ASPECTS OF ADS/CFT

By

Nilanjan Sircar

The Institute of Mathematical Sciences, Chennai.

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As members of the Viva Voce Board, we recommend that the dissertation prepared by **Nilanjan Sircar** entitled "Aspects of AdS/CFT" may be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and the work has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution or University.

Nilanjan Sircar

To,

Maa and Anneswa,

two most important persons in my life ......

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Abstract

This thesis is devoted to study of two aspects or use of AdS/CFT conjecture: One is the use of AdS/CFT in understanding the IR-cut-off appearing in the resolution of Hagedorn limiting temperature in String Theory via BFSS matrix model. And other is the use of AdS/CFT in understanding of low energy condensed matter system. A brief abstract of these studies are written separately below.

Hagedorn Phase Transition and Matrix Model for String Theory:
Hagedorn Temperature is the temperature at which the partition function diverges due to the exponential growth in the density of states, which overtakes the Boltzmann suppression factor. BFSS matrix model which describes M-theory in some limit, has membrane degrees of freedom. String like configurations can be constructed from basic degrees of freedom of the matrix model. Hagedorn temperature can be reinterpreted in terms of new degrees of freedom as a phase transition. In the first part of this thesis, we will construct two phases of matrix model ("string" phase and "clustered phase"), and compare their free energies to find signature of phase transition. It will be shown that, there exist phase transitions between these two configurations -but only in presence of an IR cut-off. The low temperature phase corresponds to a string (wrapped membrane) phase and so we call this the Hagedorn phase transition. While the presence of an IR cut-off seemingly is only required for perturbative analysis to be valid, the physical necessity of such a cut-off can be seen in the dual super-gravity side *using AdS/CFT conjecture.* Interestingly the perturbative analysis also shows a second phase transition back to a string phase. This is reminiscent of the Gregory-Laflamme instability.

Duality between Charged BTZ black hole and Luttinger liquids:
In the second part of thesis, we study properties of strongly coupled CFT's with non-zero background electric charge in 1+1 dimensions by studying the dual gravity theory - which is a charged BTZ black hole. We will calculate correlators of operators dual to scalars, gauge fields and fermions are studied at both $T = 0$ and $T \neq 0$. In the $T = 0$ case we will also able to compare with analytical results based on AdS_2 and find reasonable agreement. In particular the correlation between log periodicity and the presence of finite spectral density of gapless modes will be seen. The real part of the conductivity (given by the current-current correlator) also vanishes as $\omega \to 0$ as expected. The fermion Green's function shows

quasiparticle peaks with approximately linear dispersion but the detailed structure is neither Fermi liquid nor Luttinger liquid and bears some similarity to a "Fermi-Luttinger" liquid. This is expected since there is a background charge and the theory is not Lorentz or scale invariant. A boundary action that produces the observed non-Luttinger-liquid like behavior (k-independent non-analyticity at $\omega = 0$) in the Green's function is discussed.

List of Publications:

1. B. Sathiapalan and Nilanjan Sircar,
   "Can the Hagedorn Phase Transition be explained from Matrix Model for Strings?"
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   "Properties of CFTs dual to Charged BTZ black-hole"
   Nucl.Phys.B 839:526-551 (2010), arXiv:0909.4051 [hep-th].

Contents

List of Figures

List of Tables

The grand aim of all science is to cover the greatest number of empirical facts by logical deduction from the smallest number of hypotheses or axioms.

Albert Einstein

1 Introduction

Human mind always has tended to describe complex phenomena in Nature as collective phenomena of some basic elements and their interaction. The other tendency of human mind is to describe seemingly diverse phenomena in terms of a unique framework. This craving for simplicity and unification, has led human beings to an indefinite search for ultimate set of objects which underlie everything. This quest has lead to various possibilities, amongst which String Theory has been the leading candidate for the past few decades.

1.1 String Theory

The elementary objects in string theory are one dimensional (Fig.(1.1)) in contrast to usual zero dimensional elementary particles. All observed particles are various vibrational states of this fundamental string (open or closed). The theory also contains zero and higher dimensional surfaces, called D-branes (Dirichlet-branes), where open strings can end (Fig.(1.1)). Thus string theory provides a unique framework for studying various interactions found in nature. It is the leading candidate for a theory that unifies the four basic forces of nature *viz.* Electromagnetic, Weak, Strong and Gravitation. *World sheets* (2-dimensional surfaces) describe time evolution of strings and *world volume* for general higher dimensional D-branes (compared to *world line* for point particle)(Fig.(1.2)).

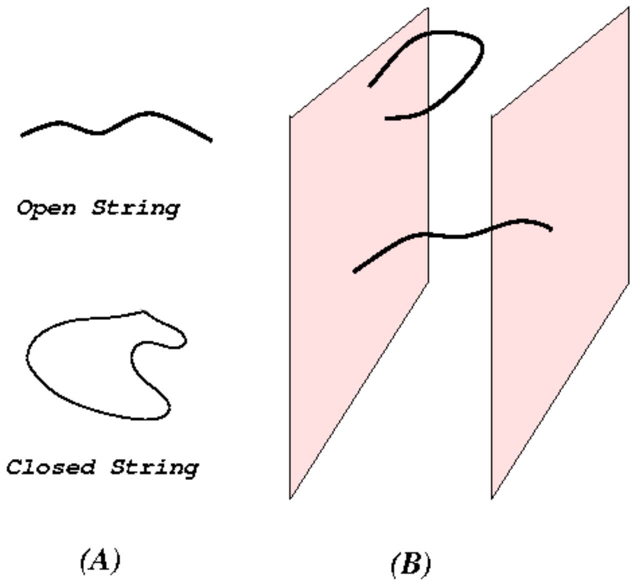

Figure 1.1: (A) Open and Closed String. (B) Open Strings ending at D2-branes.

We know *elementary particles* in our universe (*viz.* electron, quark.) are 0-dimensional-particle-like and are very well described by *Quantum Field Theory* (QFT, theory of *Fields* living in 3+1 dimensional space-time. *Quantization* of which gives elementary particles). Most successful example of this is *Standard Model*, proposed in mid 20th Century, describes all known particles and three of the four fundamental forces very well up to the accuracy of modern day experiments. So why do we need String Theory? Also, how do we explain the fact that experimentally “electrons“ are not one-dimensional? Firstly, usual methods of Quantum Field Theory does not work for gravity. It suffers from *ultra-violet* (UV) or short distance divergences. Avoiding this UV divergences is one of the triumphs of string theory. Furthermore it naturally has a spin-2 particle (candidate for gravitons, mediator of gravity) in the spectrum of closed strings. Spectrum of string theory also naturally has various other particles, which makes it a candidate for unified theory, which answers our first question. The size of these strings (l_s) could be of the order of 10^{-33} cm $\sim (10^{19} GeV)^{-1}$, which is much smaller than the distance scale probed by modern experiments. So strings of such small magnitude appears as point-like to modern *microscopes*; which answers our second question. Also to define a well defined string theory we have to demand a symmetry between *fermions* (half-integer spin particles) and *bosons* (integer spin particles), known as

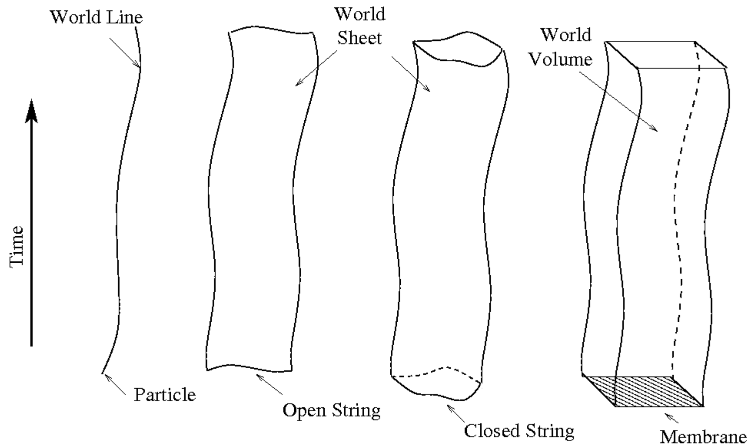

Figure 1.2: Time evolution of point particle, open strings, closed strings and Membrane/D2-brane

Supersymmetry. Strings with this symmetry are called Superstrings. Also consistency requires these superstrings to live in $9+1$ dimensional space-time, instead of $3+1$ dimensional space-time in which we live. Usual way out from this large number of space dimensions is to assume that they are curled up into very small size compared to typical length scale probed by current experiments ($\sim$ TeV). Consistency requirement also demands existence of five possible superstring theories : type IIA, type IIB, type I, SO(32) heterotic, and $E_8 \times E_8$ heterotic. But we expect to have a unique theory which describes our Nature. We will come back to the resolution of uniqueness in the next section. There are also various choices for compact spaces for each of these string theories, which leads to many different string theories in $3+1$ dimension! But, some of them are very close to observed universe, although a choice which has complete quantitative agreement with observations is yet to be found.

There is yet another paradox : infinite tower of vibrational states of string corresponds to infinite number of elementary particles! Present day experiment provides with an finite set of elementary particles. Again we are saved by the fact that string length scale is much smaller than the present day experimental scale ($\sim$ TeV). It turns out that there exists a finite set of vibrational states which corresponds to massless particles. The infinite tower of vibrational states corresponds to massive particles, with mass $\sim \frac{1}{l_s} \sim 10^{19} GeV$, which is out of the reach of modern experiments. So current set of known elementary particles belongs to the massless sector of string vibrations, and will be described very well by the *low energy effective field theory (EFT)* obtained from the low energy limit of string theory. The massless elementary particles can be given small mass compared to the string scale by methods of *symmetry breaking* available in QFT, as many of the particles found in nature are massive.

1.2 M(atrix) theory

As promised, let us address the issue of uniqueness. The five different superstring theories are related to one another by *duality symmetries viz. T-duality and S-duality.*

T duality: This duality usually maps theories compactified on different types of spaces. For example, Type IIA theory compactified on a circle of radius R is dual to Type IIB theory compactified on circle of radius $\frac{1}{R}$. In this duality the "momentum modes" (the momentum along the compact direction are forced to be discrete due to periodic nature) of one theory maps to "winding modes" (number of times the string wraps the periodic or compact direction) in the other and vice-verse. $SO(32)$ heterotic string theory compactified on a torus T^4 is dual to Type IIA string theory compactified on a compact manifold known as $K3$.

S duality: This duality maps a weakly coupled theory to a strongly coupled theory. If g and $\tilde{g}$ are coupling constants of two theories related by S duality, often have a simple relation of the form: $g = \tilde{g}^{-1}$. Thus perturbative expansion in one theory contains information about non-perturbative effects in the dual theory. For example, Type IIB theory is found to be S dual to itself. Type I theory is S dual to $SO(32)$ heterotic theory in $9+1$ dimensions.

As all the five different string theories are related to one another by these duality maps, it strongly suggests presence of a single underlying theory known as "M-theory" or sometimes referred to as "U-theory" or universal theory. In special limits, M-theory is described by one of the (compactified) five different weakly coupled string theories. Besides these five theories there exists a sixth one, which is $(10+1)$-dimensional supergravity theory (SUGRA). M-theory in low energy limit reduces to this 11 dimensional supergravity theory and the term M-theory sometimes refers to this low energy supergravity theory. 11 dimensional supergravity contains extended objects called $M2$ (2-dimensional membranes) and $M5$-branes which are electrically and magnetically charged under 3-form potential present in the supergravity theory.

Another way of defining M-theory is as the strong coupling limit of Type IIA string theory. The Type IIA string theory is taken equivalent to the M-theory compactified on a circle, where the radius of compactification is proportional to the string coupling. There exists correspondence between various objects in compactified M-theory and Type IIA string theory, *viz.* wrapped M2-brane on the compact direction corresponds to the fundamental string of the Type IIA theory.

Witten had suggested that "M" of M theory should stand for "Magic", "Mystery" or "Membrane" according to taste! For the purpose of this thesis "M" will stand for Matrix. BFSS (Banks, Fischler, Shenker and Susskind) matrix model [1] provides a systemic procedure for computing corrections to the low energy limit of M-theory or the 11 dimensional supergravity theory. The proposal was that large N limit of matrix quantum mechanics model should describe all of M-theory in an infinite momentum frame(IMF). Let us consider M-theory compactified on a circle (say, 11th direction). The quanta corresponding to the momenta along the compact direction corresponds to $D0$-branes of Type IIA string theory. In the limit where radius of compactification R and the compact momentum p_{11} are both taken to large, this correspondence relates M-theory in the IMF to the non-relativistic theory of many Type IIA $D0$-branes which is described by matrix quantum mechanics. This proposal provides a non-perturbative formulation of M-theory. In Matrix Model, all other entities like Dp-branes and string are viewed as composites of these fundamental $D0$-branes. A detailed description of this model is given in the main section of the thesis.

1.3 AdS/CFT

One of the major conceptual breakthroughs in string theory is *AdS/CFT Conjecture*. Since it has been proposed by Maldacena in 1997 for specific case of string theory [2, 3, 4], it has emerged as a more general class of dualities between two apparently different physical systems and has find its use in fields ranging from high energy physics to low energy condensed matter systems. The term AdS/CFT Conjecture (a.k.a. Holography or Gauge/Gravity duality) is sometimes used loosely to denote this large class of dualities. The generic form of this duality is that a string theory/ M-theory (or in the low energy limit, Supergravity theories) on a manifold K (a.k.a "Bulk") is equivalent to a quantum field theory on the boundary of K. The conjecture has survived all the non-trivial tests performed. In some limit, this conjecture relates quantities in a pure classical supergravity theory to a full quantum strongly coupled field theory quantities (which do not contain gravity). This implies that various highly non-perturbative and "difficult" problems can be addressed by mapping it to a "simple" computation in a classical gravitational theory. The conjecture is assumed to very general and also widely used in systems which is not even derived from any string / M / supergravity theories but with considerable promise of being derived.

Originally the conjecture was proposed by studying N coincident $D3$-branes in Type IIB string theory in the large N limit. In this limit the system two possible dual descriptions: as solutions of classical supergravity / string theory and also equivalently by quantum field theory living on the D-brane system. This led to the conjecture : *Type IIB string theory on $AdS_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills' theory in (3+1) dimensions.* This is the most famous and rigorously studied example of the conjecture. $AdS_5 \times S^5$ is a solution of classical supergravity in $9+1$ dimensions and Type IIB strings can be assumed to be moving in this background geometry. According to the conjecture it can be equivalently described by studying a quantum field theory living on the $(3+1)$ dimensional boundary of AdS_5. For this particular case, the field theory is a special supersymmetric gauge theory known as $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills' theory, which is also an *Conformal Field Theory (CFT)* (i.e. field theory having a special class of symmetry called *Conformal symmetry*, or naively speaking the action is symmetric under scaling.) and thus the origin of the name AdS/CFT conjecture. This is also a classic and concrete example of "Holography Principle" [5, 6] *i.e. for a consistent quantum theory of gravity, the fundamental degrees of freedom reside at the boundary of space-time and not in the interior.* As a direct generalization to other systems, we can consider various deformations in the either side which will reflect as a deformation in the dual theory. In particular, AdS/CFT finds application in studying renormalization group (RG) flows in the perturbed boundary theory. CFTs do not have any intrinsic scale, and as a consequence the effective physical coupling constants is independent of energy scale. But if the CFTs are perturbed, it can have energy scale dependent couplings and this behavior is described by RG flow equations. In the dual gravity theory this RG flow is projected onto a deformation in $AdS_5 \times S^5$ such that the direction or coordinate along which this geometry changes maps to the energy of the boundary theory. So in general we can have dualities between non-AdS and non-CFTs, but still will be termed as "AdS/CFT duality". The origin of this duality can be related to the world sheet "Open-Closed string duality" *i.e. exchange of world-sheet parameters implies a duality between 1-loop open string "Feynman diagram" and a closed string tree diagram.* Also massless spectrum of closed strings contain gravity and open string does not, which implies some kind of Gauge/Gravity duality. A proper understanding of the origin of AdS/CFT conjecture or a proof is still lacking.

In this thesis we will encounter two different application of this conjecture.

1.4 Hagedorn temperature and Matrix Model for strings

String theory shows an exponential growth in the density of single particle states as a function of mass or energy. The density of states at high energy E grows as $\rho(E) \sim Exp(\beta_H E)$ where $\beta_H \sim l_s$ (l_s is the string length). The canonical partition function is defined as $Z = \sum_E \rho(E) Exp(-\beta E)$, where $T = \frac{1}{\beta}$ is the temperature. The expression clearly shows that for $T \geq \frac{1}{\beta_H}$ the canonical partition function diverges. Similar behavior was noticed in theories for hadronic matter [7] and in the same spirit, in the case of strings also, it is thought to be as a "limiting temperature". $T_H = \frac{1}{\beta_H}$ is called *Hagedorn temperature.*

There is some evidence to interpret this temperature as a phase transition temperature [8, 9, 10, 11]. Yang-Mills theories are known to have confinement-deconfinement phase transition, and also they are known to be dual to String Theories with $1/N$ (for gauge group $SU(N)$) interpreted as the string coupling constant, g_s. It is then natural to identify this Hagedorn Transition temperature with confinement-deconfinement transition [12, 13, 14, 15, 16, 17, 18, 19]. Above the deconfinement transition the gluon flux tube disintegrates. Correspondingly one would expect that the string disintegrates above the Hagedorn phase transition and is replaced by something else - perhaps a black hole.

It is difficult to study the disintegration of string theory using the perturbative string formalism. One needs a non-perturbative description where a string can be described in terms of some other entity. One such description is the BFSS matrix model. In this model one can construct a classical configuration that looks like a membrane. As a 10-dimensional object it is a D2 brane of IIA string theory. If one of the space dimensions is compactified, the D2 brane wrapped around it, is T-dual to a D-string (D1 brane) of IIB string theory. This in turn is S-dual to an F (fundamental)-string. We can consider specific configuration of these $D1$-branes or $D0$-branes, which forms "Long D-string". We can pretend that this long D-string is the fundamental string whose phase transition we are interested in. At the phase transition this classical membrane configuration can be expected to disintegrate so that we end up with just a bunch of localized D0-branes. This is thus the "S-dual" of the Hagedorn transition. This was what was investigated in in a qualitative way in [20] . It was shown by computing (in a high temperature approximation) the one loop free energy, that there is a phase transition from a membrane phase to a clustered phase. However an IR cutoff was crucial for the calculation. The motivation for the IR cutoff is roughly that the $D0$ branes are actually bound (albeit marginally) and so one expects them to be localized. Two $D0$ brane potential, at finite temperature, studied in [21, 22, 23] shows possibility of bound state. In this thesis we will describe the work [24], where the one-loop analysis was done carefully without approximations. The net result reaffirms the result of [20] , but with some modifications in the analytical expressions. Interestingly an additional phase transition is found *back to the string phase*. The one-loop partition function of strings from matrix model [25, 26] was also studied extensively in [27, 28, 29].

Holography [30, 2] has given some new insights into the dynamics and phase structure of super-symmetric Yang-Mills [14, 31, 32]. This should also be taken into account. Deformations of these theories have also been studied [33, 34, 35, 36, 37]. In fact, motivated by the analysis in [33, 34], the thermal Hawking-Page phase transition [38, 39] in AdS with a "hard wall" [40] has been studied - the hard wall removes a portion of the AdS near $r = 0$, which in the gauge theory corresponds to the IR region. In these models the cutoff is a way to simulate a boundary gauge theory that is not conformally invariant, i.e. confining. The cutoff radius is related to the mass parameter of the gauge theory. It has in fact been shown [41] that an

IR cutoff is crucial for the existence of two phases: BPS Dp-branes (more precisely their near horizon limit which for $p = 3$ is AdS_5) and black Dp-branes (for $p = 3$, AdS_5 black brane), separated by a finite temperature phase transition. Also [42] has argued (from entropy considerations) that $O(g_s)$ higher derivative corrections to super-gravity must induce a finite horizon to develop for a configuration of extremal D0-branes. This also acts as an IR cutoff. Based on all this, the main conclusion of the paper [24] was that *an infrared cutoff needs to be included in the BFSS matrix model if it is to describe string theory.*

This is quite understandable from another point of view. Simple parameter counting shows that the BFSS model, as it stands, cannot be equivalent to string theory. It has only one dimensionless parameter N. The other parameter is g_{YM} - which is dimensionful and just sets the overall mass scale. String theory describing N D0 branes has two dimensionless parameter N and g_s, and a dimensionful parameter l_s. The two theories thus cannot be equivalent, except in some limit $g_s \to 0$ (or $g_s \to \infty$ - M-theory). For finite g_s one needs an additional parameter on the Yang-Mills side. Thus an IR cutoff L_0 introduces a second scale into the Yang-Mills theory, and thus the ratio of the two scales is a dimensionless parameter and now the parameter counting agrees. Furthermore, low energies in the matrix model that describes D0 branes, corresponds in the super-gravity description to short distances, i.e. regions near the D0 branes, where the dilaton profile corresponds to a large value of g_s. If we include the low energy (IR) region in the configuration space of the Yang-Mills (matrix model) theory, we are forced to include the effects of finite g_s. The IR cutoff is an additional parameter that signifies our ignorance of (large) strong coupling effects. With an IR cutoff we can maintain g_s at a finite (small) but non-zero value. What actually happens in this region (i.e. close to D0 branes) due to strong coupling effects of large g_s is not fully known, but as mentioned above [42] has a plausible proposal based on entropy arguments. The suggestion is that a finite size horizon develops. This if true, vindicates the introduction of an IR cutoff. Note that another way of making the functional integral finite is to give a mass to the scalar fields - thereby removing the zero mode. The matrix model corresponding to the BMN pp wave limit [43] in fact has such a mass term with the mass being a free parameter. This is reminiscent of the N=1* theories studied in [33, 34] and these techniques have been applied to the BMN matrix model [44]. We can therefore take an agnostic attitude regarding the origin of the cutoff and treat it as the extra parameter necessary to match with string theory for finite g_s. It should correspond to the fact that the D0-brane bound state must have a finite size. This is not easy to see in the BFSS matrix model, in perturbation theory, because of the flat directions in the potential.

As mentioned above there seems to be another high temperature string phase. This is reminiscent of the $T \to 1/T$ symmetry that has been discussed by many authors [45, 46, 47, 48, 49, 50, 51, 52, 53]. The perturbative analysis shows a second phase transition to a string phase at very high temperatures. This is reminiscent of Gregory-Laflamme instability [54, 55, 56, 57]. It will be described briefly in end of Chapter 5. It will shown that the entropic arguments that motivate the Gregory-Laflamme transition can also be made for extremal black holes with finite size horizon such as the Reissner Nordstrom black holes.

1.5 Applications of AdS/CFT in Condensed Matter Physics

A variety of ill-understood strongly coupled field theories are expected to describe phenomena such as High-T_c superconductivity [58, 59, 60]. A technique that seems tailor made for this is the AdS/CFT correspondence [2, 3, 4], where a string theory in AdS background is dual to a conformal field theory on the boundary. If the CFT is strongly coupled, then the curvature of the dual gravity background is small and the massive string excitations do not play much of a role, and so pure gravity is a good approximation. This is the reason this technique is ideal for strongly coupled theories.[1] Historically the correspondence was developed in Euclidean signature in [2, 3, 4, 61, 62, 63, 64] to name a few. Some of the references where real time formalism of the correspondence was studied is [65, 66, 67, 68, 69, 70].

If one wants to study the finite temperature behavior of the CFT, one can study dual backgrounds that are asymptotically AdS but have a finite temperature. This could be just pure AdS with a thermal gas of photons or other massless particles; or backgrounds that contain a black hole in the interior. There is typically a critical temperature above which the black hole is the favored background [39]. The critical

[1] The same fact also makes it difficult to study standard continuum QCD, which is asymptotically free and therefore weakly coupled.

temperature is in fact zero if the boundary is taken to be flat R^n rather than S^n. This is what happens when we represent the AdS by a Poincare patch. Equivalently there is a scaling limit that gives R^n starting from the S^n boundary of global AdS. Recently such backgrounds with a charged Reisner Nordstrom black hole in 3+1 dimensions have been studied to understand 2+1 dimensional CFT's with charge. In particular fermionic fields have been studied and interesting "non-Fermi liquid behavior" [71] has been discovered [72, 73, 74, 75]. The same system was studied with magnetic field turned on in [76, 77].

In this thesis we will discuss the work done in [78], which studies analogous situation in 2+1 dimensions[2]. There are charged BTZ black holes that are asymptotically AdS_3 [81]. We extract various boundary correlators for scalar operators, currents and also fermionic operators in this theory. One of the advantages of studying this system is that since they describe various strongly coupled CFT's in 1+1 dimension, one has some idea of what to expect in the boundary theory. At the UV end it is dual to AdS_3 and in the IR it flows to another fixed point CFT. The presence of the background charge breaks Lorentz Invariance and also scale invariance of the boundary theory. Thus one expects deviations from the Luttinger liquid [3] behavior due to irrelevant perturbations of various types. These deviations in general change the linear dispersion to a non linear one and the result bears some similarity to a Fermi liquid. A class of these theories have been studied in perturbation theory and have been called "Fermi-Luttinger" liquids [82]. Thus our holographic Green's functions could be a strong coupling non perturbative version of the Green's function studied in [82] .

We should note that the AdS/CFT correspondence works in full string theory, so ideally we should consider embeddings of $2+1$ dimensional black hole in full string theory in $9+1$ dimensions. In our present study we assume the existence of such an embedding. By analogy with the best studied example of AdS/CFT correspondence, where Type IIB string theory on $AdS_5 \times S^5$ corresponds to $SU(N)$ super Yang Mills, one expects that tree level calculation in the bulk corresponds to

[2]A similar situation was studied in [79]. Hydrodynamic aspects of $1+1$ field theories using AdS/CFT was studied in [80]

[3]The term Luttinger liquids (a.k.a. Tomonaga-Luttinger liquids) generally refers to $1+1$ dimensional interacting many-body (electrons) systems. Critical exponents of Luttinger liquids can be studied using $1+1$-dimensional CFT. In this thesis we will use the term Luttinger liquid and $1+1$-dim CFT interchangeably.

the $N \to \infty$ planar limit of the boundary theory. In our case there is no identifiable parameter N. We can take it generically as a measure of the number of species of particles in the boundary theory. Thus when we refer to Luttinger liquid we are actually referring to a $c = \infty$ conformal field theory with a large (infinite) number of scalar fields. Similarly the gravity approximation (neglecting stringy modes) corresponds to large 't Hooft coupling in Yang-Mills. In our case we do not have an action for the boundary theory and hence no well defined notion of coupling constant. However we can assume that the theory describes some non trivial "strong coupling" fixed point with consequent large *anomalous* dimensions. This can be taken as the operational meaning of "strong coupling".

In the above approximation (planar, strong coupling) used here the full string theory embedding is not required. However the full embedding would in principle determine the charges and dimensions of the operators of the boundary theory, which for us are free parameters. Also to go beyond this approximation would require a knowledge of the embedding.

We study Green's functions corresponding to scalars, fermions and gauge fields at both zero and non zero temperature. The gauge field calculation gives the conductivity. This is studied at both zero and non zero temperatures. At zero temperature the small ω behavior is universal (as shown in [75]) and we see that our numerical results are consistent with this expectation. The same analysis can also be done for fermions and scalars and again there is consistency with the analytic results at zero temperature and low frequencies. This gives some confidence in the numerical calculations. The fermion Green's function does show quasiparticle peaks at specific momenta and these can be identified as Fermi surfaces. The dispersion relation is approximately linear. The "log periodicity" in the Green's function are also observed and is consistent with the analytical expectations. Qualitatively the curves are also similar to the Fermi-Luttinger liquid curves.

We also attempt to reproduce in the boundary theory the intriguing non analyticity in the $T = 0$ fermion Green's function at $\omega = 0$ for *any* k. We show that it can be explained if one assumes that there are modes that have their velocity renormalized to zero, interacting with fermions. These could be thus some localized modes representing impurities.

1.6 Organization of thesis

This thesis is divided into two parts. Part I studies Hagedorn phase transition in the matrix model for string theory. This is addressed in Chapter 2-6. Background materials required for this topic is briefly discussed in Chapter 2 and Chapter 3. Chapter 4 describes the calculation of 1-loop partition function for matrix model for two different phases and the comparison of free energy. Chapter 5 addresses the physical origin of IR cut-off required for 1-loop analysis by use of AdS/CFT Conjecture. Chapter 6 summarizes the results of Part I. In second part of the thesis we study the connection between Charged BTZ black hole and Luttinger liquid by use of AdS/CFT conjecture. This is addressed in Chapter 7-12. Background materials required for this part of the thesis is described in Chapter 7-8. Chapter 9 describes the background solution and its near horizon limit. Chapter 10 contains the calculations and results for Spinor Green's function. Chapter 11 contains the calculations and results for Vector and Scalar Green's function. Chapter 12 contains summary of results of Part II of the thesis. Chapter 13 contains summary of the full thesis and future directions.

Part I

Backgroud material: Part I

String Theorists Find a Rosetta Stone

Gary Taubes

2 M(atrix) Theory

2.1 M-Theory

M-theory plays a crucial role in understanding duality relations among various string theory. A complete understanding of M-theory is still lacking, till now it refers to a wish-list and some collection of facts for a suggested theory. M-theory is conjectured to be the strong coupling limit of type IIA string theory. In this limit, it behaves as an eleven dimensional theory in an infinite flat space background. At low energy, it behaves as a eleven dimensional super-gravity, which is a very well defined classical theory with field content [83]:
e^a_I: a vielbein field (bosonic, 44 components)
ψ_I: a Majorana fermion gravitino (fermionic, 128 components)
A_{IJK}: a 3-form potential (bosonic, 84 components)
where (I, J, K) run from $0, 1 \cdots 9, 11$. M-theory contains a 2-dimensional membrane (M2) and a 5-dimensional hypersurface (M5) which couple electrically and magnetically to the 3-form field respectively. The tension of these membranes (M2) is given by $\frac{1}{(l_p^{(11)})^3}$, where $l_p^{(11)}$ is eleven dimensional Planck length. M-theory compactified on a circle S^1 is equivalent to Type IIA string theory with the radius of the circle (in 11th direction) R_{11} is related to the string coupling g_s through $R_{11} = g_s^{\frac{2}{3}} l_p^{(11)} = g_s l_s$. $l_s = \sqrt{\alpha'}$ is the string length. The decompactification limit $R_{11} \to \infty$ corresponds to strong coupling limit of the string theory. Various objects of M-theory on a circle and Type IIA string theory are mapped to one another as shown in the Table(2.1).

KK photon ($g_{\mu 11}$)	RR gauge field A_μ
super-graviton with $p_{11} = \frac{1}{R_{11}}$	D0-brane
wrapped M2-brane	IIA F-string (Fundamental string)
unwrapped M2-brane	IIA D2-brane
wrapped M5-brane	IIA D4-brane
unwrapped M5-brane	IIA NS5-brane

Table 2.1: Correspondence between M-theory and IIA string theory. (super-graviton: particles corresponding to 11d SUGRA fields,$g_{\mu 11}$: components of 11-dimensional metric, KK: Kaluza-Klein, RR: Ramond-Ramond)

2.2 BFSS Matrix Model

In 1996, a seminal paper by Banks, Fischler, Shenker and Susskind [1] proposed that M-theory in infinite momentum frame [84] is described by large N limit of a supersymmetric matrix ($N \times N$) quantum mechanics model. The infinite momentum frame (IMF) was considered by first compactifying M-theory on an S^1 along the 11th direction with radius R_{11}. The system is then given an infinite boost along this 11th direction. As discussed before compactified M-theory gives Type IIA string theory. The compactification gives discrete momentum modes (Kaluza-Klein photon) along 11th direction which correspond to Ramond-Ramond (RR) photon of IIA string theory. Perturbative string states do not carry RR charge *i.e.* string perturbative states carry vanishing momentum along 11th direction. $D0$ branes, which are non-perturbative solitons, can only carry RR charge. $D0$-branes are point particles which carry a single unit of RR charge equivalently they have momentum along 11th direction given by $P_{11}^{D0branes} = \frac{1}{R_{11}}$. The $D0$-branes carry quantum numbers of first massive KK modes of the 11-dimensional supergravity multiplet (super-gravitons). As 11 dimensional objects they are massless, but the 10-d mass of $D0$-branes is $\frac{1}{R_{11}}$. Super-gravitons with KK momentum $p_{11} = \frac{N}{R_{11}}$ (N is an integer) corresponds to bound state composites of $D0$-branes. In the limit where $R_{11} \to \infty$ (Decompactification, M-theory limit) and $p_{11} = \frac{N}{R_{11}} \to \infty$ (in IMF), M-theory is described by non-relativistic theory [1] of N $D0$-branes where

[1]In IMF, the existence of a transverse (11th direction is called longitudinal and the rest transverse) Galilean symmetry leads to a non-relativistic system. The role of non-relativistic mass is played by longitudinal momentum p_{11} [1]

$N \to \infty$. The low-energy Lagrangian for a system of N IIA $D0$-branes is given by a Lagrangian arising from dimensional reduction to $(0+1)$ dimension of the $(9+1)$ dimensional $U(N)$ Super Yang-Mills Lagrangian. The BFSS conjecture is that in IMF M-theory is described by this $(0+1)$ dimensional matrix quantum mechanics Lagrangian given by [85, 1],

$$\mathcal{L} = \frac{1}{2g_s l_s} Tr \left[(D_t X^i)^2 + \frac{1}{(2\pi l_s^2)^2} \sum_{i>j} [X^i, X^j]^2 + \frac{1}{2\pi l_s^2} \theta^T i D_t \dot{\theta} - \frac{1}{(2\pi l_s^2)^2} \theta^T \gamma_i [X^i, \theta] \right] \tag{2.1}$$

Where nine spatial coordinates of the system of N $D0$-branes are roughly given by eigenvalues of the Harmitian $N \times N$ scalar matrices, $X^i_{a,b}$ $(i = 1, \cdots 9)$. The matrices X are accompanied by 16 component fermioninc superpartners $\theta_{a,b}$ which transform as spinors under $SO(9)$ group of transverse rotations. γ^i are 16×16 matrices satisfying $SO(9)$ Clifford Algebra. $D_t = \partial_t - i[A_0, \cdots]$ and A_0 is the gauge field along the time direction surviving the dimensional reduction. The theory has 16 supercharges and thus is a $\mathcal{N} = 16$ supersymmetric matrix quantum mechanics. This theory was discussed many years before the development of D-branes [86, 87, 88]. A detailed discussion of this theory was found in D-brane context in [89, 90, 91, 92]. Exhaustive review of BFSS Matrix model can be found in [93, 85, 94, 95]. A detailed description of dimensional reduction of SYM is given in Appendix (A) of this thesis.

The classical vacuum configuration of (eqn.(2.1)) corresponds to solution of equation of motion where potential energy of the system is minimized, which occurs when $[X^i, X^j] = 0$ for all (i, j) in addition to zero fermion fields and the fields X^i are covariantly constant. So, the fields X^i can be simultaneously diagonalized by gauge transformation. The N diagonal elements of the matrix X^i corresponds to the positions of N $D0$-branes in the $i-th$ direction [89]. For general configurations this geometric interpretation of diagonal elements as positions of D-branes are no longer valid.

As discussed in [94], matrix theory can encode, even in finite N, a configuration of multiple objects. In this sense, it is natural to interpret matrix theory as an *second quantized* theory from the view of target space compared to string theory which gives a first quantized theory in target space.

The original BFSS conjecture was defined in the large N limit. It was later argued by Susskind [96], that finite N version of the matrix model can be shown equivalent to the *discrete light-cone quantized (DLCQ)* sector of M-theory with N units of compact momentum. In DLCQ a light-like direction X^- is compactified instead of X^{11}, where $X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{11})$ are the light-cone directions. So the momentum along light-like circle is quantized, $P^+ = \frac{N}{R^-}$, with N being an integer. With a light-like circle the value of R^- can be changed by a boost. So uncompactified limit can not be obtained by simply considering $R^- \to \infty$. Instead it is obtained by taking both R^- and N to infinity with P^+ kept constant. Arguments due to Seiberg [97] and Sen [98] made the connection more precise and justified the use of the low-energy $D0$-brane action in BFSS conjecture. We will review this argument in the following section.

2.2.1 DLCQ M-theory and BFSS matrix model

In DLCQ, we compactify a light-like circle which corresponds to,

$$\begin{pmatrix} X^{11} \\ X^0 \end{pmatrix} \sim \begin{pmatrix} X^{11} \\ X^0 \end{pmatrix} + \begin{pmatrix} \frac{R^-}{\sqrt{2}} \\ -\frac{R^-}{\sqrt{2}} \end{pmatrix} \tag{2.2}$$

where X^{11} is the longitudinal space-like direction and X^0 is the time-like direction in the 11 dimensional space-time ($X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{11})$). This theory has a quantized momentum in the compact direction,

$$P^+ = \frac{N}{R^-} \tag{2.3}$$

We can consider the compactification (eqn.(2.2)) as a limit of compactification on a space-like circle which is almost light like,

$$\begin{pmatrix} X^{11} \\ X^0 \end{pmatrix} \sim \begin{pmatrix} X^{11} \\ X^0 \end{pmatrix} + \begin{pmatrix} \sqrt{\frac{R^-}{2} + R_{11}^2} \\ -\frac{R^-}{\sqrt{2}} \end{pmatrix} \simeq \begin{pmatrix} X^{11} \\ X^0 \end{pmatrix} + \begin{pmatrix} \frac{R^-}{\sqrt{2}} + \frac{R_{11}^2}{\sqrt{2}R^-} \\ -\frac{R^-}{\sqrt{2}} \end{pmatrix} \tag{2.4}$$

with $R_{11} << R^{-}$. The light-like compactification (2.2) is obtained from (2.4) as $R_{11} \to 0$. This compactification is related by a large boost,

$$\beta_v = \frac{R^-}{\sqrt{(R^-)^2 + 2R_{11}^2}} \simeq 1 - \left(\frac{R_{11}}{R^-}\right)^2 \tag{2.5}$$

to a spatial compactification on

$$\begin{pmatrix} X^{11'} \\ X^{0'} \end{pmatrix} \sim \begin{pmatrix} X^{11'} \\ X^{0'} \end{pmatrix} + \begin{pmatrix} R_{11} \\ 0 \end{pmatrix} \tag{2.6}$$

where prime denotes boosted coordinates given by,

$$\begin{pmatrix} X^{11'} \\ X^{0'} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\beta_v^2}} & \frac{\beta_v}{\sqrt{1-\beta_v^2}} \\ \frac{\beta_v}{\sqrt{1-\beta_v^2}} & \frac{1}{\sqrt{1-\beta_v^2}} \end{pmatrix} \begin{pmatrix} X^{11} \\ X^{0} \end{pmatrix} \tag{2.7}$$

So a M-theory which has been compactified on an light-like circle, R^- (DLCQ M-theory) can be viewed as a limit of a (boosted) M-theory compactified on a space-like circle, R_{11}, where the radius of the space-like circle vanishes in this limit ($R_{11} \to 0$). As discussed before space-like compactification of M-theory can be identified Type IIA string theory. And in the limit of small radius the IIA string theory is weakly coupled but the theory apparently becomes complicated as string tension vanishes. The type IIA coupling and string length are related to compactification radius and 11D Plank length as(Sec.(2.1)),

$$\begin{aligned} g_s &= (\frac{R_{11}}{l_p^{(11)}})^{\frac{3}{2}} \\ l_s^2 &= \frac{(l_p^{(11)})^3}{R_{11}} \end{aligned} \tag{2.8}$$

So in the limit $R_{11} \to 0$, $g_s \to 0$ and $\alpha' = l_s^2 \to \infty$. To summarize the DLCQ M-theory compactified on a finite radius when boosted can be identified with type IIA string theory with vanishing string coupling and tension. So it seems in this limit the string theory is complicated and will not provide an useful alternative description of M-theory.

