RSK BASES IN INVARIANT THEORY AND REPRESENTATION THEORY

By

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As members of the Viva Voce Board, we recommend that the dissertation prepared by **Preena Samuel** entitled "RSK Bases in invariant theory and representation theory" may be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and the work has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution or University.

Preena Samuel

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Abstract

From the combinatorial characterizations of the right, left, and two-sided Kazhdan-Lusztig cells of the symmetric group, 'RSK bases' are constructed for certain quotients by two-sided ideals of the group ring and the Hecke algebra. Applications to invariant theory, over various base rings, of the general linear group and representation theory, both ordinary and modular, of the symmetric group are discussed.

Contents

List of Publication(s)

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Chapter 1

Introduction

The starting point of the work carried out in this thesis is a question in classical invariant theory. It leads naturally to questions about representations of the symmetric group over the complex numbers and over fields of positive characteristic, and in turn to the computation of the determinant of a certain matrix encoding the multiplication of Kazhdan-Lusztig basis elements of the Hecke algebra, using which one can recover a well-known criterion for the irreducibility of Specht modules over fields of positive characteristic.

Let n denote a fixed integer and $\mathfrak{S}_n$ be the symmetric group on n letters. Let k denote a commutative ring with unity and V a free module of finite rank d over k.

Consider the group of k-linear automorphisms of V, denoted $GL(V)$, acting diagonally on the space $V^{\otimes n}$ of n-tensors. The symmetric group $\mathfrak{S}_n$ also acts on $V^{\otimes n}$ by permuting the factors: the action of $\sigma \in \mathfrak{S}_n$ on pure tensors is given by

$$(v_1 \otimes \cdots \otimes v_n)\sigma := v_{1\sigma} \otimes \cdots \otimes v_{n\sigma},$$

where $i\sigma$ denotes the image of i under the action of σ (which we assume to act from the right). This action commutes with the $GL(V)$-action and so, the map $\phi_n : k\mathfrak{S}_n \to \mathrm{End}_k V^{\otimes n}$ defining the action of $\mathfrak{S}_n$ on $V^{\otimes n}$ has image in $\mathrm{End}_{GL(V)} V^{\otimes n}$ – the space of $GL(V)$-invariant endomorphisms of $V^{\otimes n}$. A classical result in invariant theory (see [dCP76, Theorems 4.1, 4.2]) states that this map is a surjection onto $\mathrm{End}_{GL(V)} V^{\otimes n}$. The result further states that, under a mild condition on k (which holds for example when k is an infinite field), the kernel is equal to $J(n,d)$ — the two-sided ideal generated by the element $y_d := \sum_{\tau \in \mathfrak{S}_{d+1}} \mathrm{sign}(\tau)\tau$ where $\mathfrak{S}_{d+1}$ is the subgroup of $\mathfrak{S}_n$ consisting of the permutations that fix point-wise the elements $d+2, \ldots, n$; when $n \leq d$, $J(n,d)$ is defined to be 0. Thus, the quotient $k\mathfrak{S}_n/J(n,d)$ gets identified with the algebra of $GL(V)$-endomorphisms of $V^{\otimes n}$.

It is, therefore, of invariant-theoretic interest to obtain a basis for the quotient $k\mathfrak{S}_n/J(n,d)$. Indeed our first main result provides such a basis (see Theorem 4.4.1):

Theorem 1.0.1 *Let k be an arbitrary commutative ring with unity. With notations as* *above, the permutations σ of $\mathfrak{S}_n$ such that the sequence $1\sigma, \ldots, n\sigma$ has no decreasing subsequence of length more than d, form a basis for $k\mathfrak{S}_n/J(n,d)$.* $\square$

The proof of the theorem involves the Hecke algebra of the symmetric group and its Kazhdan-Lusztig C-basis (see §2.2 for definitions). The Hecke algebra of $\mathfrak{S}_n$ over A, where A is the Laurent polynomial ring $\mathbb{Z}[v, v^{-1}]$, is a "deformation" of the group ring of $\mathfrak{S}_n$ over A. We denote it as $\mathcal{H}$. For a commutative ring k with unity and a an invertible element in k, we denote by $\mathcal{H}_k$ the k-algebra $\mathcal{H} \otimes_A k$ defined by the unique ring homomorphism $A \to k$ given by $v \mapsto a$. The Hecke algebra is a deformation of the group algebra of $\mathfrak{S}_n$ in the sense that the k-algebra $\mathcal{H}_k$, defined by the map $v \mapsto 1$, is isomorphic to the group algebra $k\mathfrak{S}_n$.

The main technical part of the proof of Theorem 1.0.1 is in describing an A-basis for an appropriate two-sided ideal in $\mathcal{H}$ that specializes, under the map $v \mapsto 1$, to $J(n,d)$ (see Lemma 4.4.2). The key ingredient, in turn, in proving this is the combinatorics associated to the Kazhdan-Lusztig cells (§2.2.2).

If we take the base ring k in the above discussion to be the field $\mathbb{C}$ of complex numbers. Then the ideal $J(n,d)$ as defined above has a representation theoretic realization namely, let $\lambda(n,d)$ be the unique partition of n with at most d parts that is smallest in the dominance order (§2.3.1); consider the linear representation of $\mathfrak{S}_n$ on the free vector space $\mathbb{C}\mathcal{T}_{\lambda(n,d)}$ generated by tabloids of shape $\lambda(n,d)$ (§3.2), the ideal $J(n,d)$ is the kernel of the $\mathbb{C}$-algebra map $\mathbb{C}\mathfrak{S}_n \to \mathrm{End}_{\mathbb{C}}\mathbb{C}\mathcal{T}_{\lambda(n,d)}$ defining this representation.

Replacing the special partition $\lambda(n,d)$ above by an arbitrary partition of n, say λ, and considering the $\mathbb{C}$-algebra map $\rho_\lambda : \mathbb{C}\mathfrak{S}_n \to \mathrm{End}_{\mathbb{C}}\mathbb{C}\mathcal{T}_\lambda$ defining the linear representation of $\mathfrak{S}_n$ on the space $\mathbb{C}\mathcal{T}_\lambda$ generated by tabloids of shape λ, we ask:

> Is there a natural set of permutations that form a $\mathbb{C}$-basis for the group ring $\mathbb{C}\mathfrak{S}_n$ modulo the kernel of the map ρ_λ?

This question is addressed by the following theorem (see Theorem 4.2.1):

Theorem 1.0.2 *Permutations of* RSK-shape μ, *as μ varies over partitions that dominate λ, form a $\mathbb{C}$-basis of $\mathbb{C}\mathfrak{S}_n$ modulo the kernel of $\rho_\lambda : \mathbb{C}\mathfrak{S}_n \to \mathrm{End}\mathbb{C}\mathcal{T}_\lambda$.*

The dominance order on partitions is the usual one (§2.3.1). The RSK-shape of a permutation is defined in terms of the RSK-correspondence (§2.3.2). As follows readily from the definitions, the shape of a permutation σ dominates the partition $\lambda(n,d)$ precisely when $1\sigma, \ldots, n\sigma$ has no decreasing sub-sequence of length exceeding d. Thus, in the case when the base ring is the complex field, Theorem 1.0.1 follows from Theorem 1.0.2.

The observation mentioned in §4.1.2 plays a key role in proving the above theorem. The above result holds, as we observe in §4.2.2, even after extending scalars to any field of characteristic 0 essentially owing to the fact that such a field is flat over $\mathbb{Z}$. Further, by an example (§4.2.3) we illustrate that these results are not true in general over fields of positive characteristic.

The question raised above can be modified to one of a more intrinsic appeal which, in turn, can be posed in a more general setting.

Given a partition λ of n, consider, instead of the action of $\mathfrak{S}_n$ on tabloids of shape λ, the Specht module $S^\lambda_{\mathbb{C}}$ (§3.2). The Specht modules are irreducible and every irreducible $\mathbb{C}\mathfrak{S}_n$-module is isomorphic to $S^\mu_{\mathbb{C}}$ for some $\mu \vdash n$ (§3.3.5, §3.4).

The irreducibility of $S^\lambda_{\mathbb{C}}$ implies, by a well-known result of Burnside (see, e.g., [Bou73, Chapter 8, §4, No. 3, Corollaire 1]), that the defining $\mathbb{C}$-algebra map $\mathbb{C}\mathfrak{S}_n \to \mathrm{End}_{\mathbb{C}} S^\lambda_{\mathbb{C}}$ is surjective. The dimension of $S^\lambda_{\mathbb{C}}$ equals the number $d(\lambda)$ of standard tableaux of shape λ (§2.3.1, §3.3.5). Thus there exist $d(\lambda)^2$ elements of $\mathbb{C}\mathfrak{S}_n$ (or even of $\mathfrak{S}_n$ itself) whose images in $\mathrm{End}_{\mathbb{C}} S^\lambda_{\mathbb{C}}$ form a basis (for $\mathrm{End}_{\mathbb{C}} S^\lambda_{\mathbb{C}}$). We ask:

Is there a natural choice of such elements of $\mathbb{C}\mathfrak{S}_n$, even of $\mathfrak{S}_n$?

More generally, in the setting of the Hecke algebra we consider the right cell modules (§3.3.5), denoted $R(\lambda)$ for $\lambda \vdash n$, as introduced by Kazhdan-Lusztig [KL79]. It is well-known ([MP05]; see also Proposition 3.4.3) that these modules are isomorphic to the Specht modules of $\mathcal{H}$ as defined by Dipper and James (§3.2). However, in this thesis, we choose to work with the cell modules because of their combinatorial appeal which is of significance to us. When k is a field for which the algebra $\mathcal{H}_k$ (§2.2) is semisimple the right cell modules $R(\lambda)_k := R(\lambda) \otimes k$, $\lambda \vdash n$, are irreducible and as λ varies over all partitions of n they give a complete set of irreducibles (Theorem 3.3.18). Then arguing as earlier we ask:

Is there a natural choice of elements of $\mathcal{H}_k$ which form a basis for $\mathrm{End}_k(R(\lambda)_k)$?

Our answer (see Proposition 5.1.1):

Theorem 1.0.3 *Assume $\mathcal{H}_k$ is semisimple. For λ a partition of n, the images in $\mathrm{End}_k R(\lambda)_k$ of the Kazhdan-Lusztig basis elements C_x, $\text{RSK-shape}(x) = \lambda$ form a basis (for $\mathrm{End}_k R(\lambda)_k$).*

The RSK-shape of a permutation is defined in terms of the RSK-correspondence (§2.3.2). The right cell module $R(\lambda)_{\mathbb{C}}$ in the sense of Kazhdan-Lusztig (§3.3.5), being equivalent (see Proposition 3.4.3; [GM88]; [Nar89]) to the Specht module $S^\lambda_{\mathbb{C}}$ (§3.2), the above theorem also answers the former question (about $\mathrm{End}_{\mathbb{C}} S^\lambda_{\mathbb{C}}$).

Theorem 1.0.3 shows in particular that the Kazhdan-Lusztig C-basis behaves well with respect to irreducible representations. The more obvious candidate for a basis of $\mathrm{End}_k(R(\lambda)_k)$, namely that consisting of T_w, RSK-shape$(w)=\lambda$ (the analogues of group elements corresponding to permutations of RSK-shape λ, in the group algebra of $\mathfrak{S}_n$, see Definition 2.2.3), does not work. This is seen by an example in §5.1.1.

The proof of the above theorem depends on the Wedderburn structure theory. When $\mathcal{H}_k$ is not semisimple the argument, of course, falls apart. So we deal with this case by a head-on approach which involves the construction of a matrix, denoted as $\mathcal{G}(\lambda)$ (§5.2.1). From the very definition of $\mathcal{G}(\lambda)$, it will be obvious that if the determinant of $\mathcal{G}(\lambda)$ does not vanish, then the C_x's as in Theorem 1.0.3 will continue to be a basis for $\mathrm{End}_k(R(\lambda)_k)$.

Using this idea we explore conditions under which the above theorem (Theorem 1.0.3) can be extended to the non-semisimple case. One such condition is described below. Note that the irreducibility of $R(\lambda)_k$ is essential for the statement as in Theorem 1.0.3 to be true.

Let k be a field and let $A \to k$ be the unique homomorphism given by $v \mapsto a$ where $a \in k$ is an invertible element. Let e be the smallest positive integer such that $1+a+\cdots+a^{e-1} = 0$ in k; if no such integer exists then $e = \infty$. When $a = 1$, the value of e is just the characteristic of the field.

A partition λ is called *e-regular* if the number of parts of λ of any given length, is less than e. Then with the above notation, we have

Theorem 1.0.4 (see Theorem 5.1.3) *For an e-regular partition λ such that $R(\lambda)_k$ is irreducible, the Kazhdan-Lusztig basis elements C_w, w of* RSK-shape λ*, thought of as operators on $R(\lambda)_k$ form a basis for* $\mathrm{End}_k R(\lambda)_k$.

When $\mathcal{H}_k$ is not semisimple the cell module $R(\lambda)_k$ is not necessarily irreducible. An issue of wide interest in the modular representation theory of $\mathfrak{S}_n$ (and of $\mathcal{H}$) is to indicate for a given field k the partitions $\lambda \vdash n$ for which $R(\lambda)_k$ is irreducible as a $k\mathfrak{S}_n$-module[1] (resp. as an $\mathcal{H}_k$-module). There is much literature available addressing this issue (see for example, [Mat99, Chapter 5, §4], [Fay05]). We take an approach, which is new to the best of our knowledge, by pursuing the ideas alluded to earlier, via the matrix $\mathcal{G}(\lambda)$ (§5.2.1). The matrix $\mathcal{G}(\lambda)$ encodes the action of the C-basis elements of RSK-shape λ on the cell module $R(\lambda)$ in a "nice" way and we notice that

Theorem 1.0.5 (see Theorem 5.3.1) *If the determinant* $\det \mathcal{G}(\lambda)|_{v=a}$ *does not vanish in k, then $R(\lambda)_k$ is irreducible.*

It would be worthwhile, therefore to study the $\det \mathcal{G}(\lambda)$ more closely. Towards this, we give a combinatorial formula for the determinant of the matrix $\mathcal{G}(\lambda)$. Though this formula is computed over the ring $A = \mathbb{Z}[v, v^{-1}]$, for the sake of simplicity we state it here only in the special case when $v = 1$. In this case we have:

Theorem 1.0.6 (Hook Formula) *For a partition λ of n,*

$$\det \mathcal{G}(\lambda)|_{v=1} = \prod \left(\frac{h_{ac}}{h_{bc}}\right)^{d(\beta_1,\ldots,\beta_a+h_{bc},\ldots,\beta_b-h_{bc},\ldots,\beta_r)} \tag{1.1}$$

with notation as above, where h_{ab} is the hook length of the node (a, b) in a tableau of shape λ (the hook length *of a node is the total number of nodes below it and to the right of it, including itself), $\beta_1 > \ldots > \beta_r$ is the β-sequence of λ (see §6.2 for definition), and the product runs over $\{(a, b, c)|(a, c), (b, c)$ are nodes such that $a < b\}$.*

[1]Note that $R(\lambda)_k := R(\lambda) \otimes k$ with scalars extended via the homomorphism given by $v \mapsto 1$ is a $\mathcal{H}_k \cong k\mathfrak{S}_n$- module (see §2.2).

