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TOPICS IN MODERN MATHEMATICS

By
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THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS - 20, (INDIA)

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P.K.G.

CONTENTS

CHAPTER I

ELEMENTS OF POINT-SET TOPOLOGY

1. PRELIMINARIES.

DEFINITION 1.1. Let X be a non-empty set. A class $\mathfrak{I}$ of subsets of X is a topology on X if and only if it satisfies the following axioms:

(i) X, ϕ belong to $\mathfrak{I}$, where ϕ is the empty set.

(ii) The union of any number of sets in $\mathfrak{I}$ belongs to $\mathfrak{I}$.

(iii) The intersection of any two sets in $\mathfrak{I}$ belongs to $\mathfrak{I}$.

The sets belonging to $\mathfrak{I}$ are called $\mathfrak{I}$-open sets and the pair $(X, \mathfrak{I})$ is called a topological space. When the underlying topology is understood, we simply speak of 'open sets' relative to that topology and denote the corresponding topological space by X.

EXAMPLES 1.2.

(i) The set of all real numbers denoted by $\mathbb{R}$, together with the usual topology made up of all the open intervals is a topological space. The two-dimensional space $\mathbb{R}^2$, together with the topology constituted by all the open discs is another topological space.

(ii) Consider $X = \{a,b,c,d,e\}$. Define

$$\mathfrak{I}_1 = \{X, \phi, \{a\}, \{b\}, \{a,b.c\}\}$$

$$\mathfrak{I}_2 = \{X, \phi, \{a\}, \{a,c,d\}, \{b,c,d,e\}\}$$

$$\mathfrak{I}_3 = \{X, \phi, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,c,d\}\}$$

$\mathcal{T}_1$ is not a topology, since $\{a\} \cup \{b\} = \{a,b\} \notin \mathcal{T}_1$, while $\mathcal{T}_2$ fails to be a topology as $\{a,c,d\} \cap \{b,c,d,e\} = \{c,d\} \notin \mathcal{T}_2$. On the other hand, $\mathcal{T}_3$ is a topology.

(iii) If $\mathcal{T}$ is made up of all the subsets of X, then $\mathcal{T}$ is called the discrete topology, and X, together with $\mathcal{T}$, is called a discrete space.

(iv) If $\mathcal{T}_1 = \{X, \phi\}$, then it is called the trivial or indiscrete topology and the corresponding topological space is called an indiscrete space.

(v) If $\mathcal{T}$ consists of ϕ and all those subsets of X whose compliments are finite, then $\mathcal{T}$ is a topology and it is known as the co-finite topology on X.

EXERCISE 1.3. Given X, is there a prescribed rule to enumerate the number of possible topologies that can be defined on it ?

DEFINITION 1.4. Let X be a topological space. A point $x \in X$ is a <u>limit point</u> of a subset A of X if and only if every open set G containing x, contains a point of A different from x. Symbolically, G open, $x \in G \Rightarrow A \cap (G \setminus \{x\}) \neq \phi$. The set of all limit points of A is called the derived set of A and denoted by A^d.

Thus, the set of positive integers, denoted by N, does not have any limit points, since for every real number a, there exists a $\delta > 0$ such that the open interval $(a-\delta, a+\delta)$ contains no point of N other than a. In the case of a closed interval $[a,b]$, every point $x \in [a,b]$ is a limit point of the semi-closed interval $(a,b]$. For the set of rationals and the set of irrationals, every real number will be a limit point. For $X = \{a,b,c,d,e\}$, $\mathcal{J} = \{X, \phi, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,c,d\}\}$, if A is the subset of X defined by $\{a,b,c\}$, then $a \in X$ is not a limit point of A, since one of the open sets containing a, namely $\{a\}$, does not contain a point of A different from a. On the other hand, $b \in X$ is a limit point of A, as the open sets containing b are $\{a,b\}$, $\{a,b,c,d\}$ and X, each of which contains a point of A different from b. Likewise c,d,e are limit points. Thus $A^d = \{b,c,d,e\}$. In the case of the indiscrete space, X and ϕ are the only open subsets of X. Then X is the only open set containing any point $x \in X$. Hence x is an accumulation point of every subset of X except the empty set ϕ and $\{x\}$. Accordingly, the derived set of any subset A of X is equal to ϕ if $A = \phi$, $X - \{x\}$ if $A = \{x\}$ and X if A contains two or more points.

EXERCISE 1.5. Analyse the limit points of subsets of the other topological spaces that we have listed in 1.2.

DEFINITION 1.6. In a topological space X, a subset A of X is a <u>closed set</u> if and only if its complement is open.

In 1.2, (ii), $\mathcal{J}_3$ defined a topology. The closed sets are given by $\mathcal{C} = \{\phi, X, \{b,c,d,e\}, \{c,d,e\}, \{b,e\}, \{e\}\}$.

Notice that X and ϕ are both open and closed and there are subsets such as $\{b,c\}$ which are neither open nor closed. In the case of the discrete space, every subset of X being open, its complement is always closed. Thus, all the subsets of X are both open and closed. The interval $(a,b]$ is neither open nor closed.

DEFINITION 1.7. The <u>closure</u> of a subset A of X, is the intersection of all closed supersets containing A and is denoted by $\bar{A}$.

Note that $\bar{A} = A \cup A^d$. Thus, for the rationals denoted by Q, $\bar{Q} = \mathbb{R}$. In the case of the co-finite topology, if A is a finite subset, $\bar{A} = A$, since A itself is closed, while if A is an infinite subset, X is the only closed superset of A and so $\bar{A} = X$.

<u>Kuratowski Closure Axioms 1.8</u>. (i) $\bar{\phi} = \phi$, (ii) $A \subset \bar{A}$, (iii) $\overline{A \cup B} = \bar{A} \cup \bar{B}$, (iv) $\bar{(\bar{A})} = \bar{A}$, where A and B are subsets of X.

<u>Proof.</u> (i) and (iv) are true since ϕ and $\bar{A}$ are closed sets and therefore equal to their closures.

(ii) $\bar{A} = A \cup A^d$ and hence $A \subset \bar{A}$. Equality holds only when A is closed.

(iii) $A \subset \bar{A}$, $B \subset \bar{B}$, $A \cup B \subset \bar{A} \cup \bar{B}$. But $\bar{A} \subset \overline{A \cup B}$, $\bar{B} \subset \overline{A \cup B}$ implies $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$. Consider $x \in \overline{A \cup B}$. Then $x \in A \cup B$ or $x \in (A \cup B)^d$ which implies that $x \in A$ or B, or every open set G containing x contains a point $y (\neq x) \in A \cup B$ which in turn implies that $x \in \bar{A}$ or $\bar{B}$ or $y \in A$ or B. Thus x is a limit point for A or B which gives $x \in \bar{A}$ or $\bar{B}$ or $y \in A$ or B. Thus x is a limit point for A or B which gives $x \in \bar{A}$ or $\bar{B}$ and therefore $x \in \bar{A} \cup \bar{B}$. Hence $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$, which together with the preceding inclusion gives $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

DEFINITION 1.9. Let A be a subset of a topological space X. A point $x \in A$ is an <u>interior point</u> of A, if there exists an open set G satisfying the condition $p \in G \subset A$. The set of all interior points of A is called the <u>interior of A</u> and denoted by A^{o}. The <u>exterior of A</u>, written ext.(A), is the interior of the complement A' of A. The <u>boundary of A</u>, written bdy (A), is the set of points which do not belong to either the interior or the exterior of A.

In example 1.2, (ii), the points a, b are interior points of a subset $\{a,b,c\}$ of X since $\{a,b\}$ is an open set containing them which is also contained in A. Notice that c is not an interior point of A, since c does not belong to any open set contained in A. Hence $A^{o} = \{a,b\}$. The complement of A is $\{d,e\}$. Neither d nor e is an interior point of A', since neither belongs to any open subset of A'. Hence $(A')^{o} = \phi$. Thus, the boundary of $A = \{c,d,e\}$. In the case of the indiscrete space, X and ϕ being the only open sets, the boundary of any subset A of X is X itself. Considering the rationals Q, since every open interval of $\mathbb{R}$ contains both rational and irrational points, Q has no interior or exterior points and therefore the boundary of Q is the entire real line.

DEFINITION 1.10 Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies defined on X. If each $\mathcal{T}_1$-open subset of X is also $\mathcal{T}_2$-open, i.e., if $\mathcal{T}_1 \subset \mathcal{T}_2$, then $\mathcal{T}_1$ is coarser (smaller, weaker) than $\mathcal{T}_2$ or $\mathcal{T}_2$ is finer (larger, stronger) than $\mathcal{T}_1$.

Remark 1.11. The intersection of two topologies is a topology, but their union need not be a topology. For, $X, \phi \in \mathcal{T}_1$ and $\mathcal{T}_2$ and therefore to their intersection. Suppose G,H are open sets in $\mathcal{T}_1 \cap \mathcal{T}_2$. Then, in particular, $G,H \in \mathcal{T}_1$ and $G,H \in \mathcal{T}_2$. Further, since $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies, $G \cup H$ and $G \cap H$ belong to both $\mathcal{T}_1$ and $\mathcal{T}_2$ and therefore to their intersection, which proves that $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology. This result can be generalized to any number of topologies and thus $\bigcap_i \mathcal{T}_i$ is also a topology on X. On the other hand, if $X = \{a,b,c\}$, $\mathcal{T}_1 = \{X, \phi, \{a\}\}$ and $\mathcal{T}_2 = \{X, \phi, \{b\}\}$, then $\mathcal{T}_1 \cap \mathcal{T}_2 = \{X, \phi, \{a\}, \{b\}\}$ which is not a topology since $\{a\} \cup \{b\} = \{a,b\} \notin \mathcal{T}_1 \cap \mathcal{T}_2$.

DEFINITION 1.12. Let A be a non-empty subset of X. Every set which is obtained as the intersection of A with the $\mathcal{T}$-open subsets of X is said to be open relative to A. and the class $\mathcal{T}_A$ of all such sets is a topology called the relative topology on A. The corresponding topological space $(A, \mathcal{T}_A)$ is called a subspace of $(X, \mathcal{T})$.

Consider $X = \{a,b,c\}$, $\mathfrak{T} = \{X, \phi, \{a\}, \{b\}, \{a,b\}\}$, $A = \{a,b\}$. Then $\mathfrak{T}_A = \{A, \phi, \{a\}, \{b\}, \{a,b\}\}$ is the relative topology induced by A. Consider the usual topology on $\mathbb{R}$ and the relative topology $\mathfrak{T}_A$ on the closed interval $A = [3,8]$. Now $[3,5) = (2,5) \cap A$ and thus $[3,5)$ is open in the relative topology on A but not in the usual topology of the real line. Thus, a set may be open relative to a subspace but be neither open nor closed in the entire space. A relative topology is therefore useful in that its sets may be open without being open relative to the whole space.

DEFINITION 1.13. A sequence $\{x_n\} \in X$ is said to <u>converge</u> to a point $x \in X$ if and only if for each open set G containing x, there exists a positive integer n_o such that $n > n_o \Rightarrow x_n \in G$. In other words, $\{x_n\}$ is convergent to x if and only if G contains all except a finite number of the terms of the sequence. We say $\{x_n\} \rightarrow x$ or $\lim\limits_{n \to \infty} x_n = x$.

In the case of a discrete space, $\{x_n\}$ converges to x if and only if the sequence is of the form $\{x_1, x_2, \ldots, x_{n_o}, x, x, \ldots\}$, for every point $x \in X$ is contained in the open set $\{x\}$ and if $\{x_n\} \rightarrow x$, then $\{x\}$ must contain all but a finite number of terms of $\{x_n\}$. For the indiscrete space, X is the only open set containing any point $x \in X$ and X contains every term of $\{x_n\}$ and hence $\{x_n\}$ converges to every $x \in X$.

DEFINITION 1.14. A mapping $f: (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)$, taking a topological space $(X, \mathcal{T}_1)$ into a topological space $(Y, \mathcal{T}_2)$ is an <u>open mapping</u> if $f(G)$ is open in $(Y, \mathcal{T}_2)$ whenever G is open in $(X, \mathcal{T}_1)$. f is a <u>continuous mapping</u> if $f^{-1}(G)$ is open in $(X, \mathcal{T}_1)$ whenever G is open in $(Y, \mathcal{T}_2)$. Any image $f(X)$ of a topological space X under a continuous mapping f is called a <u>continuous image of X</u>.

DEFINITION 1.15. <u>A homeomorphism</u> is a one-to-one, open, continuous mapping of one topological space $(X, \mathcal{T}_1)$ onto another topological space $(Y, \mathcal{T}_2)$. Two topological spaces are homeomorphic if there exists a homeomorphism of one space onto the other.

Thus, if $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ are homeomorphic, their points can be put into one-to-one correspondence in such a way that their open sets also correspond to each other.

DEFINITION 1.16. A <u>topological property</u> is one which remains invariant over all homeomorphic transformations of a given topological space. Topology as a subject, investigates all the topological properties of a topological space.

2. COMPACT SPACES.

DEFINITION 2.1. Let X be a topological space. A class $\{G_i\}$ of open subsets of X is an <u>open cover</u> of X if each point in X belongs to at least one G_i. In other words, $X \subset \bigcup_i G_i$. A subclass of an open cover, which is itself an open cover is called a <u>subcover</u>.

DEFINITION 2.2. A <u>compact space</u> is a topological space in which every open cover has a finite subcover. A <u>compact subspace</u> of a topological space is a subspace which is compact as a topological space in its own right.

<u>Example 2.3.</u> Consider the class $\{ D_p : p \in Z \times Z \}$, where D_p is an open disc in $\mathbb{R}^2$ with centre $p = (m,n)$ and radius 1, Z being the set of integers. Then this class constitutes a cover of $\mathbb{R}^2$. On the other hand, the class of open discs with centre p and radius $\frac{1}{2}$ will not cover $\mathbb{R}^2$, since there exist points like $(\frac{1}{2}, \frac{1}{2})$ which do not belong to any member of the class.

THEOREM 2.4. Any closed subspace of a compact space is compact.

<u>Proof.</u> Let X be a compact space and let Y be a closed subspace of X. We have to show that Y is compact. Let $\{G_i\}$ be an open cover of Y. Each G_i, being open in the relative topology on Y, is obtained as the intersection of Y with an open subset H_i of X. Since Y is closed, Y' is open and Y' together with the H_i's constitutes an open cover of X. But X is compact and therefore this open cover admits of a finite subcover. If Y' occurs in this subcover, we omit it and the remaining sets constitute a finite subclass $\{H_{i_1}, H_{i_2}, \dots, H_{i_m}\}$ whose union covers X. Taking the intersections of each of these H_{i_m}'s with Y, we obtain a finite subcover $\{G_{i_1}, G_{i_2}, \dots, G_{i_m}\}$ of the original cover $\{G_i\}$ of Y which implies that Y is compact.

THEOREM 2.5. Any continuous image of a compact space is compact.

Proof. Let $f: X \rightarrow Y$ be a continuous mapping taking a compact space X into an arbitrary topological space Y. Let the image of X under f be $f(X)$. We must prove that $f(X)$ is a compact subspace of Y. Let $\{G_i\}$ be an open cover of $f(X)$. Then, each G_i is obtained as the interesection of $f(X)$ with an open subset H_i of Y. Since f is continuous, $f^{-1}(H_i)$ is open in X, for each H_i and $\{f^{-1}(H_i)\}$ forms an open cover of X. By the compactness of X, it yields a finite subcover. The union of the corresponding H_i's, of which these are inverse images clearly contains $f(X)$ and therefore the associated G_i's constitute a finite subcover of the original open cover of $f(X)$. This implies that $f(X)$ is compact as a subspace of Y.

DEFINITION 2.6. A topological space is compact if and only if every class of closed sets with empty intersection has a finite subclass with empty intersection.

This definition is equivalent to the original one, since a class of open sets will form an open cover if and only if the class of all their complements has empty intersection.

DEFINITION 2.7. A class of subsets of a nonempty set is said to have the finite intersection property if every finite subclass has non-empty intersection.

Exercise 2.8. Prove that a topological space is compact if and only if every class of closed sets which has the finite intersection property has a non-empty intersection.

DEFINITION 2.9. A subset A of a topological space X is sequentially compact if and only if every sequence in A has a subsequence which converges to a point in A.

If A is a finite subset of X, then it is sequentially compact. For, if $\{x_n\}$ is a sequence in A, then at least one of the elements $x \in A$ must appear an infinite number of times in the sequence. Thus $\{x, x, \ldots\}$ is a subsequence of $\{x_n\}$ and it converges to $x \in A$. The open interval $(0,1)$ is not sequentially compact, since the sequence $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ converges to 0 and so does every subsequence. But $0 \notin (0,1)$.

DEFINITION 2.10. A subset A of a topological space X is countably compact if and only if every infinite subset B of A contains a limit point belonging to A.

Every closed and bounded interval is countably compact. For if B is an infinite subset of $A = [a,b]$, then B is also bounded and hence contains a limit point (by the Bolzano-Weierstrass theorem which states that every infinite bounded set has at least one limit point) which belongs to A since A is closed. The open interval $(0,1)$ is not countably compact since $B = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ has only one limit point, 0, which does not belong to $(0,1)$.

THEOREM 2.11. Let X be a topological space. If X is compact or sequentially compact, then it is also countably compact.

Proof. We have to show that compact $\Longrightarrow$ countably compact $\Longleftarrow$ sequentially compact. First, assume X is compact. Let A be a subset of X with no limit point belonging to X. Then each point $x \in X$ belongs to an open set G_x which contains at most one point of A. The class $\{G_x\}$ is an open cover of X and by the compactness of X, there exists a finite subcover $\{G_{x_1}, \ldots, G_{x_m}\}$ such that $A \subset X \subset \bigcup_{i=1}^{m} G_{x_i}$. But each G_{x_i} contains at most one point of A and hence A being a subset of $\bigcup_{i=1}^{m} G_{x_i}$ can contain at most m points which in turn implies that A is finite. Thus, every infinite subset of X will contain at least one limit point in X which proves that X is countably compact.

Suppose X is sequentially compact and let A be an infinite subset of X. Then there exists a sequence $\{x_n\} \in A$ with distinct terms and this contains a subsequence $\{x_{n_i}\}$, also with distinct terms. This subsequence converges to a point $x \in X$. Hence, every open set containing x, contains an infinite number of points of A. Accordingly, $x \in X$ is a limit point of A which implies that X is countably compact.

Remark 2.12. The continuous image of a sequentially (countably) compact set is sequentially (countably) compact.

DEFINITION 2.13. A subset N of a topological space X is a <u>neighbourhood</u> of a point $x \in X$, if and only if there exists an open set G such that $x \in G \subset N$.

Notice that the relation 'N is a neighbourhood' of x' is the inverse of the relation 'x is an interior point of N'. Each closed interval $[x-\delta, x+\delta]$ is a neighbourhood of $x \in \mathbb{R}$, since it contains the open interval $(x-\delta, x+\delta)$ which is centred on x. Similarly, if $x \in \mathbb{R}^2$, every closed disc with centre x, is a neighborhood of x, since it contains the open disc with centre x.

<u>Exercise 2.14</u>. A set is open if and only if it contains a neighbourhood of each of its points.

The union U of all open subsets of a set A is an open subset of A. If A contains a neighbourhood of each of its points, then each point $x \in A$ belongs to some open subset of A which implies $x \in U$. On the other hand, if A is open, it contains a neighbourhood (A itself) of each of its points.

DEFINITION 2.15. The class of all neighbourhoods of a point $x \in X$ is called the <u>neighbourhood system of x</u> and denoted by $\mathcal{N}_x$.

THEOREM 2.6. The following properties characterize $\mathcal{N}_x$

(i) $\mathcal{N}_x$ is non-empty and x belongs to each member of $\mathcal{N}_x$.

(ii) The intersection of any two members of $\mathcal{N}_x$ belongs to $\mathcal{N}_x$.

(iii) Every superset of a member of $\mathcal{N}_x$ belongs to $\mathcal{N}_x$.

(iv) Each member $N \in \mathcal{N}_x$ is a superset of a member $N_1 \in \mathcal{N}_x$, where N_1 is a neighbourhood of each of its point, i.e., $N_1 \in \mathcal{N}_y$ for every $y \in N_1$.

Proof. (i) is obvious. To prove (ii), let N_1 and N_2 be two neighbourhoods of x. Then, there exist open sets G_1 and G_2 such that $x \in G_1 \subset N_1$, $x \in G_2 \subset N_2$. Hence $x \in G_1 \cap G_2 \subset N_1 \cap N_2$ and since $G_1 \cap G_2$ is open, $N_1 \cap N_2 \in \mathcal{N}_x$. To establish (iii), let $N \in \mathcal{N}_x$ and suppose $N_1 \supset N$. Since N is a neighbourhood of x, there exists an open set G such that $x \in G \subset N \subset N_1$ which implies $N_1 \in \mathcal{N}_x$. To show that (iv) is true, let $N \in \mathcal{N}_x$. Then, $x \in G \subset N$. If N_1 is chosen such that $G \subset N_1 \subset N$, then $N_1 \in \mathcal{N}_x$.

DEFINITION 2.17. A topological space is locally compact if and only if every point $x \in X$ has a compact neighbourhood.

Consider the real line $\mathbb{R}$, with its usual topology. Each point $x \in \mathbb{R}$ is interior to the closed interval $[x-\delta, x+\delta]$ which is bounded and therefore compact (by the Heine-Borel theorem which guarantees that a closed and bounded subset of $\mathbb{R}$ is compact). Thus each point has a compact neighbourhood which implies $\mathbb{R}$ is locally compact. But $\mathbb{R}$ is not a compact space since the class $\{\ldots,(-3,-1),(-2,0), (-2,0),(-1,1),(0,2),(1,3),\ldots\}$ is an open cover of $\mathbb{R}$ which does not contain a finite subcover. This shows that a locally compact space need not be compact. The converse however is true.

THEOREM 2.18. Every compact space is locally compact.

Proof. Let X be a compact space. Since X is also a topological space, it is a neighbourhood of each of its points which shows that X is locally compact.

DEFINITION 2.19. A topological space X is said to be embedded in a topological space, Y, if X is homeomorphic to a subspace of Y. Further, if Y is compact, then it is known as a compactification of X.

Usually, the compactification of a topological space X is accomplished by adjoining one or more points to X and then defining an appropriate topology on the enlarged set such that the enlarged space becomes compact or contains X as a subspace. The extended real number system $\mathbb{R} \cup \{-\infty, \infty\}$ provides a compactification for $\mathbb{R}$.

Let $\mathbb{C}$ denote the x-y plane in $\mathbb{R}^3$, the Euclidean space of dimension three. Let S denote the sphere with centre $(0,0,1)$ and unit radius. The line joining $(0,0,2) \in S$ and any point $p \in \mathbb{C}$ intersects S in exactly one point p', distinct from $(0,0,2)$. If $f : \mathbb{C} \to S$ defined by $f(p) = p'$, then f is a homeomorphism of $\mathbb{C}$, which is not compact, onto the subset $S \setminus \{\infty\}$ of S. But S is compact and hence is a compactification of $\mathbb{C}$.

DEFINITION 2.20. Let $(X, \mathcal{J})$ be a topological space. The Alexandrov or one-point compactification of $(X, \mathcal{J})$ is denoted by $(X_\infty, \mathcal{J}_\infty)$, where $X_\infty = X \cup \{\infty\}$, $\{\infty\}$ being called the point of infinity, which is distinct from every other point in X and $\mathcal{J}_\infty$ consists of every member of $\mathcal{J}$ as well as the complements in X_∞ of every closed and compact subset of X.

3. SEPARABLE SPACES, SOME SEPARATION AXIOMS.

DEFINITION 3.1. A subset A of a topological space X is everywhere dense or dense if and only if $\overline{A} = X$.

Observe that the set of rationals Q is dense in $\mathbb{R}$. In the case of the indiscrete space, every non-empty subset is dense in X.