Let us consider the behavior of the energy of the states we are interested in the limiting theory. Consider a state with light cone energy P^- and compact momentum $P^+ = \frac{N}{R^-}$. The longitudinal momentum in the spatially compactified boosted theory is given by $P'_{11} = \frac{N}{R_{11}}$. The energy in the spatially compactified theory is given by

$$E' = \frac{N}{R_{11}} + \Delta E' \tag{2.9}$$

where $\Delta E' << \frac{N}{R_{11}}$ is the energy scale of the $10D$ string theory. The term $\frac{N}{R_{11}}$ is the mass-energy of N $D0$-branes or the longitudinal momentum in the compactified M-theory. Relating this back to almost light like compactified (eqn.(2.4)) theory we have,

$$\begin{pmatrix} P_{11} \\ E \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\beta_v^2}} & -\frac{\beta_v}{\sqrt{1-\beta_v^2}} \\ -\frac{\beta_v}{\sqrt{1-\beta_v^2}} & \frac{1}{\sqrt{1-\beta_v^2}} \end{pmatrix} \begin{pmatrix} P'_{11} \\ E' \end{pmatrix} \tag{2.10}$$

so,

$$P^- = \frac{1}{\sqrt{2}}(E - P_{11}) = \delta_v \Delta E' \simeq \frac{R^-}{R_{11}} \Delta E' \tag{2.11}$$

where $\delta_v = \frac{1}{\sqrt{2}}\sqrt{\frac{1+\beta_v}{1-\beta_v}}$ and β_v is given by (eqn.(2.5)). In the limit $R_{11} \to 0$, $\delta_v \simeq \frac{R^-}{R_{11}}$. So the $10D$ energy scale $\Delta E'$ of the corresponding state of type IIA string theory is given by $\Delta E' \simeq P^- \frac{R_{11}}{R^-}$. We know $l_s \to \infty$ in this limit where $R_{11} \to 0$. So in units of string energy scale($\sim \frac{1}{l_s}$) the energy scale of the state becomes (using (eqn.(2.8))),

$$\frac{\Delta E'}{(1/l_s)} = \frac{P^-}{R^-} \sqrt{R_{11}(l_p^{(11)})^3} \tag{2.12}$$

which vanishes as $R_{11} \to 0$. So although the string energy scale vanishes, the energy scale of interest (*i.e.* the energy scale of the sector in string theory that maps to DLCQ M-theory in the limit considered) vanishes faster. Such a sector with energy scale much smaller than string scale gives rise to a simple theory and can formulate an alternative description of DLCQ M-theory.
Let us rewrite the expression for $\Delta E'$ by putting in the units of measurement ($l_p^{(11)} = M_p$, M_p is $11d$ Plank mass),

$$\frac{\Delta E'}{M_p} = \frac{(P^-/M_p)}{R^- M_p}(R_{11} M_p) \tag{2.13}$$

Let us now consider the M-theory compactified spatially (R_{11}) is replaced by

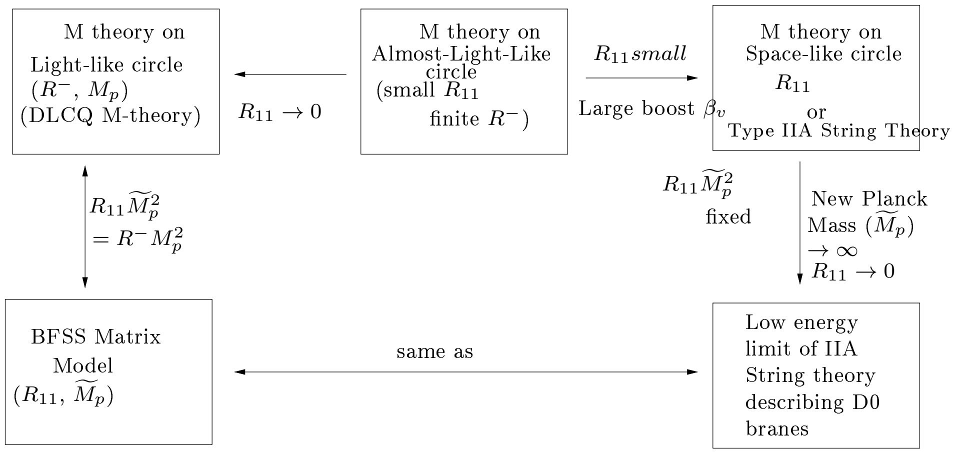

Figure 2.1: Relation between DLCQ M theory and BFSS matrix model

$\widetilde{M}$-theory with Planck mass, $\widetilde{M}_p$, on a spatial circle $\widetilde{R}_{11}$ (where, $\widetilde{R}_{11}\widetilde{M}_p = R_{11}M_p$). $\widetilde{M}_p$ is different from M_p in which the DLCQ M-theory is defined. Then the above expression becomes ($\frac{\widetilde{\Delta E'}}{\widetilde{M}_p} = \frac{\Delta E'}{M_p}$),

$$\frac{\widetilde{\Delta E}'}{\widetilde{M}_p} = \frac{(P^-/M_p)}{R^- M_p}(\widetilde{R}_{11}\widetilde{M}_p) \tag{2.14}$$

implies,

$$\widetilde{\Delta E'} = P^-(\frac{\widetilde{R}_{11}\widetilde{M}_p^2}{R^- M_p^2}) \tag{2.15}$$

Let us consider $\widetilde{\Delta E}'$ to be independent of $\widetilde{R}_{11}$. Then as P^-, M_p are independent of $\widetilde{R}_{11}$, the quantity $\widetilde{R}_{11}\widetilde{M}_p^2$ should be a constant independent of $\widetilde{R}_{11}$. So as $\widetilde{R}_{11} \to 0$ we should take $\widetilde{M}_p \to \infty$, such that the quantity $\widetilde{R}_{11}\widetilde{M}_p^2$ is held fixed and in particular we can choose (let us also drop the tilde index on R_{11}, as R_{11} is assumed to be always the radius of spatial circle on which $\widetilde{M}$-theory is compactified.),

$$R^{11}\widetilde{M_p}^2 = R^- M_p^2 \tag{2.16}$$

which is equivalent to demanding $\widetilde{\Delta E}' = P^-$ *i.e.* the energy does not scale under boost if we change the units appropriately ($\Delta E'$ is the light-cone energy ($P^{-'}$) apart from $\sqrt{2}$ factor in the boosted theory). As boost does not affect transverse directions,

$$M_p R_i = \widetilde{M_p}\widetilde{R_i} \tag{2.17}$$

where R_i are any length parameter in transverse direction. If we compactify one of the transverse direction with radius R_i and consider the T dual along that direction, the T dual radius is given by $R_i^* = \frac{\alpha'}{R_i}$. So the scaling gives,

$$\widetilde{R}_i^* = R_i^* \tag{2.18}$$

So relation between the parameters of the two theory is given by,

$$\begin{aligned}
\frac{\widetilde{M_p}}{M_p} &= \delta_v^{\frac{1}{2}} \\
\frac{\widetilde{g_s}}{g_s} &= \delta_v^{-\frac{3}{4}} \\
\frac{\tilde{\alpha}'}{\alpha'} &= \delta_v^{-\frac{1}{2}} \\
\frac{\widetilde{R_i}}{R_i} &= \delta_v^{-\frac{1}{2}} \\
\widetilde{R}_i^* &= R_i^*
\end{aligned} \tag{2.19}$$

So with finite N, R^- and M_p, the DLCQ M-theory gets mapped to $\widetilde{M}$-theory compactified on spatial circle. The corresponding string theory for $\widetilde{M}$-theory compactified on spatial circle is weakly coupled and with very large string tension (Notice as $R_{11} \to 0$ or $\delta \to \infty$, both $\widetilde{g}_s$ and $\widetilde{l}_s$ goes to zero). In this limit $\widetilde{g}_s$, $\widetilde{l}_s \to 0$, type IIA string theory is described by lowest excitations in the the theory, and as these objects has N units of RR-charge or momentum in $11th$ direction, the theory is described by N $D0$-branes. In the low energy limit, the theory of these $D0$-branes is given by supersymmetric matrix model like $BFSS$ obtained from dimensional reduction of 10d SYM or equivalently the non-relativistic limit of non-abelian Dirac-Born-Infield (DBI) action for N $D0$-branes. Non-abelian DBI action is not well defined, except in the non-relativistic limit. But in the limit $R_{11} \to 0$, it can be shown that higher order corrections to DBI action vanish and is given by matrix model for N non-relativistic $D0$-branes. To summarize the DLCQ M-theory with N units of light-cone momentum, compactification radius R^- and Planck mass scale M_p is equivalent to matrix model (BFSS model) for N non-relativistic $D0$-branes with Plank mass scale $\widetilde{M_p}$ [2].The correspondence between the two theories is given by (eqn.(2.19)) (Fig.(2.1)). Although various parameters of the BFSS matrix model for DLCQ M-theory are vanishing or divergent, the theory is non-trivial as the $1d$ Yang-Mills coupling $\widetilde{g}^2_{YM} \sim \frac{\widetilde{g}_s}{\widetilde{l}_s^3} = \frac{g_s}{l_s^3}$ (Appendix A) is finite.

In the original BFSS conjecture, the matrix model (eqn.(2.1)) describes M-theory in the limit $R_{11} \to \infty$ (Decompactification, M-theory limit) and $p_{11} = \frac{N}{R_{11}} \to \infty$ (in IMF). Seiberg-Sen-Susskind arguments claims DLCQ M-theory with a fixed compact momentum $P^+ = \frac{N}{R^-}$ and Plank mass M_p is described by the BFSS matrix model (eqn.(2.1)) (with Plank mass $\widetilde{M_p}$) in the limit $R_{11} \to 0$ and $\widetilde{M_p} \to \infty$ such that $R^{11}\widetilde{M_p}^2 = R^- M_p^2$ and $M_p R_i = \widetilde{M_p}\widetilde{R_i}$ (Fig.(2.1)). Now to get uncompactified M-theory in the DLCQ matrix model we have to consider $N \to \infty$ such that P^+ is held fixed and we will get an sector of M-theory with fixed light-cone momentum. But from (eqn.(2.9)), we see that rest mass of $D0$-branes become infinite in this limit and will change the background flat geometry of the matrix model. We will discuss the subtleties of large N limit in more detail in the Chapter(3). In the next section we will discuss how membranes naturally appear in the matrix model.

In the thesis the parameters for BFSS matrix model even if not explicitly denoted by $\widetilde{}$, it will always be implied. Also we will use BFSS matrix model not as a model for M-theory, but in the spirit it describes DLCQ M-theory

2.3 Duality in Matrix Model

2.3.1 T duality

We will here consider T-duality of low energy $D0$-brane action compactified on a circle, which is equivalent to consider T-duality of the BFSS action (eqn.(2.1))

[2]This can also be argued as follows. The relevant degrees of freedom are strings stretched between the D0-branes. The infinite string scale decouples all the oscillators on the strings. Therefore the full theory is the minimal SYM theory. Note that closed strings or gravitons in the bulk of space time decouple both because the string scale becomes large and because the string coupling vanishes.

compactified on a circle. Let us compactify X^9 on a circle of radius $\widetilde{L}_9$, giving a space time $\mathbb{R}^{8,1} \times S^1$. T duality maps type IIA string theory compactified on a circle of radius $\widetilde{L}_9$ to type IIB string theory compactified on a circle of radius $\widetilde{L}_9^* = \frac{\widetilde{\alpha}'}{\widetilde{L}_9}$.

The circle S^1 is just the quotient of an infinite line $\mathbb{R}$ by the discrete group $\Gamma = 2\pi\widetilde{L}_9\mathbb{Z}$. Thus, we can describe the physics of $D0$-branes moving on $\mathbb{R}^{8,1} \times S^1$ by making a copy of each $D0$-brane for each element of $\mathbb{Z}$, and then imposing the symmetry under Γ. We will summarize the results of this construction of [99] in this section. The $D0$-branes on the cover of S^1 are each represented by an infinite

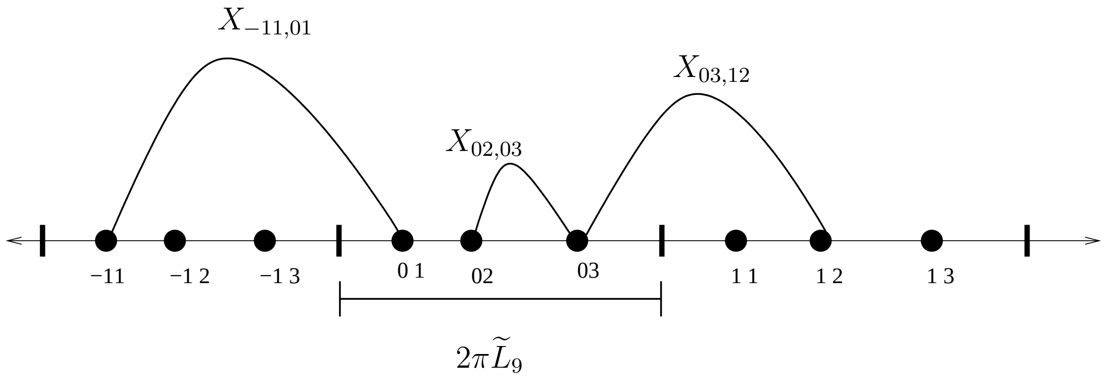

Figure 2.2: $D0$-branes on cover of S^1

number of copies of $D0$-branes labeled by integers $n \in \mathbb{Z}$. We can thus describe the dynamics of N $D0$-branes on $\mathbb{R}^{8,1} \times S^1$ by a set of infinite matrices $X^i_{ma,nb}$ (Here we call $A^0_{ma,nb} = X^0_{ma,nb}$ the matrix corresponding A^0 for simplicity) where $a, b \in \{1, \cdots, N\}$ are $U(N)$ indices and $m, n \in \mathbb{Z}$ index copies of each $D0$-brane which differ by translation in the covering space (See Fig.2.2). Matrices $X^i_{ma,nb}$ corresponds to a string stretching from $m-th$ copy of $D0$-brane number a to the $n-th$ copy of $D0$-brane number b. Let us suppress these a, b indices and write these infinite matrices whose blocks X^i_{mn} are themselves $N \times N$ matrices. In terms of infinite block matrices the quotient condition becomes a set of constraint on the allowed matrices which can be written as ,

$$
\begin{aligned}
X^i_{mn} &= X^i_{(m-1)(n-1)} \quad , \qquad i < 9 \\
X^9_{mn} &= X^9_{(m-1)(n-1)} \quad , \qquad m \neq n \\
X^9_{nn} &= 2\pi \widetilde{L}_9 \mathbb{1} + X^9_{(n-1)(n-1)}
\end{aligned}
\tag{2.20}
$$

Also since the infinite matrices $X^i_{ma,nb}$ are hermitian, we have,

$$
X^i_{mn} = X^{i\dagger}_{nm} \tag{2.21}
$$

Above constrains on the infinite matrices implies all the dynamical information of the system is contained in the matrices,

$$
X^i_n = X^i_{0n} \tag{2.22}
$$

where $(X^i_n)^\dagger = X^i_{-n}$. It can be shown that the classical vacuum configuration of this system corresponds to $X^i_n = 0$ for $n \neq 0$ and $[X^i_0, X^j_0] = 0$. So the hermitian matrices X^i_0 are simultaneously diagonolizable and the eigenvalues corresponds to position of the N $D0$-branes. The matrices X^i_n can be interpreted as Fourier modes of a theory on a dual circle (x represents coordinate along the dual circle) of radius $\widetilde{L}^*_9$,

$$
\begin{aligned}
\mathcal{A}^0 &= \sum_n e^{in\frac{x}{\widetilde{L}^*_9}} A^0_n \\
\mathcal{A}^9 &= \frac{1}{2\pi\widetilde{\alpha}'} \sum_n e^{in\frac{x}{\widetilde{L}^*_9}} X^9_n \\
\mathcal{X}^i &= \sum_n e^{in\frac{x}{\widetilde{L}^*_9}} X^i_n \quad , \qquad i = 1, \cdots 8
\end{aligned}
\tag{2.23}
$$

With above relation it can be shown that the BFSS action (eqn.(2.1)) compactified on a circle is equivalent to a action obtained from dimensional reduction of $10d$ $U(N)$ SYM to $1+1$ dimension compactified on a dual circle which gives system of N $D1$-branes (D-strings) wrapped on the dual circle. The relationship basically relates winding modes X^i_{mn} in the $D0$-brane theory (Type IIA string theory) to $(n-m)$ units of momentum in the dual theory, which is precisely the mapping from winding to momentum modes in the closed string theory under T-duality. The dual action is given by,

$$\begin{aligned}\mathcal{S} &= \frac{1}{2\widetilde{g}_s\widetilde{l}_s}\int dt\int\frac{dx}{2\pi\widetilde{L}_9^*}\,Tr[(D_t\mathcal{X}_i)^2-(D_x\mathcal{X}_i)^2+\frac{1}{(2\pi\widetilde{l}_s^2)^2}\sum_{i>j}[\mathcal{X}^i,\mathcal{X}^j]^2\\ &+ (2\pi\widetilde{l}_s^2)^2\mathcal{F}_{tx}^2+\frac{i}{2\pi\widetilde{l}_s^2}\Theta^T\not{D}\Theta-\frac{1}{(2\pi\widetilde{l}_s^2)^2}\Theta^T\gamma_i[\mathcal{X}_i,\Theta]]\end{aligned} \tag{2.24}$$

where BFSS matrix model action (eqn.(2.1)) is,

$$S=\frac{1}{2\widetilde{g}_s\widetilde{l}_s}\int dt\,Tr\left[(D_tX^i)^2+\frac{1}{(2\pi\widetilde{l}_s^2)^2}\sum_{i>j}[X^i,X^j]^2+\frac{1}{2\pi\widetilde{l}_s^2}\theta^T iD_t\dot{\theta}-\frac{1}{(2\pi\widetilde{l}_s^2)^2}\theta^T\gamma_i[X^i,\theta]\right] \tag{2.25}$$

$\mathcal{S}$ and S are related by (eqn.(2.23)) and $\not{D}=\mathbb{1}D_t-\gamma_9D_x$. The fermioinc fields Θ and θ are also related by Fourier transform like (eqn.(2.23)). $\mathcal{F}$ is the gauge field corresponding to the potential $\mathcal{A}$. The $1+1$ dimensional Yang-Mills coupling from as follows from the coefficient of $\mathcal{F}^2$ term in (eqn.(2.24)) is given $g^2_{YM,IIB}=2\pi\widetilde{L}_9^*\widetilde{l}_s\widetilde{g}_s$. Now as this theory describes $D1$-branes, $g_{YM,IIB}$ should be related to $D1$ tension, and this in turn gives the IIB D-string coupling as $\widetilde{g}_s^{IIB}=\widetilde{g}_s\frac{\widetilde{l}_s}{\widetilde{L}_9}=\frac{R_{11}}{\widetilde{L}_9}$ [3], as expected from T-duality. As Seiberg-Sen [97, 98] argument is compatible with above construction, Eqn.2.24 is dual to sector of M-theory compactified on a torus with N units of light-cone momentum.

2.3.2 S duality

The T-duality symmetry discussed above is a perturbative symmetry which take type IIA to type IIB string theory. In this subsection we will discuss the effect of an non-perturbative symmetry existing in string theory called S-duality. Under S-duality IIB theory is mapped onto itself according to the group $SL(2,\mathbb{Z})$. At the level of the low-energy IIB supergravity, the dilaton and axion from fundamental $SL(2,\mathbb{Z})$ multiplet. Also NS-NS and R-R two forms form a fundamental multiplet. String coupling is given by expectation values of the dilaton, the S-duality is an non-perturbative symmetry which exchange strong and weak coupling. Because

[3] Low energy effective action of a Dp-brane is given by $10d$ SYM reduced to $(p+1)$ dimension [89]. The $(p+1)$-dimensional Yang-Mills coupling (g_{YM}) is related to the Dp-brane tension τ_p as $g^2_{YM}=\frac{1}{(2\pi l_s^2)^2\tau_p}$, where $\tau_p=\frac{1}{g_sl_s}\frac{1}{(2\pi l_s)^p}$.

S-duality exchanges NS-NS and R-R two forms, under $SL(2,\mathbb{Z})$ the F-strings (fundamental strings) and D-strings are interchanged. In the last subsection we have obtained $\widetilde{g}_s^{IIB} = \frac{R_{11}}{\widetilde{L}_9}$, so under $R_{11} \leftrightarrow \widetilde{L}_9$, $\widetilde{g}_s^{IIB} \to \frac{1}{\widetilde{g}_s^{IIB}}$ also $D1$-branes wrapping the circle $\widetilde{L}_9^*$ is mapped to F-strings wrapping R_{11}.

2.4 Membranes and Strings in Matrix model

In order for matrix models to be a proper description of M-theory, it should contain membranes. Almost a decade before the proposal of matrix model for M-theory, de Witt et. all proposed an "matrix regularization" [100] for supermembranes. According to the regularization procedure, functions on membranes are mapped to finite size $N \times N$ matrices. In the review [93, 85, 94, 95] and references therein shows how membranes can be constructed from $U(N)$ matrix models of M-theory even at finite N. The connection of this "matrix regularization" and Lagrangian of $D0$-branes [89] was shown by Townsend [101]. We will here review the construction of infinite membranes considered in [1], which will be useful in this thesis. In the large N limit it is possible to construct membranes of infinite spatial extent. Infinite membranes are of particular interest because they can be Bogomolnyi-Prasad-Sommerfeld (BPS) states which solve the classical equation of motion of matrix theory.

2.4.1 Membrane/Matrix Correspondence

Let us consider 't Hooft-Schwinger-Von Neumann-Weyl pair of conjugate unitary operators U and V satisfying,

$$\begin{aligned} U^N &= V^N = 1 \\ UV &= e^{\frac{2\pi i}{N}} VU \end{aligned} \tag{2.26}$$

These operators can be represented on a N dimensional Hilbert space as clock and shift operators. They form a basis for all operators in the space. Any $N \times N$

matrix Z can be written in the form

$$Z = \sum_{n,m=1}^{N} Z_{nm} U^m V^n \tag{2.27}$$

In the large N limit these matrices can be conveniently written in the form,

$$\begin{aligned} U &= e^{iq} \\ V &= e^{ip} \end{aligned} \tag{2.28}$$

where p, q satisfy commutation relation [4],

$$[p, q] = \frac{2\pi i}{N} \tag{2.29}$$

though of course the operators p, q do not exist for finite N.

Let us define the adjoint action of an matrix O an another X as $[O, X] = \bar{O}X$. $\bar{O}$ is called the adjoint representation of O. Then with this definition of adjoint operators, we get for p, q pair from Eqn.2.29,

$$\begin{aligned} \bar{p} &= \frac{2\pi i}{N}\partial_q \\ \bar{q} &= -\frac{2\pi i}{N}\partial_p \end{aligned} \tag{2.30}$$

Above realization is valid only in large N limit, and was discussed in matrix model context in [102].

In large N limit, Z_{nm} can be treated as smooth functions of p, q which are treated as c-number. It is clear that only periodic functions of p, q are allowed and the space defined by p, q forms a torus. Similarity of nature of commutation of p, q and that of quantum phase space leads us to draw a correspondence between $\hbar$ and $\frac{1}{N}$. So $N \to \infty$ limit can be compared with usual "classical limit", where the operators are replaced by smooth functions of phase space coordinates. Although we should bear in mind that it is map from classical matrix model theory to a classical membrane theory and nothing is quantized. The large N limit the correspondence between matrices and function on membranes can be summarized

[4]Note our conversion for p, q is different from that in [1]. To match with [1], $p \leftrightarrow q$.

as follows,

- The operators or matrices Z can be treated as a smooth function $Z(p,q)$ where p,q are treated as c-numbers.
- The operation of taking trace of an operator goes over to N times the integral over the torus,
$$Tr[Z] \to N \int_0^{2\pi} \frac{dp}{2\pi} \int_0^{2\pi} \frac{dq}{2\pi} Z(p,q) \tag{2.31}$$
- The operation of commuting two matrices is replaced by $\frac{1}{N}$ times the classical Poisson bracket,
$$[Z,W] \to \frac{2\pi i}{N}\{\partial_p Z \partial_q W - \partial_q Z \partial_p W\} \tag{2.32}$$

Using the above correspondence, action of BFSS matrix model (eqn.(2.1)) can be shown equivalent to DLCQ of $11d$ supermembrane. Also in appendix of [1] it was shown the membrane tension from M-theory or String theory analysis and that from matrix model analysis exactly match.

2.4.2 Infinite membranes

Let us now consider the matrix model equation of motion for static configuration from (eqn.(2.1)) ,
$$[X^i,[X^j,X^i]] = 0 \tag{2.33}$$

We can consider an infinite stretched membrane which satisfies above classical equation,

$$\begin{aligned} X^8 &= R_8 p \\ X^9 &= R_9 q \\ X^i = 0 \quad \text{for all} \quad i &\neq 8,9 \end{aligned} \tag{2.34}$$

These describe classical macroscopic infinite membranes of M-theory. These membranes has a dual description in terms of collective excitation of $D0$-branes. It can be argued from the commutator of p,q and the fact that in light-cone quantization of membranes P^+ is uniformly distributed on the membrane, that $D0$-branes represent "quanta" of the area of the membrane. As $M2$-branes of M-theory, which are not wrapping the compact direction of M-theory can be mapped to $D2$-brane of type IIA string theory (See Table(2.1)), it is natural to expect this infinite membranes to be $D2$-branes. It can be argued as follows, as seen in Section (2.3.1), $D0$-branes compactified on a circle (X^9) gives $D1$-brane wrapping the dual circle (X^{9*}) . Now if we consider another direction compact (X^8), a T dual in the direction of X^8 of a system of $D1$-branes wrapping X^{9*} will give $D2$-brane wrapping torus formed by X^{9*} and X^{8*}. This actually can be shown to be equivalent by considering X^8 and X^9 to be compact directions, *i.e.* matrix model on a torus T^2. Then the classical configuration (eqn.(2.34)) describes a membrane wrapping the torus by Membrane/Matrix duality. On the other hand, if we now consider T-duality twice, (as described in Section (2.3.1)) once on each of two directions X^8 and X^9, we can show that the relation from (eqn.(2.34)), $Tr\ [X^8, X^9] = i\frac{A}{2\pi}$ (A area of the torus) gets mapped to a dual condition $\frac{1}{2\pi}\int_{T^{2**}} Tr\ \mathcal{F} = 1$ (T^{2**} is the dual torus, $\mathcal{F}$ is the gauge on the world volume of the torus by Fourier transform of original matrices), *i.e.* these classical membrane solutions can be described by N $D2$-brane charge wrapping a torus with unity of magnetic flux.

2.4.3 Wrapped membranes as IIA fundamental strings

As described in the previous sub-section, M-theory membranes which are unwrapped in the longitudinal direction appear as $D2$-branes in the IIA string theory. We can also consider wrapped membranes in M-theory which will correspond to F-strings in IIA theory. The charge in matrix model which measures the number of F-strings present is proportional to

$$\frac{i}{R_{11}} Tr\ ([X^i, X^j] D_0 X^j + [[X^i, \theta], \theta]) \tag{2.35}$$

Configurations with non-zero values of this charge were studied in [103]. To realize a configuration with fundamental string charge, we have to consider membranes extended in X^i and X^j giving a local membrane charge given by $[X^i, X^j]$ and also $D0$ branes should have velocity along X^j direction. The configuration in [94, 95] was infinite flat branes in X^1 and X^2 directions (considered in previous sub-section) which is also sliding along itself along the X^1 direction. This corresponds in IIA language to $D2$-branes wrapping the compact direction as periodic functions of X^1 with infinite fundamental strings in X^2 direction. As follows from the commutator term $[X^i, X^j]$, we cannot construct a classical string theory solution which is truly one dimensional and no local membrane charge.

2.5 Matrix String theory

A matrix formulation of string theory comes from the T-duality of matrix model action. The resulting matrix theory describes light-cone description of type IIA string theory in flat space and is given by $(1+1)$-dimensional SYM. This matrix string theory was first described in [104], which was further refined in [105, 25, 26]. We will here review the construction qualitatively. The model can be derived from matrix model for M-theory in the following way: Consider matrix theory compactified on a circle S^1 in the dimension 9 with radius $\widetilde{L}_9$. As described in Section (2.3.1), this theory has a dual description in terms of $(1+1)$-dimensional SYM on the dual circle $\hat{S}^1$. In BFSS formulation this corresponds to M-theory compactified on a T^2. If we consider the dimension 9 rather than the dimension 11 as the dimension which is compactified to get type IIA string theory, the fundamental objects carrying momentum P^+ are no longer $D0$-branes, but they are strings with longitudinal momentum $\frac{N}{R_{11}}$. Explicit calculation can show that the matrix model Hamiltonian gives *Green-Schwarz light-front type IIA string Hamiltonian* under this reidentification, with a modification that the fields are now $N \times N$ matrices. The theory due to its matrix nature automatically contains multi-string objects living in second quantized Hilbert space. It is also possible to construct extended objects in string theory in terms of non-commuting matrix variables, by a simple translation from original matrix model. Also in IIA Matrix string theory, individual string bits carrying single unit of longitudinal momentum combine to form "long strings". As the string coupling goes to zero, the coefficient of the term $[X^i, X^j]$ becomes large, which forces the matrices to be simultaneously diagonolizable. Although the string configuration is defined over a circle S^1, the matrix configuration need not be periodic, but can be related by an arbitrary permutation of the diagonal elements of the matrix. The length of the cycle of this permutation

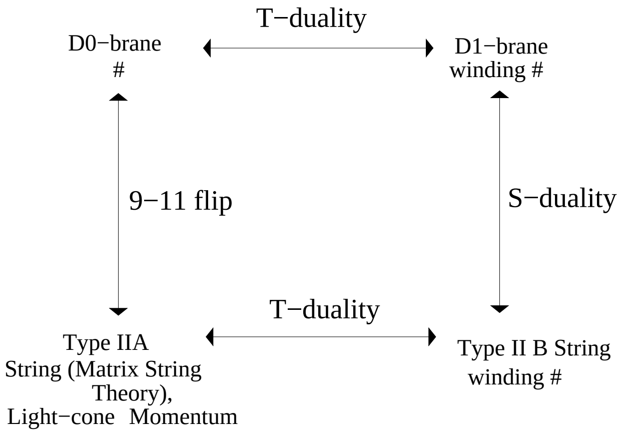

Figure 2.3: Relation between various matrix models and interpretation of number of $D0$-branes (N) in various models

determines the number of string bits which combine into a long string. Fig.2.3 shows the relationship between various matrix models.

3 AdS/CFT Correspondence for Matrix Theory

D-branes are solitonic objects in string theory which has two kinds of excitation modes: open string modes propagating on the D-brane world-volume and closed string modes propagating in the bulk (directions transverse to D-branes). These modes in general interact with each other, but there exists a limit [4, 30, 2] in which these modes decouple from each other. This limit is basically the low energy limit where D-branes can be described by massless excitations of the open string (*viz.* Gauge field), giving rise to an effective field theory description of dimensionally reduced $10d$ SYM theory described for $D0$-branes and $D1$-branes in previous sections [89, 90, 91, 92]. Separating D-branes by some distance corresponds in the field theory to giving Higgs expectation values to some fields. In the field theory limit considered is taken in such a way that this Higgs expectation values remain finite, amounting to studying branes in "sub-stringy" distances. In the decoupling limit, the D-branes possess an alternative description via the closed string massless excitations (*viz.* Gravition), *i.e.* can be studied as a solution of super-gravity. This decoupling or field theory limit was considered in [2] for N $D3$-branes and AdS/CFT correspondence was proposed. It was argued from dual nature of descriptions of $D3$-branes, that $4D$ $\mathcal{N}=4$ $U(N)$ SYM theory is dual to type IIB string theory (in low energy case, IIB super-gravity) on $AdS_5 \times S^5$ (which is a vacuum solution of type IIB super-gravity) in large N limit. $4D$ $\mathcal{N}=4$ $U(N)$ SYM theory is conformal, and thus the name AdS/CFT correspondence. It was later generalized by [30] for general class of Dp-branes and non-conformal field theories.

We will review here [30], and concentrate on the case of $D0$ and $D1$-branes. In the super-gravity description of D-branes, solutions are found with same mass and charge of D-branes. We can trust these super-gravity description as long as the curvature is locally small compared to string scale (or the Plank scale). A careful analysis shows for a system of large number of D-branes or in the large N limit, curvatures are small and supergravity results can be trusted even at sub-stringy distances in the decoupling limit considered above.

3.0.1 General Dp-brane case

Consider the Dp-brane field theory (in units of $10d$ Planck mass $\widetilde{M_p}^{(10)}$, we consider there tilde parameters to make a connection with BFSS matrix model) limit given by,

$$\widetilde{g}_{YM}^2 = (2\pi)^{(p-2)}\widetilde{g}_s\widetilde{l}_s^{(p-3)} = \text{fixed} \qquad \text{as} \qquad \widetilde{l}_s \to 0 \tag{3.1}$$

where $\widetilde{g}_s = e^{\phi_\infty}$ and ϕ_∞ is the dilaton field value at infinite distance from D-branes. This is exactly the same limit (Seiberg-Sen) considered in the Sec.(2.2.1), for matrix model description to be valid for $p < 3$. We will drop the tilde in future expressions in this section for convenience and will always consider $p < 3$. The connection between Seiberg-Sen limit and the decoupling limit discussed here is analyzed in detail in [106]. $g_s \to 0$ implies the theory decouples from the bulk, since $10d$ Newton's constant ($G_N^{(10)} = (M_p^{(10)})^{-8} \sim g_s^2 l_s^8$) [1] goes to zero. Also higher order stringy corrections are suppressed in the limit $l_s \to 0$.

Along with the limit (eqn.(3.1)), to consider finite Higgs expectation value, following limit is considered,

$$U = \frac{r}{l_s^2} = \text{fixed}, \qquad l_s \to 0 \tag{3.2}$$

where r is the radial co-ordinate defined in the transverse space of the D-branes. Note that in this limit $\frac{r}{l_s} \to 0$, which implies the system is studied at sub-stringy

[1] It should be noted that $10d$ Newton's constant is related to $11d$ by $G_N^{(11)} \sim R_{11}G_N^{(10)}$. Also $10d$ and $11d$ Planck mass is related by $(M_p^{(10)})^8 \sim R_{11}(M_p^{(11)})^9$. As in Seiberg-Sen limit $R_{11}(M_p^{(11)})^2$ is held fixed $R_{11} \to 0$ and $M_p^{(11)} \to \infty$, $M_p^{(10)} \to \infty$. Also note, $M_p^{(11)} = g_s^{-\frac{1}{3}}l_s^{-1}$ and $M_p^{(10)} = g_s^{-\frac{1}{4}}l_s^{-1}$.

level. At an given energy scale, U, the effective dimensionless coupling is $g_{eff} \simeq g_{YM}^2 N U^{p-3}$ ($\lambda = g_{YM}^2 N$ is the t'Hooft coupling). Thus perturbative Yang-Mills' calculation can be trusted when $g_{eff}^2 << 1$, which implies,

$$U >> \lambda^{\frac{1}{3-p}} \tag{3.3}$$

The Type II supergravity solution with N coincident extremal Dp-branes is given by (in string frame [2]) [107],

$$\begin{aligned}
ds^2 &= f_p^{-1/2}(-dt^2 + dx_1^2 + \cdots + dx_p^2) + f_p^{1/2}(dx_{p+1}^2 + \cdots + dx_9^2) \\
e^{-2(\phi-\phi_\infty)} &= f_p^{(p-3)/2} \\
A_{0\cdots p} &= -f12(f_p^{-1} - 1) \\
f_p &= 1 + \frac{d_p g_{YM}^2 N}{U^{7-p}} \\
d_p &= 2^{7-2p}\pi^{\frac{9-3p}{2}}\Gamma(\frac{7-p}{2})
\end{aligned} \tag{3.4}$$

In the decoupling limit given by 3.1 and 3.2, we get,

$$\begin{aligned}
ds^2 &= \alpha'\Big(\frac{U^{(7-p)/2}}{g_{YM}\sqrt{d_p N}}(-dt^2 + dx_\parallel^2) + \frac{g_{YM}\sqrt{d_p N}}{U^{(7-p)/2}}dU^2 + g_{YM}\sqrt{d_p N}U^{(p-3)/2}d\Omega_{8-p}^2\Big) \\
e^\phi &= (2\pi)^{2-p} g_{YM}^2 (\frac{g_{YM}^2 d_p N}{U^{7-p}})^{\frac{3-p}{4}} \sim \frac{g_{eff}^{(7-p)/2}}{N}
\end{aligned} \tag{3.5}$$

Note that the effective string coupling e^ϕ is finite in the decoupling limit. The curvature associated with the metric in decoupling limit is given by,

$$\alpha' R \simeq \frac{1}{g_{eff}} \sim \sqrt{\frac{U^{3-p}}{\lambda}} \tag{3.6}$$

From field theory point of view U is an energy scale and from gravity side it is characteristic length scale. Thus going to UV in field theory means taking the limit $U \to \infty$. In this limit the effective coupling vanishes and theory becomes UV free and the perturbative SYM is a valid description.

[2]The string frame and Einstein frame are related by an Weyl scaling of the metric, which in $10d$ is given by $g_{\mu\nu}^s = e^{\frac{\phi}{2}} g_{\mu\nu}^E$, where ϕ is the dilaton field.

We can also consider black Dp-brane solutions in decoupling limit which corresponds to field theory at finite temperature, given by,

$$ds^2 = \alpha'\Big(\frac{U^{(7-p)/2}}{g_{YM}\sqrt{d_pN}}(-(1-\frac{U_H^{7-p}}{U^{7-p}})dt^2+dx_{\parallel}^2)+\frac{g_{YM}\sqrt{d_pN}}{U^{(7-p)/2}(1-\frac{U_H^{7-p}}{U^{7-p}})}dU^2+g_{YM}\sqrt{d_pN}U^{(p-3)/2}d\Omega_{8-p}^2\Big) \tag{3.7}$$

The dilaton has same expression as in (eqn.(3.5)). Also for near-extremal branes,

$$U_H^{7-p} = a_p g_{YM}^4 \epsilon, \qquad a_p = \frac{\Gamma(\frac{9-p}{2})2^{11-2p}\pi^{\frac{13-3p}{2}}}{(9-p)} \tag{3.8}$$

where ϵ is the energy density of the brane above extremality and corresponds to energy density of the SYM theory.

In order to trust the type II supergravity solution (eqn.(3.5)) we need both curvature (eqn.(3.6)) and the dilaton to be small. This implies,

$$1 << g_{eff}^2 << N^{\frac{4}{7-p}} \tag{3.9}$$

So the perturbative SYM description and supergravity description do not overlap. Supergravity description can be extended in the region $g_{eff}^2 > N^{\frac{4}{7-p}}$ in terms of a dual theory.

The isometry group of metric (eqn.(3.5)) is $ISO(1,p) \times SO(9-p)$. In SYM side $ISO(1,p)$ is the Poincare symmetry and $SO(9-p)$ is the R-symmetry.

3.0.1.1 $D1$-brane case

From (eqn.(3.9)) we can show that Type IIB supergravity description of extremal N $D1$-branes is valid in the range,

$$g_{YM}N^{1/6} << U << g_{YM}\sqrt{N} \tag{3.10}$$

In the region $U >> g_{YM}\sqrt{N}$, the $1+1$ dimensional SYM theory is weakly coupled and trusted. In the region $U << g_{YM}N^{1/6}$ the string coupling is large. We can apply a S-duality, which takes $g_s \to \hat{g}_s = \frac{1}{g_s}$ and $\alpha' \to \hat{\alpha}' = g_s\alpha'$ (as $10d$ Newton's constant is invariant). In the decoupling limit considered $\hat{\alpha}' \to 0$ (as $\frac{g_s}{\alpha'}$ is held fixed). As described in [30], under this duality (eqn.(3.5)) with $p = 1$ maps to small r region of F-string. In the IR limit ($U \to 0$) the F-string coupling vanishes and the curvature in new string units is,

$$\hat{\alpha}' R \sim \frac{g_{YM}^2}{U^2} \tag{3.11}$$

Note that this curvature do not depend on N. The above relation shows that there is curvature singularity in the IR and supergravity picture breaks down. So supergravity description can be extended to a region

$$g_{YM} << U << g_{YM}\sqrt{N} \tag{3.12}$$

In the IR, $1 + 1$ dimensional SYM theory, which is super-renormalizable, flows to an trivial orbifold $((\mathbb{R}^8)^N/S_N)$ conformal field theory, which was studied in [26]. It was described in [30], the calculation of partition function shows that this orbifold theory is a good approximation if the temperature satisfies $T << g_{YM}/N^{1/2}$ or in terms of U_H (eqn.(3.7)) it translates to $U_H << g_{YM}$.