The formula is proved by showing that the matrix $\mathcal{G}(\lambda)$ is related to the matrix of a well-studied bilinear form on the Specht module S^λ (of $\mathcal{H}$) and then using known formulas for its determinant. All these calculations are carried out in the general setting of the Hecke algebra over the ring A (Theorem 6.2.1).

Using the hook formula, we obtain a combinatorial criterion for the irreducibility of S_k^λ. We prove the following: Let p denote the smallest positive integer such that $p = 0$ in k; if no such integer exists, then $p = \infty$. For an integer h, define $\nu_p(h)$ as the largest power of p (possibly 0) that divides h in case p is positive, and as 0 otherwise. The integer e is as defined prior to Theorem 1.0.4. For an integer h, define

$$\nu_{e,p}(h) := \begin{cases} 0 & \text{if } e = \infty \text{ or } e \nmid h \\ 1 + \nu_p(h/e) & \text{otherwise} \end{cases}$$

The (e,p)-*power diagram* of λ is the filling up of the nodes of a tableau of shape λ (§2.3.1) by the $\nu_{e,p}$'s of the respective hook lengths. The Hook formula and Theorem 5.3.1 put together provides us with a new proof of

Theorem 1.0.7 ([Jam78], [JM97]) *If the (e,p)-power diagram of λ has either no column or no row containing different numbers, then S_k^λ is irreducible.*

For the specialization $v \mapsto 1$, the above criterion turns out to be exactly the criterion conjectured by Carter which gives a sufficiency condition on λ for irreducibility of S_k^λ to hold. Thus, by means of our approach we are led to a new proof of the conjecture which was proved by G. D. James in 1978 [Jam78].

Organization of the thesis

The thesis consists of five chapters with a specific aspect being covered in each. The proofs of the main results, mentioned above, are covered in Chapters 4, 5 and 6. Chapters 2 and 3 are intended to be introductory and none of the results mentioned there are original.

The first part of Chapter 2 introduces the concept of Hecke algebra and all the associated preliminaries like the K-L basis, cells etc. The second part introduces all the combinatorial objects that play an important role in the thesis. In particular, the RSK correspondence is described there (§2.3.2). Many combinatorial results that are fundamental to the arguments used in the thesis are also listed and proved there (§2.3.4). In §2.3.3 the combinatorial characterizations of left, right and two-sided cells and pre-orders, as in [Gec06], are stated without proof. These combinatorial characterizations play a key role in most of the arguments in the thesis.

The main goal of Chapter 3 is to introduce all the representation theoretic objects that are of importance in the thesis. The Specht modules, cell modules, permutation modules etc are briefly introduced in §3.3. Also stated and proved there are many basic results regarding them, which are invoked in later chapters. In particular, proved in this chapter is the isomorphism of the Specht module corresponding to a shape and the cell module corresponding to the same shape following the exposition as in [MP05] (§3.4). This result enables us to use the combinatorics of cells to handle questions involving the Specht modules (see for example, §4.2.1).

The next three chapters present the main results of the thesis along with detailed proofs.

In Chapter 4, we present the proof of Theorem 1.0.1 mentioned above. Apart from a description of a basis of multilinear invariants for $(\mathrm{End}V)^{\times n}$ that we obtain almost immediately by rephrasing Theorem 1.0.1, it also enables us to:

- obtain a k-basis, closed under multiplication, for the subring of $GL(V)$-invariants of the tensor algebra of V (§4.4.2).
- when k is a field of characteristic 0, to limit the permutations in the well-known description ([Pro76], [Raz74]) of a spanning set for polynomial $GL(V)$-invariants of several matrices (§4.4.3); or, more generally, to limit the permutations in the description in [DKS03] of a spanning set by means of 'picture invariants' for polynomial $GL(V)$-invariants of several tensors (§4.4.3).

These are discussed in detail through §4.4.

Also stated and proved in Chapter 4 are Theorem 1.0.2 and the analogous statement obtained by changing scalars to an arbitrary field of characteristic 0. An example given there illustrates the failure of the same to hold over fields of positive characteristic. The following formulation of the $\mathcal{H}$-analogue of Theorem 1.0.2 with scalars in A, is also proved in the same chapter:

Theorem 1.0.8 (see Theorem 4.3.1) *The elements T_w,* RSK-shape$(w) \trianglerighteq \lambda$*, form a basis for $\mathcal{H}$ modulo the kernel of the map $\mathcal{H} \to \mathrm{End}_{\mathcal{H}} M^\lambda$ defining the permutation module M^λ as a right $\mathcal{H}$-module.*

Here M^λ (§3.3.1) denotes the $\mathcal{H}$-analogue of the tabloid representation corresponding to $\lambda \vdash n$.

Chapter 5 focuses primarily on the cell modules (§3.3.5). Presented there is the RSK bases for the endomorphism ring of the cell module, $\mathrm{End}_k R(\lambda)_k$ (where k denotes a field), for the case when $\mathcal{H}_k$ is semisimple and when it is not. The matrix $\mathcal{G}(\lambda)$ is also defined and introduced in this chapter as a means of proving Theorem 1.0.4 (stated above). In the course of the chapter a proof of Theorem 1.0.5 is also given. It is seen through Sections 5.2.2, 5.2.3 that the matrix encoding the action of the C-basis elements corresponding to permutations of shape λ is related to the matrix of a bilinear form on $R(\lambda)$ (§5.2.3). This turns out to be the principal step in arriving at Theorems 1.0.4, 1.0.5 above.

Finally, in Chapter 6, we prove the Hook Formula (6.2) for the determinant of $\mathcal{G}(\lambda)$ in the general setting of the Hecke algebra $\mathcal{H}$. As indicated earlier, we arrive at this formula by using the fact that the bilinear form on $R(\lambda)$ (mentioned above) is just the pull-back via the MP-isomorphism (§3.4) of the Dipper-James bilinear form (§3.2) on the Specht module

S^λ. Following [DJ87] we then compute a precise formula for the determinant of the Dipper-James bilinear form on S^λ. This formula can be further simplified using a combinatorial result recalled from [JM97]. Finally we deduce the formula (6.2) for $\det \mathcal{G}(\lambda)$ by using the explicit relation between $\det \mathcal{G}(\lambda)$ and the determinant of the Dipper-James bilinear form on S^λ. In the same chapter, we use this Hook Formula to arrive at a proof of Theorem 1.0.7 and as noted earlier, thus also arriving at a new proof for Carter's conjecture [Jam78] about the irreducibility of S_k^λ.

Postscript

After the initial submission of this thesis, it was learnt from John Graham and Andrew Mathas (at an ICM-2010 satellite conference held in Bangalore) that many of the results in Chapter 5 can be deduced from the fact that the Kazhdan - Lusztig basis of the Hecke algebra of the symmetric group is cellular in the sense of [GL96]. In particular, the description of the matrix $\mathcal{G}(\lambda)$ given in §5.2.3 follows from the cellularity property of the Kazhdan-Lusztig basis - a fact that we reprove using results of Geck [Gec06].

Chapter 2

Kazhdan-Lusztig cells and their combinatorics

In this chapter we begin with some background material on Coxeter groups and Hecke algebras associated to them. We soon specialize to $\mathfrak{S}_n$ and after introducing the basic combinatorial objects there, we describe the RSK-correspondence in §2.3.2. In §2.3.3, we recall certain combinatorial results which play a significant role in the rest of the chapters and finally in §2.3.4 we introduce some notations and discuss a few preliminary results that will be used repeatedly later.

All the results mentioned in §§2.1-2.2 can be found in [Lus03] or [Hum90]. In §2.3 we gather together various preliminaries which deal specifically with the combinatorial aspects of $\mathfrak{S}_n$ that play a key role in the thesis. The concepts discussed in §§2.3.1-2.3.2 are covered in greater detail in [Sag01]. The last sub-section is fairly self-contained, while the results mentioned in §2.3.3 forms the main theme of [Gec06] and we do not undertake the task of going through their proofs here.

2.1 Coxeter System $(W,\ S)$

Let W denote a group (written multiplicatively), with identity element denoted as 1, and S be a set of generators for W. Then the pair $(W,\ S)$ is a *Coxeter system* if there exists a matrix $(m(s,s'))_{(s,s')\in S\times S}$ with entries in $\mathbb{N}\cup\{\infty\}$ satisfying the conditions:

i) $m(s,s)=1$ for all $s\in S$ and

ii) $m(s,s')=m(s',s)\geq 2$ for all $s\neq s'$

and such that the natural map from the free group generated by the set S to W has kernel the normal subgroup generated precisely by the elements

$$(ss')^{m(s,s')} \text{ where } s,s'\in S \text{ and } m(s,s')<\infty.$$

We assume familiarity with the basic notions regarding Coxeter systems. However, we briefly recall here the notions that we will use. (See [Bou02, Chapter 4] for more details).

Each $w \in W$ can be written as a product of a finite sequence of elements of S. For a given w, the smallest possible integer $r \geq 0$ such that w is a product of a sequence of r elements from S is called the *length* of w, denoted as $l(w)$. Thus, $l(1) = 0$ and $l(s) = 1$ for all $s \in S$ (it is easily seen that $1 \notin S$). Let $w = s_1 \cdots s_r$ for some s_i (not necessarily distinct) in S. Then $s_1 \cdots s_r$ is said to be a *reduced expression* for w if $r = l(w)$. A *subexpression* of a given reduced expression $s_1 \cdots s_r$ is a product of the form $s_{i_1} \cdots s_{i_l}$ where $1 \leq i_1 < \cdots < i_l \leq r$. An element $u \in W$ is a *prefix* of $w \in W$ if there is a reduced expression $s_1 \cdots s_r$ for w such that the subexpression $s_1 \cdots s_j$ for some $j \leq r$ gives a reduced expression for u.

Bruhat order: For $w,\ w' \in W$, we write $w \leq w'$ if w can be obtained as a subexpression of some reduced expression for w'. This defines a partial ordering on W called the *Bruhat order*. We sometimes also write $w' \geq w$ to mean $w \leq w'$. By $w < w'$ or $w' > w$, we mean $w \leq w'$ and $w \neq w'$.

Deletion condition: ([Hum90, §5.8]) Suppose $w = s_1 \cdots s_r$, ($s_i \in S$), with $l(w) < r$. Then there exist indices $i < j$ for which $w = s_1 \cdots \hat{s_i} \cdots \hat{s_j} \cdots s_r$, where $\hat{s_k}$ means that s_k is omitted.

Remark 2.1.1

(a) For a given Coxeter system (W, S) where W is finite, there is a unique element in W which is of maximal length. (See [Lus03, §9.8], for example).

(b) For $J \subset S$, let W_J denote the subgroup of W generated by J. Then (W_J, J) is a Coxeter system in its own right. Further, if we denote the length function on W_J as l_J then $l_J(w) = l(w)$ for all $w \in W_J$. The subgroup W_J is called a *parabolic subgroup of* W (with respect to S). (See [Hum90, §5.5], for example).

2.2 Hecke Algebra corresponding to $(W,\ S)$

Let R be a commutative ring with unity. We begin with the definition of a generic Iwahori-Hecke algebra associated to a Coxeter system. By the Hecke algebra, we shall mean a particular case of a generic Iwahori-Hecke algebra.

Definition 2.2.1 Let (W, S) be a Coxeter system. Let $a_s, b_s \in R$ $(s \in S)$ be such that $a_s = a_t$ and $b_s = b_t$ whenever $s,\ t$ are conjugate under W. Then the *generic Iwahori-Hecke algebra* associated with (W, S) over R with parameters $\{a_s,\ b_s \mid s \in\ S\}$ is the free R-module $\mathcal{E}$ with basis $\{T_x \mid x \in W\}$ and multiplication given by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w) + 1 \\ a_s T_w + b_s T_{sw} & \text{if } l(sw) = l(w) - 1 \end{cases} \tag{2.1}$$

for $s \in S$, $w \in W$, making it an associative algebra with T_1 as identity.

That an algebra structure on $\mathcal{E}$ as in the above definition, exists and that it is unique is guaranteed by the following result:

Theorem 2.2.2 ([Hum90, p.146]) *Let (W, S) be a Coxeter system. Given $a_s, b_s \in R$ ($s \in S$) satisfying the conditions as in the above definition, there exists a unique structure of an associative algebra on the free R-module $\mathcal{E}$ with basis $\{T_x \mid x \in W\}$ such that T_1 acts as identity and the conditions as in (2.1) are satisfied.* $\square$

The group algebra of W over R is an example of a generic Iwahori-Hecke algebra, where the parameters are chosen to be $a_s = 0,\ b_s = 1$ for all $s \in S$

To obtain the *Hecke algebra* associated to (W, S), we take in Definition 2.2.1 the ring R to be $\mathbb{Z}[v, v^{-1}]$, the ring of Laurent polynomials with coefficients in $\mathbb{Z}$, and the parameters to be $a_s = v - v^{-1},\ b_s = 1$ for all $s \in S$.
Written explicitly, we have

Definition 2.2.3 The *Hecke algebra* associated to (W, S), denoted as $\mathcal{H}$, is a free $\mathbb{Z}[v, v^{-1}]$-module with basis T_w, $w \in W$, and multiplication being given by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w) + 1 \\ (v - v^{-1})T_w + T_{sw} & \text{if } l(sw) = l(w) - 1 \end{cases} \tag{2.2}$$

for $s \in S$, $w \in W$.