DEFINITION 3.2. A subset A of a topological space X is nowhere dense if the interior of the closure of A is empty, i.e., $(\overline{A})^{o} = \emptyset$.

If $A = \{ 0 < x < 1, x \in Q \}$, $\overline{A} = [0,1]$, $(\overline{A})^{o} = (0,1) \neq \emptyset$. Hence A is a set which is not nowhere dense in $\mathbb{R}$. If $A = \{\frac{1}{n}\}$, $\overline{A} = \{0, 1, \frac{1}{2}, \ldots\}$ and $(\overline{A})^{o} = \emptyset$. Thus $\{\frac{1}{n}\}$ is nowhere dense in $\mathbb{R}$.

DEFINITION 3.3. A topological space is separable if it contains a countable dense subset.

The real line $\mathbb{R}$ with the usual topology is separable, since the set Q is countable and dense in $\mathbb{R}$. On the other hand, the real line together with the discrete topology is not separable, since every subset of $\mathbb{R}$ is both open and closed relative to the topology and so the only dense subset of $\mathbb{R}$ is $\mathbb{R}$ itself and it is not countable.

DEFINITION 3.4. A topological space X is a T_0-space if and only if for every pair of distinct points $x, y \in X$, there exists an open set G containing one of them but not the other.

Every subspace of a T_0-space is also a T_0-space. For, if Y is a subspace of a T_0-space X, for each pair $x,y \in Y$, there exists an open set G containing x but not y, since $x,y \in X$ also. $G \cap Y$ is an open set relative to $(Y, \mathcal{T}_Y)$ which contains x but not y and hence Y is also a T_0-space, with the relative topology

DEFINITION 3.5. A topological space is a T_1-space if and only if for every pair of distinct points $x,y \in X$, each belongs to an open set which does not contain the other, i.e., $x,y \in X$ implies that there exist G,H which are open such that $x \in G$, $x \notin H$, $y \in H$, $y \notin G$.

Every subspace of a T_1-space is T_1, for whenever Y is a subspace of a T_1-space X, $x\ y \in Y \Rightarrow x,y \in X$, there exist open sets G,H such that $x \in G$, $x \notin H$, $y \in H$, $y \notin G \Rightarrow x \in G \cap Y$, $x \notin H \cap y$, $y \in H \cap y$, $y \notin G \cap Y \Rightarrow (Y, \mathcal{T}_Y)$ is a T_1-space.

Example 3.6. Consider $X = \{a,b,c\}$ and define

$$\mathcal{T}_1 = \{X, \Phi, \{a\}\}$$
$$\mathcal{T}_2 = \{X, \Phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}\}.$$

Then $(X, \mathcal{T}_1)$ is not a T_1-space, since b and c belong only to X which also contains a, whereas $(X, \mathcal{T}_2)$ is a T_1-space.

THEOREM 3.7. A topological space X is a T_1-space if and only if every singleton set is closed.

Proof. Let $x,y \in X$, $x \neq y$. Since X is T_1, there exist open sets G,H such that $x \in G$, $y \in H$, $x \notin H$, $y \notin G$. As y varies the union of all the corresponding open sets H is open and does not contain x. Clearly this union is $X - \{x\}$ and therefore $\{x\}$ is closed.

Conversely, if $\{x\}$ and $\{y\}$ are closed, $X - \{x\}$ and $X - \{y\}$ are open and contain y and x respectively which implies that X is a T_1-space.

DEFINITION 3.8. A topological space is a T_2-space or a Hausdorff space if and only if for every pair of points $x, y \in X$, there exist open sets G,H such that $x \in G$, $y \in H$, $G \cap H = \emptyset$.

Notice that every T_2-space is also a T_1-space and every subspace of a T_2-space is again a T_2-space. To prove the latter statement, let Y be a subspace of a T_2-space X. Let $x, y \in Y$. Then $x, y \in X$ also and since X is T_2, there exist open sets G,H such that $x \in G$, $y \in H$, $G \cap H = \emptyset$. $G \cap Y$, $H \cap Y$ are open in $\mathcal{J}_Y$ and $(G \cap Y) \cap (H \cap Y) = Y \cap (G \cap H) = Y \cap \emptyset = \emptyset$.

Exercise 3.9. Every metric space is a Hausdorff space.

A metric space is a pair (X,d), where X is a nonempty set and d is a real-valued function satisfying the conditions

(i) $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$.

(ii) $d(x,y) = d(y,x)$.

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ where $x,y,z \in X$.

Let (X,d) be a metric space and let $x,y \in (X,d)$. Consider two open spheres G,H with centres x,y and radius $\varepsilon/3$. Since x and y are distinct points, $d(x,y) = \varepsilon > 0$. Assume that $G \cap H \neq \emptyset$. Let $z \in G \cap H$. Then $d(x,z) < \varepsilon/3$, $d(y,z) < \varepsilon/3$ and by the triangular inequality $d(x,y) \leq d(x,z) + d(z,y) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}$ which contradicts the fact that $d(x,y) = \varepsilon$. Hence $G \cap H = \emptyset$ or (X,d) is Hausdorff.

THEOREM 3.10. Let A be a compact subset of a Hausdorff space. Let $x \in X \setminus A$. Then open sets G and H can be found such that $x \in G$, $A \subset H$, $G \cap H = \emptyset$.

Proof. Suppose X is a Hausdorff space and let A be a compact subset of X. Let $y \in A$. Since X is Hausdorff, there exist open sets G_1 and H_1 such that $x \in G_1$, $y \in H_1$, $G_1 \cap H_1 = \emptyset$. As y varies over A, open sets $H_1, H_2, \ldots$ can be found such that each y belongs to one of the H_i's and $\{H_i\}$ constitutes an open cover of A. Since A is compact, $\{H_i\}$ has a finite subcover $\{H_{i_1}, H_{i_2}, \ldots, H_{i_m}\}$. Corresponding to these H_{i_m}, a finite class $\{G_{i_1}, G_{i_2}, \ldots, G_{i_m}\}$ can be obtained such that x belongs to each G_{i_m}. Choose $G = \bigcap_{j=1}^{m} G_{i_j}$ and $H = \bigcup_{j=1}^{m} H_{i_j}$. Then $x \in G$, $A \subset H$ and $G \cap H = \emptyset$.

THEOREM 3.11. In a Hausdorff space, every compact subset is closed.

Proof. Let X be a Hausdorff space and let A be a compact subset of X. If $x \in X \setminus A$, there exist open sets G, H such that $x \in G$, $A \subset H$ and $G \cap H = \emptyset$. Hence $G \cap A = \emptyset$ which implies $x \in G \subset X \setminus A$. Thus, as x varies over $X \setminus A$, we obtain a class $\{G_x\}$ of open sets, each of which is disjoint from A. Hence $X \setminus A$ is open which shows that A is closed.

THEOREM 3.12. A one-to-one, continuous mapping of a compact space X onto a Hausdorff space Y is a homeomorphism.

Proof. Let f be the one-to-one, continuous mapping of X onto Y. We have to prove that f is an open mapping, or that $f(G)$ is open in Y whenever G is open in X. It suffices to show that $f(F)$ is closed in Y whenever F is closed in X. Since F is a closed subset of the compact space X, it iscompact by Theorem 2.4. The continuous image $f(F)$ of the compact F is compact by Theorem 2.5. But $f(F)$ is compact and a subset of the Hausdorff space Y. $f(F)$ is therefore closed by Theorem 3.11.

4. COMPLETELY REGULAR AND NORMAL SPACES.

DEFINITION 4.1. A topological space X is regular if for every closed subset F of X and for every $x \in X$, $x \notin F$, there exist disjoint open sets G, H such that $x \in G$, $F \subset H$. Notice that every subspace of a regular space is regular.

Suppose $X = \{a, b, c\}$, $\mathfrak{J} = \{X, \phi, \{a\}, \{b, c\}\}$. The class $\mathcal{C}$ of closed sets is given by $\mathcal{C} = \{\phi, X, \{b, c\}, \{a\}\}$. $(X, \mathfrak{J})$ is a regular space but it is not a T_1-space since $\{b\}$ and $\{c\}$ are not closed sets.

DEFINITION 4.2. A regular space which is also a T_1-space is called a T_3-space.

THEOREM 4.3. Every T_3-space is a T_2-space.

Proof. Let X be a T_3-space. Since X is T_1, if $x \in X$, $\{x\}$ is a closed set. If $y \neq x$, by the regularity of X, there exist open sets G,H such that $y \in G$, $\{x\} \subset H$, $G \cap H = \phi$, which implies that every pair $x, y \in X$ satisfies the criterion for a space to be Hausdorff.

DEFINITION 4.4. A topological space X is completely regular if and only if for every closed subset F of X and for every $x \in X$, $x \notin F$, there exists a continuous real-valued function $f : X \to [0,1]$ such that $f(x) = 0$, $f(F) = 1$.

Remark 4.5. Every subspace of a completely regular space is completely regular. Let X be a completely regular space. Let $F \subset X$ be a closed set. Then $X \setminus F$ is $\mathcal{J}$-open and $Y \cap (X \setminus F)$ is $\mathcal{J}_Y$-open, where Y is a subspace of X. This implies that $(Y \cap X) \setminus F$ is $\mathcal{J}_Y$-open which shows that $Y \setminus F$ is $\mathcal{J}_Y$-open. Assume that $x \in Y$, $x \notin F$. Then $x \in X$ also. Since X is completely regular, there exists a continuous real-valued function $f : X \to [0,1]$ such that $f(x) = 0$, $f(F) = 1$ and since $Y \setminus F$ is $\mathcal{J}_Y$-open, F is $\mathcal{J}_Y$-closed which proves that Y is also completely regular.

THEOREM 4.6. Every completely regular space is regular.

Proof. Let X be a completely regular space. Let F be a closed subset of X and let $x \in X$, $x \notin F$. Then, there exists a real-valued continuous function $f : X \to [0,1]$ such that $f(x) = 0$,

$f(F) = 1$. Since the real line is Hausdorff, the subspace $[0,1]$ of $\mathbb{R}$ is also Hausdorff. Therefore, there exist open sets G, H such that $0 \in G$, $1 \in H$, $G \cap H = \phi$. $f^{-1}(G)$ and $f^{-1}(H)$ are open in X, since f is continuous and their intersection is empty. Further $x \in f^{-1}(G)$ and $F \subset f^{-1}(F)$. Hence X is a regular space.

DEFINITION 4.7. A completely regular space which is also T_1 is called a <u>Tychonoff space</u>.

DEFINITION 4.8. Let $F = \{f_i\}$ be a class of functions from any set X to a set Y. Then F <u>separates points</u> if and only if every pair of distinct points $x, y \in X$, there exists an $f_i \in F$ such that $f_i(x) \neq f_i(y)$.

If $F = \{\sin x, \sin 2x, \sin 3x, \ldots\}$ is a class of function defined on $\mathbb{R}$, then $f_i(0) = f_i(\pi) = 0$. Thus F does not separate points of $\mathbb{R}$.

THEOREM 4.9. The space $\mathscr{C}(X, \mathbb{R})$ of all continuous real-valued functions defined on a Tychonoff space X separates points.

<u>Proof.</u> Let X be a Tychonoff space and let $x, y \in X$. Since X is T_1, $\{x\}$ and $\{y\}$ are closed. Since X is completely regular, there exists a real-valued continuous function $f : X \rightarrow [0,1]$ such that $f(x) = 0$, $f(\{y\}) = 1$ which implies that $f(x) \neq f(y)$. Thus $\mathscr{C}(X, \mathbb{R})$ separates points of X.

DEFINITION 4.10. A topological space X is <u>normal</u> if and only if for every pair F_1, F_2 of disjoint closed subsets of X, there exist disjoint open sets G and H such that $F_1 \subset G$ and $F_2 \subset H$.

If $X = \{a,b,c\}$, $\mathcal{J} = \{X,\phi, \{a\}, \{b\}, \{a,b\}\}$, then $\mathcal{C} = \{\phi,X, \{b,c\}, \{c,a\}, \{c\}\}$. If F_1 and F_2 are disjoint closed subsets of X, one of them, say F_1, must necessarily be ϕ. Hence ϕ and X are disjoint open sets and $F_1 \subset \phi$, $F_2 \subset X$. Thus $(X,\mathcal{J})$ is a normal space. But it is not a T_1-space since $\{a\}$ and $\{b\}$ are not closed; neither is it regular since $a \notin \{c\}$ and the only open set containing c is X itself, which also contains a. Any space with the discrete topology or the trivial topology is normal, for in the first case every subset is open and closed while in the second, the only two subsets are X and ϕ, which are both open and closed.

THEOREM 4.11. Let X be a topological space. Then the following are equivalent:

(i) X is normal

(ii) If H is an open superset of a closed set F, there exists an open set G such that $F \subset G \subset \overline{G} \subset H$.

Proof. (i) $\Rightarrow$ (ii). Let $F \subset H$, with F closed and H open. Then the complement of H denoted by H' is closed and $F \cap H' = \phi$. By the normality of X, there exist open sets G_1 and G_2 such that $F \subset G_1$, $H' \subset G_2$ and $G_1 \cap G_2 = \phi$. But $G_1 \cap G_2 = \phi$ implies $G_1 \subset G_2'$ and $H' \subset G_2$ implies $G_2' \subset H$. In addition, G_2' is closed. Therefore $F \subset G_1 \subset \overline{G}_1 \subset G_2' \subset H$.

(ii) $\Rightarrow$ (i). Let F_1, F_2 be disjoint, closed sets. Then $F_1 \subset F_2'$ and F_2' is open. By (ii), there exists an open set G such that $F_1 \subset G \subset \overline{G} \subset F_2'$. But $\overline{G} \subset F_2'$ implies $F_2 \subset \overline{G}'$ and

$G \subset \bar{G}$ implies $G \cap \bar{G}' = \phi$. Furthermore, $\bar{G}'$ is open. Thus $F_1 \subset G$, $F_2 \subset \bar{G}'$ with $G, \bar{G}'$ being disjoint open sets. This proves that X is normal.

THEOREM 4.12. A compact Hausdorff space is normal.

Proof. Let X be a compact Hausdorff space and let F_1, F_2 be two disjoint closed subsets of X. Since X is compact, F_1 and F_2 are compact by Theorem 3.4. Let $x \in F_1$. By Theorem 3.10, there exist disjoint open sets G_1, H_1 such that $x \in G_1$, $F_2 \subset H_1$. As x varies over F_1, we obtain a class $\{G_i\}$ of open sets whose union contains F_1. Since F_1 is compact, there is a finite subcover $\{G_{i_1}, G_{i_2}, \ldots, G_{i_m}\}$ which contains F_1. If $\{H_{i_1}, H_{i_2}, \ldots, H_{i_m}\}$ is the corresponding finite subclass of open sets containing F_2, define $G = \bigcup_{j=1}^{m} G_{i_j}$, $H = \bigcap_{j=1}^{m} H_{i_j}$. Then $F_1 \subset G$, $F_2 \subset H$ and $G \cap H = \phi$ which proves that X is normal.

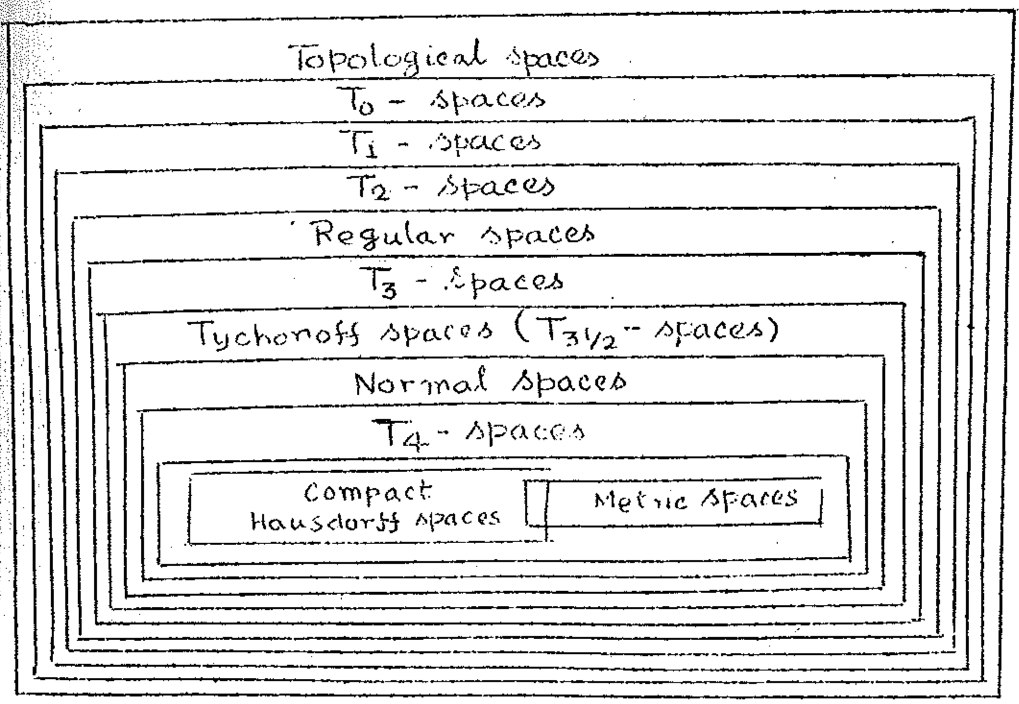
DEFINITION 4.13. A normal space which is also a T_1-space is a T_4-space.

EXERCISE 4.14. Every T_4-space is also a T_3-space.

Let X be a T_4-space. Let $x \in X$ and let F be a closed subset of X disjoint from x. Since X is T_1, $\{x\}$ is closed. By normality, there exist open sets G,H such that $\{x\} \subset G$, $F \subset H$, $G \cap H = \phi$ which implies that X is a T_3-space.

LEMMA (Urysohn) 4.15. Let X be a normal space. If F_1, F_2 are closed subspaces of X, there exists a real-valued continuous function $f: X \to [0,1]$ such that $f(F_1) = 0, f(F_2) = 1$.

By virtue of Urysochn's lemma, a T_4-space is Tychonoff. Since a completely regular space is also regular, a Tychonoff space is a T_3-space.



5. <u>CONNECTED SPACES</u>.

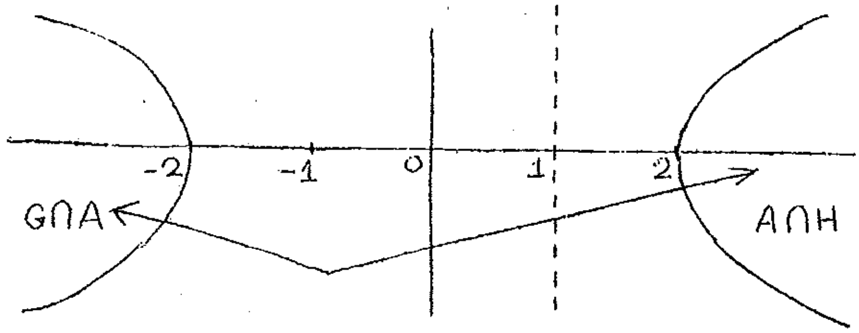
DEFINITION 5.1. Two subsets A and B of a topological space X are <u>separated</u> if and only if (i) A and B are disjoint (ii) each contains no limit point of the other. In other words, $A \cap \bar{B} = \phi$ and $\bar{A} \cap B = \phi$.

Consider $A = (2,3)$, $B = (3,4)$, $C = [4,5)$. Then $\bar{A} = [2,3]$ $\bar{B} = [3,4]$ and $\bar{C} = [4,5]$. A and B are separated while B and C are not, since $\bar{B} \cap C = \{4\}$.

DEFINITION 5.2. A subset A of a topological space X is disconnected if there exist open subsets G, H of X such that $A \cap G$ and $A \cap H$ are disjoint, non-empty sets whose union is A. $G \cup H$ is called a disconnection of A.

DEFINITION 5.3. A subset A which is not disconnected is said to be connected.

Example 5.4. Let $A \subset \mathbb{R}^2$ defined by $\{(x,y) : x^2 - y^2 \geq 4\}$. If $G = \{(x,y) : x < 1\}$ and $H = \{(x,y) : x > 1\}$, then $A \cap G$ and $A \cap H$ are disjoint and nonempty. Further, $(A \cap G) \cup (A \cap H) = A$ and $G \cup H$ constitutes a disconnection of A.



EXERCISE 5.5. A subset A of a topological space is disconnected if and only if $A \cap G \neq \phi$, $A \cap H \neq \phi$, $A \subset G \cup H$, $G \cap H \subset A'$.

For A to be disconnected, in addition to the first two conditions, $(A \cap G) \cup (A \cap H) = A$ and $(A \cap G) \cap (A \cap H) = \phi$. But $(A \cap G) \cup (A \cap H) = A \cap (G \cup H)$. Thus if $A = (A \cap G) \cup (A \cap H)$, $A \subset G \cup H$ and if $A \subset G \cup H$, $A \cap (G \cup H) = A$. Again, $(A \cap G) \cap (A \cap H) = A \cap (G \cap H)$. Therefore $A \cap (G \cap H) = \phi$ implies $G \cap H \subset A'$ and if $G \cap H \subset A'$, $A \cap (G \cap H) = \phi$.

THEOREM 5.6. A set A is connected if and only if it is not the union of two nonempty separated sets.

Proof. We show equivalently that A is disconnected if and only if it is the union of two nonempty separated sets. Suppose A is disconnected and let $G \cup H$ be a disconnection of A. Then A is the union of two non-empty sets $A \cap G$ and $A \cap H$, which are also disjoint. Suppose they are not separated and let x be a limit point of $A \cap G$ which belongs to $A \cap H$. Then $x \in H$ and H being an open set, contains a point of $A \cap G$ different from x, which implies that $(A \cap G) \cap H \neq \phi$. This gives a contradiction, since $G \cup H$ being a disconnection implies $(A \cap G) \cap (A \cap H) = \phi$. Conversely, let $A = B \cup C$, where B and C are nonempty separated Sets. Then $B \cap \overline{C} = \phi$, $\overline{B} \cap C = \phi$. Choose $G = \overline{B}'$ and $H = \overline{C}'$. Then G and H are open and $(B \cup C) \cap G = C$, $(B \cup C) \cap H = B$ are nonempty disjoint sets whose union is $B \cup C$. Thus $G \cup H$ is a disconnection of $A = B \cup C$.

COROLLARY 5.7. If A and B are connected sets which are not separated, then $A \cup B$ is connected.

Proof. Suppose $A \cup B$ is disconnected and let $G \cup H$ be a disconnection of $A \cup B$. By 5.5, $A \cup B \subset G \cup H$, $G \cap H \subset (A \cup B)'$. Since A is a subset of $A \cup B$, $A \subset G \cup H$ and $G \cap H \subset A'$. Thus, if $A \cap G$ and $A \cap H$ are nonempty, then $G \cup H$ is a disconnection of A. But A is connected, hence either $A \cap G$ or $A \cap H$ is empty, which implies that either $A \subset H$ or $A \subset G$. Similarly, either $B \subset G$ or $B \subset H$. Now, if $A \subset G$, $B \subset H$ (or $A \subset H$, $B \subset G$) by Theorem 5.6,

$(A \cup B) \cap G = A$ and $(A \cup B) \cap H = B$ are separated sets which contradicts the hypothesis that A and B are not separated. Hence either $A \cup B \subset G$ or $A \cup B \subset H$ and so $G \cup H$ fails to be a disconnection of $A \cup B$. Thus $A \cup B$ is connected.

DEFINITION 5.8. Let X be a topological space. Then X is disconnected if and only if there exist nonempty open sets G and H such that $X = G \cup H$ and $G \cap H = \emptyset$. X is connected if and only if it cannot be expressed as the union of two disjoint, nonempty open (or closed) sets.

EXERCISE 5.9. In a connected space X, the only subsets which are both open and closed are X and $\emptyset$.