3.0.1.2 $D0$-brane case

Perturbation theory in the matrix model or (0+1)-dimensional SYM can be trusted when $U > g_{YM}^{2/3} N^{1/3}$ (eqn.(3.3)). The Type IIA supergravity description is valid in the range,

$$g_{YM}^{2/3} N^{1/7} << U << g_{YM}^{2/3} N^{1/3} \tag{3.13}$$

As described in [30, 106], in the IR region $g_{YM}^{2/3} N^{1/9} << U << g_{YM}^{2/3} N^{1/7}$, it is described by uplifting near extremal $D0$ brane solution to 11 dimension and the solution is translationally invariant along the 11th-dimensional circle. The solution can be generated by starting with an uncharged black string along x_{11} and then boosting infinitely it along x_{11} while taking the limit,

$$\gamma \to \infty, \qquad \gamma\mu = \frac{N}{2\pi R_{11}^2} = \text{fixed} \tag{3.14}$$

where μ is mass per unit length of the black string in its rest frame. For lower value of energy the system behaves more as single graviton. This region is described by matrix model black holes described in [108].

3.1 Hagedorn/Deconfinement/Hawking-Page Transition

In this section we will describe the connection between Hawking-Page phase transition [38], Confinement-Deconfinement transition and Hagedorn transition. Connection between Hawking-Page transition and deconfinement is discussed in [39] and connection with Hagedorn transition is discussed in [11, 14, 109, 15, 18] and references therein. We will first summarize the known results for Hawking-Page Transition, Confinement-Deconfinement transition and Hagedorn transition, and then summarize the connection between them.

3.1.1 Hagedorn temperature in String theory

We will summarize the notion of Hagedorn temperature in string theory. The detail review of the subject can be found in [11, 12, 110]. As discussed in the introduction of the thesis, the finite temperature string partition function shows a singularity at a temperature of order square root of string tension due to exponential growth in the number of states with energy. Another interpretation of the Hagedorn temperature was given in [11], where it was shown that that Hagedorn temperature corresponds to the temperature at which a winding mode on the thermal circle becomes tachyonic (negetive mass squared). Thermal partition function $Z(T)$ of a field theory may be computed by the considering the Euclidean action with time direction compactified on a circle of radius β ($\beta = \frac{1}{T}$ corresponds to inverse temperature T). So for free string theory $Z(T)$ can be computed by the string theory torus partition function on Euclidean space-time, which is again related to the 1-loop vacuum energy of a string theory in one lower dimension. This vacuum energy diverges if there exists a tachyonic field in its' spectrum and indeed a superstring winding a thermal cycle odd number of times becomes tachyonic at high temperature. The ground state of a superstring that winds once around the thermal circle has mass,

$$m_w^2 = \frac{2}{\alpha'}\Big(\frac{T_H^2}{T^2} - 1\Big) \tag{3.15}$$

which becomes tachyonic ($m_w^2 < 0$) for $T > T_H$. $T_H = \frac{1}{2\pi\sqrt{2\alpha'}}$ for a type II superstring. T_H matches with the alternative calculation from density of states at a given mass level. So this provides an alternative description of Hagedorn temperature. As at $T = T_H$, a string mode becomes massless, which can be then interpreted as a phase transition [8, 9, 45, 11]. This can interpreted either as a first order which happens at at temperature slightly lower than Hagedorn temperature or a second order transition which happens at Hagedorn temperature, depending on the sign of finite g_s corrections. However, for strictly free strings ($g_s = 0$) such an interpretation of phase transition is not possible, and the partition function is simply ill-defined for $T > T_H$. Also as shown in [11], the free energy above Hagedorn temperature should have genus zero contribution, but in case of string theory the free energy should begin from 1-loop order as the genus zero world-sheet is simply connected. This implies above at above Hagedorn temperature the genus zero world sheet is not simply connected. It was also suggested [11], in condensation of the thermal winding modes would makes this genus zero world-sheet not simply connected and should be replaced by quantized Riemann surfaces.

3.1.2 Confinement-Deconfinement transition in Gauge theory at large N

QCD (In general $SU(N)$ Yang-Mills' theory) is believed to undergo deconfining phase transition [111, 112, 10, 113] [3] at a non-zero temperature T_{dec} for any N. The deconfinement transition is first order for both large N and $N = 3$ (QCD). Below deconfinement temperature the theory is described by glueballs and mesons. The free energy is order 1 in the large N limit. But as the theory is strongly coupled at low temperature regime and the perturbative analysis fails. The confinement for this case is a speculation as the theory is known to be confined at zero temperature. At high temperature the theory is weakly coupled, and the perturbative analysis ("asymptotic freedom") shows that the theory is deconfined, and the the-

[3] In confined phase the physical states of the system are charged zero under the gauge group, in deconfined phase the physical states are in general charged

ory is described in terms of free gluons. The number of gluons is of order N^2. Therefore the free energy is order N^2 in the large N limit. Lattice calculations show that this phase transition is first order. In the large N limit, $\lim_{N\to\infty} \frac{F(T)}{N^2}$ is one order parameter, where $F(T)$ is the free energy. The order parameter vanishes in confined phase and is finite in the deconfined phase. There exists another order parameter which is also present in finite N analysis, given by expectation value of the Polyakov-Susskind loop [111, 112].

For supersymmetric Yang-Mills' like $\mathcal{N} = 4$ theory in flat space analysis can be done both at strong and weak coupling limit via AdS/CFT. Let us consider the limit $N \to \infty$ with 't Hooft coupling $\lambda = g_{YM}^2 N$ held fixed. Since the theory is conformal the free energy density is given by $F = -\frac{\pi^2}{6} N^2 f(\lambda) T^4$ (the dependence on N is obtained from large-N counting). $f(\lambda)$ can be obtained in small λ limit via perturbative analysis and in the large λ limit via AdS/CFT computations. It is generally assumed $f(\lambda)$ is analytic in the real line λ. So there exists no phase transition in pure $\mathcal{N} = 4$ SYM theory in flat space. But modifications of $\mathcal{N} = 4$ SYM theory, such as $\mathcal{N} = 1^*$ theory [4] of Polchinski-Strassler [33] shows promise of deconfinement transition. Finite temperature aspects of $\mathcal{N} = 1^*$ model was studied in [114], but a complete understanding of the phase structure is still lacking. Also $\mathcal{N} = 4$ SYM theory on a sphere shows deconfinement phase transition at a temperature of order inverse radius of the sphere as discussed in detail in [3, 39, 14].

3.1.3 Hawking-Page Transition

Hawking-Page transition was originally found in [38], as a first order phase transition between thermal AdS_4 space and Schawrzschild black-hole in AdS_4 space. Later it was generalized by Witten [39] for AdS_{n+1} spaces. Euclidean AdS_{n+1} can have two possible boundaries $S^{n-1} \times S^1$ or $\mathbb{R}^{n-1} \times S^1$. It was shown that Hawking-Page transition occurs at finite temperature only in the case of $S^{n-1} \times S^1$. It was also shown in [39], AdS space with boundary $\mathbb{R}^{n-1} \times S^1$ can be obtained from $S^{n-1} \times S^1$ by a scaling limit, which also shows that the transition occurs at zero temperature. It was argued in [41], a phase transition can be realized in the case

[4] $\mathcal{N} = 1^*$ theories are obtained by mass perturbations of $\mathcal{N} = 4$ and differs from pure $\mathcal{N} = 1$ theory in terms of field content.

of flat boundary by introduction of an IR-cutoff near the origin of the thermal AdS bath.

3.1.4 Connection between the transitions

It was argued by 't Hooft in [115], that any gauge theory in the limit of large N with fixed $\lambda = g_{YM}^2 N$, can be recast as a string theory with string coupling $g_s \sim \frac{1}{N}$. The color factors in the gluon loops can be filled in a natural way to give Riemann surfaces. Thus as the vacuum diagram of genus-k is given by $g_s^{2(k-1)}$ in string theory, this corresponds to large-N $SU(N)$ theories to $N^{2(1-k)}$. Thus the free energy analysis near deconfinement transition at large N shows that, below T_{dec}, the free energy receives contribution from only genus $k \geq 1$. But for $T > T_{dec}$, the free energy receives contribution from genus zero giving a dependence of N^2. This exactly the case of Hagedorn transition for string theories where above Hagedorn temperature there is genus zero contribution. From this duality it is natural to expect $T_{dec} = T_H$ [10, 11, 14].

It was argued in [39], the the Hawking-Page transitions in AdS_5 space can be mapped to deconfinement transition in its' dual $\mathcal{N} = 4$ SYM theory living on the boundary of AdS_5 given by $S^3 \times S^1$. Also the result that there is no finite temperature Hawking-Page transition for boundary $\mathbb{R}^3 \times S^1$, maps to a dual well known statement that $\mathcal{N} = 4$ SYM theory on flat space does not undergo any phase transition. In [33], it was shown that a mass perturbation in the $\mathcal{N} = 4$ SYM corresponds to removing a region of space in the dual AdS space. In the framework of gauge/gravity correspondence, the radial coordinate on the gravity side corresponds to the energy scale on the field theory side. Thus the small radius cutoff on the gravity side implies introducing an IR cutoff in the field theory. Motivated by this, Cai et. all [41] studied Hawking-Page transition for AdS_5 with an IR cut-off, equivalently removing a portion of AdS space near small radius. It was found that with an IR cut-off, Hawking-Page transitions for flat boundaries can be realized even at finite temperature. AdS/CFT correspondence was first noticed by Maldacena when studying the decoupling limit of N coincident $D3$-branes. In the case of coincident Dp-branes ($p \neq 3$), there are also correspondences of this kind between certain supergravity solutions and $SU(N)$ supersymmetric field theories with sixteen supercharges in $p+1$ dimensions as discussed in the Section (3.0.1). In the decoupling limit, the geometry of supergravity solutions is no longer AdS and in these cases the field theories are no longer conformal field theories. The deconfinement transitions of such non-conformal theories can be mapped to Hawking-Page transitions between BPS Dp-brane solution and black Dp-branes. It was demonstrated in [41], deconfinement transition in these $SU(N)$ theories with 16 supercharges or the corresponding Hawking-Page transition can be realized only by introduction of an IR cut-off.

Wiiten [39] showed that the deconfinement transition in $3+1$-dim $\mathcal{N}=4$ SYM theory can be mapped to Hawking-Page transition of AdS_5 black holes by AdS/CFT. BFSS matrix model describing M-theory is a dimensional reduction of $3+1$-dim SYM theory to $0+1$ dimension. So the deconfinement transition in matrix models should correspond to "Hawking-Page" like transition for solutions of dual supergravity theories. Also we believe deconfinement transitions in matrix model can be mapped to Hagedorn transition in corresponding string theory. So, it is natural to expect "Hawking-Page" transitions in this lower dimension theories to be related to Hagedorn transition in the string theory described by matrix model. So, this duality relations between various theories can be applied in analysis of phase transitions in M-theory or Matrix string theory, and may through light on the Hagedorn transition which can be realized in matrix model.

Part II

Hagedorn Phase Transition and Matrix Model for Strings

4

Phase transitions in Matrix Model for Strings and Hagedorn Temperature

4.1 Construction of Long D-strings

We will consider the classical configuration of the BFSS action (eqn.(2.1)) given by,

$$X^9 = \widetilde{L}^9 p \tag{4.1}$$

where X^9 is a compact direction of the BFSS model (See Section (2.2)), and p is defined in Section (2.4). This configuration is a solution of classical equation of motion and describes a membrane wrapped around X^9 with other edges free. If we consider $X^i, i \neq 9$ as periodic functions of q, all of form $\exp(imq)$, then we get a closed string. Let us construct this string action in matrix model.

Consider T-duality (Section (2.3.1)) of BFSS matrix model along X^9, then the dual action is given by Eqn.(2.24), which is a dimensionally reduced $1+1$ SYM theory describing N Type IIB D-strings. Let us assume that the radius of the compact direction X^9 is $\widetilde{L}^9$, assumed to be small. A described in Section (2.3.1), the type IIB F-string length and coupling is given by $\widetilde{l}_s$ (same as Type IIA F-string, string tension= $\frac{1}{2\pi\widetilde{l}_s^2}$ and $\widetilde{\alpha}' = \widetilde{l}_s^2$) and $\widetilde{g}_s^{IIB} = \widetilde{g}_s \frac{\widetilde{l}_s}{\widetilde{L}_9} = \frac{R_{11}}{\widetilde{L}_9}$. Now the radius of the compact direction is $\widetilde{L}_9^* = \frac{\widetilde{l}_s^2}{\widetilde{L}_9}$. The D-string tension is $\frac{1}{2\pi\widetilde{\beta}'}$, where $\widetilde{\beta}' = \widetilde{l}_{s\beta}^2 = \frac{(\widetilde{l}_p^{11})^3}{\widetilde{L}^9} = R_{11}\widetilde{L}_9^*$. Now consider a S-duality (Sec. (2.3.2)) transformation. Under S-duality type IIB theory is mapped to itself where the D-string and F-string interchanges and the string coupling is inverted. So we get a theory of N type IIB

F-strings string length $(\widetilde{g}_s^{IIB})^{1/2}\widetilde{l}_s = \widetilde{l}_{s\beta} = \sqrt{R_{11}\widetilde{L}_9^*}$ and coupling $\hat{\widetilde{g}}_s^{IIB} = \frac{1}{\widetilde{g}_s^{IIB}} = \frac{\widetilde{L}_9}{R_{11}}$. The compact direction is still given by $\widetilde{L}_9^*$. Now consider again a T-duality along this circle of radius $\widetilde{L}_9^*$, and the theory gets mapped to Type IIA matrix string theory (Section (2.5)) on the dual circle of radius $\widetilde{L}_9^{**} = \frac{\widetilde{l}_{s\beta}^2}{\widetilde{L}_9^*} = R_{11}$. The F-string inverse tension is given by $\widetilde{\beta}' = \widetilde{l}_{s\beta}^2 = R_{11}\widetilde{L}_9^* = \frac{(\widetilde{l}_p^{11})^3}{\widetilde{L}^9}$ (String tension does not change under T-duality) and the string coupling $\widetilde{g}_{s\beta}^{IIA} = \frac{\hat{\widetilde{g}}_s^{IIB}\widetilde{l}_{s\beta}}{\widetilde{L}_9^*} = \frac{(\widetilde{L}^9)^{3/2}}{(\widetilde{l}_p^{11})^{3/2}}$. So we get the relation $\widetilde{g}_{s\beta}^{IIA}\widetilde{l}_{s\beta} = \widetilde{L}_9$ (Compared to the original BFSS relation $\widetilde{g}_s\widetilde{l}_s = R_{11}$).

As shown in [20] , $1+1$-dimensional matrix model action (after T-dual along X^9), at zero temperature, with background configuration given by membranes constructed in the above way, matches exactly with the string theory action in light cone frame. Similar calculation was done in [1] to show that Green-Schwarz IIA string can be constructed from BFSS matrix model. Consider the bosonic part of the $1+1$-dimensional action given by Eqn.(2.24) ($A_0 = 0$ gauge),

$$\begin{aligned}\mathcal{S} &= \frac{1}{2R_{11}}\int dt\int\frac{dx}{2\pi\widetilde{L}_9^*}\,Tr[(\partial_t\mathcal{X}_i)^2 - (D_x\mathcal{X}_i)^2 + \frac{1}{(2\pi\widetilde{l}_s^2)^2}\sum_{i>j}[\mathcal{X}^i,\mathcal{X}^j]^2 \\ &+ (2\pi\widetilde{l}_s^2)^2(\partial_t\mathcal{A}_9)^2 + \cdots] \end{aligned} \tag{4.2}$$

where $\cdots$ denotes other terms which is not purely bosonic. We have also used the relation $\widetilde{g}_s\widetilde{l}_s = R_{11}$. $\mathcal{A}_9$ is related to original $0+1$ matrix model by Taylor's construction, given by (Sec.(2.3.1)),

$$\mathcal{A}^9 = \frac{1}{2\pi\widetilde{\alpha}'}\sum_{n=-\infty}^{\infty}\exp(inx\frac{L_9}{\widetilde{\alpha}'})X_n^9 \tag{4.3}$$

where X_0^9 is the original $0+1$ matrix of uncompactified theory representing $D0$-brane co-ordinate along the compact 9-th direction. $D_x = \partial_x - i[\mathcal{A}_9, \bullet]$ is the covariant derivative in a direction X_9^*, which is T-dual to X_9, and has a radius $\widetilde{L}_9^* = \frac{\widetilde{\alpha}'}{L_9}$. x is the co-ordinate along a $D1$ brane wound around X_9^*. Then if we choose the configuration (eqn.(4.1)), then,

$$\mathcal{A}^9 = \frac{1}{2\pi\widetilde{L}_9^*}p + \hat{A}^9(x,t,q) \tag{4.4}$$

As mentioned we choose the configuration such that all the fields $\mathcal{X}^i$ are independent of p except $\mathcal{A}^9$ which has p-dependence in above fashion. Now for large N, we can use Matrix/Membrane correspondence (Sec. (2.4.1)) for the action (4.2), which gives,

$$\begin{aligned} \mathcal{S} &= \frac{1}{2R_{11}}\int dt \int_0^{2\pi\widetilde{L}_9^*}\frac{dx}{2\pi\widetilde{L}_9^*}N\int_0^{2\pi}\frac{dp}{2\pi}\int_0^{2\pi}\frac{dq}{2\pi}[(\partial_t\mathcal{X}_i)^2-(\partial_x\mathcal{X}_i+\frac{1}{N\widetilde{L}_9^*}\{p,\mathcal{X}_i\}_{P.B} \\ &+ \{\hat{A}^9(x,t,q),\mathcal{X}_i\}_{P.B})^2-\frac{4\pi^2}{N^2}\frac{1}{(2\pi\widetilde{l}_s^2)^2}\sum_{i>j}\{\mathcal{X}^i,\mathcal{X}^j\}^2_{P.B} \\ &+ (2\pi\widetilde{l}_s^2)^2(\partial_t\hat{A}_9)^2+\cdots] \end{aligned} \tag{4.5}$$

Now as the fields are independent of p, the Poisson brackets will vanish except $\{p,\mathcal{X}_i\}_{P.B}=\partial_q\mathcal{X}_i$. So we get,

$$\begin{aligned} \mathcal{S} &= \frac{1}{2R_{11}}\int dt\int_0^{2\pi\widetilde{L}_9^*}\frac{dx}{2\pi\widetilde{L}_9^*}N\int_0^{2\pi}\frac{dq}{2\pi}[(\partial_t\mathcal{X}_i)^2-(\partial_x\mathcal{X}_i+\frac{1}{N\widetilde{L}_9^*}\partial_q\mathcal{X}_i)^2 \\ &+ (2\pi\widetilde{l}_s^2)^2(\partial_t\hat{A}_9)^2+\cdots] \end{aligned} \tag{4.6}$$

So in the large N limit, D_x is given by

$$D_x=\partial_x\otimes I+I\otimes\frac{1}{N\widetilde{L}_9^*}\partial_q \tag{4.7}$$

which acts on eigenfunctions,

$$e^{ir\frac{x}{\widetilde{L}_9^*}}e^{inq} \tag{4.8}$$

with eigenfunctions values $\frac{rN+n}{N\widetilde{L}_9^*}$. Fig.(4.1) illustrates the $x-q$ space. Thus the effective radius is $N\widetilde{L}_9^*$. These are the long strings described in [116, 117, 26, 104]. We can replace $\bar{D}_x$ by ∂_σ and eigenfunctions (eqn.(4.8)) by $e^{\frac{in\sigma}{N\widetilde{L}_9^*}}$. Here σ has range $0-2\pi N\widetilde{L}_9^*$. Also $\int\frac{dx}{2\pi\widetilde{L}_9^*}\int_0^{2\pi}\frac{dq}{2\pi}\rightarrow\int_0^{2\pi N\widetilde{L}_9^*}\frac{d\sigma}{2\pi N\widetilde{L}_9^*}$. So we have,

$$\mathcal{S}=\frac{1}{4\pi\widetilde{\beta}'}\int dt\int_0^{2\pi N\widetilde{L}_9^*}d\sigma[(\partial_t\mathcal{X}_i)^2-(\partial_\sigma\mathcal{X}_i)^2+(2\pi\widetilde{l}_s^2)^2(\partial_t\hat{A}_9)^2+\cdots] \tag{4.9}$$

where $\widetilde{L}_9^*=\frac{\widetilde{\alpha}'}{\widetilde{L}_9}=\frac{\widetilde{\beta}'}{R_{11}}$. In the limit $\widetilde{L}_9\rightarrow 0$, $\hat{A}_9$ can be gauged away, and the action becomes,

$$\mathcal{S} = \frac{1}{4\pi\widetilde{\beta}'}\int dt \int_0^{2\pi\widetilde{\beta}' P_{11}} d\sigma[(\partial_t \mathcal{X}_i)^2 - (\partial_\sigma \mathcal{X}_i)^2 + \cdots] \tag{4.10}$$

where, $P_{11} = \frac{N}{R_{11}}$. Using the connection between DLCQ M-theory and BFSS matrix model (Sec. (2.2.1)), we can rewrite the action as,

$$\mathcal{S} = \frac{1}{4\pi\beta'}\int dt \int_0^{2\pi\beta' P^+} d\sigma[(\partial_t \mathcal{X}_i)^2 - (\partial_\sigma \mathcal{X}_i)^2 + \cdots] \tag{4.11}$$

where $P^+ = \frac{N}{R^-}$ is the light-cone momentum. Above action is same as the light-cone gauge type IIA Green-Scharwz string with string tension β'. If we include p dependence to be non-zero then the commutator terms are non-zero, and the corresponding fluctuations are not string-like. Full non-perturbative effective action will presumably make these fluctuations massive.

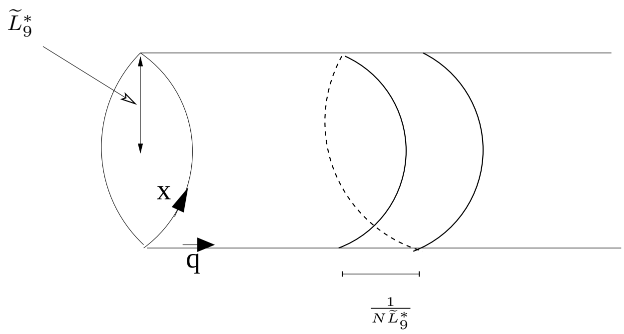

Figure 4.1: $x - q$ space

4.1.1 Two Phases

In this thesis we will calculate the free energy corresponding to two different configuration of $D0$-branes as done in our paper [24].

Phase 1: The background $X^9 = \widetilde{L}^9 p$ gives a configuration where the $D0$ branes spread out to form a string wound in the compact direction.

Phase 2: The background $X^9 = 0$ gives a phase where the $D0$-branes are clustered.

We will consider these two backgrounds to calculate free energy up to one loop level, and compare to find any signature of phase transition. It is important to have a precise definition of the measure in the functional integral. This is described in the next section and as an example we calculate partition function for $\mathcal{N}=2$ SUSY harmonic oscillator. For convenience we will drop the tilde sign on the parameters and put it back in the end.

4.2 Defining Measure for $\mathcal{N}=2$ SUSY in $1D$

We define measure such that

$$\int DxD\psi^* D\psi exp[-\pi \int_0^\beta dt(\frac{x^2}{2}+\psi^*\psi)] = 1 \tag{4.12}$$

Where $x(t)$ is a bosonic variable,and $\psi(t)$ is its super-partner. The SUSY transformation is given by,

$$\delta x = \epsilon^*\psi + \psi^*\epsilon; \delta\psi^* = -\epsilon^* x; \delta\psi = -\epsilon x. \tag{4.13}$$

where ϵ^* and ϵ are two infinitesimal anti commuting parameter. From these definition we can define the measure

$$DxD\psi^* D\psi \equiv dx_0 \prod_{n=1}^{\infty}(dx_n dx_{-n}) d\psi_0^* d\psi_0 \prod_{m>0} d\psi_m^* d\psi_{-m}^* d\psi_m d\psi_{-m} \tag{4.14}$$

The Fermionic measure in terms of ψ_1 and ψ_2,where $\psi = \psi_1 + i\psi_2$is given by,

$$D\psi^* D\psi \equiv id\psi_{10}d\psi_{20} \prod_{m>0} d\psi_{1m} id\psi_{1m}^* d\psi_{2m} id\psi_{2m}^* \tag{4.15}$$

where $x(t+\beta) = x(t)$; x_n are Fourier expansion co-efficient for $x.\psi_m$ are Fourier expansion co-efficient for $\psi(t)$, m runs over all integers for periodic boundary condition, but takes only odd values for anti-periodic boundary condition.

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{-\frac{2\pi i}{\beta}nt} \tag{4.16}$$

$$\begin{aligned}\psi(t) &= \sum_{n=-\infty}^{\infty} \psi_n e^{-\frac{2\pi i}{\beta}nt} \text{for periodic boundary condition} \quad (4.17)\\ &= \sum_{n=-\infty,odd}^{\infty} \psi_n e^{-\frac{\pi i}{\beta}nt} \text{for anti-periodic boundary condition}\end{aligned} \tag{4.18}$$

4.2.1 Zeta Function

We will need the following results [118] for our calculation,

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} n^{-s}\\ \zeta(s)' &= -\sum_{n=1}^{\infty} n^{-s}\ ln(n)\\ \zeta_{odd}(s) &= \sum_{n=1,n=odd}^{\infty} n^{-s}\\ &= (1-2^{-s})\zeta(s)\\ \zeta_{odd}(s)' &= -\sum_{n=1,n=odd}^{\infty} n^{-s}\ ln(n)\\ &= 2^{-s}\ ln2\ \zeta(s) + (1-2^{-s})\zeta(s)'\end{aligned} \tag{4.19}$$

and $\zeta(0) = -\frac{1}{2}$, $\zeta(0)' = -\frac{1}{2}ln(2\pi)$. Which gives $\zeta_{odd}(0) = 0$ and $\zeta_{odd}(0)' = -\frac{1}{2}ln(2)$.

4.2.2 Example: Super-symmetric ($\mathcal{N} = 2$) 1D Harmonic Oscillator

Consider the SUSY Harmonic Oscillator at finite temperature with action,

$$S = \int_0^\beta dt(\frac{\dot{x}^2}{2} - \psi^*\dot{\psi} + \frac{x^2}{2} + \psi^*\psi) \tag{4.20}$$

Where the SUSY transformation is given by,

$$\delta x = \epsilon^*\psi + \psi^*\epsilon; \delta\psi^* = -\epsilon^*(\dot{x} + x); \delta\psi = -\epsilon(-\dot{x} + x). \tag{4.21}$$

With periodic boundary condition on both x and ψ,

$$\begin{aligned} S &= \frac{1}{2}\beta x_0^2 + \beta\sum_{n=1}^{\infty}(1+\frac{4\pi^2 n^2}{\beta^2})x_n x_{-n} + \beta\psi_0^*\psi_0 + \sum_{n=1}^{\infty}(\beta+2\pi i n)\psi_n^*\psi_n \\ & +\sum_{n=1}^{\infty}(\beta-2\pi i n)\psi_{-n}^*\psi_{-n} \end{aligned} \tag{4.22}$$

So integrating by using the measure defined the partition function is,

$$\begin{aligned} Z &= \int Dx\ D\psi^* D\psi\ e^{-S} \\ &= \sqrt{\frac{2\pi}{\beta}} \times \prod_{n=1}^{\infty}\Big\{\frac{2\pi}{\beta(1+\frac{4\pi^2 n^2}{\beta^2})}\Big\} \times \beta \times \\ & \prod_{n=1}^{\infty}(\beta+2\pi i n) \times \prod_{n=1}^{\infty}(\beta-2\pi i n) \\ &= 1 \end{aligned} \tag{4.23}$$

Using $\prod_{n=1}^{\infty} C = C^{\zeta(0)} = C^{-1/2}$, where C is any constant number.

If we keep periodic boundary condition on x, but take anti-periodic boundary condition on ψ, the super-symmetry breaks and

$$\begin{aligned} S &= \frac{1}{2}\beta x_0^2 + \beta\sum_{n=1}^{\infty}(1+\frac{4\pi^2 n^2}{\beta^2})x_n x_{-n} + \sum_{n=1,n=odd}^{\infty}(\beta+\pi i n)\psi_n^*\psi_n \\ & + \sum_{n=1,n=odd}^{\infty}(\beta-\pi i n)\psi_{-n}^*\psi_{-n} \end{aligned} \tag{4.24}$$

So integrating by using the measure defined the partition function is,

$$\begin{aligned} Z' &= \int Dx\ D\psi^* D\psi\ e^{-S} \\ &= \sqrt{\frac{2\pi}{\beta}} \times \prod_{n=1}^{\infty}\Big\{\frac{2\pi}{\beta(1+\frac{4\pi^2 n^2}{\beta^2})}\Big\} \times \\ & \prod_{m=1,odd}^{\infty}(\beta+\pi i m) \times \prod_{m=1,odd}^{\infty}(\beta-\pi i m) \end{aligned} \tag{4.25}$$

Now rearranging the products and using ζ function to regularize the infinite products, (i.e. using $\prod_{n=1}^{\infty} C = C^{-1/2}$, $\prod_{n=1}^{\infty} n = \sqrt{2\pi}$, $\prod_{n=1,odd}^{\infty} C = 1$, $\prod_{n=1,odd}^{\infty} n = \sqrt{2}$) we get,

$$Z' = \frac{\prod_{k=0}^{\infty}(1 + \frac{4(\beta/2)^2}{\pi^2(2k+1)^2})}{\frac{\beta}{2}\prod_{n=1}^{\infty}(1 + \frac{(\beta/2)^2}{\pi^2 n^2})} = \coth \beta/2 \tag{4.26}$$

Using,$\prod_{k=0}^{\infty}(1+\frac{4x^2}{\pi^2(2k+1)^2}) = \cosh x$ and $x\prod_{n=1}^{\infty}(1+\frac{x^2}{\pi^2 n^2}) = \sinh x$ [118]. Also notice as $\beta \to \infty$, $Z' \to 1$ i.e. super-symmetry is restored in the zero temperature limit.

4.3 SUSY Scalar Field theory on $S^1 \times S^1$

We will first calculate action for a SUSY scalar field on $S^1 \times S^1$, which is then related to the action (eqn.(2.24)) we are concerned, in the following section. The Minkowski action is given by,

$$S_M = \frac{1}{g_s}\int \frac{dt_M}{l_s}\int_0^{2\pi L_9^*} \frac{dx}{2\pi L_9^*}\{(\partial_{t_M}X)^2 - (\partial_x X)^2 + \bar{\psi}(i\gamma^\mu)\partial_\mu\psi\} \tag{4.27}$$

ψ_α, $\alpha = 1, 2$ are two components (real) of two dimensional Majorana Spinor ψ.
γ matrices are given by,

$$\gamma^0 = \gamma_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \gamma^1 = -\gamma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{4.28}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}; \quad g^{00} = -g^{11} = 1 \tag{4.29}$$

Fermionic part of the action can be rewritten as,

$$\bar{\psi}(i\gamma^\mu)\partial_\mu\psi = i\psi_1(\partial_{t_M} - \partial_x)\psi_1 + i\psi_2(\partial_{t_M} + \partial_x)\psi_2 \tag{4.30}$$

Let us consider $t_M \to -it$, and t compact with periodicity β. The Euclidean action is given by $S = -iS_M$, we get,

$$S = \frac{1}{g_s}\int_0^\beta \frac{dt}{l_s}\int_0^{2\pi L_9^*} \frac{dx}{2\pi L_9^*}\{(\partial_t X)^2 + (\partial_x X)^2 + \psi_1(\partial_t + i\partial_x)\psi_1 + \psi_2(\partial_t - i\partial_x)\psi_2\} \tag{4.31}$$

Where X is periodic in both t and x, ψ_α is anti-periodic in t and periodic in x.

$$X = (2\pi L_9^*\beta)^{1/2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} X_{nm} e^{-\frac{2\pi i}{\beta} nt} e^{-\frac{i}{L_9^*} mx} \tag{4.32}$$

$$\psi_\alpha = (2\pi L_9^*\beta)^{1/4} \sum_{n=-\infty, n=odd}^{\infty} \sum_{m=-\infty}^{\infty} \psi_{\alpha,nm} e^{-\frac{\pi i}{\beta} nt} e^{-\frac{i}{L_9^*} mx} \tag{4.33}$$

X and ψ_α is real implies $X^*_{nm} = X_{-n-m}$ and $\psi^*_{\alpha,nm} = \psi_{\alpha,-n-m}$. X_{nm} is a dimensionless c-number and $\psi_{\alpha,nm}$ is a dimensionless Grassmann number.
So the action becomes,

$$\begin{aligned} S = & \tfrac{\beta}{2g_s l_s} \textstyle\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (2\pi L_9^*\beta)[(\tfrac{2\pi n}{\beta})^2 + (\tfrac{m}{L_9^*})^2] X_{nm} X_{-n-m} \\ & + \tfrac{\beta}{2g_s l_s} \textstyle\sum_{n=-\infty, n=odd}^{\infty} \sum_{m=-\infty}^{\infty} \{ i\sqrt{(2\pi L_9^*\beta)} [\tfrac{\pi n}{\beta} + i\tfrac{m}{L_9^*}] \psi_{1,nm}\psi_{1,-n-m} + \\ & i\sqrt{(2\pi L_9^*\beta)} [\tfrac{\pi n}{\beta} - i\tfrac{m}{L_9^*}] \psi_{2,nm}\psi_{2,-n-m} \} \end{aligned} \tag{4.34}$$

So we can write $S = S_B + S_F = S_B + S_{F1} + S_{F2}$. S_B is the bosonic part. S_F is the fermionic part of the action, which again receives contribution from two parts S_{F1} and S_{F2} corresponding to ψ_1 and ψ_2. Each ψ_α contributes same amount to partition function, $Z_F = Z_{F1} Z_{F2} = Z^2_{F1} = Z^2_{F2}$. The full partition function is given by

$$Z = Z_B Z_F \tag{4.35}$$

4.3.1 Bosonic Part

We will consider the bosonic part first.

$$S_B = \frac{\beta}{2g_s l_s} (2\pi L_9^*\beta) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [(\frac{2\pi n}{\beta})^2 + (\frac{m}{L_9^*})^2] X_{nm} X_{-n-m} \tag{4.36}$$

which implies,

$$S_B = (2\pi L_9^*\beta)\frac{\beta}{2g_sl_s}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}[\frac{4\pi^2n^2}{\beta^2}+\frac{m^2}{L_9^{*2}}]X_{nm}X_{nm}^* \tag{4.37}$$

$$\begin{aligned} &= (2\pi L_9^*\beta)\frac{\beta}{g_sl_s}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{[\frac{4\pi^2n^2}{\beta^2}+\frac{m^2}{L_9^{*2}}][X_{nm}X_{nm}^*+X_{n-m}X_{n-m}^*]\} \\ &+(2\pi L_9^*\beta)\frac{\beta}{g_sl_s}\sum_{n=1}^{\infty}\frac{4\pi^2n^2}{\beta^2}X_{n0}X_{n0}^*+(2\pi L_9^*\beta)\frac{\beta}{g_sl_s}\sum_{m=1}^{\infty}\frac{m^2}{L_9^{*2}}X_{0m}X_{0m}^* \end{aligned} \tag{4.38}$$

Now we can calculate partition function easily,

$$\begin{aligned} Z = &(\int_{-\infty}^{\infty}dX_{00})(\prod_{n=1}^{\infty}\prod_{m=1}^{\infty}\int_{-\infty}^{\infty}dX_{nm}dX_{nm}^*e^{-(2\pi L_9^*\beta)\frac{\beta}{g_sl_s}[\frac{4\pi^2n^2}{\beta^2}+\frac{m^2}{L_9^{*2}}]X_{nm}X_{nm}^*}) \\ &(\prod_{n=1}^{\infty}\prod_{m=1}^{\infty}\int_{-\infty}^{\infty}dX_{n-m}dX_{n-m}^*e^{-(2\pi L_9^*\beta)\frac{\beta}{g_sl_s}[\frac{4\pi^2n^2}{\beta^2}+\frac{m^2}{L_9^{*2}}]X_{n-m}X_{n-m}^*}) \\ &(\prod_{n=1}^{\infty}\int_{-\infty}^{\infty}dX_{n0}dX_{n0}^*e^{-(2\pi L_9^*\beta)\frac{\beta}{g_sl_s}\frac{4\pi^2n^2}{\beta^2}X_{n0}X_{n0}^*}) \\ &(\prod_{m=1}^{\infty}\int_{-\infty}^{\infty}dX_{0m}dX_{0m}^*e^{-(2\pi L_9^*\beta)\frac{\beta}{g_sl_s}\frac{m^2}{L_9^{*2}}X_{0m}X_{0m}^*}) \end{aligned} \tag{4.39}$$

The zero mode integral diverges. So we put a cut-off $\frac{L_0}{\sqrt{2\pi L_9^*\beta}}$ as the value of zero mode integral.

$$\begin{aligned} Z_B = &(\frac{L_0}{\sqrt{2\pi L_9^*\beta}})\prod_{n=1}^{\infty}\prod_{m=1}^{\infty}\Big\{\frac{2\pi}{(2\pi L_9^*\beta)\frac{\beta}{g_sl_s}(\frac{4\pi^2n^2}{\beta^2}+\frac{m^2}{L_9^{*2}})}\Big\}^2 \\ &\times\prod_{n=1}^{\infty}\Big\{\frac{2\pi}{(2\pi L_9^*\beta)\frac{\beta}{g_sl_s}\frac{4\pi^2n^2}{\beta^2}}\Big\}\times\prod_{m=1}^{\infty}\Big\{\frac{2\pi}{(2\pi L_9^*\beta)\frac{\beta}{g_sl_s}\frac{m^2}{L_9^{*2}}}\Big\} \end{aligned} \tag{4.40}$$

Using $\prod_{n=1}^{\infty}c = c^{\zeta(0)} = c^{-1/2}$, if we take out the constant factor $(2\pi L_9^*\beta)$ from the products, it cancels nicely with the factor in zero mode integral. These products can be rearranged to get,

$$Z_B = \Big\{ L_0 \prod_{n=1}^{\infty} \frac{2\pi}{\frac{\beta}{g_s l_s}(\frac{4\pi^2 n^2}{\beta^2})} \Big\}_{\text{free particle with mass}=\frac{1}{g_s l_s}} \times$$
$$\Big\{ \prod_{m=1}^{\infty} \Big[\sqrt{\frac{2\pi}{\frac{\beta}{g_s l_s}(\frac{m^2}{L_9^{*2}})}} \prod_{n=1}^{\infty} \frac{2\pi}{\frac{\beta}{g_s l_s}(\frac{4\pi^2 n^2}{\beta^2} + \frac{m^2}{L_9^{*2}})} \Big]_{\text{SHO,mass}=\frac{1}{g_s l_s}\text{freq.}=\frac{m}{L_9^*}} \Big\}^2 \tag{4.41}$$

therefore,

$$Z_B = L_0 \sqrt{(\frac{M}{2\pi\beta})} \prod_{m=1}^{\infty} \Big\{ \frac{1}{2\sinh(\beta\omega_m/2)} \Big\}^2 \tag{4.42}$$

where,

$$M = \frac{1}{g_s l_s},\ \omega_m = \frac{m}{L_9^*} \tag{4.43}$$

Using,

$$\eta(ix) = \prod_{k=1}^{\infty} (2\sinh(\pi k x)) \tag{4.44}$$

where $\eta(z)$ is Dedekind's eta function. We get,

$$Z_B = \frac{L_0}{\sqrt{(2\pi g_s l_s \beta)}}\ \eta(\frac{i\beta}{2\pi L_9^*})^{-2} \tag{4.45}$$

The Partition function in terms of ratio of radii of two S^1, *i.e.*. $x = \frac{\beta}{2\pi L_9^*}$,

$$Z_B = \frac{L_0}{\sqrt{4\pi^2 g_s l_s L_9^*}} \frac{1}{\sqrt{x}} \eta(ix)^{-2} \tag{4.46}$$

Dedekind's Eta Function has a symmetry given by,

$$\eta(ix) = \frac{1}{\sqrt{x}} \eta(i/x) \tag{4.47}$$

Which makes the partition function invariant under the transformation $x \to 1/x$ For low temperature, $\frac{\beta}{2\pi L_9^*} >> 1$, the free energy takes the form,

$$F_B(T) = -\frac{1}{\beta} ln(Z) \simeq -\frac{1}{12 L_9^*} - \frac{1}{2} T\ ln(\frac{L_0^2}{2\pi g_s l_s} T) \tag{4.48}$$

which shows $F_B(0) \neq 0$ due to the presence of zero-point energy,

$$F_B(0) = -\frac{1}{12L_9^*} = \sum_{n=1}^{\infty} \frac{n}{L_9^*} \tag{4.49}$$

using Zeta function regularization. The high temperature expansion, $\frac{\beta}{2\pi L_9^*} << 1$ is given as,

$$F_B(T) = -\frac{\pi^2 L_9^* T^2}{3} + \frac{T}{2}\, ln(\frac{8\pi^3 g_s l_s L_9^{*2}}{L_0^2} T) \tag{4.50}$$

4.3.2 Fermionic Part

Let us consider the fermionic part of the action,

$$\begin{aligned} S_F &= S_{F1} + S_{F2} = \frac{\beta}{2g_s l_s} \sum_{n=-\infty, n=odd}^{\infty} \sum_{m=-\infty}^{\infty} \Big\{ i\sqrt{(2\pi L_9^*\beta)} \Big[\frac{\pi n}{\beta} + i\frac{m}{L_9^*}\Big] \psi_{1,nm}\psi_{1,-n-m} \\ &+ i\sqrt{(2\pi L_9^*\beta)} \Big[\frac{\pi n}{\beta} - i\frac{m}{L_9^*}\Big] \psi_{2,nm}\psi_{2,-n-m} \Big\} \end{aligned} \tag{4.51}$$

By rearranging the sum (we have dropped the index 1 or 2 in ψ),

$$\begin{aligned} S_{F1} &= \frac{\beta}{g_s l_s}\sqrt{2\pi L_9^*\beta} \Big\{ \sum_{n,m=1,n=odd}^{\infty} (\frac{\pi n}{\beta} + i\frac{m}{L_9^*})\psi_{nm}\; i\psi_{-n-m} \\ &+ \sum_{n,m=1,n=odd}^{\infty} (\frac{\pi n}{\beta} - i\frac{m}{L_9^*})\psi_{n-m}\; i\psi_{-nm} \\ &+ \sum_{n=1,odd}^{\infty} \frac{\pi n}{\beta}\psi_{n0} i\psi_{-n0} \Big\} \end{aligned} \tag{4.52}$$

Therefore,

$$Z_{F1} = \Big\{ \prod_{n=1,odd}^{\infty} C\, \frac{\pi n}{\beta} \Big\} \Big\{ \prod_{n=1,odd}^{\infty} \prod_{m=1}^{\infty} C^2 \Big(\frac{\pi^2 n^2}{\beta^2} + \frac{m^2}{L_9^{*2}} \Big) \Big\} \tag{4.53}$$

where, $C = \frac{\beta}{g_s l_s}\sqrt{2\pi L_9^*\beta}$
Using, $\prod_{n=1,odd}^{\infty} C = C^{\zeta_{odd}(0)} = 1$, $\prod_{n=1,odd}^{\infty} n = e^{-\zeta'_{odd}(0)} = \sqrt{2}$ and rearranging

the products we get,

$$Z_{F1} = \sqrt{2} \prod_{n=1,odd}^{\infty} \prod_{m=1}^{\infty} (1 + \frac{(\frac{\pi^2 L_9^* n}{\beta})^2}{\pi^2 m^2}) \tag{4.54}$$

Using, $\frac{\sinh(x)}{x} = \prod_{k=1}^{\infty}(1 + \frac{x^2}{\pi^2 n^2})$ we get,

$$\begin{aligned} Z_{F1} &= \prod_{n=1,odd}^{\infty} \sinh(\frac{\pi^2 L_9^* n}{\beta}) \\ &= \frac{\prod_{n=1}^{\infty} 2\sinh(\frac{\pi^2 L_9^* n}{\beta})}{\prod_{n=1}^{\infty} 2\sinh(\frac{2\pi^2 L_9^* n}{\beta})} \\ &= \frac{\eta(i\frac{\pi L_9^*}{\beta})}{\eta(i\frac{2\pi L_9^*}{\beta})} \\ &= \frac{\eta(i\frac{1}{2x})}{\eta(i\frac{1}{x})} \end{aligned} \tag{4.55}$$

Where $x = \frac{\beta}{2\pi L_9^*}$, and using property of Dedekind eta function,we get,

$$Z_{F1} = \sqrt{2}\frac{\eta(2ix)}{\eta(ix)} \tag{4.56}$$

So we get ,

$$Z_F = Z_{F1} Z_{F2} = Z_{F1}^2 = 2\Big(\frac{\eta(2ix)}{\eta(ix)}\Big)^2 \tag{4.57}$$

Where $x = \frac{\beta}{2\pi L_9^*}$.