The Hecke algebra associated with the Coxeter system (W, S) is a "deformation" of the group algebra of W over $\mathbb{Z}$: taking $v = 1$ in Definition 2.2.3 we recover the group algebra of W over $\mathbb{Z}$. (See also Specializations of the Hecke algebra below).

In the next lemma we summarize a few basic facts about $\mathcal{H}$.

Lemma 2.2.4

1. *Let $s_1 \cdots s_r$ be a reduced expression for $w \in W$. Then $T_w = T_{s_1} \cdots T_{s_r}$.*

2. *Let $s \in S$ and $w \in W$. Then,*

$$T_w T_s = \begin{cases} T_{ws} & \text{if } l(ws) = l(w) + 1 \\ (v - v^{-1})T_w + T_{ws} & \text{if } l(ws) = l(w) - 1 \end{cases} \tag{2.3}$$

3. *For $w \in W$, the element T_w is invertible in $\mathcal{H}$ with inverse $T_w^{-1} = T_{s_k}^{-1} \cdots T_{s_1}^{-1}$ where $s_1 \cdots s_k$ is a reduced expression for w. For $s \in S$, the element $T_s^{-1} = T_s - (v - v^{-1})T_1$.*

4. *In the case[1] when $W = \mathfrak{S}_n$ and S is the set of simple transpositions (§2.3) we have for x, $y \in W$, let $T_x T_y = \sum_{w \in W} r_w T_w$, $r_w \in \mathbb{Z}[v, v^{-1}]$. Then $r_{xy} = 1$; and $r_w \neq 0$ only if $xy \leq w$. In particular, $r_1 \neq 0$ iff $x = y^{-1}$.*

[1]It can be verified, in the light of Remark 2.1.1(b), that if W is a parabolic subgroup of $\mathfrak{S}_n$ then the statement still holds. In fact the statement can be proved more generally for any Coxeter system (see for example, [Mat99, Exercise 1.15])

PROOF: ([DJ86, Lemma 2.1]; note the Remark on notation below)

1. This is elementary using the relations in (2.2).

2. This is proved easily by inducting on $l(w)$. When $l(w) \leq 1$ the relations in (2.3) are just those in (2.2). Also, by (1) we know that if w has a reduced expression of the form $s_1 \cdots s_r$, then $T_{s_1 \cdots s_r} = T_{s_1} \cdots T_{s_r}$. So, if $ws > w$ for $s \in S$, we have the required relation by (1). In the case when $ws < w$, since $l(ws) < l(w)$, we apply induction hypothesis to ws to get $T_{ws}T_s = T_w$. Now multiplying both sides of this equation by T_s and using the expression for T_s^2, we obtain the required relation even in this case.

3. By (2.2), we have $T_s^2 = (v - v^{-1})T_s + T_1$ which immediately gives the second part of (3). The first part now follows immediately from (1).

4. A routine inductive argument on $l(x)$ along with relations in (2.2) and (2) above proves (4). (The description of the set S as the simple transpositions in $\mathfrak{S}_n$ is used; See also [Mat99, §1.16])

□

Henceforth, by A we mean the ring $\mathbb{Z}[v, v^{-1}]$ and by $\mathcal{H}$ the Hecke algebra associated to (W, S) as in Definition 2.2.3.

Notation 2.2.5 We repeatedly use the following short-hand notation:

- $\epsilon_w := (-1)^{l(w)}$
- $v_w := v^{l(w)}$

Remark on notation: The notation that we use is as in [Lus03] (also in [Gec06]), while in [DJ86], [KL79], [MP05] (also [Hum90], [Shi86]) the notation used is slightly different. To pass from our notation to that of [DJ86], [KL79] or [MP05], we need to replace v by $q^{1/2}$ and T_w by $q^{-l(w)/2}T_w$.

Specializations of the Hecke algebra

Let k be a commutative ring with unity and a an invertible element in k. There is a unique ring homomorphism $A \to k$ defined by $v \mapsto a$. We denote by $\mathcal{H}_k$ the k-algebra $\mathcal{H} \otimes_A k$ obtained by extending the scalars to k via this homomorphism. We have a natural A-algebra homomorphism $\mathcal{H} \to \mathcal{H}_k$ given by $h \mapsto h \otimes 1$. By abuse of notation, we continue to use the same symbols for the images in $\mathcal{H}_k$ of the elements of $\mathcal{H}$. If M is a (right) $\mathcal{H}$-module, $M \otimes_A k$ is naturally a (right) $\mathcal{H}_k$-module.

An important special case is when we take a to be the unit element 1 of k. We then have a natural identification of $\mathcal{H}_k$ with the group ring kW, under which T_w maps to the element w in kW.

Convention: When the value of a is not specified, by the "specialization of $\mathcal{H}$ to k", we mean the algebra $\mathcal{H}_k$ defined via the map $v \mapsto 1$.

An involution on $\mathcal{H}$

We introduce an involution (order 2 ring automorphism) on $\mathcal{H}$ defined as follows:

$$\overline{\sum a_w T_w} := \sum \overline{a_w} T_{w^{-1}}^{-1} \tag{2.4}$$

where $a \mapsto \overline{a}$ on A is defined by $v \mapsto \overline{v} := v^{-1}$ extended $\mathbb{Z}$-linearly to give an involution on A. This is called the *bar involution.*

2.2.1 Kazhdan - Lusztig bases of $\mathcal{H}$

Apart from the basis T_w, $w \in W$, there is another A-basis of $\mathcal{H}$ which is of interest to us. It is determined uniquely by the conditions:

$$\overline{C_w} = C_w \quad \text{and} \quad C_w \equiv T_w \mod \mathcal{H}_{>0} \qquad (\ddagger)$$

where $\mathcal{H}_{>0} := \sum_{w \in W} A_{>0} T_w$, $A_{>0} := v\mathbb{Z}[v]$. This is called the C-*basis* of $\mathcal{H}$.

The existence and uniqueness of a basis as above follows from:

Theorem 2.2.6 [KL79, Theorem 1.1] *For each $w \in W$ there exists a unique element $C_w \in \mathcal{H}$ having the following properties:*

1. $\overline{C_w} = C_w$,
2. $C_w = \sum_{y \in W,\ y \leq w} \epsilon_y \epsilon_w \overline{p_{y,w}} T_y$ *where* $p_{y,w} \in v^{-1}\mathbb{Z}[v^{-1}]$ *for* $y < w$ *and* $p_{w,w} = 1$.

The existence of these elements is proved by constructing C_w by induction on the length of w. Set $C_1 := T_1$. For $w = su$ such that $l(w) = l(u) + 1$ define

$$C_w := C_s C_u - \sum_{z: sz<z<u} \nu(z,u) C_z \tag{2.5}$$

where $\nu(z,u)$ is the coefficient of v^{-1} in $p_{z,u}$ and $C_s := T_s - vT_1$. One then verifies that the properties (1) and (2) hold. That these elements form a basis follows then from property (2) above.

Remark 2.2.7 As is verified easily, the involutive anti-automorphism of the algebra $\mathcal{H}$ given by $h \mapsto h^*$, where $(\sum a_w T_w)^* := \sum a_w T_{w^{-1}}$, commutes with the bar involution on $\mathcal{H}$. Therefore,

i) it follows from the defining conditions ($\ddagger$) that $C_w^* = C_{w^{-1}}$.

ii) applying $*$ to the relation $C_w = T_w + \sum_{y \in W,\ y<w} \epsilon_y \epsilon_w \overline{p_{y,w}} T_y$ (Theorem 2.2.6(2) above), we get $C_{w^{-1}} = T_{w^{-1}} + \sum_{y \in W,\ y<w} \epsilon_y \epsilon_w \overline{p_{y,w}} T_{y^{-1}}$ (using (i)). Thus by the uniqueness condition in Theorem 2.2.6 we note that $p_{y,w} = p_{y^{-1},w^{-1}}$.

Using the above remark it can be deduced, by applying * to (2.5), that for $w = us$ such that $l(w) = l(u) + 1$ we have

$$C_w = C_u C_s - \sum_{z:zs<z<u} \mu(z,u) C_z \tag{2.6}$$

where $\mu(z,u) := \nu(z^{-1}, u^{-1})$.

From the above equation we obtain the following basic result.

Lemma 2.2.8 [KL79, § 2.3] *Let $s \in S$ and $ws < w$. Then $C_w T_s = -v^{-1} C_w$.*

PROOF: We induct on $l(w)$. If $w = s$ then inserting the expression $C_s = T_s - vT_1$ in $C_s T_s$ and simplifying it using (2.3), we get $C_s T_s = -v^{-1} C_s$. Under induction hypothesis assume the statement for z such that $l(z) < l(w)$ and $zs < z$. Replacing u by ws in (2.6) and using the expression thus obtained we get

$$C_w T_s \quad = \quad C_{ws}(C_s T_s) - \textstyle\sum_{zs<z<ws} \mu(z, ws)(C_z T_s)$$

Applying the induction hypothesis, we then have

$$\begin{aligned} C_w T_s &= -v^{-1} C_{ws} C_s + \textstyle\sum_{zs<z<ws} v^{-1} \mu(z, ws) C_z \\ &= -v^{-1} C_w \end{aligned}$$

as required. □

The above observation leads us to some useful properties of the polynomials $p_{x,w}$ which appear in statement (2) of Theorem 2.2.6.

Corollary 2.2.9 [KL79, §2.3] *Let $s \in S$ and $x,\, w \in W$.*

1. *If $x < w$, $ws < w$ and $xs > x$ then $p_{x,w} = v^{-1} p_{xs,w}$.*
2. *Let w_0 be the longest element of W. Then we have $p_{x,w_0} = v^{l(x)-l(w_0)}$ for all $x \in W$.*

PROOF: By Lemma 2.2.8, we have the relation $C_w T_s = -v^{-1} C_w$ whenever $ws < w$. Inserting into it the expression for C_w given by Theorem 2.2.6(2) and then comparing the coefficient of T_{xs} on both sides of the relation thus obtained, we deduce that $p_{x,w} + (v - v^{-1}) p_{xs,w} = v p_{xs,w}$ which readily yields (1).

For $x \in W$, we can find a finite sequence of elements $s_1, \ldots, s_r \in S$ such that $x < xs_1 < \ldots < xs_1 \cdots s_r = w_0$. Since the longest element w_0 satisfies the condition $w_0 s < w_0$ for all $s \in S$, we apply (1) repeatedly to get $p_{x,w_0} = v^{-1} p_{xs_1,w_0} = \ldots = (v^{-1})^r p_{xs_1 \cdots s_r, w_0} = (v^{-1})^r$ (by Theorem 2.2.6(2), $p_{w_0,w_0} = 1$). Note that $r = l(w_0) - l(x)$, proving (2) □

Kazhdan-Lusztig C'-basis

Consider the ring involution $j : \mathcal{H} \to \mathcal{H}$, defined by

$$j(\sum a_w T_w) := \sum \epsilon_w \overline{a_w} T_w$$

where $\epsilon_w := (-1)^{l(w)}$, and the element $\overline{a} \in A$ is as defined in (2.4). Then define,

$$C'_w := \epsilon_w j(C_w) \tag{2.7}$$

Since j is an involution on $\mathcal{H}$, the elements C'_w, $w \in W$, also form a basis for $\mathcal{H}$. It is called the C'*-basis*. By Theorem 2.2.6 and the fact that the bar involution commutes with the involution j defined above, it is clear that the elements C'_w, $w \in W$, are also determined uniquely by the two properties

$$\overline{C'_w} = C'_w \quad \text{and} \quad C'_w \equiv T_w \mod \mathcal{H}_{<0}$$

All the other properties of the C-basis can also be carried over to the C'-basis via the involution j. An instance of this, which we shall use later, is the following:

Lemma 2.2.10 *Let $s \in S$ and $ws < w$. Then $C'_w T_s = v C'_w$.*

Proof: Applying the involution to both sides of the relation $C_w T_s = -v^{-1} C_w$, we get the required relation for C'_w. □

Before ending this subsection, in the light of Corollary 2.2.9(2) and Theorem 2.2.6(2), we note that if w_0 denotes the longest element in W, then

$$C_{w_0} = \epsilon_{w_0} v_{w_0} \sum_{w \in W} \epsilon_w v_w^{-1} T_w, \qquad C'_{w_0} = v_{w_0}^{-1} \sum_{w \in W} v_w T_w \tag{2.8}$$

The above expressions together with Lemma 2.2.8 and Lemma 2.2.10 gives

$$C_{w_0}^2 = (\epsilon_{w_0} v_{w_0}^{-1} \sum_{w \in W} v_w^2)\ C_{w_0} \tag{2.9}$$

$$C'^2_{w_0} = (v_{w_0} \sum_{w \in W} v_w^{-2})\ C'_{w_0} \tag{2.10}$$

2.2.2 Kazhdan-Lusztig orders and cells

The central goal of introducing the Kazhdan-Lusztig bases is to understand the representations of the Hecke algebra $\mathcal{H}$. The advantage of the C-basis (analogously C'-basis) is that it leads to a systematic construction of certain representations. This is done by defining a pre-order on W the equivalence classes of which gives a partition of W into cells (left, right, two-sided). Later in §3.3.5, we construct representations of W associated to these cells. We now give the definition of these cells of W.

Let y and w be elements in W. Write $y \leftarrow_{\mathrm{L}} w$ if, for some element s in S, the coefficient of C_y is non-zero in the expression of $C_s C_w$ as an A-linear combination of the basis elements C_x. Replacing all occurrences of 'C' by 'C'' in this definition would make no difference. The Kazhdan-Lusztig *left pre-order* is defined by: $y \leq_{\mathrm{L}} w$ if there exists a chain $y = y_0 \leftarrow_{\mathrm{L}} \cdots \leftarrow_{\mathrm{L}} y_k = w$; the *left equivalence* relation by: $y \sim_{\mathrm{L}} w$ if $y \leq_{\mathrm{L}} w$ and

$w \leq_{\mathrm{L}} y$. Left equivalence classes are called *left cells*. Note that by the above definition the A-module $\sum_{x \leq_{\mathrm{L}} w} AC_x$ is a left ideal of $\mathcal{H}$ containing the left ideal $\mathcal{H}C_w$.

