = \exp(z\lambda)$. We obtain a holomorphic vector field v_λ on L where

$$v_\lambda(p) = \sum_{1 \leq j \leq N} \lambda_j v_j(p) \in \mathcal{T}_p L.$$

The flow of the vector field v_λ yields a holomorphic action of $\mathbb{C}$ which is just the restriction of the T-action to $\mathbb{C}_\lambda$. This $\mathbb{C}$-action on L is free and the $\mathbb{C}$-orbits are the same as the leaves of the holomorphic foliation defined by the integral curves of the vector field v_λ. By Lemma 3.2.3 each leaf is biholomorphic to $\mathbb{C}$. It turns out that the leaf space $L/\mathbb{C}$ is a Hausdorff complex analytic manifold and the projection $L \longrightarrow L/\mathbb{C}_\lambda$ is the projection of a holomorphic principal bundle with fibre and structure group the additive group $\mathbb{C}$. The underlying differentiable manifold of the leaf space is diffeomorphic to $S(L) = S(L_1) \times S(L_2)$. These statements will be proved in Theorem 3.3.3 below. We shall denote the complex manifold $L/\mathbb{C}_\lambda$ by $S_\lambda(L)$. The complex structure so obtained on $S(L)$ by scalar type λ is referred as *scalar type* and those by diagonal type λ is referred as *diagonal type*.

We shall denote by $D(\bar{L}) \subset \bar{L} = \bar{L}_1 \times \bar{L}_2$ the product of the unit disk bundles $D(\bar{L}_i) = \{p \in \bar{L}_i \mid ||p|| \leq 1\} \subset \bar{L}_i, i = 1, 2$. Also we denote by $\Sigma(\bar{L}) \subset \bar{L}$ the boundary of $D(\bar{L})$. Thus $\Sigma(\bar{L}) = D(\bar{L}_1) \times S(L_2) \cup S(L_1) \times D(\bar{L}_2)$. Observe that $S(L) = D(\bar{L}_1) \times S(L_2) \cap S(L_1) \times D(\bar{L}_2) \subset \Sigma(\bar{L})$.

Theorem 3.3.3. *With the above notations, suppose that $\alpha_\lambda : \mathbb{C} \longrightarrow T$ defines an admissible action of $\mathbb{C}$ of diagonal type on L. Then $L/\mathbb{C}$ is a (Hausdorff) complex analytic manifold and the quotient map $L \longrightarrow L/\mathbb{C}$ is the projection of a holomorphic principal $\mathbb{C}$-bundle. Furthermore, each $\mathbb{C}$-orbit meets $S(L)$ transversely at a unique point so that $L/\mathbb{C}$ is diffeomorphic to $S(L)$.*

Proof of the above theorem, which is along the same lines as the proof of [19, Theorem 1] with suitable modifications to take care the more general setting we are in, will be based on the following two lemma.

Lemma 3.3.4. *Each $\mathbb{C}$-orbit in L meets $S(L)$ at exactly one point.*

Proof. We first show that each orbit meets $S(L)$ at not more than one point. Let $p = (p_1, p_2) \in S(L)$. Suppose that $0 \neq z \in \mathbb{C}$ is such that $q := z.p = \alpha(z).p \in S(L)$. This means that, writing $q = (q_1, q_2)$, we have

$$q_i = \alpha_i(z)(p_i) = (\prod_{n_{i-1} < j \leq n_{i-1}+n_i} \exp(\lambda_j z)\epsilon_j)p_i, i = 1, 2,$$

(where $n_0 = 0$). Now $||q_i|| = ||p_i|| = 1, i = 1, 2$, and $p \neq q$. Since the hermitian metric on L_1 is invariant under $(\mathbb{S}^1)^{n_1}$, we see that $||p_1|| = ||q_1|| = ||(\prod_{1 \leq j \leq n_1}(\exp(t_j)\epsilon_j))p_1||$ where $t_j = \mathrm{Re}(\lambda_j z)$. Standardness of the T_1-action implies that either $\mathrm{Re}(\lambda_i z) = 0$ for all $i \leq n_1$ or there exist indices $1 \leq i_1 < i_2 \leq n_1$ such that $\mathrm{Re}(z\lambda_{i_1}).\mathrm{Re}(z\lambda_{i_2}) < 0$. In the latter case there exists positive reals a_1, a_2 such that $a_1\mathrm{Re}(z\lambda_{i_1}) + a_2\mathrm{Re}(z\lambda_{i_2}) = 0$. Similarly, either $\mathrm{Re}(z\lambda_j) = 0$ for all $n_1 < j \leq N$ or there exist indices $n_1 < j_1 < j_2 \leq N$ and positive reals b_1, b_2 such that $b_1\mathrm{Re}(z\lambda_{j_1}) + b_2\mathrm{Re}(z\lambda_{j_2}) = 0$. Suppose $\mathrm{Re}(a_1\lambda_{i_1}z + a_2\lambda_{i_2}z) = 0 = \mathrm{Re}(b_1\lambda_{j_1}z + b_2\lambda_{j_2}z)$. This implies that

$a_1\lambda_{i_1}+a_2\lambda_{i_2}=r(b_1\lambda_{j_1}+b_2\lambda_{j_2})$ for some positive number r. This contradicts the weak hyperbolicity condition (2). Similarly we obtain a contradiction in the remaining cases as well.

Next we show that $\mathbb{C}p\cap\Sigma(\bar{L})$ is path-connected. We shall write D_- and D_+ to denote the bounded and unbounded components of $L\setminus\Sigma(\bar{L})$.

Without loss of generality, suppose that $p=(p_1,p_2)\in\Sigma(\bar{L})$ and let $q=(q_1,q_2)\in\Sigma(\bar{L})\cap\mathbb{C}p$ be arbitrary. Say, $q=z_1.p$ with $z_1\neq 0$. Then $r\mapsto rz_1.p$ defines a path $\sigma: I\longrightarrow\mathbb{C}p$ with end points in $\Sigma(\bar{L})$. We modify the path σ to obtain a new path which lies in $\Sigma(\bar{L})$. For this purpose choose $z_0\in\mathbb{C}$, $\arg(z_0)>\frac{\pi}{2}$ such that $z_0C_1^\circ\cup z_0C_2^\circ$ is contained in the left-half space $\mathcal{R}_-=\{z\in\mathbb{C}\mid \mathrm{Re}(z)<0\}$ and $(-z_0)C_1^\circ\cup(-z_0)C_2^\circ$ is contained in the right-half space $\mathcal{R}_+=\{z\in\mathbb{C}\mid\mathrm{Re}(z)>0\}$. In particular, $\lim_{r\to\infty}|\exp(rz_0\lambda_j)|=0$ and $\lim_{r\to\infty}|\exp(-rz_0\lambda_j)|=\infty,\forall j\leq N$, where r varies in $\mathbb{R}_+$. By the second statement of Lemma 3.1.9, we see that for $i=1,2$, and any $x_i\in L_i$, $||\alpha_i(rz_0).x_i||\to 0$ and $||\alpha_i(-rz_0).x_i||\to\infty$ as $r\to+\infty$ in $\mathbb{R}$.

For any $r\in I$, let $\gamma(r)\in\mathbb{R}$ be least (resp. largest) such that $\gamma(r)z_0.\sigma(r)\in\Sigma(\bar{L})$ when $\sigma(r)\in D_+$ (resp. $\sigma(r)\in D_-$). Then γ is a well-defined continuous function of r. Now $r\mapsto\alpha(\gamma(r)z_0+rz_1).p$ is a path in $\mathbb{C}p\cap\Sigma(\bar{L})$ joining p to q.

To complete the proof, we shall show that there exist points $q'=(q'_1,q'_2)$, $q''=(q''_1,q''_2)\in\mathbb{C}p\cap\Sigma(\bar{L})$ such that $||q'_1||\leq 1, ||q'_2||=1$ and $||q''_1||=1, ||q''_2||\leq 1$. Then any path in $\mathbb{C}p\cap\Sigma(\bar{L})$ joining q' and q'' must contain a point of $S(L)$.

Choose $w_k\in\mathbb{C}^*, 1\leq k\leq 4$, such that the deleted cones $w_1C_i^\circ\subset\mathcal{R}_+, w_2C_i^\circ\subset\mathcal{R}_-$, for $i=1,2$, and, $w_3C_1^\circ, w_4C_2^\circ\subset R_-$, $w_3C_2^\circ, w_4C_1^\circ\subset\mathcal{R}_+$. Then $|\exp(rw_k\lambda_j)|\to 0$ (resp. ∞) as $r\to+\infty$ ($r\in\mathbb{R}_+$) if $\lambda_j\in C_i^\circ$ and $w_kC_i^\circ\subset\mathcal{R}_-$ (resp. $\mathcal{R}_+$). Now $||\alpha_i(rw_1)p_i||<1, ||\alpha_i(rw_2)p_i||>1, i=1,2$ for $r\in\mathbb{R}_+$ sufficiently large. It follows that any path in $\mathbb{C}p$ joining $\alpha(rw_k)(p), k=1,2$, must meet $\Sigma(\bar{L})$ for some $r=r_0$. Thus we may as well assume that $p\in\Sigma(\bar{L})$. Suppose that $||p_1||=1, ||p_2||<1$. For $r>0$ sufficiently large, $||\alpha_1(rw_3).p_1||<1$ and $||\alpha_2(rw_3).p_2||>1$. Therefore there must exist an r_1 such that setting $q'_i:=\alpha_i(r_1w_3).p_i$, we have $||q'_1||\leq 1$ and $||q'_2||=1$. Then $q'=(q'_1,q'_2)\in\mathbb{C}p\cap\Sigma(\bar{L})$ and $q'':=p$ meet our requirements.

If $||p_1||<1, ||p_2||=1$, we set $q':=p$ and find a $q''\in\mathbb{C}p\cap\Sigma(\bar{L})$ by the same argument using w_4 in the place of w_3. □

Lemma 3.3.5. *Every $\mathbb{C}_\lambda$-orbit $\mathbb{C}p$, $p\in S(L)$, meets $S(L)$ transversely.*

Proof. Denote by $\pi: L\longrightarrow S(L)$ the projection of the principal $(\mathbb{C}^*/\mathbb{S}^1)^2\cong\mathbb{R}_+^2$-bundle. Evidently, the inclusion $j: S(L)\hookrightarrow L$ is a cross-section and so $L\cong S(L)\times\mathbb{R}_+^2$. The second projection $\nu: L\longrightarrow\mathbb{R}_+^2$ is just the map $L\ni p=(p_1,p_2)\mapsto(\nu_1(p),\nu_2(p))$ where $\nu_i(p)=||p_i||\in\mathbb{R}_+$. One has therefore an isomorphism $\mathcal{T}_pL|_{S(L)} \cong \mathcal{T}_pS(L) \oplus \mathbb{R}^2$, and the corresponding second projection map $\mathcal{T}_pL \longrightarrow \mathbb{R}^2$ is the differential of ν. Therefore $\mathbb{C}p$ is *not* transverse to $S(L)$ if and only if $av_\lambda(p) \in \mathcal{T}_pS(L)$ for some complex number $a \neq 0$; equivalently, if and only if $d\nu_i(av_\lambda(p)) = 0, i = 1, 2$, for some $a \neq 0$.

By Remark 3.3.2 we have:

$$d\nu_i(av_\lambda(p)) = \sum_{1\leq j\leq n_1} d\nu_i(a\lambda_j v_j(p)) = \sum_{1\leq j\leq n_1} \mathrm{Re}(a\lambda_j)\nu'_{j,p_1}(1).$$

Similarly, $av_\lambda(p)(\nu_2) = \sum_{n_1<j\leq N} \mathrm{Re}(a\lambda_j)\nu'_{j,p_2}(1)$. Therefore, $\mathbb{C}p$ is not transverse to $S(L)$ if and only if $\sum_{1\leq j\leq n_1} \mathrm{Re}(a\lambda_j)r_j = 0 = \sum_{n_1<j\leq N} \mathrm{Re}(a\lambda_j)s_j$ for some complex number $a \neq 0$ and reals $r_j, s_k \geq 0$ (not all zero). This means that $\sqrt{-1}\mathbb{R} \subset aC_1^\circ \cap aC_2^\circ$ and hence $C_1^\circ \cap C_2^\circ \neq \emptyset$, contradicting the weak hyperbolicity condition. $\square$

Proof of Theorem 3.3.3 : We shall first show that $L/\mathbb{C}$ is Hausdorff by showing that $\pi_\lambda : L \longrightarrow S(L)$ which sends $p \in L$ to the unique point in $\mathbb{C}p \cap S(L)$ is continuous.

Let (p_n) be a sequence in L that converges to a point $p_0 \in L$. Let $q_n := \pi_\lambda(p_n) \in S(L)$ and choose $z_n \in \mathbb{C}$ such that $z_n.p_n = q_n$. Since $||p_n||, ||q_n||, n \geq 1$, are bounded, it follows by an argument similar to the proof of Lemma 3.2.3 that (z_n) is bounded, and, passing to a subsequence if necessary, we may assume that it converges to a $z_0 \in \mathbb{C}$. By the continuity of $\mathbb{C}$-action, $z_m.p_n \to z_0.p_0$ as $m, n \to \infty$. Therefore $z_n.p_n = q_n \to z_0.p_0$ and $\pi_\lambda(p_0) = q_0$ and so π_λ is continuous and that the restriction of π_λ to $S(L)$ is a homeomorphism whose inverse is the composition $S(L) \hookrightarrow L \longrightarrow L/\mathbb{C}$.