4.3.3 SUSY scalar field partition function

So SUSY partition function for the action (eqn.(4.31)) is given by (eqn.(4.35)),

$$Z = Z_B Z_F = \frac{2L_0}{\sqrt{4\pi^2 g_s l_s L_9^*}} \frac{1}{\sqrt{x}} \frac{\eta^2(2ix)}{\eta^4(ix)} \tag{4.58}$$

using Eqns. (4.46) and (4.57). Compare with (Eqn.(4.46)),

$$Z_B = \frac{L_0}{\sqrt{4\pi^2 g_s l_s L_9^*}} \frac{1}{\sqrt{x}} \eta(ix)^{-2} \tag{4.59}$$

Now Z does not have the $x \to 1/x$ symmetry, which is natural as two directions are not similar due to different boundary condition. At low temperature, i.e. $x \to \infty$, $Z = 2L_0 \sqrt{\frac{(\frac{1}{g_s l_s})}{2\pi\beta}}$, which is the partition function for a super-symmetric free particle (the zero mode).

4.4 Free energy for two phases of Matrix Model

We use Background Gauge Fixing Method (see Appendix B) to calculate the free energy up to one loop for the action ((eqn.(2.24))), which is a $1+1$ dimensional SYM action. From Appendix B we have,

$$lnZ^{(1)} = -\beta F = 5Tr(ln\ \bar{D}_E^2)_{bosonic} - 4Tr(ln\ \bar{D}_E^2)_{fermionic} - Tr(ln\ \bar{D}_E^2)_{ghost} \tag{4.60}$$

where, $(\bar{D}_E)^2 = (\partial_\tau)^2 + (\bar{D}_9)^2$ and $\bar{D}_9 = \partial_x - i[a_9, \bullet]$. Background of A_0 is zero [1] and background for A_9 is a_9. As described in Section (4.1), for String Phase $a_9 = \widetilde{L}_9 p$ and for Clustered phase $a_9 = 0$. Therefore,

$$\begin{aligned} \bar{D}_9 &= \partial_x - i\widetilde{L}_9[p, \bullet] = \partial_x \otimes I + I \otimes \frac{1}{N\widetilde{L}_9^*}\partial_q \qquad \text{for String Phase} \\ &= \partial_x \qquad\qquad\qquad\qquad\qquad\qquad \text{for Clustered Phase} \end{aligned} \tag{4.61}$$

in the large N limit. The derivative acts on eigenfunctions functions of form given in (eqn.(4.8)) and can be replaced by ∂_σ as described in Section (4.1). Where σ is a effective periodic coordinate with period $2\pi N\widetilde{L}_9^*$ and x has a period $2\pi\widetilde{L}_9^*$. Thus $\bar{D}_9 = \partial_\sigma$ for String Phase and $\bar{D}_9 = \partial_x$ for Clustered Phase.

[1] At finite temperature zero mode of A_0 is a physical variable (as it can not be gauged away due to periodic nature of Euclidean time) and corresponds to chemical potential. This has been set to zero for our case. More generally one can consider free energy as a function of zero mode of A_0 (as considered in [21] in a different context). Analysis similar to [21], could be done in our case, and keep this for future. Also note that non-zero value of the chemical potential will remove the necessity of IR cut-off and as it introduces a scale and the parameter counting (as discussed in introduction) will match even without the IR cut-off.

The ghost terms (eqn.(4.60)) effectively cancels the two gauge fields, and remaining theory is effectively that of 8 SUSY scalar fields, except the fields are now $U(N)$ matrices in adjoint representation and the derivatives are little complicated than that for scalar fields. For clustered phase at 1-loop the partition function is equivalent to $8N^2$ SUSY scalar fields on $S^1 \times S^1$ with radii β and $\widetilde{L}_9^*$. And for string phase it is equivalent to $8N$ (factor of N comes as the fluctuations are considered to be independent of p and there is trace over p direction. The trace over q and integral over x gives the effective integral over σ) SUSY scalar fields on $S^1 \times S^1$ with radii β and $N\widetilde{L}_9^*$.

4.4.1 Phase 1: String Phase

The partition function for phase 1 (string) is given by,

$$Z_1 = e^{-\beta F_1} = \Big\{ \frac{2\widetilde{L}_0}{\sqrt{4\pi^2 \widetilde{g}_s \widetilde{l}_s N \widetilde{L}_9^*}} \frac{1}{\sqrt{\frac{x}{N}}} \frac{\eta^2(i\frac{2x}{N})}{\eta^4(i\frac{x}{N})} \Big\}^{8N} \tag{4.62}$$

$$\begin{aligned} \beta F_1 &= -8\ N\ ln(b) + 8\ N\ ln(\sqrt{x}) - 16\ N\ ln(\eta(i\frac{2x}{N})) \\ &+ 32\ N\ ln(\eta(i\frac{x}{N})) \end{aligned} \tag{4.63}$$

where $x = \frac{\beta}{2\pi \widetilde{L}_9^*}$, $b = \frac{2\widetilde{L}_0}{\sqrt{4\pi^2 \widetilde{g}_s \widetilde{l}_s \widetilde{L}_9^*}} = \frac{2\widetilde{L}_0}{\sqrt{4\pi^2 \widetilde{\beta}'}}$.

4.4.2 Phase2: Clustered Phase

The partition function for phase 2 (clustered) is given by,

$$Z_2 = e^{-\beta F_2} = \Big\{ \frac{2\widetilde{L}_0}{\sqrt{4\pi^2 \widetilde{g}_s \widetilde{l}_s \widetilde{L}_9^*}} \frac{1}{\sqrt{x}} \frac{\eta^2(2ix)}{\eta^4(ix)} \Big\}^{8N^2} \tag{4.64}$$

$$\begin{aligned} \beta F_2 &= -8\ N^2\ ln(b) + 8\ N^2\ ln(\sqrt{x}) - 16\ N^2\ ln(\eta(2ix)) \\ &+ 32\ N^2\ ln(\eta(ix)) \end{aligned} \tag{4.65}$$

4.4.2.1 Low temperature expansion

Consider $x >> 1$ and $x/N >> 1$,

$$\begin{aligned} \beta F_1 &\simeq -8Nln(b) + 8Nln(\sqrt{x}) && (4.66) \\ \beta F_2 &\simeq -8N^2 ln(b) + 8N^2 ln(\sqrt{x}) && (4.67) \end{aligned}$$

Using,

$$\eta(ix) \simeq e^{-\frac{\pi x}{12}} \text{ for } x >> 1 \tag{4.68}$$

So $\beta F_1 < \beta F_2$, i.e. the string phase will be favored at low temperature if $b/\sqrt{x} \leq 1$. We can expect a phase transition from the string phase to the clustered phase as the temperature is increased from zero, at $x = b^2$. For the transition temperature to lie in the validity region of the low temperature expansion : $b >> \sqrt{N}$. Let us call this temperature as T_H.

$$T_H = \frac{\pi R^{11}}{2\widetilde{L}_0^2} \tag{4.69}$$

In terms of the DLCQ parameters,

$$T_H = \frac{\pi R^-}{2L_0^2} \tag{4.70}$$

using the scaling properties discussed in section (2.3). If we now take the limit $L_0 \to \infty$, the transition temperature $T_H \to 0$. *So, it is essential to have a finite value of* L_0 *to get phase transition at finite temperature.* This is the temperature at which there is a deconfinement transition in the Yang-Mills' model, which should be same as *Hagedorn transition.*

4.4.2.2 High temperature expansion

Consider $x << 1$,

$$\begin{aligned} \beta F_1 &\simeq 8Nln(\frac{2N}{b}) - 8Nln(\sqrt{x}) - \frac{2N^2\pi}{x} && (4.71) \\ \beta F_2 &\simeq 8N^2 ln(\frac{2}{b}) - 8N^2 ln(\sqrt{x}) - \frac{2N^2\pi}{x} && (4.72) \end{aligned}$$

Using,

$$\eta(ix) \simeq e^{-(\frac{\pi}{12x}+ln\sqrt{x})} \text{ for } x << 1 \tag{4.73}$$

We see that, at $x > \frac{4}{b^2}$, the clustered phase is favored but at very high temperature we can again have a string phase. This "Gregory-Laflamme" kind of transition will occur at $x = \frac{4}{b^2}N^{-1/N} \simeq \frac{4}{b^2}$ (For large N, $N^{-1/N} \sim 1$). This is also consistent with $b >> \sqrt{N}$. Let us call this temperature as T_G.

$$T_G = \frac{\widetilde{L}_0^2}{8\pi^3 R^{11}\widetilde{L}_9^{*2}} \tag{4.74}$$

In terms of the DLCQ parameters,

$$T_G = \frac{L_0^2}{8\pi^3 R^- L_9^{*2}} \tag{4.75}$$

using the scaling properties discussed in section 2.3. In this case, note $L_0 \to \infty$ implies $T_G \to \infty$.

Let us express this in terms of the parameters of Yang-Mills theory : The infrared cutoff on $A = \frac{X}{\tilde{l}_s^2}$ is $\frac{\widetilde{L}_0}{\tilde{l}_s^2} = \frac{1}{\widetilde{L}_0^*}$. Thus we get (up to factors of 2π)

$$T_G = \frac{\widetilde{l}_s^4}{R_{11}\widetilde{L}_9^{*2}\widetilde{L}_0^{*2}} = \frac{\widetilde{l}_s^3}{\widetilde{g}_s\widetilde{L}_9^{*2}\widetilde{L}_0^{*2}} = \frac{1}{\widetilde{g}_{YM0+1}^2}\frac{1}{\widetilde{L}_9^{*2}\widetilde{L}_0^{*2}} = \frac{1}{\widetilde{g}_{YM1+1}^2}\frac{1}{\widetilde{L}_9^{*}\widetilde{L}_0^{*2}} \tag{4.76}$$

4.4.3 Numerical results for Phases

Fig.4.2 shows the division of parameter space (N, x, b) into regions where each of the phases are preferred, which is obtained by numerically comparing the free energies obtained in previous sections. The numerical plot re-affirms our analytic result that there exists two phase transition.

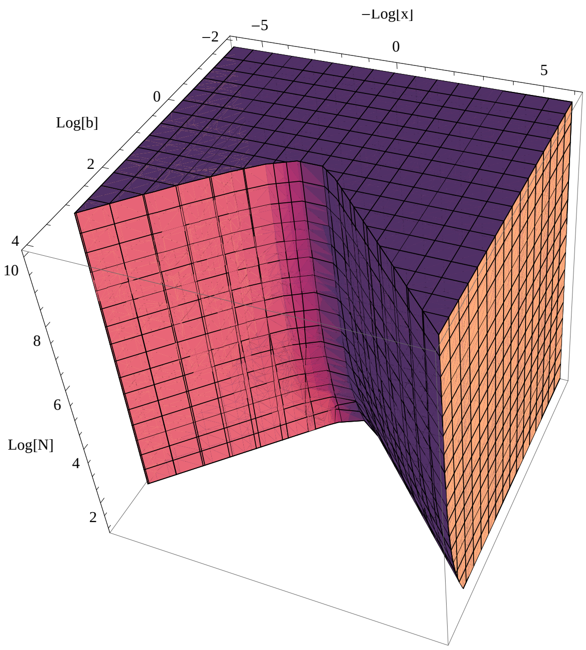

Figure 4.2: Solid region represents string phase and empty region is clustered phase.

5
IR cut-off and AdS/CFT

The BFSS matrix model with tilde parameters is supposed to be a non-perturbative description of M-theory (or IIA String Theory). But the idea of gravity dual [30] of Yang-Mills' theories allows us to relate the matrix model to super-gravity (with tilde parameters). This issue is discussed in the Chapter (3). One can use this to infer properties of the matrix model.

Two classical solutions of Type IIA super-gravity are : (1) the decoupling limit of black $D0$ branes and (2) the BPS $D0$ branes. In a recent study [41] phase transition of these two solutions were discussed. It was shown that the IR cut-off plays a crucial role in phase transition. As mentioned earlier this is motivated by the work of [33, 34, 40]. We will redo the analysis of this paper [41] here and try to explain the physical origin of IR cutoff used in [41] . The super-gravity solutions have tilde parameters, but for convenience we will drop the tilde signs and put them back at the end.

5.1 Hawking-Page transition in D-branes

Ideally, we should construct the super-gravity solution corresponding to the wrapped membrane. We reserve this for the future. Here we are only interested in understanding the nature of the phase transition and the role of the IR cutoff, so we will just use the solution for N coincident D0-branes used also in [41]. In the decoupling limit, (with $U = \frac{r}{l_s^2}$ = fixed, where r is radial co-ordinate defined in the transverse space of the brane. U also sets the energy scale of the dual Yang-Mills theory.) the solution for N coincident black $D0$-branes in Einstein frame is given by (eqn.(3.7)),

$$\begin{aligned} ds^2_{Ein} &= \frac{\alpha'}{2\pi g_{YM}}\Bigg\{ -\frac{U^{\frac{49}{8}}}{(g^2_{YM}d_0N)^{\frac{7}{8}}}\left(1-\frac{U_H^7}{U^7}\right)dt^2 \\ &+ \frac{(g^2_{YM}d_0N)^{\frac{1}{8}}}{U^{\frac{7}{8}}}\left[\frac{dU^2}{1-\frac{U_H^7}{U^7}}+U^2d\Omega_8^2\right]\Bigg\}, \qquad (5.1) \\ e^{\phi} &= 4\pi g^2_{YM}\left(\frac{g^2_{YM}d_0N}{U^7}\right)^{\frac{3}{4}}, \qquad (5.2) \\ F_{U0} &= -\alpha'^{\frac{1}{2}}\frac{7U^6}{4\pi^2 d_0 N g^4_{YM}}. \qquad (5.3) \end{aligned}$$

where $d_0 = 2^7\pi^{9/2}\Gamma(7/2)$ is a constant. Simply setting $U_H = 0$ gives the solution for N coincident BPS D0-branes.

The Euclidean action can be obtained by setting $t = -i\tau$. The Euclidean time τ has a period

$$\beta = \frac{4\pi g_{YM}\sqrt{d_0N}}{7U_H^{\frac{5}{2}}} \qquad (5.4)$$

in order to remove the conical singularity. This is the inverse Hawking temperature of the black $D0$ brane in the decoupling limit.

Now the on-shell Euclidean action for the two solutions can be calculated and gives,

$$\begin{aligned} I_{black} &= \frac{7^3}{16}\frac{V(\Omega_8)\beta}{16\pi G'_{10}}\int_{U_H \ or \ U_{IR}}^{U_{uv}} U^6 dU \qquad (5.5) \\ I_{bps} &= \frac{7^3}{16}\frac{V(\Omega_8)\beta'}{16\pi G'_{10}}\int_{U_{IR}}^{U_{uv}} U^6 dU \qquad (5.6) \end{aligned}$$

where $G'_{10} = \alpha'^{-7}G_{10} = 2^7\pi^{10}g^4_{YM}$ is finite in the decoupling limit. U_{uv} is introduced to regularize the action and is taken to ∞ in the end. The temperature of BPS branes β' is arbitrary and can be fixed by demanding the temperature of both the solutions to be same at the UV boundary U_{uv}, which gives $\beta' = \beta\sqrt{1-\frac{U_H^7}{U_{uv}^7}}$. U_{IR} is a IR cut-off which removes the region $U < U_{IR}$ of the geometry. The integration in the action starts from U_{IR} for BPS solution and, U_{IR} or U_H for the black brane solution depending on $U_H < U_{IR}$ or $U_H > U_{IR}$ respectively. If we put $U_{IR} = 0$ i.e. in absence of the IR cut-off, comparison of the actions (eqns.(5.5),(5.6)) shows that there is no phase transition, and the black brane phase is always favored. Let us consider the case $U_H > U_{IR}$,

$$\Delta I_{bulk} = \lim_{U_{uv}\to\infty}(I_{black} - I_{bps}) = \frac{7^2}{16}\frac{V(\Omega_8)\beta}{16\pi G'_{10}}(-\frac{1}{2}U_H^7 + U_{IR}^7) \tag{5.7}$$

Which shows a change in sign as we increase the temperature i.e. U_H (eqn.(5.4)). The system will undergo a phase transition ("Hawking-Page Phase transition") from BPS brane to Black brane solution at $U_H^7 = 2U_{IR}^7$. Actually, we should also consider Gibbons-Hawking surface term for careful analysis (as done in [41]) which corrects the transition temperature by some numerical factor given by,

$$\beta_{crit} = \frac{4\pi\widetilde{g}_{YM}\sqrt{d_0 N}}{7(\frac{49}{20})^{5/14}\ \widetilde{U}_{IR}^{5/2}} \tag{5.8}$$

We see a IR cutoff is essential to realize a phase transition, (as $\widetilde{U}_{IR} \to 0$, $\beta_{crit} \to \infty$) so to get confinement-deconfinement phase transition in dual super Yang-Mills theory we have to introduce a IR cutoff.

As mentioned in the introduction, one possible mechanism for the origin of the cutoff for $D0$ branes can be understood from the analysis of [42]. It was shown that the higher derivative corrections to super-gravity introduce a finite horizon area for extremal $D0$ brane solution which is otherwise zero. The multigraviton states (with total N units of momentum in the 11th direction) and the single graviton state seem to both be microstates of the same black hole when interaction effects higher order in g_s are included. Radius of the horizon developed due to higher derivative corrections is $R \sim \widetilde{l}_s\widetilde{g}_s^{1/3}$. So we can get an estimate of IR cutoff by identifying R with the IR cutoff in our case, $\widetilde{U}_{IR} = \frac{R}{\widetilde{l}_s^2} \sim \frac{\widetilde{g}_s^{1/3}}{\widetilde{l}_s} \sim \widetilde{g}_{YM}^{2/3}$, which is finite in the scaling limit. If we plug in this value of $\widetilde{U}_{IR}$ in eqn.(5.8), we get

$$\beta_{crit} = \frac{4\pi\sqrt{d_0 N}}{7(\frac{49}{20})^{5/14}\ \widetilde{g}_{YM}^{2/3}} \sim \frac{1}{\widetilde{U}_{IR}} \tag{5.9}$$

In case of $D1$ brane system, which is just T dual to the system studied above also shows that a IR cutoff is required for phase transition [41]. We were unable to find a analysis like [42] corresponding to wrapped membrane system, which we need to get an estimate of the IR cutoff.

5.2 Gregory-Laflamme Transition

In our calculation we find a temperature T_G, where the $D0$-branes spread out uniformly along the compact space. This configuration is just the one that is favored at very low temperatures. It is not clear whether this perturbative result is reliable. However, a similar phase transition exists in the dual super-gravity theory, known as "black hole-black string" transition or Gregory-Laflamme transition [31, 32, 54, 55, 56, 119, 120]. It is shown in [31], that the near horizon geometry of a charged black string in $R^{8,1} \times S^1$ (winding around the S^1) develops a Gregory-Laflamme instability at a temperature $T_{GL} \sim \frac{1}{L^2 g_{YM}\sqrt{N}}$, where L is the radius of S^1 and g_{YM} is $1+1$ dimensional Yang-Mills' coupling. Below this temperature the system collapses to a black-hole. In the weak coupling limit, the dual $1+1$ SYM theory also shows a corresponding phase transition by clustering of eigenvalues of the gauge field in the space-like compact direction below the temperature, $T'_{GL} \sim \frac{1}{L^3 g^2_{YM} N}$, as shown numerically in [31]. This should be compared to the perturbative result (eqn.(4.76)) $T_G \sim \frac{1}{g^2_{YM}(L^*_0)^2 L^*_9}$. So the presence of high temperature string phase in our model must correspond to some kind of "Gregory-Laflamme" transition in dual super-gravity. This is (at least superficially) independent of the issue of any classical instability. This is because both solutions may be locally stable, but at finite temperatures it is possible to have a first order phase transition to the global minimum.

In our perturbative result, the high temperature phase is a string rather than a black string i.e. it is the same as the low temperature phase. The question thus arises whether Gregory-Laflamme transitions can happen for extremal objects. We can give a heuristic entropy argument to show Gregory-Laflamme kind of transition is also possible for extremal system. In the original argument [57], it was shown that for extremal branes there is no instability. However, these systems had zero horizon area. Instead, we will here consider a 5 dimensional extremal RN black hole with a large compact direction, and the same solution with the mass smeared uniformly along the compact direction ("RN black ring"). The metric, ADM mass (M_5) and entropy (S_5) for a $5d$ extremal RN black hole solution is given by (where the compact direction is approximated by a non-compact one),

$$\begin{aligned} ds_5^2 &= -(1-\frac{r_e^2}{r^2})^2 dt^2 + (1-\frac{r_e^2}{r^2})^{-2} dr^2 + r^2 d\Omega_3^2 && (5.10) \\ M_5 &= \frac{3\pi}{4G_5} r_e^2 && (5.11) \\ S_5 &= \frac{2\pi^2 r_e^3}{4G_5} = \frac{\pi^2}{2}(\frac{4}{3\pi})^{3/2} G_5^{1/2} M_5^{3/2} && (5.12) \end{aligned}$$

where G_5 is $5d$ Newton's constant. Similarly, we can write the metric for $5d$ extremal RN Black ring, which when dimensionally reduced gives a $4d$ extremal RN black hole. The metric, ADM mass ($M_{(4\times1)}$) and entropy ($S_{(4\times1)}$) is given by,

$$\begin{aligned} ds_{(4\times1)}^2 &= -(1-\frac{R_e}{r})^2 dt^2 + (1-\frac{R_e}{r})^{-2} dr^2 + dx^2 + r^2 d\Omega_2^2 && (5.13) \\ M_{(4\times1)} &= \frac{R_e}{G_4} && (5.14) \\ S_{(4\times1)} &= \frac{4\pi R_e^2 \times 2\pi L}{4G_5} = \frac{1}{2} G_5 \frac{M^2}{L} && (5.15) \end{aligned}$$

where $G_4 = \frac{G_5}{2\pi L}$ is $4d$ Newton's constant and L is the radius of the compact direction x. If we consider $M_5 = M_{(4\times1)} = M$ and compare the entropy,

$$\frac{S_5}{S_{(4\times1)}} = \frac{16}{9}\frac{L}{r_e} \qquad (5.16)$$

So, when radius of the compact direction is greater than the radius of the $5d$ black hole horizon, the black hole solution is entropically favored. As we increase the horizon radius, there may be phase transition when horizon size becomes of the order of the radius of the compact dimension, above which the "string" solution is entropically favored. Our analysis is a simple entropy comparison. As mentioned above, this is independent of the classical stability issue, that was studied in detail in [57] . Therefore, extremal solutions with a *finite horizon size* may also show a Gregory-Laflamme kind of transition. This needs further study.

6
Conclusions I

In part 1 of the thesis, the finite temperature phase structure of string theory has been studied using the BFSS matrix model which is a $0+1$ Super-symmetric Yang-Mills (SYM) theory. This is first studied in perturbation theory. This was actually a refinement of an earlier calculation [20] where some approximations were made. The result of this study is that there is a finite and non zero phase transition temperature T_H below which the preferred configuration is where the $D0$ branes are arranged in the form of a wrapped membrane and above which the $D0$ branes form a localized cluster. It is reasonable to identify this temperature with the "Hagedorn" temperature, which was originally defined for the free string. We have found that $T_H \sim \frac{1}{L_0}$, where L_0 is the IR cutoff of the Yang-Mills theory which needs to be introduced to make the calculations well defined.

The $0+1$ SYM has a dual super-gravity description. Here also it is seen that in the presence of an IR cutoff there is a critical temperature above which the BPS D0 brane is replaced by a black hole.

Simple parameter counting shows that the BFSS matrix model needs one more dimensionful parameter if it is to be compared with string theory, so the IR cut-off L_0 can be thought of as one choice for this extra parameter. It makes the comparison well defined by effectively removing the strongly coupled region of the configuration space in SYM as well as in super-gravity. A physical justification for this (beyond parameter counting) comes from the work of [42]. It is shown there that the entropy matching requires even the extremal BPS configuration of D0-branes to develop a horizon, due to higher derivative string loop corrections to super-gravity. This is an issue that deserves further study.

Finally, the perturbative result shows a second phase transition at a higher temperature, back to a string like phase. This could be an artifact of perturbation theory. On the other hand, it is very similar to the Gregory-Laflamme instability and there is also some similarity in the expressions obtained in [31] for the critical temperature. This also requires further study.

Part III

Background material: Part II

7

AdS/CFT and Real Time Green's Function

The main prescription of AdS/CFT correspondence [2] allows us to calculate quantum correlators in the boundary CFT by classical computations in the supergravity dual. It was originally formulated for Euclidean signature [3, 121]. In principle Minkowski formulation may be avoided by working in Euclidean space and then analytically continuing to find real-time propagators. But one needs to know Euclidean correlators for all Matsubara frequencies. In general it is difficult, and a direct computation in Minkowski space is preferable. Subtleties of Lorentzian signature AdS/CFT was discussed in [65, 66, 122]. A working recipe of Minkowski space correlators was given by Son-Starinets in [67] and later treated rigorously in [123]. An alternative formulation of the prescription was given by Iqbal-Liu in[124, 70]. We will review here the prescription of [67, 124, 70] and summarize their results.

7.1 Definitions of Correlators

We will consider here general properties of Minkowski Correlators at finite temperature. A retarded propagator for an arbitrary hermitian bosonic operator $\hat{\mathcal{O}}$ in thermal equilibrium (*i.e.*QFT at finite temperature) is defined as,

$$G^R(k) = -i \int d^4x e^{-ik.x} \theta(t) < [\mathcal{O}(x), \mathcal{O}(0)] >_\rho \tag{7.1}$$

where we have considered the $(-+++)$ metric convention. $\theta(t)$ is the Heaviside step function. The $<\mathcal{A}>_\rho = Tr(\rho\mathcal{A})$ denotes average over thermal ensemble, where ρ is the thermal density function. The advanced propagator is defined similarly,

$$G^A(k) = i\int d^4x e^{-ik.x}\theta(-t) < [\mathcal{O}(x), \mathcal{O}(0)] >_\rho \tag{7.2}$$

from these definitions we can show

$$G^R(k)^* = G^R(-k) = G^A(k) \tag{7.3}$$

If the system is parity invariant, then $Re(G^{A,R})$ are even functions of $\omega = k^0$ and $Im(G^{A,R})$ are odd functions of ω. For the Euclidean formulation, one considers Matsubara correlators, given by,

$$G^E(k) = \int d^4x_E e^{-ik_E.x_E} < T_E\mathcal{O}(x_E), \mathcal{O}(0) >_{S_E} \tag{7.4}$$

where T_E denotes Euclidean time ordering, and $<\cdots>_{S_E}$ denotes the vacuum expectation value w.r.t. Euclidean action S_E, where the time direction is compact with radius $\beta = \frac{1}{T}$. The Matsubara propagator is defined only at discrete values of the frequency ω_E. For bosonic $\mathcal{O}$ these Matsubara frequencies are multiples of $2\pi T$.

The Euclidean and Minkowski propagators are related to each other by analytic continuation. The retarded propagator $G^R(k)$ can be analytically continued to whole upper half of the complex plane, and the relation is given by,

$$G^R(2\pi iTn, \mathbf{k}) = -G^E(2\pi Tn, \mathbf{k}) \tag{7.5}$$

The advanced propagator can be analytically continued to lower half plane and the relation to G^E is given as,

$$G^A(-2\pi iTn, \mathbf{k}) = -G^E(-2\pi Tn, \mathbf{k}) \tag{7.6}$$

In particular for $n = 0$,

$$G^R(0, \mathbf{k}) = G^A(0, \mathbf{k}) = -G^E(0, \mathbf{k}) \tag{7.7}$$

7.2 AdS/CFT prescription for Minkowski Correlators

Let us first recall the formulation of AdS/CFT Correspondence in Euclidean space [3]. We will here consider the famous example of the duality: Type IIB superstring theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ $SU(N)$ SYM theory on the 3+1-dimensional boundary of AdS_5. Let g_{YM} be the Yang-Mills coupling. Then in the limit of large N and large 't Hooft coupling $\lambda = g_{YM}^2 N$, the correspondence reduces to a duality between strongly coupled $\mathcal{N} = 4$ SYM theory and classical supergravity on $AdS_5 \times S^5$. The Euclidean version of the metric for $AdS_5 \times S^5$ is given by,

$$ds^2 = \frac{R^2}{z^2}(d\tau^2 + dx_i^2 + dz^2) + R^2 d\Omega_5^2 \tag{7.8}$$

which is a solution to Type II B supergravity with 5-form flux turned on. R is the AdS radius and is related to the dual parameters by $R^4 = 4\pi g_s \alpha'^2 N$. This metric can be obtained as a field theory limit or near horizon limit of N $D3$ branes solution described in the Section (3.0.1) for $p = 3$. In AdS/CFT correspondence, the $3+1$-dimensional QFT lives on the boundary of AdS_5 space given by $z = 0$. Suppose a bulk field ϕ is coupled to an operator $\mathcal{O}$ on the boundary in such a way that the interaction Lagrangian is $\phi\mathcal{O}$. Then the AdS/CFT correspondence can be formally write as,

$$< e^{\int_{\partial M} \phi_0 \mathcal{O}} >_{CFT} = e^{-S_{cl}[\phi]} \tag{7.9}$$

where the left-hand side is the generating functional for correlators of $\mathcal{O}$ in the boundary field theory and right-hand side is the action of classical solution to the equation of motion for ϕ in the bulk metric (eqn.(7.8)) with the boundary condition $\phi|_{z=0} = \phi_0$. The metric (eqn.(7.8)) corresponds to zero temperature field theory at the boundary. To compute Matsubara correlator at finite temperature it has to be replaced with a non-extremal one,

$$ds^2 = \frac{R^2}{z^2}(f(z)d\tau^2 + dx_i^2 + \frac{dz^2}{f(z)}) + R^2 d\Omega_5^2 \tag{7.10}$$

where $f(z) = 1 - \frac{z^4}{z_H^4}$ and $z_H = (\pi T)^{-1}$, and T is the Hawking temperature. The Euclidean time τ is periodic $\tau \sim \tau + T^{-1}$ and z now runs from 0 to z_H.

The temperature T then corresponds to the Euclidean boundary field theory with compact time direction with same period. The correlators can be found as,

$$< \mathcal{O}(x_1)\mathcal{O}(x_2) >= \frac{\delta S_{cl}\,[\phi]}{\delta\phi_0(x_1)\delta\phi_0(x_2)}\bigg|_{\phi_0=0} \tag{7.11}$$

One can formally write Minkowski version of AdS/CFT as,

$$< e^{i\int_{\partial M}\phi_0\mathcal{O}} >_{CFT} = e^{iS_{cl}[\phi]} \tag{7.12}$$

One of the problem with this formulation is the boundary condition of ϕ at horizon. For Minkowski space, the regularity of ϕ can not be demanded at $z = z_H$, as the solution wildly oscillates near the horizon and has two modes: ingoing and outgoing. For the in-coming wave boundary condition, where waves can only travel to the region inside the horizon, one may suspect to get Retarded Correlators for a finite temperature field theory and for outgoing wave Advanced correlator. But this simple correspondence fails to work.

Let us consider the Minkowski version of the AdS part of the metric (eqn.(7.10)),

$$ds^2 = \frac{R^2}{z^2}(-f(z)d\tau^2 + dx_i^2 + \frac{dz^2}{f(z)}) = g_{zz}dz^2 + g_{\mu\nu}(z)dx^\mu dx^\nu \tag{7.13}$$

with the following action for a bulk scalar field ϕ with mass m,

$$S = -\frac{1}{2}\int dx^4 \int_{z_B}^{z_H} \sqrt{-g}(g^{zz}(\partial_z\phi)^2 + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2) \tag{7.14}$$

where for our case $z_B = 0$. The linearized field equation for ϕ is given by,

$$\frac{1}{\sqrt{-g}}\partial_z(\sqrt{-g}g^{zz}\partial_z\phi) + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi = 0 \tag{7.15}$$

The equation of motion has to be solved with a fixed value of ϕ at z_B, and can be write in the form,

$$\phi(z, x) = \int \frac{d^4k}{(2\phi)^4}e^{ik_\mu x^\mu} f_k(z)\phi_0(k) \tag{7.16}$$

where $f_k(z_B) = 1$ and $\phi_0(k)$ is determined by the Fourier transform of source field $\phi(z_B, x) = \phi_0(x)$ at the boundary. the effective equation of motion for $f_k(z)$ is given by,

$$\frac{1}{\sqrt{-g}}\partial_z(\sqrt{-g}g^{zz}\partial_z f_k(z)) - (g^{\mu\nu}k_\mu k_\nu + m^2)\phi = 0 \tag{7.17}$$

with the boundary condition $f_k(z_B) = 1$ and satisfying incoming-wave boundary condition at $z = z_H$. The on-shell action reduces to

$$S = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\phi_0(-k)\mathcal{F}(k,z)\phi_0(k)\Big|_{z=z_B}^{z=z_H} \tag{7.18}$$

where

$$\mathcal{F}(k,z) = \sqrt{-g}g^{zz}f_{-k}(z)\partial_z f_k(z) \tag{7.19}$$

If we assume the formal definition of Eqn.(7.12), the retarded Green's function is given by taking two derivatives of the classical on-shell (eqn.(7.18)) and is given by,

$$G^R(k) = \frac{1}{2}\mathcal{F}(k,z)\Big|_{z_B}^{z_H} + \frac{1}{2}\mathcal{F}(-k,z)\Big|_{z_B}^{z_H} \tag{7.20}$$

It can be shown using $f_k^*(z) = f_{-k}(z)$ (which also solves Eqn.(7.17) with in-going boundary condition), that the imaginary part of $\mathcal{F}$ is conserved flux and is independent of z. This means the imaginary part of the $\mathcal{F}$ cancels for each term in the expression of G^R, and the retarded Green's function is real. One suggestion of avoiding the problem may be just dropping the contribution from horizon, but still the imaginary part cancels out as $\mathcal{F}(-k,z) = \mathcal{F}^*(k,z)$. So by this prescription one can not get a retarded Green's function which is in general complex.

7.2.1 Son-Starinets Prescription

Son and Starinets gave an ad hoc resolution to this problem of real time correlators from AdS/CFT in [67]. They gave a prescription and various checks on the validity of the formula. The prescription is given as follows,

- Solve the mode equation (7.17) with the boundary condition $f_k(z_B) = 1$ and the asymptotic solution is ingoing (outgoing) wave at the horizon for time-like momentum. For space-like momentum, the horizon boundary condition is similar to Euclidean case, and is taken to be the regular solution.

- The retarded (advanced) propagator is given by,

$$G^R(k) = \mathcal{F}(k,z)|_{z_B} = \sqrt{-g}g^{zz}\partial_z f_k(z)\Big|_{z=z_B} \tag{7.21}$$

  where $\mathcal{F}$ is given by (7.19). notice the surface terms coming from IR or the horizon is dropped. This part of the metric influences the correlator through the boundary condition imposed on bulk field ϕ

Since imaginary part of $\mathcal{F}$ is independent of the radial coordinate z, $Im(G^{R,A})$ can be computed by evaluating $Im(\mathcal{F}(k,z))$ at any convenient value of z. This ad hoc prescription was later confirmed rigorously in [123].