Right pre-order, equivalence, and cells are defined similarly. The *two sided pre-order* is defined by: $y \leq_{\mathrm{LR}} w$ if there exists a chain $y = y_0, \ldots, y_k = w$ such that, for $0 \leq j < k$, either $y_j \leq_{\mathrm{L}} y_{j+1}$ or $y_j \leq_{\mathrm{R}} y_{j+1}$. Two sided equivalence classes are called *two sided cells.*

Remark 2.2.11 Since C_s is defined as $T_s - vT_1$, it may be easily seen that, if for $y, w \in W$ we set $y \leftarrow_{\mathrm{L}}' w$ whenever there exists an element s in S such that the coefficient of C_y is non-zero in the expression of $T_s C_w$ as a A-linear combination of the C-basis elements, then the pre-order $\leq_{\mathrm{L}}$ obtained from the relation $\leftarrow_{\mathrm{L}}'$ is the same as the left pre-order defined above (using the C-basis instead of T).

Lemma 2.2.12 *Let w_0 be the longest element in W. Then for elements $w, w' \in W$ we have, $w \leq_{\mathrm{L}} w'$ if and only if $w_0 w' \leq_{\mathrm{L}} w_0 w$. Similarly for $\leq_{\mathrm{R}}$. Moreover, $\nu(w, w') = \nu(w_0 w', w_0 w)$.* (See [KL79, Corollary 3.2] for proof; [Shi86, Lemma 1.4.6(iii)]) □

Remark 2.2.13 By Remark 2.2.7(ii), we have $p_{y,w} = p_{y^{-1},w^{-1}}$ and hence $\nu(y, w) = \nu(y^{-1}, w^{-1})$. By definition, the latter term is $\mu(y, w)$. Thus, by Lemma 2.2.12, we conclude that $\mu(w, w') = \mu(w_0 w', w_0 w)$ for $w,\ w' \in W$.

For any $y \in W$, we associate to it two sets defined as follows:

$$\mathfrak{R}(y) := \{s \in S |\ ys < y\} \qquad \mathfrak{L}(y) := \{s \in S |\ sy < y\}$$

Then we have,

Lemma 2.2.14 *Let $w, w' \in W$.*

1. *If $w \leq_{\mathrm{R}} w'$ then $\mathfrak{L}(w') \subset \mathfrak{L}(w)$.*
2. *If $w \leq_{\mathrm{L}} w'$ then $\mathfrak{R}(w') \subset \mathfrak{R}(w)$.*

   (See [Lus03, Lemma 8.6] for a proof) □

2.3 Combinatorics of cells in $\mathfrak{S}_n$

From now on, we fix A to be the Laurent polynomial ring in one indeterminate $\mathbb{Z}[v, v^{-1}]$. Let n be a fixed integer and $\mathfrak{S}_n$ the symmetric group on n letters. Let S denote the subset consisting of simple transpositions $(1, 2), (2, 3) \ldots (n-1, n)$. Then $(\mathfrak{S}_n, S)$ is a Coxeter system and its Hecke algebra defined over A is denoted as $\mathcal{H}$.

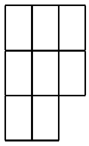
In $\mathfrak{S}_n$, there are certain combinatorial descriptions for cells as defined in the earlier section. Before beginning with this description we introduce some basic definitions.

2.3.1 Basic notions

Partitions and shapes

By a *partition* λ of n, written $\lambda \vdash n$, is meant a sequence $\lambda_1 \geq \ldots \geq \lambda_r$ of positive integers such that $\lambda_1 + \ldots + \lambda_r = n$. The integer r is the *number of parts* in λ. We often write $\lambda = (\lambda_1, \ldots, \lambda_r)$; sometimes even $\lambda = (\lambda_1, \lambda_2, \ldots)$. When the latter notation is used, it is to be understood that $\lambda_t = 0$ for $t > r$.

Partitions of n are in bijection with *shapes of Young diagrams* (or simply *shapes*) with n boxes: the partition $\lambda_1 \geq \ldots \geq \lambda_r$ corresponds to the shape with λ_1 boxes in the first row, λ_2 in the second row, and so on, the boxes being arranged left- and top-justified. This diagram of boxes is sometimes also called the *Young diagram of shape* λ denoted as λ. Here for example is the shape corresponding to the partition $(3, 3, 2)$ of 8:



Partitions are thus identified with shapes and the two terms are used interchangeably.

Dominance order on partitions

Given partitions $\mu = (\mu_1, \mu_2, \ldots)$ and $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n, we say μ *dominates* λ, and write $\mu \trianglerighteq \lambda$, if

$$\mu_1 \geq \lambda_1, \quad \mu_1 + \mu_2 \geq \lambda_1 + \lambda_2, \quad \mu_1 + \mu_2 + \mu_3 \geq \lambda_1 + \lambda_2 + \lambda_3, \quad \ldots.$$

We write $\mu \triangleright \lambda$ if $\mu \trianglerighteq \lambda$ and $\mu \neq \lambda$. The partial order $\trianglerighteq$ on the set of partitions (or shapes) of n will be referred to as the *dominance order*.

Tableaux and standard tableaux

A *Young tableau*, or just *tableau*, of shape $\lambda \vdash n$ is an arrangement of the numbers 1, …, n in the boxes of shape λ. There are, evidently, $n!$ tableaux of shape λ. A tableau is *row standard* (respectively, *column standard*) if in every row (respectively, column) the entries are increasing left to right (respectively, top to bottom). A tableau is *standard* if it is both row standard and column standard. An example of a standard tableau of shape $(3, 3, 2)$:

1	3	5
2	6	8
4	7	

The number of standard tableaux: The number of standard tableaux of a given shape $\lambda \vdash n$ is denoted $d(\lambda)$. There is a well-known 'hook length formula' for it (see

[Sag01, p.124]; [FRT54]): $d(\lambda) = n!/\prod_\beta h_\beta$, where β runs over all boxes of shape λ and h_β is the *hook length* of the box β which is defined as one more than the sum of the number of boxes to the right of β and the number of boxes below β.

The hook lengths for the shape $(3,3,2)$ are shown below:

$$\begin{array}{|c|c|c|}\hline 5 & 4 & 2 \\ \hline 4 & 3 & 1 \\ \hline 2 & 1 \\ \cline{1-2}\end{array}$$

Thus $d(3,3,2) = 8!/(5.4.2.4.3.1.2.1) = 42$.

Row and Column stabilizers:

Given a tableau T of shape $\lambda \vdash n$, we obtain two collections of subsets of the set $\{1, 2, \ldots, n\}$:

R : Each subset consists of numbers appearing along each row of T.

C : Each subset consists of numbers appearing along each column of T.

The *row stabilizer* of T is the subgroup of $\mathfrak{S}_n$ which leaves each subset in the collection **R** invariant. It is the set of permutations in $\mathfrak{S}_n$ that permute the numbers appearing in each row of T among themselves. Similarly, the *column stabilizer* of T is the subgroup of $\mathfrak{S}_n$ which leaves each subset in the collection **C**, invariant. It is the set of permutations in $\mathfrak{S}_n$ that permute the numbers in each column of T among themselves.

For example, let

$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$$

Then the row stabilizer of T is the subgroup $\mathfrak{S}_{\{1,3\}} \times \mathfrak{S}_{\{2,5\}} \times \mathfrak{S}_{\{4\}}$ and its column stabilizer is $\mathfrak{S}_{\{1,2,4\}} \times \mathfrak{S}_{\{3,5\}}$.

2.3.2 RSK-correspondence

We recall, in this subsection, the combinatorial algorithm which goes under the name of *Robinson-Schensted-Knuth*. It is a well-known procedure that sets up a bijection between the symmetric group $\mathfrak{S}_n$ and ordered pairs of standard tableaux of the same shape with n boxes. The aim of the algorithm was to provide a purely combinatorial proof that the number of elements in $\mathfrak{S}_n$ is equal to the number of pairs of standard tableaux of the same shape λ, as λ varies over all partitions of n,*i.e.*,

$$\sum_{\lambda \vdash n} d(\lambda)^2 = n!$$

The genesis of the above relation lies in the representation theory of $\mathfrak{S}_n$ over $\mathbb{C}$: the partitions, $\lambda \vdash n$, parametrize all the non-isomorphic irreducible representations of $\mathbb{C}\mathfrak{S}_n$;

the number $d(\lambda)$ of standard tableaux of shape λ is the same as the dimension of the irreducible representation of $\mathbb{C}\mathfrak{S}_n$ associated to λ (see §3.2).

In order to describe the algorithm, which sets up the bijection as mentioned earlier, we will need the *insertion algorithm* which is described as follows: Let P be a tableau consisting of an arbitrary set of distinct numbers. If all the numbers from 1 to n appear then it is a tableau in the sense defined in §2.3.1. Let x be a number not appearing in P. Then the insertion algorithm to insert x in P, denoted as $P \leftarrow x$, is given as follows:

I1 Set R to be the first row of P.

I2 **While** x is less than some element in R, **do**

> **I2a** Let y be the smallest number in R greater than x and replace y by x.
>
> **I2b** Set $x := y$ and R as the next row down.

I3 Now x is greater than every element in R, so place x at the end of the row and **stop**.

The *Robinson-Schensted-Knuth correspondence* (or *RSK-correspondence*, for short) is a bijection between $\mathfrak{S}_n$ and pairs of standard tableaux of the same shape with n boxes. The bijection is given by an algorithm that takes a permutation π and produces from it a pair $(P(\pi), Q(\pi))$ of standard tableaux of a certain shape. This is done as follows: Let π be written in two-line notation as

$$\pi = \begin{matrix} 1 & 2 & \cdots & n \\ x_1 & x_2 & \cdots & x_n \end{matrix}$$

We construct $Q(\pi)$ as

$$((\ldots((\ \boxed{x_1}\ \leftarrow x_2) \leftarrow x_3)\cdots) \leftarrow x_{n-1}) \leftarrow x_n$$

The tableaux $P(\pi)$, called the recording tableau, is obtained by simply placing the integer k in the box that is added at the k-th step of the construction of $Q(\pi)$.

Example 2.3.1 Consider the permutation $(14253) \in \mathfrak{S}_5$, written in two-line notation as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$$

Applying the above algorithm, we get the pair

$$P = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array} \qquad Q = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}$$

That the map defined by the above algorithm is a bijection is proved by reversing the algorithm step by step to recover the permutation associated to a pair of standard tableaux.

Denote by (P_k, Q_k) the tableaux obtained at the k-th step. Then, to go from (P_k, Q_k) to (P_{k-1}, Q_{k-1}) consider the number appearing in Q_k in the box which contains the largest number in P_k and apply the reverse row-insertion algorithm to Q_k with this number. The element that is bumped out is the image of k under the resulting permutation. And the tableau obtained by removing the originating box in Q_k is the tableau Q_{k-1}. Removing the largest number in P_k we get P_{k-1}.

It should be noted that the algorithm presented above is slightly different from those given in standard texts ([Ful97], [Sag01]). For reasons that will be explained later (see Remark 2.3.8), we have modified the procedure by associating to a permutation the same pair of standard tableaux as obtained by the standard procedure but with their positions interchanged, *i.e.*, if $(A(w), B(w))$ is the pair associated to $w \in \mathfrak{S}_n$ by the algorithm as in [Ful97] or [Sag01], then by the RSK-correspondence we mean the bijection that associates the pair $(B(w), A(w))$ to the permutation w. In the light of the following result of Schützenberger, the modification amounts to associating the permutation w^{-1} to the pair $(A(w), B(w))$.

Theorem 2.3.2 (Schützenberger) (see [Sag01, Theorem 3.6.6]) *If $w \in \mathfrak{S}_n$ then $A(w^{-1}) = B(w)$ and $B(w^{-1}) = A(w)$.* □

Notation 2.3.3 We write $(P(w),\ Q(w))$ for the ordered pair of standard Young tableaux associated to the permutation w by the RSK-correspondence. Call $P(w)$ the *P-symbol* and $Q(w)$ the *Q-symbol* of w. It will be convenient also to use $(P(w), Q(w))$ for the permutation w, $C_{(P(w),Q(w))}$ or $C(P(w), Q(w))$ for the Kazhdan-Lusztig C-basis element C_w.

Definition 2.3.4 The *RSK shape* of a permutation w is defined to be the shape of the tableau $P(w)$ (which is the same as that of $Q(w)$).

An example

The permutation $(1542)(36)$ (written as a product of disjoint cycles) has RSK-shape $(3, 2, 1)$. Indeed it is mapped under the RSK correspondence in our sense to the ordered pair (A, B) of standard tableaux, where:

$$A = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \qquad B = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

2.3.3 Cells and RSK-correspondence

We now recall the combinatorial characterizations of left, right and two sided cells in terms of the RSK correspondence and the dominance order on partitions (§2.3.1). These statements are the foundation on which most of our arguments rest.

The following statement characterizes the relation $\leq_{\mathrm{LR}}$. (Also see comments in [Gec06] about [Ari00]).

Proposition 2.3.5 ([Gec06, Theorem 5.1]) *Let w, $w' \in \mathfrak{S}_n$. Then $w \leq_{\mathrm{LR}} w'$ if and only if* RSK-shape$(w) \trianglelefteq$ RSK-shape(w'). □

The next statement establishes the "unrelatedness" of distinct one-sided (left/right) cells within a two-sided cell. Though stated only for $\leq_{\mathrm{L}}$ and left cells, the analogous statement is true also for $\leq_{\mathrm{R}}$ and right cells.

Proposition 2.3.6 ([Gec06, Theorem 5.3]) *Let w, $w' \in \mathfrak{S}_n$. If $w \leq_{\mathrm{L}} w'$ and $w \sim_{\mathrm{LR}} w'$ then $w \sim_{\mathrm{L}} w'$.* (See also [Lus81, Lemma 4.1]) □

Finally, the following proposition gives a combinatorial characterization of the left, right and two-sided equivalence. Statements (1) and (2) of the Proposition can be found also in [KL79] or [Ari00].