By what has just been shown, $L/\mathbb{C}$ is in fact a Hausdorff manifold and that π_λ is a diffeomorphism. The orbit space $L/\mathbb{C}$ has a natural structure of a complex analytic space with respect to which the projection $L \longrightarrow L/\mathbb{C}$ is analytic. Using Lemma 3.3.5 we see that $L \longrightarrow L/\mathbb{C}$ is a submersion. It follows that L is the total space of a complex analytic principal bundle with fibre and structure group $\mathbb{C}$. The last statement of the theorem follows from Lemmata 3.3.4 and 3.3.5. $\square$

We get the classical examples of elliptic curve, Hopf manifolds and Calabi-Eckmann manifolds by the above construction and more precisely complex structures on these manifolds are that of scalar type (cf. Section 2.3.1). We generalize the Loeb and Nicolau [19] construction only in the case corresponding to diagonal vector fields.

Example 3.3.6. The action of the structure group $T_i = \mathbb{C}^*$ is a standard action on the holomorphic principal $\mathbb{C}^*$-bundle $L_i \longrightarrow X_i, i = 1, 2$. Given any complex number τ such that $\mathrm{Im}\tau > 0$, one obtains a proper holomorphic imbedding $\mathbb{C} \longrightarrow \mathbb{C}^* \times \mathbb{C}^*$ defined as $z \mapsto (\exp(2\pi iz), \exp(2\pi i\tau z))$. We shall denote the image by $\mathbb{C}_\tau$. The action of the structure group $\mathbb{C}^* \times \mathbb{C}^*$ on $L := L_1 \times L_2$ can be restricted to $\mathbb{C}$ via the above imbedding to obtain a holomorphic principal $\mathbb{C}$-bundle with total space L and base space the quotient space $L/\mathbb{C}_\tau$. The projection $L \longrightarrow X$ factors through $S_\tau(L)$ to yield a principal bundle $S_\tau(L) \longrightarrow X$ with fibre and structure group $\mathbb{E} := (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{C}_\tau$. Since $\mathbb{E}$ is a compact Riemann surface with fundamental group isomorphic to $\mathbb{Z}^2$, it is an elliptic curve. It can be seen that $\mathbb{E} \cong \mathbb{C}/\Gamma$ where Γ is the lattice $\mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$. The quotient space $L/\mathbb{C}$ is diffeomorphic to $S(L) = S(L_1) \times S(L_2)$. The resulting complex structure on $S(L)$ is of *scalar* type and denoted by $S_\tau(L)$.

Example 3.3.7. When X_1 is a point, one has $X \cong X_2, L \cong \mathbb{C}^* \times L_2$. In this case, the orbit space $L/\mathbb{C}$ is readily identified with $L_2/\mathbb{Z}$ where the $\mathbb{Z}$ action is generated by $v \mapsto \prod_{2\leq j\leq N} \exp(2\pi\sqrt{-1}\lambda_j/\lambda_1)\epsilon_j.v$ where $v \in L_2$. The projection $L_2 \longrightarrow S_\lambda(L)$ is a covering projection with deck transformation group $\mathbb{Z}$.

Remark 3.3.8. (i) When λ is of scalar type, the projection $L \longrightarrow X$ factors through $S_\lambda(L)$ and yields a complex analytic bundle $S_\lambda(L) \longrightarrow X$ with fibre and structure group the elliptic curve $(\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{C}$. When endowed with diagonal type complex structure the projection $S_\lambda(L) \longrightarrow X$ of the principal $\mathbb{S}^1 \times \mathbb{S}^1$-bundle, which is smooth, is not complex analytic in general. (Cf. Theorem 5.2.1.)

(ii) When the X_i do not admit any non-trivial T_i- action, we obtain only scalar type complex structures on $S(L)$. For example, this happens when the X_i are compact Riemann surfaces of genus at least 2, as $Aut(X_i)$ is finite.

We conclude this section with the following observation.

Theorem 3.3.9. *Suppose that $H^1(X_1;\mathbb{R}) = 0$ and that $c_1(\bar{L}_1) \in H^2(X_1;\mathbb{R})$ is non-zero. Then $S(L)$ is not symplectic and hence non-Kähler with respect to* any *complex structure.*

Proof. In the Leray-Serre spectral sequence over $\mathbb{R}$ for the $\mathbb{S}^1$-bundle with projection $q : S(L_1) \longrightarrow X_1$ the differential $d : E_2^{0,1} \cong H^1(\mathbb{S}^1;\mathbb{R}) \cong \mathbb{R} \longrightarrow E_2^{2,0} = H^2(X_1;\mathbb{R})$ is non-zero. It follows that $E_3^{0,1} = E_\infty^{0,1} = 0$. Since $H^1(X_1;\mathbb{R}) = 0$, we see that $H^1(S(L_1);\mathbb{R}) = 0$. Hence, by the Künneth formula, $H^2(S(L);\mathbb{R}) = H^2(S(L_1);\mathbb{R}) \oplus H^2(S(L_2);\mathbb{R})$.

Let $u_i \in H^2(S(L_i);\mathbb{R}), i = 1, 2$, be arbitrary. Since $\dim S(L_i)$ is odd for $i = 1, 2$, $u_1^r u_2^s = 0$ for any $r, s \geq 0$ such that $r+s = n$, where $2n := \dim_{\mathbb{R}} S(L)$. Hence $\omega^n = 0$ for any $\omega \in H^2(S(L);\mathbb{R})$. $\square$

3.4 Complex Structures on Product of Sphere Bundles

Let $\pi : \bar{E} \longrightarrow X$ be a vector bundle over a compact complex manifold X. Let $\mathbb{P}(\bar{E})$ be the compact complex manifolds obtain by identifying one dimensional subspace of each fibre $\bar{E}_x := \pi^{-1}(x)$ for $x \in X$. Let $\bar{L} \subset \bar{E} \times \mathbb{P}(\bar{E})$ be defined as

$$\bar{L} = \{(e, V) \mid e \in V \in \mathbb{P}(\bar{E})\}. \tag{3}$$

The projection map $\bar{L} \longrightarrow \mathbb{P}(\bar{E}); (e, V) \mapsto V$, is the projection of a holomorphic line bundle. We have the following identification:

$$\bar{E} \setminus X = \bar{L} \setminus \mathbb{P}(\bar{E}), \text{ where } \bar{E} \ni e \longleftrightarrow (e, \mathbb{C}e). \tag{4}$$

A hermitian metric on the vector bundle $\bar{E} \longrightarrow X$ yields a hermitian metric on the line bundle $\bar{L} \longrightarrow \mathbb{P}(\bar{E})$. The total space $S(\bar{L})$ of the circle bundle associated to the line bundle $\bar{L} \longrightarrow \mathbb{P}(\bar{E})$ is same as the total space $S(\bar{E})$ of the sphere bundle associated to the vector bundle $\bar{E} \longrightarrow X$ under the identification (4).

Let $\bar{E}_i \longrightarrow X_i$ be holomorphic vector bundle of rank k_i over compact complex manifolds $X_i, i = 1, 2$. Fix a hermitian metric on the bundle $\bar{E}_i \longrightarrow X_i$. Let $S(\bar{E}_i)$ be the associated unit sphere bundle over $X_i, i = 1, 2$. In the previous section we showed that in the case when $k_i = 1$ complex structures on $S(\bar{E}_1) \times S(\bar{E}_2)$ always exist such as complex structures of scalar type. See the Example 3.3.6. For arbitrary k_i the existence of complex structures on $S(\bar{E}_1) \times S(\bar{E}_2)$ follow readily as $S(\bar{E}_i)$ is same as $S(\bar{L}_i)$, where $\bar{L}_i$ is as defined as in (3). Thus we obtained a family of complex structures on the product of sphere bundles associated to holomorphic vector bundles over compact complex manifolds. We call the complex structures thus obtained as scalar type. We have the projection $S(\bar{E}_1) \times S(\bar{E}_2) \longrightarrow X_1 \times X_2$ of holomorphic fibre bundle with fibres a Calabi-Eckmann manifolds. This implies that in the case $k_i > 1$, the complex manifolds $S(\bar{E}_1) \times S(\bar{E}_2)$ thus obtained are non-Kähler and hence non-algebraic.

The analogue of the diagonal type complex structures on $S(\bar{E}_1) \times S(\bar{E}_2)$ are obtained by considering an action of a torus $T_i := (\mathbb{C}^*)^{n_i}$ on the fibre bundle $E_i := \bar{E}_i \setminus X_i \longrightarrow X_i$ which satisfies the condition similar to that in the Definition 1.2.1. We call this action again 'standard' actions. See the Definition 3.4.1 below.

The matrix group $GL(n,\mathbb{C})$ is the structure group for any vector bundle $\bar{E} \longrightarrow X$. The torus subgroup $\mathbb{C}^*$, the subgroup ($\cong \mathbb{C}^*$) of scalar matrices, is the centre of $GL(n,\mathbb{C})$. The scalar action of $\mathbb{C}^*$ on the vector space $\mathbb{C}^n$ extends to give the scalar action of $\mathbb{C}^*$ on the vector bundle $\bar{E} \longrightarrow X$.

Definition 3.4.1. *Let $\bar{E} \longrightarrow X$ be a hermitian vector bundle over X. We say that the T-action on E is d-*standard *(or more briefly* standard*) if the following conditions hold:*
(i) the restricted action of the diagonal subgroup $\Delta \subset T$ on E is via the d-fold covering projection $\Delta \longrightarrow \mathbb{C}^*$, where $\mathbb{C}^*$ action is that of scalar multiplication on the bundle $E \longrightarrow X$.
(ii) For any $e \in E$ and $1 \le j \le n$, let $\nu_{e,j} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be defined as $t \mapsto ||t\epsilon_j.e||$. Then $\nu'_{e,j}(t) > 0$ for all t unless $\mathbb{R}_+\epsilon_j$ is contained in the isotropy at e.

Example 3.4.2. Let $\bar{E} \longrightarrow X$ be a holomorphic hermitian vector bundle over compact complex manifold X. The scalar action of the torus $\mathbb{C}^*$ on the fibre bundle $E \longrightarrow X$ is a standard action with $d = 1$.

Example 3.4.3. Let $\gamma_{n,k}$ be the k-plane bundle over the Grassmannian $\mathbb{G}_{n,k}$ of k-dimensional subspace of $\mathbb{C}^n$. The total space $\bar{E}(\gamma_{n,k})$ of $\gamma_{n,k}$ is the subset of $\mathbb{G}_{n,k} \times \mathbb{C}^n$ define as :

$$\bar{E}(\gamma_{n,k}) = \{(V, v) \mid v \in V\}.$$

The natural action of the torus $T = (\mathbb{C}^*)^n$ on the vector space $\mathbb{C}^n$ given by

$$t.(z_1, z_2, \ldots, z_n) = (t_1 z_1, t_2 z_2, \ldots, t_n z_n), \text{for } t = (t_1, t_2, \ldots, t_n).$$

In the case when $k = 1$ we have seen in the Example 3.1.2 that this action of $T = (\mathbb{C}^*)^n$ extends to give a d-standard action of T of the fibre bundle $E(\gamma_{n,1}) \longrightarrow \mathbb{G}_{n,k} = \mathbb{P}^{n-1}$ with $d = 1$. The general case of arbitrary k is similar.

Let $\bar{E}_i \longrightarrow X_i$ be a hermitian vector bundle. Let $T_i := (\mathbb{C}^*)^{n_i}$ be a d_i-standard action on the fibre bundle $E_i := \bar{E}_i \setminus \{0\} \longrightarrow X_i, i = 1, 2$. Such T_i action gives a d_i-standard action of T_i (in the sense of the Definition 1.2.1) on the principal $\mathbb{C}^*$-bundle $L_i := \bar{L}_i \setminus \mathbb{P}(\bar{E}_i) \longrightarrow \mathbb{P}(E_i)$. For any $\lambda \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ satisfying weak hyperbolic condition of type (n_1, n_2), let α_λ be the admissible $\mathbb{C}$-action on the principal $(\mathbb{C}^* \times \mathbb{C}^*)$-bundle $E_1 \times E_2 = L_1 \times L_2 \longrightarrow \mathbb{P}(\bar{E}_1) \times \mathbb{P}(\bar{E}_2)$. We obtain a complex structures of a diagonal type on $S(\bar{E}_1) \times S(\bar{E}_2)$ by identifying it with the orbit space $(E_1 \times E_2)/\mathbb{C}$. See the Theorem 3.3.3. Furthermore, the quotient map $E_1 \times E_2 \longrightarrow S(\bar{E}_1) \times S(\bar{E}_2)$ is the projection of a principal $\mathbb{C}$-bundle.

Remark 3.4.4. The above construction of complex manifolds $S(L)$ is valid even in the more general setting when the complex manifolds $X_i, i = 1, 2$ are non-compact.

Chapter 4

Complex Structures of Linear Type

In the previous chapter, we constructed a family of complex structures on $S(L_1) \times S(L_2)$ where $S(L_i)$ is the circle bundle associated to a principal $\mathbb{C}^*$-bundle L_i over an arbitrary compact complex manifolds $X_i, i = 1, 2$. Complex manifolds thus obtained are of diagonal type. The basic construction involves the notion of standard action of a torus T_i on L_i. In this chapter we shall construct the complex structure of *linear type* on $S(L_1) \times S(L_2)$ in the case when the complex manifolds X_i are flag manifolds and the $\mathbb{C}^*$-bundle $L_i \longrightarrow X_i, i = 1, 2$ are associated to a negative ample line bundle over X_i. These linear type complex structures will correspond to complex structures on $\mathbb{S}^{2m-1} \times \mathbb{S}^{2n-1}, m, n \geq 1$ obtained by Loeb and Nicolau [19] by using linear resonant vector fields. (refer Section 2.3). We first start with constructing some non-trivial standard action of a torus on the principal $\mathbb{C}^*$-bundle over a flag manifold and then we shall use this set up to construct linear type complex structures on $S(L_1) \times S(L_2)$.