7.2.2 Iqbal-Liu Prescription

An alternative equivalent prescription was given in [124, 70]. According to the prescription the retarded Green's function for a boundary operator $\mathcal{O}$ corresponding to a bulk field ϕ is given by,

$$G^R(k_\mu) = \left(\lim_{z\to z_B} \frac{\Pi(z;k_\mu)|_{\phi_R}}{\phi_R(z;k_\mu)}\right)\Bigg|_{\phi_0=0} \tag{7.22}$$

where Π is the canonical momentum conjugate to ϕ with respect to radial (z) foliation. $\phi_R(z,k_\mu)$ is the solution to the equations of motion with in-falling boundary condition at horizon and $\lim_{z\to z_B}\phi_R(z,k_\mu) \to \phi_0(k_\mu)$. The notation $\Pi(z;k_\mu)|_{\phi_R}$ indicates that Φ has to be evaluated on the classical solution ϕ_R. It was shown in [124, 70], that this definition is equivalent to the Son-Starinets prescription at level of linear approximation. Recall, the standard result from linear response theory, that if one considers a system in equilibrium at $t\to-\infty$, and then perturbs its action with the term $\int d^dx\phi_0(x)\mathcal{O}(x)$, the one point function in presence of the source is given at linearized level by,

$$< \mathcal{O}(k_\mu) >_{\phi_0} = G^R(k_\mu)\phi_0(k_\mu) \tag{7.23}$$

Then the equivalent statement of the prescription (7.22) is,

$$< \mathcal{O}(k_\mu) >_{\phi_0} = \lim_{z \to z_B} \Pi(z; k_\mu) \Big|_{\phi_R} \tag{7.24}$$

Above is true in general for Euclidean version of AdS/CFT given by (7.9), where,

$$< \mathcal{O}(x) >_{\phi_0} = -\frac{\delta S_{cl}}{\delta \phi_0(x)} = -\lim_{z \to z_B} \Pi_E(z, x) \Big|_{\phi_E} \tag{7.25}$$

where Π_E is the canonical momentum conjugate to ϕ w.r.t radial (z) foliation in Euclidean signature evaluated at the classical solution ϕ_E. The last equality comes from the well known fact in classical mechanics that derivative of an on-shell action with respect to the boundary value of a field is equal to the canonical momentum conjugate to the field, evaluated at the boundary. Eqn.(7.23) is Lorentzian version of the statement (7.25), but in absence of a proper statement equivalent to (7.9), status of the statement (7.22) is at the level of conjecture, and it was applied for known results in [124, 70], and so far has passed all the tests. Also in [70], it was shown by analytic continuation that (7.22) gives the correct Euclidean action. We will show the equivalence of the statement (7.22) with that of Son-Starinets (7.21) for bulk scalar field and move on to applications.

Consider the action for bulk scalar field given by (7.14), the canonical momentum conjugate to ϕ is given by,

$$\Pi(z; x) = \sqrt{-g(z)} g^{zz}(z) \partial_z \phi(z; x) \tag{7.26}$$

and its Fourier transform given by (7.16),

$$\Pi(z; k) = \sqrt{-g(z)} g^{zz}(z) (\partial_z f_k(z)) \phi_0(k) \tag{7.27}$$

where $\phi_0(k)$ is the Fourier transform of $\phi_0(z; x)$ which is the boundary value of a classical solution when f_k satisfies (7.17) with ingoing boundary condition and proper normalization $f_k(z_B) = 1$. Then using the prescription (7.22), the retarded Green's function is same as that obtained by Son-Starinets prescription (eqn.(7.20)).

Although we mainly talked about AdS_5/CFT_4 correspondence, the conjecture of AdS/CFT has expanded its horizon to general duality between systems in background of classical gravity solutions and QFT living on the boundary of this gravity solution. In particular, this duality is assumed to be true for asymptotically AdS space times. Consider bulk fields in a background metric,

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + g_{ii}(dx^i)^2 \tag{7.28}$$

The boundary is taken to be at $r = \infty$ (to match with previous discussion $r = \frac{1}{z}$), where various components of the metric have asymptotic AdS behavior,

$$g_{tt}, g^{rr}, g_{ii} \simeq r^2, \qquad r \to \infty \tag{7.29}$$

Also assume the theory is translationally invariant in x^μ directions, *i.e.*the metric components are function of r only and work in momentum space along these directions,

$$\phi(r, x^\mu) = \phi(r, k_\mu)e^{-i\omega t + ik.x}, \qquad k_\mu = (-\omega, k) \tag{7.30}$$

where ϕ represents a general bulk field.

7.2.2.1 Example: Massive Scalar field

Let us consider a massive scalar field action (7.14) in a background metric (7.28) in $(d+1)$-dimensional space-time. Then ϕ_R has asymptotic behavior,

$$\phi_R(r, k_\mu) \sim A(k_\mu)r^{\Delta - d} + B(k_\mu)r^{-\Delta}, \quad r \to \infty \tag{7.31}$$

where $\Delta = \frac{d}{2} + \sqrt{m^2 + \frac{d^2}{4}}$ is to be interpreted as the scaling dimension of the boundary operator. For massless case $(m = 0)$, $r \to \infty$, $\phi = A(k_\mu)$, which corresponds to the boundary value or the dual field theory source. But in general it differs by a power of r, and the $r \to \infty$ has to be taken carefully to compute G^R. This implies,

$$< \mathcal{O}(k_\mu) >_A = \lim_{r\to\infty} r^{\Delta - d}\Pi(z; k_\mu)\Big|_{\phi_R} \tag{7.32}$$

where as before $\Pi = -\sqrt{-g}g^{rr}\partial_r\phi$ is the momentum conjugate to ϕ. So asymptotic expansion of Π becomes,

$$\Pi(r,k_\mu) \simeq -(\Delta-d)A(k_\mu)r^{\Delta} + \Delta B(k_\mu)r^{d-\Delta} \tag{7.33}$$

Then by prescription (7.22),

$$G_R(k_\mu) = \lim_{r\to\infty} r^{2(\Delta-d)} \frac{\Pi(r,k_\mu)\Big|_{\phi_R}}{\phi_R(r,k_\mu)} = (2\Delta-d)\frac{B(k_\mu)}{A(k_\mu)} \tag{7.34}$$

Note there is change in (7.22), a power of r multiplied to extract the finite piece in $r\to\infty$ limit. The general folklore is to take the limit in such a way that we extract finite piece and which is in general also non-analytic in k_μ. As finite terms which are analytic in momentum corresponds to local terms in correlators (delta-function terms in the position space) and is of not much interest. Also note in the asymptotic solution of ϕ_R, has two independent components given by coefficients A and B, which corresponds to normalizable and non-normalizable modes in Lorentzian signature discussed in [66]. According to the identification in [66], coefficient of the solution A corresponds to the source in dual theory and B, the normalizable mode corresponds to operator expectation value. So the prescription (7.22) correctly reproduces the Green's function defined in [66].

7.2.2.2 Example: Vector Field

Consider a bulk vector field A_M with Maxwell action and gauge coupling g_{eff}. This is dual to a conserved current $\mathcal{J}^\mu$, and the prescription (7.23) becomes,

$$< \mathcal{J}^\mu > = -\lim_{r\to\infty} \frac{1}{g_{eff}^2}\sqrt{-g}F^{r\mu} \tag{7.35}$$

where $F_{\mu\nu}$ is the field strength corresponding to A_μ. The right hand side of the equation is momentum conjugate to A_μ.

7.2.2.3 Example: Fermion field

As seen in previous subsections the Iqbal-Liu prescription is directly applicable in case of scalars and vector operators, applicability to fermionic case is little subtle as discussed in [70], we will briefly summarize their results here. A very nice review and detailed analysis for fermions can also be found in [125]. Fermionic real time propagators was constructed in [70], by first applying the prescription (7.22) in Euclidean space, and then analytically continuing to real time. Consider the Minkowski action for fermions in the asymptotically AdS metric (eqn.(7.28)),

$$S = i\int d^{d+1}x\sqrt{-g}\; i(\bar{\Psi}\Gamma^M \mathcal{D}_M\Psi - m\bar{\Psi}\Psi) + S_{\partial M} \tag{7.36}$$

where

$$\bar{\Psi} = \Psi^\dagger\Gamma^{\underline{t}}, \quad \mathcal{D}_M = \partial_M + \frac{1}{4}\omega_{abM}\Gamma^{ab} \tag{7.37}$$

and ω_{abM} is the spin connection. We will denote M and a, b to denote bulk space-time and tangent space indices respectively, and $\mu, \nu \cdots$ to denote indices along the boundary directions, i.e. $M = (r, \mu)$. The Γ^a obey $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$, where $\eta = Diagonal(-1, 1, \cdots, 1)$ and $\Gamma^M = e^M_a\Gamma^a$, where e^M_a is the vielbein. Also (t, r, x_μ) denote space time indices and $(\underline{t}, \underline{r}, \underline{x}_\mu)$ the tangent space indices.

$S_{\partial M}$ is required to make the action stationary under variations of the spinor field. It was shown in [63], that $S_{\partial M}$ can be fixed uniquely (demanding stationary action, locality, absence of derivatives and invariance under asymptotic AdS symmetry) to,

$$S_{\partial M} = -i\int_{\partial M} d^dx\sqrt{-gg^{rr}}\,\bar{\Psi}_+\Psi_- \tag{7.38}$$

where $\Psi_\pm = \frac{1}{2}(1 \pm \Gamma^{\underline{r}})\Psi$. The addition of the boundary term makes the on-shell action independent of Ψ_-. So with fixing $\Psi_\pm$ *in-going*, we have the freedom to fix only half of the components at the boundary for the fermionic fields. As discussed in [70, 125], this related to the fact that equation of motion of fermions are first order.

We will be working in momentum space with the Fourier transform of Ψ denoted by $\Psi(r, k_\mu)$. Consider the momentum conjugate to Ψ_+ with respect to r-foliation,

$$\Pi_+(r, k_\mu) = i\sqrt{-gg^{rr}}(r)\bar{\Psi}_-(r, k_\mu) \tag{7.39}$$

The scaling dimension of an boundary fermionic operator (Δ) is given in terms of the mass of the bulk fermion field (in asymptotically AdS_{d+1} space-time),

$$\Delta = \frac{d}{2} + m \tag{7.40}$$

Then the boundary value for Ψ_- or the source has a scaling dimension $\Delta - d$ (as the boundary action $-i\int d^dx(\bar{\chi_0}\mathcal{O} + \bar{\mathcal{O}}\chi_0)$ should be scale invariant). Then,

$$\chi_0(k_\mu) = \lim_{r\to\infty} r^{d-\Delta}\Psi_+(r, k_\mu) \tag{7.41}$$

where $\chi_0(k_\mu)$ is the boundary source. It will be related to the boundary value of Ψ_- (which will correspond to operator expectation value) by equation of motion, as with the action (7.36) and *in-going boundary condition for both* $\Psi_\pm$, only Ψ_+ is independent. In general we expect a relation of the form,

$$\psi_0(k_\mu) = \mathcal{S}(k_\mu)\chi_0(k_\mu) \tag{7.42}$$

where $\mathcal{S}$ is a matrix and

$$\psi_0(k_\mu) = \lim_{r\to\infty} r^{\Delta}\Psi_-(r, k_\mu) \tag{7.43}$$

Now we should take care in defining Gamma matrices in the boundary theory which is one dimension less compared to the bulk. A detailed analysis of relation between Gamma matrices of various dimension can be found in Appendix B of [126]. We will here just state the result (capital Gamma denotes bulk matrices and small Gamma denotes boundary matrices),

- For d even, $\Gamma^\mu = \gamma^\mu$ and $\Gamma^{\underline{r}} = \gamma^{d+1}$, where γ^{d+1} is the chirality operator in d-dimension. Thus from boundary point of view $\Psi_\pm$ transform as d-dimensional Weyl spinor of opposite chirality. Then χ_0 and $\mathcal{O}$ are d-dimensional boundary spinor of definite chirality, *i.e.*a bulk Dirac spinor is mapped to a chiral spinor operator at boundary.

- For d odd, $\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix}$ and $\Gamma^{\underline{r}} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$ In this basis the two components $\Psi_\pm$ each transform as a d-dimensional Dirac spinor; thus χ_0 and $\mathcal{O}$ are both Dirac spinors.

Notice that in all dimensions the number of components of $\mathcal{O}$ is always half of that of Ψ.

For d odd, let us the spinor field as,

$$\begin{aligned} \Psi_+ &= \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix} \\ \Psi_- &= \begin{pmatrix} 0 \\ \psi_- \end{pmatrix} \end{aligned} \tag{7.44}$$

And the corresponding boundary values of $\psi_\pm$ be $\widetilde{\chi}_0$ and $\widetilde{\psi}_0$ respectively as defined in (7.41) and (7.43). Then the relation corresponding to (7.42), takes the form,

$$\widetilde{\psi}_0(k_\mu) = \widetilde{\mathcal{S}}(k_\mu)\widetilde{\chi}_0(k_\mu) \tag{7.45}$$

Also write the conjugate momentum as,

$$\pi_+(r, k_\mu) = i\sqrt{-gg^{rr}}(r)\bar{\psi}_-(r, k_\mu) \tag{7.46}$$

So the Iqbal-Liu prescription takes the form,

$$< \bar{\mathcal{O}} >_{\chi_0} = \lim_{r\to\infty} r^{\Delta-d}\pi_+ = i\bar{\widetilde{\psi}}_0(k) = i\widetilde{\chi}_0(k)^\dagger\widetilde{\mathcal{S}}(k)^\dagger\gamma^{\underline{t}} \tag{7.47}$$

which implies,

$$< \mathcal{O} >_{\widetilde{\chi}_0} = -i\widetilde{\mathcal{S}}(k)\widetilde{\chi}_0(k) \tag{7.48}$$

By definition of retarded correlator $G^R \sim < \mathcal{O}\mathcal{O}^\dagger >$ (from action $-i\int d^dx(\bar{\chi}_0\mathcal{O} + \bar{\mathcal{O}}\chi_0)$),

$$< \mathcal{O}(k) >_{\widetilde{\chi}_0} = -iG^R(k)\ \gamma^{\underline{t}}\ \widetilde{\chi}_0(k) \tag{7.49}$$

So comparing above two equations, we have,

$$G^R(k) = -i\widetilde{\mathcal{S}}(k)\gamma^{\underline{t}} \tag{7.50}$$

where $\widetilde{\mathcal{S}}$ is given by (7.45). There is an overall sign ambiguity, the sign is chosen demanding unitarity or demanding imaginary part of the Green's function is positive for all ω [70].
For d even, we do not need such an decomposition, and similarly, the Green's function is given as,

$$G^R(k) = -i\mathcal{S}(k)\gamma^{\underline{t}} \tag{7.51}$$

where $\mathcal{S}$ is given by (7.42).

7.2.2.3.1 Spinor in pure AdS_3: Consider the bulk geometry to be pure AdS_3 given by,

$$ds^2 = r^2(-dt^2 + dx^2) + \frac{dr^2}{r^2} \tag{7.52}$$

We will use the prescription described in previous subsection to calculate the boundary ($r \to \infty$) retarded Green's function. Since we have considered pure AdS_3, the boundary field theory will be a $1+1$-dimensional CFT at zero temperature. The equation of motion in this background geometry is given by,

$$\begin{aligned} \Psi_+ &= -i\frac{\gamma.k}{k^2}\hat{\mathcal{A}}(-m)\Psi_- \\ \Psi_- &= i\frac{\gamma.k}{k^2}\hat{\mathcal{A}}(m)\Psi_+ \end{aligned} \tag{7.53}$$

where $\gamma.k = -\gamma^0\omega + \gamma^1\kappa$, $k^2 = -\omega^2 + \kappa^2$, $k_\mu = (-\omega, \kappa)$ and

$$\hat{\mathcal{A}}(m) = r(r\partial_r + 1 - m) \tag{7.54}$$

and,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7.55}$$

The equation of motion for $\psi_\pm$, which in $2+1$-dimensions are just two complex functions is given by,

$$\hat{\mathcal{A}}(\pm m)\psi_\pm = i(\omega \mp k_1)\psi_\mp \tag{7.56}$$

and

$$\hat{\mathcal{A}}(-m)\hat{\mathcal{A}}(m)\psi_+ = k^2\psi_+ \tag{7.57}$$

Let us consider the case of $m = 0$. The scaling dimension of the boundary operator is $\Delta = 1$ (as $m = 0$ and $d = 2$) and $\hat{\mathcal{A}}(0) = r(r\partial_r + 1)$. We have,

$$r^2\partial_r^2\psi_+ + 4r\partial_r\psi_+ + (2 - \frac{k^2}{r^2})\psi_+ = 0 \tag{7.58}$$

which has a solution of the form,

$$\psi_+ = \frac{A(k)}{r}e^{i\frac{\sqrt{-k^2}}{r}} + \frac{B(k)}{r}e^{-i\frac{\sqrt{-k^2}}{r}} \tag{7.59}$$

we will assume time-like momentum $i.e. k^2 < 0$. If we now demand in-going boundary condition at $r = 0$ (Poincare horizon, as described in next section), we have,

$$\psi_+ = \frac{A(k)}{r}e^{i\ sgn(\omega)\frac{\sqrt{-k^2}}{r}} \tag{7.60}$$

Now putting the solution of ψ_+ back in (7.56), we get a relation between $\psi_\pm$,

$$\begin{aligned} \psi_- &= -\sqrt{\frac{\omega+\kappa}{\omega-\kappa}}\psi_+, \quad \omega > |\kappa| \\ \psi_- &= \sqrt{\frac{\omega+\kappa}{\kappa-\omega}}\psi_+, \quad \omega < -|\kappa| \end{aligned} \tag{7.61}$$

Let us consider the case $\omega > |\kappa|$. Also from definitions of previous section,

$$\psi_0 = \lim_{r\to\infty} r\Psi_-; \qquad \chi_0 = \lim_{r\to\infty} r\Psi_+ \tag{7.62}$$

So we get using (7.53) ,

$$\mathcal{S} = -\frac{\gamma.k}{\sqrt{-k^2}} \tag{7.63}$$

So the Green's function is given by (Eqn.(7.50), but with an overall negetive sign),

$$G^R(k) = i\mathcal{S}\gamma^{\underline{t}} = i\frac{\gamma.k}{\sqrt{-k^2}}\gamma^{\underline{t}} = \begin{pmatrix} i\sqrt{\frac{\omega-\kappa}{\omega+\kappa}} & 0 \\ 0 & i\sqrt{\frac{\omega+\kappa}{\omega-\kappa}} \end{pmatrix} \tag{7.64}$$

Notice $G^R_{11}(\omega,\kappa) = \frac{1}{G^R_{22}(\omega,\kappa)} = G^R_{22}(\omega,-\kappa)$. It is sufficient to study one of the components. In particular for $2+1$ dimensional case we can formally define the retarded Green's function as,

$$G^R(k_\mu) = i \lim_{r\to\infty} \frac{\psi_-}{\psi_+} \tag{7.65}$$

with overall sign to be fixed demanding $Im(G^R) > 0$ for all $\omega > 0$. Above definition is true for any asymptotically AdS_3 space-time, and we will use this definition to calculate Green's function in later chapters.

7.3 In-going boundary condition

Consider the metric,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + a(r)^2 d\overrightarrow{x}^2 \tag{7.66}$$

with $f(r), a(r) \to r^2$ as $r\to\infty$, Horizon at $f(r_0)=0$, Boundary at $r\to\infty$. The in-falling boundary condition is given by,

- Non-degenerate Horizon: Simple pole at $r = r_0$ i.e. $f(r) \sim \frac{4\pi}{\beta}(r-r_0)$. Solution near horizon: $e^{-i\omega t}(r-r_0)^{-i\frac{\omega\beta}{4\pi}}$.
- Degenerate horizon: Double pole at $r = r_0$ i.e. $f(r)\sim c(r-r_0)^2$. Solution near horizon: $e^{-i\omega t}e^{\frac{i\omega}{c(r-r_0)}}$
- Poincare Horizon: (near $z\to\infty$) Consider the metric (with $r=\frac{1}{z}$),

$$ds^2 = \frac{1}{z^2}[-dt^2 + d\overrightarrow{x}^2 + dz^2] \tag{7.67}$$

Solution near horizon: $e^{-i\omega t}e^{-kz}$ where

$$k = \begin{cases} -i\sqrt{\omega^2 - \overrightarrow{k}^2} & \omega > |\overrightarrow{k}| \\ i\sqrt{\omega^2 - \overrightarrow{k}^2} & \omega < |\overrightarrow{k}| \end{cases} \tag{7.68}$$

Assuming time-like momentum $|\omega| > |\overrightarrow{k}|$.

8

Lessons from Condensed Matter

Recently a lot of work has been initiated in studying strongly coupled Condensed matter systems by using AdS/CFT conjecture discovered in String theory. As described in the earlier chapters, the term AdS/CFT conjecture refer to a general class of symmetries observed in systems including gravity and those without gravity. In particular in these conjecture classical gravity theories in $d+1$-dimension space-time are dual to strongly coupled Quantum Field Theories living on boundary of this space-time *i.e.*in d-dimension. The extra dimension in gravity system or the bulk acts as a energy scale in the boundary theory, and renormalization group flows in the boundary theory can be studied by studying variation of the classical geometry in this extra dimension. As classical calculations are more tractable, this "AdS/CFT tool" becomes extremely useful in extracting quantum strongly coupled theories, where usual perturbation technique fails. In this thesis we will study one such system using this tool-kit. A review of application of AdS/CFT to Condensed matter systems can be found in [60, 127]. In this chapter we will introduce various Condensed matter terms which we will use in this thesis.

8.1 Fermi Gas

It is a non interacting translationally invariant system of non interacting electrons, with the single-particle eigenstates are plane waves with energy $\epsilon_k = \frac{k^2}{2m}$ (k is the momentum vector and m is the mass of the particles). The ground state of an N non-interacting particle system is the well-known *Fermi sea*: all states up to the *Fermi wave vector* k_F are filled, all the other states are empty. The energy of the last occupied state is the *Fermi energy*, $E_F = \frac{k_F^2}{2m} = \mu(T=0)$ (μ is the chemical potential). The elementary excitations are creation of *particles* ($|k| > k_F$) and destruction of particles at $|k| < k_F$ called *holes*. We can also construct *particle-hole excitations* which keep the total particle number fixed, *i.e.*one takes one particle from some state k, with $|k| < k_F$, and puts it into a state k' , with $|k'| > k_F$. These particle-hole excitations are parametrized by the two quantum numbers k, k' and thus form a continuum.

8.2 Fermi and Non-Fermi Liquids

Landau Fermi liquid is essentially a Fermi gas when interactions of the electrons with each other are included. Landau's theory of Fermi liquid is based on the idea of a continuous and one-to-one correspondence between eigen states of non-interacting and interacting system. Fermi Liquid Theory is expected to break down in many situations involving "strongly correlated electrons". A nice review on this subject can be found in [128, 129]. Let us start with a filled fermi sphere and imagine adding one more fermion in a momentum eigenstate. One can imagine doing this with a Fermi gas first, and then quickly increasing ("adiabatic") the strength of the interaction parameter. The particle becomes a *quasi-particle* with the same momentum. But it is not an energy eigenstate and can decay. Due to phase space limitations the life time increases as $\frac{1}{(k-k_F)^2}$ as $k \to k_F$. So near k_F (k_F remains same as the Fermi gas) these are legitimate excitations. It is not possible in frame work of Landau's to derive microscopic parameters, but a general form of one-particle Green's function can be predicated based on the existence of quasi-particle *i.e.*quasiparticle poles in the correlators. Consider one-particle Euclidean Green's function [1],

$$\mathcal{G}(\mathbf{k}, \omega) = \frac{1}{i\omega - \epsilon^{00}_{\mathbf{k}} - \Sigma(\mathbf{k}, \omega)} \tag{8.1}$$

where $\epsilon^{00}_{\mathbf{k}}$ is the bare particle energy. Excitation energies of the system are given by the poles of the correlator. Landau's assumption of existence of quasi-particles close to Fermi surface, amounts to demanding regular behavior of the self-energy correction $\Sigma(\mathbf{k}, \omega)$. A expansion of the Green's function up to second order in

[1]The retarded Green's function is given by $G^R(\mathbf{k}, \omega) = \mathcal{G}(\mathbf{k}, i\omega \to \omega + i\epsilon)$, where $\epsilon \to 0$.

$(|k| - k_F)$ gives,

$$\mathcal{G}(\mathbf{k}, \omega) = \frac{z_{\mathbf{k}}}{i\omega - \epsilon^0_{\mathbf{k}} + i\ sgn(\omega)\tau(\omega)^{-1}} \tag{8.2}$$

where $z_{\mathbf{k}}$, quasi-particle weight gives the jump in momentum distribution function at k_F compared to unity for non-interacting case. $\tau \sim \frac{1}{(k-k_F)^2}$ gives the quasi-particle lifetime and $\epsilon^0_{\mathbf{k}}$ is the effective energy of the quasi-particle including effective chemical potential (renormalized by $\Sigma(k_F, 0)$). As the energy of the quasiparticle on the Fermi surface is zero, and we are concerned with the excitations near the Fermi surface, dispersion relation is taken to be linear of form $\epsilon^0_{\mathbf{k}} = \omega = \frac{k_F}{m^*}(|k| - k_F)$. m^* is called the effective mass of this quasiparticles. The difference between the bare mass m and m^* is due to interaction effects which in principle can be calculated from a microscopic theory of the system in question. Given that the non-interacting particles obey Fermi-Dirac statistics, the quasiparticles also obey the same. In practice, this means that Landau's theory is useful for phenomena at energy scales much smaller than the Fermi energy, but inapplicable otherwise. Also ground state energy E_F receives contribution from states well below the Fermi energy, so excitation above the ground state are the fundamental objects of Landau's theory. When the Green's function does not have the properties of Fermi liquid it is called a *non-Fermi liquid*.

8.3 Luttinger Liquid

The best understood example of an Non-Fermi Liquid is $1 + 1$ dimensional interacting electrons, also known as *Luttinger Liquid*. Extensive review of this subject can be found in [130, 128, 129]. Luttinger Liquid is an exactly solvable model, in particular it can be solved by "Bosonization formalism". In one dimension, the Fermi surface is replaced by two Fermi points at k_F and $-k_F$. The assumption in Luttinger liquid model is that the spectrum is linear about Fermi points. So the particle-hole excitations with a given momentum has same kinetic energy, compared to higher dimensional electron gas. In Luttinger liquid, the elementary excitations are not quasi-particles but collective oscillations of the charge and spin density, propagating coherently, but in general with different velocities. The correlation function in contrary to Fermi liquid shows non-universal power laws with interaction-dependent exponents *i.e.*the quasiparticle pole is replaced by power-law edge singularity. Also even at zero temperature the momentum distribution function does not show a jump like Fermi liquid, but the derivative becomes infinite at Fermi points. Another characteristic of the Luttinger liquid is particle-hole symmetry, *i.e.*identical behavior of the particle and hole part of the spectral function. The spinless Luttinger liquid has a spectral function (given by imaginary part of retarded Green's function) of the form,

$$A(k,\omega) \sim \frac{1}{|\omega - v_c\widetilde{k}|^{1-\gamma}} \tag{8.3}$$

near the singularity where γ measures the interaction strength and $\widetilde{k} = (k - k_F)$. $\gamma = 0$ corresponds to the non-interacting case. In the case with both spin and charge, there are two such singularities, with different velocities for spin (v_s) and charge (v_c) excitations (Spin-charge separation). A schematic comparison of Spectral functions of non-interacting electrons, fermi Liquid and Luttinger liquid is given in Fig.(8.1).

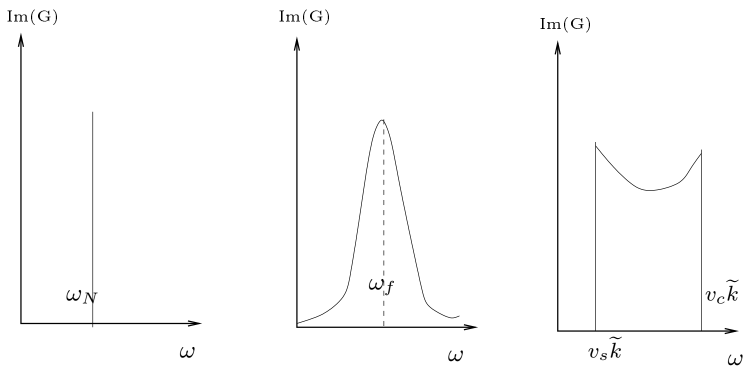

Figure 8.1: Spectral function or imaginary part of green's function for non-interacting electrons ($\omega_N = \frac{k_F}{m}(k - k_F)$), Fermi Liquid ($\omega_f = \frac{k_F}{m^*}(k - k_F)$), and a Luttinger liquid. v_s and v_c are spin and charge velocity respectively.

8.3.1 Bosonization

In $1+1$ dimensions fermions can be bosonized *i.e.*there exists an duality between fermionic degrees of freedom and corresponding bosonic degrees. This was first applied to show equivalence of sine-Gordon and Thirring Model in [131], where a map was given between bosonic degrees of freedom of a interacting scalar field action (sine-Gordon) in $1+1$ dimension with fermionic degrees of freedom of a theory (Thirring Model) of single Dirac field action in $1+1$-dimension. This technique can be used to solve various $1+1$ models, like Luttinger liquid. A review of bosonization technique can be found in [132]. The Bosonization map is given by:

$$\begin{aligned}
\psi_L &\approx e^{i\phi_L} \qquad \psi_R \approx e^{-i\phi_R} \\
\bar{\psi}\gamma^\mu\partial_\mu\psi &\approx (\partial_\mu\phi)^2 = (\partial_t\phi)^2 - v^2(\partial_x\phi)^2 \\
\psi_L^\dagger\psi_R &\approx e^{-i(\phi_L+\phi_R)} \\
\bar{\psi}\gamma^\mu\psi &\approx \epsilon^{\mu\nu}\partial_\mu\phi
\end{aligned} \tag{8.4}$$

where v is velocity of the excitation. ψ is a fermionic field and ϕ is bosonic field. L/R represents left-moving and right-moving fields. This has a Lorentz invariant form with v as velocity of light. So an interaction of type $(\bar{\psi}\gamma^\mu\psi)^2$ renormalises the kinetic term and hence also renormalises the dimension of the operator $e^{i\phi_L}$. The correlation becomes $(\omega - v(k-k_F))^\alpha$ where α depends on this interaction strength. Non Lorentz invariant ρ^2 ($\rho \sim \bar{\psi}\gamma^0\psi$,"charge-charge" interaction) term adds $(\partial_x\phi)^2$ in the dual bosonic theory. This renormalizes the velocity of the excitation.

8.4 Fermi-Luttinger Liquid

Deviation from Luttinger liquid behavior can be seen in modified Luttinger models *e.g.* Fermi-Luttinger liquid [82, 133]. The formulation and exact solvability of Luttinger liquid relies on the assumption of linear dispersion. In [82] the spectral function was modified by an non-linear term for spinless Luttinger liquid.

$$\omega_\pm = \pm v\widetilde{k} + \frac{\widetilde{k}^2}{2m} + \cdots \tag{8.5}$$

where $\omega_\pm$ represents left-moving and right-moving particles. The presence of finite mass m breaks the particle-hole symmetry, and modifies the spectral function near particle and hole differently. As a consequence the edge singularity for particle is replaced by a Lorentzian peak like Fermi-liquid with finite decay rate, but corresponding behavior for hole remains same (*"Particle-hole asymmetry"*). The scaling exponent now also becomes function of momentum k. Very close to the singularity the behavior of the particle spectral function become similar to a Fermi-liquid, but far away it behaves like an Luttinger liquid.

Part IV

Duality between Charged BTZ black hole and Luttinger liquids

9

Charged BTZ Black-hole

9.1 The Background Geometry: Charged $2+1$ dimensional Black Hole

In our analysis we will consider background of a charged black hole in $2+1$ dimension. The Einstein-Maxwell action is given by,

$$S_{EM} = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left(R + \frac{2}{l^2} - 4\pi G F_{\mu\nu}F^{\mu\nu}\right) \tag{9.1}$$

Where G is $3D$ Newton constant, $-\frac{1}{l^2}$ is the cosmological constant (l is the AdS-length).

A solution of $2+1$ dimensional Einstein-Maxwell action (9.1) is given by the following metric [81, 134],

$$ds^2 = -r^2 f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\theta^2 \tag{9.2}$$

where, $f(r) = \frac{1}{l^2} - \frac{8GM}{r^2} - \frac{8\pi GQ^2}{r^2}ln(\frac{r}{l})$ and $F_{tr} = \frac{Q}{r}$. Also let r_+ be the horizon radius determined by the largest real root of $f(r_+) = 0$. The coordinates have following range: $-\infty < t < \infty$, $r_+ \leq r < \infty$ and $0 \leq \theta < 2\pi$. We scale the coordinates and redefine parameters in the following way such that θ can be replaced by x, where

$-\infty < x < \infty$,

$$r = \lambda r',\ t = \frac{t'}{\lambda}, \lambda d\theta = \frac{dx}{l},\ Q = \lambda Q',\ M = \lambda^2(M' - \pi Q'^2 ln\ \lambda) \tag{9.3}$$

We can also define $r_+ = \lambda r'_+$. Then the metric is given by ,

$$ds^2 = -r'^2 f(r')dt'^2 + \frac{dr'^2}{r'^2 f(r')} + \frac{r'^2}{l^2}dx^2 \tag{9.4}$$

with $f(r') = \frac{1}{l^2} - \frac{8GM}{r'^2} - \frac{8\pi GQ'^2}{r'^2} ln(\frac{r'}{l})$. Note that $f(r'_+) = 0$ now, and the vector potential is given by,

$$A = -Q' ln(\frac{r'}{r'_+})dt' \tag{9.5}$$

We have chosen a gauge such that potential at horizon is zero. We now introduce a further redefinition of coordinates to make all the parameters and coordinates dimensionless[1]. This is convenient for calculations.

Thus consider the redefinitions:

$$\begin{aligned} r' = r'_+ r'', \quad Q' = \frac{r'_+}{l\sqrt{16\pi G}} Q'', \\ (t', x') = \frac{l^2}{r'_+}(t'', x''), \quad A'_0 = \frac{r'_+}{l\sqrt{16\pi G}} A''_0 \end{aligned} \tag{9.6}$$

After which the metric becomes (dropping primes for simplicity),

$$\frac{ds^2}{l^2} = -r^2 f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 dx^2 \tag{9.7}$$

where,

$$f(r) = (1 - \frac{1}{r^2} - \frac{Q^2}{2r^2} ln(r)) \tag{9.8}$$

and,

$$A = -Q ln(r) dt \tag{9.9}$$

We now choose a coordinate $z = \frac{1}{r}$ and set $l = 1$, we get the metric,

[1] Note that the parameter Q has dimensions $length^{-\frac{1}{2}}$

$$
\begin{aligned}
ds^2 &= \frac{1}{z^2}\left[-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^2\right] \\
f(z) &= 1 - z^2 + \frac{Q^2}{2} z^2 \; ln(z) \\
A &= Q \; ln(z) \; dt
\end{aligned} \tag{9.10}
$$

As $z \to 0$, the metric asymptotes to AdS_3 metric and is called boundary. The metric is also singular at $z = 1$ called horizon. The black hole temperature is given by $T = -\frac{f'(1)}{4\pi} = \frac{(1-\frac{Q^2}{4})}{2\pi}$ ($|Q| \leq 2$).

9.1.1 Near-horizon Geometry

As is described in [75, 135], at $T = 0$ ($Q = 2$) the low frequency limit divides the full $d+1$-dimensional bulk space-time into two distinct regions. The analysis was done for $d > 2$. As shown in our paper [78], it was shown that similar analysis can be done for $d = 2$ also. Near the horizon called *inner region* we have an AdS_2 factor where as *outer region* is described by AdS_3 black-hole. For inner region, we define new coordinates (τ, ζ) such that,

$$
1 - z = \frac{\omega}{2\zeta} \qquad : \qquad \tau = \omega t \tag{9.11}
$$

such that as $\omega \to 0$ and $(1 - z) \to 0$, both ζ and τ are finite. In this limit the background metric (9.10) in new co-ordinates is $AdS_2 \times \mathbb{R}$, given by,

$$
ds^2 = \frac{1}{2\zeta^2}(-d\tau^2 + d\zeta^2) + dx^2 \qquad ; \qquad A_\tau = -\frac{1}{\zeta} \tag{9.12}
$$

where, radius of AdS_2 subspace $R_2 = 1/\sqrt{2}$ in the unit of AdS_3 radius which we have set to unity.

The *inner region* is given by $\epsilon < \zeta < \infty$, and the metric is given by (9.12). The *outer region* is given by $\frac{\omega}{2\epsilon} < 1 - z$, where we will use the z coordinate and the metric (9.10) (See Fig.(9.1)). We will consider the limit $\omega \to 0$ and $\epsilon \to 0$, such that $\frac{\omega}{2\epsilon} \to 0$. So that the outer region is given by $z > 1$.

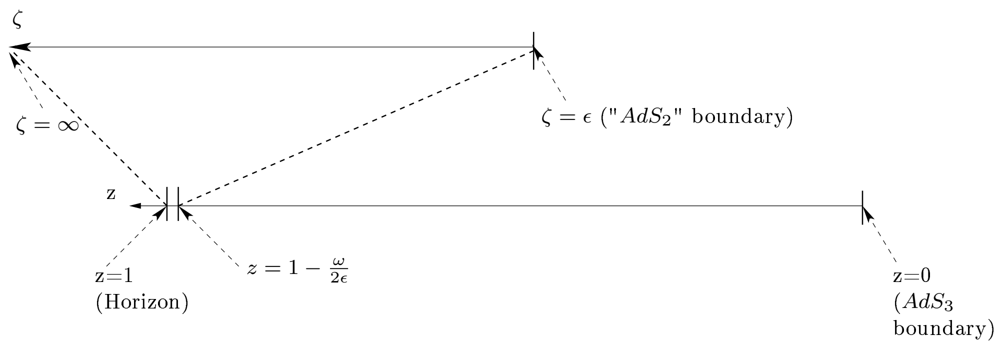

Figure 9.1: ζ and z space: Inner and Outer Region

9.1.2 Near-horizon CFT

A bulk field Φ in asymptotic AdS_3 black-hole (eqn.(9.10)) maps to a tower of fields Φ_k in the inner AdS_2 region, where k is the momentum along $\mathbb{R}$. Similarly the boundary operator $\mathcal{O}(k)$ maps to an IR operator $\mathcal{O}_k^{IR}$ in IR CFT. The retarded Green's function ($\mathcal{G}_k(\omega)$) for IR operator dual to Φ_k can be obtained analytically by solving bulk equation of motion of Φ_k in AdS_2 with constant electric field (9.12). The scaling dimension of the operator is in general function of k and is given by $\delta_k = \frac{1}{2} + \nu_k$. The form of the function ν_k (which also depends on Q and charge and mass of the field Φ) will be given for specific cases in the later chapters. However, for current discussion we will not consider detailed nature of the function ν_k. The IR Green's function has a form,

$$\mathcal{G}_k(\omega) = c(k)\omega^{2\nu_k} \tag{9.13}$$

where $c(k)$ is a complex function *i.e.* can be written in general as $c(k) = |c(k)|e^{i\gamma_k}$. Small ω limit of the full Green's function can be obtained by matching the inner and outer region solution [75] at the overlapping region ($\zeta \to 0$ and $1-z = \frac{\omega}{2\zeta} \to 0$). The perturbative expansion for the retarded Green's function of full theory can be written for a generic value of k as [75],

$$G_R(\omega, k) = K\frac{(b_+^{(0)}(k) + \omega b_+^{(1)(k)} + \mathcal{O}(\omega^2)) + \mathcal{G}_k(\omega)\left(b_-^{(0)}(k) + \omega b_-^{(1)}(k) + \mathcal{O}(\omega^2)\right)}{(a_+^{(0)}(k) + \omega a_+^{(1)}(k) + \mathcal{O}(\omega^2)) + \mathcal{G}_k(\omega)\left(a_-^{(0)}(k) + \omega a_-^{(1)}(k) + \mathcal{O}(\omega^2)\right)}, \tag{9.14}$$

where, $a_\pm^{(n)}$ and $b_\pm^{(n)}$ are unknown functions to be obtained by matching solutions of inner and outer region. K is a positive constant depending on overall normalization of the action. We will give a summary of various possible cases as discussed in[75],

- For ν_k real, all the functions $a_\pm^{(n)}$ and $b_\pm^{(n)}$ are real. Now if $a_\pm^{(0)} \neq 0$, then

$$Im(G_R(\omega = 0, k)) = 0 \quad , \quad Re(G_R(\omega = 0, k)) = K\frac{b_+^{(0)}(k)}{a_+^{(0)}(k)} \tag{9.15}$$

  Also near $\omega = 0$, the denominator of (9.14) can be expanded,

$$Im(G_R(\omega, k)) = G_R(\omega = 0, k)d_0 Im(\mathcal{G}_k(\omega)) + \cdots \propto \omega^{2\nu_k} \tag{9.16}$$

  where $d_0 = \frac{b_-^{(0)}}{b_+^{(0)}} - \frac{a_-^{(0)}}{a_+^{(0)}}$. The spectral function of the full theory has a non-trivial scaling dependence near low frequency and the scaling behavior is completely determined by the IR CFT. The amplitude or the k-dependent prefactor depends non-trivially on the behavior of the outer region and thus on the UV physics.