Proposition 2.3.7 ([Gec06, Corollary 5.6]) *Let w, $w' \in \mathfrak{S}_n$. Then the following hold:*

1. $w \sim_{\mathrm{L}} w' \Leftrightarrow Q(w) = Q(w')$.
2. $w \sim_{\mathrm{R}} w' \Leftrightarrow P(w) = P(w')$.
3. $w \sim_{\mathrm{LR}} w' \Leftrightarrow$ RSK-shape(w) = RSK-shape(w'). (This follows easily from Proposition 2.3.5 and the definition of $\sim_{\mathrm{LR}}$). □

Remark 2.3.8 The proofs of the above statements can be found in [Gec06]. However, it should be noted that in [Gec06] permutations act from the left while for us permutations always act from the right. Also, in [Gec06], the RSK-algorithm as given in [Ful97] (or [Sag01]) is used. So, in the light of Theorem 2.3.2, the statements of the above propositions hold verbatim even in our setup, under the assumption that the (modified) RSK-correspondence as described here is used to obtain the P, Q symbols of a permutation.

2.3.4 Some notes and notations

For λ a partition of n,

- λ' denotes the *transpose* of λ which is defined to be the shape obtained by taking the transpose of the Young diagram of shape λ. E.g., $\lambda' = (3, 2, 2, 1)$ for $\lambda = (4, 3, 1)$.
- t^λ denotes the standard tableau of shape λ in which the numbers 1, 2, ..., n appear in order along successive rows; t_λ is defined similarly, with 'columns' replacing 'rows'. E.g., for $\lambda = (4, 3, 1)$, we have:

$$t^\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3} 8 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \qquad t_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 8 \\ \hline 2 & 5 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}$$

- Permutations in $\mathfrak{S}_n$ act naturally on tableaux of shape $\lambda \vdash n$ by acting entry-wise. Denote by w_λ the permutation in $\mathfrak{S}_n$ such that $t^\lambda w_\lambda = t_\lambda$.

The parabolic subgroup W_λ and its coset representatives

Let W_λ denote the row stabilizer of t^λ. It is a parabolic subgroup of $\mathfrak{S}_n$ generated by $W_\lambda \cap S$. Let $w_{0,\lambda}$ denote the longest element of W_λ. E.g., for $\lambda = (4,3,1)$, W_λ is isomorphic to the product $\mathfrak{S}_4 \times \mathfrak{S}_3 \times \mathfrak{S}_1$. The longest element of W_λ is given by the sequence $(1w_{0,\lambda}, \ldots, nw_{0,\lambda}) = (4,3,2,1,7,6,5,8)$.

Remark 2.3.9 *The longest element $w_{0,\lambda}$ of W_λ has* RSK-shape *λ'; by the definition of the RSK-correspondence it is obvious that $w_{0,\lambda}$ corresponds to the pair $(t_{\lambda'}, t_{\lambda'})$.*

Define $\mathfrak{D}_\lambda := \{w \in \mathfrak{S}_n \mid t^\lambda w \text{ is row standard}\}$. Clearly, w_λ as defined above, is an element of $\mathfrak{D}_\lambda$. The next proposition lists out a few properties of the elements in $\mathfrak{D}_\lambda$.

Proposition 2.3.10 ([DJ86, Lemma 1.1]) *For λ a partition of n,*

1. *$\mathfrak{D}_\lambda$ is a set of right coset representatives of W_λ in $\mathfrak{S}_n$.*
2. *the element $d \in \mathfrak{D}_\lambda$ is the unique element of minimal length in $W_\lambda d$.*
3. *$l(wd) = l(w) + l(d)$, for $w \in W_\lambda$ and $d \in \mathfrak{D}_\lambda$.*
4. *$\mathfrak{D}_\lambda = \{d \in \mathfrak{S}_n \mid l(sd) > l(d) \text{ for all } s \in W_\lambda \cap S\}$.*

Proof: If for each element $w \in \mathfrak{S}_n$ we associate the tableau $t^\lambda w$ then under this association the elements of the coset $W_\lambda d$, $d \in \mathfrak{D}_\lambda$ correspond to the collection of tableaux which vary from each other by a permutation of the row-wise entries. Now noting that $d \in \mathfrak{D}_\lambda$ corresponds, under the above association, to the unique tableaux in the collection of $t^\lambda w$, $w \in W_\lambda d$, which is row-standard (*i.e.*, increasing along rows but not necessarily increasing along columns), the bijection as in (1) is immediate.

To prove (2) we use the fact that the length l counts the number of inversions, so $l(d) = \#\{(i,j) | 1 \leq i < j \leq n,\ i.d > j.d\}$ (see [BB05, Proposition 1.5.2]). Let $s \in W_\lambda$ such that $s = (i, i+1)$. Let a, b appear in $t^\lambda d$ in the positions where i, $i+1$ appear in t^λ, then $t^\lambda sd$ is obtained by inverting the positions of a and b in $t^\lambda d$. Also, since $t^\lambda d$ is row-standard we have $a < b$. Thus, we see that the number of inversions in sd is $>$ number of inversions in d *i.e.*, $l(sd) > l(d)$. Since this is true for all $s \in W_\lambda \cap S$ the uniqueness in (2) can be deduced by method of contradiction, as outlined in the proof of item (4) given below.

For $w \in W_\lambda$, $d \in \mathfrak{D}_\lambda$, we know that $l(wd) \leq l(w) + l(d)$. Suppose that $l(wd) < l(w) + l(d)$ then by the deletion condition (see §2.1), we get $w' < w$, $d' < d$ such that $w'd' = wd$. Hence we get $d \in W_\lambda d' = W_\lambda d$ but $l(d') < l(d)$ which contradicts (2). Thus $l(wd) = l(w) + l(d)$, as claimed in (3).

For establishing the equality in (4), we first notice using (3) that $\mathfrak{D}_\lambda$ is contained in the set on the right-hand side of (4). To prove the other way inclusion, suppose $d \in \mathfrak{S}_n$ such that $l(sd) > l(d)$ for all $s \in S \cap W_\lambda$, we claim that d is of minimal length in the coset $W_\lambda d$, which then by (2) implies that $d \in \mathfrak{D}_\lambda$. We prove the claim by assuming the contrary, as follows:

Suppose d is not of minimal length in the coset $W_\lambda d$, then let $x \in W_\lambda d$ be one such. We can express d as a product of the form wx for some $w \in W_\lambda$ so that $l(d) \leq l(w) + l(x)$. If $l(d) < l(w)+l(x)$ then by deletion condition, we find elements w', x' such that $w' < w$ and $x' < x$ with $d = w'x'$, which will contradict the minimality of $l(x)$ unless $x' = x$. Thus, we conclude that $l(d) = l(w) + l(x)$. Now since $x \neq d$, we get $w \neq 1$, which implies there is a $u \in S \cap W_\lambda$ such that $uw < w$, so that $uwx < wx$ which contradicts the hypothesis that $sd > d$ for all $s \in W_\lambda \cap S$.

□

We now prove a useful lemma that enables us to characterize elements xs, for $x \in \mathfrak{D}_\lambda$ and $s \in S$. The lemma is true more generally for "distinguished" coset representatives of an arbitrary parabolic subgroup where the distinguished coset representatives are defined by property (4) in Proposition 2.3.10. The proof presented here holds true verbatim even in this general setup.

Lemma 2.3.11 (Deodhar's lemma) *Let $x \in \mathfrak{D}_\lambda$, $s \in S$. Then either $xs \in \mathfrak{D}_\lambda$ or $xs = ux$ for $u \in W_\lambda \cap S$.*

Proof: Suppose $xs \notin \mathfrak{D}_\lambda$, then property (4) in Proposition 2.3.10 does not hold, *i.e.*, there exists an element $u \in W_\lambda \cap S$ such that $l(uxs) < l(xs)$. On the other hand, $x \in \mathfrak{D}_\lambda$ implies that $l(ux) > l(x)$. Thus we have,

$$l(x) < l(ux) = l(uxs) \pm 1 \leq l(xs)$$

Let $s_1 \cdots s_r$ be a reduced expression for uxs. Then $us_1 \cdots s_r$ is a reduced expression for xs since $l(uxs) + 1 = l(xs)$. As $l(xs) > l(x)$, we should be able find a reduced expression for x as a suitable subexpression of $us_1 \cdots s_r$. If s_i is dropped then $x = us_1 \cdots \hat{s}_i \cdots s_r$ leading us to the contradiction that $ux < x$. Hence $x = s_1 \cdots s_r$ so that $xs = us_1 \cdots s_r = ux$, as required.

□

The set $\mathfrak{D}_\lambda$ can be described entirely based on just one element in it, namely the longest coset representative. We have,

Proposition 2.3.12 ([DJ86, Lemma 1.4]) *Let d_λ be an element of maximal length in $\mathfrak{D}_\lambda$. Then,*

1. *d_λ is unique.*

2. *$\mathfrak{D}_\lambda$ is precisely the set of all the prefixes of d_λ.*

PROOF: Let w_0 be the longest element in $\mathfrak{S}_n$. Let $x \in \mathfrak{D}_\lambda$ such that $w_0 \in W_\lambda x$. Then by maximality of length and Proposition 2.3.10(3), we have $w_0 = w_{0,\lambda}x$ and x should be of maximal length in $\mathfrak{D}_\lambda$. By uniqueness of the longest element in $\mathfrak{S}_n$, we have $w_{0,\lambda}x = w_{0,\lambda}d_\lambda$, proving (1) above.

For proving (2), first we observe that if $y \in \mathfrak{D}_\lambda$ and $s \in S$ such that $ys < y$ then $ys \in \mathfrak{D}_\lambda$. This follows immediately by Deodhar's lemma (Lemma 2.3.11), since if $ys \notin \mathfrak{D}_\lambda$ then $ys > y$. Thus, all prefixes of d_λ are in $\mathfrak{D}_\lambda$. The other way inclusion is proved by reverse induction on $l(y)$ where $y \in \mathfrak{D}_\lambda$. If $l(y)$ is maximal then $y = d_\lambda$. Assume $y \neq d_\lambda$. For $s \in S$, if $l(ys) > l(y)$ and $ys \in \mathfrak{D}_\lambda$ then by induction ys is a prefix of d_λ, so is y and we are done. So, assume for every $s \in S$ either $ys \notin \mathfrak{D}_\lambda$ or $l(ys) < l(y)$. Suppose $ys \notin \mathfrak{D}_\lambda$ where $s \in S$ then by Deodhar's lemma $ys = uy$ for some $u \in W_\lambda \cap S$. Then,

$$l(w_{0,\lambda}ys) = l(w_{0,\lambda}uy) = l(w_{0,\lambda}) - 1 + l(y) = l(w_{0,\lambda}y) - 1$$

On the other hand, if $l(ys) < l(y)$ and $ys \in \mathfrak{D}_\lambda$ then

$$l(w_{0,\lambda}ys) = l(w_{0,\lambda}) + l(ys) = l(w_{0,\lambda}) + l(y) - 1 = l(w_{0,\lambda}y) - 1$$

Thus, we obtain that for all $s \in S$, we get $w_{0,\lambda}ys < w_{0,\lambda}y$. This readily implies $w_{0,\lambda}y = w_0$, a contradiction to the assumption that $y \neq d_\lambda$. Hence the proof is complete.

$\square$

We had already seen that w_λ is an element of $\mathfrak{D}_\lambda$. By the above proposition, it is a prefix of d_λ and every prefix of w_λ is also in $\mathfrak{D}_\lambda$. The next lemma characterizes these prefixes.

Lemma 2.3.13 ([DJ86, Lemma 1.5]) *The set of $w \in \mathfrak{S}_n$ such that $t^\lambda w$ is a standard tableau is the same as the set of prefixes of w_λ.*

PROOF: We begin with an observation. Suppose $w \in \mathfrak{S}_n$ such that $t^\lambda w$ is standard. If $l(w(i,i+1)) < l(w)$ then $(i+1).w^{-1} < i.w^{-1}$, since l counts the number of inversions. So, if i occurs in node (r,c) and $(i+1)$ occurs in node (r',c') of $t^\lambda w$ then $r > r'$ and $c < c'$ (because $t^\lambda w$ is standard). Now it is obvious that $t^\lambda w(i,i+1)$ is also standard.

The observation made in the last paragraph shows that for every prefix w of w_λ, $t^\lambda w$ is also standard. Conversely, let $t^\lambda w$ is standard and assume $w \neq w_\lambda$. Then there exists $i,\ j$, with $j > i+1$, occurring in consecutive boxes in some column of $t^\lambda w$. It can be seen easily that there is a k, $i \leq k < j$ such that k occurs in a node (a,b) and $k+1$ occurs in node (a',b') such that $a < a'$ and $b > b'$ of $t^\lambda w$. Then $t^\lambda w(k,k+1)$ is standard and $l(w(k,k+1)) = l(w) + 1$. By reverse induction on $l(w)$, $w(k,k+1)$ is a prefix of w_λ, and so is w. $\square$

More on coset representatives

After all this discussion about the right coset representatives of W_λ, it is easy to verify the following remark about the left coset representatives,

Remark 2.3.14 *The set $\mathfrak{D}_\lambda^{-1}$ is a set of minimal length left coset representatives of W_λ in $\mathfrak{S}_n$. This set has similar properties as that of $\mathfrak{D}_\lambda$ as listed in Proposition 2.3.10.*

Let $\lambda, \mu \vdash n$. We now describe a set of double coset representatives for W_λ–W_μ in the following lemma:

Lemma 2.3.15 (see [GP00, Proposition 2.1.7] *Let $\lambda, \mu \vdash n$. Then for each element $w \in \mathfrak{S}_n$ there is a $u \in W_\lambda$, $v \in W_\mu$ and a unique $d \in \mathfrak{D}_\lambda \cap \mathfrak{D}_\mu^{-1}$ such that $w = udv$ and $l(w) = l(u) + l(d) + l(v)$. In particular, the set $\mathfrak{D}_\lambda \cap \mathfrak{D}_\mu^{-1}$ is a set of double coset representatives of W_λ–W_μ.*

Proof: Let $w \in \mathfrak{S}_n$ and write $w = ux$, where $u \in W_\lambda$ and $x \in \mathfrak{D}_\lambda$, and $l(w) = l(u) + l(x)$ by Proposition 2.3.10. Write $x = dv$ where $d \in \mathfrak{D}_\mu^{-1}$ and $v \in W_\mu$ and $l(x) = l(d) + l(v)$. Since d is a prefix of $x \in \mathfrak{D}_\lambda$, we get $d \in \mathfrak{D}_\lambda$. Thus, $w = udv$ with $u \in W_\lambda$, $d \in \mathfrak{D}_\lambda \cap \mathfrak{D}_\mu^{-1}$, $v \in W_\mu$ and $l(w) = l(u) + l(d) + l(v)$, as required. The uniqueness follows by noting that d, as obtained above, is the unique element of minimal length in the double coset $W_\lambda d W_\mu = W_\lambda w W_\mu$. □