For notations and basic fact of Lie groups and flag manifolds we shall refer to the Section 2.1 of Chapter 2.

4.1 Standard Action of a Torus on Principal $\mathbb{C}^*$-Bundles over Flag Manifolds

Let G be a simply connected complex simple Lie group. Let P be a parabolic subgroup of G. Let $\bar{L}$ be a G-equivariant non-trivial line bundle over the flag manifold G/P. Let $L \longrightarrow G/P$ be the associated principal $\mathbb{C}^*$-bundle over G/P. Note that G acts almost effectively on G/P and hence on L. (Almost effective means that the subgroup of G which fixes every element of L is finite, and since G simple, is contained in $Z(G)$, the centre of G). Let T be any maximal torus of G. We claim that the T-action is not d-standard (with respect to any isomorphism $T \cong (\mathbb{C}^*)^l$) for any $d \geq 1$. For, if the T-action were d-standard, then T would contain a subgroup $\Delta \cong \mathbb{C}^*$ whose restricted action is as described in the Definition 1.2.1(i). Since the G-action commutes with that of the structure group $\mathbb{C}^*$ of $\bar{L}$, it follows that $z.g(v) = g.z(v)$ for all $v \in \bar{L}, z \in \Delta, g \in G$. Since the G-action on L is almost effective, we see $g^{-1}zg = \zeta z$ where $\zeta \in Z(G)$, the centre of G, which is a finite group. This implies that $\Delta/Z(G)$ is in the centre of $G/Z(G)$ contradicting our hypothesis that G is simple. However, in the case when $\bar{L}$ is a negative ample line bundle over G/P, we shall show that there is a d-standard action of the torus $T \times T_0$ on the principal $\mathbb{C}^*$-bundle $L \longrightarrow G/P$, where the $T_0 \cong \mathbb{C}^*$ acts via a d_0 fold covering projection $T_0 \longrightarrow \mathbb{C}^*$ on the structure group of $L \longrightarrow G/P$. Here $d = (l+1)d_0$, where l being rank of G.

In the Section 3.1, we constructed some examples of standard action of the torus $(\mathbb{C}^*)^n$ on certain principal $\mathbb{C}^*$-bundle over flag manifolds $SL(n, \mathbb{C})/P$. We shall extend the same idea to obtain a standard action of a torus on principal $\mathbb{C}^*$-bundle over more general flag manifolds G/P.

Fix a maximal torus T in a simply connected simple complex Lie group G. Let B be a Borel subgroup containing T. For a dominant weight ϖ, let $V(\varpi)$ be the corresponding finite dimensional irreducible G-module with the highest weight ϖ. Let P_ϖ be the parabolic subgroup of G obtained by 'omitting' ϖ. Let $\bar{L}_\varpi \longrightarrow G/P_\varpi$ be the G-equivariant ample line bundle. In this case we have an embedding of G/P_ϖ into the projective space $\mathbb{P}(V(\varpi))$ such that the G-equivariant line bundle $\bar{L}_{-\varpi}$, dual to $\bar{L}_\varpi$, is the pullback of the tautological line bundle.

Let $\Lambda(\varpi) \subset \Lambda$ denote the set of all weights of $V(\varpi)$. For $\mu \in \Lambda(\varpi))$ we denote the multiplicity of μ in $V(\varpi)$ by m_μ i.e. $m_\mu = \dim V_\mu(\varpi)$, where $V_\mu(\varpi) := \{v \in V(\varpi) \mid t(v) = \mu(t)v, \forall\ t \in T\}$, is the μ-weight space in $V(\varpi)$. We put a hermitian inner product on $V(\varpi)$ with respect to which the decomposition $V(\varpi) = \bigoplus_{\mu \in \Lambda(\varpi))} V_\mu(\varpi)$ is an orthogonal. Such an hermitian product is invariant under the compact torus $K \subset T$. Indeed, without loss of generality we may assume that the inner product is invariant under a maximal compact subgroup of G that contains K.

Let $\varpi_1, \varpi_2, \ldots, \varpi_l$ be the fundamental weights. Consider the homomorphism $\psi : T \longrightarrow (\mathbb{C}^*)^l$ of algebraic groups defined as $t \mapsto (\varpi_1(t), \ldots, \varpi_l(t))$. It is an isomorphism since $\varpi_1, \ldots, \varpi_l$ is $\mathbb{Z}$-basis for $\chi(T)$. We shall identify T with $(\mathbb{C}^*)^l$ via ψ. Let $\varpi \in \Lambda^+$ be a dominant weight. Write $\mu =$

$\sum_{1\leq j\leq l} a_{\mu,j}\varpi_j$ for $\mu \in \Lambda(\varpi)$ so that

$$\mu(t) = \prod_{1\leq j\leq l} t_j^{a_{\mu,j}} \quad \text{where } t = (t_1, t_2, \cdots, t_l) \in T.$$

If $v \in V_\mu(\varpi)$, then $t.v = \prod t_j^{a_{\mu,j}}.v$. It is not difficult to see that the T-action on $V(\varpi) \setminus 0 \longrightarrow P(V(\varpi))$ is *not* standard since $w_0(\varpi) \in \Lambda(\varpi)$ is *negative* dominant, i.e., $-w_0(\varpi) \in \Lambda^+$, as in this case all $a_{w_0(\varpi),j}, 1 \leq j \leq l$ will be negative. Set $d' := 1 + \sum |a_{\mu,j}|$ where the sum is over $\mu \in \Lambda(\varpi), 1 \leq j \leq l$. The group $T' := T \times \mathbb{C}^*$ acts on $V(\varpi)$ where the last factor acts via the covering projection $\mathbb{C}^* \longrightarrow \mathbb{C}^*$, $z \mapsto z^{-d'}$, where the target $\mathbb{C}^*$ acts as scalar multiplication. Thus $(t,z).v = \mu(t)z^{-d'}v$ where $v \in V_\mu(\varpi), (t,z) \in T'$. Now consider the $(l+1)$-fold covering projection $\widetilde{T} := (\mathbb{C}^*)^{l+1} \longrightarrow T'$, defined as $(t_1, \ldots, t_{l+1}) \mapsto (t_{l+1}^{-1}t_1, \ldots, t_{l+1}^{-1}t_l, \prod_{1\leq j\leq l+1} t_j^{-1})$. The torus $\widetilde{T}$ acts on the principal $\mathbb{C}^*$-bundle $V(\varpi) \setminus \{0\} \longrightarrow \mathbb{P}(V(\varpi))$ via the above surjection.

Denote by $\widetilde{\epsilon}_j : \mathbb{C}^* \longrightarrow \widetilde{T}$ the jth coordinate imbedding. For any $\mu \in \Lambda(\varpi)$, and any $v \in V_\mu(\varpi)$, we have $z\widetilde{\epsilon}_{l+1}.v = z^{d'} \prod_{1\leq j\leq l} z^{-a_{\mu,j}}v = z^{d'-\sum a_{\mu,j}}v$, and, when $j \leq l$, we have $z\widetilde{\epsilon}_j.v = z^{d'+a_{\mu,j}}v$. Also, if $z = (z_0, \ldots, z_0) \in \widetilde{T}$, then $z.v = z_0^{(l+1)d'}v = z_0^d v$, where $d = (l+1)d'$. Observe that the exponent of z that occurs in the above formula for $z\widetilde{\epsilon}_j.v$ is positive for $1 \leq j \leq l+1$ by our choice of d'. We shall denote this exponent by $d_{\mu,j}$, that is,

$$d_{\mu,j} = \left\{ \begin{array}{ll} d' + a_{\mu,j}, & 1 \leq j \leq l, \\ d' - \sum_{1\leq i\leq l} a_{\mu,i}, & j = l+1, \end{array} \right. \tag{5}$$

where $\mu = \sum_{1\leq j\leq l} a_{\mu,j}\varpi_j \in \Lambda(\varpi)$.

Next note that the compact torus $\widetilde{K} := K \times \mathbb{S}^1 \subset T \times \mathbb{C}^*$ preserves the hermitian product on $V(\varpi)$ and hence the (induced) hermitian metric on the tautological line bundle over $\mathbb{P}(V(\varpi))$. From the explicit description of the action just given, it is clear that conditions (i) and (ii) of Definition 1.2.1 hold. Thus we have extended the T-action to an action of $\widetilde{T}$-action which is standard. We are ready to prove

Proposition 4.1.1. *We keep the above notations. Let $\varpi \in \Lambda^+$ be any dominant weight of G. Then the T-action can be extended to a d-standard action of $\widetilde{T} := T \times \mathbb{C}^*$ on $L_{-\varpi} \longrightarrow G/P_\varpi$ where $d = d'(l+1)$.*

Proof. Since $\bar{L}_\varpi$ is a very ample line bundle over G/P_ϖ, one has a G - equivariant embedding $G/P_\varpi \longrightarrow \mathbb{P}(V(\varpi))$ where $V(\varpi) = H^0(G/P_\varpi, \bar{L}_\varpi)^*$. By our discussion above, the T-action on the tautological bundle over the projective space $\mathbb{P}(V(\varpi))$ has been extended to a d-standard action of $\widetilde{T}$ for an appropriate $d > 1$. The tautological bundle over $\mathbb{P}(V(\varpi))$ restricts to $L_{-\varpi}$ on G/P_ϖ. Clearly the $L_{-\varpi}$ is $\widetilde{T}$-invariant. Put any $\widetilde{K}$-invariant hermitian metric on $V(\varpi)$ where $\widetilde{K}$ denotes the maximal compact subgroup of $\widetilde{T}$. As observed above, $z\widetilde{\epsilon}_j.v = z^{d_{\mu,j}}v$ where $d_{\mu,j} > 0$ for $v \in V_\mu(\varpi)$, it follows that condition (ii) of Definition 1.2.1 holds. Therefore the $\widetilde{T}$-action on $L_{-\varpi}$ is d-standard. $\square$

Let $\widetilde{G} = G \times \mathbb{C}^*$. Let $\pi : \widetilde{G} = G \times \mathbb{C}^* \longrightarrow G \times \mathbb{C}^*$ be the $(l+1)$-fold covering obtained from the $(l+1)$-fold covering of the last factor and identity on the first. The maximal torus $\pi^{-1}(T \times \mathbb{C}^*)$ of $\widetilde{G}$ can be identified with $\widetilde{T}$. With respect to an appropriate choice of identification $\widetilde{T} \cong (\mathbb{C}^*)^{l+1}$, we see that the action of G on $L_{-\varpi}$ extends to $\widetilde{G}$ in such a manner that the $\widetilde{T}$-action is d-standard where $d = (l+1)d'$ as above. Since $G/P_\varpi = \widetilde{G}/\widetilde{P}$, where $\widetilde{P} = \pi^{-1}(P_\varpi \times \mathbb{C}^*)$, the $\mathbb{C}^*$-bundle $L_{-\varpi} \longrightarrow X$ is $\widetilde{G}$-equivariant. The parabolic subgroup $\widetilde{P}$ contains the Borel subgroup $\widetilde{B} := \pi^{-1}(B \times \mathbb{C}^*)$. We shall refer to $\bar{L}_{-\varpi} \longrightarrow X$ as a *d-standard $\widetilde{G}$-homogeneous line bundle.*

4.2 Linear Type Complex Structures on $S(L_1) \times S(L_2)$

Let $\bar{L}_i \longrightarrow X_i$, $i = 1, 2$, be d_i-standard $\widetilde{G}_i$-homogeneous line bundles over $X_i = \widetilde{G}_i/\widetilde{P}_i$ which are negatively ample. Let $\widetilde{G} = \widetilde{G}_1 \times \widetilde{G}_2$, $(\mathbb{C}^*)^N \cong \widetilde{T} = \widetilde{T}_1 \times \widetilde{T}_2$ where $N = rank(\widetilde{G}) = n_1 + n_2$ with $n_i := l_i + 1$ and $\widetilde{B} = \widetilde{B}_1 \times \widetilde{B}_2$. The torus $\widetilde{T}$ is a maximal torus contained in the Borel subgroup $\widetilde{B}$ of $\widetilde{G}$. Denote by $\widetilde{\Phi}^+$ the set of simple positive roots determined by $\widetilde{T} \subset \widetilde{B} \subset \widetilde{G}$. It is clear that $\widetilde{\Phi}^+ = \widetilde{\Phi}_1^+ \cup \widetilde{\Phi}_2^+$ where $\widetilde{\Phi}_i^+$ is the set of simple positive roots determined by $\widetilde{T}_i \subset \widetilde{B}_i \subset \widetilde{G}_i, i = 1, 2$. Here we considered an element $\gamma \in \chi(\widetilde{T}_i)$ as an element of $\chi(\widetilde{T}_1 \times \widetilde{T}_2)$ by composing it with the projection $\widetilde{T}_1 \times \widetilde{T}_2 \longrightarrow \widetilde{T}_i$.