If possible, let $A \subset X$ be both open and closed. Then A' is also both open and closed. $A = \bar{A}$ and $A' = \overline{A'}$ and $A \cap \overline{A'} = A \cap A'$ $= \phi$, $\bar{A} \cap A' = A \cap A' = \phi$ which shows that A and A' are separated. Now $X = A \cup A'$ and therefore not connected since A, A' are nonempty separated sets. This gives a contradiction.

Remark 5.10. A subset of $\mathbb{R}$ is connected if and only if it is an interval.

THEOREM 5.11 Any continuous image of a connected space is connected.

Proof. Let $f : X \to Y$ be a continuous mapping of a connected space X into an arbitrary topological space Y. We must show that $f(X)$ is connected as a subspace of Y. Suppose $f(X)$ is disconnected. Then, there exist open sets G, H of Y such that $f(X) \subset G \cup H$, $G \cap H \subset (f(X))'$, $f(X) \cap G \neq \phi$, $f(X) \cap H \neq \phi$. As f is continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are open sets of X and

$f^{-1}(G) \cup f^{-1}(H) = X$ gives a disconnection of X, which contradicts the connectedness of X. Thus $f(X)$ is connected as a subspace of Y.

THEOREM 5.12. A topological space X is disconnected if and only if there exists a continuous mapping f of X onto the discrete two-point space $\{0,1\}$.

Proof. Suppose X is disconnected. Then there exist open sets G, H such that $X = (X \cap G) \cup (X \cap H)$. Define a mapping f by $f(x) = 0$ if $x \in X \cap G$, $f(x) = 1$ if $x \in X \cap H$. f is clearly onto and continuous since $X \cap G$ and $X \cap H$ are nonempty, open and disjoint.

Conversely, if there exists such a continuous mapping of X onto $\{0,1\}$, then X is disconnected. For, if X were connected, f being continuous, its image $\{0,1\}$ should be connected by Theorem 5.11 and $\{0,1\}$ is certainly disconnected.

DEFINITION 5.12. A maximal connected subspace of a topological space X is called a component.

If X is connected, it has only one component, namely itself. In a discrete space, every singleton is a component.

LEMMA 5.14. Let X be a topological space. If $\{C_i\}$ is a nonempty class of connected subspaces of X with $\bigcap_i C_i \neq \phi$, then $C = \bigcup_i C_i$ is also a connected subspace.

Proof. Suppose C is disconnected. Then there exist open sets G,H such that $C \subset G \cap H$ and $G \cap H \subset C'$. Each of the C_i's being connected will be contained either in G or in H. Since

$\bigcap_i C_i \neq \phi$, all the C_i's must belong only to G or only to H, which implies that $C = \bigcup_i C_i$ belongs completely to G or to H and thus $G \cup H$ fails to be a disconnection for C. Thus C is connected.

LEMMA 5.15. Let X be a topological space and let A be a connected subspace of X. If B is a subspace of X such that $A \subseteq B \subseteq A'$, then B is also a connected subspace. In particular, $\bar{A}$ is connected.

Proof. Assume B is disconnected. Then there exist open sets G, H of X such that $B \subset G \cup H$, $B \cap G \neq \phi$, $B \cap H \neq \phi$, $G \cap H \subset B'$. Since A is connected and $A \subseteq B$ implies $A \subset G \cup H$, A is contained either in G or in H and is disjoint from the other. Assume that $A \cap H = \phi$. Then $\bar{A} \cap H = \phi$ and since $B \subseteq \bar{A}$, $B \cap H = \phi$ which contradicts the fact that B is disconnected. Hence B is connected In particular, since $A \subseteq \bar{A}$, $\bar{A}$ is connected.

THEOREM 5.15. Let X be a topological space. Then the following are true:

(i) Each point $x \in X$ is contained in exactly one component of X.

(ii) Each connected subspace is contained in a component of X.

(iii) If A is connected subspace of X which is both open and closed, then it is a component of X.

(iv) Each component of X is closed.

Proof. (i) Let $x \in X$. Let $\{C_i\}$ be the class of all connected subspaces of X which contain x. This class is nonempty, since $\{x\}$ belongs to it and is connected. By Lemma 5.14, $C = \bigcup_i C_i$

is a connected subspace of X which contains x. Clearly C is maximal and the only component of X which contains x.

(ii) follows from (i) since a connected subspace of X is contained in the component which contains any one of its points.

(iii) Since A is a connected subspace of X, by (ii), $A \subset C$, where C is a component of X. But $C = (C \cap A) \cup (C \cap A')$, $A \cap C \neq \phi$, $C \cap A' \neq \phi$, A being both open and closed, A' is also both open and closed. Hence C has a disconnection which contradicts the fact that it is a component and therefore connected. Thus A cannot be contained in C and therefore A = C.

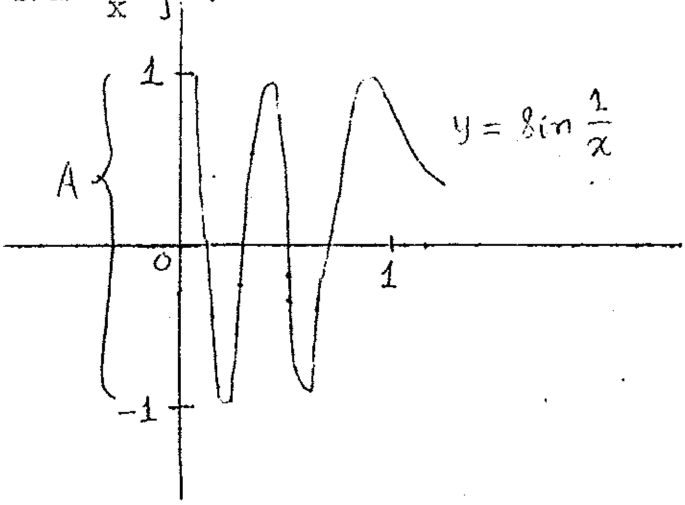
(iv) Let C be a component of X. If C is not closed, $C \subset \overline{C}$. By Lemma 5.15, since C is connected, $\overline{C}$ is also connected which contradicts the maximality of C as a connected subspace of X. Hence C is closed.

6. TYPES OF CONNECTEDNESS.

DEFINITION 6.1. A topological space X is locally connected if and only if for every point $x \in X$ and every open set G containing x, there exists an open connected neighbourhood of x which is contained in G.

In the case of a discrete space, each singleton set $\{x\}$ is an open connected set containing x which is contained in every open set containing x.

Example 6.2. A connected space need not be locally connected. Let $A = \{(0,y) : -1 \leq y \leq 1\}$ and $B = \{(x,y) : x > 0,\ y = \sin \frac{1}{x}\}$.



A is an interval and therefore by 5.12 connected. B is the continuous image of a connected set and hence connected. A and B are not separated, since every point of A is a limit point of B. Thus $A \cup B$ is connected by 5.7. But $A \cup B$ is not locally connected, for if the point $(0,1)$ is considered, the open disc with centre $(0,1)$ and radius $\frac{1}{4}$ does not contain any connected neighbourhood of $(0,1)$.

Example. 6.3. A locally connected space need not be connected. If $A = \{x : a < x < b\}$ and $B = \{x : c < x < d\}$, $A \cup B$ is locally connected since A and B are locally connected. But $A \cup B$ is not connected by Theorem 5.6, since $A \cap \overline{B} = \Phi$, $B \cap \overline{A} = \Phi$ implies that A and B are separated.

THEOREM 6.4. Let X be a locally connected space and let C be a component of X. Then C is open.

<u>Proof.</u> Let $x \in C$. Since X is locally connected, there exists at least one open connected neighbourhood N of x. Since C is a component, $N \subseteq C$. This implies that C is a neighbourhood of x. But x being an arbitrary point of C, C is a neighbourhood of each of its points. Using 2.14, we conclude that C is open.

<u>Exercise 6.5.</u> Show that in a compact locally connected space, the number of components is finite.

<u>Proof.</u> Let X be a locally connected space, which is also compact. Let $x \in X$. Then x belongs to one and only one component, say C_x. Each C_x is open by Theorem 6.4 and hence $\bigcup_{x \in X} C_x = X$ is also open. Thus $\{C_x\}$ constitutes a cover for X. By the compactness of X, this open cover admits of a finite subcover. However, the removal of any C_x from the class $\{C_x\}$ implies that it is no longer a cover, since a corresponding $x \in X$ will be left out. Thus, $\{C_x\}$ itself should be a finite cover which proves that the number of components is finite.

DEFINITION 6.6. Let I be the closed unit interval $[0,1]$ Let X be an arbitrary topological space and let $x, y \in X$. A continuous function $f : I \to X$ is called an <u>arc</u> (or <u>path</u>) from x to y if $f(o) = x$ and $f(1) = y$. Here x is called the initial point and y is the terminal point.

Notice that if $f : I \to X$ is an arc from x to y, $\tilde{f} : I \to X$ defined by $\tilde{f}(s) = f(1-s)$ is a path from y to x.

DEFINITION 6.7. For any $x \in X$, the constant function e_x: $I \to X$ defined by $e_x(s) = x$ is continuous and hence an arc, known as the <u>constant path</u> at x.

DEFINITION 6.8. A topological space X is <u>arcwise connected</u> if and only if for each pair of points, $x, y \in X$, there exists an arc joining x and y which belongs to X.

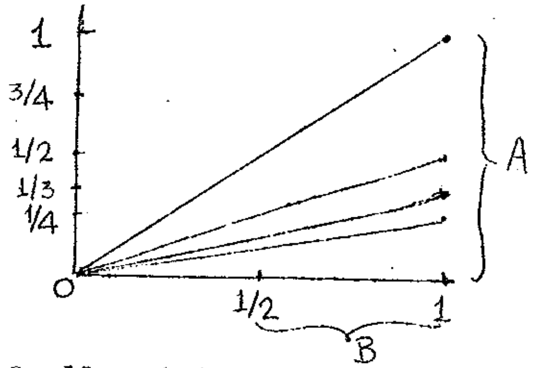
DEFINITION 6.9. The maximal arcwise connected subsets of X are called the <u>arcwise connected components</u> of X.

Observe that these arcwise connected components of X constitute a partition of X.

THEOREM 6.10. Every arcwise connected space X is connected.

<u>Proof.</u> Let $x \in X$. For each $y \in X$, $y \neq x$, define I_{xy} as the arc joining x and y. Then, each I_{xy} is the continuous image of $[0,1]$ and is therefore connected, by 5.10 and Theorem 5.11. The intersection of all the I_{xy}'s is nonempty since x belongs to each I_{xy}. Thus, $\bigcup_{y \in X} I_{xy}$ is connected by Lemma 5.14, which implies that X is connected.

<u>Example 6.11.</u> Every connected space need not be arcwise connected. Let $A = \{(x,y) : 0 \leq x \leq 1, y = \frac{x}{n}, n$ being an integer$\}$ and $B = \{(x,0) : \frac{1}{2} \leq x \leq 1\}$



A is the set of all points on the line segments joining (0,0) to the points of the form $(1,\frac{1}{n})$ where n is an integer. A and B being arcwise connected are also connected. Furthermore, each point $x \in B$ is a limit point of A and hence A and B are not arcwise connected, because there exists no arc joining any point of A to any point of B.

THEOREM 6.12. Arcwise-connectedness is a topological property.

Proof. Let X be an arcwise connected space and let h be a homeomorphism of X onto an arbitrary topological space Y. We have to prove that Y is also arcwise connected. Since X is arcwise connected, there exists a continuous function $f : I \to X$ such that if $x, y \in X$, $f(o)=x, f(1)=y$. Let a, b be the points of Y which correspond to $x, y \in X$ under the homeomorphism h. Then $h(x) = a$, $h(y) = b$. Define $g = hf$. Then clearly $g : I \to Y$ and $g(o) = h(f(o)) = h(x) = a$, $g(1) = h(f(1)) = h(y) = b$. g is continuous, being the product of two continuous functions and is an arc joining a and b which implies that Y is arcwise connected.

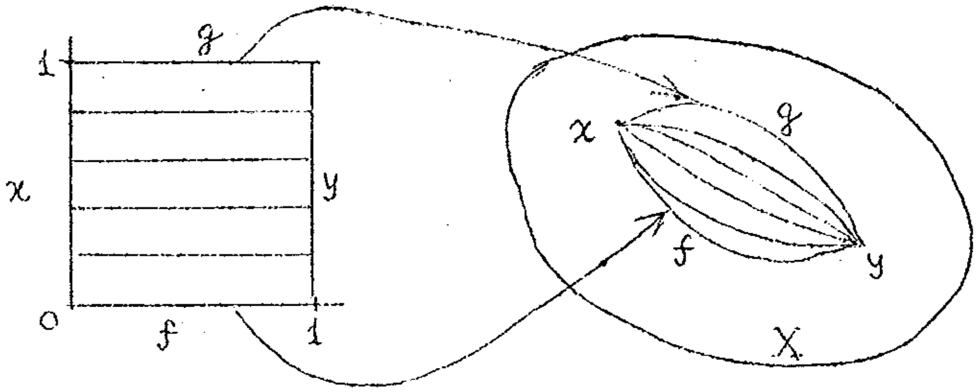
DEFINITION 6.13. A closed path is one for which the initial and terminal points coincide and the path is then said to be closed at that point.

In particular, the constant path e_x is a closed path at x.

DEFINITION 6.14. Let $f : I \to X$, $g : I \to X$ be two paths with the same initial point $x \in X$ and the same terminal point $y \in X$. Then f is said to be homotopic to g if there exists a continuous function $h : I^2 \to X$ such that

$$h(s,o) = f(s) \qquad h(o,t) = x$$
$$h(s,1) = g(s) \qquad h(1,t) = y$$

as indicated in the figure below



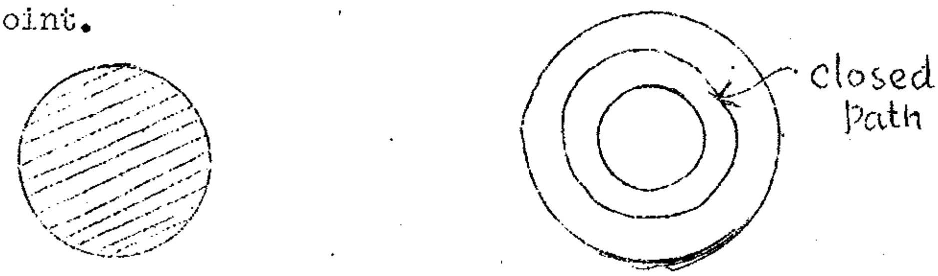
h is called a homotopy from f to g.

Example 6.51. Let X be the set of points between two concentric circles. Then the paths f and g in the diagram below on the left are homotopic, whereas the paths f and g are not homotopic as seen on the right hand side.

DEFINITION 6.16. A closed path $f : I \to X$ which is homotopic to the constant path is said to be contractable to a point.

DEFINITION 6.17. A topological space X is simply connected if and only if every closed path in X is contractable to a point.

Example 6.18. An open disc in $\mathbb{R}^2$ is simply connected, whereas an annulus is not, since there are closed curves as indicated in the figure on the right, which are not contractable to a point.



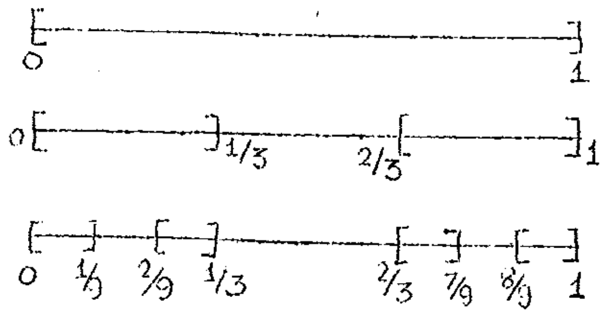
simply connected   not simply connected

DEFINITION 6.19. A topological space X is <u>totally disconnected</u> if for every pair of distinct points, $x, y \in X$, there exists a disconnection of X.

The discrete space is the simplest example of a totally disconnected space.

<u>Example 6.20.</u> The rationals and the irrationals are totally disconnected spaces. For if $x, y \in Q$, there exists an irrational number a such that $x < a < y$. Choose $G = \{x \in Q : x < a\}$ $H = \{x \in Q : x > a\}$. Then $G \cup H$ is a disconnection of Q and $x \in G$, $y \in H$. Thus Q is totally disconnected. Similarly, the set of all irrationals is also totally disconnected.

Consider the Cantor set F defined as follows. Let $F_1 = [0,1]$, $F_2 = [0,\frac{1}{3}] \cup [\frac{2}{3}, 1]$, $F_3 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ and so on. Define $F = \bigcap_{n=1}^{\infty} F_n$.



The Cantor set is an outstanding example of a compact totally disconnected subspace of the real line $\mathbb{R}$.

Remark 6.21. The components of a totally disconnected space are its points.

For, if X is a totally disconnected space, it suffices to show that every subspace Y of X which contains more than one point is disconnected. Let $x, y \in Y$, $x \neq y$ and let $X = G \cup H$ be a disconnection of X with $x \in G$, $y \in H$. Then $Y = (Y \cap G) \cup (Y \cap H)$ is obviously a disconnection of Y, thus proving that every singleton is maximal connected and therefore a component.

7. BASES AND PRODUCT TOPOLOGIES.

DEFINITION 7.1. Let X be a topological space. A class of open subsets of X is a base for the topology $\mathcal{T}$ defined on X if and only if every open set $G \in \mathcal{T}$ is the union of members of $\mathcal{B}$. In other words, $\mathcal{B} \subset \mathcal{T}$ is a base for $\mathcal{T}$ if and only if for every point $x \in X$ which is contained in an open set $G \in \mathcal{T}$, there exists $B \in \mathcal{B}$ such that $x \in B \subset G$.

Observe that $\emptyset = \cup \{ B_\lambda : \lambda \in \emptyset \}$ and hence belongs to $\mathcal{B}$.

Example 7.2. The open intervals of $\mathbb{R}$ form a base for the usual topology on $\mathbb{R}$. For if $G \subset \mathbb{R}$ is open and $x \in G$, there exists an open interval (a,b) with $x \in (a,b) \subset G$. The open discs form a base for the usual topology on $\mathbb{R}^2$. The open rectangles in $\mathbb{R}^2$, with their sides parallel to the axes, also constitute a base $\mathcal{B}$ for the usual topology on $\mathbb{R}^2$. For, if $G \subset \mathbb{R}^2$ is open and $x \in G$, there exists an open disc with centre x such that the disc is contained in G. Any rect··gle $B \in \mathcal{B}$ whose vertices lie on the boundary of this disc will satisfy the condition $x \in B \subset G$.

In the case of a discrete space, all the singleton sets form a base for the corresponding discrete topology.

Given any class $\mathcal{B}$ of subsets of a set X, is it possible for this class to be a base for some topology on X ? Since X is open in every topology on X, the condition $X = \bigcup \{B : B \in \mathcal{B}\}$ is necessary. In fact, if $X = \{a,b,c\}$ and $\mathcal{B} = \{ \{a,b\} , \{a,c\} \}$, then $\mathcal{B}$ cannot be a base for any topology on X, since their intersection $\{a\}$ should be open, as the members of a base are open subsets of X, and $\{a\}$ is not the union of members of $\mathcal{B}$. We can therefore formulate a set of necessary and sufficient conditions for a class of subsets of X to be a base for some topology on X.

THEOREM 7.3. Let $\mathcal{B}$ be a class of subsets of a nonempty set X. Then $\mathcal{B}$ is a base for a topology on X if and only if

(i) $X = \bigcup \{ B : B \in \mathcal{B}\}$

(ii) for every $B_1, B_2 \in \mathcal{B}$ and every $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$, (i.e.) the intersection of any two members of $\mathcal{B}$ is also a union of members of $\mathcal{B}$.

Proof. If $\mathcal{B}$ is a base for some topology on X, then the members of $\mathcal{B}$ are open sets and the intersection of two open sets being open can be expressed as a union of members of $\mathcal{B}$.

On the contrary, suppose $\mathcal{B}$ satisfies conditions (i) and (ii). Define $\mathcal{J}$ to be the collection of all subsets of X which are the unions of members of $\mathcal{B}$. We shall prove that $\mathcal{J}$ is a topology on X. 1.1 (i) is obvious. Since each members of $\mathcal{J}$ is a union of members of $\mathcal{B}$, the union of any number of members of $\mathcal{J}$ is a union of members of $\mathcal{B}$ and therefore belongs to $\mathcal{J}$. Thus 1.1 (ii) is verified. To establish 1.1 (iii), let $G_1, G_2 \in \mathcal{J}$. If $x \in G_1 \cap G_2$, then $x \in G_1$ and G_2. By the definition of $\mathcal{J}$, G_1 and G_2 are unions of members of $\mathcal{B}$ and so there exist sets $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset G_1$ and $x \in B_2 \subset G_2$. Since $x \in B_1 \cap B_2$, by condition (ii) in the hypothesis of the theorem, there exists $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$ But $B_1 \cap B_2 \subset G_1 \cap G_2$. Thus, we have shown that every point of $G_1 \cap G_2$ is contained in a member $B \in \mathcal{B}$, which is itself contained in $G_1 \cap G_2$. Thus $G_1 \cap G_2$ is the union of members of $\mathcal{B}$ and therefore belongs to $\mathcal{J}$.

Remark. 7.4. The definition of a base has been so chosen that 1.1 (ii) is automatically satisfied if we start with any class of subsets of X. It is consequently natural to ask if there is a method of obtaining all the open sets of $\mathcal{J}$, which automatically satisfy both axioms 1.1 (ii) and 1.1(iii). This is answered by the following definition.

DEFINITION 7.5. A class S of open subsets of a topological space X is a <u>subbase</u> for the corresponding topology $\mathfrak{T}$ if and only if the class of all finite intersections of members of S forms a base for $\mathfrak{T}$.

Note that $X = \bigcap \{ S_\lambda : \lambda \in \emptyset \}$ and so automatically belongs to S.

<u>Example 7.6.</u> The infinite open intervals of $\mathbb{R}$ form a subbase for the usual topology on $\mathbb{R}$, since every open interval $(a,b) = (a,\infty) \cap (-\infty, b)$ and the open intervals (a,b) form a base for that topology. Similarly, the infinite horizontal and vertical open strips give a subbase for the usual topology on $\mathbb{R}^2$, as the intersection of a horizontal and a vertical open strip is an open rectnagle and these open rectangles constitute a base for this topology.

<u>Remark 7.7.</u> If $\mathcal{A}$ is any class of subsets of a non-empty set X, it may not be a base for a topology on X, but it will always be a subbase for a unique topology $\mathfrak{T}$ on X by Theorem 7.3. Thus, if $X = \{a,b,c,d\}$ and $\mathcal{A} = \{\{a,b\}, \{a,c\}, \{d\}\}$, then finite intersection of $\mathcal{A}$ gives rise to the class $\mathcal{B} = \{\{a,b\}, \{a,c\}, \{d\}, \{a\}, \emptyset, X\}$, where $X \in \mathcal{B}$ since it is by definition the empty intersection of members of $\mathcal{A}$. Forming the unions of members of $\mathcal{B}$ leads to the class $\mathfrak{T} = \{\{a,b\}, \{a,c\}, \{d\}, \{a\}, \emptyset, X, \{a,b,d\}, \{a,c,d\}, \{a,d\}, \{a,b,c\}\}$ which is the topology on X generated by $\mathcal{A}$.

DEFINITION 7.8. If X_1 and X_2 are two nonempty sets, their <u>product</u> $X_1 \times X_2$ is defined to be the set of all ordered pairs (x_1,x_2), $x_1 \in X_1$, $x_2 \in X_2$. This definition can be

extended to the case of n sets, for any positive integer n. If $X_1, X_2, \ldots, X_n$ are nonempty sets, then their <u>product</u> $X_1 \times X_2 \times \ldots \times X_n$ is the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$, where $x_i \in X_i$ for each i.