The following observation turns out to be useful later,

Lemma 2.3.16 (see [GP00, Theorem 2.1.12]) *Let $\lambda, \mu \vdash n$. Let $d \in \mathfrak{D}_\lambda \cap \mathfrak{D}_\mu^{-1}$. Then the subgroup $d^{-1} W_\lambda d \cap W_\mu$ is generated by $d^{-1} W_\lambda d \cap W_\mu \cap S$.*

Proof: Define $L := W_\lambda \cap S$, $K := W_\mu \cap S$ and $J := d^{-1} L d \cap K$. Clearly, $W_J \subset d^{-1} W_\lambda d \cap W_\mu$. It therefore, suffices to prove that $W_\lambda d \cap d W_\mu \subset d W_J$. Let $w \in W_\lambda d \cap d W_\mu$. Then $w = ud = dv$, where $u \in W_\lambda$ and $v \in W_\mu$ and $l(w) = l(u) + l(d) = l(d) + l(v)$. In particular, $l(u) = l(v)$. Let $v = v_0 \cdots v_r$, where $v_i \in K$. Set $d_0 = d$ and define $d_i \in \mathfrak{D}_\lambda$ using Deodhar's Lemma, so that $d_{i-1} v_i = x_i d_i$ where $d_i \in \mathfrak{D}_\lambda$ and $x_i \in L$ – if $d_{i-1} v_i \in \mathfrak{D}_\lambda$ then $x_i = 1$ otherwise $d_i = d_{i-1}$. Then we have $ud = dv = x_1 \cdots x_r d_r$ where $x_1 \cdots x_r \in W_\lambda$ and $d_r \in \mathfrak{D}_\lambda$. By uniqueness of the expression ud, we have $u = x_1 \cdots x_r$ and hence $r = l(u) = l(v)$. Therefore, $x_i \neq 1$ for all i and so, $d_i = d$ for all i. This means $d v_i = x_i d$ for all i equivalently, $v_i \in J$ from which we conclude that $w \in d W_J$. □

Remark 2.3.17 Lemma 2.3.15, Lemma 2.3.16 and Remark 2.3.14 hold true more generally for W_λ, W_μ replaced by any parabolic subgroups of $\mathfrak{S}_n$ and $\mathfrak{D}_\lambda$, $\mathfrak{D}_\mu$ replaced by the respective sets of minimal length right coset representatives (defined as in Proposition 2.3.10(4) which implies it is the unique element of minimal length in the right coset containing it). The proofs are verbatim those given above.

Few more combinatorial results

We now gather together a few combinatorial results regarding the Kazhdan-Lusztig relations, which will be used in the sequel:

Lemma 2.3.18 ([MP05, Lemma 3.3]) *Let $\lambda \vdash n$. Define $w = w_0 w_{0,\lambda} w_\lambda$. Then,*

1. $w_{0,\lambda}\mathfrak{D}_\lambda = \{y \in \mathfrak{S}_n \mid y \leq_{\mathrm{R}} w_{0,\lambda}\}$. *Thus, $y \leq_{\mathrm{R}} w_{0,\lambda}$ if and only if for every row of $t^\lambda y$ the entries are decreasing to the right.*
2. $\{w_{0,\lambda} d \mid d \text{ a prefix of } w_\lambda\} = \{y \in \mathfrak{S}_n \mid y \sim_{\mathrm{R}} w_{0,\lambda}\}$.
3. *The element w, as defined above, is in the same left cell as $w_{0,\lambda'}$.*
4. *w is a prefix of every element in the right cell containing it.*

Proof: (1) By Proposition 2.3.10(3) and (2.6), it is easy to check that $w_{0,\lambda}\mathfrak{D}_\lambda \subset \{y \in \mathfrak{S}_n \mid y \leq_{\mathrm{R}} w_{0,\lambda}\}$. To prove the other way inclusion, notice that if $y \leq_{\mathrm{R}} w_{0,\lambda}$ then by Lemma 2.2.14 $\mathfrak{L}(w_{0,\lambda}) \subset \mathfrak{L}(y)$, so, $sy < y$ for all $s \in S \cap W_\lambda$. Expressing y as ud where $u \in W_\lambda$, $d \in \mathfrak{D}_\lambda$ we have for all $s \in S \cap W_\lambda$, $sud < ud$, as we just observed, and $sd > d$, by Proposition 2.3.10(3). Therefore we conclude that $su < u$ for all $s \in S \cap W_\lambda$ and so, $u = w_{0,\lambda}$. Hence $y = w_{0,\lambda} d$ for some $d \in \mathfrak{D}_\lambda$, as required. The second part now follows immediately, noticing also that by its definition $w_{0,\lambda}$ reverses the entries in each row of t^λ.

(2) By (1), we have $w_{0,\lambda} d \leq_{\mathrm{R}} w_{0,\lambda}$. Further, using (2.6) it can be seen that for any prefix d of w_λ, $w_{0,\lambda} w_\lambda \leq_{\mathrm{R}} w_{0,\lambda} d$. However, an easy verification shows that $w_{0,\lambda} w_\lambda$ corresponds under the RSK-correspondence to the pair $(t_{\lambda'}, t^{\lambda'})$ (compare [MP05, Lemma 3.2]) while, $w_{0,\lambda}$ corresponds to $(t_{\lambda'}, t_{\lambda'})$. By Proposition 2.3.7(2) this means $w_{0,\lambda} w_\lambda \sim_{\mathrm{R}} w_{0,\lambda}$. Thus, for each prefix d of w_λ, we get, $w_{0,\lambda} d \sim_{\mathrm{R}} w_{0,\lambda}$, thereby proving one-way inclusion. Now using Lemma 2.3.13 and the characterisation of right cells given by Proposition 2.3.7, a counting argument proves the equality of the two sets.

(3) Applying the RSK-correspondence to $w_{0,\lambda} w_\lambda$ we get the pair $(t_{\lambda'}, t^{\lambda'})$ (compare [MP05, Lemma 3.2]). Notice that the pair corresponding under RSK to $w_0 w_{0,\lambda} w_\lambda$ is just the transpose of the tableaux corresponding to $w_{0,\lambda} w_\lambda$. So, $w = w_0 w_{0,\lambda} w_\lambda$ corresponds to (t^λ, t_λ). Proposition 2.3.7 then proves (3), as the Q-symbols of both w and $w_{0,\lambda'}$ are the same.

(4) Putting together Lemma 2.2.12 and statement (2) above, we deduce that the right cell containing w is given by $\{w_0 w_{0,\lambda} d \mid d \text{ is a prefix of } w_\lambda\}$, which can then be easily identified with the set $\{wb \mid b \text{ is a prefix of } w_{\lambda'}\}$ since $w_\lambda^{-1} = w_{\lambda'}$. Also, $l(w_0) = l(w) + l(w_{0,\lambda}) + l(w_{\lambda'})$ — observe that $l(w_0 w) = l(w_0) - l(w)$. This fact along with the relation $w_0 = w w_{\lambda'} w_{0,\lambda}$, implies that $l(wb) = l(w) + l(b)$ for all prefixes b of $w_{\lambda'}$. Hence the claim. $\square$

Chapter 3

Specht modules, Cell modules: An introduction

In this chapter we introduce the basic representation theoretic objects of interest to us and present some preliminary results about them to be used later.

The tabloid representations and Specht modules for $\mathfrak{S}_n$, introduced in §3.2, are classical objects in its representation theory. These modules have analogues also in the setup of the Hecke algebra, namely the permutation modules and Specht modules for $\mathcal{H}$. In §3.3, we give a short introduction to these $\mathcal{H}$-modules and by Proposition 3.3.12 provide justification to calling them as the analogues of their $\mathfrak{S}_n$ counterparts. There is one more module which is of primary importance to our study – the cell module. As $\mathcal{H}$-modules the cell modules are not different from the Specht modules, the proof of which forms the content of §3.4. Given there is an explicit isomorphism between the cell module corresponding to a partition and the Specht module corresponding to the same partition. In §3.5 we indicate a relation between the Specht modules corresponding to a partition and its transpose partition (refer §2.3.4), a fact that we will need later.

Section 3.2 is a very sketchy introduction to the $\mathfrak{S}_n$-modules that are of interest to us. More detailed introduction can be found in [Sag01], for example. The content of §3.3 is gathered from [DJ86], [DJ87] and [MP05].

We begin by recalling some elementary results from the structure theory of semisimple algebras.

3.1 A short recap of the structure theory of semisimple algebras

The facts recalled here are all well known: see for example [Bou73], [GP00, pp. 218, 247]. Let k denote a field. Let V be a simple (right) module for a semisimple algebra $\mathfrak{A}$ of finite dimension over k. Then the endomorphism ring $\mathrm{End}_{\mathfrak{A}}V$ is a division algebra (Schur's Lemma), say E_V. Being a subalgebra of $\mathrm{End}_k V$, it is finite dimensional as a vector space over k, and V is a finite dimensional vector space over it. Set $n_V := \dim_{E_V} V$. The ring $\mathrm{End}_{E_V}V$ of endomorphisms of V as a E_V-vector space can be identified (non-canonically, depending upon a choice of basis) with the ring $\mathcal{M}_{n_V}(D_V)$ of matrices of size $n_V \times n_V$ with entries in the opposite algebra D_V of E_V. The natural ring homomorphism $\mathfrak{A} \to \mathrm{End}_{E_V}V$ is a surjection (density theorem).

There is an isomorphism of algebras (Wedderburn's structure theorem):

$$\mathfrak{A} \simeq \prod_V \mathrm{End}_{E_V}V \simeq \prod_V \mathcal{M}_{n_V}(D_V), \tag{3.1}$$

where the product is taken over all (isomorphism classes of) simple modules. There is a single isomorphism class of simple modules for the simple algebra $\mathrm{End}_{E_V}V$, namely that of V itself, and its multiplicity is n_V in a direct sum decomposition into simples of the right regular representation of $\mathrm{End}_{E_V}V$. Thus n_V is also the multiplicity of V in the right regular representation of $\mathfrak{A}$. And of course

$$\dim_k V = n_V(\dim_k E_V) \geq n_V \tag{3.2}$$

The hypothesis of the following proposition admittedly appears contrived at first sight, but it will soon be apparent (in §3.3.5) that it is tailor-made for our purpose.

Proposition 3.1.1 *Let $W_1, \ldots, W_s$ be $\mathfrak{A}$-modules of respective dimensions $d_1, \ldots, d_s$ over k. Suppose that the right regular representation of $\mathfrak{A}$ has a filtration in which the quotients are precisely $W_1^{\oplus d_1}, \ldots, W_s^{\oplus d_s}$. Then*

1. *$\mathrm{End}_{\mathfrak{A}}W_i = k$ and W_i is absolutely irreducible, $\forall\ i$, $1 \leq i \leq s$.*
2. *W_i is not isomorphic to W_j for $i \neq j$.*
3. *W_i, $1 \leq i \leq s$, are a complete set of simple $\mathfrak{A}$-modules.*
4. *$\mathfrak{A} \simeq \prod_{i=1}^{s} \mathrm{End}_k W_i$.*

Proof: Let V be a simple submodule of W_i. Then $\dim_k V \leq d_i$. The hypothesis about the filtration implies that the multiplicity of V in the right regular representation is at least d_i. From Eq. (3.2), we conclude that $d_i = n_V$ and $\dim_k E_V = 1$. So $V = W_i$ is simple and $E_V = k$. If $\overline{k}$ denotes an algebraic closure of k, then

$$\mathrm{End}_{\mathfrak{A}\otimes_k \overline{k}}(V \otimes_k \overline{k}) = (\mathrm{End}_{\mathfrak{A}}V) \otimes_k \overline{k} = k \otimes_k \overline{k} = \overline{k}.$$

So V is absolutely irreducible and (1) is proved.

If $W_i \simeq W_j$ for $i \neq j$, then the multiplicity of W_i in the right regular representation would exceed d_i contradicting Eq. (3.2). This proves (2). Since every simple module has positive multiplicity in the right regular representation, (3) is clear. Finally, (4) follows from (1) and Eq. (3.1). $\square$

3.2 Some $\mathfrak{S}_n$-modules

Let $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_r) \vdash n$. A *tabloid* of shape λ is a partition of the set $[n] := \{1, \ldots, n\}$ into an ordered r-tuple of subsets, the first consisting of λ_1 elements, the second of λ_2 elements, and so on. The members of the first subset are arranged in increasing order in the first row, those of the second subset in the second row, and so on. Depicted below are two tabloids of shape $(3, 3, 2)$:

$$\begin{array}{ccc} \hline 1 & 3 & 5 \\ \hline 7 & 8 & 9 \\ \hline 4 & 6 & \\ \cline{1-2} \end{array} \qquad \begin{array}{ccc} \hline 3 & 5 & 8 \\ \hline 1 & 6 & \\ \hline 2 & 7 & \\ \cline{1-2} \end{array}$$

Notation 3.2.1 Given a Young tableau T of shape λ (§2.3.1), it determines, in the obvious way, a tabloid of shape λ denoted $\{T\}$: the first subset consists of the elements in the first row, the second of those in the second row, and so on.

Tabloid representations

The defining action of $\mathfrak{S}_n$ on $[n]$ induces, in the obvious way, an action on the set $\mathcal{T}_\lambda$ of tabloids of shape λ. The free $\mathbb{Z}$-module $\mathbb{Z}\mathcal{T}_\lambda$ with $\mathcal{T}_\lambda$ as a $\mathbb{Z}$-basis becomes therefore a linear representation of $\mathfrak{S}_n$ over $\mathbb{Z}$. By base change, we get such a representation over any commutative ring with unity k: $k\mathcal{T}_\lambda := \mathbb{Z}\mathcal{T}_\lambda \otimes_{\mathbb{Z}} k$. We call it the *tabloid representation* corresponding to the shape λ.