Let $\lambda \in Lie(\widetilde{B})$ and let $\lambda = \lambda_s + \lambda_u$ be its Jordan decomposition, where $\lambda_s = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N = Lie(\widetilde{T})$ satisfies the weak hyperbolicity condition (2) of type (n_1, n_2) and $\lambda_u \in Lie(\widetilde{B}_u)$, the Lie algebra of the unipotent radical $\widetilde{B}_u$ of $\widetilde{B}$. Thus $[\lambda_u, \lambda_s] = 0$ in $Lie(\widetilde{B})$. The analytic imbedding $\alpha_\lambda : \mathbb{C} \longrightarrow \widetilde{B}$ where $\alpha_\lambda(z) = \exp(z\lambda) = \exp(z\lambda_s).\exp(z\lambda_u)$ defines an action, again denoted α_λ, of $\mathbb{C}$ on $L := L_1 \times L_2$ and an action $\widetilde{\alpha}_\lambda$ on $V(\varpi_1) \times V(\varpi_2)$. Denote by $\mathbb{C}_\lambda$ the image $\alpha_\lambda(\mathbb{C}) \subset \widetilde{B}$. We shall now give an explicit description of these actions. Let $v_i \in V(\varpi_i)$ and write $v_i = \sum_{\mu \in \Lambda(\varpi_i)} v_\mu$ where $v_\mu \in V_\mu(\varpi_i)$. Set

$$\lambda_\mu := \sum \lambda_j d_{\mu,j} \qquad (6)$$

where the sum ranges over $n_{i-1} < j \leq n_{i-1} + n_i$ with $n_0 = 0$. Then

$$\widetilde{\alpha}_{\lambda_s}(z)(v_1, v_2) = (u_1, u_2), \qquad \text{where}$$

$$u_i = \sum_{\mu \in \Lambda(\varpi_i)} \prod_j \exp(z\lambda_j)\widetilde{\epsilon}_j.v_\mu = \sum_\mu \prod_j (\exp(z\lambda_j d_{\mu,j})v_\mu = \sum_\mu \exp(z\lambda_\mu)v_\mu, \tag{7}$$

where the product is over j such that $n_{i-1} < j \leq n_{i-1} + n_i$.

The $\mathbb{C}$-action α_{λ_s} on L is just the restriction to $L \subset V(\varpi_1) \times V(\varpi_2)$ of the $\mathbb{C}$-action $\widetilde{\alpha}_{\lambda_s}$. Since the λ_μ are all positive linear combination of the λ_j, the action of $\mathbb{C}$ on $V(\varpi_1, \varpi_2) := (V(\varpi_1) \setminus \{0\}) \times (V(\varpi_2) \setminus \{0\})$, the total space of the product of tautological bundles, is admissible.

Fix a basis for $V(\varpi_i)$ consisting of weight vectors so that $GL(V(\varpi_i))$ is identified with invertible $r_i \times r_i$-matrices, where $r_i := \dim V(\varpi_i)$. Note that action of the diagonal subgroup of $GL(V(\varpi_i))$ on $V(\varpi_i)\setminus\{0\}$ is standard and that $\widetilde{T}_i$ is mapped into D, the diagonal subgroup of $GL(V(\varpi_1)) \times GL(V(\varpi_2))$. We put a hermitian metric on $V(\varpi_1) \times V(\varpi_2)$ which is invariant under the compact torus $(\mathbb{S}^1)^{r_1+r_2} \subset D$. Considered as a subgroup of $GL(V(\varpi_1)) \times GL(V(\varpi_2))$, the $\mathbb{C}$-action $\widetilde{\alpha}_{\lambda_s}$ on $V(\varpi_1, \varpi_2)$ is the *same* as that considered by Loeb-Nicolau corresponding to $\lambda_s(\varpi_1, \varpi_2) := (\lambda_\mu, \lambda_\nu)_{\mu \in \Lambda(\varpi_1), \nu \in \Lambda(\varpi_2)} \in Lie(D) = \mathbb{C}^{r_1} \times \mathbb{C}^{r_2}$, where it is understood that each λ_μ occurs as many times as $\dim V_\mu(\varpi_1), \mu \in \Lambda(\varpi_1)$, and similarly for $\lambda_\nu, \nu \in \Lambda(\varpi_2)$.

Observation: The $\lambda_s(\varpi_1, \varpi_2)$ satisfy the weak hyperbolicity condition of type (r_1, r_2) since the λ_μ are *positive* integral linear combinations of the λ_j.

The differential of the Lie group homomorphism $\widetilde{G}_1 \times \widetilde{G}_2 \to GL(V(\varpi_1)) \times GL(V(\varpi_2))$ maps λ_s to the diagonal matrix $diag(\lambda_s(\varpi_1, \varpi_2))$ and λ_u to a nilpotent matrix $\lambda_u(\varpi_1, \varpi_2)$ which commutes with $\lambda_s(\varpi_1, \varpi_2)$. Indeed $\lambda(\varpi_1, \varpi_2) := \lambda_s(\varpi_1, \varpi_2) + \lambda_u(\varpi_1, \varpi_2)$ has a block decomposition compatible with weight-decomposition of $V(\varpi_1) \times V(\varpi_2)$ where the μ-th block is $\lambda_\mu I_{m(\mu)} + A_\mu$, where A_μ is nilpotent and $I_{m(\mu)}$ is the identity matrix of size $m(\mu)$, the multiplicity of $\mu \in \Lambda(\varpi_i), i = 1, 2$.

Recall that, for the $\mathbb{C}$-action $\widetilde{\alpha}_\lambda$ on $V(\varpi_1, \varpi_2)$, the orbit space $S_\lambda(\varpi_1, \varpi_2) := V(\varpi_1, \varpi_2)/\mathbb{C}$ is a complex manifold diffeomorphic to the product of spheres $\mathbb{S}^{2r_1-1} \times \mathbb{S}^{2r_2-1}$ by [19, Theorem 1]. Indeed, the canonical projection $V(\varpi_1, \varpi_2) \longrightarrow S_\lambda(\varpi_1, \varpi_2)$ is the projection of a holomorphic principal bundle with fibre and structure group $\mathbb{C}$. When $\lambda_u = 0$, these statements also follow from Theorem 3.3.3.

Since $rank(\widetilde{G}) > |\widetilde{\Phi}^+|$, for any $\varepsilon > 0$ we can find $t_\varepsilon \in \widetilde{T}$ such that $\gamma(t_\varepsilon) = \varepsilon$ for all $\gamma \in \widetilde{\Phi}^+$.

Theorem 4.2.1. *We keep the above notations.*
(i) The orbit space, denoted $L/\mathbb{C}_\lambda$, of the $\mathbb{C}$-action on $L = L_1 \times L_2$ defined

by $\lambda = \lambda_s + \lambda_u$ is a Hausdorff complex manifold and the canonical projection $L \longrightarrow L/\mathbb{C}_\lambda$ is the projection of a principal $\mathbb{C}$-bundle. Furthermore, $L/\mathbb{C}_\lambda$ is analytically isomorphic to $L/\mathbb{C}_{\lambda_\varepsilon}$ where $\lambda_\varepsilon := Ad(t_\varepsilon)(\lambda)$ and $t_\varepsilon \in \widetilde{T}$ is such that $\gamma(t_\varepsilon) = \varepsilon$ for all $\gamma \in \widetilde{\Phi}^+$.
(ii) If $|\varepsilon|$ is sufficiently small, then each orbit of $\mathbb{C}_{\lambda_\varepsilon}$ on L meets $S(L)$ transversely at a unique point. In particular, the restriction of the projection $L \longrightarrow L/\mathbb{C}_{\lambda_\varepsilon}$ to $S(L) \subset L$ is a diffeomorphism.

Proof. When $\lambda_u = 0$, the theorem is a special case of Theorem 3.3.3. So assume $\lambda_u \neq 0$.

Since the $\mathbb{C}$-action $\alpha_{\lambda_\varepsilon}$ is conjugate by the analytic automorphism $t_\varepsilon : L \longrightarrow L$ to α_λ, we see that $L/\mathbb{C}_\lambda \cong L/\mathbb{C}_{\lambda_\varepsilon}$ as a complex analytic space. Thus, it is enough to prove the theorem for $|\varepsilon| > 0$ sufficiently small.

Consider the projective embedding $\phi'_i : X_i = G_i/P_i \hookrightarrow \mathbb{P}(V(\varpi_i))$ defined by the ample line bundle $\bar{L}_i^*$. The circle-bundle $S(L_i) \longrightarrow X_i$ is just the restriction to X_i of the circle-bundle associated to the tautological bundle over $\mathbb{P}(V(\varpi_i))$. Thus ϕ'_i yields an imbedding $\phi_i : S(L_i) \longrightarrow \mathbb{S}^{2r_i-1}$. Let $\phi : S(L) \longrightarrow S(V(\varpi_1, \varpi_2)) = \mathbb{S}^{2r_1-1} \times \mathbb{S}^{2r_2-1}$ be the product $\phi_1 \times \phi_2$.

Set $\lambda_{u,\varepsilon} = Ad(t_\varepsilon)\lambda_u$ so that $\lambda_\varepsilon = \lambda_s + \lambda_{u,\varepsilon}$. Let $\beta \in R^+$ be a positive root. Let $X_\beta \in Lie(B_u)$ denotes a weight vector of weight β. Note that, if $\beta = \prod_{\gamma \in \Phi^+} \gamma^{k_{\beta,\gamma}}$ (i.e. in additive notation $\beta = \sum_{\gamma \in \Phi^+} k_{\beta,\gamma}\gamma$), then

$$Ad(t_\varepsilon)X_\beta = \beta(t_\varepsilon)X_\beta = \prod_{\gamma \in \Phi^+} \gamma^{k_{\beta,\gamma}}(t_\varepsilon)X_\beta = \varepsilon^{|\beta|}X_\beta$$

where $|\beta| = \sum k_{\beta,\gamma} \geq 1$. This implies that $\lambda_\varepsilon \to \lambda_s$ as $\varepsilon \to 0$, and, furthermore, $\lambda_\varepsilon(\varpi_1, \varpi_2) \to \lambda_s(\varpi_1, \varpi_2)$ as $\varepsilon \to 0$. By [19, Theorem 1], for $|\varepsilon|$ sufficiently small, each $\mathbb{C}$-orbit for the $\widetilde{\alpha}_{\lambda_\varepsilon}$-action on $V(\varpi_1, \varpi_2)$ is closed and properly imbedded in $L(\varpi_1, \varpi_2)$ and intersects $\mathbb{S}^{2r_1-1} \times \mathbb{S}^{2r_2-1}$ at a unique point. In particular, each orbit of the $\mathbb{C}$-action corresponding to λ_ε meets $S(L) \subset \mathbb{S}^{2r_1-1} \times \mathbb{S}^{2r_2-1}$ at a *unique* point when $|\varepsilon| > 0$ is sufficiently small.

Consider the map $\pi_{\lambda_\varepsilon} : L \longrightarrow S(L)$ which maps each $\alpha_{\lambda_\varepsilon}$ orbit to the unique point where it meets $S(L)$. This is just the restriction of the map $V(\varpi_1, \varpi_2) \longrightarrow \mathbb{S}^{2r_1-1} \times \mathbb{S}^{2r_2-1}$ and hence continuous. It follows that the orbit space $L/\mathbb{C}_{\lambda_\varepsilon}$ is Hausdorff and that the map $\bar{\pi}_{\lambda_\varepsilon} : L/\mathbb{C} \longrightarrow S(L)$ induced by $\pi_{\lambda_\varepsilon}$ is a homeomorphism, whose inverse is just the composition $S(L) \hookrightarrow L \longrightarrow L/\mathbb{C}$. Since each $\mathbb{C}$-orbit for α_{λ_s}-action meets $S(L)$ transversely by Lemma 3.3.5, and since $S(L)$ is compact, the same is true for the $\alpha_{\lambda_\varepsilon}$-action provided $|\varepsilon|$ is sufficiently small. For such an ε, the $\pi_{\lambda_\varepsilon}$ is a submersion and $\bar{\pi}_{\lambda_\varepsilon}$ is a diffeomorphism. The orbit space $L/\mathbb{C}_{\lambda_\varepsilon}$ has a natural structure of a complex analytic space with respect to which $\pi_{\lambda_\varepsilon}$ is analytic. We have shown above that $L/\mathbb{C}_{\lambda_\varepsilon}$ is a Hausdorff manifold and that $\pi_{\lambda_\varepsilon}$ is a submersion. It follows that $\pi_{\lambda_\varepsilon}$ is the projection of a principal complex analytic bundle with fibre and structure group $\mathbb{C}$. □

Definition 4.2.2. *The manifold $S(L)$, with the complex structure induced from $L/\mathbb{C}_\lambda$ will be said to be of* linear type*, and will be denoted $S_\lambda(L)$.*

Remark 4.2.3. (i) Loeb and Nicolau [19] consider more general $\mathbb{C}$-actions on $\mathbb{C}^m \times \mathbb{C}^n$ in which the corresponding vector field is allowed to have higher order resonant terms. In our setup we have only to consider linear actions—the corresponding vector fields can at most have terms corresponding to resonant relations of the form "$\lambda_i = \lambda_j$".

(ii) When the P_i are maximal parabolics and L_i the negative generators of the Picard group of G_i/P_i, the smooth manifold $S(L_i)$ is simply-connected and is a homogeneous space H_i/Q_i where $H_i \subset G_i$ is simply-connected compact and Q_i is the centralizer of a circle-subgroup contained in H_i. Thus $S(L) = S(L_1) \times S(L_2)$ are among the manifolds classified by Wang [32] which admit *homogeneous* complex structures. In fact $S(L)$ admits complex structures invariant under the action of $H_1 \times H_2$. The homogeneous complex structures on $S(L)$ correspond to the scalar type. The more general linear type should be thought of as a deformation of the homogeneous complex structure constructed by Wang.