If the sets X_1 and X_2 have topologies associated with them, we can define a topology for their product, as the following theorem indicates.

THEOREM 7.9. If X_1, X_2 are topological spaces, the class $\mathcal{B}$ of all sets of the form $G_1 \times G_2$, with G_1 open in X_1 and G_2 open in X_2 is a base for a topology for $X_1 \times X_2$.

<u>Proof.</u> The set $X_1 \times X_2$ is itself of the required form and is therefore the union of all the members of the class $\mathcal{B}$. Thus (i) of Theorem 7.3 is satisfied. To verify (ii), let $(x_1, x_2) \in (V_1 \times W_1) \cap (V_2 \times W_2)$, where V_1, V_2 are open in X_1, W_1, W_2 are open in X_2. Then $x = (x_1, x_2)$ also belongs to $(V_1 \cap V_2) \times (W_1 \cap W_2) = (V_1 \times W_1) \cap (V_2 \times W_2)$, where $V_1 \cap V_2$ is open in X_1 and $W_1 \cap W_2$ is open in X_2. Therefore $x \in B \subseteq B_1 \cap B_2$ where $B, B_1, B_2 \in \mathcal{B}$, which proves that $\mathcal{B}$ is a base for a topology on $X_1 \times X_2$.

The <u>product topology</u> for $X = X_1 \times X_2$ is obtained by using the base provided by Theorem 7.9.

<u>DEFINITION 7.10.</u> The <u>projections</u> P_{X_1} and P_{X_2} of the product of two sets X_1 and X_2 are the mappings of $X_1 \times X_2$ onto X_1 and X_2 respectively, defined by setting $P_{X_1}((x_1, x_2)) = x_1$, $P_{X_2}((x_1, x_2)) = x_2$.

THEOREM 7.11. The projections P_{X_1} and P_{X_2} are continuous and open mappings and the product topology is the smallest topology for which the projections are continuous.

<u>Proof.</u> If G_1 is an open set in X_1, then $P_{X_1}^{-1}(G_1) = G_1 \times X_2$, which is an open set in $X_1 \times X_2$. Therefore P_{X_1} is continuous. Likewise P_{X_2} is also continuous. To prove that P_{X_1} is open, if G is an open subset of $X_1 \times X_2$, then it is the union of the base elements $G_1 \times G_2$, for which $P_{X_1}(G_1 \times G_2) = G_1$. Thus, $P_{X_1}(G)$ is the union of the open sets G_1 and hence an open mapping as it takes open sets to open sets. Finally, suppose $\mathcal{T}$ is a topology for $X_1 \times X_2$ in which the projections are continuous. Then, for each pair of open sets G_1, G_2 in X_1 and X_2 respectively, the set $G_1 \times G_2 = (G_1 \times X_2) \cap (X_1 \times G_2) = P_{X_1}^{-1}(G_1) \cap P_{X_2}^{-1}(G_2)$ must be open in $\mathcal{T}$, since the projections are continuous in $\mathcal{T}$. Thus, every set which is open relative to the product topology is also open relative to $\mathcal{T}$, which implies that the product topology is the smallest topology for which the projections are continuous.

The needs of topology insist on an extension of these ideas to an arbitrary class of nonempty sets. We adopt the convention that the product is empty if any of the coordinate sets is empty. The notion of the product of an arbitrary number of nonempty sets was introduced by Tychonoff in 1935 and is known as the <u>Tychonoff topology</u>.

Remark 7.12. Consider $X = X_1 \times X_2$. A point $x = (x_1, x_2) \in X_1 \times X_2$ can be thought of as a function on $\{1,2\}$ such that $x(i) = x_i \in X_i$, $i = 1,2$. Now introduce a topology on $X_1 \times X_2$ by insisting that each projection P_{X_i}, $i = 1,2$ is continuous. This implies that $P_{X_i}^{-1}(G_i)$ is open in X whenever G_i is open in X_i. Thus, the product space $X_1 \times X_2$ consists of all functions x defined on $\{1,2\}$, whose values at 1,2 are x_1, x_2 with the topology generated by the subbase given by

$$\{P_{X_i}^{-1}(G_i) : G_i \in \mathfrak{J}_i,\ \mathfrak{J}_i \text{ being the class of open sets of } X_i,\ i = 1,2\}.$$

A typical generalization is as follows.

DEFINITION 7.13. Let I be an index set and let $\{X_i\}$ be a class of topological spaces. The product space $X = \prod_{i \in I} X_i$ is defined as the set of all functions x defined on I such that the value of x at $i \in I$, namely, x_i belong to X_i and its topology is that generated by the subbase.

$$\{P_{X_i}^{-1}(G_i) : G_i \in \mathfrak{J}_i,\ \mathfrak{J}_i \text{ being the family of open sets of } X_i, i \in I\}$$

where P_{X_i} is the projection mapping of X onto the i^{th}-coordinate space X_i.

Remark 7.14. Notice that if G_i is open in X_i, it does not necessarily follow that $\prod_{i \in I} G_i$ is open in X, since it is not in general possible to write $\prod_{i \in I} G_i$ as a union of finite intersections of sets belonging to the subbase. If I is finite, $\prod_{i \in I} G_i$ is certainly open, but if I is an infinite set, this need not necessarily be the case.

CHAPTER II

MEASURE AND INTEGRATION

1. PRELIMINARIES.

DEFINITION 1.1. Let X be a set. A nonempty class $\mathcal{R}$ of subsets of X is a ring if it is closed under finite unions and set theoretic differences. In other words, if $A, B \in \mathcal{R}$, then $A \cup B \in \mathcal{R}$, $A - B \in \mathcal{R}$. In particular, if $X \in \mathcal{R}$, then $\mathcal{R}$ is an algebra.

Notice that $\phi = A - A \in \mathcal{R}$, $A \triangle B = (A-B) \cup (B-A) \in \mathcal{R}$, $A \cap B = (A \cup B) - (A \triangle B) \in \mathcal{R}$.

DEFINITION 1.2. A nonempty class $\mathcal{S}$ of subsets of X is a σ-ring if it is closed under countable unions and set theoretic differences. Thus, if $A, B, \{A_n\} \in \mathcal{S}$ then $\bigcup_1^\infty A_n \in \mathcal{S}$, $A-B \in \mathcal{S}$. If $X \in \mathcal{S}$, then $\mathcal{S}$ is called a σ-algebra.

As $A \cup B = A \cup B \cup \phi \cup \phi \cup \ldots$, it is clear that every σ-ring is a ring. Conversely, any ring which is closed under countable unions is a σ-ring. A σ-ring is also closed under countable intersections since $\bigcap_1^\infty A_n = A_1 - \bigcup_1^\infty (A_1 - A_n)$. Observe that if $\{\mathcal{S}_i\}$ is a family of σ-rings of subsets of X, the intersection of the $\mathcal{S}_i$ is also a σ-ring.

DEFINITION 1.3. Let $X \in \mathcal{R}$, the set of all real numbers and let $\mathcal{E}$ be the class of all semi-closed intervals $[a,b)$. The Borel class $\mathcal{B}$ is the σ-ring generated by $\mathcal{E}$. The intervals $(a,b]$, (a,b), $[a,b]$ can also be used and the generated σ-ring is the same in each case.

DEFINITION 1.4. A <u>set function</u> γ is a function defined on a nonempty class of sets and its range is the extended real number system $\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$.

DEFINITION 1.5. A <u>measure</u> μ is a set function defined on a ring $\mathcal{R}$ with the following properties:

(i) μ is nonnegative. If $E \in \mathcal{R}$, $\mu(E) \geq 0$.

(ii) μ is additive. If $E, F \in \mathcal{R}$, $E \cap F = \phi$, $\mu(E \cup F) = \mu(E) + \mu(F)$.

(iii) $\mu(\phi) = 0$.

(iv) If $\{E_n\}$ is an increasing sequence belonging to $\mathcal{R}$, whose union E is also in $\mathcal{R}$, then $\mu(E) = \sup \mu(E_n)$.

PROPERTIES 1.6.

(i) μ is monotone. If $E, F \in \mathcal{R}$, $E \subset F$, $\mu(E) \leq \mu(F)$.
Since $F = (F-E) \cup E$, $\mu(F) = \mu(F-E) + \mu(E)$ and μ being nonnegative, $\mu(F) \geq \mu(E)$, using the fact that $E \subset F$.

(ii) μ is conditionally subtractive: $\mu(F-E) = \mu(F) - \mu(E)$, if $E, F \in \mathcal{R}$, $E \subset F$, and $\mu(E)$ is finite.

(iii) μ is finitely additive: If $E_1, E_2, \ldots, E_n$ are mutually disjoint sets in $\mathcal{R}$, then $\mu(\bigcup_1^n E_k) = \sum_1^n \mu(E_k)$.

(iv) μ is countably additive: If $\{E_n\}$ is a sequence of mutually disjoint sets in $\mathcal{R}$ such that $\bigcup_1^\infty E_n \in \mathcal{R}$, then $\mu(\bigcup_1^\infty E_n) = \sum_1^\infty \mu(E_n)$ in the sense that the supremum of the increasing sequence of partial sums $\sum_1^n \mu(E_k) = \mu(\bigcup_1^\infty E_k)$.

(v) μ is countably subadditive: $\mu(\bigcup_1^{\infty} E_n) \leq \sum_1^{\infty} \mu(E_n)$ if the condition of mutual disjointness on $\{E_n\}$ is relaxed.

DEFINITION 1.7. If $E \in \mathcal{R}$ and $\mu(E) < \infty$ for every E, then μ is a <u>finite measure.</u> If, given $E \in \mathcal{R}$, there exists a sequence $\{E_n\} \in \mathcal{R}$ such that $E \subset \bigcup_1^{\infty} E_n$ and $\mu(E_n) < \infty$ for all n, then μ is a <u>σ-finite measure</u>. A finite measure is called <u>totally finite</u> if its domain of definition is an algebra of sets and a <u>σ-finite</u> measure is <u>totally σ-finite</u> if its domain of definition is again an algebra of sets.

DEFINITION 1.8 If F is a fixed set in $\mathcal{R}$, the set function μ_F defined by the formula $\mu_F(E) = \mu(F \cap E)$, $E \in \mathcal{R}$ is called the <u>contraction of μ by F</u>.

<u>Remark 1.9</u>. μ_F is a measure on $\mathcal{R}$. If $\mu(F) < \infty$, then μ_F is a finite measure.

DEFINITION 1.10. Let $\mathcal{H}$ be the class of all sets A which satisfy the condition $A \subset \bigcup_1^{\infty} E_n$, $E_n \in \mathcal{R}$. If $A \in \mathcal{H}$ and $B \subset A$, $B \in \mathcal{H}$. For this reason, $\mathcal{H}$ is called a <u>hereditary class</u>.

It is clear that $\mathcal{H}$ is a ring and hence a σ-ring. Thus, we have the concept of a <u>hereditary σ-ring</u>. For any set X, the class of all subsets of X is a hereditary σ-ring and since the intersection of any family of hereditary σ-rings is a hereditary σ-ring, it follows that given any class of subsets of X, there is a smallest hereditary σ-ring containing $\mathcal{E}$. This is denoted by $\mathcal{H}(\mathcal{E})$ and is called the hereditary σ-ring generated by $\mathcal{E}$.

DEFINITION 1.11. Let μ be a measure on a ring $\mathcal{R}$. A set function μ^* can be defined on $\mathcal{H}(\mathcal{R})$ by the formula $\mu^*(A) = \inf \quad \{ \sum_1^\infty (E_n) : A \subset \bigcup_1^\infty E_n, E_n \in \mathcal{R} \}$.

DEFINITION 1.12. An <u>outer measure</u> is a set function ν whose domain of definition is a hereditary σ-ring and it is positive, monotone and countably subadditive, with $\nu(\phi) = 0$.

DEFINITION 1.13. A set $E \in \mathcal{H}$ is <u>ν-measurable</u> if $\nu(A) = \nu(A \cap E) + \nu(A \cap E')$ for all $A \in \mathcal{H}$, where E' denotes the complement of E.

The subadditivity of ν implies that $\nu(A) \leq \nu(A \cap E) + \nu(A \cap E')$ for any $E \in \mathcal{H}$ and every $A \in \mathcal{H}$. Thus, in order to establish the ν-measurability of E, it suffices to prove that $\nu(A) \geq \nu(A \cap E) + \nu(A \cap E')$ for every $A \in \mathcal{H}$.

THEOREM 1.14. If ν is an outer measure on a hereditary σ-ring $\mathcal{H}$, the class $\mathcal{M}$ of all ν-measurable sets is a ring.

<u>Proof.</u> Suppose $E, F \in \mathcal{M}$. Then E and F split A additively, for every $A \in \mathcal{H}$. Thus

$$\nu(A) = \nu(A \cap E) + \nu(A \cap E'), \quad \nu(A) = \nu(A \cap F) + \nu(A \cap F') \tag{1.1}$$

We have to prove that $E-F$ and $E \cup F$ are ν-measurable. Notice that $E-F = E \cap F'$, $(E-F)' = (E \cap F')' = E' \cup F$. We have to establish that

$$\nu(A) = \nu[A \cap (E-F)] + \nu[A \cap (E-F)'] \tag{1.2}$$

and

$$\nu(A) = \nu[A\cap(E\cup F)] + \nu[A\cap(E\cup F)'] \tag{1.3}$$

Since $A\cap(E'\cup F)\subset A$, $A\cap(E'\cup F)\in \mathcal{H}$. Consider

$$\nu[A\cap(E-F)'] = \nu[A\cap(E'\cup F)] = \nu[A\cap(E'\cup F)\cap F] + \nu[A\cap(E'\cup F)\cap F']$$

$$= \nu[A\cap F] + \nu[A\cap(E'\cap F')].$$

Thus, using the fact that $A\cap F'\subset A$, we have

$$\nu[A\cap(E\cap F')] + \nu[A\cap(E\cap F')'] = \nu[A\cap(E\cap F')] + \nu(A\cap F) + \nu[A\cap(E'\cap F')]$$

$$= \nu(A\cap F) + \nu(A\cap F') \text{ using } (1.1)$$

which proves that $E-F\in\mathcal{M}$.

Again, $A\cap(E\cup F)\in A$ and

$$\nu[A\cap(E\cup F)] = \nu[A\cap(E\cup F)\cap F] + \nu[A\cap(E\cup F)\cap F']$$

$$= \nu(A\cap F) + \nu[A\cap(E\cap F')]$$

which gives

$$\nu[A\cap(E\cap F)] + \nu[A\cap(E\cup F)'] = \nu(A\cap F) + \nu[A\cap(E\cap F')] + \nu[A\cap(E'\cap F')]$$

$$= \nu(A\cap F) + \nu(A\cap F') = \nu(A).$$

Thus (1.3) is true.

THEOREM 1.15. If ν is an outer measure defined on a hereditary σ-ring $\mathcal{H}$ and $\mathcal{M}$ is the class of all ν-measurable sets, then

(i) $\mathcal{M}$ is a σ-ring

(ii) If $\{E_n\}$ is a sequence of mutually disjoint sets in M whose union is E, then
$\nu(\Lambda \cap E) = \sum_1^\infty \nu(\Lambda \cap E_n)$ for every $\Lambda \in \mathcal{H}$.

(iii) The restriction of ν to $\mathcal{M}$ is a measure.

<u>Proof</u>. Define

$$\nu_A(E) = \nu(\Lambda \cap E), \quad \Lambda \in \mathcal{H}, \quad E \in \mathcal{M}.$$

It is easy to verify that ν_A is a measure on $\mathcal{M}$. Suppose $\{E_n\}$ is a sequence of mutually disjoint sets in $\mathcal{M}$, whose union is E and suppose $\Lambda \in \mathcal{H}$. We shall show simultaneously that $\mathcal{M}$ is a σ-ring and that (ii) holds.

Since the E_n's are disjoint, $E_1 \cap E_2 = \phi$ and hence $E'_1 \cap E_2 = E_2$. As E_1 splits $\Lambda \cap (E_1 \cup E_2)$ additively, we have

$$\nu[\Lambda \cap (E_1 \cup E_2)] = \nu[\Lambda \cap (E_1 \cup E_2) \cap E_1] + \nu[\Lambda \cap (E_1 \cup E_2) \cap E_1']$$
$$= \nu(\Lambda \cap E_1) + \nu(\Lambda \cap E_2).$$

Thus $\nu_A(E_1 \cup E_2) = \nu_A(E_1) + \nu_A(E_2)$, which shows that ν_A is additive. By a process of induction, ν_A is finitely additive. Define $F_n = \bigcup_1^n E_k$. Then, by Theorem 1.14, $F_n \in \mathcal{M}$, since $\mathcal{M}$ is a ring and is therefore closed under finite unions. Now

$\nu_A(F_n) = \nu_A(\bigcup_1^n E_k) = \sum_1^n \nu_A(E_k)$. Since $F_n \uparrow E$, $F_n' \downarrow E'$ and in particular, $\Lambda \cap F_n' \supset \Lambda \cap E'$ for every n.

Since F_n is ν-measurable and ν is monotone,

$$\nu(\Lambda) = \nu(\Lambda \cap F_n) + \nu(\Lambda \cap F_n') \geq \nu(\Lambda \cap F_n) + \nu(\Lambda \cap E')$$
$$= \nu_A(F_n) + \nu(\Lambda \cap E') = \sum_1^n \nu_A(E_k) + \nu(\Lambda \cap E') \tag{1.4}$$

As n is arbitrary, (1.4) yields

$$\nu(\Lambda) \geq \sum_1^\infty \nu_A(E_k) + \nu(\Lambda \cap E') \tag{1.5}$$

But, $\sum_1^\infty \nu_A(E_n) = \sum_1^\infty \nu(\Lambda \cap E_n) \geq \nu(\bigcup_1^\infty \Lambda \cap E_n)$, by countable subadditivity of ν.
$= \nu(\Lambda \cap E)$.

Substituting in (1.5), we have $\nu(\Lambda) \geq \nu(\Lambda \cap E) + \nu(\Lambda \cap E')$, which proves that E is ν-measurable. Thus, $E \in \mathcal{M}$ and E being equal to $\bigcup_1^\infty E_n$, $\mathcal{M}$ is closed under countable unions and therefore a σ-ring.

From (1.5), it follows that

$$\nu(\Lambda) \geq \sum_1^\infty \nu_A(E_n) + \nu(\Lambda \cap E') \geq \nu(\Lambda \cap E) + \nu(\Lambda \cap E') = \nu(\Lambda).$$

Therefore $\nu(\Lambda) = \sum_1^\infty \nu_A(E_n) + \nu(\Lambda \cap E')$ (1.6)

In (1.6), replacing Λ by $\Lambda \cap E$, we have

$$\nu(\Lambda \cap E) = \sum_1^\infty \nu_A(E_n) + \nu(\Lambda \cap E \cap E') = \sum_1^\infty \nu_A(E_n) + \nu(\phi) = \sum_1^\infty \nu_A(E_n).$$

In particular, when $\Lambda = E$, (1.6) becomes

$$\nu(E) = \sum_1^\infty \nu_E(E_n) + \nu(E \cap E') = \sum_1^\infty \nu(E_n)$$

which proves that ν is countably additive on $\mathcal{M}$. Therefore, the restriction of ν to $\mathcal{M}$ is a measure.

DEFINITION 1.16. If $\mu^*(A)=\inf\{\mu(F).\ A\subset F\in \mathcal{S}\}$,then the <u>inner measure</u> μ_* is defined on $\mathcal{H}(\mathcal{S})$ by $\mu_*(A)=\sup\{\mu(F).\ A\supset F\in\mathcal{S}\}$.

DEFINITION 1.17. If X is a set and $\mathcal{S}$ is a σ-ring of subsets of X, then the pair $(X,\mathcal{S})$ is called a <u>measurable space</u>. If $E\in\mathcal{S}$, E is said to be <u>measurable with respect to</u> $\mathcal{S}$, or simply a <u>measurable set</u>.

DEFINITION 1.18. Let X be a set. If A is a subset of X, it is <u>locally measurable</u> if $A\cap E$ is measurable, for every measurable set E. The class of all locally measurable sets is denoted by $\mathcal{S}_\lambda$.

DEFINITION 1.19. Let $(X,\mathcal{S})$ be a measurable space and let f be a real-valued function defined on X. Let $N(f) = \{x\in X: f(x)\neq 0\}$. A function $f: X\to\mathbb{R}$ is a <u>measurable function</u> if $N(f)\cap f^{-1}(M)$ is a measurable set for every Borel set M.

We emphasize that the concept of measurability is defined here only for the functions which have values in $\mathbb{R}$. If f is allowed to take the values $\pm\infty$, the sets $f^{-1}(\{\infty\})$ and $f^{-1}(\{-\infty\})$ should also be assumed to be measurable.

DEFINITION 1.20. If A is any subset of X, the <u>characteristic function of A</u>, denoted by χ_A, is the function defined on X, whose values are 1 at the points belonging to A and 0 at the points of X-A. Symbolically

$$\chi_A=\begin{cases}1\ , & x\in A\\ 0\ , & x\in X-A.\end{cases}$$

2. <u>PROPERTIES OF MEASURABLE FUNCTIONS AND MEASURABLE SETS</u>.

THEOREM 2.1. A function $f: X\to\mathbb{R}$ is measurable if and only if

(i) N(f) is measurable

(ii) $f^{-1}(M)$ is locally measurable, for every Borel set M.

Proof. Suppose $f: X \to \mathbb{R}$ is measurable. Then $N(f) \cap f^{-1}(M)$ is a measurable set for every Borel set M. Now, $\mathbb{R}$ is a Borel set and $f^{-1}(\mathbb{R}) = X$. Therefore $N(f) \cap f^{-1}(\mathbb{R}) = N(f) \cap X$ is a measurable set. This implies that $N(f)$ is measurable thus establishing (i). To prove that (ii) is necessary, suppose M is a Borel set and let $\{0\} \in M$. Then $f^{-1}(M) \supset f^{-1}(\{0\}) = [N(f)]'$ and $f^{-1}(M) - N(f) = f^{-1}(M) \cap [N(f)]' = [N(f)]'$. If $\{0\} \notin M$, $f^{-1}(M) \subset N(f)$ and $f^{-1}(M) - N(f) = f^{-1}(M) \cap [N(f)]'$

$$\subset N(f) \cap [N(f)]' = \phi .$$

In either case, $f^{-1}(M) - N(f)$ is locally measurable. Since $f^{-1}(M) \cap N(f)$ is a measurable set by hypothesis, it follows that the set

$$f^{-1}(M) = [f^{-1}(M) \cap N(f)] \cup [f^{-1}(M) - N(f)]$$

is locally measurable.

On the other hand, if (i) and (ii) are satisfied and if M is a Borel set, by the definition of local measurability, $N(f) \cap f^{-1}(M)$ is a measurable set which implies that f is a measurable function.

THEOREM 2.2. If $f: X \to \mathbb{R}$ is a function such that $N(f)$ is measurable, then each of the following conditions is necessary and sufficient for the function f to be measurable.

(i) $\{x: f(x) < c\}$ is locally measurable, for each real number c.

(ii) $\{x: f(x) \leq c\}$ " " "

(iii) $\{x: f(x) > c\}$ " " "

(iv) $\{x: f(x) \geq c\}$ " " "

Proof. The sets specified in conditions (i)-(iv) are the inverse images under f, of the sets of the form $(-\infty, c)$, $(-\infty, c]$, (c, ∞) and $[c, \infty)$. These are Borel sets since

$$(-\infty, c) = \bigcup_{n=1}^{\infty} [c-n, c)$$

$$(-\infty, c] = (-\infty, c) \cup \bigcap_{n=1}^{\infty} [c, c + \frac{1}{n})$$

$$(c, \infty) = \bigcup_{n=1}^{\infty} [c + \frac{1}{n}, c + n)$$

$$[c, \infty) = \bigcup_{n=1}^{\infty} [c, c+n).$$

Using Theorem 2.1, we conclude that each of the conditions (i) through (iv) is necessary for the measurability of f.