Specht modules

The Specht module corresponding to a partition $\lambda \vdash n$ is a certain $\mathfrak{S}_n$-submodule of the tabloid representation $\mathbb{Z}\mathcal{T}_\lambda$ defined as follows:

Define the elements $\mathfrak{e}_T$ in $\mathbb{Z}\mathcal{T}_\lambda$ for tableaux T of shape λ as:

$$\mathfrak{e}_T := \sum \epsilon(\sigma)\{T\sigma\}$$

where the sum is taken over permutations σ of $\mathfrak{S}_n$ in the column stabiliser of T, $\epsilon(\sigma)$ denotes the sign of σ, and $\{T\sigma\}$ denotes the tabloid corresponding to the tableau $T\sigma$ (refer Notation 3.2.1). The *Specht module* $\mathcal{S}^\lambda$ is the linear span of the $\mathfrak{e}_T$ as T runs over all tableaux of shape λ. It is an $\mathfrak{S}_n$-submodule of $\mathbb{Z}\mathcal{T}_\lambda$ with $\mathbb{Z}$-basis $\mathfrak{e}_T$, as T varies over standard tableaux ([Sag01, Theorem 2.6.5]). It is therefore a free $\mathbb{Z}$-module of rank equal to the number $d(\lambda)$ of standard tableaux of shape λ (§2.3.1). By base change, we get the Specht module $\mathcal{S}_k^\lambda$ over any commutative ring with identity k: $\mathcal{S}_k^\lambda := \mathcal{S}^\lambda \otimes_{\mathbb{Z}} k$, which evidently is free over k .

Over the field $\mathbb{C}$, the Specht modules are irreducible and in fact they are all the irreducible modules of $\mathfrak{S}_n$ (see for example, [Sag01, Theorem 2.4.6]). Moreover, for the tabloid module $\mathbb{C}\mathcal{T}_\lambda$, we have a decomposition into irreducibles as stated below (see [Sag01, Theorem 2.11.2] for proof):

Proposition 3.2.2 *There are positive integers $k_{\mu\lambda}$ for $\mu \trianglerighteq \lambda$, such that $\mathbb{C}\mathcal{T}_\lambda \cong \oplus_{\mu\trianglerighteq\lambda} k_{\mu\lambda}\mathcal{S}^\mu_{\mathbb{C}}$.* $\square$

3.3 Some $\mathcal{H}$-modules: Definitions and preliminaries

Let $\lambda \vdash n$ be fixed. Recall from §2.3.4 that λ' denotes the transpose of λ; the permutation w_λ takes t^λ to t_λ. Let W_λ be the row stabilizer of t^λ. Then $\mathfrak{D}_\lambda := \{w \in \mathfrak{S}_n \mid t^\lambda w \text{ is row standard}\}$ is a set of right coset representatives of W_λ in $\mathfrak{S}_n$ (Proposition 2.3.10(1)).

We set,

$$x_\lambda := \sum_{w\in W_\lambda} v_w T_w \qquad y_\lambda := \sum_{w\in W_\lambda} \epsilon_w v_w^{-1} T_w \qquad z_\lambda := v_{w_\lambda} x_\lambda T_{w_\lambda} y_{\lambda'}$$

Using Corollary 2.2.9(2), we note that

$$x_\lambda = v_{w_{0,\lambda}} C'_{w_{0,\lambda}} \qquad\qquad y_\lambda = \epsilon_{w_{0,\lambda}} v^{-1}_{w_{0,\lambda}} C_{w_{0,\lambda}} \tag{3.3}$$

By means of these elements we define certain $\mathcal{H}$-modules which will be of importance in our study. Before we begin with the definitions of these modules, we make the following combinatorial observation that will see us through many proofs later:

Lemma 3.3.1 [DJ86, Lemma 4.1]) *Let $\lambda \vdash n$ and $w \in \mathfrak{S}_n$ such that $x_\lambda T_w y_{\lambda'} \neq 0$. Then $x_\lambda T_w y_{\lambda'} = \pm v^i x_\lambda T_{w_\lambda} y_{\lambda'} \neq 0$ for some non-negative integer i.*

Proof: In view of Lemma 2.3.15, we may assume that $w \in \mathfrak{D}_\lambda \cap \mathfrak{D}^{-1}_{\lambda'}$. Suppose that $w^{-1}W_\lambda w \cap W_{\lambda'} \neq (1)$, then by Lemma 2.3.16 there exists an element $s \in w^{-1}W_\lambda w \cap W_{\lambda'} \cap S$. Let D be a set of minimal length coset representatives of the subgroup $\{1, s\}$ in $W_{\lambda'}$. Then $y_{\lambda'} = (T_1 - vT_s)\sum_{d\in D} \epsilon_d v_d T_d$. Inserting this expression in $x_\lambda T_w y_{\lambda'}$, we get

$$x_\lambda T_w y_{\lambda'} = (x_\lambda T_w T_1 - v^{-1} x_\lambda T_w T_s) \sum_{d\in D} \epsilon_d v_d^{-1} T_d$$

Since $s \in w^{-1}W_\lambda w \cap W_{\lambda'}$, there exists a $u \in W_\lambda$ such that $ws = uw$, and hence $u \in S$ and $T_w T_s = T_u T_w$. Now using the fact that for $u \in W_\lambda \cap S$, $x_\lambda T_u = v x_\lambda$ we deduce that $(x_\lambda T_w T_1 - v^{-1} x_\lambda T_w T_s)$ in the above expression is 0, leading to a contradiction to the hypothesis that $x_\lambda T_w y_{\lambda'} \neq 0$. Hence we have $w^{-1}W_\lambda w \cap W_{\lambda'} = (1)$. Noticing that $w^{-1}W_\lambda w$ is the row stabilizer of the (row-standard) tableau $t^\lambda w$ the last condition holds only if every element in a column in t_λ occurs in different rows of $t^\lambda w$. Now it is easy to conclude that this is possible if and only if $t^\lambda w = t_\lambda$, so that $w = w_\lambda$. Hence the claim. $\square$

Remark 3.3.2 Let $\mu = (\mu_1, \ldots, \mu_r)$ be a *composition* of n, *i.e.*, $\mu_i > 0$ for all $i = 1, \ldots r$ and $\mu_1 + \cdots + \mu_r = n$. Then μ defines a subgroup W_μ of all permutations in $\mathfrak{S}_n$ which fix the subsets $S_1, \ldots, S_r$ of sizes $\mu_1, \mu_2 \ldots, \mu_r$ respectively consisting of numbers $1, \ldots, \mu_1$; $\mu_1 + 1, \ldots, \mu_1 + \mu_2; \ldots$ in that order. Define μ'_i to be the number of subsets S_j having atleast i elements. Clearly, $(\mu'_1, \mu'_2, \ldots)$ is a partition of n. Now, let $y_\mu := \sum_{u \in W_\mu} \epsilon_u v_u T_u$ then the proof of the above lemma can be imitated to show that if $x_\lambda T_w y_\mu \neq 0$ then $w^{-1} W_\lambda w \cap W_\mu = (1)$ ([DJ86, Lemma 4.1]). This condition means that the elements of S_i are all in different rows of the row-standard tableau $t^\lambda w$. Then, noticing that the numbers in S_i are all smaller than the numbers in S_{i+1}, we conclude that $\mu' \trianglerighteq \lambda$.

Notation 3.3.3 It will be convenient for us to fix the following notation. For a subset $\mathfrak{S}$ of $\mathfrak{S}_n$, denote by $\langle C_y | y \in \mathfrak{S} \rangle_A$ the A-span of $\{C_y | y \in \mathfrak{S}\}$ in $\mathcal{H}$. For an A-algebra k, denote by $\langle C_y | y \in \mathfrak{S} \rangle_k$ the k-span of $\{C_y | y \in \mathfrak{S}\}$ in $\mathcal{H}_k$. Similar meanings are attached to $\langle T_y | y \in \mathfrak{S} \rangle_A$ and $\langle T_y | y \in \mathfrak{S} \rangle_k$.

Notation 3.3.4 Let k be a ring, a an invertible element in k. We already defined $\mathcal{H}_k$ (§2.2). Also, for any $\mathcal{H}$-module N, we denote by N_k the $\mathcal{H}_k$-module $N \otimes_A k$ obtained by extending scalars via the homomorphism $A \to k$ given by $v \mapsto a$. It will also be convenient to denote by n, the element $n \otimes 1 \in N_k$.

3.3.1 Permutation modules M^λ

Following [DJ86], we define the *permutation module* M^λ to be the right ideal $x_\lambda \mathcal{H}$. The basic properties of M^λ are presented in the following lemma ([DJ86, Lemma 3.2]):

Proposition 3.3.5 *With notations as above,*

1. *The module M^λ is a free A-module with basis $\{x_\lambda T_d \mid d \in \mathfrak{D}_\lambda\}$.*
2. *If $d \in \mathfrak{D}_\lambda$ and $s \in S$, then*

$$x_\lambda T_w T_s = \begin{cases} x_\lambda T_{ws} & \text{if } l(ws) = l(w) + 1 \text{ and } ws \in \mathfrak{D}_\lambda \\ (v - v^{-1}) x_\lambda T_w + x_\lambda T_{ws} & \text{if } l(ws) = l(w) - 1 \text{ and } ws \in \mathfrak{D}_\lambda \\ v_s x_\lambda T_w & \text{if } ws \notin \mathfrak{D}_\lambda \end{cases} \tag{3.4}$$

PROOF: Since T_w, $w \in \mathfrak{S}_n$ form a basis of $\mathcal{H}$, it is obvious that the elements $x_\lambda T_w$, $w \in \mathfrak{S}_n$ span the module $x_\lambda \mathcal{H}$ over A. On the other hand, every element $w \in \mathfrak{S}_n$ can be expressed uniquely as a product ud where $u \in W_\lambda$, $d \in \mathfrak{D}_\lambda$ such that $l(ud) = l(u) + l(d)$ (Proposition 2.3.10). So, we have $T_{ud} = T_u T_d$. By Lemma 2.2.10 we know that $C'_{w_{0,\lambda}} T_s = v C'_{w_{0,\lambda}}$ for $s \in W_\lambda \cap S$. Putting all this together, justifies the claim that the collection $x_\lambda T_d$, $d \in \mathfrak{D}_\lambda$ is enough to span $x_\lambda \mathcal{H}$. Computing $x_\lambda T_d$ by inserting the expression for x_λ in it, we get $x_\lambda T_d = \sum_{w \in W_\lambda} v_w T_w T_d$. Since $l(wd) = l(w) + l(d)$ for all $w \in W_\lambda$, we have $T_w T_d = T_{wd}$. The linear independence now follows from the linear independence of T_x, $x \in \mathfrak{S}_n$ and the fact that the cosets $W_\lambda d$, $d \in \mathfrak{D}_\lambda$ are disjoint. This proves (1).

By Deodhar's lemma and its proof, we know that for $s \in S$ and $d \in \mathfrak{D}_\lambda$, the element ds is either in $\mathfrak{D}_\lambda$ or else $ds > d$ and $ds = s'd$ for some $s' \in W_\lambda \cap S$. Statement (2) follows from these observation along with the multiplication rule given in (2.3). $\square$

The bilinear form $\langle \ , \ \rangle$ on M^λ:

As in [DJ86, page 34], define a A-bilinear form $\langle \ , \ \rangle$ on M^λ by defining it on the A-basis described by Proposition 3.3.5 as follows: set $\langle x_\lambda T_d, x_\lambda T_e \rangle$ equal to 1 or 0 accordingly as elements d, e of $\mathfrak{D}_\lambda$ are equal or not. The form is evidently symmetric.

Recall the A-linear, anti-automorphism of the algebra $\mathcal{H}$: $h \mapsto h^*$ given by $T_w \mapsto T_{w^{-1}}$ (see Remark 2.2.7). We have,

Lemma 3.3.6 *Let m_1, $m_2 \in M^\lambda$ and $h \in \mathcal{H}$. Then*

$$\langle m_1 h, m_2 \rangle = \langle m_1, m_2 h^* \rangle \tag{3.5}$$

Proof: We may assume that $h = T_s$ for some $s \in S$ and that $m_1 = x_\lambda T_d$, $m_2 = x_\lambda T_e$ for some $d, e \in \mathfrak{D}_\lambda$. Then we are reduced to showing that $\langle x_\lambda T_d T_s, x_\lambda T_e \rangle = \langle x_\lambda T_d, x_\lambda T_e T_s \rangle$. By the definition of the bilinear form, both sides of this equation turns out to be 0 unless $e = d$ or $e = ds$. In the case when $e = d$, the equation is valid owing to the symmetry of the bilinear form while if $e = ds$ we can assume that $d < ds$. Then as $ds = e \in \mathfrak{D}_\lambda$ we use the appropriate relation in (3.4) to verify the validity of the equation. $\square$

3.3.2 Specht modules S^λ

The *Specht module* S^λ is defined to be the right ideal $z_\lambda \mathcal{H}$. By Lemma 3.3.1 we know that this is a non-zero submodule of M^λ. It is a free A-module with basis given by,

Theorem 3.3.7 *The set* $\{v_d z_\lambda T_d \mid t^\lambda w_\lambda d$ is a standard tableau$\}$ *forms an A basis for S^λ.* (see [DJ86, Theorem 5.6] for proof)[1] $\square$

The basis given by the above theorem is called the *standard basis* of S^λ. The Specht modules were first defined in the seminal work of Dipper and James [DJ86] where they also had proved that the above set forms a basis. Showing the linear independence over A of this set is almost straight forward, given the Lemma 3.3.8 below. However, the proof that these elements span the space $z_\lambda \mathcal{H}$ is much more involved and we don't give it here.

Lemma 3.3.8 *For d prefix of $w_{\lambda'}$, the coefficient of $x_\lambda T_{w_\lambda d}$ in $z_\lambda T_d$ is a unit in A and the other terms $x_\lambda T_u$, $u \in \mathfrak{D}_\lambda$, involved in $z_\lambda T_d$ satisfy $l(u) > l(w_\lambda d)$.*

[1] This basis is exactly the same as that given in [DJ86]. The leading scalar factor v_d in the above basis appears due to the notational difference with [DJ86] (Refer §2.2, Remark on notation).

PROOF: Inserting the expression for $y_{\lambda'}$ in z_λ (see beginning of §3.3 for definition of $y_{\lambda'}$), we get that

$$z_\lambda = x_\lambda T_{w_\lambda} + \sum_{w \in W_{\lambda'} \setminus \{1\}} \epsilon_w v_w^{-1} x_\lambda T_{w_\lambda w}$$

Since $w_\lambda \in \mathfrak{D}_{\lambda'}^{-1}$, we have $l(w_\lambda w) = l(w_\lambda) + l(w)$ for all $w \in W_{\lambda'}$ (Proposition 2.3.10(3)).