(iii) One has a commuting diagram

$$\begin{array}{ccc} L & \hookrightarrow & V(\varpi_1, \varpi_2) \\ \pi_\lambda \downarrow & & \downarrow \pi_{\lambda(\varpi_1,\varpi_2)} \\ S_\lambda(L) & \hookrightarrow & \mathbb{S}^{2r_1-1} \times \mathbb{S}^{2r_2-1} \end{array}$$

in which the horizontal maps are holomorphic and the vertical maps, projections of holomorphic principal $\mathbb{C}$-bundles.

Chapter 5

Cohomology of $S_\lambda(L)$

In this Chapter we shall compute the cohomology groups of the complex manifold $S_\lambda(L)$ with values in the structure sheaf. Using this we shall compute the Picard group and the algebraic dimension of $S_\lambda(L)$.

Let $L_i \longrightarrow X_i, i = 1, 2$ be holomorphic principal $\mathbb{C}^*$-bundle over complex projective manifolds X_i, with $\dim X_i \geq 1$. We assume that the principal $\mathbb{C}^*$-bundle $L_i \longrightarrow X_i$ is associated to a negative ample line bundle $\bar{L}_i \longrightarrow X_i$. Further we assume that X_i is arithmetically Cohen-Macaulay for the projective embedding determined by the ample line bundle $\bar{L}_i^*$. This means that the cone $\hat{L}_i$ over X_i is a Cohen-Macaulay affine analytic space.

We apply the Künneth formula established by A. Cassa [7, Teorema 3] to obtain the following lemma. We refer to Theorem 2.2.8 for the Künneth formula. Let $L = L_1 \times L_2$. Then $\mathcal{O}_{L_1 \times L_2} = pr_1^* \mathcal{O}_{L_1} \hat{\otimes} pr_2^* \mathcal{O}_{L_2}$, where pr_i^* denotes the projection $L_1 \times L_2 \longrightarrow L_i, i = 1, 2$. Here we denote the structure sheaf of an analytic variety Y by $\mathcal{O}_Y$.

Lemma 5.0.4. *Let $\bar{L}_i \longrightarrow X_i$ be negative ample holomorphic line bundle over a smooth projective variety $X_i, i = 1, 2$. Assume that X_i is arithmetically Cohen-Macaulay for the projective embedding determined by the ample line bundle $\bar{L}_i^*$. Then, $H^i(L; \mathcal{O}_L) = 0$ for $i \neq 0, \dim X_1, \dim X_2, \dim X_1 + \dim X_2$. Also, $H^0(L; \mathcal{O}_L) \cong H^0(L_1; \mathcal{O}_{L_1}) \hat{\otimes} H^0(L_2; \mathcal{O}_{L_2})$.*

Proof. By Corollary 2.1.8 we have $H^i(L_i, \mathcal{O}_{L_i}) = 0$ for $i \neq 0, \dim X_i$. Rest of the proof now follow readily by the Künneth formula 2.2.8.

□

Remark 5.0.5. We remark that the vanishing of the cohomology groups $H^q(L; \mathcal{O}_L)$ for $0 < q < \min\{\dim X_1, \dim X_2\}$ in the Lemma 5.0.4 (ii) follows from [2, Ch. I, Theorem 3.6]. To see this, set $\hat{L} := \hat{L}_1 \times \hat{L}_2 \setminus A$ where A is the closed analytic space $A = \hat{L}_1 \times \{a_2\} \cup \{a_1\} \times \hat{L}_2$. The ideal

$\mathcal{I} \subset \mathcal{O}_{\hat{L}}$ of A equals $\mathcal{I}_1.\mathcal{I}_2$ where $\mathcal{I}_1, \mathcal{I}_2$ are the ideals of the components $A_1 := \hat{L}_1 \times \{a_2\}, A_2 := \{a_1\} \times \hat{L}_2$ of A. Then $\operatorname{depth}_A \mathcal{O}_L = \operatorname{depth}_{\mathcal{I}} \mathcal{O}_{\hat{L}} = \min_j \{\operatorname{depth}_{\mathcal{I}_j} \mathcal{O}_{\hat{L}}\} = \min_j \{\operatorname{depth}_{a_j} \mathcal{O}_{\hat{L}_j}\} = \min\{\dim X_1 + 1, \dim X_2 + 1\}$. Thus we see that $\operatorname{depth}_A \mathcal{O}_{\hat{L}} = \min\{\dim X_1 + 1, \dim X_2 + 1\}$. Therefore $H^q(\hat{L}_1 \times \hat{L}_2; \mathcal{O}_{\hat{L}_1 \times \hat{L}_2}) \cong H^q(L; \mathcal{O}_L)$ if $q < \min\{\dim X_1, \dim X_2\}$ by [2, Ch. I, Theorem 3.6] where the isomorphism is induced by the inclusion. Since $\hat{L}_1 \times \hat{L}_2$ is Stein, the cohomology groups $H^q(L; \mathcal{O}_L)$ vanish for $1 \le q < \min\{\dim X_1, \dim X_2\}$.

Note that the hypothesis of the above lemma are satisfied in the case when $X_i, i = 1, 2$ are flag manifolds G_i/P_i where G_i is a semi-simple complex Lie group over $\mathbb{C}$, P_i is a parabolic subgroup and $\bar{L}_i$ any negative ample line bundle, over G_i/P_i. This follows because the flag manifold X_i are arithmetically Cohen-Macaulay for the projective embedding determined by any ample line bundles. For this fact refer to the Section 2.1.2. If we assume that L itself is very ample, then it is not possible to blow-down X. However, in this case, the following lemma allows one to compute the cohomology groups of L.

Lemma 5.0.6. *Let E be any holomorphic principal $\mathbb{C}^*$-bundle over a complex manifold X. Let E^* be the dual to E. Then $E \cong E^*$ as complex manifolds. In particular, $H^{p,q}_{\bar{\partial}}(E) \cong H^{p,q}_{\bar{\partial}}(E^*)$.*

Proof. Let $\psi : E \longrightarrow E^*$ be the map $v \mapsto v^*$ where $v^*(\lambda v) = \lambda \in \mathbb{C}$. Then ψ is a biholomorphism. □

Suppose that α_λ is an admissible $\mathbb{C}$-action on $L \longrightarrow X$ of scalar type, or diagonal type, or linear type. It is understood that in the case of diagonal type, there is a standard T_i-action on $X_i, i = 1, 2$, and that $X_i = G_i/P_i$ and L_i negative ample in the case of linear type action. Denote by v_λ (or more briefly v) the holomorphic vector field on L associated to the $\mathbb{C}$-action. Thus the $\mathbb{C}$-action is just the flow associated to v. We shall denote by $\mathcal{O}_v^{\mathrm{tr}}$ the sheaf of germs of local holomorphic functions which are constant along the $\mathbb{C}$-orbits. Thus $\mathcal{O}_v^{\mathrm{tr}}$ is isomorphic to $\pi_\lambda^*(\mathcal{O}_{S_\lambda(L)})$. One has an exact sequence of sheaves

$$0 \to \mathcal{O}_v^{\mathrm{tr}} \to \mathcal{O}_L \xrightarrow{v} \mathcal{O}_L \to 0. \tag{8}$$

Since the fibre of $\pi_\lambda : L \longrightarrow S_\lambda(L)$ is Stein, we see that $H^q(L; \mathcal{O}_v^{\mathrm{tr}}) \cong H^q(S_\lambda(L); \mathcal{O}_{S_\lambda(L)})$ for all q. Thus, the exact sequence (8) leads to the following long exact sequence:

$$0 \to H^0(S_\lambda(L); \mathcal{O}_{S_\lambda(L)}) \to H^0(L; \mathcal{O}_L) \to H^0(L; \mathcal{O}_L) \to H^1(S_\lambda(L); \mathcal{O}_{S_\lambda(L)}) \to$$

$$\cdots \to H^{q-1}(L;\mathcal{O}_L) \to H^q(S_\lambda(L);\mathcal{O}_{S_\lambda(L)}) \to H^q(L;\mathcal{O}_L) \to H^q(L;\mathcal{O}_L)$$
$$\to H^{q+1}(S_\lambda(L);\mathcal{O}_{S_\lambda(L)}) \to H^{q+1}(L;\mathcal{O}_L) \to H^{q+1}(L;\mathcal{O}_L) \to \cdots \qquad (9)$$

Theorem 5.0.7. *Suppose that $L = L_1 \times L_2$ where the L_i satisfy the hypotheses of Lemma 5.0.4. Suppose that $1 \le \dim X_1 \le \dim X_2$. Then $H^q(S_\lambda(L);\mathcal{O}) = 0$ provided $q \notin \{0, 1, \dim X_i, \dim X_i + 1, \dim X_1 + \dim X_2, \dim X_1 + \dim X_2 + 1; i = 1, 2\}$. Moreover one has $\mathbb{C} \subset H^1(S_\lambda(L);\mathcal{O})$, given by the constant functions in $H^0(L;\mathcal{O})$.*

Proof. The only assertion which remains to be explained is that the constant function 1 is not in the image of $v_* : H^0(L;\mathcal{O}) \longrightarrow H^0(L;\mathcal{O})$. All other assertions follow trivially from the long exact sequence (9) and the Lemma 5.0.4.

Suppose that $f : L \longrightarrow \mathbb{C}$ is such that $v(f) = 1$. This means that $\frac{d}{dz}|_{z=0}(f \circ \mu_p)(z) = 1$ for all $p \in L, z \in \mathbb{C}$, where $\mu_p : \mathbb{C} \longrightarrow L$ is the map $z \mapsto \alpha_\lambda(z).p = z.p$. Since $\mu_{w.p}(z) = z.(w.p) = (z+w).p = \mu_p(z+w)$, it follows that $\frac{d}{dz}|_{z=w}(f \circ \mu_p) = 1 \ \forall w \in \mathbb{C}$. Hence $f \circ \mu_p(z) = z + f(p)$. This means that the complex hypersurface $Z(f) := f^{-1}(0) \subset L$ meets each fibre at exactly one point. It follows that the projection $L \longrightarrow S_\lambda(L)$ restricts to a bijection $Z(f) \longrightarrow S_\lambda(L)$.

In fact, since $v(f) \neq 0$ we see that $Z(f)$ is smooth and since v_p is tangent to the fibres of the projection $L \longrightarrow S_\lambda(L)$ for all $p \in Z(f)$, we see that the bijective morphism of complex analytic manifolds $Z(f) \longrightarrow S_\lambda(L)$ is an immersion. It follows that $Z(f) \longrightarrow S_\lambda(L)$ is a biholomorphism. Thus $Z(f)$ is a *compact* complex analytic sub manifold of $L \subset \hat{L}$. Since $\hat{L}$ is Stein, this is a contradiction. □

5.1 Picard Group

For a complex manifold Y, the group of isomorphism class of line bundle on Y is isomorphic to the cohomology group $H^1(Y,\mathcal{O}^*) =: Pic(Y)$. We denote the kernel of the natural map $H^1(Y,\mathcal{O}) \longrightarrow H^1(Y,\mathcal{O}^*)$ by $Pic^0(Y)$. The vector space $Pic^0(Y)$ is isomorphic to the class of line bundles with trivial Chern class. Our next result concerns the Picard group $Pic(S_\lambda(L))$.

Proposition 5.1.1. *Let $L_i \longrightarrow X_i$ be as in the Theorem 5.0.4 . Suppose that X_i is simply connected. Then $Pic^0(S_\lambda(L)) \cong \mathbb{C}^l$ for some $l \ge 1$.*

Proof. Since $\bar{L}$ is negative ample, $c_1(L_i) \in H^2(X_i;\mathbb{Z})$ is a non-torsion element. Clearly $H^1(S_\lambda(L);\mathbb{Z}) = 0$ by a straightforward argument involving the Serre spectral sequence associated to the principal $\mathbb{S}^1 \times \mathbb{S}^1$-bundle with projection $S(L) \longrightarrow X_1 \times X_2$. Using the exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 1$

we see that $Pic^0(S_\lambda(L)) \cong H^1(S_\lambda(L);\mathcal{O}) \cong \mathbb{C}^l$. Now $l \geq 1$ by the Theorem 5.0.7. $\square$

The above proposition is applicable when $X_i = G_i/P_i$ and $\bar{L}_i$ are negative ample. However, in this case we have the following stronger result.

Theorem 5.1.2. *Let $X_i = G_i/P_i$ where P_i is any parabolic subgroup and let $\bar{L}_i \longrightarrow X_i$ be a negative line bundle, $i = 1,2$. We assume that, when $X_i = \mathbb{P}^1$, the bundle $\bar{L}_i$ is a generator of $Pic(X_i)$. Then $Pic^0(S_\lambda(L)) \cong \mathbb{C}$. If the P_i are maximal parabolics and the L_i are generators of $Pic(X_i) \cong \mathbb{Z}$, then $Pic(S_\lambda(L)) \cong Pic^0(S_\lambda(L)) \cong \mathbb{C}$.*

Proof. It is easy to see that $H^1(S(L);\mathbb{Z}) = 0$ and that, when P_i are maximal parabolics and L_i generators of $Pic(X_i) \cong \mathbb{Z}$, $S(L)$ is 2-connected. If $\dim X_i > 1$ for $i = 1,2$, then $H^1(L;\mathcal{O}) = 0$ by Theorem 5.0.4 and so we need only show that $coker(H^0(L;\mathcal{O}) \xrightarrow{v_*} H^0(L;\mathcal{O}))$ is isomorphic to $\mathbb{C}$. In case $\dim X_i = 1$—equivalently $X_i = \mathbb{P}^1$—$\bar{L}_i$ is the tautological bundle by our hypothesis. Thus $L_i = \mathbb{C}^2 \setminus \{0\}$. In this case we need to also show that $\ker(H^1(L;\mathcal{O}) \xrightarrow{v_*} H^1(L;\mathcal{O}))$ is zero. Note that the theorem is known due to Loeb and Nicolau [19, Theorem 2] when both the X_i are projective spaces and the $\bar{L}_i$ are negative ample generators—in particular when both $X_i = \mathbb{P}^1$.