To establish the sufficiency, suppose (i) holds. Since $[a,b) = (-\infty, b) - (-\infty, a)$, the intervals of the form $(-\infty, c)$ generate a Borel class $\mathcal{B}$, which implies that f is measurable. Again, if (ii) holds, $(-\infty, c) = \bigcup_{n=1}^{\infty} (-\infty, c - \frac{1}{n}]$ and since the intervals $(-\infty, c)$ generate $\mathcal{B}$, the intervals $(-\infty, c]$ also generate $\mathcal{B}$, thus yielding the measurability of f. As $[a, \infty) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, \infty)$, the intervals (c, ∞) generate $\mathcal{B}$. Hence (iii) implies that f is measurable. Finally, since $[a,b) = [a, \infty) - [b, \infty)$, the intervals $[c, \infty)$ generate $\mathcal{B}$, which shows that if (iv) is satisfied, then f is measurable.

THEOREM 2.3. If f and g are measurable functions defined on X and c is any real number, then each of the following sets

(i) $A = \{x : f(x) < g(x) + c\}$

(ii) $B = \{x : f(x) \leq g(x) + c\}$

(iii) $C = \{x : f(x) = g(x) + c\}$

is locally measurable.

Proof. Suppose $x \in X$ and r is a rational number. Then, the condition $f(x) < g(x) + c$, holds if and only if $f(x) < r$ and $r < g(x) + c$ or $g(x) > r-c$. Now A can be written as

$$A = \bigcup_{r} [\{x : f(x) < r\} \cap \{x : g(x) > r-c\}].$$

Since f and g are measurable functions and c is any real number, using Theorem 2.2, $\{x : f(x) < r\}$ and $\{x : g(x) > r-c\}$ are locally measurable and so is their intersection. Thus A is locally measurable. Interchanging the roles of f and g in A and making use of the fact that the complement of the locally measurable set A is also locally measurable, we have $\{x : g(x) \geq f(x) + c\}$ is locally measurable which implies that $\{x : f(x) \leq g(x) - c\}$ is locally measurable. c being any real number, we conclude that B is locally measurable. Finally, $C = B - A$ is locally measurable.

DEFINITION 2.4. If c is a real number, the function cf is defined by $(cf)(x) = cf(x)$. $x \in X$.

THEOREM 2.5. If f is measurable and c is a real number, then cf is measurable.

<u>Proof</u>. If $c=0$, then $cf\equiv 0$, in which case $N(cf)=\phi$ which is certainly measurable. If $c\neq 0$, $N(cf) = N(f)$ which is a measurable set since f is a measurable function. Thus, in either case $N(cf)$ is measurable. Now, if $c > 0$, let $cf=h$. Then the set

$$\{x:h(x) < a\} = \{x:cf(x) < a\} = \{x:f(x) < \tfrac{a}{c}\}$$

is locally measurable, since f is measurable using Theorem 2.2 for f. If $c < 0$, the set

$$\{x:h(x) < a\} = \{x:cf(x) < a\} = \{x:f(x) > \tfrac{a}{c}\}$$

which is once again locally measurable, since f is measurable. Thus the function cf satisfies the conditions of Theorem 2.2 and is therefore a measurable function.

DEFINITION 2.6. If f and g are measurable functions, their sum is defined by $(f+g)(x) = f(x)+g(x)$.

THEOREM 2.7. If f and g are measurable functions, then so is $f+g$.

<u>Proof</u>. Notice that $N(f+g) \subset N(f) \cup N(g)$. Since $-g = (-1).g$, and g is measurable, by Theorem 2.5, $-g$ is also measurable. Thus, the set

$$\{x: f(x) + g(x) \leq c\} = \{x:f(x) \leq c-g(x)\}$$

is locally measurable by Theorem 2.3. If, in particular, $c=0$, then $\{x:f(x) + g(x) = 0\}$ is locally measurable, which implies that $[N(f+g)]'$ is locally measurable. Therefore, $N(f+g) \cap E$ is measurable for every measurable E, since the complement of a locally measurable set is locally measurable. In particular, $N(f+g) \cap [N(f) \cup N(g)] = N(f+g)$ is measurable. Thus, $f+g$ satisfies the conditions of Theorem 2.2 and is consequently a measurable function.

REMARK 2.8. If f is a measurable function, its positive and negative parts defined by $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$ are also measurable. In particular, $|f| = f^+ + f^-$ is measurable.

REMARK 2.9. If f and g are measurable functions,

$$(f \cup g)(x) = \max\{f(x), g(x)\} = \frac{f(x)+g(x) + |f(x)-g(x)|}{2}$$

$$(f \cap g)(x) = \min\{f(x), g(x)\} = \frac{f(x)+g(x) - |f(x)-g(x)|}{2}$$

are also measurable functions.

THEOREM 2.10. If f and g are measurable functions, their product fg is also a measurable function, where $(fg)(x) = f(x)g(x)$.

Proof. Since f is measurable, $N(f)$ is measurable and $N(f^2) = N(f)$ is also measurable. Likewise $N(g^2)$ is measurable. Consider $N(f^2) \cap \{x : f^2(x) \geq c\}$. If $c=0$, this set becomes $N(f^2) \cap \{x : f^2(x) \geq 0\} = \{x : f^2(x) > 0\}$. Now

$$\{x : f^2(x) > 0\} = \{x : f(x) < 0\} \cup \{x : f(x) > 0\}$$

and therefore locally measurable, since the function f is measurable and $N(f)$ is a measurable set. If $c > 0$, $N(f^2) \cap \{x : f^2(x) \geq c\}$ is once again locally measurable. If $c < 0$,

$$N(f^2) \cap \{x : f^2(x) \geq c\} = \{x : f^2(x) > c\} = \{x : f(x) > c\} \cup \{x : f(x) < - c\}$$

and this implies that $\{x : f^2(x) \geq c\}$ is locally measurable. Thus, f^2 satisfies the conditions of Theorem 2.2 and is therefore a measurable function. Similarly, g^2 is a measurable function and

$$fg = \frac{(f+g)^2 - (f-g)^2}{4}$$

is a measurable function by Theorems 2.5 and 2.7.

DEFINITION 2.11. A measure space is a triple $(X, \mathcal{S}, \mu)$ where $(X, \mathcal{S})$ is a measurable space and μ is a measure defined on $\mathcal{S}$.

DEFINITION 2.12. A measure space is finite or σ-finite if the corresponding measure μ is finite or σ-finite.

DEFINITION 2.13. If $\{E_n\}$ is a sequence of sets, the limit superior of the sequence is the set of all points x such that $x \in E_n$ for infinitely many n. In other words, $\limsup E_n = \bigcap_{n \geq 1} \bigcup_{k \geq n} E_k$. The limit inferior of the sequence is the set of points x for which there exists an index n such that $x \in E_k$ for all $k \geq n$. Symbolically, $\liminf E_n = \bigcup_{n \geq 1} \bigcap_{k \geq n} E_k$.

REMARK 2.14. If $\{E_n\}$ is a sequence of measurable sets, then $\limsup E_n$ and $\liminf E_n$ are also measurable.

THEOREM 2.15. (Arzela-Young). If $\{E_n\}$ is a sequence of measurable sets in a finite measure space $(X, \mathcal{S}, \mu)$ with $\mu(E_n) \geq \varepsilon$ for every n, where ε is an arbitrary positive number, then $\mu(\limsup E_n) \geq \varepsilon$.

Proof. Let $E = \limsup E_n$ and define $F_n = \bigcup_{k \geq n} E_k$. Now $F_n \downarrow E$. Since μ is a finite measure, $\mu(F_n) \downarrow \mu(E)$. But $F_n \supset E_n$ and this yields $\mu(F_n) \geq \mu(E_n) \geq \varepsilon$ for all n. Hence $\mu(E) = \inf \mu(F_n) \geq \varepsilon$.

3. TYPES OF CONVERGENCE.

DEFINITION 3.1. A sequence $\{f_n\}$ of real-valued functions defined on a measure space $(X, \mathcal{S}, \mu)$ converges almost everywhere to the real-valued function $f(x)$ if there exists a null set ϕ such that $x \notin \phi$ implies that $f_n(x)$ converges to $f(x)$. We use the abbreviation a.e. for 'almost everywhere'.

DEFINITION 3.2. A function $f(x)$ defined on $(X, \mathcal{S}, \mu)$ is essentially bounded if $|f| \leq M$ a.e., where M is a positive constant.

DEFINITION 3.3. A sequence $\{f_n\}$ of real-valued functions is fundamental almost everywhere if there exists a null set ϕ such that $x \notin \phi$ implies $\{f_n\}$ is a Cauchy sequence.

PROPERTIES 3.4.

(i) If $\{f_n\} \to f$ a.e., then f is fundamental a.e.

(ii) If $\{f_n\} \to f$ a.e.. $\{f_n\} \to g$ a.e., then $f=g$ a.e.

(iii) If $\{f_n\} \to f$ a.e. and g is a real-valued function such that $f=g$ a.e., then $\{f_n\} \to g$ a.e.

(iv) If $\{f_n\} \to f$ a.e., $\{g_n\}$ is a sequence of real-valued functions such that $f_n = g_n$ a.e. for each n, then $\{g_n\} \to f$ a.e.

(v) If $\{f_n\} \to f$ a.e., $\{g_n\} \to g$ a.e., c is a real number and $A \subset X$, then

$cf_n \to cf$ a.e.

$f_n + g_n \to f+g$ a.e.

$|f_n| \to |f|$ a.e.

$f_n \cup g_n \to f \cup g$ a.e. and $f_n \cap g_n \to f \cap g$ a.e.

$f_n^+ \to f^+$ a.e. and $f_n^- \to f^-$ a.e.

$\chi_A f_n \to \chi_A f$ a.e.

$f_n g_n \to fg$ a.e.

DEFINITION 3.5. Let $\{f_n\}$ and f be measurable functions defined on a finite measure space $(X, \mathcal{S}, \mu)$. f_n converges to f in measure if, for each $\varepsilon > 0$, $\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \to 0$ as $n \to \infty$. Briefly, $\{f_n\} \to f$ in (m).

DEFINITION 3.6. A sequence $\{f_n\}$ of measurable functions defined on a finite measure space $(X, \mathcal{S}, \mu)$ is fundamental in measure if, for each $\varepsilon > 0$, $\mu(\{x : |f_m(x) - f_n(x)| \geq \varepsilon\}) \to 0$ as $n \to \infty$. Briefly, $\{f_n\}$ is fundamental in (m).

THEOREM 3.7. Let $\{f_n\}$ and f be measurable functions defined on a finite measure space $(X, \mathcal{S}, \mu)$. Suppose $\{f_n\}$ converges to f a.e. Then, for each $\varepsilon > 0$, $\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \to 0$ as $n \to \infty$.

Proof. Let $E_n = \{x : |f_n(x) - f(x)| \geq \varepsilon\}$. Assume that $\mu(E_n) \not\to 0$ as $n \to \infty$. Then, there exists $\delta > 0$ and a subsequence $\{E_{n_k}\}$ of E_n, such that $\mu(E_{n_k}) \geq \delta$ for all k. Define $E = \limsup E_{n_k}$. Since μ is finite, by Theorem 2.15, $\mu(E) \geq \delta$. Suppose E is not a null set. Since $f_n \to f$ a.e., it is possible to find at least one point $x \in E$ such that $f_n(x) \to f(x)$. But, $|f_{n_k}(x) - f(x)| \geq \varepsilon$, for infinitely many k, by the definition of E and this contradicts the fact that $f_n(x) \to f(x)$ for $x \in E$. Thus, $\mu(E_n)$ must converge to 0 as $n \to \infty$.

THEOREM 3.8. If $\{f_n\}$ converges to f in (m), then $\{f_n\}$ is fundamental in (m).

<u>Proof.</u> Given $\varepsilon > 0$, define $E_{mn} = \{x : |f_m(x) - f_n(x)| \geq \varepsilon\}$, $E_m = \{x : |f_m(x) - f(x)| \geq \varepsilon/2\}$ and $E_n = \{x : |f_n(x) - f(x)| \geq \varepsilon/2\}$. Then

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f_n(x) - f(x)| ,$$

which implies that $E_{mn} \subset E_m \cup E_n$. Thus, $\mu(E_{mn}) \leq \mu(E_m) + \mu(E_n) \to 0$ as $m,n \to \infty$. This shows that $\{f_n\}$ is fundamental in measure.

EXERCISE 3.9.

(i) If $\{f_n\}$ is fundamental in (m), there exists a measurable function f such that $\{f_n\} \to f$ in (m).

(ii) If $\{f_n\} \to f$ in ⓜ and $\{f_n\} \to g$ in ⓜ, then $f=g$ a.e.

(iii) If $\{f_n\} \to f$ in ⓜ and g is a measurable function such that $f=g$ a.e., then $\{f_n\} \to g$ in ⓜ.

(iv) If $\{f_n\} \to f$ in ⓜ and $\{g_n\}$ is a sequence of measurable functions such that $f_n=g_n$ a.e., then $\{g_n\} \to f$ in ⓜ.

LEMMA 3.10. If $\{f_n\}$ converges to f in ⓜ and $f_n \geq 0$ a.e., then $f \geq 0$ a.e.

<u>Proof</u>. $f_n \geq 0$ for all n, everywhere except for a set of measure zero. By a suitable modification, $f_n \geq 0$ everywhere. Given $\varepsilon > 0$, define $E = \{x: f(x) \leq -\varepsilon\}$ and $E_n = \{x: |f_n(x)-f(x)| \geq \varepsilon\}$ By hypothesis, $\mu(E_n) \to 0$. If $x \in E$, $f(x) \leq -\varepsilon$. Since $f(x) = [f(x) - f_n(x)] + f_n(x) \geq f(x) - f_n(x)$, we conclude that $f(x) - f_n(x) \leq -\varepsilon$ and hence $|f(x)-f_n(x)| \geq \varepsilon$. Thus, if $x \in E$, x also belongs to E_n, which shows that $E \subset E_n$ for all n. Therefore $\mu(E) = 0$ and it follows that $\{x: f(x) < 0\} = \bigcup_{m=1}^{\infty} \{x: f(x) \leq -\frac{1}{m}\}$ is a null set, which proves that $f \geq 0$ a.e.

THEOREM 3.11. If $\{f_n\}$, f and g are measurable functions, $\{f_n\} \to f$ in ⓜ and $f_n \leq g$ a.e. for all n, then $f \leq g$ a.e.

Proof. We have $g - f_n \geq 0$ a.e. and $g - f_n \to g-f$ in (m). Hence $g-f \geq 0$ a.e. by Lemma 3.10.

COROLLARY 3.12. If $\{f_n\}$, f and g are measurable, $\{f_n\} \to f$ in (m) and $|f_n| \leq |g|$ a.e. for all n, then $|f| \leq |g|$ a.e.

Proof. Since $|f_n| \to |f|$ in (m), $|f| \leq |g|$ a.e. by Theorem 3.11.

DEFINITION 3.13. A sequence $\{f_n\}$ of measurable functions converges almost uniformly to a measurable function f, if given $\varepsilon > 0$, a measurable set E can be found such that $\mu(E) < \varepsilon$ and $\{f_n\}$ converges to f uniformly on $X \setminus E$. Briefly, $\{f_n\} \to f$ a.u.

THEOREM 3.14. (Egoroff). Let $(X, \mathcal{S}, \mu)$ be a finite measure space and let $\{f_n\}$ be a sequence of measurable functions converging almost everywhere to a measurable function f. Then $\{f_n\}$ converges to f almost uniformly.

Proof. We shall assume that $\{f_n(x)\} \to f(x)$ for all x, without any loss of generality and establish at the end that this relaxation is valid. For fixed m and $n=1,2,\ldots$, define $F_n^m = \bigcup_{k \geq n} \{x : |f_k(x) - f(x)| \geq \frac{1}{m}\}$. Since each $f_k - f$ is a measurable function, the sets $\{F_n^m\}$ are measurable by Theorem 2.3. Further, for fixed m, the sequence $\{F_k^m\}$ is decreasing. By our assumption, given any positive integer m and any $x \in X$, $f_k(x) \to f(x)$ implies the existence of an index r such that $|f_k(x) - f(x)| < \frac{1}{m}$ whenever $k \geq r$. This, in turn, implies that $x \notin F_r^m$ and therefore $x \notin \bigcap_{n=1}^{\infty} F_n^m$. Thus, for each fixed m, $\bigcap_{n=1}^{\infty} F_n^m$ is empty, which gives that $\mu(F_n^m) \downarrow \phi$ as $n \to \infty$, μ being a finite measure. For each m, $\mu(F_n^m) \to 0$ as $n \to \infty$. Hence there exists an index n(m) such that $\mu(F_{n(m)}^m) \leq \frac{\varepsilon}{2^m}$. Defining $E = \bigcup_{m=1}^{\infty} F_{n(m)}^m$, we have E is a measurable set and $\mu(E) \leq \sum_{m=1}^{\infty} \mu(F_{n(m)}^m) \leq \varepsilon \sum_{1}^{\infty} 2^{-m} = \varepsilon$.

It remains to prove that $f_n(x) \to f(x)$ uniformly on X - E. Given $\varepsilon >' 0$, we should find an index r such that $k \geq r$ implies that $|f_k(x) - f(x)| < \varepsilon$ for all $x \in X-E$. Now,

$$X-E = X - \bigcup_{m=1}^{\infty} F_{n(m)}^m = \bigcap_{m=1}^{\infty} X-F_{n(m)}^m = \bigcap_m \bigcap_{k \geq n(m)} \{x : |f_k(x)-f(x)| < \tfrac{1}{m}\}$$

$$= \bigcap_m \{x : |f_k(x)-f(x)| < \tfrac{1}{m} \text{ for all } k \geq n(m)\}.$$

Choose m such that $\frac{1}{m} < \varepsilon$. Since

$$X-E \subset \{x : |f_k(x)-f(x)| < \tfrac{1}{m} \text{ for all } k \geq n(m)\} .$$

we have $|f_k(x)-f(x)| < \varepsilon$ for all $x \in X-E$, provided $k \geq n(m)$.

Finally, consider the general case, where $f_n \to f$ a.e. Let ϕ be a null set on whose complement $f_n(x) \to f(x)$. Define $g_n = \chi_{X-\phi} f_n$ and $g = \chi_{X-\phi} f$. We shall show that g_n and g are measurable. In general, $\chi_A f$ will be measurable for every measurable function f and for every locally measurable set A. Assume $h = \chi_A f$. Then, $N(h) = N(\chi_A) \cap N(f) = A \cap N(f)$. But A being locally measurable and N(f) being measurable,

$A \cap N(f)$ is measurable which proves that $N(h)$ is measurable. Let M be a Borel set. By Theorem 2.1, $f^{-1}(M)$ is locally measurable. If $\{0\} \in M$, then $h^{-1}(M) = (\chi_A f)^{-1}(M) = [A \cap f^{-1}(M)] \cup (X-A)$ and is therefore locally measurable. If $\{0\} \notin M$, then $h^{-1}(M) = A \cap f^{-1}(M)$ and once again locally measurable. Thus h satisfies the conditions of Theorem 2.1 and is therefore a measurable function.

Since g_n and g are of the same form as h, $X-\phi$ being locally measurable, g_n and g are measurable and $g_n(x) \to g(x)$ for all x. Given $\varepsilon > 0$, by the first part of the proof, there exists a measurable set G such that $\mu(G) < \varepsilon$ and $g_n(x) \to g(x)$ uniformly on $X-G$. Defining $E = \phi \cup G$, we have, $\mu(E) \leq \mu(0) + \mu(G) < \varepsilon$. Since $\chi_{X-\phi} f_n(x) \to \chi_{X-\phi} f(x)$ uniformly for $x \in X-G$ and $x \in (X-\phi) \cap (X-G)$ implies $x \in X-(\phi \cup G) = X-E$, we have $f_n(x) \to f(x)$ uniformly on $X-E$.

4. INTEGRABLE SIMPLE FUNCTIONS.

DEFINITION 4.1. A real-valued function f defined on a measurable space $(X, \mathcal{S})$ is <u>simple</u> if

(i) f is measurable

(ii) the range of f is a finite set of real numbers.

The simplest example of a simple function is the characteristic function of a measurable set.

THEOREM 4.2. Let f be a real-valued function. Then f is simple if and only if there exist a finite set of real numbers $C_1, C_2, \ldots, C_n$ and a finite number of measurable sets $E_1, E_2, \ldots, E_n$ such that $f = \sum_{k=1}^{n} C_k \chi_{E_k}$.

Proof. Suppose f is imple and let $C_1, C_2, \ldots, C_n$ be the finite non-zero values of f. Define $E_k = f^{-1}(\{C_k\})$. Since f is measurable, $f^{-1}(\{C_k\})$ is locally measurable and since $E_k \subset N(f)$, E_k is measurable. Evidently, $f = \sum_{k=1}^{n} C_k \chi_{E_k}$. On the contrary, if $f = \sum_{k=1}^{n} C_k \chi_{E_k}$, since χ_{E_k} is measurable, $C_k \chi_{E_k}$ is measurable and thus f is measurable. Further, f can take at most 2^n values and therefore f is a simple function.

Notice that the representation $f = \sum_{k=1}^{n} C_k \chi_{E_k}$ is not unique. However, if the E_k's are pairwise disjoint and the C_k's are distinct, the representation becomes unique.

THEOREM 4.3. Every extended, real-valued measurable function is the limit of a sequence $\{f_n\}$ of simple functions. In particular, if $f \geq 0$, then $f_n \geq 0$ and f_n may be assumed to be increasing.

Proof. Suppose $f_n \geq 0$. Then, for every $x \in X$, define

$$f_n(x) = \begin{cases} \frac{i-1}{2^n}, & \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}, \ i = 1, 2, \ldots, 2^n n \\ n, & f(x) \geq n. \end{cases}$$

Clearly, $\{f_n\}$ is an increasing sequence of non-negative functions, which are simple. In the interval $[0, \frac{1}{2^n})$, $|f(x) - f_n(x)| \leq \frac{1}{2^n}$. Thus, we can choose n so that $|f(x) - f_n(x)| < \frac{1}{2^n}$ as long as $f(x) < \infty$. If $f(x)= \infty$, then $f_n(x) = \infty$, for each n.

DEFINITION 4.4. A simple function $f = \sum_{k=1}^{n} C_k \chi_{E_k}$ defined on a measure space $(X, \mathcal{S}, \mu)$ is said to be <u>integrable</u>, if $\mu(E_k) < \infty$ for every k for which $C_k \neq 0$. The <u>integral</u> of f denoted by $\int f(x)d\mu(x)$ or $\int fd\mu$ is defined as $\int fd\mu = \sum_{k=1}^{n} C_k \mu(E_k)$.

Observe that the definition of the integral of a simple function is independent of the mode of representation. Briefly, ISF stands for an integrable simple function.

PROPERTIES 4.5.

(i) If f and g are ISF, C is a real number and A is a locally measurable set, then cf, $f+g$, $|f|$, $f \cup g$, $f \cap g$, f^+, f^-, $\chi_A f$ and fg are ISF.

(ii) If f is an ISF and $a, b \in \mathbb{R}$, then $\int(af+bg)d\mu = a\int fd\mu + b\int gd\mu$.

(iii) If f is an ISF and $f \geq 0$ a.e., then $\int fd\mu = 0$.

(iv) If f and g are ISF and $f \geq g$ a.e., then $\int fd\mu \geq \int gd\mu$.

(v) If f and g are ISF, then $\int|f+g|d\mu \leq \int|f|d\mu + \int|g|d\mu$.

(vi) If f is an ISF, $|\int fd\mu| \leq \int|f|d\mu$.