- $a_\pm^{(0)}(k) = 0$ is possible for only discrete values of k and real values of ν_k. Again all the functions $a_\pm^{(n)}$ and $b_\pm^{(n)}$ are real. Say the $a_\pm^{(0)}(k)$ vanishes at some $k = k_F$. We can expand (9.14) near $k = k_F$ and $\omega = 0$, and to leading order the Green's function has the form,

$$G_R(\omega, k) = \frac{h_1}{(k - k_F) - \frac{\omega}{v_F} - h_2\mathcal{G}_{k_F}(\omega)} \tag{9.17}$$

  where h_1, h_2 and v_F depends on k_F and the details of outer solution. General discussions on various properties of these quantities can be found in [75]. The Green's function shows *quasi-particle-like-pole* similar to Fermi liquid discussed in Chapter (8).

- Consider ν_k purely imaginary *i.e.* $\nu_k = -i\lambda_k$ where λ_k is real.The leading small ω behavior of the Green's function becomes,

$$G_R(\omega, k) \simeq \frac{b_+^{(0)} + b_-^{(0)} c(k)\omega^{-2i\lambda_k}}{a_+^{(0)} + a_-^{(0)} c(k)\omega^{-2i\lambda_k}} + \mathcal{O}(\omega) \tag{9.18}$$

where it can be shown that $b_-^{(0)} = (b_+^{(0)})^*$ and $a_-^{(0)} = (a_+^{(0)})^*$ [75]. Then (9.18) is invariant under discrete scaling $\omega \to e^{n\tau_k}\omega$ where $n \in \mathbb{Z}$ and $\omega \to 0$, *i.e.* the Green's function is *log-periodic* in ω with period $\tau_k = \frac{\pi}{\lambda_k}$. So again the leading non-analytic behavior of the Green's function is completely given by IR CFT.

10

Fermion Green's Function

10.1 Spinor Green's Function

The action for bulk spinor field is given by,

$$S_{spinor} = \int d^3x\sqrt{-g}\, i(\bar{\Psi}\Gamma^M\mathcal{D}_M\Psi - m\bar{\Psi}\Psi) \tag{10.1}$$

where

$$\bar{\Psi} = \Psi^\dagger\Gamma^{\underline{t}}, \quad \mathcal{D}_M = \partial_M + \frac{1}{4}\omega_{abM}\Gamma^{ab} - iqA_M \tag{10.2}$$

and ω_{abM} is the spin connection[1]. The non-zero components of spin connection are given by,

$$\begin{aligned} \omega_{\underline{t}\,\underline{z}} &= (\tfrac{f}{z} - \tfrac{f'}{2})dt \\ \omega_{\underline{z}\,\underline{x}} &= \tfrac{\sqrt{f}}{z}dx \end{aligned} \tag{10.3}$$

To analyse Dirac equation following from (10.1) in the background (9.10), we use the following basis,

[1]We will use M and a, b to denote bulk space-time and tangent space indices respectively, and $\mu, \nu \cdots$ to denote indices along the boundary directions, i.e. $M = (z, \mu)$. The Γ^a obey $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$, where $\eta = diagonal(-1, 1, 1)$ and $\Gamma^M = e^M_a\Gamma^a$, where e^M_a is the vielbein. Also (t, z, x) denote space time indices and $(\underline{t}, \underline{z}, \underline{x})$ the tangent space indices.

$$\Gamma^{\underline{t}} = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma^{\underline{x}} = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\Gamma^{\underline{z}} = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} \tag{10.4}$$

where $\Psi_\pm$ are complex functions given by,

$$\Psi_\pm = \frac{1}{2}(1 \pm \Gamma^{\underline{z}})\Psi \tag{10.5}$$

Now writing

$$\Psi_\pm = e^{-i\omega t + ikx} \ \psi_\pm(z) \tag{10.6}$$

the Dirac equation becomes,

$$\begin{aligned}(zf(z)\partial_z - f(z) + \frac{1}{4}zf'(z) \mp m\sqrt{f(z)})\psi_\pm \\ = iz(\omega + \mu \ ln(z) \mp k\sqrt{f(z)})\psi_\mp\end{aligned} \tag{10.7}$$

where $\mu = qQ$. Let us consider the following transformation

$$\tilde{\psi}_\pm = e^{-\int_a^z \frac{dt}{tf(t)}\left[f(t) - \frac{1}{4}tf'(t)\right]}\psi_\pm \tag{10.8}$$

where a is some constant. The equations for $\tilde{\psi}_\pm$ becomes,

$$\left[zf(z)\partial_z \mp m\sqrt{f(z)}\right]\tilde{\psi}_\pm = iz(\omega + \mu \ ln(z) \mp k\sqrt{f(z)})\tilde{\psi}_\mp \tag{10.9}$$

We will drop the tilde for simplicity in following calculations.

A bulk Dirac spinor field Ψ with charge q is mapped to a fermionic operator $\mathcal{O}$ in CFT of the same charge and conformal dimension $\Delta = 1 \pm m$. In 2+1 dimension $\mathcal{O}$ is a chiral spinor. By studying the Dirac equation in 2+1 dimension asymptotically AdS space (eqn.(10.9)), we can find the retarded Green's Function of the $1+1$ dimensional boundary CFT [75, 74, 70, 73](Also as described in Chapter(7)). We will study behavior for simple case where bulk fermion is massless (*i.e.* $\Delta =$ 1). Following the calculation of [70], the boundary Green's function is given by

(Eqn.7.65 in Chapter(7)),

$$G_R = \lim_{z\to 0} i\frac{\psi_-(z)}{\psi_+(z)} = \lim_{z\to 0} G(z) \tag{10.10}$$

$G(z)$ defined in the above equation satisfies a first order non-linear differential equation which follows directly from bulk Dirac equation (eqn.(10.9)),

$$\begin{aligned} zf(z)\partial_z G(z) + G(z)^2 z(\omega + \mu\ ln(z) - k\sqrt{f(z)}) \\ +z(\omega + \mu\ ln(z) + k\sqrt{f(z)}) = 0, \end{aligned} \tag{10.11}$$

where $\mu = qQ$.

10.1.1 Boundary Condition

In order get the retarded correlation function for the dual fermionic operator $\mathcal{O}$, we need to impose ingoing boundary condition for ψ at the horizon. We substitute $\psi_+(z) = e^S$ in (10.9) to obtain the leading behavior as $z \to 1$,

$$\psi_- = \frac{zfS' - m\sqrt{f}}{iz(\omega + \mu\ ln(z) - k\sqrt{f})} e^S \tag{10.12}$$

Assume that the dominant term is the one with the z-derivative acting on e^S. This gives

$$\partial_z\psi_- = S'\frac{zfS' - m\sqrt{f}}{iz(\omega + \mu\ ln\ z - k\sqrt{f})} e^S \tag{10.13}$$

Plug this into other equation of (10.9) to get an equation for S, which is given by:

$$S'(z) = \pm\frac{\sqrt{m^2 + z^2(k^2 - (\frac{\omega+\mu\ ln\ z}{\sqrt{f}})^2)}}{z\sqrt{f}} \tag{10.14}$$

and putting above in Eqn.10.11, we get,

$$G(z) = \frac{zfS' - m\sqrt{f}}{z(\omega + \mu ln(z) - k\sqrt{f})} \tag{10.15}$$

This can be used to derive the asymptotic behavior (near $z = 1$) of the Green's functions.

Case 1 $\mathbf{Q^2} \neq \mathbf{4}$: $f(z) \approx 4\pi T(1-z)$, and $\psi_+ = e^{S(z)} \approx e^{\mp i\frac{\omega}{4\pi T} ln(1-z)}$. The negative sign in the exponent gives in-going wave. Putting back $S(z)$ with this choice of solution in equation (10.15), gives $G(z=1) = i$.
Case 2 $\mathbf{Q^2} = \mathbf{4}$: In the extremal case where $f(z) \approx 2(1-z)^2$, we get

$$S'(z) = \pm \frac{\sqrt{m^2 + k^2 - \frac{1}{2}(\frac{\omega}{(1-z)} - \mu)^2}}{\sqrt{2}(1-z)} \tag{10.16}$$

for $\omega \neq 0$, we can neglect the other terms (keeping next to leading order term),

$$\psi_+ = (1-z)^{\pm\frac{i\mu}{2}} e^{\pm\frac{i\omega}{2(1-z)}} \tag{10.17}$$

The positive sign in the exponent is the ingoing wave. Also near $z = 1$ using (10.15) we conclude that near $z = 1$, for ingoing waves,

$$G = i\frac{\psi_-}{\psi_+} = \frac{\sqrt{m^2 + k^2 - \frac{1}{2}(\frac{\omega}{(1-z)} - \mu)^2} - m}{\frac{1}{\sqrt{2}}(\frac{\omega}{(1-z)} - \mu - k\sqrt{2})} \tag{10.18}$$

This gives the boundary condition for $\omega \neq 0$

$$G(z=1) = i \tag{10.19}$$

In the case where $\omega = 0$ we get,

$$G(z=1) = \frac{m - \sqrt{m^2 + k^2 - \frac{\mu^2}{2} - i\epsilon}}{(k + \frac{\mu}{\sqrt{2}})} \tag{10.20}$$

where $m^2 \to m^2 - i\epsilon$ to choose the correct sign in square root.

10.1.2 AdS_2 limit

Near horizon, zero frequency limit of (eqn.(10.9)) in new co-ordinates (eqn.(9.11)),

$$\left[2\zeta\partial_\zeta \mp m\sqrt{2}\right]\tilde{\psi}_\pm = i(2\zeta - \mu \mp k\sqrt{2})\tilde{\psi}_\mp \tag{10.21}$$

where $\mu = 2q$ (As $Q = 2$). These equations are basically the equations for a fermion in AdS_2 with an additional "mass" term in the 2-D action of the form $ik\bar{\psi}\Gamma\psi$ where $\Gamma = \sigma_1$. We can analytically calculate AdS_2 dual CFT correlation function from the above equation [75]. The asymptotic behavior of the above set of equations can be obtained by taking ($\zeta \to 0$) giving

$$\left[2\zeta\partial_\zeta \mp m\sqrt{2}\right]\tilde{\psi}_\pm = i(-\mu \mp k\sqrt{2})\tilde{\psi}_\mp \tag{10.22}$$

The solutions are

$$\psi_+ = C\zeta^{-\nu_k} + \tilde{C}\zeta^{\nu_k} \tag{10.23}$$

$$\psi_- = \frac{i}{2}\left[-\frac{C(2\sqrt{2}m + 4\nu_k)\zeta^{-\nu_k}}{(\mu + \sqrt{2}k)} + \frac{\tilde{C}(-2\sqrt{2}m + 4\nu_k)\zeta^{\nu_k}}{(\mu + \sqrt{2}k)}\right], \tag{10.24}$$

where,

$$\nu_k = \frac{1}{2}\sqrt{2(m^2 + k^2) - \mu^2}, \tag{10.25}$$

C and $\tilde{C}$ are the integration constants and are related to each other by imposing "in-going" boundary condition as $\zeta \to \infty$. The scaling dimension of the operator $\mathcal{O}_k$ is $\delta_k = 1/2 + \nu_k$. So, the leading order ω dependence (going back to z coordinates) is $\omega^{\pm\nu_k}$. Using which we can get the expression for IR Green's function

$$\mathcal{G}_k(\omega) = \mathcal{H}(\mu, k)\omega^{2\nu_k}, \tag{10.26}$$

where $\mathcal{H}(\mu, k)$ can be determined by solving (10.21) analytically with in-going boundary condition at $\zeta \to \infty$.

10.2 What to expect

10.2.0.1 Symmetry properties

As a consistency check of our numerics we can use the following symmetry properties of the Green's function obtained by direct inspection of the equation of motion (10.9) with $m = 0$, $G(\omega, -k, Q, \mu) = -\frac{1}{G(\omega,k,Q,\mu)}$, $G(\omega, 0, Q, \mu) = i$, $G(-\omega, -k, Q, -\mu) = -G^*(\omega, k, Q, \mu)$, $G(\omega, k, -Q, \mu) = G(\omega, k, Q, \mu)$.

10.2.0.2 UV behavior

Since our background geometry (9.10) asymptotes to AdS_3, in the ultra violet ($\omega >> (T, \mu)$) we expect the effects of finite density and temperature become negligible and it will recover conformal invariance. If we choose our background geometry as pure AdS_3, the Green's function (massless bulk fermion) can be easily obtained as [70](See Sec. (7.2.2.3.1) in Chapter(7)),

$$G_{AdS}(\omega, k) = i\sqrt{\frac{(\omega + k + i\epsilon)}{(\omega - k + i\epsilon)}} \tag{10.27}$$

where $\epsilon \to 0$. This is Green's function for a dimension 1 chiral operator in $1+1$ dimensional CFT. $Im(G_{AdS})$ or the spectral function has a symmetry under $(\omega, k) \to (-\omega, -k)$ ("Particle-hole symmetry') and has an edge-singularity along $\omega = k$. $Im(G_{AdS})$ is zero in the range $\omega = (-k, k)$. Also $\omega \to \pm\infty$, $Im(G_{AdS}) \to 1$ and $G_{AdS}(\omega, k = 0) = i$.

In the ultra violet (UV) we expect same scaling behavior ($\Delta = 1$) and linear dispersion with velocity unity. Note that the scaling dimension of the fermionic operator in the boundary is 1 compared to usual dimension $\frac{1}{2}$ fermionic operators (*viz.* electron operator) in $1+1$ dimension. The scaling dimension of the operator in the IR of the boundary theory may be very different from 1 as the boundary theory may flow to a different fixed point in IR as described in next section.

10.2.0.3 IR behavior

As shown in [75, 135], theories dual to charged extremal black holes (in $d+1$ dimensions, $d > 2$) have a universal IR behavior controlled by the AdS_2 region in the bulk. A similar analysis goes through in $d = 2$ case (See Sec.(9.1.2) and (10.1.2)).

At zero temperature, the the background geometry (Sec.(9.1.1)) is described by $AdS_2 \times \mathbb{R}$ in the near horizon limit. From the boundary field theory point of view, although even at $T = 0$ the conformal invariance was broken by μ, the theory will have an scale invariance in the IR limit ($\omega << \mu$) and will be controlled by an IR CFT dual to AdS_2. The IR behavior suggests that the spectral function $A(k, \omega) = Im(G_R) \sim \omega^{2\nu_k}$ where $\nu_k = \frac{1}{2}\sqrt{2(m^2 + k^2) - \mu^2}$ and ν_k is real. The spectral function in this case vanishes as $\omega \to 0$. If ν_k is imaginary, the Green's function is "log-periodic" in ω ($G_R(\omega, k) = G_R(\omega e^{n\xi(k,\mu)}, k)$ where $n \in \mathbb{Z}$ with $\xi(k,\mu) = \frac{2\pi}{\sqrt{\mu^2 - 2(m^2+k^2)}}$). There may also exist particular values of $k = k_F$ where we can have a quasi-particle-like pole as $\omega \to 0$. The spectral function at those values of k is given by (Eqn.(9.17)),

$$A(\omega, k) = \frac{h_1 \Sigma_2}{(k - k_F - \frac{\omega}{v_F} - \Sigma_1)^2 + \Sigma_2^2} \tag{10.28}$$

where $\Sigma_{1,2}$ are real and imaginary part of $\Sigma \sim \omega^{2\nu_{k_F}}$ respectively (As discussed in Sec.(9.1.2)). As mentioned in [75, 135], that the form of IR Green's function suggests that the IR CFT is a chiral sector of $1+1$ dimensional CFT.

10.2.0.4 Condensed Matter Systems

For $1+1$ dimension boundary theory one typically expects to get Luttinger Liquid behavior. The condensed matter related terms are explained in more detail in Chapter (8). "Particle-hole symmetry'($A(\omega, k_F + \tilde{k}) = A(-\omega, k_F - \tilde{k})$, $\tilde{k} = k - k_F$ is the deviation of momentum from the Fermi momentum k_F), absence of quasi-particle peak (*i.e.* Lorentzian peak known as quasi-particle peak in usual Landau Fermi liquid theory is replaced by power law edge singularity) , linear dispersion, spin-charge separation are the hallmarks of Luttinger liquid in $1+1$ dimension. In spinless Luttinger liquid the spectral function for electron (dimension $\frac{1}{2}$ operator) has a behavior [136, 137] (for $\tilde{k} > 0$),

$$A(k_F + \tilde{k}) \sim (\omega - v_F \tilde{k})^{\gamma - 1} (\omega + v_F \tilde{k})^{\gamma} e^{-a|\omega|} \tag{10.29}$$

γ is Luttinger liquid exponent, gives the IR scaling dimension which is related to "anomalous scaling dimension" of the fermion operator (See Fig.(10.1(a))). a is a constant depending on γ. The expression (10.29) is a simplified version of the expression given in [136, 137], but has the essential essence of it. The expression for spectral function for Luttinger liquid given in Chapter (8), can be shown equivalent to (10.29) near the singularity $\omega = v_F \tilde{k}$. Deviation from these behavior can be seen in modified Luttinger models *e.g.* Fermi-Luttinger liquid [82] . In Fermi-Luttinger liquid the dispersion is modified by a non-linear term, and as a consequence the edge singularity for particle (Fig.(10.1(a))) is replaced by a Lorentzian peak like Fermi-liquid, but corresponding behavior for hole remains same (*"Particle-hole asymmetry"*). The scaling exponent also becomes function of k. Very close to the singularity the behavior of the particle spectral function become similar to (10.28), which resembles a Fermi-liquid. We should note that in our model, $b$oth scale invariance and Lorentz invariance is broken for the boundary theory, so we can expect deviations from Luttinger Liquid. In a Luttinger liquid one could construct correlators like,

$$G(\omega,k)=\frac{(\omega-v_1k+i\Delta_1)^{\alpha}}{(\omega-v_2k+i\Delta_2)^{\beta}}\qquad \alpha,\beta>0 \tag{10.30}$$

where ω, k are defined as deviations from Fermi point. At a fixed point one expects $\Delta_{1,2}=\epsilon\to 0$, and has a edge singularity like (Fig.(10.1(a))). But at a generic point on the RG between fixed points one expects some finite imaginary part which smooths out the singularity to a peak (*viz.* Fermi-Luttinger liquid [82]). The Green's function have branch cut singularity is along $\omega=v_1k$ and $\omega=v_2k$; the peak is along the second line of singularity. The Green's function along the peak becomes,

$$G(\omega)=\frac{(\omega^2(1-\frac{v_1}{v_2})^2+\Delta_1^2)^{\frac{\alpha}{2}}}{\Delta_2^{\beta}}e^{i(\alpha\theta-\beta\frac{\pi}{2})} \tag{10.31}$$

where $\tan\theta=\frac{\Delta_1}{\omega(1-\frac{v_1}{v_2})}$. $Im\ G$ clearly increases with ω $(\alpha,\beta<1)$ as long as Δ does not have an ω dependence - as at a fixed point Luttinger liquid. In general if Δ_1 is sub linear then one expects $Im\ G\approx\frac{\Delta_1^{\alpha}}{\Delta_2^{\beta}}$. If $\Delta_1\approx\omega^x$ with $x>1$ then $Im\ G\approx\frac{\omega^{\alpha}}{\Delta_2^{\beta}}$.

On the other hand for a Fermi liquid one expects a behavior like (10.28) but with $\Sigma\sim\omega^2$. But the IR AdS_2 dictates that the scaling exponent of Σ is generically different from usual Fermi liquid. In analyzing the AdS_3 example we must keep these points in mind. We expect on general grounds that at least for weak coupling it should behave as a Luttinger liquid. More generally it could go into a massive phase. But we do not find evidence of a mass gap in the numerics below. Any deviation from a Luttinger liquid therefore is something noteworthy.

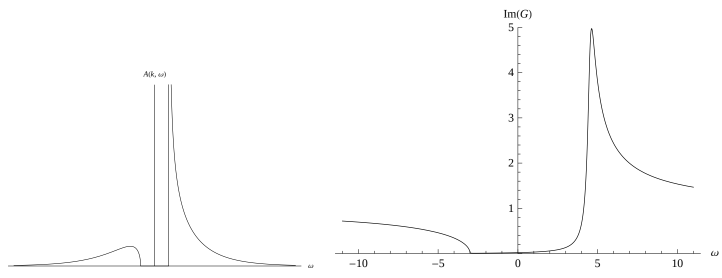

Figure 10.1: Left:Spectral function of spinless Luttinger Liquid $A(k,\omega)$ vs ω for $\tilde{k} > 0$ and $\gamma = 1/2$. It vanishes the range $\omega = (-v_F\tilde{k}, v_F\tilde{k})$ Right:Spectral function given by eqn.(10.30) for $k = 5$, $\alpha = \beta = 1/2$, $\Delta_1 = .01, \Delta_2 = .2$, $v1 = -.6, v2 = .9$

10.3 Numerical results for Spinor Green's Function

Since charge q of the bulk field fixes the charge of boundary operator under global $U(1)$, we must keep q fixed to study a particular boundary operator. For $T = 0$ ($Q = \pm 2$), $\mu = \pm 2q$, we will study the Fermi-momentum and velocity for fixed q. We will also vary q or μ to study the "log-periodicity" discussed in section (10.2.0.3). For finite temperature case, we will fix q and study the Green's function for various values of temperature (equivalently Q). A brief overview of numerics used for Green's function analysis is given in Appendix (C).

10.3.0.5 Zero temperature

We will mainly study the qualitative features of zero temperature Green's function as the numerics is not very precise. Quantitative studies are made at finite temperature where we have very good numerical handle.

- **UV behavior:** For $q = 0$, the Green's function matches over a large range with pure AdS Green's Function (10.27). For finite q, the $G_R \to 1$ as $\omega \to \pm\infty$ for fixed k which implies the UV scaling dimension of the operator is 1 (Fig.(10.2)). For fixed ω, the Green's function always matches near $k = 0$ which is consistent with symmetry properties (Section (10.2.0.1)). The qualitative nature of the peak ($q \neq 0$) matches with Luttinger Liquid

Green's function considered in eq.(10.30) (Compare Fig.(10.1(b)) & (10.3)). As expected for $k >> \mu$ from pure AdS behavior (eqn.10.27), the spectral function should be zero in the range $\omega+\mu \in (-k, k)$ and a peak near $\omega+\mu \sim k$ [73], is approximately seen in Fig.10.3.

- **IR behavior:** For a fixed q density plot of the spectral function (fig.(10.4)) shows a sharp **quasi-particle like peak** at $\omega = 0$ and some value of momentum $k = k_F$, called Fermi momentum. At Fermi point the peak height goes to infinity and the width goes to zero. The behavior is as described in section (10.2.0.3), this can be seen in modified Luttinger liquids as describe in the paragraph below the equation (10.31). We expect that the theory flow to a different fixed point in IR. But at small q, along the dispersion curve we see the spectral function has a minima at $\omega = 0$ compared to the quasi-particle like peak, as expected in Luttinger liquid (Fig.(10.7)). Fermi momentum changes from k_F to $-k_F$ for $q \to -q$ due to the symmetry (section 10.2.0.1). For $q = 1/2$, $k_F = -1.367$ and $v_F = .43$ obtained from the dispersion plot which is **linear**.(fig.10.5). If similar analysis was done for $q = 1$, where $k_F = 1.644$ and $v_F = -.16$. Note k_F changes sign if we change q. Now, as we can see from the expression of scaling dimension of IR operator $\mathcal{O}_k$ at Fermi momentum ($q = 1/2$) turns out to be $\nu_{k_F} = \frac{1}{2}\sqrt{2k_F^2 - 4q^2} = .83 > \frac{1}{2}$. In this regime according to analysis [75] the dispersion relation should be linear. Also as described in section (10.2.0.3), the spectral function should go to zero if ν_k real (Fig.(10.6(b))) and is "log-periodic" where ν_k is imaginary (fig.10.6(a)). We found a very good match of the numerical and analytical results for periodicity. Non-analyticity of Green's function at $\omega = 0$ independent of k is not typical of Luttinger liquids and in Section (10.4) we have discussed a possible resolution of this.

- **Particle-Hole (A)symmetry:** We find "particle-hole asymmetry" for $q = \frac{1}{2}$, *i.e.* the spectral function behavior is different under reflection at Fermi point (fig.(10.4)). This is again very different from Luttinger liquid behavior, but it can be observed in Fermi-Luttinger Liquids as discussed in [82]. The particle hole symmetry is restored for $\mu = 2q = 0$ (fig.(10.7)), and the asymmetry slowly increases with q.

- **Gapless phase:** Also a closer look at density plot (fig.(10.4)) shows the system is in **gapless** phase which is as expected because in the IR the theory is expected to have conformal invariance.

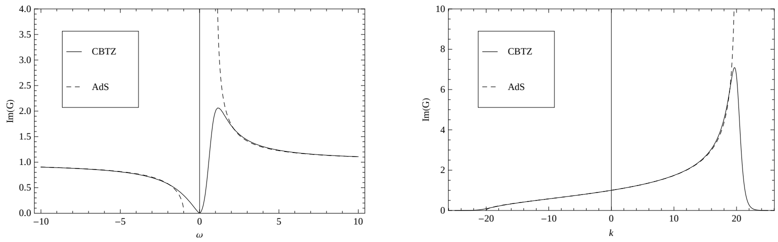

Figure 10.2: $T = 0$: Imaginary part of Fermion Green's function for $q = 0$.Left:$k = 1$. Right:$\omega = 20$

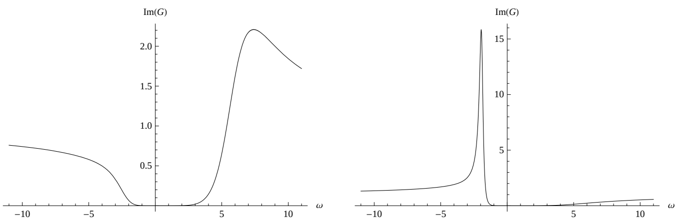

Figure 10.3: $T = 0$: Imaginary part of Fermion Green's function for $q = 1/2$.Left:$k = 4$ (Compare with what is expected, Figure (10.1)). Right:$k = -4$

10.3.0.6 Finite temperature

The behavior of the spectral function at finite temperature can be summarized as follows:

- **IR behavior:** At sufficiently small temperatures, the quasi-particle like peak still survives. But the width gets broadened with temperature and also the Fermi frequency shifts to some non-zero value. Table (10.1) shows various peak properties at various temperature with $q = 1/2$. Data shows that $\omega_F \neq 0$ is a finite temperature effect and goes to zero as $T \to 0$. Also Δ_F (FWHM or "Full Width at Half Maximum" at Fermi point) goes to 0 with T. Fermi velocity v_F is independent of temperature. The FWHM

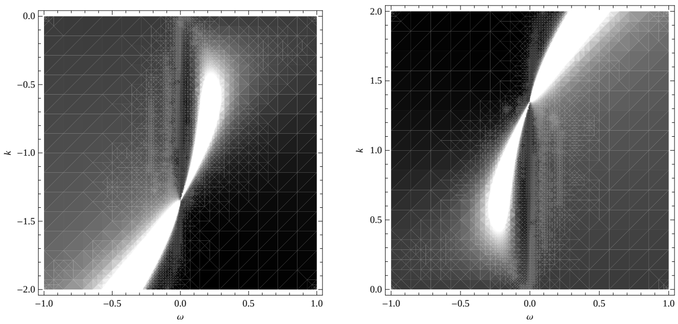

Figure 10.4: $T = 0$: Density plot of imaginary part of Fermion Green's function for $q = 1/2$ and $q = -1/2$. Lighter regions have higher value of the function

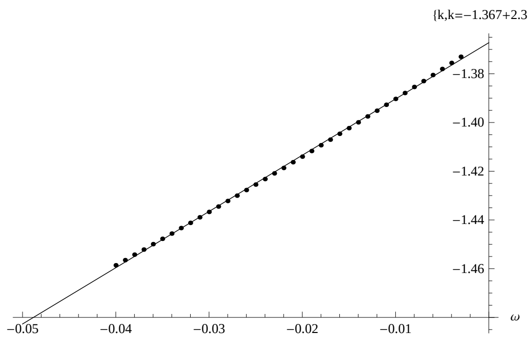

Figure 10.5: $T = 0$: Dispersion relation near Fermi point for $q = 1/2$

along the dispersion varies as $\Delta - \Delta_F = A(\omega_F - \omega)^x$ where $x = 2.25(\omega < \omega_F), 1.8(\omega > \omega_F)$ at $T = .00159$ (fig.(10.8) ,fig.(10.9)). k_F varies linearly with T as $k_F = -1.3671 + 13.49T$. Which implies at $T = 0$,$k_F = -1.3671$ which matches with the zero temperature result.

- **Dispersion:** The dispersion curve (fig.(10.10),fig.(10.8)) is linear near the Fermi point, but deviates from linearity away from the Fermi point. v_F approaches 1 as $(k - k_F) << 0$ as expected from UV behavior, but v_F becomes large $(k - k_F) > 0$. We expect v_F would go to 1 for $(k - k_F) >> 0$, but we were unable to explore that region as the peak gets very broad in that range and numerical error increases.

- **Particle-Hole (A)symmetry:** Again in finite temperature case also the spectral function shows "particle-hole asymmetry" at low temperature, but

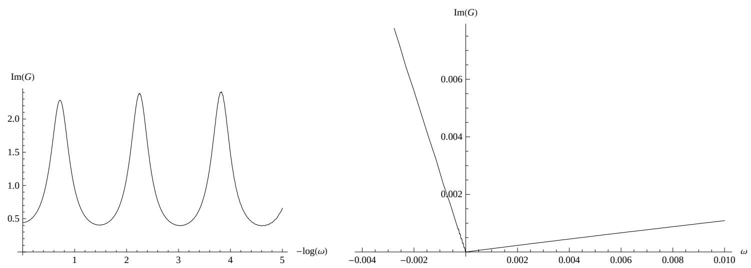

Figure 10.6: $T = 0$: IR behavior of imaginary part of fermion Green's function for Left:$\mu = 4$, $k = .5$, Right: $\mu = 1$, $k = 1$

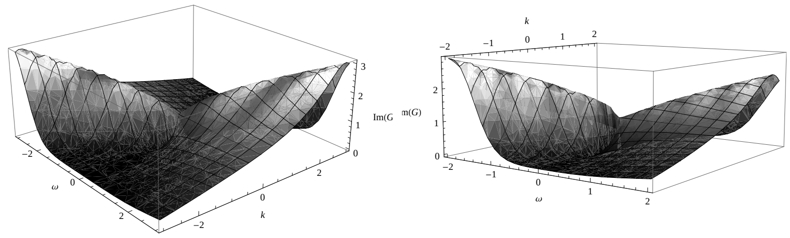

Figure 10.7: $T = 0$: Imaginary part of fermion Green's function for $q = 0$ and $q = 10^{-1}$ respectively

the symmetry gets restored for $Q \to 0$ (large temperature) with fixed q. We can conclude that the particle-hole asymmetry is controlled by the parameter $\mu = qQ$.

- **Large Temperature:** For large temperature, $Q \to 0$, the Green's function approaches to that for uncharged non-rotating BTZ (UBTZ) as given in [70] (with $T_L = T_R = \frac{1}{2\pi}$, $G_{UBTZ} = i\frac{\Gamma(\frac{1}{4}-i\frac{\omega-k}{2})\Gamma(\frac{3}{4}-i\frac{\omega+k}{2})}{\Gamma(\frac{3}{4}-i\frac{\omega-k}{2})\Gamma(\frac{1}{4}-i\frac{\omega+k}{2})}$). The quasi-particle peak is completely lost at large temperature and has a minima along the line of dispersion at $\omega = 0$.

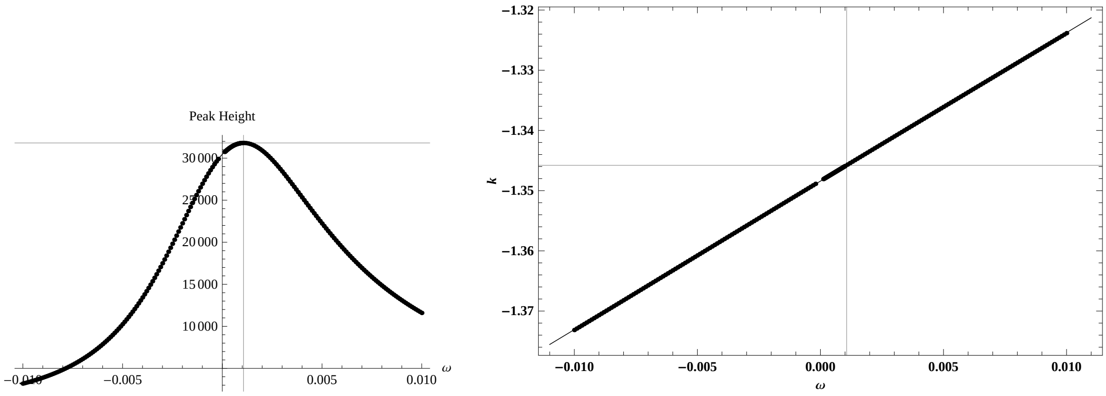

Figure 10.8: $q = 1/2, Q = 1.99$ Right: Variation of peak height of imaginary part of (Fermion) Green's function with frequency shows the Fermi point is shifted to $\omega = 0.00106$. Left: Dispersion is linear near Fermi point, k_F is obtained by finding k corresponding to$\omega = 0.00106$ in the linear fit.

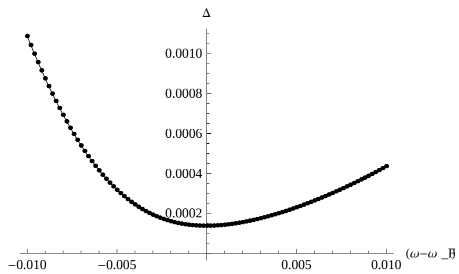

Figure 10.9: $q = 1/2, Q = 1.99$. Variation of FWHM with $(\omega - \omega_F)$ (Fermion)

10.4 Non Fermi Liquid Behavior in the Boundary Field Theory

Since our boundary theory lies on an RG trajectory connecting a UV fixed point CFT dual to AdS_3 and a CFT in the IR (because it is gapless) we can expect it to be described by the action of a Luttinger liquid like theory modified by the addition of some irrelevant perturbations. These terms are allowed because the background charge density (implied by the chemical potential $A_0(z = 0)$) breaks scale invariance. It also breaks Lorentz invariance, which means that one cannot assume that all propagating modes propagate with the same velocity. The effect of

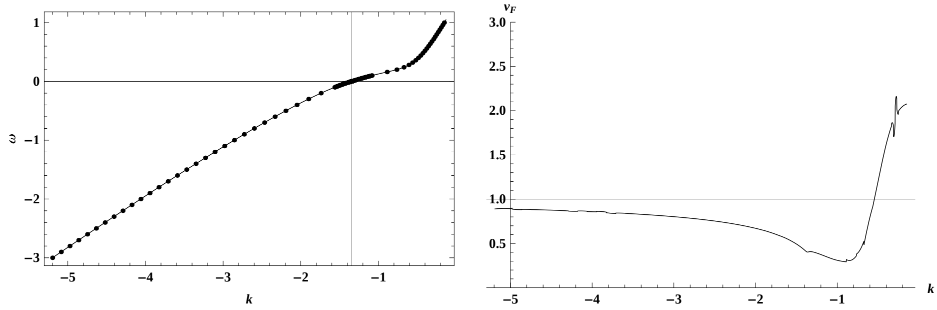

Figure 10.10: $q = 1/2, Q = 1.99$. Left: Dispersion curve Right: v_F vs k (Fermion)

q=1/2						
Q	T	ω_F	k_F	Δ_F	H_F	v_F
1.9	0.0155	-0.00342	-1.1968	4.534×10^{-1}	392	0.386
1.95	0.007858	0.00181	-1.2712	2.735×10^{-3}	1616	0.401
1.99	0.00159	0.00106	-1.3458	1.381×10^{-4}	3.2×10^{4}	0.406
1.992	0.00127	0.00089	-1.3499	9.34×10^{-5}	4.7×10^{4}	0.405
1.995	0.00079	0.000597	-1.3563	4.139×10^{-5}	1.1×10^{5}	0.404
1.998	0.00032	0.000278	-1.3628	8.729×10^{-6}	5×10^{5}	0.403
1.999	0.000159	0.00013	-1.3651	2.726×10^{-6}	1.6×10^{6}	0.400

Table 10.1: Peak properties of the Green's function: $\omega_F \rightarrow$ Fermi frequency, $k_F \rightarrow$ Fermi momentum, $\Delta_F \rightarrow$ FWHM at Fermi point, $H_F \rightarrow$ Peak Height, $v_F \rightarrow$ Fermi velocity

these irrelevant terms have been investigated in [82] in perturbation theory. One expected effect is that the linear dispersion relation is corrected by non linear terms. This has the consequence that the interaction terms induce a finite imaginary part to the self energy correction and in some approximation it introduces a Lorentzian peak modifying the edge singularity of the Luttinger liquid. This is called a Fermi-Lutttinger liquid in [82] and was discussed in Sec.(10.2.0.4).

The holographic analysis also showed an interesting non Luttinger behavior in that the zero frequency behavior is peculiar. It has a singularity at $\omega = 0$ that is k-independent. In this section we attempt to reproduce this in the boundary theory.

Let us consider the following action:

$$S = \int dxdt[i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi] \tag{10.32}$$

ψ is the quasiparticle fermion field whose Green's function we are interested in determining. It has dimension half unlike the fermion operator that was studied in the last section, which had dimension one. Nevertheless this is a matter of detail and will not affect the main point of this section. ϕ is the bosonized version of some other charged fermion, Ψ. They are related by $\bar{\Psi}\gamma^{\mu}\Psi = \epsilon^{\mu\nu}\partial_{\nu}\phi$ (discussed in Chapter (8)).

The above action assumes Lorentz invariance and we have set $v = 1$. But because Lorentz invariance is absent one can expect the two quasiparticles to have different velocities:

$$S = \int dxdt[i\bar{\psi}(\gamma^0\partial_0 + \gamma^1 v_F\partial_1)\psi - \frac{1}{2}(\partial_0\phi\partial^0\phi + v_S^2\partial_1\phi\partial^1\phi)] \tag{10.33}$$

Both v_S and v_F can get renormalized by interactions. For instance adding a charge-charge interaction term for Ψ adds $(\partial_x\phi)^2$ to the action. This modifies v_S. In fact we will assume that v_S is renormalized to zero. This will be crucial as we will see below.

We now add an interaction term [2]:

$$\Delta S = g\int dxdt\ i\bar{\psi}\psi cos\ \beta\phi \tag{10.34}$$

We can perform a Wick rotation ($it_M = t_E$) to Euclidean space for calculations. Using the fact that $\langle\phi(x,t)\phi(0,0)\rangle = -\frac{1}{4\pi}ln\ (r^2 + a^2)$ we see that $cos\ \beta\phi = (a)^{\frac{\beta^2}{4\pi}} : cos\beta\phi :$ and has dimension $\frac{\beta^2}{4\pi}$. The interaction term that has been added is thus irrelevant when $\frac{\beta^2}{4\pi} > 1$. One also sees that

$$\langle: cos\beta\phi(x,t) :: cos\beta\phi(0,0) :\rangle = \frac{1}{2}\frac{1}{(x^2+t^2)^{\frac{\beta^2}{4\pi}}} \tag{10.35}$$

[2] Instead of $\bar{\psi}\psi$ one can add $\psi^{\dagger}\psi$ which would correspond to a density. This term is allowed since we do not have Lorentz invariance and also has an interpretation of some spin density interacting with some projection of a magnetic field, for instance. We thank Thomas Vojta for suggesting this.