The validity of the theorem for the case when λ is of diagonal type implies its validity in the linear case as well. This is because one has a family $\{L/\mathbb{C}_{\lambda_\epsilon}\}$ of complex manifolds parametrized by $\epsilon \in \mathbb{C}$ defined by $\lambda_\epsilon = \lambda_s + \lambda_{u,\epsilon}$, where $S_{\lambda_\epsilon}(L) = L/\mathbb{C}_{\lambda_\epsilon} \cong L/\mathbb{C}_\lambda$ if $\epsilon \neq 0$ and $\lambda_0 := \lambda_s$ is of diagonal type. (See §3.) The semi-continuity property ([15, Theorem 6, §4]) for $\dim H^1(S_{\lambda_\epsilon}(L);\mathcal{O})$ implies that $\dim H^1(S_\lambda(L);\mathcal{O}) \leq \dim H^1(S_{\lambda_s}(L);\mathcal{O})$. But Theorem 5.0.7 says that $\dim H^1(S_\lambda(L);\mathcal{O}) \geq 1$ and so equality must hold, if $H^1(S_{\lambda_s}(L);\mathcal{O}) \cong \mathbb{C}$. Therefore we may (and do) assume that the complex structure is of diagonal type.

First we show that $coker(v_* : H^0(L;\mathcal{O}) \longrightarrow H^0(L;\mathcal{O}))$ is 1-dimensional, generated by the constant functions. Consider the commuting diagram where $\widetilde{v}$ is the holomorphic vector field defined by the action of $\mathbb{C}$ given by $\lambda(\varpi_1,\varpi_2)$ on $V(\varpi_1,\varpi_2)$. Note that $\widetilde{v}_x = v_x$ if $x \in L$.

$$\begin{array}{ccc} H^0(V(\varpi_1,\varpi_2);\mathcal{O}) & \xrightarrow{\widetilde{v}_*} & H^0(V(\varpi_1,\varpi_2);\mathcal{O}) \\ \downarrow & & \downarrow \\ H^0(L;\mathcal{O}) & \xrightarrow{v_*} & H^0(L;\mathcal{O}) \end{array}$$

By Hartog's theorem, $H^0(V(\varpi_1,\varpi_2);\mathcal{O}) \cong H^0(V(\varpi_1) \times V(\varpi_2);\mathcal{O})$. Also, since $\hat{L}_i$ is normal at its vertex [25], again by Hartog's theorem, $H^0(L;\mathcal{O}) \cong H^0(\hat{L}_1 \times \hat{L}_2;\mathcal{O})$. Since $\hat{L}_i \subset V(\varpi_i)$ are closed sub varieties, it follows that the both vertical arrows, which are induced by the inclusion of L in $V(\varpi_1, \varpi_2)$, are surjective. From what has been shown in the proof of Theorem 5.0.7, we know that the constant functions are not in the cokernel of v_*. So it suffices to show that $coker(\widetilde{v}_*)$ is 1-dimensional. This was established in the course of proof of Theorem 2 of [19]. For the sake of completeness we sketch the proof. We identify $V(\varpi_i)$ with $\mathbb{C}^{r_i}$ where $r_i := \dim V(\varpi_i)$, by choosing a basis for $V(\varpi_i)$ consisting of weight vectors. Let $r = r_1 + r_2$ so that $\mathbb{C}^r \cong V(\varpi_1) \times V(\varpi_2)$. The problem is reduced to the following: Given a holomorphic function $f : \mathbb{C}^r \longrightarrow \mathbb{C}$ with $f(0) = 0$, solve for a holomorphic function ϕ satisfying the equation

$$\sum_i b_i z_j \frac{\partial \phi}{\partial z_i} = f, \tag{10}$$

where we may (and do) assume that $\phi(0) = 0$. In view of the *Observation* made preceding the statement of Theorem 4.2.1, we need only to consider the case where $(b_i) = (\lambda_\mu, \lambda_\nu)_{\mu \in \Lambda(\varpi_1), \nu \in \Lambda(\varpi_2)} \in \mathbb{C}^r$ satisfies the weak hyperbolicity condition of type (r_1, r_2). Denote by $z^{\mathbf{m}}$ the monomial $z_1^{m_1} \dots z_n^{m_r}$ where $\mathbf{m} = (m_1, \dots, m_r)$ and by $|\mathbf{m}|$ its degree $\sum_{1 \le j \le r} m_j$. Let $f(z) = \sum_{|\mathbf{m}|>0} a_{\mathbf{m}} z^{\mathbf{m}} \in H^0(\mathbb{C}^r; \mathcal{O})$. Then $\phi(z) = \sum a_{\mathbf{m}}/(b.\mathbf{m}) z^{\mathbf{m}}$ where $b.\mathbf{m} = \sum b_j m_j$ is the unique solution of Equation (10). Note that weak hyperbolicity and the fact that $|\mathbf{m}| > 0$ imply that $b.\mathbf{m} \neq 0$, and, $b.\mathbf{m} \to \infty$ as $\mathbf{m} \to \infty$. Therefore ϕ is a convergent power series and so $\phi \in H^0(\mathbb{C}^r; \mathcal{O})$.

It remains to show that, when $X_1 = \mathbb{P}^1$, $L_1 = \mathbb{C}^2 \setminus \{0\}$, and $\dim X_2 > 1$, the homomorphism $v_* : H^1(L; \mathcal{O}) \longrightarrow H^1(L; \mathcal{O})$ is injective. Let $z_j, 1 \le j \le r$, denote the coordinates of $\mathbb{C}^2 \times V(\varpi_2)$ with respect to a basis consisting of $\widetilde{T}$-weight vectors. Since $\dim X_2 > 1$, we have $H^1(L_2, \mathcal{O}) = 0$. Also $H^1(L_1; \mathcal{O}) = H^1(\mathbb{C}^2 \setminus \{0\}; \mathcal{O})$ is the space $\mathcal{A}$ of convergent power series $\sum_{m_1, m_2 < 0} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}$ in z_1^{-1}, z_2^{-1} without constant terms.

By Lemma 5.0.4 and Künneth formula 2.2.8, $H^1(L; \mathcal{O}) = \mathcal{A} \hat{\otimes} H^0(L_2; \mathcal{O}) \cong \mathcal{A} \hat{\otimes} H^0(\hat{L}_2; \mathcal{O})$. Let $\mathcal{I} \subset H^0(V(\varpi_2); \mathcal{O})$ denote the ideal of functions vanishing on $\hat{L}_2$ so that $H^0(\hat{L}_2; \mathcal{O}) = H^0(V(\varpi_2); \mathcal{O})/\mathcal{I}$. One has the commuting diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{A}\hat{\otimes}\mathcal{I} & \to & \mathcal{A}\hat{\otimes}H^0(V(\varpi_2);\mathcal{O}) & \to & \mathcal{A}\hat{\otimes}H^0(L_2;\mathcal{O}) & \to 0 \\
 & & \widetilde{v}_* \downarrow & & \widetilde{v}_* \downarrow & & v_* \downarrow & \\
0 & \to & \mathcal{A}\hat{\otimes}\mathcal{I} & \to & \mathcal{A}\hat{\otimes}H^0(V(\varpi_2);\mathcal{O}) & \to & \mathcal{A}\hat{\otimes}H^0(L_2;\mathcal{O}) & \to 0
\end{array}$$

where the rows are exact. Theorem 2 of [19] implies that

$$\widetilde{v}_* : \mathcal{A}\hat{\otimes}H^0(V(\varpi_2);\mathcal{O}) \longrightarrow \mathcal{A}\hat{\otimes}H^0(V(\varpi_2);\mathcal{O})$$

is an isomorphism. As before, this is equivalent to showing that Equation (10) has a (unique) solution ϕ without constant term when $f = \sum_{\mathbf{m}} c_{\mathbf{m}} z^{\mathbf{m}} \in \mathcal{A}\hat{\otimes}H^0(V(\varpi_2);\mathcal{O})$, is any convergent power series in $z_1^{-1}, z_2^{-1}, z_j, 3 \leq z_j \leq r$, where the sum ranges over $\mathbf{m} = (m_1, m_2, \ldots, m_r) \in \mathbb{Z}^r, m_1, m_2 < 0, m_j \geq 0,\ \forall j \geq 3$. It is clear that $\phi(z) = \sum c_{\mathbf{m}}/(b.\mathbf{m})z^{\mathbf{m}}$ is the unique formal solution. Note that weak hyperbolicity condition implies that $b.\mathbf{m} \neq 0$ and $b.\mathbf{m} \to \infty$ as $\sum_{j\geq 1} |m_j| \to \infty$. So $\phi(z)$ is a well-defined convergent power series in the variables $z_1^{-1}, z_2^{-1}, z_j, j \geq 3$ and is divisible by $z_1^{-1}z_2^{-1}$. Hence $\phi \in \mathcal{A}\hat{\otimes}H^0(V(\varpi_2);\mathcal{O})$ and so $\widetilde{v}_* : \mathcal{A}\hat{\otimes}H^0(V(\varpi_2);\mathcal{O}) \longrightarrow \mathcal{A}\hat{\otimes}H^0(V(\varpi_2);\mathcal{O})$ is an isomorphism. The ideal $\mathcal{I}$ is stable under the action of $\widetilde{T}_2$, and so is generated as an ideal by (finitely many) polynomials in $z_3, \ldots, z_n$ which are $\widetilde{T}_2$-weight vectors. In particular, the generators are certain homogeneous polynomials $h(z_3, \ldots, z_n)$ such that $\widetilde{v}_*(z_1^{m_1}z_2^{m_2}h) = b.\mathbf{m}z_1^{m_1}z_2^{m_2}h\ \forall m_1, m_2 \in \mathbb{Z}$ where $z^{\mathbf{m}}$ is any monomial that occurs in $z_1^{m_1}z_2^{m_2}h$. It follows easily that $\widetilde{v}_*$ maps $\mathcal{A}\hat{\otimes}\mathcal{I}$ isomorphically onto itself. A straightforward argument involving diagram chase now shows that $v_* : \mathcal{A}\hat{\otimes}H^0(L_2;\mathcal{O}) \longrightarrow \mathcal{A}\hat{\otimes}H^0(L_2;\mathcal{O})$ is an isomorphism. This completes the proof. $\square$

Remark 5.1.3. In the case when X_1 is any projective space $\mathbb{P}^{r_1}$ and L_1 is the tautological bundle over X_1, then the map $v^* : H^{r_1}(L, \mathcal{O}) \longrightarrow H^{r_1}(L, \mathcal{O})$ is a isomorphism. This can be showed as similar to the case $r_1 = 1$ which is done in the last part of the previous theorem. Using this, as in the Theorem 5.0.7, we can deduce that $H^{r_1}(S_\lambda(L), \mathcal{O}) = 0, r_1 > 1$. Moreover if $|r_1 - r_2| > 2$ then we can further deduce that $H^{r_1+1}(S_\lambda(L), \mathcal{O}) = 0$. This is slight improvement of the Theorem 5.0.7 .

Assume that $P_i \subset G_i$ are maximal parabolics and the $\bar{L}_i$ are the negative ample generators of the $Pic(G_i/P_i) \cong \mathbb{Z}$. We have the following description of the principal $\mathbb{C}$-bundles over $S_\lambda(L)$. Let $z \neq 0$. Let $\{g_{ij}\}$ be a 1-cocyle defining the principal $\mathbb{C}$-bundle $L \longrightarrow S_\lambda(L)$. Then the $\mathbb{C}$-bundle L_z representing the element $z[L] \in H^1(S_\lambda(L);\mathcal{O})$ is defined by the cocylce $\{zg_{i,j}\}$ for any $z \in \mathbb{C}$. We denote the corresponding $\mathbb{C}$-bundle by L_z. Note that the total space and the projection are the same as that of L. The $\mathbb{C}$-action on L_z is related to that on L where $w.v \in L_z$ equals $(w/z).v = \alpha_\lambda(w/z)(v) \in L$ for $w \in \mathbb{C}, v \in L$. The vector field corresponding to the $\mathbb{C}$-action on L_z is given by $(1/z)v_\lambda$. Of course, when $z = 0$, L_z is just the product bundle.

We shall denote the line bundle (i.e. rank 1 vector bundle) corresponding to L_z by E_z. Observe that if $z \neq 0$

$$E_z = L_z \times_{\mathbb{C}} \mathbb{C}, \text{ where } (w.v, t) \sim (v, \exp(2\pi\sqrt{-1}w)t),\ w, t \in \mathbb{C}, v \in L_z.$$

If $z \neq 0$, any cross-section $\sigma : S_\lambda(L) \longrightarrow E_z = L_z \times_{\mathbb{C}} \mathbb{C}$ corresponds to a holomorphic function $h_\sigma : L \longrightarrow \mathbb{C}$ which satisfies the following:

$$h_\sigma(w.v) = \exp(-2\pi\sqrt{-1}w)h_\sigma(v) \tag{11}$$

for all $v \in L_z, w \in \mathbb{C}$. Equivalently, this means that

$$h_\sigma(\alpha_\lambda(w)v) = \exp(-2\pi\sqrt{-1}wz)h_\sigma(v) \text{ for } w \in \mathbb{C} \text{ and } v \in L.$$

This implies that

$$v_\lambda(h_\sigma) = -2\pi\sqrt{-1}zh_\sigma. \tag{12}$$

Conversely, if h satisfies (11), then it determines a unique cross-section of E_z over $S_\lambda(L)$.