DEFINITION 4.6. If f is an ISF and E is a measurable set, then $\chi_E f$ is also ISF and the <u>integral of</u> f <u>w.r.t.</u> E is defined as $\int_E f d\mu = \int \chi_E f d\mu$.

THEOREM 4.7. If f is an ISF, $a, b \in \mathbb{R}$ and E is a measurable set such that for $x \in E$, $a \leq f(x) \leq b$, then $a\mu(E) \leq \int_E f d\mu \leq b\,\mu(E)$.

<u>Proof</u>. Notice that the given condition implies that $a\chi_E \leq f\chi_E \leq b\chi_E$ and hence on integrating and using definition 4.6 we get the result.

DEFINITION 4.8. The <u>indefinite integral</u> of an ISF f is the finite valued set function ν defined for every measurable set E by $\nu(E) = \int_E f d\mu$.

THEOREM 4.9. If f is an ISF which is nonnegative a.e., then its indefinite integral is monotone.

<u>Proof</u>. If $E \subset F$, $\chi_E f \leq \chi_F f$. Therefore, $\int \chi_E f d\mu \leq \int \chi_F f d\mu$ which implies that $\int_E f d\mu \leq \int_F f d\mu$. Thus, $\nu(E) \leq \nu(F)$, which proves that ν is monotone.

DEFINITION 4.10. A finite valued set function defined on the class of all measurable sets is said to be <u>absolutely continuous</u> if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ for every measurable set E for which $\mu(E) < \delta$.

THEOREM 4.11. The indefinite integral of an ISF is absolutely continuous.

Proof. Suppose C is a positive number greater than all the possible values of $|f|$. Then $|\int_E f\,d\mu| \leq \int_E |f|\,d\mu <$ $C\int_E d\mu = C\,\mu(E)$ which implies that $|\nu(E)| < C\,\mu(E)$. Thus, $\mu(E) < \delta$ implies $|\nu(E)| < C\,\delta(E)$ for every measurable set E.

THEOREM 4.12. The indefinite integral of an ISF is countably additive.

Proof. If f is the characteristic function of a measurable set E of finite measure, then the assertion of the theorem is a consequence of the countable additivity of the measure μ. Every simple function being a finite linear combination of characteristic functions, the countable additivity of ν follows in the case of an arbitrary integrable simple function.

DEFINITION 4.13. The distance between the ISF f and the ISF g is given by $\rho(f,g) = \int|f-g|\,d\mu$.

DEFINITION 4.14. A sequence $\{f_n\}$ of ISF is <u>fundamental in the mean</u> or <u>mean fundamental</u> if $\rho(f_m,f_n) \to 0$ as $m,n \to \infty$.

THEOREM 4.15. A mean fundamental sequence $\{f_n\}$ of ISF is fundamental in measure.

Proof. Given $\varepsilon > 0$, define $E_{mn} = \{x : |f_m(x) - f_n(x)|\} \geq \varepsilon$

Then $\rho(f_m, f_n) = \int |f_m - f_n| d\mu \geq \int_{E_{mn}} |f_m - f_n| d\mu \geq \varepsilon \int_{E_{nm}} d\mu = \varepsilon\, \mu(E_{mn})$

Since $\{f_n\}$ is mean fundamental, $\rho(f_m, f_n) \to 0$ as $m, n \to \infty$, which implies that $\mu(E_{mn}) \to 0$ as $m, n \to \infty$. Therefore $\{f_n\}$ is fundamental in (m).

THEOREM 4.16. If $\{f_n\}$ is a mean fundamental sequence of ISF and if the indefinite integral of f_n is ν_n, $(n=1,2,\ldots)$ then $\nu(E) = \lim_n \nu_n(E)$ exists for every measurable set E and the set function ν is finite valued and countably additive.

Proof. Considering $\nu_m(E) - \nu_n(E)$, we find that since $\{f_n\}$ is mean fundamental,

$$|\nu_m(E) - \nu_n(E)| = \left| \int_E [f_m(x) - f_n(x)] d\mu \right| \leq$$

$$\int_E |f_m - f_n| d\mu \to 0 \quad \text{as } m, n \to \infty,$$

from which the uniformity, existence and finiteness of the limit are clear. Using the finite additivity of limits, we conclude that ν is finitely additive. To show that ν is countably additive, consider a disjoint sequence $\{E_n\}$ of measurable sets, whose union is E. Then, for every pair of positive integers n and j,

$$\begin{aligned} \left|\nu(E) - \sum_{i=1}^{j} \nu(E_i)\right| &\leq |\nu(E) - \nu_n(E)| + \left|\nu_n(E) - \sum_{i=1}^{j} \nu_n(E_i)\right| \\ &\quad + \left|\sum_{i=1}^{j} \nu_n(E_i) - \sum_{i=1}^{j} \nu(E_i)\right| \\ &< 2\varepsilon + \left|\nu_n(E) - \sum_{i=1}^{j} \nu_n(E_i)\right| \text{ for sufficiently large } n \\ &< 3\varepsilon \text{ for sufficiently large } n \text{ and } j, \end{aligned}$$

which proves that $\nu(E) = \lim_j \sum_{i=1}^{j} \nu(E_i) = \sum_{i=1}^{\infty} \nu(E_i)$. Thus ν is countably additive.

DEFINITION 4.17. If $\{\nu_n\}$ is a sequence of finite valued set functions defined on the class of all measurable sets of a measurable space $(X, \mathcal{S})$, then the terms of the sequence are uniformly absolutdy continuous if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|\nu_n(E)| < \varepsilon$ for every measurable set E for which $\mu(E) < \delta$ and for every positive integer n.

THEOREM 4.18. If $\{f_n\}$ is a mean fundamental sequence of ISF and if the indefinite integral of f_n is ν_n, $(n=1,2,..)$ then the set functions ν_n are uniformly absolutely continuous.

Proof. Let $\varepsilon > 0$ and let n_o be a positive integer such that for $m,n \geq n_o$, we have $\int |f_m - f_n| d\mu < \varepsilon/2$. Let δ be a positive number such that $\int_E |f_n| d\mu < \varepsilon/2$, $n=1,2,\ldots,n_o$, for every measurable set E for which $\mu(E) < \delta$. If $n < n_o$, $|\nu_n(E)| = |\int_E f_n d\mu| < \int_E |f_n| d\mu < \varepsilon/2$. If $n \geq n_o$,

$$|\nu_n(E)| \leq \int |f_n - f_{n_o} + f_{n_o}| d\mu \leq \int |f_n - f_{n_o}| d\mu + \int |f_{n_o}| d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Thus, the set functions ν_n are uniformly absolutely continuous.

5. INTEGRABLE FUNCTIONS.

Let $M = M(X, \mathcal{S})$ be the class of all measurable functions from $X \to \mathbb{R}^*$ and let $M^+ = M^+(X, \mathcal{S})$ be the class of all non-negative measurable functions from $X \to \mathbb{R}^*$.

DEFINITION 5.1. Let $f \in M^+$. Then f is <u>integrable w.r.t. μ</u> if there exists a sequence $\{f_n\}$ of ISF such that $0 \leq f_n \uparrow f$ and $\int f_n d\mu$ is finite. The <u>integral of</u> f <u>w.r.t.</u> μ is defined by $\int f d\mu = \sup_n \int f_n d\mu$.

DEFINITION 5.2. If E is a measurable set, then $f\chi_E \in M^+$ and the <u>integral of</u> f <u>over</u> E <u>w.r.t.</u> μ is defined as $\int_E f d\mu = \int f \chi_E d\mu$.

LEMMA 5.3. (i) If f, g M^+ and $f \leq g$, then $\int f d\mu \leq \int g d\mu$. (ii) If E, F are measurable sets and $E \subseteq F$, then $\int_E f d\mu \leq \int_F f d\mu$.

<u>Proof</u>. Choose two simple functions ϕ and ψ such that $0 \leq \phi \leq f \leq \psi \leq g$. Since $\phi . \psi$ are simple functions and $\phi \leq \psi$, $\int \phi d\mu \leq \int \psi d\mu$ which implies that $\sup \int \phi d\mu \leq \sup \int \psi d\mu$. This gives $\int f d\mu \leq \int g d\mu$, as f, g M^+.

To prove (ii), since $E \subsetneq F$, $f\chi_E \leq f\chi_F$. Therefore, $\int f\chi_E d\mu \leq \int f \chi_F d\mu$ by (i). Thus, $\int_E f d\mu \leq \int_F f d\mu$.

THEOREM 5.4. (Monotone Convergence Theorem). If $\{f_n\}$ is a monotone increasing sequence of functions belonging to M^+ and if $\{f_n\}$ converges to f, then $\int f d\mu = \lim_n \int f_n d\mu$.

<u>Proof</u>. Since f is the limit of a sequence of measurable functions, f is also measurable. As $f_n \leq f_{n+1} \leq f$, it follows that $\int f_n d\mu \leq \int f_{n+1} d\mu \leq \int f d\mu$ by Lemma 5.3, for

all n. Therefore

$$\lim_n \int f_n d\mu \leq \int f d\mu . \tag{5.1}$$

To establish the opposite inequality, let c be a real number such that $0 < C < 1$ and let ϕ be a simple function with $0 \leq \phi \leq f$. Let $A_n = \{ x \in X: f_n(x) \geq c\, \phi(x) \}$. Notice that $A_n \subseteq A_{n+1}$ and $X = \bigcup_1^{\infty} A_n$. Now

$$\int_{A_n} c\phi d\mu \leq \int_{A_n} f_n d\mu \leq \int f_n d\mu \tag{5.2}$$

and

$$\int \phi d\mu = \lim_n \int_{A_n} \phi d\mu .$$

Taking limits as $n \to \infty$ in (5.2) we get

$$C.\lim_n \int_{A_n} \phi d\mu \leq \lim_n \int f_n d\mu$$

which implies that $C. \int \phi d\mu \leq \lim_n \int f_n d\mu$. This result holds for any simple function ϕ satisfying the condition $0 \leq \phi \leq f$. Using the fact that $C < 1$, $C \int \phi\, d\mu < \int \phi d\mu \leq \lim_n \int f_n d\mu$ and $\sup_\phi \int \phi d\mu \leq \lim_n \int f_n d\mu$ which shows that

$$\int f d\mu \leq \lim_n \int f_n d\mu \tag{5.3}$$

Combining (5.1) and (5.3) we have $\int f d\mu = \lim_n \int f_n d\mu$.

LEMMA 5.5. (Fatou). If $\{f_n\}$ is a sequence of functions belonging to M^+, then

$$\int (\lim\inf f_n) d\mu \leq \lim\inf \int f_n d\mu .$$

<u>Proof</u>. Let $g_m = \inf \{f_m, f_{m+1}, \ldots\}$ so that when $m \leq n$, $g_m \leq f_n$. Then $\{g_m\}$ is an increasing sequence and $\int g_m d\mu \leq \int f_n d\mu$ for $m \leq n$ by Lemma 5.3, so that $\int g_m d\mu \leq \liminf \int f_n d\mu$. Since $\{g_m\}$ is increasing and converges to $\liminf f_n$, by Theorem 5.4, we have

$$\int(\liminf f_n) d\mu = \lim_n \int g_m d\mu \leq \liminf \int f_n d\mu .$$

COROLLARY 5.6. Suppose $f \in M^+$. Then $f(x) = 0$ a.e. if and only if $\int f d\mu = 0$.

<u>Proof</u>. Suppose $\int f d\mu = 0$. Let $E_n = \{ x : X : f(x) > \frac{1}{n} \}$. Then $f > \frac{1}{n} \cdot \chi_{E_n}$. Now $0 = \int f d\mu \geq \frac{1}{n} \int \chi_{E_n} d\mu \geq \frac{1}{n} \mu(E_n) \geq 0$ implies $\mu(E_n) = 0$. Hence, writing $\{x \in X : f(x) > 0\} = \bigcup_1^\infty E_n$, we have $\mu(\bigcup_1^\infty E_n) \leq \sum_1^\infty \mu(E_n) = 0$. Thus $\mu(\bigcup_1^\infty E_n) = 0$ which establishes that $f(x) = 0$ a.e. Conversely, given $f(x) = 0$ a.e., define $E = \{ x \in X : f(x) > 0 \}$. Then $\mu(E) = 0$, since $f(x) = 0$ a.e. Let $f_n = n \chi_E$. Since $f \leq \liminf f_n$, using Lemma 5.5, $\int f d\mu \leq \int \liminf f_n d\mu \leq \liminf \int f_n d\mu$. Thus $\int f d\mu \leq \liminf \int n \chi_E d\mu = \liminf [n. \ (E)] = 0$. Thus, as $f \in M^+$, $0 \leq \int f d\mu \leq 0$ which implies that $\int f d\mu = 0$.

REMARK 5.7. The Monotone Convergence theorem holds if convergence is replaced by convergence a.e.

THEOREM 5.8. If $0 \leq f \leq g$, where g is integrable and f is measurable, then f is also integrable and $\int f d\mu \leq \int g d\mu$.

<u>Proof</u>. Let $\{g_n\}$ be a sequence of ISF such that $0 \leq g_n \uparrow g$ and $\int g_n d\mu$ is bounded. Let $\int g_n d\mu < M < \infty$. Let f_n be a sequence of simple functions such that $0 \leq f_n \uparrow f$. We have to prove that $\int f_n d\mu$ is bounded. Let $h_n = f_n \cap g_n$. Then h_n is a simple function and $0 \leq h_n \leq g_n$. Further, $N(h_n) \leq N(g_n)$. For an integrable simple function $F = \sum_1^n C_k \chi_{E_k}$, $\mu(E_k) < \infty$ for every k for which $C_k \neq 0$ by 4.4. This is equivalent to the condition $\mu(N(F)) < \infty$. Thus, $\mu(N(k_n)) \leq \mu(N(g_n))$ which is bounded by virtue of the fact that g is integrable and $\{g_n\}$ is a sequence of simple functions converging to g, whose integrals are bounded. Therefore h_n is an ISF. Also, $\int h_n d\mu \leq \int g_n d\mu < M$ for all n. Since $0 \leq h_n = f_n \cap g_n$ which converges to $f \cap g = f$, we find that f is integrable and $\int f d\mu = \sup_n \int h_n d\mu \leq \sup_n \int g_n d\mu = \int g d\mu$.

COROLLARY 5.9. If $f \geq 0$ is integrable and $g \geq 0$ is measurable, then $f \cap g$ is integrable.

<u>Proof</u>. $0 \leq f \cap g \leq f$ and $f \cap g$ is measurable. Using Theorem 5.8, $f \cap g$ is integrable.

THEOREM 5.10. If $f \geq 0$ is integrable, then for each $\varepsilon > 0$, the set $\{x : f(x) \geq \varepsilon\}$ has finite measure. In particular, $N(f)$ is the union of a sequence of measurable sets of finite measure.

<u>Proof</u>. Let $\{f_n\}$ be a sequence of ISF f and let $\int f_n d\mu < M < \infty$ for all n. Define $E_n = \{x : f_n(x) > \varepsilon\}$ and $E = \{x : f(x) > \varepsilon\}$. Then $E_n \uparrow E$ and hence $\mu(E_n) \uparrow \mu(E)$. For each n, $\varepsilon \chi_{E_n} < \chi_E f_n \leq f_n$ and hence $\varepsilon \mu(E_n) < \int_E f_n d\mu < \int f_n d\mu$ which is bounded for all n. Thus, $\mu(E_n) < M/\varepsilon$ for all n, which implies that $\mu(E)$ is finite. In particular, $N(f) = \bigcup_1^\infty \{x : f(x) > \frac{1}{n}\}$ and thus $N(f)$ is the union of a sequence of measurable sets of finite measure.

NOTATION 5.11. Let $\mathcal{P}^1(\mu)$ be the class of all functions belonging to M^+ and which are integrable.

DEFINITION 5.12. A measurable function f <u>belonging to</u> M <u>is integrable w.r.t.</u> μ if there exist functions $g, h \in \mathcal{P}^1$ such that $f = g - h$. The <u>integral of</u> f <u>w.r.t.</u> μ is defined by $\int f d\mu = \int g d\mu - \int h d\mu$.

DEFINITION 5.13. If E is a measurable set, $f\chi_E$ belongs to M and the <u>integral of</u> f <u>over</u> E <u>w.r.t.</u> μ is defined by $\int_E f d\mu = \int \chi_E f d\mu$.

NOTATION 5.14. Let $\mathcal{L}^1(\mu)$ be the class of all functions belonging to M and which are integrable. Then $\mathcal{L}^1(\mu) \supset \mathcal{P}^1(\mu)$.

LEMMA 5.15. If $f \in \mathcal{L}^1$, $f \geq 0$, then $f \in \mathcal{P}^1$. Moreover, if g is measurable with $g \leq f$, then $g \in \mathcal{L}^1$ and $\int g d\mu \leq \int f d\mu$.

Proof. The first part is obvious from the definition of $\mathcal{P}^1$. Using Theorem 5.8, since $f \in \mathcal{P}^1$, g is measurable and $g \le f$, $g \in \mathcal{P}^1$ and $\int g d\mu \le \int f d\mu$. To prove integrability for the class $\mathcal{L}^1$, we have only to extend g to that larger class.

LEMMA 5.16. If $f, g \in \mathcal{L}^1$, $c \in \mathbb{R}$ and A is a locally measurable set, then

(i) $cf \in \mathcal{L}^1$ and $\int cf d\mu = c\int f d\mu$.

(ii) $f+g \in \mathcal{L}^1$, and $\int (f+g) d\mu = \int f d\mu + \int g d\mu$

(iii) $\chi_A f \in \mathcal{L}^1$.

Proof. The results follow easily from the corresponding ones in the case of functions belonging to $\mathcal{P}^1$ and an extension is immediate.

THEOREM 5.17. If f is measurable, then the following are equivalent.

(i) $f \in \mathcal{L}^1$

(ii) $|f| \in \mathcal{L}^1$

(iii) $f^+, f^- \in \mathcal{L}^1$ and $|\int f d\mu| \le \int |f| d\mu$.

Proof. (i) $\Rightarrow$ (ii). Since $f \in \mathcal{L}^1$, $f = g-h$, $g, h \in \mathcal{P}^1$ and since f is measurable, $|f|$ is also measurable. Now $0 \le |f| \le g+h$ and thus $|f| \in \mathcal{L}^1$ by Lemma 5.15. To prove that (ii) $\Rightarrow$ (iii), since $|f| \in \mathcal{L}^1$ and $0 \le f^+ \le |f|$, $0 \le f^- \le |f|$, we have $f^+, f^- \in \mathcal{L}^1$ as f^+, f^- are measurable by Lemma 5.15. Further,

$$|\int f d\mu| = |\int (f^+ - f^-) d\mu| \le |\int f^+ d\mu| + |\int f^- d\mu| \le \int f^+ d\mu + \int f^- d\mu = \int |f| d\mu .$$

To show that (iii) $\Rightarrow$ (i), let $f^+, f^- \in \mathscr{L}^1$. Then, there exist functions, $g_1, g_2, h_1, h_2 \in \mathcal{P}^1$ such that $f^+ = g_1 - g_2$, $f^- = h_1 - h_2$. Therefore $f = f^+ - f^-$ can be expressed as $(g_1 - g_2) - (h_1 - h_2)$, which implies that $f \in \mathscr{L}^1$.

THEOREM 5.18. (Lebesgue Dominated Convergence Theorem). If $\{f_n\} \in \mathscr{L}^1$ such that $\{f_n\}$ converges a.e. to a measurable function f and if $|f_n| \leq g$ for all n, where $g \in \mathscr{L}^1$, then $f \in \mathscr{L}^1$ and $\int f d\mu = \lim_n \int f_n d\mu$.

Proof. Without loss of generality, $\{f_n\}$ may be assumed to converge to f everywhere by suitably defining these functions on a set of measure zero. By Lemma 5.15, $f \in \mathscr{L}^1$. Notice that $g+f_n \geq 0$, $g-f_n \geq 0$. An application of Lemmas 5.16 and 5.5 yields

$$\int f d\mu + \int g d\mu = \int (f+g) d\mu \leq \int \liminf (g+f_n) d\mu \leq \int g d\mu + \int (\liminf f_n) d\mu$$
$$\leq \int g d\mu + \liminf \int f_n d\mu .$$

Thus, $\int f d\mu \leq \liminf \int f_n d\mu$. To obtain the opposite inequality, using the same lemmas,

$$\int g d\mu - \int f d\mu = \int (g-f) d\mu \leq \int \liminf (g-f_n) d\mu \leq \int g d\mu - \limsup \int f_n d\mu$$
$$\leq \int g d\mu - \limsup \int f_n d\mu$$

which gives $\int f d\mu \geq \limsup \int f_n d\mu$. Thus

$$\int f d\mu \leq \liminf \int f_n d\mu \leq \limsup \int f_n d\mu \leq \int f d\mu$$

which implies that $\int f d\mu = \lim_n \int f_n d\mu$.

COROLLARY 5.19. (Lebesgue Bounded Convergence Theorem). If $\{f_n\} \in \mathcal{L}^1$ such that $\{f_n\}$ converges a.e. to a measurable function f and if $|f_n| < M$ for all n, where M is a positive constant, then $f \in \mathcal{L}^1$ and $\int f d\mu = \lim \int f_n d\mu$.

Proof. The proof is readily available by a direct application of Theorem 5.18.

6. SIGNED MEASURES AND DECOMPOSITIONS.

DEFINITION 6.1. A signed measure defined on the class of all measurable sets of a measurable space $(X, \mathcal{S})$ is an extended, real-valued, countably additive set function μ such that $\mu(\phi) = 0$ and μ assumes at most only one of the values $+\infty$ and $-\infty$.

DEFINITION 6.2. A signed measure μ is finite, σ-finite, totally finite or totally σ-finite if the corresponding conditions in the case of a measure are satisfied by μ, with the restriction that '$\mu(E)$' is replaced by $|\mu(E)|$, in definition 1.7.

THEOREM 6.3. If E, F are measurable sets with $E \subset F$ and if μ is a signed measure with $|\mu(F)| < \infty$, then $|\mu(E)| < \infty$.

Proof. Since $F = (F-E) \cup E$, $\mu(F) = \mu(F-E) + \mu(E)$ by virtue of the fact that $F-E$ and E are disjoint. If one of $\mu(E)$ or $\mu(F-E)$ is infinite and the other is finite, then $\mu(F) = \infty$

which contradicts the fact that $|\mu(F)| < \infty$. If both $\mu(E)$ and $\mu(F-E)$ are infinite, since μ can take at most only one of the values $+\infty$ and $-\infty$, their sum is infinite which again gives a contradiction. Therefore, the only possibility is that both $\mu(E)$ and $\mu(F-E)$ are finite. Thus $|\mu(E)| < \infty$ whenever $E \subset F$ and $|\mu(F)| < \infty$, which shows that every measurable subset of a set F of finite signed measure has finite signed measure.

THEOREM 6.4. If $\{E_n\}$ is a sequence of disjoint measurable sets with $|\mu(\bigcup_1^\infty E_n)| < \infty$, then the series $\sum_1^\infty |\mu(E_n)|$ is convergent.