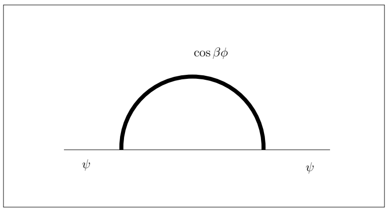

Figure 10.11: 1 loop correction to ψ propagator

The propagator in momentum space can be obtained by dimensional analysis and is (in Minkowski space) $G(\omega,k)_{cos\beta\phi} \approx [\omega^2 - v_s^2 k^2]^{\frac{\beta^2}{4\pi}-1}$. If we now assume v_s is renormalized to zero, we get $[\omega^2]^{\frac{\beta^2}{4\pi}-1}$

Thus a one loop correction (Fig.(10.11)) to the ψ two point function (inverse propagator) is:

$$\Sigma(p) \approx \int \frac{d\omega dk}{(2\pi)^2} \frac{(\omega + v_F k)}{(\omega^2 - v_F^2 k^2)((p^0-\omega)^2)^{\frac{\beta^2}{4\pi}-1}} \tag{10.36}$$

We can do the k integral to get $ln\ \omega$ and then do the ω integral to get $\Sigma(p^0,p^1) \approx \frac{[ln\ p^0 - C - \psi(p^0)]}{(p^0)^{\frac{\beta^2}{4\pi}-3}}$. This is a p^1 independent non-analyticity that one is seeing in the AdS computation. The crucial element that made this happen was the fact that the velocity of the Ψ was zero. This could correspond to some effective non propagating or localized particle.

11

Vector and Scalar Green's Function

11.1 Vector Green's Function

In this section we consider aspects of the vector field and its perturbation. This gives the current-current correlators of the boundary theory. We will calculate Green's function as a function of the frequency, keeping no x dependence (i.e.zero wave vector). This is obtained as the ratio of the two solutions in time dependent perturbations δA_x. The perturbations in A_x and g_{tx}, which we call $a(z,t)$ and $\delta(z,t)$ respectively obeys (we are assuming a time dependence of $e^{-i\omega t}$ for both a and δ) following set of equations (obtained from Einstein-Hilbert-Maxwell action (eqn.(9.1))):

$$\begin{aligned} f\partial_z(zf\partial_z a) + z\omega^2 a + Qf\partial_z(z^2\delta) &= 0 \\ zQa + \partial_z(z^2\delta) &= 0 \end{aligned} \tag{11.1}$$

Where the $f(z)$ is given by (9.10). We can combine the two equations (11.1) to get an equation for $a(r,t)$ - which is:

$$f^2a'' + f(\frac{f}{z} + f')a' + (\omega^2 - 2Q^2 f)a = 0 \tag{11.2}$$

We can now solve the above equation numerically for $a(z)$. It is convenient (to make the numerics stable) to factorize the oscillatory behavior at the horizon ($z = 1$) where $f(z) = 0$.

- **For Q $\neq$ 2:**

We can define $a(z) = (f(z)^{\frac{i\omega}{c}})\rho(z)$. The differential equation for $\rho(z)$ is:

$$f^2\rho'' + [\frac{f^2}{z} + (1 + 2i\frac{\omega}{c})ff']\rho'$$
$$+[\frac{i\omega}{c}ff'' + \frac{i\omega}{cz}ff' - \frac{\omega^2}{c^2}f'^2 + \omega^2 - Q^2 f]\rho = 0 \tag{11.3}$$

c gets fixed by requiring that $\rho(1)$ be non-zero. One finds $c^2 = (f'(1))^2$. The sign is fixed by requiring that the solution for a be ingoing at the horizon. This gives $c = -f'(1)$.

- **For $\mathbf{Q = 2}$:**
Now $f(z)$ has a double pole at $z = 1$, and the oscillatory behavior is different (which is determined by the near horizon limit of (11.2)). We define $a(z) = e^{\frac{i\omega}{2(1-z)}}(1-z)^{\frac{-i\omega}{6}}g(z)$. The positive sign in the exponential is chosen to make the solution in-going at the horizon. The differential equation for $g(z)$ is given by,

$$g''(z) + \Big(\frac{1}{z} + \frac{f'(z)}{f(z)} + \frac{i\omega}{(1-z)^2} + \frac{i\omega}{3(1-z)}\Big)g'(z)$$
$$+\Big[\frac{i\omega}{6}(1 + \frac{i\omega}{6})\frac{1}{(1-z)^2} + \frac{i\omega}{6(1-z)}\Big\{\frac{1}{z} + \frac{f'(z)}{f(z)} + \frac{i\omega}{(1-z)^2}\Big\}$$
$$+\frac{\omega^2}{f(z)^2} - \frac{\omega^2}{4(1-z)^4} + \frac{i\omega}{2(1-z)^2}\Big\{\frac{f'(z)}{f(z)} + \frac{2}{1-z}\Big\}$$
$$-\frac{8}{f(z)} + \frac{i\omega}{2z(1-z)^2}\Big]g(z) = 0 \tag{11.4}$$

Analyzing the leading behavior at $z = 0$ we see that the solution has degenerate eigenvalues and is given by $a(z) = a^+ + a^- \; ln(z)$. The Green's function for the conjugate operator $J(x,t)$ is given by the ratio $\frac{a^+}{a^-}$ (Some issues regarding the ambiguity in considering this ratio is discussed in Sec.(11.1.2)). The ratio between J^x and $\partial_t A_x = -i\omega A_x$ is the conductivity σ. So the Green's function divided by $-i\omega$ gives the conductivity (See fig. (11.1),(11.2)). Green's function is calculated numerically by solving the differential equations (11.3) and (11.4). We assume the functions $\rho(z)$ and $g(z)$ are power series in $(1-z)$ near horizon and also choose the normalization $\rho(1) = g(1) = 1$. Then $a^{\pm}$ is determined from the asymptotic nature $(z \to 0)$ of the numerical solution (See Appendix C). In the next subsection we will discuss about the low energy and zero temperature limit of the conductivity.

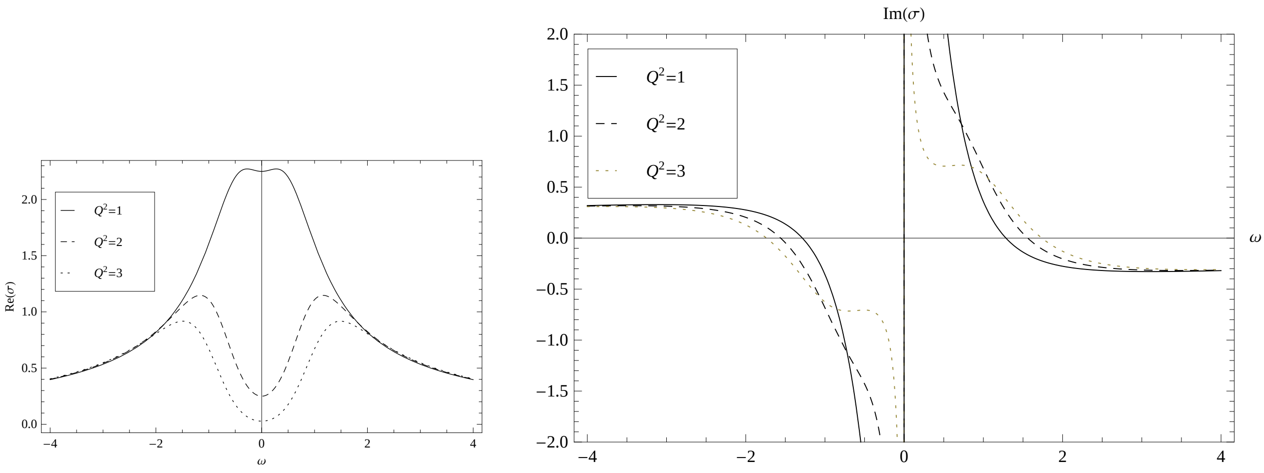

Figure 11.1: Conductivity at finite temperature

11.1.1 AdS_2 limit: $\omega \to 0$ limit of Conductivity at $T = 0$

As we have discussed for the fermion case, for this case also we do the same coordinate transformation (eqn.(9.11)) for the near horizon limit of the extremal black hole. So, in the near horizon AdS_2 the equation of motion for the gauge field perturbation (eqn.(11.2)) turns out to be,

$$\zeta^2 \frac{d^2 a}{d\zeta^2} + (\zeta^2 - 2)a = 0, \tag{11.5}$$

This equation is same as the equation for a scalar with $m^2 = 2$ and no charge. This can be analytically solved. The solution with ingoing boundary condition at the horizon becomes,

$$a = C\sqrt{\frac{2}{\pi}}\left[i\left(-\cos(\zeta) + \frac{\sin(\zeta)}{\zeta}\right) + \left(\sin(\zeta) - \frac{\cos(\zeta)}{\zeta}\right)\right], \tag{11.6}$$

where C is the integration constant. Solution near the boundary of AdS_2 $\zeta \to 0$ looks like

$$a = -\frac{1}{\zeta} - \frac{i}{3}\zeta^2 \tag{11.7}$$

So, ofter going back to the original co-ordinate the leading order ω behavior of the IR Green's function is given by,

$$\mathcal{G}_R(\omega) = \frac{i}{3}\omega^3 \tag{11.8}$$

. So, the real part of the conductivity is

$$Re\sigma(\omega \to 0) = \lim_{\omega \to 0} \frac{1}{\omega} Im\mathcal{G}_R(\omega) \sim \omega^2 \tag{11.9}$$

So, we see that a dimension 2 IR CFT operator determines the limiting behavior of the optical conductivity in low frequency limit of boundary $(1+1)$ dimensional field theory. This behavior should be contrasted with behavior of doped Mott insulator [138], where the real part of conductivity goes as ω^3. In addition, the imaginary part of σ has a pole at $\omega = 0$ which is also clear from the plot (Fig.11.2),

$$Im\sigma(\omega \to 0) \propto \frac{1}{\omega} \tag{11.10}$$

Now, from the well known Kramers-Kronig relation

$$Re(\sigma) \propto \delta(\omega) \tag{11.11}$$

at $\omega = 0$. From the numerical analysis (fig.11.2) we were not able to see the delta function peak. But this is a natural expectation that for any translational invariant theory the DC conductivity (for $\omega = 0$), should be infinity.

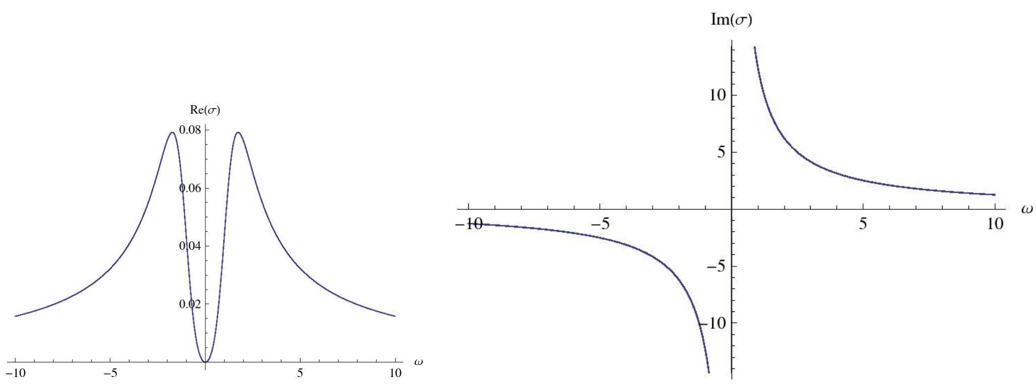

Figure 11.2: Conductivity at zero temperature

11.1.2 Comments on Vector Green's function

The solution for vector field A_t is given by the solution (eqn.(9.5)) (r is the usual radial coordinate and often we will use $z = 1/r$. $r = r_+$ is the location of the horizon).

$$A_t = -Q\ ln(r/r_+) = Q\ ln(zr_+) \tag{11.12}$$

If we consider the effective scalar given by $\phi_{eff} = l^2 \sqrt{g}^{tt} A_t \sim zA_t$ near boundary ($z \to 0$) we get

$$\phi_{eff} = Q\ z\ ln(z/z_0) + Q\ ln(z_0 r_+) z \tag{11.13}$$

z_0 is an arbitrary normalization scale - denoting violation of conformal invariance. This is a situation where the eigenvalues are degenerate so z and $z\ ln(z)$ are the two independent solution. There is some ambiguity regarding which solution corresponds to the source and which one to the expectation value of the conjugate operator, which in this case is the charge density. Thus if following [139] we take Q as the value of the source then the charge density is $Q\ ln(z_0 r_+)$. The analysis of the degenerate case for the scalar does seem to give this (as will be discussed in the end of this section). On the other hand reversing the roles gives Q as the charge density.

Let us see what thermodynamic arguments give. Consider the action (9.1) with the metric (9.4) in $r = 1/z$ coordinate,

$$ds^2 = -\frac{f(z)}{z^2} dt^2 + \frac{1}{f(z) z^2} dz^2 + \frac{1}{z^2} dx^2 \tag{11.14}$$

where $f(z) = \frac{1}{l^2} - 8GMz^2 + 8\pi G Q^2 z^2 ln(zl)^2$. Also $F_{tz} = \frac{Q}{z}$, $F^{tz} = -z^3 Q$. The equation of motion also gives:

$$R = -\frac{6}{l^2} - 4\pi G F^2 \tag{11.15}$$

Thus

$$S = \frac{1}{16\pi G} \int \frac{d^3x}{z^3} \left(-\frac{4}{l^2} - 8\pi G F^2\right) \tag{11.16}$$

Thus we get for the charge dependent part of the action :

$$S = V_1\beta Q^2 \int_\epsilon^{\frac{1}{r_+}} \frac{dz}{z^3}\ z^2 = -V_1\beta Q^2 ln(\epsilon r_+) \tag{11.17}$$

The chemical potential at the boundary is $A_t = \mu = Q\ ln(\epsilon r_+)$. Thus the free energy is $\beta\Omega = -V_1\beta Q^2 ln(\epsilon r_+) = -V_1\beta\frac{\mu^2}{ln(\epsilon r_+)}$. And the expectation value of the charge density is $-\frac{1}{V_1}\frac{\partial\Omega}{\partial\mu} = 2Q$. There is no dependence on r_+. This suggests that we should treat the coefficient of $z\ ln(z)$ as the expectation value and not the source. But we will follow the argument given in the following paragraph and consider coefficient of the log term as source.

Let us consider the case of the scalar field with degenerate eigenvalues. Thus if $f_1(z) \approx z^\lambda$ is a solution then the Frobenius prescription is to consider $\frac{df_1}{d\ \lambda}$ as the second solution. So we have a situation where $\phi(z) \approx a^+z^\lambda + a^-z^\lambda ln\ z + ...$ is the form of the solution where the dots indicate terms that are determined algebraically in terms of $a^\pm$. We have to decide which solution corresponds to the source and which one to the expectation value. The original idea was that one solution (non-normalizable) solution corresponds to the source and the other (normalizable) one gives the expectation value. This idea has been extended and it has also been pointed out that in some situations when both are normalizable one can have a duality between the two possibilities and that both are valid. In our case the solution $z^\lambda ln\ z$ is more divergent and is best taken as the source. In that case the Green's function is $\frac{a^+}{a^-}$. One can consider the case where we have $\lambda^\pm$ wit $\lambda^+ \approx \lambda^-$. Then take the two independent solutions to be $a^+z^{\lambda^+}$ and $a^-\frac{z^{\lambda^+}-z^{\lambda^-}}{\lambda^+-\lambda^-}$ - anticipating that we are going to set $\lambda^+ = \lambda^-$. If one then applies the prescription of [67] with ϕ normalized to one at the boundary one gets, on setting $\lambda^+ = \lambda^- = \lambda$,

$$G = \frac{\partial_z\phi(\epsilon)}{\phi(\epsilon)}$$

$$= \frac{a^+\lambda + a^- - a^-\lambda ln\ \epsilon}{a^+ + a^-ln\ \epsilon} \tag{11.18}$$

When $\epsilon \to 0$ we get a leading piece λ, which is uninteresting because it is analytic in momenta, and the sub leading non analytic piece $\frac{\lambda a^+}{a^-}$. In case $\lambda = 0$ we get

$$\frac{a^-}{a^+ + a^- ln\ \epsilon} = \frac{a^-}{(\frac{a^+}{a^- ln\ \epsilon} + 1)a^- ln\ \epsilon} \tag{11.19}$$

So the leading non-analytic piece is again $\frac{1}{ln^2\ \epsilon}\frac{a^+}{a^-}$. Thus we can conclude that in general for degenerate cases $G = \frac{a^+}{a^-}$ as considered in the Sec.11.1.

11.1.3 Complex Scalar Green's function

In this section we consider the complex scalar field in the background geometry (9.10) and derive the Green's function numerically for the dual operators in the boundary. We first set up the analytical equations in the following subsection and then discuss the numerical results. The action for the scalar field in the background (9.10)) is given by,

$$S = \int dtdx \int_0^1 dz \frac{1}{z}\left[f(z)|\partial_z\Phi|^2 - f(z)^{-1}|D_t\Phi|^2 + |\partial_x\Phi|^2 + \frac{m^2}{z^2}|\Phi|^2\right]$$

where, $D_t = \partial_t - iqA_t$. Considering the ansatz for $\Phi(z,x,t)$:

$$\Phi(z,x,t) = \int \frac{dkd\omega}{(2\pi)^2} e^{i(kx-\omega t)}\phi_k(z) \tag{11.20}$$

we get the following equation of motion:

$$A(z)\phi_k^{''}(z) + B(z)\phi_k^{'}(z) + V(z)\phi_k(z) = 0 \tag{11.21}$$

$$\begin{aligned} A(z) &= z^2 f(z) \\ B(z) &= z[f^{'}(z)z - f(z)] \\ V(z) &= z^2[-k^2 + \frac{1}{f(z)}(\omega + qQln(z))^2] - m^2 \end{aligned} \tag{11.22}$$

To minimise errors in numerical computations it will be helpful to factor out the oscillations near the horizon be writing $\phi_k(z) = f(z)^{\nu} S(z)$. By expanding the differential equation about the horizon value of z one can fix ν to be $-i\omega/4\pi T$ where, T is the temperature of the black hole obtained in section 2. The minus sign is chosen for the ingoing solution at the horizon. The resulting differential equation in $S(z)$ is,

$$S^{''}Af^2 + S^{'}[2\nu Aff^{'} + Bf^2] + S[(\nu(\nu-1)Af^{'2} + \nu Aff^{''} + \nu Bff^{'} + Vf^2] = 0 \tag{11.23}$$

The solution to the equation (11.21) near the boundary, $z \to 0$, for non-zero mass of the scalar field behaves as, (we have dropped the log term coming from the potential term, $V(z)$ in (11.21) assuming a finite mass for the scalar field [1])

$$\phi(z) = a^{-}(k,\omega) z^{d/2-\Delta} + a^{+}(k,\omega) z^{\Delta} \tag{11.24}$$

Here $d = 2$ is the boundary space-time dimension and $\Delta = d/2 + \sqrt{(d/2)^2 + m^2}$. As per [139] the boundary Green's function is given by, a^{+}/a^{-}. The solution for $\phi(z)$ is obtained numerically and the boundary Green's function can be written as,

$$\begin{aligned} G(k,\omega) &= \frac{a^{+}(k,\omega)}{a^{-}(k,\omega)} \\ &= \frac{z^{d-2\Delta}}{2\Delta - d}\left[(\Delta - d) + z\frac{\phi_k^{'}(z)}{\phi_k(z)}\right] \quad ; \; z \to 0 \end{aligned} \tag{11.25}$$

For the extremal case, $(Q = 2)$ a slight modification of the above procedure is needed due to the presence of degenerate horizons. We write $\phi_k(z) = e^{\nu_1/(1-z)}(1-z)^{\nu_2} S(z)$, where $\nu_1 = i\omega/2$ and $\nu_2 = i(q - \omega/6)$. The signs are fixed so that the infalling solution is chosen at the horizon. Again this is done to factor out the oscillations near the horizon. Like the finite temperature case the resulting differential equation (11.26) can now be solved numerically (See Appendix C).

[1] In the following numerical computations we have set $m^2 = 1/4$.

The boundary Green's function is given by (11.25).

$$
\begin{aligned}
S^{''}A(z-1)^4 &+ S^{'}[(z-1)^2\{2A(\nu_1+\nu_2(z-1))+B(z-1)^2\}]+ \\
S[A\{\nu_1^2 &+ 2\nu_1(\nu_2-1)(z-1)+(\nu_2-1)\nu_2(z-1)^2\} \\
&+ B\{(z-1)^2(\nu_1+\nu_2(z-1))\}+V(z-1)^4]=0
\end{aligned}
\tag{11.26}
$$

11.1.4 AdS_2 limit : $\omega \to 0$ limit of Scalar Green's function at $T=0$

The wave equation is

$$
\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}D_\nu\phi) - iqA^\mu D_\mu\phi - m^2\phi = 0, \tag{11.27}
$$

Where, $D_\mu = \partial_\mu - iqA_\mu$ is the covariant derivative. The scalar field has charge q under the background $U(1)$ gauge field. So, the explicit form of the equation of motion for the scalar field is

$$
z^3\partial_z\left(\frac{f(z)}{z}\partial_z\phi\right) + \left[\frac{z^2}{f(z)}(\omega+qA_t)^2 - m^2 - z^2k^2\right]\phi = 0 \tag{11.28}
$$

Where, we substitute as before $\phi(z,t,x)=\phi(z)e^{-i\omega t+ikx}$.

Now, we want to calculate the near extremal horizon behavior of the scalar field in the low frequency limit. For this, we first consider the scale transformation (eqn.(9.11)) which leads to the near horizon metric to be $AdS_2 \times R$. The background metric looks like (eqn.(9.12)).The radius of AdS_2 subspace $R_2 = 2$ in the unit of background AdS radius which we set to unity. Using this near horizon values for the metric components (eqn.(9.12)), the equation becomes

$$
\frac{d^2\phi}{d\zeta^2} + \left[\left(1-\frac{\mu}{2\zeta}\right)^2 - \frac{1}{2\zeta^2}(m^2+k^2)\right]\phi = 0, \tag{11.29}
$$

where, $\mu = 2q$. It is important to note that the above equation is same as a charged scalar field equation of mass $m_k^2 = m^2+k^2$ in a background AdS_2 space (9.12).

The asymptotic behavior of this equation may be obtained by taking $\zeta \to 0$

and one finds

$$\phi(r) \approx A\zeta^{\lambda_+} + B\zeta^{\lambda_-} \tag{11.30}$$

where λ are eigenvalues of

$$\begin{aligned} \lambda(\lambda-1) + \frac{1}{4}[\mu^2 - 2(m^2+k^2)] &= 0 \\ \implies \lambda_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+2(m^2+k^2)-\mu^2} \end{aligned} \tag{11.31}$$

If we now substitute $\zeta = \frac{\omega}{2(1-z)}$, we get

$$\phi(z) \approx A\omega^{\lambda_+}(1-z)^{-\lambda_+} + B\omega^{\lambda_-}(1-z)^{-\lambda_-} \tag{11.32}$$

From this we can conclude that the Green's functions in the AdS_2 (ratio of A/B) goes as $\omega^{\lambda_+-\lambda_-} = \omega^{2\nu}$ where $\nu = \sqrt{\frac{1}{4} + \frac{m^2+k^2}{2} - \frac{\mu^2}{4}}$. This is just the same as the analysis of [75], who use this to further conclude that the imaginary part of the full Green's function has this scaling behavior near $\omega = 0$ (Section (9.1.2)). If ν is not real one has also the interesting periodicity in $\log(\omega)$.

We should note that the effective mass of the charged scalar field in AdS_2 in the presence of the gauge field is $(\frac{m^2+k^2}{2} - \frac{q^2Q^2}{4})$. The condition for log-periodicity implies that this effective mass is lower than the BF bound in AdS_2 which is $-1/4$. This means that for the choice of the parameters which exhibit log-periodicity of the Green's function the scalar field is unstable.

Figure 11.3 shows that the log-periodic behavior (in ω) of the real and the imaginary parts of the Green's function. We have plotted these for $q = 2$, $k = 0$ and for $m^2 = 1/4$. Note that this value of mass is well above the BF bound for the scalar field in AdS_3 (which is -1), although the effective mass in AdS_2 is unstable. The scalar field though asymptotically stable is unstable in the near-horizon AdS_2 region. We expect that this instability will lead to the condensation of the scalar field. In the case of charged scalars in AdS_4 this instability leads to a transition to a hairy black hole phase which in the dual theory has been identified as the transition to a superconductor phase (see [140] for a nice review). This needs detailed analysis, some progress have been made recently in this direction by[141].

The imaginary part of the Green's function becomes negative for smaller values of ω. We presume that this is due to instabilities occurring from the fact that the

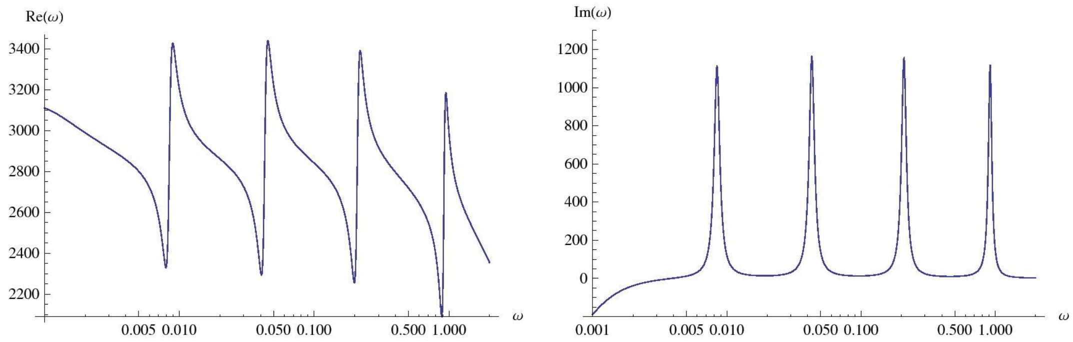

Figure 11.3: Re$G(\omega)$ and Im$G(\omega)$ for $k = 0$, $q = 2$ and $m^2 = 1/4$. The plots are with respect to $\log(\omega)$, however the markings on the horizontal axis are that of ω.

effective AdS_2 mass for the scalar is tachyonic in the region of parameter values where the Green's function shows log-periodicity. However the log-period that appears in the plot matches with the analytical value which in this case is 1.65.

11.1.5 Numerical results

Let us now note some general properties of the Green's functions before discussing the numerical results. From the differential equations it can be seen the Green's function is even in k, $G(\omega, k, qQ) = G(\omega, -k, qQ)$. For $Q = 0$, complex conjugation of the Green's function has the same effect as $\omega \to -\omega$. This implies that Im $G(\omega, k) = -$ Im $G(-\omega, k)$, and Re $G(\omega, k) =$ Re $G(-\omega, k)$. For non-zero values of qQ, this symmetry is lost due to the presence of the covariant derivative, however, $G(\omega, k, qQ) = G^*(-\omega, k, -qQ)$. The asymmetric behavior is visible in all the plots for nonzero Q. We now turn to our numerical results.

Zero temperature: Figure 11.4 shows the real and imaginary parts of the boundary Green's function for the extremal case, $Q = 2$ or $T = 0$, $q = 1$ and for various values of k. At zero temperature, the Green's function has peaks lying between $\omega = 0$ and $\omega = 0.05$. These peaks shift towards $\omega = 0$ as k is increased which disappear for larger values of k. The plot (Figure 11.4) shows a range of k for which the peaks exist. We would like to also point out that the values of the parameters considered here lie in the region where the Green's function is periodic in $\log(\omega)$ for small ω. As discussed in more detail in the Section (11.1.4).

Finite temperature: For a fixed value of k, the peaks lying in the positive ω region smoothen out beyond a particular temperature. Figure 11.5 shows the plot

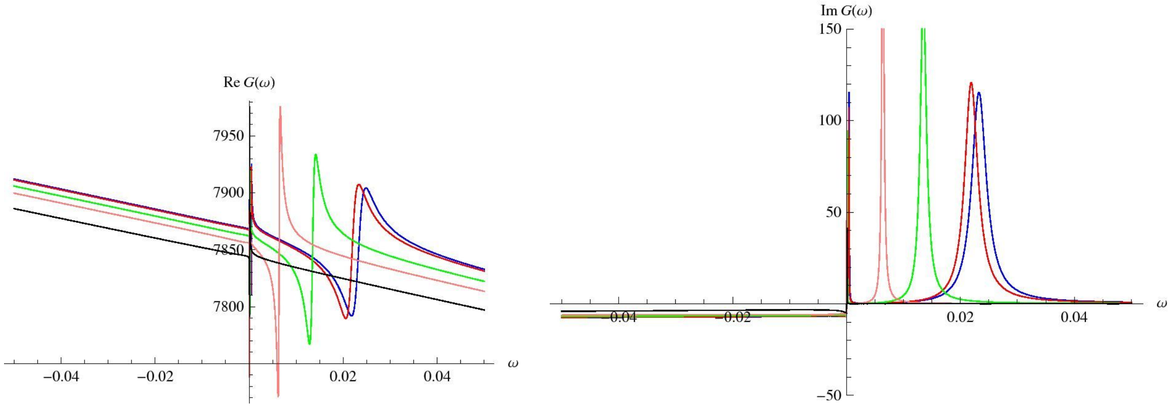

Figure 11.4: Re$G(\omega)$ and Im$G(\omega)$ for $Q = 2$ or $T = 0$ and for the following values of k: 0.1 (Blue), 0.2 (Red), 0.5 (Green), 0.7 (Pink), 1 (Black)

for the real and the imaginary parts of the Green's functions for various values of Q or T and $k = 0$ and for $q = 1$. The Green's function is very sensitive to variations in temperature near the zero temperature region. The number of peaks reduce to one in the finite temperature regions for a particular value of k. Eventually this peak smoothens out as the temperature is increased further. At the highest temperature $Q = 0$, the curves (not shown in the figure) have the symmetries mentioned at the beginning of this section. For large values of ω, the functions are monotonically increasing or decreasing. Like the zero temperature case the peaks shift towards the origin of ω and then disappear for larger values of k. In fact, like at zero temperature there exists a range of k for which there are peaks. This window of k for which the peaks exist shrinks as the temperature is increased.

Large ω: In the large ω limit for fixed k the Green's function should behave as, $|G(\omega)| \sim \omega^{\alpha(m)}$, With $\alpha(m) = 2\Delta - d$. This scaling behavior can be verified numerically for our Green's function. We get the following results from our numerical computation at finite temperature: $\alpha(0.5) = 2.229$, $\alpha(1) = 2.830$, $\alpha(1.5) = 3.627$. These may be compared to the analytical values: $\alpha(0.5) = 2.236$, $\alpha(1) = 2.828$, $\alpha(1.5) = 3.605$. We have verified that results correct to the first decimal place can be obtained for the real and imaginary parts of the Green's function both for finite and zero temperatures. Similarly the same scaling can be checked for large k with fixed finite ω. We get the numerical answers correct to the first decimal place as above.

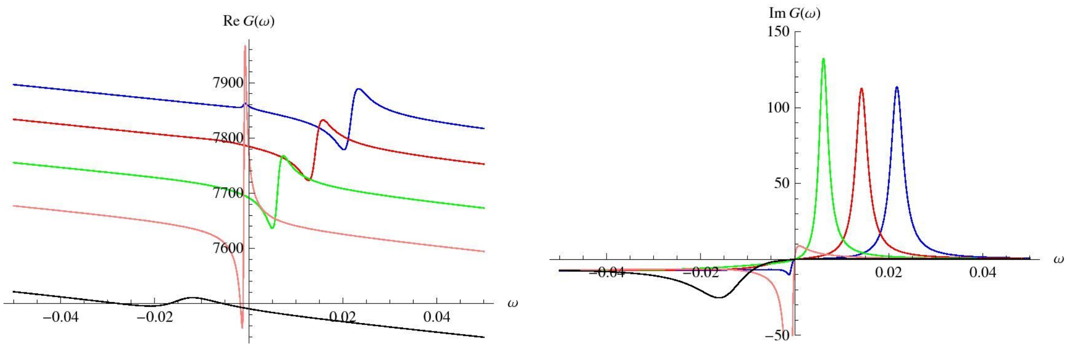

Figure 11.5: Re$G(\omega)$ and Im$G(\omega)$ for $k = 0$ and for the following values of Q: 1.998 (Blue), 1.99 (Red), 1.98 (Green), 1.97 (Pink), 1.95 (Black)

12
Conclusions II

We have attempted to understand the behavior of Green's functions of a 1+1 field theory with some background charge density using the AdS/CFT correspondence. The forms of Green's functions of scalars, fermions and currents have been obtained at zero as well as non-zero temperatures. The fermion Green's function shows interesting behavior - not typical of a Landau Fermi liquid and can be termed non-Fermi liquid. In 1+1 dimensions one may expect Luttinger liquid behavior under some circumstances. At high frequencies this is seen. However at low frequencies the behavior is quite different and seem to have similarities with Fermi-Luttinger Liquid. We have also made detailed comparison for $T = 0, \;\; \omega \to 0$ with expectations from AdS_2 and found reasonable agreement. In particular the intriguing log periodicity found in [73, 75] is also seen. We have also suggested a possible explanation for the k-independent non-analyticity at $\omega = 0$ in the fermion (or scalar) Green's function from the point of view of the boundary theory. The non-analyticity can be explained if there are modes with almost zero velocity (i.e. non propagating) that interact with these fermions. These could be impurities for instance.

Part V

Summary

13

Summary and Scope

In this thesis we have studied two applications of AdS/CFT conjecture. In the first part of the thesis we have used the conjecture to understand the IR cut-off appearing in the calculation of partition function of matrix model for strings. This apparently ad hoc introduction of IR cut-off is crucial to investigate phase transition in matrix model, and hence the Hagedorn transition. In the second part, we have used AdS/CFT conjecture to analyze the duality between Charged BTZ black hole and Luttinger liquids. We will here summarize the results and discuss some future directions.

13.1 Hagedorn transition in Matrix Model for Strings

In this part of the thesis (Chapter 4-6), we have studied 1-loop partition function of DLCQ M-theory for two phases : Long D-strings and Clustered. By comparison of free energy we found two possible phase transitions. As we increase the temperature from zero, String phase dominates, at a temperature T_H this Long D-strings "Cluster", which we identify as the Hagedorn transition. As we increase the temperature, there is a Gregory-Laflamme kind of transition to again string phase. This may be the reminiscent of $T \to \frac{1}{T}$ symmetry discussed in [45, 39], as the "second Hagedorn transition" where the thermal tachyon vanishes. It was also recently found in analysis of thermodynamics of p-adic string theory in [142, 143]. A proper understanding for this duality symmetry in temperature is still lacking.

We have found that $T_H \sim \frac{1}{L_0}$, where L_0 is the IR cutoff of the Yang-Mills theory which needs to be introduced to make the calculations well defined. This can be an artifact of the perturbation theory. We can in principle use AdS/CFT correspondence for $D0$-branes, to construct supergravity duals for the matrix model or the SYM theory configurations we have considered, which may throw light on the origin of this IR cut-off. In this thesis, we have demonstrated for simple supergravity model that such an IR cut-off has to be introduced in the dual theory if we ignore finite g_s corrections. But an exact analysis for the supergravity models for the configurations considered in this thesis is yet to be done. Also simple parameter counting shows that the BFSS matrix model needs one more dimensionful parameter if it is to be compared with string theory at finite g_s, so the IR cutoff L_0 can be thought of as one choice for this extra parameter. Higher loop calculations for the partition function may also resolve this issue.

In our analysis we have considered the background of time component of the gauge field to zero. Introduction of a finite value for the background (chemical potential) removes the necessity of IR cut-off, by introduction of another scale given by chemical potential. Our result is the special case of zero chemical potential. The effect of non-zero background of A_0 was studied in a different context in [21]. The effect of non-zero chemical potential needs to be analyzed for our case. Also we have considered only two possible configurations, a complete analysis of the phase structure for matrix model is lacking.

13.2 Duality between Charged BTZ black hole and Luttinger liquid

In the second part of the thesis (Chapter 9-12) we have studied the behavior of Green's functions of a $1+1$ field theory with some background charge density using the AdS/CFT correspondence. We have studied Green's functions of scalars, fermions and currents in the boundary theory have been obtained at zero as well as non-zero temperatures.

So far the use of AdS/CFT in Condensed matter was limited to the study of boundary theory where calculations can not be done, but in $1+1$ dimension due to presence of various analytic tools like Bosonization, our aim was to explore the duality from both sides.

The fermion Green's function shows interesting behavior of Fermi-Luttinger liquid, which is Luttinger liquid with a non-linear term in dispersion. Our hope is that this Green's may be equivalent to a strong coupling limit of this Fermi-Luttinger liquid.

We were also able to suggest a possible explanation for the k-independent non-analyticity at $\omega = 0$ in the fermion (or scalar) Green's function from the point of view of the boundary theory, by constructing a toy model. It was seen in this toy model, the non-analyticity can be explained if there are modes with almost zero velocity (i.e. non propagating) that interact with these fermions. These could be impurities for instance. Further progress can be made by incorporating more features of the Green's function in our toy model.

Another immediate extension of our work, will be to construct Green's function corresponding to massive fermions, which we were unable to presently due to some technical problems in constructing proper numerics. Bulk mass corresponds to scaling dimension of boundary operators, so massive fermion analysis will provide an opportunity to study operator with scaling dimension identical to free electrons.

As a test of our numerics we have compared it with various limits where analytical results can be obtained, and it shows a very reasonable match. In particular the intriguing log periodicity found in [73, 75] is also seen.

The conductivity behavior is seem to qualitatively match with doped Mott insulator, but a detailed comparison needs to explored.

Finally there is the obvious question of understanding what experimental setup would correspond to these theories. There is an ample amount of experimental systems available in $1+1$ dimensions, like edge modes of quantum Hall liquid, carbon nanotubes, cleaved edge semiconductor wires, antiferromagnetic spin chains, and cold atoms in 1D optical traps. Connections with these experiments may provide "experimental" verification of AdS/CFT conjecture!

Part VI

Appendix

A

Dimensional Reduction of 10-dimensional Super-Yang-Mills Theory

We will review the dimensional reduction of super Yang-Mills theory following [90, 85].