5.2 Algebraic Dimension

For a complex manifolds Y, we shall denote the field of meromorphic functions on Y by $\mathcal{M}(Y)$. The algebraic dimension of a complex manifold Y is the transcendence degree $tr.deg_{\mathbb{C}}(\mathcal{M}(Y)$. We have the following result concerning the field of meromorphic functions on $S_\lambda(L)$ with diagonal type complex structure. The proof will be given after some preliminary observations. In the end of the section we shall give explicitly the algebraic dimension of $S_\lambda(L)$.

Theorem 5.2.1. *Let L_i be the negative ample generator of $Pic(G_i/P_i) \cong \mathbb{Z}$ where P_i is a maximal parabolic subgroup of G_i, i=1,2. Assume that $S_\lambda(L)$ is of diagonal type. Then the field $\kappa(S_\lambda(L))$ of meromorphic functions of $S_\lambda(L)$ is purely transcendental over $\mathbb{C}$. The transcendence degree of $\kappa(S_\lambda(L))$ is less than* $\dim S_\lambda(L)$.

Let U_i denote the *opposite big cell*, namely the B_i^--orbit of $X_i = G_i/P_i$ the identity coset where B_i^- is the Borel subgroup of G_i opposed to B_i. One knows that U_i is a Zariski dense open subset of X_i and is isomorphic to $\mathbb{C}^{r_i}$ where r_i is the number of positive roots in the unipotent part $P_{i,u}$ of P_i. The bundle $\pi_i : L_i \longrightarrow X_i$ is trivial over U_i and so $\widetilde{U}_i := \pi_i^{-1}(U_i)$ is isomorphic to $\mathbb{C}^{r_i} \times \mathbb{C}^*$. We shall now describe a specific isomorphism which will be used in the proof of the above theorem.

Consider the projective imbedding $X_i \subset \mathbb{P}(V(\varpi_i))$. Let $v_0 \in V(\varpi_i)$ be a highest weight vector so that P_i stabilizes $\mathbb{C}v_0$; equivalently, $\pi_i(v_0)$ is the identity coset in X_i. Let $Q_i \subset P_i$ be the isotropy at $v_0 \in V(\varpi_i)$ for the G_i so that $G_i/Q_i = L_i$. The Levi part of P_i is equal to centralizer of a one-dimensional torus $\mathcal{Z}$ contained in T and projects onto $P_i/Q_i \cong \mathbb{C}^*$, the structure group of $L_i \longrightarrow X_i$.

Let $F_i \in H^0(X_i; \bar{L}_i^*) = V(\varpi_i)^*$ be the lowest weight vector such that $F_i(v_0) = 1$. Then $U_i \subset X_i$ is precisely the locus $F_i \neq 0$ and $F_i|_{\pi_i^{-1}([v])} : \mathbb{C}v \longrightarrow \mathbb{C}$ is an isomorphism of vector spaces for $v \in \widetilde{U}_i$. We denote also by F_i the restriction of F_i to $\widetilde{U}_i$.

Let Y_β be the Chevalley basis element of $Lie(G_i)$ of weight $-\beta, \beta \in R^+(G_i)$. We shall denote by $X_\beta \in Lie(G_i)$ the Chevalley basis element of weight $\beta \in R^+(G_i)$. Recall that $H_\beta := [X_\beta, Y_\beta] \in Lie(T)$ is non-zero whereas $[X_\beta, Y_{\beta'}] = 0$ if $\beta \neq \beta'$.

Let $R_{P_i} \subset R^+(G_i)$ denote the set of positive roots of G_i complementary to positive roots of Levi part of P_i and fix an ordering on it. (Thus $\beta \in R_{P_i}$ if and only if $-\beta$ is a not a root of P_i.) Let $r_i = |R_{P_i}| = \dim X_i$. Then $Lie(P_{i,u}^-) \cong \mathbb{C}^{r_i}$ where $P_{i,u}^-$ denotes the unipotent radical of the parabolic opposed to P_i. Observe that $P_i \cap P_{i,u}^- = \{1\}$. The exponential map defines an *algebraic* isomorphism $\theta : \mathbb{C}^{r_i} \cong Lie(P_{i,u})^- \longrightarrow U_i$ where $\theta((y_\beta)_{\beta \in R_{P_i}}) = (\prod_{\beta \in R_{P_i}} \exp(y_\beta Y_\beta)).P_i \in G_i/P_i$. It is understood that, here and in the sequel, the product is carried out according to the ordering on R_{P_i}.

If $v \in \mathbb{C}v_0$, then θ factors through the map $\theta_v : \mathbb{C}^{r_i} \cong Lie(P_{i,u}^-) \longrightarrow \widetilde{U}_i$ defined by $(y_\beta)_{\beta \in R_{P_i}} \mapsto \prod \exp(y_\beta Y_\beta).v$. Moreover, F_i is constant —equal to $F_i(v)$—on the image of θ_v.

We define $\widetilde{\theta} : \mathbb{C}^{r_i} \times \mathbb{C}^* \cong Lie(P_{i,u}^-) \times \mathbb{C}^* \cong P_{i,u}^- \times \mathbb{C}^* \longrightarrow \widetilde{U}_i$ to be $\widetilde{\theta}((y_\beta), z) = (\prod \exp(y_\beta Y_\beta)).zv_0 = \theta_{z.v_0}((y_\beta))$. This is an isomorphism. We obtain coordinate functions $z, y_\beta, \beta \in R_{P_i}$ by composing $\widetilde{\theta}^{-1}$ with projections $\mathbb{C}^{r_i} \times \mathbb{C}^* \longrightarrow \mathbb{C}$. Note that $F_i(\widetilde{\theta}((y_\beta), z))) = z$. Thus the coordinate function z is identified with F_i.

Since F_i is the lowest weight vector (of weight $-\varpi_i$), $Y_\beta F_i = 0$ for all $\beta \in R^+(G_i)$. Define $F_{i,\beta} := X_\beta(F_i), \beta \in R_{P_i}$. Then $Y_\beta(F_{i,\beta}) = -[X_\beta, Y_\beta]F_i = -H_\beta(F_i) = \varpi_i(H_\beta)F_i$ for all $\beta \in R_{P_i}$. Note that $\varpi_i(H_\beta) \neq 0$ as $H_\beta \in R_{P_i}$. If $\beta', \beta \in R_{P_i}$ are unequal, then $Y_{\beta'} F_{i,\beta} = 0$. It follows that $Y_{\beta'}^m(F_{i,\beta}) = 0$ unless $\beta' = \beta$ and $m = 1$.

The following result is well-known to experts in standard monomial theory. (See [18].)

Lemma 5.2.2. *With the above notations, the map $\widetilde{U}_i \longrightarrow \mathbb{C}^{r_i} \times \mathbb{C}^*$ defined as $v \mapsto ((F_{i,\beta}(v))_{\beta \in R^+_{P_i}}; F_i(v)),\ v \in \widetilde{U}_i$, is an algebraic isomorphism for $i = 1, 2$.*

Proof. It is easily verified that $\partial f/\partial y_\beta|_{v_0} = Y_\beta(f)(v_0)$ for any local holomorphic function defined in a neighborhood of v_0. (Cf. [18].)

Let $y = \widetilde{\theta}((y_\gamma), z) = \prod_{\gamma \in R_{P_i}}(\exp(y_\gamma Y_\gamma) \in P_i^-$. Denote by $l_y : \widetilde{U}_i \longrightarrow \widetilde{U}_i$ the left multiplication by y. If $v = y.v_0 \in \widetilde{U}_i$, then $(\partial/\partial y_\beta|_v)(f)$ equals

$(\partial/\partial y_\beta)|_{v_0}(f\circ l_y)$. Taking $f = F_{i,\beta}, \beta \in R_{P_i}$ a straightforward computation using the observation made preceding the lemma, we see that $(\partial/\partial y_\beta|_v)(F_{i,\gamma}) = Y_\beta|_{v_0}(F_{i,\gamma} \circ l_y) = F_i(v)\varpi_i(H_\beta)\delta_{\beta,\gamma}$ (Kronecker δ). We also have

$$(\partial/\partial y_\beta|_v)(F_i) = 0 \text{ for all } v \in \widetilde{U}_i.$$

Hence $(\partial/\partial y_\beta)|_v(F_{i,\gamma}/F_i) = \varpi_i(H_\beta)\delta_{\beta,\gamma}$. Thus the Jacobian matrix relating the $F_{i,\beta}/F_i$ and the $y_\beta, \beta \in R_{P_i}$, is a diagonal matrix of *constant functions.* The diagonal entries are non-zero as $\varpi_i(H_\beta) \neq 0$ for $\beta \in R_{P_i}$ and since F_i is nowhere vanishing, the lemma follows. □

We shall use the coordinate functions $F_i, F_{i,\beta}, \beta \in R_{P_i}$, to write Taylor expansion for analytic functions on $\widetilde{U}_i$. In particular, the coordinate ring of the affine variety $\widetilde{U}_i$ is just the algebra $\mathbb{C}[F_{i,\beta}, \beta \in R_{P_i}][F_i, F_i^{-1}]$. The projective normality [25] of G_i/P_i implies that $\mathbb{C}[\hat{L}_i] = \oplus_{r\geq 0}H^0(X_i; L_i^{-r}) = \oplus_{r\geq 0}V(r\varpi_i)^*$. Since $\widetilde{U}_i$ is defined by the non-vanishing of F_i, we see that $\mathbb{C}[\widetilde{U}_i] = \mathbb{C}[\hat{L}_i][1/F_i]$.

Example 5.2.3. In the particular case of Grassmannian, we shall describe the lemma explicitly as follow.

Let X be the Grassmannian $\mathbb{G}_{4,2}$, the space of all vector subspace of dim 2 in $\mathbb{C}^4$. Let $M_{4,2}$ be the space of 4×2 matrices of rank 2. Let $\sigma : M_{4,2} \longrightarrow \mathbb{G}_{4,2}$ be the projection map, where $\pi(A)$ is the subspace of $\mathbb{C}^4$ generated by the column vectors of A. The flag manifold X can be identified with the quotient $M_{4,2}/\sim$, where $\sim$ is the relation in which $A \sim B$ if and only if there exit $C \in GL(2,\mathbb{C})$ such that $A = BC$. We have the Plücker embedding $\theta : \mathbb{G}_{4,2} \hookrightarrow \mathbb{P}(\wedge^2\mathbb{C}^4)$; $\theta(A) = v_1 \wedge v_2$ where v_1 and v_2 are column vectors of A. Let $I(4,2) = \{I = (i_1.i_2) \mid 1 \leq i_1 < i_2 \leq 4\}$. We denote the homogeneous coordinates of points in $\mathbb{P}(\wedge^2\mathbb{C}^4)$ by $(p_I, I \in I(4,2))$. Then for $A \in \mathbb{G}_{4,2}$, p_I is 2-minor of A with row indices $I = (i_1, i_2)$. Fix $I_0 = (1,2)$. We denote by the open subset $U_{I_0} \subset \mathbb{G}_{4,2}$ the locus $p_{I_0} \neq 0$. Every element $A \in U_{I_0}$ has unique representation of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \quad ; \quad x_i \in \mathbb{C} \text{ for } 1 \leq i \leq 4.$$

This gives a structure of affine space to U_{I_0} with coordinate function x_1,x_2,x_3, and x_4. The functions p_I/p_{I_0}, $I \in I(4,2)$, gives a well defined functions on $U_{I_0} \cong \mathbb{C}^4$. These functions can be expressed in terms of coordinate x_1,x_2,x_3 and x_4 as follows: $p_{13}/p_{12} = x_2, p_{14}/p_{12} = x_4, p_{23}/p_{12} = -x_1, p_{24}/p_{12} =$

$-x_3, p_{34}/p_{12} = x_1x_4 - x_2x_3$. This give a new coordinate system on U_{I_0} with coordinate functions $p_{13}/p_{12}, p_{14}/p_{12}, p_{23}/p_{12}$ and p_{24}/p_{12}. Let $\pi : E(\wedge^2\gamma_{4,2}) \longrightarrow \mathbb{G}_{4,2}$ be the holomorphic principal $\mathbb{C}^*$-bundle over $\mathbb{G}_{4,2}$, corresponding to the tautological bundle $\wedge^2\gamma_{4,2}$. The Plücker embedding $\theta : \mathbb{G}_{4,2} \hookrightarrow \mathbb{P}(\wedge^2\mathbb{C}^4)$ can be "lifted" to an embedding $E(\wedge^2\gamma_{4,2}) \hookrightarrow \wedge^2\mathbb{C}^4 \setminus \{0\}$. Let $\widetilde{U}_{I_0} := \pi^{-1}(U_{I_0})$ be the open subset of $E(\wedge^2\gamma_{4,2})$. Then $\widetilde{U}_{I_0} \cong \mathbb{C}^* \times U_{I_0}$. The functions $p_{12}, p_{13}/p_{12}, p_{14}/p_{12}, p_{23}/p_{12}$ and p_{24}/p_{12} gives a coordinate functions on $\widetilde{U}_{I_0}$. Since $p_{12} \neq 0$ on $\widetilde{U}_{I_0}$, the functions $p_{12}, p_{13}, p_{14}, p_{23}$ and p_{24} gives a new coordinate system on $\widetilde{U}_{I_0}$.