Proof. Construct

$$E_n^+ = \begin{cases} E_n & \text{if} \quad \mu(E_n) \geq 0 \\ 0 & \text{if} \quad \mu(E_n) < 0 \end{cases}$$

$$E_n^- = \begin{cases} E_n & \text{if} \quad \mu(E_n) \leq 0 \\ 0 & \text{if} \quad \mu(E_n) > 0. \end{cases}$$

Then. $\mu(\bigcup_1^\infty E_n^+) = \sum_1^\infty \mu(E_n^+)$ and $\mu(\bigcup_1^\infty E_n^-) = \sum_1^\infty \mu(E_n^-)$. Suppose $\sum_1^\infty \mu(E_n^+)$ is divergent. Since $\mu(E_n^+) \geq 0$, $\sum_1^\infty \mu(E_n^+) = +\infty$. But $\mu(E_n^-) < 0$ which implies that $\sum_1^\infty \mu(E_n^-)$ cannot be divergent, since μ can take at most only one of the values $+\infty$ and $-\infty$. Thus, only one of the series $\sum_1^\infty \mu(E_n^+)$, $\sum_1^\infty \mu(E_n^-)$ can be divergent. Suppose $\sum_1^\infty \mu(E_n^+)$ is divergent. Since $\sum_1^\infty \mu(E_n^-)$ is convergent to $-M$ say, where $M > 0$, $\sum_1^\infty \mu(E_n) \to \sum_1^\infty \mu(E_n^+) - M$ which implies that $\sum_1^\infty \mu(E_n) \to +\infty$. Therefore, $|\mu(\bigcup_1^\infty E_n)| = |\sum_1^\infty \mu(E_n)|$ converges to $+\infty$ which contradicts the hypothesis. Thus, both the series $\sum_1^\infty \mu(E_n^+)$ and $\sum_1^\infty \mu(E_n^-)$ should be convergent and therefore $\sum_1^\infty |\mu(E_n)| = \sum_1^\infty \mu(E_n^+) + \sum_1^\infty \mu(E_n^-)$ is also convergent.

DEFINITION 6.5. A complex measure on the class of all measurable sets of a measurable space $(X, \mathcal{S})$ is a set function μ such that, for every measurable set E, $\mu(E) = \mu_1(E) + i\,\mu_2(E)$, where μ_1, μ_2 are signed measures.

DEFINITION 6.6. Let μ be a signed measure. A set E is said to be positive w.r.t. μ if for every measurable set F, $E \cap F$ is measurable and $\mu(E \cap F) \geq 0$. A set E is negative w.r.t. μ if for every measurable set F, $E \cap F$ is measurable and $\mu(E \cap F) \leq 0$.

HAHN DECOMPOSITION 6.7. If μ is a signed measure, there exist two disjoint sets A and B whose union is X, such that A is positive and B is negative w.r.t. μ. $A \cap B$ is called the Hahn decomposition of X.

REMARK 6.8. A Hahn decomposition of X is not unique and therefore necessitates a modification. Let $X = A_1 \cup B_1$, $X = A_2 \cup B_2$ be two Hahn decompositions of X. We shall prove that, for every measurable set E, $\mu(E \cap A_1) = \mu(E \cap A_2)$ and $\mu(E \cap B_1) = \mu(E \cap B_2)$. Consider $E \cap (A_1 - A_2)$.

This is clearly contained in $E \cap A_1$. Since A_1 is positive w.r.t. μ, $\mu(E \cap A_1) \geq 0$ and thus $\mu[E \cap (A_1 - A_2)] \geq 0$. Again, $E \cap (A_1 - A_2) \subset E \cap B_2$ and since B_2 is negative w.r.t. μ, $\mu[E \cap (A_1 - A_2)] \leq 0$. Combining these two results, we find that $\mu[E \cap (A_1 - A_2)] = 0$. By symmetry,

$\mu[E \cap (A_2 - A_1)] = 0$. Thus $\mu(E \cap A_1) =$
$\mu[E \cap (A_1 \cup A_2)] = \mu(E \cap A_2)$, since

$$\mu[E \cap (A_1 \cup A_2)] = \mu[E \cap (A_1 - A_2)] + \mu[E \cap (A_2 - A_1)] + \mu[E \cap (A_1 \cap A_2)].$$

DEFINITION 6.9. If E is a measurable set, the <u>upper variation of μ</u> is defined as $\mu^+(E) = \mu(E \cap A)$ and the <u>lower variation of μ</u> is $\mu^-(E) = -\mu(E \cap B)$. The set function $|\mu|$ defined by $|\mu|(E) = \mu^+(E) + \mu^-(E)$ is called the <u>total variation of μ</u>.

JORDAN DECOMPOSITION 6.10. The upper, lower and total variations of a signed measure μ being measures, the <u>Jordan decomposition of μ</u> is given by $\mu(E) = \mu^+(E) - \mu^-(E)$, for every measurable set E. If μ is totally finite or totally σ-finite, μ^+ and μ^- are also totally finite or totally σ-finite and at least one of the measures μ^+ and μ^- is always finite.

DEFINITION 6.11. If $(X, \mathcal{S})$ is a measurable space and μ, ν are signed measures defined on $\mathcal{S}$, ν is said to be <u>absolutely continuous w.r.t.</u> μ, written symbolically as $\nu << \mu$, if $\nu(E) = 0$ for every measurable set E for which $|\mu|(E) = 0$.

THEOREM 6.12. If μ, ν are signed measures, then the following are equivalent.

(i) $\nu << \mu$

(ii) $\nu^+ << \mu$ and $\nu^- << \mu$

(iii) $|\nu| << |\mu|$.

<u>Proof</u>. To prove that (i) $\Rightarrow$ (ii), let E be a measurable set and let μ, ν be signed measures with $\nu << \mu$. Then $\nu(E) = 0$ whenever $|\mu|(E) = 0$. Let $X = A \cup B$ be a Hahn decomposition w.r.t. μ. Now $0 \leq |\mu|(E \cap A) \leq |\mu|(E) = 0$ and $0 \leq |\mu|(E \cap B) \leq |\mu|(E) < 0$ and since $E \cap A$ and $E \cap B$ are measurable, this implies that $\nu(E \cap A) = \nu^+(E) = 0$ and $-\nu(E \cap B) = \nu^-(E) = 0$ whenever $|\mu|(E) = 0$.

The facts (ii) $\Rightarrow$ (iii) and (iii) implies (i) follow from the relations $|\nu|(E) = \nu^+(E) + \nu^-(E)$ and $0 \leq |\nu(E)| \leq |\nu|(E)$ respectively.

DEFINITION 6.13. If $(X, \mathcal{S})$ is a measurable space and μ, ν are signed measures defined on $\mathcal{S}$, then ν is said to be <u>mutually singular w.r.t.</u> μ, symbolically $\nu \perp \mu$, if there exist two disjoint sets A and B such that for every measurable set E, $A \cap E$ and $B \cap E$ are measurable and $|\mu|(A \cap E) = |\nu|(A \cap E) = 0$.

NOTATION 6.14. We say that $f = g[\mu]$ if $\{x : f(x) \neq g(x)\}$ is a measurable set of measure zero w.r.t. $|\mu|$.

THEOREM 6.15 (Radon-Nikodym). If $(X, \mathcal{S}, \mu)$ is a totally σ-finite measure space and if ν is a σ-finite signed measure defined on $\mathcal{S}$ which is absolutely continuous w.r.t. μ, then there exists a finite-valued measurable function f defined on X such that $\nu(E) = \int_E f\,d\mu$, for every measurable set E.

For a proof, see P.R.Halmos, "Measure Theory".

THEOREM 6.16. (Lebesgue Decomposition Theorem). If $(X, \mathcal{S})$ is a measurable space and μ, ν are totally σ-finite, signed measures on $\mathcal{S}$, there exist two uniquely determined totally σ-finite signed measures ν_0 and ν_1 whose sum is ν, such that $\nu_0 \perp \mu$ and $\nu_1 << \mu$.

Proof. Since X is a countable disjoint union of measurable sets on which both μ and ν are finite, there is no loss of generality in assuming that μ and ν are finite. Since $\nu_i (i=0,1)$ will be absolutely continuous or singular w.r.t. according as it is absolutely continuous or singular w.r.t. $|\mu|$, we may assume that μ is a measure. As ν^+ and ν^- can be treated separately, we may also assume that ν is a measure. Observing that $\nu << \nu + \mu$, there exists a measurable function f such that $\nu(A) = \int_E f\,d\mu + \int_E f\,d\nu$ for every measurable set E, by Theorem 6.15. Since $0 \leq \nu(E) \leq \mu(E) + \nu(E)$, we have $0 \leq f \leq 1 \; [\mu+\nu]$ and therefore $0 \leq f \leq 1 \; [\nu]$, using 6.14. Defining $A = \{x : f(x) = 1\}$ and $B = \{x : 0 \leq f(x) < 1\}$. we have $\nu(A) = \int_A d\mu + \int_A d\nu = \mu(A) + \nu(A)$ and therefore, since ν is finite, $\mu(A) = 0$. If $\nu_0(E) = \nu(E \cap A)$ and $\nu_1(E) = \nu(E \cap B)$ for every measurable set E, then clearly $\nu_0 \perp \mu$. To prove that $\nu_1 << \mu$, if $\mu(E) = 0$, $\int_{E\cap B} d\nu = \nu(E \cap B) = \int_{E\cap B} f d\nu$ and therefore $\int_{E\cap B} (1-f) d\nu = 0$. Since $1-f \geq 0 \; [\nu]$, it follows that $\nu_1(E) = \nu(E \cap B) = 0$. Thus, the existence of ν_0 and ν_1 has been established. To prove the uniqueness, if $\nu = \nu_0 + \nu_1$ and $\nu = \nu_0' + \nu_1'$ are two Lebesque decompositions of ν, then $\nu_0 - \nu_0' = \nu_1' - \nu_1$. Since $\nu_0 - \nu_0'$ is singular and $\nu_1' - \nu_1$ is absolutely continuous w.r.t. μ we have $\nu_0 = \nu_0'$, $\nu_1 = \nu_1'$.

7. PRODUCT MEASURES AND PRODUCT SPACES.

DEFINITION 7.1. If X and Y are any two sets, not necessarily subsets of the same space, the Cartesian product of the two sets is the set of all ordered paires (x,y), where $x \in X$ and $y \in Y$ and is denoted by $X \times Y$.

DEFINITION 7.2. If $A \subset X$, $B \subset Y$, then $E = A \times B$ is a subset of $X \times Y$ and is called a rectangle. The component sets A and B are referred to as the sides of the rectangle.

REMARKS 7.3.

(i) A rectangle is empty if and only if one of its sides is empty.

(ii) If $E_1 = A_1 \times B_1$ and $E_2 = A_2 \times B_2$ are nonempty rectangles, then $E_1 \subset E_2$ if and only if $A_1 \subset A_2$ and $B_1 \subset B_2$.

(iii) If E_1 and E_2 are nonempty rectangles and $E_1 = E_2$, then $A_1 = A_2$ and $B_1 = B_2$.

EXERCISE 7.4. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are rings of subsets of X and Y respectively, then the class $\mathcal{R}$ of all finite, disjoint unions of rectangles of the form $A \times B$, where $A \in \mathcal{R}_1$ and $B \in \mathcal{R}_2$ is also a ring.

In order to establish that $\mathcal{R}$ is a ring, it should be proved that $\mathcal{R}$ is closed under finite unions and set-theoretic differences. It is obviously closed under finite unions of disjoint sets. We shall prove that it is closed under the formation of differences. If $E_1 = A_1 \times B_1$, $E_2 = A_2 \times B_2$ and $(x,y) \in E_1 \cap E_2$, then $x \in A_1 \cap A_2$ and $y \in B_1 \cap B_2$, so that $E_1 \cap E_2 \subset (A_1 \cap E_2) \times (B_1 \cap B_2)$. On the other hand, by 7.3 (ii), $(A_1 \cap A_2) \times (B_1 \cap B_2) \subset E_1$ and $(A_1 \cap A_2) \times (B_1 \cap B_2) \subset E_2$ and therefore $(A_1 \cap A_2) \times (B_1 \cap B_2) \subset E_1 \cap E_2$. The two inclusions combine to show that $E_1 \cap E_2 = (A_1 \cap A_2) \times (B_1 \cap B_2)$. But, $A_1 \cap A_2 \in \mathcal{R}_1$, $B_1 \cap B_2 \subset \mathcal{R}_2$, since $\mathcal{R}_1$, $\mathcal{R}_2$ are rings. Thus, it follows that $\mathcal{R}$ is closed under the formation of finite intersections. Now,

$$(A_1 \times B_1) - (A_2 \times B_2) = [(A_1 \cap A_2) \times (B_1 - B_2)] \cup [(A_1 - A_2) \times B_1]$$

and since, $\bigcup_{r=1}^{n} F_r - \bigcup_{r=1}^{m} E_r = \bigcup_{r=1}^{n} \bigcap_{r=1}^{m} (F_r - E_r)$ and $\mathcal{R}$ is closed under finite intersections it follows that $\mathcal{R}$ is closed under differences of elements belonging to $\mathcal{R}$. Further, if $M_1, M_2 \in \mathcal{R}$, $M_1 \cup M_2 = (M_1 - M_2) \cup (M_2 - M_1) \cup (M_1 \cap M_2)$, a disjoint union of members of $\mathcal{R}$. Thus a finite union of members of $\mathcal{R}$ belongs to $\mathcal{R}$. Hence $\mathcal{R}$ is a ring.

DEFINITION 7.5. If X,Y are any two sets and $\mathfrak{S}, \mathfrak{T}$ are two σ-rings of subsets of X and Y respectively, then <u>$\mathfrak{S} \times \mathfrak{T}$ is the σ-ring of subsets of X x Y</u> generated by the class of all sets of the form $A \times B$, where $A \in \mathfrak{S}$, $B \in \mathfrak{T}$.

DEFINITION 7.6. If $(X, \mathfrak{S})$ and $(Y, \mathfrak{T})$ are measurable spaces, then $(X \times Y, \mathfrak{S} \times \mathfrak{T})$ is a measurable space called the <u>Cartesian product of the two given measurable spaces</u>. A rectangle $A \times B$ in this Cartesian product is a measurable rectangle if $A \in \mathfrak{S}$ and $B \in \mathfrak{T}$.

DEFINITION 7.7. The <u>class of measurable sets in the Cartesian product</u> of two measurable spaces $(X, \mathfrak{S})$ and $(Y, \mathfrak{T})$ is the σ-ring generated by the class of all measurable rectangles.

DEFINITION 7.8. Let $(X, \mathfrak{S})$ and $(Y, \mathfrak{T})$ be measurable spaces and let $(X \times Y, \mathfrak{S} \times \mathfrak{T})$ be their Cartesian product.

If E is any subset of $X \times Y$ and x is any point of X, the set $E_x = \{y:(x,y) \in E\}$ is called an X-section of E or a section of E determined by x. Similarly, for any point $y \in Y$, the set $E_y = \{x:(x,y) \in E\}$ is called a Y-section of E or a section of E determined by y.

REMARK 7.9. A section of a set in the product space is not a set in that space, but a subset of one of the components.

DEFINITION 7.10. If f is any function defined on E and $x \in X$, the function f_x defined on the section E_x by $f_x(y) = f(x,y)$ is called an X-section of f or a section of f determined by x. Dually, the function f_y, defined on the section E_y determined by any $y \in Y$ and given by $f_y(x) = f(x,y)$ is called a Y-section of f or a section of f determined by y.

EXERCISE 7.11. Every section of a measurable set is measurable.

In other words, we have to prove that if $E \in \mathfrak{S} \times \mathfrak{T}$, then $E_x \in \mathfrak{T}$ and $E_y \in \mathfrak{S}$, for every $x \in X$, $y \in Y$. Fix $x \in X$ and define $g: Y \rightarrow X \times Y$ by $g(y) = (x,y)$. Then, $g^{-1}(F) = F_x$, for all $F \subset X \times Y$. The class $\mathfrak{E} = \{F \subset X \times Y: g^{-1}(F) \in \mathfrak{T}\}$ is clearly a σ-ring. If $A \in \mathfrak{S}$ and $B \in \mathfrak{T}$, then $g^{-1}(A \times B) = A$ or ϕ. In either case, $g^{-1}(A \times B) \in \mathfrak{T}$. Thus, $\mathfrak{E}$ is a σ-ring containing every measurable rectangle, and therefore $\mathfrak{S} \times \mathfrak{T} \subset \mathfrak{E}$. In particular, $E \in \mathfrak{E}$, that is,

$E_x \in \mathcal{T}$. Similarly, every Y-section of a measurable set is measurable.

EXERCISE 7.12. Every section of a measurable function is a measurable function.

Let f be a measurable function defined on $X \times Y$ and let f_x be an X-section of f. Now f_x will be a measurable function if $N(f_x) \cap f_x^{-1}(M)$ is a measurable set, for every Borel set M. Obviously $N(f_x) = (N(f))_x$ and if M is any Borel set on $\mathbb{R}$,

$$f_x^{-1}(M) = \{y : f_x(y) \in M\} = \{y : f(x,y) \in M\} = \{y : (x.y) \in f^{-1}(M)\}$$
$$= (f^{-1}(M))_x.$$

The result is now immediate on using 7.11 and the fact that f is a measurable function. In a similar manner, every Y-section of f is also a measurable function.

DEFINITION 7.13. If $E \in \mathcal{S} \times \mathcal{T}$, the functions $f_E : X \to \mathbb{R}^*$ and $g_E : Y \to \mathbb{R}^*$ are defined by $f_E(x) = \nu(E_x)$ and $g_E(y) = \mu(E_y)$.

If the component spaces of a Cartesian product are measure spaces, we have the notion of a product measure.

DEFINITION 7.14. If $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are finite measure spaces, there exists a unique measure λ on $\mathcal{S} \times \mathcal{T}$ such that $\lambda(A \times B) = \mu(A)\,\nu(B)$ for every measurable rectangle $A \times B$ and this is called the <u>product</u> of μ and ν.

REMARK 7.15. λ is finite and $\lambda(E) = \int f_E d\mu = \int g_E d\nu$ for every $E \in \mathcal{S} \times \mathcal{T}$.

DEFINITION 7.16. If $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are measure spaces, their <u>Cartesian product</u> is the measure space $(X \times Y, \mathcal{S} \times \mathcal{T}, \mu \times \nu)$.

We shall now indicate the relations between integrals on a product space and integrals on the component spaces.

DEFINITION 7.17. Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be σ-finite measure spaces and let λ be the product measure $\mu \times \nu$ on $\mathcal{S} \times \mathcal{T}$. If a function h on $X \times Y$ has its integral defined, then the integral is denoted by $\int h(x,y)\, d\lambda(x,y)$ or $\int h(x,y)\, d(\mu \times \nu)(x,y)$ and is called the <u>double integral</u> of h. If h_x is such that $\int h_x(y) d\nu(y) = f(x)$ is defined and if $\int f d\mu$ is also defined, then

$$\int f d\mu = \iint h(x,y) d\nu(y) d\mu(x) = \int d\mu(x) \int h(x,y) d\nu(y).$$

Similarly, if h_y is such that $\int h_y(x) d\mu(x) = g(y)$ is defined and if $\int g d\nu$ is also defined, then

$$\int g d\nu = \iint h(x,y) d\mu(x) d\nu(y) = \int d\nu(y) \int h(x,y) d\mu(x).$$

The integrals $\iint h d\mu\, d\nu$ and $\iint h d\nu\, d\mu$ are called the <u>iterated integrals</u> of h.

LEMMA 7.18. A measurable subset E of $X \times Y$ has measure zero if and only if almost every X-section (or almost every Y-section) has measure zero.

Proof. By remark 7.15,

$$\lambda(E) = \int f_E d\mu = \int g_E d\nu \quad .$$

If $\lambda(E) = 0$, the two integrals are in particular finite and hence by Corollary 5.6 their non-negative integrands must vanish a.e. On the other hand, if either of the integrands vanishes a.e., $\lambda(E) = 0$.

LEMMA 7.19. If h is a non-negative measurable function on $X \times Y$, then $\int h d(\mu \times \nu) = \iint h d\mu\, d\nu = \iint h d\nu\, d\mu$.

Proof. If h is the characteristic function of a measurable set E, then $\int h(x,y) d\nu(y) = \nu(E_x)$ and $\int h(x,y) d\mu(x) = \mu(E_y)$ and the result is immediate. In general, we can find an increasing sequence $\{h_n\}$ of nonnegative simple functions converging to h everywhere, by Theorem 4.3. Since a simple function can be expressed as a finite linear combination of characteristic functions, the theorem follows for every h_n which is substituted instead of h. Now, $\lim_n \int h_n d\lambda = \int h d\lambda$. If $f_n(x) = \int h_n(x,y) d\nu(y)$, then it follows that $\{f_n\}$ is an increasing sequence of nonnegative measurable functions converging to $f(x) = \int h(x,y) d\nu(y)$ for every x. Hence f is measurable and is obviously nonnegative. Thus $\lim_n \int f_n d\mu = \int f d\mu$. This proves the equality of the double integral and one of the iterated integrals. The other part follows in a similar way.

THEOREM (Fubini) 7.20. If h is an integrable function on $X \times Y$, then almost every section of h is integrable. If the functions f and g are defined by

$f(x) = \int h(x,y)d\nu(y)$ and $g(y) = \int h(x,y)d\mu(x)$,

then f and g are integrable and

$$\int hd(\mu\times\nu) = \int fd\mu = \int gd\nu \qquad (7.1)$$

Proof. A real-valued function f is integrable if and only if f^+ and f^- are integrable. Hence, it suffices to consider nonnegative functions h. The equality (7.1) is then obvious using Lemma 7.19. Since f and g are non-negative, measurable and have finite integrals, they are also integrable. Finally, since this implies f and g are finite valued almost everywhere, the sections of h have the desired properties.

CHAPTER III

FOURIER TRANSFORMS.

1. THE CLASSICAL APPROACH.

In this section, the definition and properties of Fourier transforms on the real line will be investigated, from the point of view of classical analysis.

DEFINITION 1.1. A complex-valued function f is measurable, if its real and imaginary parts are separately measurable.

DEFINITION 1.2. Let $1 \leq p < \infty$. The class of all complex valued, measurable functions f, defined on $\mathbb{R}$ is said to be p^{th}-power integrable or $f \in L^p(-\infty, \infty)$, if $\int_{-\infty}^{\infty} |f(x)|^p dx < \infty$. If $f \in L^p(-\infty, \infty)$, the L^p-norm of f is defined as $\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{1/p}$.

We shall only consider the Fourier transforms for the class $L^1(-\infty, \infty)$. For each $f \in L^1(-\infty, \infty)$, the integral $\int_{-\infty}^{\infty} e^{ixt} f(t)dt$ exists for all $x \in \mathbb{R}$.

DEFINITION 1.3. The Fourier transform $\hat{f}$ of $f \in L^1(-\infty, \infty)$ is defined by the relation

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt} f(t)dt, \text{ for } x \in \mathbb{R}.$$

PROPERTIES 1.4.

(i) $\hat{f}$ is continuous on $(-\infty, \infty)$. For, if x and h are any two real numbers,

$$|\hat{f}(x+h) - \hat{f}(x)| = \left| \int_{-\infty}^{\infty} e^{ixt}(e^{iht}-1)f(t)dt \right| \leq \int_{-\infty}^{\infty} |e^{ixt}(e^{iht}-1) f(t)|dt.$$

The integrand on the extreme right is bounded by $2|f(t)|$ and tends to zero as $h \to 0$. An application of the Lebesgue Dominated Convergence Theorem shows that the integral on the extreme right converges to zero as $h \to 0$. Consequently, the expression on the extreme left also tends to zero as $h \to 0$, which establishes the continuity of $\hat{f}$ at x. (For further details regarding the dominated convergence theorem, refer to E.J.McShane).

(ii) Since $\hat{f}$ is continuous, it is necessarily bounded on $(-\infty, \infty)$ and its actual upper bound is given by the inequality

$$|\hat{f}(x)| \leq \int_{-\infty}^{\infty} |e^{ixt} f(t)|dt = \int_{-\infty}^{\infty} |f(t)|dt = \|f\|_1 .$$

The right hand side being $\|f\|_1$ is a value, independent of x and therefore $\sup\limits_{-\infty < x < \infty} |\hat{f}(x)| \leq \|f\|_1$.

REMARK 1.5. If f $L^1(-\infty, \infty)$, $\lim\limits_{t \to 0} \int_{-\infty}^{\infty} |f(x+t)-f(x)|dx = 0$.