A.1 10d Super Yang Mills (SYM)

$(9+1)$-dimensional $U(N)$ $\mathcal{N}=1$ super Yang-Mills action with metric $\eta_{\mu\nu} = diag(+1,-1,\cdots,-1)$ (μ,ν runs from $0,1,\cdots,10$) is given by,

$$S = \frac{1}{g_{YM}^2}\int d^{10}x\left[-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \frac{i}{2}\bar{\Psi}^a\Gamma^\mu D_\mu\Psi^a\right] \tag{A.1}$$

where [1],

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \\ D_\mu\Psi &= \partial_\mu\Psi - i[A_\mu,\Psi] \\ A_\mu &= \mathbb{I}A^0_\mu + T^a A^a_\mu \\ \Psi &= \mathbb{I}\Psi^0 + T^a\Psi^a \end{aligned} \tag{A.2}$$

The gauge potential A_μ and Majorana-Weyl spinor Ψ are in the adjoint representation of the gauge group $U(N)$. T^a is the generator of the $SU(N)$ part of the gauge

[1] Summation over repeated index implied unless otherwise stated

group satisfying $[T^a, T^b] = if^{abc}T^c$, where f^{abc} is completely anti-symmetric structure constants. $\mathbb{I}$ is the $N \times N$ Identity matrix and (A^0_μ, Ψ^0) represents $U(1)$ part of $U(N)$. In the following analysis we will suppress the $U(N)$ indices. The 32×32 Dirac matrices Γ_μ ($\mu = 0, \cdots, 9$) satisfies $\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}$. The supersymmetry transformations that leaves the action invariant is given by,

$$\begin{aligned} \delta A_\mu &= i\bar{\epsilon}\Gamma_\mu\Psi \\ \delta\Psi &= \frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon \end{aligned} \tag{A.3}$$

where ϵ is an constant anticommuting Majorana-Weyl spinor and $\Gamma_{\mu\nu} = \frac{1}{2}[\Gamma_\mu, \Gamma_\nu]$. We will rewrite the gamma matrices in the following form,

$$\begin{aligned} \Gamma_0 = \Gamma^0 &= \mathbb{1} \otimes \sigma_2 \\ \Gamma_i = -\Gamma^i &= \gamma_i \otimes i\sigma_1 \\ \Gamma_s = \Gamma^0 \cdots \Gamma^9 &= \mathbb{1} \otimes \sigma_3 \end{aligned} \tag{A.4}$$

where, γ_i are 9 real symmetric 16×16 matrices satisfying $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. $\mathbb{1}$ is the 16-dimensional identity matrix and $\sigma_{1,2,3}$ are the usual Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.5}$$

The first 8 γ matrices can be identified with the Dirac matrices of $spin(8)$ and the last with 8 dimensional chirality. We will also use 16 component spinors ψ defined by

$$\Psi = \psi \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{A.6}$$

The action now becomes,

$$S = \frac{1}{g^2_{YM}} \int d^{10}x \; Tr\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\psi^T D_0\psi - \frac{i}{2}\psi^T\gamma_i D_i\psi\right] \tag{A.7}$$

For reduction to $0+1$-dimensions, all the fields are taken independent of all space directions. The volume term in the direction which the reduction is done is dropped. The action (eqn.(A.1)) becomes (T denotes transpose),

$$S = \frac{1}{2g_{YM}^2}\int dtTr\left[(D_0A_i)^2 + \frac{1}{2}[A_i, A_j]^2 + i\psi^T D_0\psi - \psi\gamma_i[A_i, \psi]\right] \tag{A.8}$$

Let us now define $A_i = \frac{X_i}{2\pi\alpha'}$ and $\psi = \frac{\theta}{(2\pi\alpha')^{\frac{3}{2}}}$. The action becomes,

$$S = \frac{1}{2g_sl_s}\int dt\ Tr\left[(D_0X_i)^2 + \frac{1}{2}\frac{1}{(2\pi\alpha')^2}[X_i, X_j]^2 + i\psi^T D_0\psi - \frac{1}{(2\pi\alpha')}\psi\gamma_i[A_i, \psi]\right] \tag{A.9}$$

where, $g_{YM}^2 = \frac{g_s}{4\pi^2 l_s^3}$. The above action is same as the one considered in BFSS matrix model (eqn.(2.1)).

A.2 From 10 to $p+1$

Consider all fields in 10d SYM action (eqn.(A.1)) independent of $x_{p+1}, \cdots, x_{10}$. Let us consider the indices $\mu, \nu, \cdots$ rums from $0, 1, \cdots, p$ and $i, j, \cdots$ from $(p+1), \cdots, 10$. Then the dimensional reduction gives,

$$S = \frac{1}{g_{YM}^2}\int d^{(p+1)}x\ Tr\left[\frac{1}{2}(D_\mu A_i)(D^\mu A_i) + \frac{1}{4}[A_i, A_j]^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\psi^T \not{D}\psi - \frac{1}{2}\psi^T\gamma_i[A_i, \psi]\right] \tag{A.10}$$

where $\not{D} = \mathbb{1}D_0 - \gamma_\mu D_\mu$ and $\mathbb{1}$ is 16×16 identity matrix.

B

Background Gauge Fixing

Consider dimensionally reduced maximally supersymmetric $U(N)$ Yang-Mills theory in 10-dimension to (p+1) dimensions (Appendix A),

$$S_{YM} = \int d^{p+1}\mathcal{L}_{YM} = \frac{1}{2g_{YM}^2}\int d^{p+1}xTr\{(D_\mu A_i)(D^\mu A_i) + \frac{1}{2}[A_i, A_j]^2 \\ -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + i\theta^T \not{D}\theta - \theta^T\gamma_i[A_i, \theta]\} \qquad \text{(B.1)}$$

where $\not{D} = \mathbb{1}D_0 - \sum_{j=1,\cdots,p}\gamma_j D_j$ and $\mathbb{1}$ is 16×16 identity matrix. $\eta_{\mu,\nu} = (1,-1,-1,\cdots,-1)$ and γ_i $(i = 1,\cdots,8)$ are 16×16 Dirac matrices of $spin(8)$. $\gamma_9 = \gamma_1\cdots\gamma_8$ is the corresponding chirality matrix. The field content is given by the gauge potential A_μ, 16 component Majorana-Weyl spinor θ and $10-p$ scalars A_i, all in the adjoint representation of $U(N)$. Also, $D_\mu\bullet = \partial_\mu\bullet - i[A_\mu, \bullet]$, where $\bullet$ can be replaced by any of (A, θ). We will review the background gauge fixing method in this case following [144].
Let us consider,

$$\begin{aligned} A_\mu &= a_\mu + A'_\mu \\ A_i &= a_i + A'_i \\ \theta &= \Theta + \theta' \end{aligned} \qquad \text{(B.2)}$$

Where a_μ, a_i, Θ are background fields obeying classical equation of motion. Let us define a new covariant derivative as $\bar{D}_\mu = \partial_\mu - i[a_\mu, \bullet]$. The primed fields are quantum fluctuations which is integrated out in the path integral to calculate partition function. Also,

$$F_{\mu\nu} = \bar{F}_{\mu\nu} + (\bar{D}_\mu A'_\nu - \bar{D}_\nu A'_\mu) - i[A'_\mu, A'_\nu] \tag{B.3}$$

where, $\bar{F}_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - i[a_\mu, a_\nu]$.
Now the allowed gauge transformation is the ones which keep the background unchanged, i.e. $\delta a_\mu = \delta a_i = \delta\Theta = 0$.Then the gauge transformation on the fluctuations are given by,

$$\begin{aligned} \delta A'_\mu &= \bar{D}_\mu \alpha - i[A'_\mu, \alpha] \\ \delta A'_i &= -i[A'_i, \alpha] \\ \delta\theta' &= -i[\theta', \alpha] \end{aligned} \tag{B.4}$$

The gauge fixing condition we use is,

$$\bar{D}_\mu A^{\mu\prime} = 0 \tag{B.5}$$

therefore the Gauge fixing Lagrangian,

$$\mathcal{L}_{gf} = -\frac{1}{2g_{YM}^2 \xi} Tr(\bar{D}_\mu A^{\mu\prime})^2 \tag{B.6}$$

where ξ is an arbitrary parameter ($\xi = 0$ gives Landau gauge and $\xi = 1$ Feynman gauge) and the ghost Lagrangian,

$$\mathcal{L}_{gh} = \frac{1}{2g_{YM}^2} Tr\{(\bar{D}_\mu \bar{\omega})(\bar{D}^\mu \omega - i[A^{\mu\prime}, \omega])\} \tag{B.7}$$

Where ω and $\bar{\omega}$ are ghost and anti ghost respectively. Consider the background such that $a_i = 0$, $\Theta = 0$ and $\Omega = \bar{\Omega} = 0$, where Ω is the background for ghost. Let us consider the Lagrangian up to quadratic in quantum fluctuation or the 1-loop Lagrangian. We also use the classical equation of motion for the background fields.

We get,

$$\begin{aligned}
\mathcal{L}^{(1)}_{YM} &= \frac{1}{2g^2_{YM}} Tr\{(\bar{D}_\mu A'_i)^2 - (\bar{D}_\mu A'_\nu)^2 - \frac{1}{2}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} - iA'_\mu A'_\nu \bar{F}^{\mu\nu} + (\bar{D}_\mu A^{\mu\prime})^2 + i\theta^{T\prime}\bar{\not{D}}\theta'\} \\
\mathcal{L}^{(1)}_{gf} &= -\frac{1}{2g^2_{YM}\xi} Tr(\bar{D}_\mu A^{\mu\prime})^2 \\
\mathcal{L}^{(1)}_{gh} &= \frac{1}{2g^2_{YM}} Tr\{(\bar{D}_\mu \bar{\omega}')(\bar{D}^\mu \omega')\}
\end{aligned} \tag{B.8}$$

Then the full gauge fixed 1-loop Lagrangian in Feynman gauge ($\xi = 1$) is given by,

$$\begin{aligned}
\mathcal{L}^{(1)} &= \frac{1}{2g^2_{YM}} Tr\{(\bar{D}_\mu A'_i)^2 - (\bar{D}_\mu A'_\nu)^2 - \frac{1}{2}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} - iA'_\mu A'_\nu \bar{F}^{\mu\nu} \\
&+ i\theta^{T\prime}\bar{\not{D}}\theta' + (\bar{D}_\mu \bar{\omega}')(\bar{D}^\mu \omega')\} \\
&= \frac{1}{2g^2_{YM}} Tr\{-A'_i(\bar{D})^2 A'_i + A'_\nu(\bar{D})^2 A^{\nu\prime} - \frac{1}{2}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} - iA'_\mu A'_\nu \bar{F}^{\mu\nu} \\
&+ i\theta^{T\prime}\bar{\not{D}}\theta' - \bar{\omega}'(\bar{D})^2\omega')\}
\end{aligned} \tag{B.9}$$

Now consider the case $p = 1$ *i.e.* $(1+1)$-dimensional SYM, then the above action becomes,

$$\begin{aligned}
\mathcal{L}^{(1)} &= \frac{1}{2g^2_{YM}} Tr\{-A'_i(\bar{D})^2 A'_i + A'_0(\bar{D})^2 A'_0 - A'_9(\bar{D})^2 A'_9 + \bar{F}^2_{09} + 2iA'_0 A'_9 \bar{F}_{09} \\
&+ i\theta^{T\prime}\bar{\not{D}}\theta' - \bar{\omega}'(\bar{D})^2\omega')\}
\end{aligned} \tag{B.10}$$

where $\mu = 0, 9$ are the two dimensions of the SYM, $\bar{D}^2 = D_0^2 - D_9^2$, and $i = 1, \cdots, 8$. Let $\theta' = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$, where $\theta_{1,2}$ are eight component spinors. Also consider $\mathbb{1} = \begin{pmatrix} \mathbb{1}_{8\times 8} & 0 \\ 0 & \mathbb{1}_{8\times 8} \end{pmatrix}$ and $\gamma_9 = \begin{pmatrix} \mathbb{1}_{8\times 8} & 0 \\ 0 & -\mathbb{1}_{8\times 8} \end{pmatrix}$. Then, $\theta^{T\prime}\bar{\not{D}}\theta' = \theta_1^T(D_0 - D_9)\theta_1 + \theta_2^T(D_0 + D_9)\theta_2$ where, $\theta^*_{1,2} = \theta_{1,2}$.

The Euclidean action is defined by $S_E = -iS(t = -i\tau, A'_0 = iA'_\tau)$ (where, path integral is defined as $Z = \int \mathcal{D}[\bullet]e^{iS[\bullet]} = \int \mathcal{D}[\bullet]e^{-S_E[\bullet]}$ and τ is considered real),

$$\begin{aligned}
S^{(1)}_E &= -\frac{1}{2g^2_{YM}} \int d\tau \; Tr\{A'_i(\bar{D}_E)^2 A'_i + A'_\tau(\bar{D}_E)^2 A'_\tau + A'_9(\bar{D}_E)^2 A'_9 - \bar{F}^2_{\tau 9} - 2iA'_\tau A'_9 \bar{F}_{\tau 9} \\
&- \theta_1^{T\prime}(\bar{D}_\tau - i\bar{D}_9)\theta'_1 - \theta_2^{T\prime}(\bar{D}_\tau + i\bar{D}_9)\theta'_2 + \bar{\omega}'(\bar{D}_E)^2\omega')\}
\end{aligned} \tag{B.11}$$

where $(\bar{D}_E)^2 = (\bar{D}_\tau)^2 + (\bar{D}_9)^2$. Consider the case $a_\tau = 0, a_9 = constant$, then $\bar{F}_{\tau 9} = 0$ and the action becomes,

$$\begin{aligned} S_E^{(1)} &= -\frac{1}{2g_{YM}^2}\int d\tau\ Tr\{A_i'(\bar{D}_E)^2A_i' + A_\tau'(\bar{D}_E)^2A_\tau' + A_9'(\bar{D}_E)^2A_9' \\ &- \theta_1^{T\prime}(\bar{D}_\tau - i\bar{D}_9)\theta_1' - \theta_2^{T\prime}(\bar{D}_\tau + i\bar{D}_9)\theta_2' + \bar{\omega}'(\bar{D}_E)^2\omega')\} \end{aligned} \tag{B.12}$$

Which gives the 1-loop Euclidean partition function as (Note: as θ and θ^T are not independent, the path integral of the fermionic fluctuation will also give a factor of $\frac{1}{2}$ like bosonic, but with a opposite sign),

$$\begin{aligned} lnZ^{(1)} &= \frac{10}{2}Tr(ln\ \bar{D}_E^2)_{bosonic} - \frac{8}{2}Tr(ln\ \bar{D}_L)_{fermionic} - \frac{8}{2}Tr(ln\ \bar{D}_R)_{fermionic} - Tr(ln\ \bar{D}_E^2)_{ghost} \\ &= 5Tr(ln\ \bar{D}_E^2)_{bosonic} - 4Tr(ln\ \bar{D}_E^2)_{fermionic} - Tr(ln\ \bar{D}_E^2)_{ghost} \end{aligned} \tag{B.13}$$

where $\bar{D}_L$ and $\bar{D}_R$ are operators acting on the left moving and right moving fermion, and $\bar{D}_E^2 = \bar{D}_L\bar{D}_R$.

C

Numerical Programming

In this appendix *Mathematica* 7 codes used for various calculations for the second part of my thesis are given.

C.1 Fermion

Here ω is the frequency, k is the momentum and $\mu = Qq$. ϵ is a small cut-off near $z = 0$. For zero temperature $Q = 2$.

C.1.1 Zero Temperature

The following is the Mathamatica code used in computation of Fermion Green's function calculation at zero temperature ($Q = 2$), (Greens0T[$\omega\_, k\_, \mu\_, \epsilon\_$] is the Green's function at zero temperature, Greens0Tpre[$\omega\_, k\_, \mu\_, \epsilon\_$] is the same code but with more precision. Here the precision was set to 20 for the input variables, which can be altered by changing the parameter in SetPrecision[] function and simultaneously changing the parameter WorkingPrecision in NDSolve.)

Greens0T[$\omega\_, k\_, \mu\_, \epsilon\_$]:=
Module[{$kt = \omega, kx = k, mu = \mu, \epsilon mod = \epsilon, f, z, Sol, G, bc0, bc1, \epsilon 1 = 10^{-3}\epsilon$},
(*ϵ1 is a small number in bc0*)
$f[z\_]$:=$1 - z^2 + 2z^2\ Log[z]$;
$bc0 = I$Sqrt[$\frac{mu^2}{2} - kx^2 + I\epsilon 1$]/($mu$/Sqrt[2] + kx);
(*exactly at $\omega = 0$ the b.c. is given by $bc0$ otherwise it is $bc1 = I$,$I = \sqrt{-1}$*)
$bc1 = I(1 -$ KroneckerDelta[kt, 0]) + KroneckerDelta[kt, 0] $bc0$; $Sol =$
NDSolve[{$f[z]G'[z] + G[z]^2(kt+$
mu Log[z] $- kx$ Sqrt[$f[z]$]) + ($kt + mu$ Log[z]+
kx Sqrt[$f[z]$]) == 0, $G[1-\epsilon]$ == $bc1$}, G, {$z, \epsilon, 1-\epsilon$},
MaxSteps $\to$ Infinity]; Evaluate[$G[\epsilon]$]/.Sol[[1]]]

Greens0Tpr[$\omega\_, k\_, \mu\_, \epsilon\_$]:=
Module[{kt = SetPrecision[ω, 20], kx = SetPrecision[k, 20], mu = SetPrecision[μ, 20],
$\epsilon mod = \epsilon, f, z, Sol, G, bc0, bc1,$
$\epsilon 1 = 10$^ $- 3\epsilon$}, (*ϵ1 is a small number in bc0*)f[z_]:=1 $- z$^2 + 2z^2Log[z];
$bc0 = I$Sqrt[$\frac{mu^2}{2} - kx^2 + I\epsilon 1$]/($mu$/Sqrt[2] + kx);
(*exactly at $\omega = 0$ the b.c. is given by $bc0$ otherwise it is $bc1 = I$,$I = \sqrt{-1}$*)
$bc1 = I(1 -$ KroneckerDelta[kt, 0]) + KroneckerDelta[kt, 0]$bc0$;
Sol = NDSolve[{$f[z]G'[z] + G[z]^2(kt + mu$ Log[z] $- kx$ Sqrt[$f[z]$]) + ($kt + mu$ Log[z]+
kxSqrt[$f[z]$]) == 0, $G[1-\epsilon]$ == $bc1$}, G, {$z, \epsilon, 1-\epsilon$}, WorkingPrecision $\to$ 19,
MaxSteps $\to$ Infinity];
Evaluate[$G[\epsilon]$]/.Sol[[1]]]

Now we define derivatives (fop,fkp,foppr,fkppr) of the imaginary part Green's function (fok,fokpr) with respect to ω and k. We define derivatives for finite change in

ω and k given by $\Delta\omega = \Delta k = epsilon \sim 10^{-4}$. We have set $\epsilon = 10^{-4}$ and $\mu = 1$.

$epsilon = 10^{-4}$; fok[x_, y_]:=Im[Greens0T[$x, y, 1, 10^{-4}$]];
fop[x_, y_]:=(fok[$x + epsilon, y$] − fok[x, y])/$epsilon$; fkp[x_, y_]:=
(fok[$x, y + epsilon$] − fok[x, y])/$epsilon$; fokpr[x_, y_]:=Im[Greens0Tpr[$x, y, 1, 10^{-4}$]];
foppr[x_, y_]:=(fokpr[$x + epsilon, y$] − fokpr[x, y])/$epsilon$;

maxo[] finds the maximum of fok[ω, k] with respect to ω at a fixed k. maxk[] finds the maximum of fok[ω, k] with respect to k at a fixed ω. The output of these functions are as follows: $\{\omega, k, fokmax, fokmaxp\}$, where ω or k gives the position of the maximum depending on which is kept fixed. $fokmax$ = fok[ω, k] is the value at the maximum and $fokmaxp$ = fop[ω, k] or fkp[ω, k] is the value of the respective derivative at maximum. The loops for searching maximum starts at an initial guess for the position $xstart$(or $ystart$) and terminates when either $fokmax > fmax$ or $fokmaxp < delta$. $fmax$ and $delta$ are given as input. We have to also give $astart$ as input, which determines the initial step size for maximum search in the loop.

maxo[k_, $astart$_, $xstart$_, $delta$_, $fmax$_]:=
Module[$\{x, y, a\}$, $x = xstart; y = k; a = astart$; Label[1];
While[fop[x, y] > $delta$&&fok[x, y] < $fmax$, x+=a]; $a = .5a$;
If[Abs[fop[x, y]] < $delta$||fok[x, y] > $fmax$, Goto[end]];
While[fop[x, y] < $delta$&&fok[x, y] < $fmax$, x+= − a];
$a = .5a$; If[Abs[fop[x, y]] < $delta$||fok[x, y] > $fmax$, Goto[end], Goto[1]];
Label[end]; {x, y, fok[x, y], fop[x, y]}]

maxk[omega_, astart_, ystart_, delta_, fmax_]:=
Module[$\{x, y, a\}$, $x = omega$;
$y = ystart$; $a = astart$; Label[1]; If[Abs[a] $< 10^{-5}$, Goto[end]]; While[fkp[x, y] $> delta$
&&fok[x, y] < fmax, y+=a]; $a = .5a$;
If[Abs[fkp[x, y]] $< delta$||fok[x, y] $> fmax$, Goto[end]];
While[fkp[x, y] $< delta$&&fok[x, y] $< fmax$, y+=$-a$]; $a = .5a$;
If[Abs[fkp[x, y]] $< delta$||fok[x, y] $> fmax$,
Goto[end], Goto[1]]; Label[end];
$\{x, y,$ fok[x, y], fkp[x, y]$\}$]

The module maxkloop[] searches for maximum of fok[ω, k] for a range of ω between ωst and ωe and at a interval int. yst corresponds to the initial guess for position of maximum along k at $\omega = \omega st$. ast is the value of $astart$ used in previous program.

maxkloop[ωst_, ast_, yst_, ωe_, int_]:=
Module[$\{x, y, a, data, Nsteps, i = 1\}$, $x = \omega st$; $y = yst$;
$a = ast$; $Nsteps$ = Abs[Round[$(\omega e - \omega st)/int$]]; For[$i = 1, i < Nsteps, i$++,
$d[i]$ = maxk[$x, a, y, 10^{-4}, 10^6$]; $x = x + int$; $y = N[d[i][[2]]]$]; Table[$d[i]$, $\{i, 1, Nsteps - 1\}$]]

C.1.2 Finite Temperature

The following is the Mathamatica code used in computation of Fermion Green's function calculation at finite temperature ($Q \neq 2$). The module PrecisionSet[x, pre] sets precision of x to desired value pre. The module GreensFiniteTpre[$\omega, k, Q, q, \epsilon, pre$] calculates the Green's function at finite temperature. ϵ sets the boundary cut-off near $z = 0$. fokpr[] is the imaginary part of the Green's function at $q = \frac{1}{2}$, $Q = 1.99$ and $pre = 25$. Analysis at various values of temperature are done by changing Q at fixed q. fkppr[] evaluates derivative of fokpr[] w.r.t. k at fixed ω using "finite difference approximation" (ϵ) as in zero temperature case.

PrecisionSet$[x_, pre_]$:= If[Precision$[x]$ == Infinity, $N[x,$ pre$]$, SetPrecision$[x, pre]]$;
GreensFiniteTpre$[\omega_, k_, Q_, q_, \epsilon_, pre_]$:=
Module$[\{kt =$ PrecisionSet$[\omega, pre], kx =$ PrecisionSet$[k, pre], c1 =$
PrecisionSet$[Q, pre], c2 =$ PrecisionSet$[q, pre], \epsilon mod = \epsilon, f, z, Sol, G, sg, T\}$,
$f[z\_]$:=$1 - z^2 + ((z^2 c1^2)/2)$Log$[z]$; $Sol =$
NDSolve$[\{f[z]G'[z] + G[z]^2(kt + c1c2Log[z] - kx$Sqrt$[f[z]])+$
$(kt + c1c2$Log$[z] + kx$Sqrt$[f[z]])$ == 0,
$G[1-\epsilon]$ == $I\}, G, \{z, \epsilon, 1-\epsilon\}$,
MaxSteps $\to$ Infinity, WorkingPrecision $\to N[pre - 1]]$; $sg =$ Evaluate$[G[\epsilon]]/.Sol[[1]]]$;

fokpr$[x,y,\epsilon Nd_]$:= Im[GreensFiniteTpre$[x, y, 199/100, 1/2, \epsilon Nd, 25]]$;

fkppr$[x,y,\epsilon,\epsilon Nd_]$:= (fokpr$[x, y+\epsilon, \epsilon Nd]$ − fokpr$[x, y, \epsilon Nd])/\epsilon$

The module maxkpr[] is similar to what described for zero temperature (maxk[]), with a slight modification, which alters the value of ϵ (parameter of finite difference approximation) according to the step size in the loop. maxklooppr[] is similar to the module used in zero temperature case (maxkloop[]).

maxkpr[$omega\_, astart\_, ystart\_, delta\_, fmax\_, \epsilon Nd\_$]:=
Module[$\{x, y, a, \epsilon\}, x = omega; y = ystart; a = astart;$ Label[1]; If[Abs[a] $< delta$, Goto[end]];
If[Abs[a] $> 10^{-6}, \epsilon = 10^{-10}, \epsilon = a10^{-4}$];
While[fkppr[$x, y, \epsilon, \epsilon Nd$] $> delta$&&fokpr[$x, y, \epsilon Nd$] $< fmax, y$+=a]; $a = a/2$;
If[Abs[a] $> 10^{-6}, \epsilon = 10^{-10}, \epsilon = a10^{-4}$]; If[Abs[fkppr[$x, y, \epsilon,$
ϵNd]] $< delta$||fokpr[$x, y, \epsilon Nd$] $> fmax$,
Goto[end]]; While[fkppr[$x, y, \epsilon, \epsilon Nd$] $< delta$&&fokpr[$x, y, epsilonNd$] $< fmax, y$+= $-a$];
$a = a/2$; If[Abs[a] $> 10^{-6}, \epsilon = 10^{-10}, \epsilon = a10^{-4}$];
If[Abs[fkppr[$x, y, \epsilon, \epsilon Nd$]] $< delta$||fokpr[$x, y, \epsilon Nd$] $> fmax$, Goto[end], Goto[1]];
If[Abs[a] $> 10^{-6}, \epsilon = 10^{-10}, \epsilon = a10^{-4}$]; Label[end]; {$omega$,
N[y, Abs[Log[10, $delta$]] + 1], fokpr[$x, y, \epsilon Nd$],
fkppr[$x, y, \epsilon, \epsilon Nd$]}]

maxklooppr[$\omega st\_$, ast_, yst_, ωe_, int_]:=
Module[{x, y, a, d, data, Nsteps, $i = 1$}, $x = \omega$st; y = yst; a = ast;
Nsteps = Abs[Round[(ωe − ωst)/int]];
For[$i = 1, i <$ Nsteps + 1, i++, $d[i]$ = maxkpr[x, a, y, 10^ − 7, 10^7, 10^ − 10]; $x = x$ + int;
$y = d[i][[2]]$]; Table[$d[i]$, {i, 1, Nsteps}]]

Rootfindpr[$\omega, xi, f0, \delta, \epsilon Nd$], finds solution for the equation of the form fokpr[$\omega, k, \epsilon Nd$] == $f0$ at a given ω. $k = xi$ is the initial guess for the solution. δ sets the allowed error in the solution. ϵNd is the boundary cut-off in the Green's function.

Rootfindpr[ω_, xi_, f0_, δ_, ϵNd_]:=
Module[{x = xi, Δ}, While[True, $\Delta = N$[(f0 − fokpr[ω, x, ϵNd])/fkppr[ω, x, 10^ − 10, ϵNd]];
(*Precision is MachinePrecision*)
If[Abs[Δ] $> \delta, x = x + \Delta$, Break[]]]; x]

FWHMpr[$\omega, x1i, x2i, f0, \delta, \epsilon Nd$], finds the Full-Width-at-Half-Maximum (FWHM) in the k-direction (δk) for a given ω. It uses the Rootfindpr[] function to determine two values of k for which the value of the function fokpr[] has half of its maximum value ($f0 = \frac{fmax}{2}$, where $fmax$ is the maximum value). The initial guesses for the two values are $x1i$ and $x2i$. For a range of ω, the data set for peak position and peak values are first obtained using maxklooppr[] and the output is stored as $maxkdata$. FWHMloop[] uses the $maxkadata$ list to calculate the FWHM for the range of ω.

$\text{FWHMpr}[\omega\_, x1i\_, x2i\_, f0\_, \delta\_, \epsilon Nd\_]:=$

$\text{Module}[\{x1, x2, width\}, x1 = \text{Rootfindpr}[\omega, x1i, f0, \delta, \epsilon Nd]; x2 = \text{Rootfindpr}[\omega, x2i, f0, \delta, \epsilon\text{Nd}];$

$\text{width} = \text{Abs}[\text{x1} - \text{x2}]; \{\text{x1}, \text{x2}, \text{width}\}]$

$\text{FWHMloop}[\omega\text{st}\_, \text{y1st}\_, \text{y2st}\_, \omega\text{e}\_, \text{int}\_]:=$

$\text{Module}[\{x = \omega\text{st}, \text{y1} = \text{y1st}, \text{y2} = \text{y2st}, d, \text{data}, \text{Nsteps}, i = 1\},$

$\text{Nsteps} = \text{Abs}[\text{Round}[(\omega\text{e} - \omega\text{st})/\text{int}]]; \text{For}[i = 1, i < \text{Nsteps} + 1, i{+}{+},$

$d[i] = \text{FWHMpr}[x, \text{y1}, \text{y2}, \text{maxkdata}[[i]][[3]]/2, 10^\wedge - 7, 10^\wedge - 10];$

$x = x + \text{int}; \text{y1} = d[i][[1]]; \text{y2} = d[i][[2]];$

$\text{Print}[i, d[i]]]; \text{Table}[\{\text{maxkdata}[[i]][[1]](*\omega\_**), d[i][[3]]\}, \{i, 1, \text{Nsteps}\}]]$

C.2 Vector

Solution near boundary, $z = \epsilon = ep$ (ϵ is a small number, close to zero) for gauge field perturbation is given by $a(z) \simeq a^{+} + a^{-}Log(z)$. So, $am = a^{-} = za'(z)\Big|_{z=\epsilon}$ and $ap = a^{+} = [a(z) - a^{-}Log(z)]\Big|_{z=\epsilon}$.

C.2.1 Finite Temperature

The module Cond$[\omega, q, ep]$ gives Conductivity at finite temperature, where q is the square of the black-hole charge.

Cond$[\omega_, q_, ep_]$:=Module$[\{Q = q, \epsilon = ep, eqn, kt = \omega, \nu, f, At,$
$Je, Bc, Bc1, L, E1, H1, am, ap, F, r, S, x\}$,
Clear$[d*]$; $f[r\_]$:=$1 - r^2 + (Qr^2/2)$Log$[r]$; $T = (1 - Q/4)/(2\pi)$; $\nu = I\omega/(4\pi T)$;
$d[0] = 1$; $n = 5$; $eqn = r\ f[r]^2\ F''[r] + f[r]((2\nu + 1)rf'[r] + f[r])F'[r]+$
$(r\ \nu(\nu - 1)\ f'[r]^2 + \nu\ rf''[r]f[r]+$
$(f[r] + rf'[r])\nu f'[r] + r(kt^2 - Qf[r]))F[r]$; Je$[r_]$:=Sum$[d[j](1 - r)^j, \{j, 0, n\}]$;
(* Je[r] is the near horizon (r=1) solution ansatz for the equation (eqn==0)*)
$L =$ Table[FullSimplify[SeriesCoefficient$[eqn/.F \to Je, \{r, 1, i\}]], \{i, n\}]$;
(*L[[i]]==0 is the equations obtained by putting the ansatz (Je) in the expression (eqn),
then expanding in the power series about $r = 1$, and subsequently
setting the coefficients of each power to zero *)
For$[i = 1, i < n + 1, i{+}{+}, S_i = d[i]/.$Solve$[L[[i]] == 0, d[i]][[1]]; d[i] = S_i]$;
(* The unknown series coeff of the ansatz, d[[i]] are obtained solving the coupled equations L*)
Bc$[r_]$:=Sum$[d[j](1 - r)^j, \{j, 0, n\}]$; Bc1$[r_]$:=$D[Bc[x], x]/.x \to r$;
(*Bc[r], and Bc1[r]=Bc'[r] are solutions of eqn near horizon, and used for boundary condition*)
E1 = NDSolve$[\{eqn == 0, F[1 - \epsilon] ==$ Bc$[1 - \epsilon], F'[1 - \epsilon] ==$ Bc1$[1 - \epsilon]\}$,
$F, \{r, \epsilon, 1 - \epsilon\}$, MaxSteps $\to 10^7]$; H1[r_]:=$f[r]$^(ν)Evaluate$[F[r]]$/.E1[[1]][[1]];
am = $r\ D[$H1$[r], r]/.r \to \epsilon$; ap = H1$[\epsilon]$ − am Log$[\epsilon]$; $-I/$(kt)(ap/am)]

C.2.2 Zero Temperature

The module Cond[ω, ep] gives Conductivity at zero temperature,

CondT0[$kt\_, ep\_$]:=Module[{$\epsilon = ep, eqn, eqn1, \omega = kt, T, f, At, Je,$
$Bc, Bc1, E1, H1, am, ap, g, r, S, x, n, i, L, d$},
Clear[d*]; $f[r\_]$:=$1 - r^2 + 2r^2$Log[r]; $d[0] = 1; n = 5; eqn1 = g''[r] + (1/r + f'[r]/f[r]+$
$I\omega/(1-r)^2 + I\omega/(3(1-r)))g'[r] + (\frac{i\omega}{2(1-r)^2 r} - \frac{\omega^2}{4(1-r)^4} + \frac{\omega^2}{f[r]^2} - \frac{8}{f[r]}+$
$\frac{i\omega(\frac{2}{1-r} + \frac{f'[r]}{f[r]})}{2(1-r)^2} + I\omega/6(I\omega/6+1)(1/(1-r)^2)+$
$I\omega/(6(1-r))(1/r + f'[r]/f[r] + I\omega/(1-r)^2))g[r];$
$eqn = (1-r)^3 eqn1$; Je[$r\_$]:=Sum[$d[j](1-r)^j, \{j, 0, n\}$];
L = Table[FullSimplify[SeriesCoefficient[eqn/.$g \to$ Je, {$r, 1, i$}]], {i, n}];
For[$i = 1, i < n+1, i{+}{+}, S_i = d[i]$/.Solve[$L[[i]] == 0, d[i]$][[1]]; $d[i] = S_i$];
Bc[$r\_$]:=Sum[$d[j](1-r)^j, \{j, 0, n\}$];
Bc1[$r\_$]:=D[Bc[x], x]/.$x \to r$;
E1 = NDSolve[{$eqn == 0, g[1-\epsilon]$ == Bc[$1-\epsilon$], $g'[1-\epsilon]$ == Bc1[$1-\epsilon$]}, g, {$r, \epsilon, 1-\epsilon$},
MaxSteps $\to$ Infinity]; H1[r_]:=Exp[$I\omega/(2(1-r))$](($(1-r)^{(-I\omega/6))}$Evaluate[$g[r]$]/.E1[[1]][[1]];
am = r D[H1[r], r]/.$r \to \epsilon$; ap = H1[ϵ] − am Log[ϵ];
$-I/(\omega)$(ap/am)]

C.3 Scalar

C.3.1 Finite Temperature

The following is the Mathamatica code used in Scalar Green's function calculation at finite temperature,

```
Solution[N_, epsilon_, omega_, momentum_, charge_]:=
Module[{nt = N, w = SetPrecision[omega, 30], ep = SetPrecision[epsilon, 30],
k = SetPrecision[momentum, 30],
Q = SetPrecision[charge, 30], P, delta, A, B, V, S, L, HorizonSeries, HorizonSeriesPrime},
Clear[b*, a*];
SetPrecision[a*, 30]; SetPrecision[b*, 30]; T = 1/(4Pi)(2 - Q^2/2);
SetPrecision[T, 30]; m = 1/2; SetPrecision[m, 30];
p = SetPrecision[1 + Sqrt[1 + m^2], 30]; delta[z_]:=1 - z^2 + z^2/2Q^2Log[z];
P[z_]:=1 - z^2 + z^2Q^2Log[z]/2; A[z_]:=z^2P[z]^2;
B[z_]:=zP[z](z^2(Q^2/2 - 1) + z^2Q^2Log[z]/2 - 1); V[z_, w_]:=
 - P[z]k^2z^2 + z^2(w + QLog[z])^2 - m^2P[z]; S[z_]:=
Sum[a[n](z - 1)^n, {n, 0, nt}];


        (*Series expansion near horizon*)
        d = -Iw/(4PiT); SetPrecision[d, 30]; L[z_, w_]:=
        Series[D[S[z], {z, 2}]delta[z]^2A[z] + D[S[z], z](B[z]delta[z]^2+
        2ddelta[z]D[delta[z], z]A[z]) + S[z](delta[z]^2V[z, w]+
        A[z]d(d - 1)D[delta[z], z]^2 + dB[z]delta[z]D[delta[z], z]+
        ddelta[z]D[delta[z], {z, 2}]A[z]), {z, 1, nt}];


(*Finding solutions for series coefficients a[i]*)
b[0] = a[0]; For[i = 3, i < nt + 1, i++, b[i - 2] = a[i - 2]/.Solve[SeriesCoefficient[L[z, w], {z, 1, i}]
 == 0, a[i - 2]][[1]]; a[i - 2] = b[i - 2]];
```

(*Writing Series at horizon*)
HorizonSeries[z_, w_] = Sum[$b[n](z-1)$^n, {n, 0, nt − 2}];
HorizonSeriesPrime[z_, w_] = D[HorizonSeries[z, w], z];

(*Solving the differential equation numerically*)
$a[0] = 1$; zb = ep;
PhiEqn:=D[G[z], {z, 2}]delta[z]^2A[z] + D[G[z], z](B[z]delta[z]^2+
2$d$delta[z]D[delta[z], z]A[z]) + G[z](delta[z]^2V[z, w] + A[z]$d(d-1)D$[delta[z], z]^2+
dB[z]delta[z]D[delta[z], z] + $d$delta[z]D[delta[z], {z, 2}]A[z]);
Final = NDSolve[{PhiEqn == 0, G[1 − ep] == HorizonSeries[1 − ep, w], G'[1 − ep]
== HorizonSeriesPrime[1 − ep, w]}, G, {z, zb, 1 − ep}, WorkingPrecision → 25,
MaxSteps → Infinity];
Source[z_] = delta[z]^d $G[z]z^{(p-2)}$; Fluctuation[z_] =
D[Source[z], z]$z^{(3-2p)}/(2p-2)$; gw = Fluctuation[zb]/Source[zb]/.Final]

C.3.2 Zero Temperature

The following is the Mathamatica code used in Scalar Green's function calculation at zero temperature,

nt = 7; (* number of terms *)
ep = 10^(−5); SetPrecision[ep, 30]; $Q = 2$; SetPrecision[Q, 30]; $m = 1/2$;
SetPrecision[m, 30]; $d = I - Iw/6$; p = SetPrecision[1 + Sqrt[1 + m^2], 30];
P[z_]:=1 − z^2 + $z^{2}Q$^2Log[z]/2; A[z_]:=
$z^{2}P$[z]^2; B[z_]:=zP[z](z^2(Q^2/2 − 1)+
$z^{2}Q$^2Log[z]/2 − 1); V[z_, w_]:= − P[z]$k^{2}z$^2+
z^2(w + QLog[z])^2 − $m^{2}P$[z];
S[z_]:=Sum[$a[n](z-1)$^n, {n, 0, nt}];

(*Series expansion near horizon*)

$L[\text{z\_}, \text{w\_}]$:= Series$[4S[z]V[z,w](-1+z)$^$4+$
$2B[z](-1+z)$^$2((Iw+2d(-1+z))S[z]+2S'[z](-1+z)$^$2)+$
$A[z]((-w(w+4I(-1+z))+4d(1+Iw-z)(-1+z)+ \quad 4d$^$2(-1+z)$^$2)S[z]$
$+4(Iw+2d(-1+z))(-1+z)$^$2S'[z]+4(-1+z)$^$4S''[z]),\{z,1,\text{nt}\}];b[0]=a[0];$
For$[i=6, i<\text{nt}+1, i{+}{+}, \ b[i-5]$
$= \ a[i-5]/.$ Solve[SeriesCoefficient$[L[z,w],\{z,1,i\}]{=}{=}0, a[i-5]][[1]];$
$a[i-5]=b[i-5]];$

(*Writing Series at horizon*)

HorizonSeries[z_, w_] = Sum$[b[n](z-1)$^$n,\{n,0,\text{nt}-5\}];$
HorizonSeriesPrime[z_, w_] = D[HorizonSeries$[z,w],z];$

(*Solving the differential equation numerically*)(*
$w=0.1;$*)$a[0]=1;$ zb = ep; PhiEqn:= $4G[z]V[z,w](-1+z)$^$4+$
$2B[z](-1+z)$^$2((Iw+2d(-1+z))G[z]+2G'[z](-1+z)$^$2)+$
$A[z]((-w(w+4I(-1+z))+4d(1+Iw-z)(-1+z)+ \quad 4d$^$2(-1+z)$^$2)G[z]+$
$4(Iw+2d(-1+z))(-1+z)$^$2\ G'[z]+4(-1+z)$^$4G''[z]);$
Final[omega_, momentum_]:={w = SetPrecision[omega, 30];
k = SetPrecision[momentum, 30]; NDSolve[{PhiEqn==0, G[1 − ep]==
HorizonSeries[1 − ep, w], G'[1 − ep]==HorizonSeriesPrime[1 − ep, w]}, G, {z, zb, 1 − ep},
WorkingPrecision->25, MaxSteps->Infinity]};
Source[z_] = Exp$[Iw/(2(1-z))](z-1)$^$dG[z]z$^$(p-2);$
Fluctuation[z_] = D[Source$[z],z]z$^$(3-2p)/(2p-2);$
gw = Fluctuation[zb]/Source[zb];

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