Now let $X = X_1 \times X_2$ and $\widetilde{T} = \widetilde{T}_1 \times \widetilde{T}_2 \cong (\mathbb{C}^*)^N, N = n_1 + n_2$, where the isomorphism is as chosen in §3. Let $d_i > 0, i = 1, 2$, be chosen as in Proposition 4.1.1 so that the $\widetilde{T}_i$-action on $L_i \longrightarrow G_i/P_i$ is d_i-standard. Let $\lambda = \lambda_s \in Lie(\widetilde{T})$. Suppose that λ satisfies the weak hyperbolicity condition of type (n_1, n_2).

Recall from (6) and (7) that for any weight $\mu_i \in \Lambda(\varpi_i)$, there exist elements $\lambda_{\mu_1}, \lambda_{\mu_2} \in \mathbb{C}$ such that for any $v = (v_1, v_2) \in V_{\mu_1}(\varpi_1) \times V_{\mu_2}(\varpi_2)$, the α_λ-action of $\mathbb{C}$ is given by $\alpha_\lambda(z)v = (\exp(z\lambda_{\mu_1})v_1, \exp(z\lambda_{\mu_2})v_2)$. In fact $\lambda_{\mu_i} = \sum_{n_{i-1}<j\leq n_{i-1}+n_i} d_{\mu_i,j}\lambda_j$ where $d_{\mu_i,j}$ are certain *non-negative integers.* It follows that, as observed in the discussion preceding the statement of Theorem 4.2.1, the complex numbers $\lambda_{\mu_i} \in \mathbb{C},\ \mu_i \in \Lambda(\varpi_i), i = 1, 2$ satisfy weak hyperbolicity condition:

$$0 \leq \arg(\lambda_{\mu_1}) < \arg(\lambda_{\mu_2}) < \pi,\ \forall \mu_i \in \Lambda(\varpi_i), i = 1, 2. \tag{13}$$

We observe that if $\mu = \mu_1 + \cdots + \mu_r = \nu_1 + \cdots + \nu_r$, where $\mu_j, \nu_j \in \Lambda(\varpi_i)$, then $\lambda_{\mu,r} := \sum \lambda_{\mu_j} = \sum \lambda_{\nu_j}$. (This is a straightforward verification using (5) and (6).) Therefore, if $v \in V(\varpi_i)^{\otimes r}$ is any weight vector of weight μ, we get, for the diagonal action of $\mathbb{C}$, $z.v = \exp(\lambda_{\mu,r}z)v$.

Any finite dimensional $\widetilde{G}_i$-representation space V is naturally $\widetilde{G}_1 \times \widetilde{G}_2$-representation space and is a direct sum of its $\widetilde{T}$-weight spaces V_μ. If V arises from a representation of G_i via $\widetilde{G}_i \longrightarrow G_i$, then the $\widetilde{T}$-weights of V are the same as T-weights.

Definition 5.2.4. *Let $Z_i(\lambda) \subset \mathbb{C}, i = 1, 2,$ be the abelian subgroup generated by $\lambda_\mu, \mu \in \Lambda(\varpi_i)$ and let $Z(\lambda) := Z_1(\lambda) + Z_2(\lambda) \subset \mathbb{C}$.*

The λ-weight *of an element $0 \neq f \in Hom(V_\mu(\varpi_i); \mathbb{C})$ is defined to be $wt_\lambda(f) := \lambda_\mu$. If $h \in Hom(V(\varpi_i)^{\otimes r}, \mathbb{C})$ is weight vector of weight $-\mu$, (so that $h \in Hom(V_\mu(\varpi_i)^{\otimes r}; \mathbb{C})$) we define the* λ-weight of *of h to be $\lambda_{\mu,r}$.*

If $f \in Hom(V_\mu(r\varpi_i), \mathbb{C})$ is a weight vector (of weight $-\mu$), then it is the image of a unique weight vector $\widetilde{f} \in Hom(V(\varpi_i)^{\otimes r}, \mathbb{C})$ under the surjection *induced by the $\widetilde{G}_i$-inclusion* $V(r\varpi_i) \hookrightarrow V(\varpi_i)^{\otimes r} = V(r\varpi_i) \oplus V'$ *where* $\widetilde{f}|V' = 0$. *We define the* λ-weight of f *to be* $wt_\lambda(f) := wt_\lambda(\widetilde{f})$.

If $h_i \in V(r_i\varpi_i)^* \subset \mathbb{C}[\hat{L}_i]$, $i = 1, 2$, are weight vectors, then h_1h_2 is a weight vector of $V(r_1\varpi_1)^* \otimes V(r_2\varpi_2)^* \subset \mathbb{C}[\hat{L}_1 \times \hat{L}_2]$ and we have $wt_\lambda(h_1h_2) = wt_\lambda(h_1) + wt_\lambda(h_2) \in Z(\lambda)$. Note that $wt_\lambda(f_1 \dots f_k) = \sum_{1 \le j \le k} wt_\lambda(f_j) \in Z(\lambda)$ where $f_j \in \mathbb{C}[\hat{L}_1 \times \hat{L}_2] = \oplus_{r_1, r_2 \ge 0} V(r_1\varpi_1)^* \otimes V(r_2\varpi_2)^*$ are weight vectors. Also $wt_\lambda(f) \in Z(\lambda)$ is a *non-negative* linear combination of $\lambda_j, 1 \le j \le N$ for any $\widetilde{T}$-weight vector $f \in \mathbb{C}[\hat{L}_1 \times \hat{L}_2]$.

If $f \in V(\varpi_i)^*$, it defines a holomorphic function on $V(\varpi_1) \times V(\varpi_2)$ and hence on L, and denoted by the same symbol f; explicitly $f(u_1, u_2) = f(u_i)$, $\forall (u_1, u_2) \in L$.

Lemma 5.2.5. *We keep the above notations. Assume that* $\lambda = \lambda_s \in Lie(\widetilde{T}) = \mathbb{C}^N$. *Fix* $\mathbb{C}$-*bases* $\mathcal{B}_i$ *for* $V(\varpi_i)^*$, *consisting of* $\widetilde{T}$-*weight vectors. Let* $z_0 \in Z(\lambda)$. *There are only finitely many monomials* $f := f_1 \dots f_k$, $f_j \in \mathcal{B}_1 \cup \mathcal{B}_2$ *having* λ-*weight* z_0. *Furthermore,* $v_\lambda(f) = wt_\lambda(f)f$.

Proof. The first statement is a consequence of weak hyperbolicity (see (13)). Indeed, since $0 \le \arg(\lambda_\mu) < \pi$ for all $\mu \in \Lambda(\varpi_i), i = 1, 2$, given any complex number z_0, there are only finitely many non-negative integers c_j such that $\sum c_j \lambda_{\mu_j} = z_0$.

As for the second statement, we need only verify this for $f \in \mathcal{B}_i, i = 1, 2$. Suppose that $f \in \mathcal{B}_1$ and that f is of weight $-\mu$, $\mu \in \Lambda(\varpi_1)$, say. Then, for any $(u_1, u_2) \in L$, writing $u_1 = \sum_{\nu \in \Lambda(\varpi_1)} u_\nu$, using linearity and the fact that $f(u_1, u_2') = f(u_\mu)$ we have

$$\begin{aligned} v_\lambda(f)(u_1, u_2) &= \lim_{w \to 0}(f(\alpha_\lambda(w)(u_1, u_2)) - f(u_1, u_2))/w \\ &= \lim_{w \to 0}(f(\exp(\lambda_\mu w)u_\mu) - f(u_1))/w \\ &= \lim_{w \to 0}(\tfrac{\exp(\lambda_\mu w) - 1}{w})f(u_1) \\ &= \lambda_\mu f(u_1) \\ &= \lambda_\mu f(u_1, u_2). \end{aligned}$$

This completes the proof. $\square$

We assume that $F_i, F_{i,\beta}, \beta \in R_{P_i}$, are in $\mathcal{B}_i$, $i = 1, 2$.

Let $\mathcal{M} \subset \mathbb{C}(\widetilde{U}_1 \times \widetilde{U}_2)$ denote the multiplicative group of all Laurent monomials in $F_i, F_{i,\beta}, \beta \in R_{P_i}, i = 1, 2$. One has a homomorphism $wt_\lambda : \mathcal{M} \longrightarrow Z(\lambda)$. Denote by $\mathcal{K}$ the kernel of wt_λ. Evidently, $\mathcal{M}$ is a free abelian group of rank $\dim L$.

Lemma 5.2.6. *With the above notations,* $wt_\lambda : \mathcal{M} \longrightarrow Z(\lambda)$ *is surjective. Any* $\mathbb{Z}$-*basis* $h_1, \dots, h_k$ *of* $\mathcal{K}$ *is algebraically independent over* $\mathbb{C}$.

Proof. Suppose that $\nu \in Z_i(\lambda)$. Write $\nu = \sum a_\mu \lambda_\mu$ and choose $b_\mu \in \mathcal{B}_i$ to be of weight μ. Then $wt_\lambda(\prod_\mu b_\mu^{a_\mu}) = \nu$. On the other hand, $wt_\lambda(b_\mu)$ equals the λ-weight of any monomial in the $F_i^{-1}, F_i, F_{i,\beta}, \beta \in R_{P_i}$ that occurs in $b_\mu|\widetilde{U}_i$. The first assertion follows from this.

Let, if possible, $P(z_1, \ldots, z_k) = 0$ be a polynomial equation satisfied by $h_1, \ldots, h_k$. Note that the h_j are certain Laurent monomials in a transcendence basis of the field $\mathbb{C}(\widetilde{U}_1 \times \widetilde{U}_2)$ of *rational functions* on the affine variety $\widetilde{U}_1 \times \widetilde{U}_2$. Therefore there must exist monomials $z^{\mathbf{m}}$ and $z^{\mathbf{m}'}$, $\mathbf{m} \neq \mathbf{m}'$, occurring in $P(z_1, \ldots, z_k)$ with non-zero coefficients such that $h^{\mathbf{m}} = h^{\mathbf{m}'} \in \mathbb{C}(\widetilde{U}_1 \times \widetilde{U}_2)$. Hence $h^{\mathbf{m}-\mathbf{m}'} = 1$. This contradicts the hypothesis that the h_j are linearly independent in the multiplicative group $\mathcal{K}$. $\square$

We now turn to the proof of Theorem 5.2.1.

Proof of Theorem 5.2.1: By definition, any meromorphic function on $S_\lambda(L)$ is a quotient f/g where f and g are holomorphic sections of a holomorphic line bundle E_z. Any holomorphic section $f : S(L) \longrightarrow E_z$ defines a holomorphic function on L, denoted by f, which satisfies Equation (11). By the normality of $\hat{L}_1 \times \hat{L}_2$, the function f then extends uniquely to a function on $\hat{L}_1 \times \hat{L}_2$ which is again denoted f. Thus we may write $f = \sum_{r,s \geq 0} f_{r,s}$ where $f_{r,s} \in V(r\varpi_1)^* \otimes V(s\varpi_2)^*$. Now $v_\lambda f = af$ and $v_\lambda f_{r,s} \in V(r\varpi_1)^* \otimes V(s\varpi_2)^*$ implies that $v_\lambda(f_{r,s}) = af_{r,s}$ for all $r, s \geq 0$ where $a = -2\pi\sqrt{-1}z$. This implies that $wt_\lambda(f_{r,s}) = a$ for all $r, s \geq 0$. This implies, by Lemma 5.2.5, that $f_{r,s} = 0$ for sufficiently large r, s and so f is *algebraic*.

Now writing f and g restricted to $\widetilde{U}_1 \times \widetilde{U}_2$ as a polynomial in the the coordinate functions $F_i^{\pm}, F_{i,\beta}, i = 1, 2$, introduced above, it follows easily that f/g belongs to the field $\mathbb{C}(\mathcal{K})$ generated by $\mathcal{K}$. Evidently $\mathcal{K}$—and hence the field $\mathbb{C}(\mathcal{K})$—is contained in $\kappa(S_\lambda(L))$. Therefore $\kappa(S_\lambda(L))$ equals $\mathbb{C}(\mathcal{K})$. By Lemma 5.2.6 the field $\mathbb{C}(\mathcal{K})$ is purely transcendental over $\mathbb{C}$.

Finally, since $Z(\lambda)$ is of rank at least 2 and since $wt_\lambda : \mathcal{M} \longrightarrow Z(\lambda)$ is surjective, $tr.deg(\kappa(S_\lambda(L)) = rank(\mathcal{K}) \leq rank(\mathcal{K}) - 2 = \dim(L) - 2 = \dim(S_\lambda(L)) - 1$. $\square$

Remark 5.2.7. (i) We have actually shown that the transcendence degree of $\kappa(S_\lambda(L))$ equals the rank of $\mathcal{K}$. In the case when X_i are projective spaces, this was observed by [19]. When λ is of scalar type, $tr.deg(\kappa(S_\lambda(L)) = \dim(S_\lambda(L)) - 1$.

(ii) Theorem 5.2.1 implies that any algebraic reduction of $S_\lambda(L)$ is a rational variety. In the case of scalar type, one has an elliptic curve bundle $S_\lambda(L) \longrightarrow X_1 \times X_2$. (Cf. [29].) Therefore this bundle projection yields an algebraic reduction. In the general case however, it is an interesting problem

to construct explicit algebraic reductions of these compact complex manifolds. (We refer the reader to [23] and references therein to basic facts about algebraic reductions.)
(iii) We conjecture that $\kappa(S_\lambda(L))$ is purely transcendental for $X_i = G_i/P_i$ where P_i is any parabolic and $\bar{L}_i$ is any negative ample line bundle over X_i, where $S_\lambda(L)$ has any linear type complex structure.

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