LEMMA 1.6. (Riemann-Lebesgue) If $f \in L^1(-\infty, \infty)$, then

$$\lim_{x \to \pm\infty} \hat{f}(x) = 0.$$

Proof. By Definition 1.3, $\hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt} f(t)dt$ and therefore, $-\hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt} e^{i\pi} f(t)dt = \int_{-\infty}^{\infty} e^{ix(t + \frac{\pi}{x})} f(t)dt =$ $= \int_{-\infty}^{\infty} e^{ixt} f(t - \frac{\pi}{x})dt$, by changing t to $t + \frac{\pi}{x}$ and noting that

$\frac{\pi}{x}$ is a constant as far as this integration is concerned. Thus, we have $2\hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt}\left[f(t) - f(t - \frac{\pi}{x})\right]dt$, on subtraction, and this in turn yields $2|\hat{f}(x)| \leq \int_{-\infty}^{\infty} |f(t)-f(t-\frac{\pi}{x})|dt$. Using remark 1.5, we conclude that $\lim_{x \to \pm\infty} \int_{-\infty}^{\infty} |f(t)-f(t-\frac{\pi}{x})|dt=0$. This proves that $\hat{f}(x) \to 0$ as $x \to \pm\infty$.

LEMMA 1.7. If $f \in L^1(-\infty, \infty)$, $\lim_{x \to \pm\infty} \int_{-\infty}^{\infty} f(t) \sin xt\, dt = 0$,

$$\lim_{x \to \pm\infty} \int_{-\infty}^{\infty} f(t) \cos xt\, dt = 0.$$

Proof. This lemma follows from Lemma 1.6 and the identities $\sin xt = \frac{e^{ixt}-e^{-ixt}}{2i}$, $\cos xt = \frac{e^{ixt}+e^{-ixt}}{2}$.

Question 1.8. We have observed that if $f \in L^1(-\infty, \infty)$, then $\hat{f}$ is continuous on $(-\infty, \infty)$ and $\lim_{x \to \pm\infty} \hat{f}(x) = 0$. Is every function which is continuous on $(-\infty, \infty)$ and which vanishes at $\pm\infty$, the Fourier transform of some function belonging to $L^1(-\infty, \infty)$?

This need not be the case, as the following example reveals.

EXAMPLE 1.9. Let $g(x)$ be a function defined as

$$g(x) = \begin{cases} \frac{1}{\log x}, & x > e \\ \frac{x}{e}, & 0 \leq x \leq e \\ -g(-x), & x < 0. \end{cases}$$

g is clearly continuous on $(-\infty, \infty)$ and vanishes at $\pm\infty$. However, it is not the Fourier transform of any function belonging to $L^1(-\infty, \infty)$ and this can be easily proved by contradiction.

DEFINITION 1.10. The convolution of two functions f, g $L^1(-\infty, \infty)$ is defined by $(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$.

We shall denote $(f*g)(x)$ by $h(x)$, whenever the integral exists and the convolution is defined.

We now state two theorems, which are constantly used. The proofs are omitted.

THEOREM 1.11. (Fubini). If the double integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dx\,dy$ is absolutely convergent, then $\int_{-\infty}^{\infty} f(x,y)dy$ exists for almost all x and is an integrable function of x. Moreover, $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f(x,y)dy =$
$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} f(x,y)dx$.

THEOREM 1.12. (Tonelli-Hobson). If either of the repeated integrals $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f(x,y)dy$ or $\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} f(x,y)dx$ is absolutely convergent, then the corresponding double integral is also absolutely convergent and all the three integrals are equal.

THEOREM 1.13. $h(x)$ exists for almost all x and is an integrable function of x. Moreover, $\|h\|_1 \leq \|f\|_1 \|g\|_1$ and $\hat{h} = \hat{f}\hat{g}$.

<u>Proof</u>. For every t, $\int_{-\infty}^{\infty} |f(x-t)|dx = \int_{-\infty}^{\infty} |f(u)|du < \infty$ and therefore $\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} |f(x-t)g(t)|dx = \int_{-\infty}^{\infty} |g(t)|dt \int_{-\infty}^{\infty} |f(u)| du < \infty$, since $f, g \in L^1(-\infty, \infty)$. Using Theorem 1.12, we conclude that the double integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-t)g(t)dx\, dt$ is absolutely convergent. It follows that $\int_{-\infty}^{\infty} f(x-t)g(t)dt$ exists for almost all x and is an integrable function of x by virtue of Theorem 1.11. In other words, $h \in L^1(-\infty, \infty)$, whenever the integral is defined.

Further,

$$\|h\|_1 = \int_{-\infty}^{\infty} |h(x)|dx \le \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |f(x-t)g(t)|dt = \int_{-\infty}^{\infty} |g(t)|dt \int_{-\infty}^{\infty} |f(x-t)|dx$$
$$= \|g\|_1 \, \|f\|_1 ,$$

the change in the order of integration being justified by Theorem 1.12, since we have already proved that $h \in L^1(-\infty, \infty)$.

Again, we have

$$\begin{aligned}
\hat{h}(x) &= \int_{-\infty}^{\infty} e^{ixt} h(t)dt = \int_{-\infty}^{\infty} e^{ixt} dt \int_{-\infty}^{\infty} f(t-u)g(u)du \\
&= \int_{-\infty}^{\infty} g(u)du \int_{-\infty}^{\infty} e^{ixt} f(t-u)dt \qquad \text{by Theorem 1.12,} \\
&= \int_{-\infty}^{\infty} g(u)du \int_{-\infty}^{\infty} e^{ix(t+u)} f(t)dt = \int_{-\infty}^{\infty} g(u)e^{ixu}du \int_{-\infty}^{\infty} e^{ixt} f(t)dt \\
&= \hat{g}(x)\hat{f}(x).
\end{aligned}$$

REMARK 1.14. Convolution is commutative and associative. The commutativity is immediate from the definition of convolution, by merely effecting a change of variable. The associativity makes use of the fact that $h \in L^1(-\infty, \infty)$.

LEMMA 1.15. If $f \in L^1(-\infty, \infty)$ and if T is the transformation which associates with each f, its Fourier transform $\hat{f}$, then, for any real number a

(i) $T[f(t+a)] = e^{-iax}\hat{f}(x)$

(ii) $T[e^{iat}f(t)] = \hat{f}(x+a)$.

Proof. (i) $T[f(t+a)] = \int_{-\infty}^{\infty} e^{ixt}f(t+a)dt = \int_{-\infty}^{\infty} e^{ix(t-a)}f(t)dt$

$= e^{-iax}\hat{f}(x)$.

(ii) $T[e^{iat}f(t)] = \int_{-\infty}^{\infty} e^{ixt}[e^{iat}f(t)]dt = \int_{-\infty}^{\infty} e^{i(x+a)t}f(t)dt$

$= \hat{f}(x+a)$.

LEMMA 1.16. If $a, a_1 \in \mathbb{R}$; $f, f_1 \in L^1(-\infty, \infty)$ and T is the transformation defined as before, then

(i) $T[af + a_1f_1] = a(Tf) + a_1(Tf_1)$.

(ii) $T[f(at)] = \frac{1}{a}\hat{f}(\frac{x}{a})$

(iii) $T[\bar{f}(t)] = \overline{\hat{f}(-x)}$, where the bar denotes the complex conjugate of f.

Proof. (i) follows easily, on applying the definition. Since $|af + a_1f_1| \leq |af| + |a_1f_1|$, $(af+a_1f_1) \in L^1(-\infty, \infty)$ and $T[af + a_1f_1]$ is given by

$$\int_{-\infty}^{\infty} e^{ixt}[(af + a_1f_1)(t)]dt = a\int_{-\infty}^{\infty} e^{ixt}f(t)dt + a_1\int_{-\infty}^{\infty} e^{ixt}f_1(t)dt$$

$$= a(Tf) + a_1(Tf_1).$$

(ii) $T[f(at)] = \int_{-\infty}^{\infty} e^{ixt}f(at)dt = \frac{1}{a}\int_{-\infty}^{\infty} e^{\frac{ixt}{a}} f(t)dt = \frac{1}{a}\hat{f}(\frac{x}{a})$.

(iii) As $|\overline{f}| = |f|$, $\overline{f} \in L^1(-\infty, \infty)$ and its Fourier transform is well defined. Let $f(t) = f_1(t) + if_2(t)$, where f_1, $f_2 \in L^1(-\infty, \infty)$. Then, $\overline{f}(t) = f_1(t) - if_2(t)$ and

$$T[\overline{f}(t)] = \int_{-\infty}^{\infty} e^{ixt}[f_1(t) - if_2(t)]dt = \int_{-\infty}^{\infty} e^{ixt} f_1(t)dt$$

$$- i \int_{-\infty}^{\infty} e^{ixt} f_2(t)dt$$

$$= \overline{\left(\int_{-\infty}^{\infty} e^{-ixt} f_1(t)dt\right)} + \overline{\left(\int_{-\infty}^{\infty} i \quad e^{-ixt} f_2(t)dt\right)} =$$

$$= \overline{\left(\int_{-\infty}^{\infty} e^{-ixt}[f_1(t) + if_2(t)]dt\right)}$$

$$= \overline{\hat{f}(-x)} .$$

LEMMA 1.17. If a sequence of functions $\{f_n\}$ converges to f in L^1-norm, then the sequence of their Fourier transforms $\{\hat{f}_n\}$ converges uniformly to $\hat{f}$ in the interval $(-\infty, \infty)$.

<u>Proof</u>. If $\{f_n\}$, $f \in L^1$, by hypothesis, $\|f_n - f\|_1 = \int_{-\infty}^{\infty} |f_n - f| dt \to 0$. Now,

$$|\hat{f}_n(x) - \hat{f}(x)| = \left| \int_{-\infty}^{\infty} e^{ixt} f_n(t)dt - \int_{-\infty}^{\infty} e^{ixt} f(t)dt \right| =$$

$$= \left| \int_{-\infty}^{\infty} e^{ixt}[f_n(t) - f(t)]dt \right|$$

$$\leq \quad |f_n(t) - f(t)|dt \to 0, \ x \in (-\infty, \infty).$$

Thus, the sequence $\{\hat{f}_n\}$ converges uniformly to $\hat{f}$ for all $x \in (-\infty, \infty)$.

THEOREM 1.18.(i) If $f \in L^1(-\infty, \infty)$ and $itf \in L^1(-\infty, \infty)$, then $\hat{f}'(x)$ exists and $T[itf] = \hat{f}'(x)$, where the dash denotes differentiation.

(ii) If $f \in L^1$ and $f' \in L^1$, then $T[f'(t)] = -ix\hat{f}(x)$.

Proof. (i) Consider $\frac{\hat{f}(x+h)-\hat{f}(x)}{h}$. By Lemmas 1.15 and 1.16, this is equal to $T[f(t)\frac{e^{ith}-1}{h}] = T[F(h,t)]$, say. Now, $F(h,t)$ converges to $itf(t)$ at every point t as $h \to 0$ and $|F(h,t)| = |f(t)| \left|\frac{e^{ith}-1}{h}\right| \to |t|\ |f(t)| \in L^1(-\infty, \infty)$. Thus, $F(h,t)$ converges to $itf(t)$ in L^1-norm. Using Lemma 1.17, $T[F(h,t)] \to T[itf(t)]$ uniformly as $h \to 0$. Hence, the derivative of $\hat{f}(x)$ exists at every point x and $\hat{f}'(x) = T[itf]$.

(ii) Since $f' \in L^1$, its Fourier transform exists and using Lemmas 1.15 and 1.16,

$$T[f'(t)] = T[\lim_{h \to 0} \frac{f(t+h)-f(t)}{h}] = \lim_{h \to 0} [(\frac{e^{-ihx}-1}{h})\hat{f}(x)]$$
$$= -ix\hat{f}(x).$$

Thus, the theorem shows the relationship between the function f, its Fourier transform, and their derivatives.

QUESTION 1.19. Suppose $f \in L^1(-\infty, \infty)$ and $\hat{f}$ is its Fourier transform, can the function f be determined from known values $\hat{f}(x)$ of $\hat{f}$?

This cannot always be done, but under <u>suitable conditions</u>, such an inversion is possible. The following theorems provide such conditions, under which the inversion will be permissible.

THEOREM 1.20. If f, $\hat{f} \in L^1(-\infty, \infty)$ and if f is continuous at a point t, then

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} \hat{f}(x)dx.$$

For a proof, refer to R.R.Goldberg.

THEOREM 1.21. (Riemann Localization Theorem). If $f \in L^1(-\infty, \infty)$, $S_R(t) = \frac{1}{2\pi} \int_{-R}^{R} e^{-ixt} \hat{f}(x)dx$ and $g_t(v) = \frac{f(t+v)+f(t-v)}{2} - f(t)$, then for sufficiently small $\delta > 0$, $\int_0^{\delta} \left| \frac{g_t(v)}{v} \right| dv = M < \infty$ implies $\lim_{R \to \infty} S_R(t) = f(t)$.

Proof. For fixed t,

$$S_R(t) = \frac{1}{2\pi} \int_{-R}^{R} e^{-ixt} f(x)dx = \frac{1}{2\pi} \int_{-R}^{R} e^{-ixt}dx \int_{-\infty}^{\infty} e^{ixu} f(u)du$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-R}^{R} e^{ixv} f(t+v)dvdx \qquad \text{by Theorem 1.12}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t+v)dv \int_{-R}^{R} e^{ixv}dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin Rv}{v} f(t+v)dv$$

$$= \frac{1}{\pi} \left[\int_{-\infty}^{0} \frac{\sin Rv}{v} f(t+v)dv + \int_{0}^{\infty} \frac{\sin Rv}{v} f(t+v)dv \right]$$

$$= \frac{1}{\pi} \left[\int_{0}^{\infty} \frac{\sin Rv}{v} f(t-v)dv + \int_{0}^{\infty} \frac{\sin Rv}{v} f(t+v)dv \right]$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin Rv}{v} \left[\frac{f(t+v) + f(t-v)}{2} \right] dv$$

Then, $S_R(t) - f(t) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin Rv}{v} g_t(v)dv$

$$= \frac{2}{\pi} \left[\int_0^{\delta} \frac{\sin Rv}{v} g_t(v)dv + \int_{\delta}^{\infty} \frac{\sin Rv}{v} g_t(v)dv \right] = I_1 + I_2, \text{say.}$$

By hypothesis, since $\int_0^{\delta} \left| \frac{g_t(v)}{v} \right| dv = M$, integrating I_1 by parts, we find that $I_1 \to 0$ as $\delta \to 0$. Using Lemma 1.7, $I_2 \to 0$ as $R \to \infty$. Thus, $\lim_{R \to \infty} [S_R(t) - f(t)] = 0$, which shows that $\lim_{R \to \infty} S_R(t) = f(t)$.

Theorems 1.20 and 1.21 indicate that the convergence of $S_R(t)$ to $f(t)$ at a point depends only on the behaviour of $f(t)$ in a neighbourhood of that point.

EXAMPLE 1.22. If $f(t) = e^{-t}$ for $t \geq 0$ and 0 for $t < 0$, then its Fourier transform $\hat{f}$ is $\int_0^{\infty} e^{ixt} e^{-t} dt = \int_0^{\infty} e^{t(ix-1)} dt$ and this integral is convergent for $(ix-1) < 0$ and its value is $\frac{1}{1-ix}$. But $\hat{f}$ does not belong to $L^1(-\infty, \infty)$. Therefore, the inversion is not valid.

2. THE ABSTRACT HARMONIC ANALYSIS APPROACH.

We now pass on to a generalization of the concepts of section 1, by employing the notion of a locally compact abelian group, which is fundamental to the theory of abstract harmonic analysis.

DEFINITION 2.1. A locally compact abelian group G, is an abelian group, which is also a locally compact Hausdorff space, such that the group operations are continuous. In other words, if addition is the group operation, then the mappings $x \to -x$ of G onto G and $(x,y) \to x+y$

of $G \times G$ onto G are both continuous. If multiplication is the group operation, the mappings $x \to \frac{1}{x}$ of G onto G and $(x,y) \longrightarrow xy$ of $G \times G$ onto G are both continuous.

We shall use the abbreviation LCAG for a locally compact abelian group.

EXAMPLES 2.2.

(i) Any abelian group G is essentially an LCAG when endowed with the discrete topology.

(ii) The group $\mathbb{R}$ of real numbers, with addition as the group operation and with the usual topology is an LCAG.

(iii) The group $\mathbb{Z}$ of all integers $0, \pm 1, \pm 2, \ldots,$ with addition as the group operation is an LCAG if a topology is defined by specifying that every set of integers should be open.

(iv) The circle group T of all complex numbers with absolute value unity and with multiplication as the group operation is an LCAG, whose topology is the relative topology induced by the usual topology of $\mathbb{R}^2$. As T is in 1:1 correspondence with the set of all points in $[0, 2\pi)$, T can also be looked upon as the group of real numbers belonging to $[0, 2\pi)$ with the group operation given by addition modulo 2π.

(v) If G_1 and G_2 are two locally compact abelian groups, their sum G_1+G_2 is also an LCAG, with the corresponding product topology.

REMARK 2.3. A discrete group is an LCAG in which every set is open. Notice that $\mathbb{Z}$ is discrete and T is compact. $\mathbb{R}$ is neither discrete nor compact.

DEFINITION 2.4. If G is an LCAG, a <u>character</u> of G is a continuous homomorphism of G into the group T. In other words, $\hat{x}$ is a character of G if

(i) $\hat{x}$ is a continuous function on G

(ii) $\hat{x}(x+y) = \hat{x}(x)\hat{x}(y)$ for all $x, y \in G$ if addition is the group operation and $\hat{x}(xy) = \hat{x}(x)\hat{x}(y)$ if multiplication is the group operation.

(iii) $|\hat{x}(x)| = 1$ for all $x \in G$.

The set of all characters of G will be denoted by $\hat{G}$ and is called <u>the character group of G</u> or <u>the dual group of G</u>.

REMARK 2.5. If $\hat{x}_1, \hat{x}_2 \in \hat{G}$, define $\hat{x}_1\hat{x}_2(x) = \hat{x}_1(x)\hat{x}_2(x)$ for all $x \in G$. Then $\hat{G}$ is a group with respect to this multiplication. A topology can be defined for $\hat{G}$ by choosing as a basis, the class of all open sets $O(\hat{x}_0, \varepsilon, K)$ such that $O(\hat{x}_0, \varepsilon, K) = \{\hat{x} \in G : |\hat{x}(x) - \hat{x}_0(x)| < \varepsilon\}$, $x \in K$, $\hat{x}_0 \in \hat{G}$, $\varepsilon > 0$ and K being a compact subset of G. With this topology, $\hat{G}$ becomes an LCAG. This topology is called <u>the topology of uniform convergence on compact subsets</u>.

EXAMPLES 2.6.

(i) For $G = \mathbb{R}$, the mapping $x \to e^{-i\hat{x}x}$ where $x, \hat{x} \in \mathbb{R}$, is a character since it is continuous being an exponential function, $|e^{-i\hat{x}x}| = 1$ and $e^{-i\hat{x}(x+y)} = e^{-i\hat{x}x}\, e^{-i\hat{x}y}$. There is a 1:1 correspondence between the elements of $\mathbb{R}$ and the character group $\hat{\mathbb{R}}$. In fact $\mathbb{R} = \hat{\mathbb{R}}$.

(ii) For $G = \mathbb{Z}$, the mapping $x \to e^{-i\hat{x}x}$ where $x \in \mathbb{Z}$ and $0 \leq \hat{x} < 2\pi$ is a character and $\hat{\mathbb{Z}} = T$.

(iii) When $G = T$, $\hat{G} = \mathbb{Z}$.

REMARK 2.7. From example 2.6, we notice that the character group of a discrete group is a compact group and conversely the character group of a compact group is discrete.

DEFINITION 2.8. If G is an LCAG with dual group $\hat{G}$, then each $x \in G$ defines an element of $\hat{\hat{G}}$ and the value of $\hat{x}$ at x is equal to the value of x at $\hat{x}$. This duality enables us to use the symbol $< x, \hat{x} >$ to denote this common value.

A striking result is due to Pontryagin and is known after him.

THEOREM 2.9. (<u>Pontryagin Duality Theorem</u>). If $\hat{G}$ is the dual group of an LCAG G, then $G = \hat{\hat{G}}$.

For a proof, refer to W.Rudin.

DEFINITION 2.10. The <u>Borel subsets of a set X</u> are the members of the smallest σ-ring of subsets of X which contains every compact set. If X is a locally compact Hansdorff space, a <u>positive Borel measure</u> on X is a function μ which assigns to every Borel subset of X a non-negative real number or $+\infty$ such that $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ whenever $A_1, A_2, \ldots$ is a sequence of pairwise disjoint Borel sets in X.

DEFINITION 2.11. A <u>Borel measure</u> μ <u>is regular</u> if for each Borel set A, $\mu(A) = \inf \mu(U_n)$, where the infimum is taken over all the open sets $\{U_n\}$ containing A.

DEFINITION 2.12. Let G be an LCAG. A <u>Haar measure</u> on G is a positive, regular Borel measure m with $m(E) < \infty$ if E is compact and $m(E+x) = m(E)$ for every measurable set $E \subset G$ and every $x \in G$.

REMARK 2.13. Von Newmann has established that the Haar measure m is unique upto multiplication by a positive constant and it is finite if and only if G is compact.

EXAMPLES 2.14.

(i) If $G = \mathbb{R}$, for every integrable function f, the Lebesque measure dx defined on $\mathbb{R}$ is the invariant measure and $\int_{\mathbb{R}} f(x+a)dx = \int_{\mathbb{R}} f(x)dx, a \in \mathbb{R}$.

(ii) If $G = \mathbb{Z}$, the invariant measure is that which assigns as measure to any set, the number of elements in that set.

(iii) If $G = T$, the invariant measure is the Lebesque measure defined on the circle and it is normalized to have the value unity.

(iv) If $G = G_1+G_2$, the Haar measure of G is the product measure of the Haar measures of G_1 and G_2.

DEFINITION 2.15. The class $L^1(G)$ is made up of all the functions which are integrable with respect to the invariant measure m. The norm of $f \in L^1(G)$ is defined by $\|f\|_1 = \int_G |f(t)|\,dm(t)$.

DEFINITION 2.16. If G is an LCAG with dual group $\hat{G}$, the <u>Fourier transform</u> $\hat{f}$ of $f \in L^1(G)$ is given by $\hat{f}(\hat{x}) = \int_G \overline{\langle x,\hat{x} \rangle}\, f(x)\, dm(x)$ for $x \in G$, $\hat{x} \in \hat{G}$.

REMARK 2.17. Notice that the Definition 2.16 provides a generalization of 1.3 apart from a constant factor. If $G = \mathbb{Z}$, $L^1(\mathbb{Z})$ is the set of all sequences $\{a_n\}_{n=-\infty}^{\infty}$ such that $\sum_{n=-\infty}^{\infty} |a_n| < \infty$ and the Fourier transform of such a sequence is $\sum_{n=-\infty}^{\infty} a_n e^{in\hat{x}}$, $0 \leq \hat{x} < 2\pi$. When $G=T$, the Fourier transform of a function f in $L^1(T)$ is the sequence $\frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)dx$, $n \in \mathbb{Z}$, of Fourier coefficients of f.

REMARK 2.18. As observed in 1.4 and Lemma 1.6, we have a generalization here and $\hat{f}$ is a continuous function on G, which vanishes at infinity if $\hat{G}$ is not compact.

DEFINITION 2.19. The <u>convolution</u> of $f,g \in L^1(G)$ is defined as $(f * g)(x) = \int_G f(x-y)g(y)dm(y)$ if addition is the group operation in G and as $(f * g)(x) = \int_G f(xy^{-1})g(y)dm(y)$

if multiplication is the group operation.

Certainly, $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ and $\widehat{f * g} = \hat{f}\hat{g}$.

Thus, the theory of abstract harmonic analysis sheds a great deal of light on the corresponding classical theory setting and extensive studies have been made to effect good generalizations of the latter. For detailed comparisons, consult L.H.Loomis or W.Rudin.

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