By Lemma 2.2.4(4) we know that the product $T_{w_\lambda w} T_d$ expressed in terms of the T-basis consists only of terms T_u such that $u \geq w_\lambda w d$. Note that, if $x > y$ then every reduced expression for x contains a subexpression which is reduced for y (see [Lus03, Proposition 2.4] for example). Using this and the properties of $\mathfrak{D}_\lambda$ (Proposition 2.3.10) it is easy to see that, if $x > y$ and $x \in W_\lambda d'$, $y \in W_\lambda d''$ where $d', d'' \in \mathfrak{D}_\lambda$ then $l(d') > l(d'')$. Applying this observation to u such that T_u appears in the product $T_{w_\lambda w} T_d$, it can be deduced that $x_\lambda T_{w_\lambda w} T_d$ can be expressed as a linear combination of $x_\lambda T_{d'}$, $d' \in \mathfrak{D}_\lambda$ such that $l(d') \geq l(w_\lambda w) - l(d)$ (notice that $w_\lambda w \in \mathfrak{D}_\lambda$ for $w \in W_{\lambda'}$). The fact that $w_{\lambda'} = w_\lambda^{-1}$, in turn implies that $l(w_\lambda w) - l(d) = l(w_\lambda d) + l(w)$. Thus we conclude that, by applying T_d to the relation for z_λ obtained above, we get

$$z_\lambda T_d = x_\lambda T_{w_\lambda d} + \sum_{l(u) > l(w_\lambda d)} \epsilon_u v_u^{-1} x_\lambda T_u$$

as required. □

The bilinear form $\langle\ ,\ \rangle$ on S^λ:

Owing to the fact that S^λ is a submodule of M^λ, we can restrict the bilinear form $\langle , \rangle$ defined on M^λ (§3.3.1) to the Specht module. Moreover, we have the following remarkable property of this form, that goes under the name "The submodule theorem".

For a submodule U of M^λ, we define $U^\perp := \{m \in M^\lambda | \langle m, u \rangle = 0 \text{ for all } u \in U\}$.

Theorem 3.3.9 (The Submodule Theorem) *Let F be a field, $a \in F$ be invertible. Let $\mathcal{H}_F$ be the specialization of $\mathcal{H}$ via $v \mapsto a$. Let U be a submodule of M_F^λ. Then $S_F^\lambda \subset U$ or $U \subset S_F^{\lambda^\perp}$.*

PROOF: Let $m \in U \subset M^\lambda$. Then by Lemma 3.3.1, $m y_{\lambda'} = r z_\lambda$ for some $r \in F$. Therefore, if $m y_{\lambda'} \neq 0$ for some $m \in U$ then $z_\lambda \in U$ and hence $S_F^\lambda \subset U$. If $m y_{\lambda'} = 0$ for all $m \in U$ then

$$\langle z_\lambda h, m \rangle = \langle x_\lambda T_{w_\lambda}, m h^* y_\lambda^* \rangle \ = \langle x_\lambda T_{w_\lambda}, 0 \rangle \ = 0$$

for all $h \in \mathcal{H}_F$. Hence $U \subset S_F^{\lambda^\perp}$. □

Corollary 3.3.10 *Let $F = \mathbb{Q}(v)$. Then S_F^λ is irreducible for all partitions $\lambda \vdash n$.*

PROOF: We begin with the observation that $S_F^\lambda \cap S_F^{\lambda^\perp}$ is either S_F^λ or the unique maximal proper submodule of S_F^λ. Indeed, if U is a proper submodule of S_F^λ then applying the above theorem to U we obtain that $U \subset S_F^{\lambda^\perp}$ hence in the intersection.

In order to prove the irreducibility of S_F^λ it would therefore suffice to show that $S_F^\lambda \cap S_F^{\lambda^\perp} = (0)$. Suppose $m \in S_F^\lambda \cap S_F^{\lambda^\perp}$, so that $\langle m, m\rangle = 0$. Since $m \in M_F^\lambda$, we can write $m = \sum_{d\in\mathfrak{D}_\lambda} r_d x_\lambda T_d$, where $r_d \in \mathbb{Q}(v)$ so by the definition of the form on M_F^λ, $\langle m, m\rangle = \sum_{d\in\mathfrak{D}_\lambda}(r_d)^2 = 0$. Multiplying by a suitable polynomial in v to clear denominators, we can modify m to assume that $r_d \in \mathbb{Q}[v]$ so that $\sum_{d\in\mathfrak{D}_\lambda}(r_d)^2 \in \mathbb{Q}[v]$. By inserting $v = a \in \mathbb{Q}^+$ we notice that the sum of positive rationals is 0. Hence each $r_d(a) = 0$ for all $a \in \mathbb{Q}^+$ which is possible only if the polynomials $r_d(v)$ $(d \in \mathfrak{D}_\lambda)$ were identically 0. Thus $m = 0$, as required. □

A criterion for $\langle , \rangle$ to be non-zero on S_F^λ

The next proposition gives a criteria for the bilinear form on S^λ, defined above, to be non-zero over a field F. A shape λ is *e-regular* if for each $i \in \mathbb{N}$, the number of parts of λ that equal i is less than e.

Proposition 3.3.11 ([DJ86, Theorem 6.3(i)]) *Let F be a field. Let $a \in F$ be an invertible element. Denote by e the smallest integer such that $1 + a^2 + \cdots + a^{2(e-1)} = 0$. Let $\lambda \vdash n$ be e-regular. Then there exist elements e_1, $e_2 \in S_F^\lambda$ such that $\langle e_1, e_2\rangle \neq 0$.* □

The proof of the above proposition involves the construction of elements e_1, $e_2 \in S_F^\lambda$ such that the only terms $x_\lambda T_u$ $(u \in \mathfrak{D}_\lambda)$ common to both e_1 and e_2, when expressed as a linear combination of the basis of M_F^λ, are of the form $x_\lambda T_{w_\lambda w}$ where $w \in W_{\lambda'}$ such that w permutes the numbers appearing in rows of the same length in t_λ among themselves. The details of the construction can be found in [DJ86].

3.3.3 Relating M^λ with $\mathbb{Z}\mathcal{T}_\lambda$

Before proceeding further, it would be appropriate to pause and note that the modules M^λ and S^λ defined above are analogues of $\mathbb{Z}\mathcal{T}_\lambda$ and $\mathcal{S}^\lambda$ for $\mathfrak{S}_n$ in the sense that, if we specialize M^λ and S^λ to $\mathbb{Z}$ via the map given by $v \mapsto 1$ (refer to §2.2 and Notation 3.3.4 to recall what we mean by specialization) and denote the modules obtained thus by $\tilde{M^\lambda}$ and $\tilde{S^\lambda}$, then we have the following:

Proposition 3.3.12 *There exists an $\mathfrak{S}_n$-isomorphism between the modules $\tilde{M^\lambda}$ and $\mathbb{Z}\mathcal{T}_\lambda$. Further, the restriction of this isomorphism to $\tilde{S^\lambda}$ gives an isomorphism of $\tilde{S^\lambda}$ with the $\mathfrak{S}_n$-module $\mathcal{S}^\lambda$.*

Proof: Let $\rho_\lambda{:=}\sum_{w\in W_\lambda} w$ and $\kappa_\lambda{:=}\sum_{w\in W_\lambda} \epsilon_w w$. Under the identification of $\mathcal{H}\otimes_A \mathbb{Z}$ with $\mathbb{Z}\mathfrak{S}_n$, the elements $x_\lambda \otimes 1 \mapsto \rho_\lambda$ and $y_\lambda \otimes 1 \mapsto \kappa_\lambda$. So $\tilde{M^\lambda}$ is identified with $\rho_\lambda\mathbb{Z}\mathfrak{S}_n$. Now define

$$\theta : \rho_\lambda\mathbb{Z}\mathfrak{S}_n \to \mathbb{Z}\mathcal{T}_\lambda$$

as the mapping $\rho_\lambda w \mapsto \{t^\lambda\}w$ for $w \in \mathfrak{S}_n$. This is clearly an $\mathfrak{S}_n$-isomorphism of the right ideal $\rho_\lambda\mathbb{Z}\mathfrak{S}_n$ with $\mathbb{Z}\mathcal{T}_\lambda$. This proves the first part of the Proposition.

Suppose κ_t denotes the signed sum over the elements of the column stabilizer of a tableau t. Then it is easy to see that for $w \in \mathfrak{S}_n$

$$\kappa_{tw} = w^{-1}\kappa_t w \tag{3.6}$$

Therefore, $\kappa_\lambda = \kappa_{t^\lambda} = w_\lambda \kappa_{t^\lambda w_\lambda} w_\lambda^{-1}$, which leads us to the relation $\rho_\lambda\kappa_\lambda = \rho_\lambda w_\lambda \kappa_{t^\lambda w_\lambda} w_\lambda^{-1}$. Theorem 3.3.7 gives a basis for $\tilde{S^\lambda}$, which under its identification as a right ideal in $\mathbb{Z}\mathfrak{S}_n$ is given by $\rho_\lambda w_\lambda \kappa_{t^\lambda w_\lambda} d$ where d is a prefix of $w_{\lambda'}$. By (3.6) , we get $\rho_\lambda w_\lambda \kappa_{t^\lambda w_\lambda} d = \rho_\lambda \kappa_{t^\lambda} w_\lambda d$. The image of the latter element under the map θ is $\{t^\lambda\}\kappa_{t^\lambda} w_\lambda d$ which by (3.6) can be rewritten as $\{t^\lambda\} w_\lambda d \kappa_{t^\lambda w_\lambda d} = \mathfrak{e}_{t^\lambda w_\lambda d}$. However, for each prefix d of $w_{\lambda'}$ the element $w_\lambda d$ denotes a prefix of w_λ. Hence by Lemma 2.3.18 the tableau $t^\lambda w_\lambda d$ is a standard. Thus the image of the standard basis elements of $\tilde{S^\lambda}$ is precisely the basis $\{\mathfrak{e}_t |\ t \text{ standard } \lambda\text{-tableau}\}$ of $\mathcal{S}^\lambda$, as required. □

3.3.4 Monomial module

Let $\lambda \vdash n$. The monomial module corresponding to λ is defined to be the right ideal $y_\lambda\mathcal{H}$ (see beginning of §3.3 for definition of y_λ). By (3.3), it is obvious that this is the same as the right ideal, $C_{w_{0,\lambda}}\mathcal{H}$. This module again is free over A, as will be seen below. It has two A-bases which are of interest to us. The next two Propositions present for us these bases. Note that since $j(C_{w_{0,\lambda}}) = \epsilon_{w_{0,\lambda}} C'_{w_{0,\lambda}}$, by Equation (3.3) we have $j(M^\lambda) = j(x_\lambda)\mathcal{H} = y_\lambda\mathcal{H}$. Thus, Proposition 3.3.5 readily leads us to the T-*basis* of $y_\lambda\mathcal{H}$, described explicitly below:

Proposition 3.3.13 *The set $\{C_{w_{0,\lambda}} T_d |\ d \in \mathfrak{D}_\lambda\}$ forms a basis for $C_{w_{0,\lambda}}\mathcal{H}$ over A. It is called the T-basis of $C_{w_{0,\lambda}}\mathcal{H}$.* □

Considering the definition of the relation $\leq_{\mathrm{R}}$ and Remark 2.2.11 given there, it is clear that $C_w\mathcal{H} \subset \langle C_y | y \leq_{\mathrm{R}} w\rangle_A$. In case of the longest element $w_{0,\lambda}$ of W_λ, in fact, equality holds. This is done in the Proposition below and we call this basis the C-*basis* of $C_{w_{0,\lambda}}\mathcal{H}$.

Proposition 3.3.14 ([MP05, Lemma 2.11]) *The A-span $\langle C_w | w \leq_{\mathrm{R}} w_{0,\lambda}\rangle_A$ of the elements C_w, $w \leq_{\mathrm{R}} w_{0,\lambda}$, equals the right ideal $C_{w_{0,\lambda}}\mathcal{H}$.*

PROOF: By the definition of $\leq_{\mathrm{R}}$, the inclusion $C_{w_{0,\lambda}}\mathcal{H} \subset \langle C_w | w \leq_{\mathrm{R}} w_{0,\lambda}\rangle_A$ is immediate. To prove the reverse inclusion we use Proposition 2.3.12 and Lemma 2.3.18 together to deduce that the set $\{w |\ w \leq_{\mathrm{R}} w_{0,\lambda}\} = \{w_{0,\lambda} d \mid d \text{ prefix of } d_\lambda\}$, which we denote as $\mathfrak{C}_\lambda$. Inducting on the $l(d)$, we show that $C_{w_{0,\lambda} d} \in \langle C_w | w \leq_{\mathrm{R}} w_{0,\lambda}\rangle_A$ for all prefixes d of d_λ as follows:

When $d = 1$ the claim is obvious. Assuming the induction hypothesis for all $d' \in \mathfrak{C}_\lambda$ such that $l(d') < l(d)$, we consider the element $C_{w_{0,\lambda} d}$ for some prefix d of d_λ. Writing $d = es$, where e is itself a prefix of d_λ and $l(es) = l(e) + 1$, we get from (2.6)

$$C_{w_{0,\lambda} d} = C_{w_{0,\lambda} e} C_s - \sum_{zs < z < w_{0,\lambda} e} \mu(z, w_{0,\lambda} e) C_z$$

Every z for which C_z appears in the last term of the above expression has to satisfy $z \leq_{\mathrm{R}} w_{0,\lambda}e \leq_{\mathrm{R}} w_{0,\lambda}$ and $z < w_{0,\lambda}e$. Since $l(e) < l(d)$, by using the induction hypothesis we conclude that all terms in the right hand side of the above expression are in $C_{w_{0,\lambda}}\mathcal{H}$ implying that $C_{w_{0,\lambda}d}$ is also in $C_{w_{0,\lambda}}\mathcal{H}$, as required. $\square$

Order the elements of $\mathfrak{D}_\lambda$ refining the partial order given by their lengths. This naturally induces an ordering on the C-,T-bases of $\mathcal{H}$ since they are both indexed by $\mathfrak{D}_\lambda$. Then the proposition below shows that with respect to this ordering the two bases are in unitriangular relationship with each other.

Lemma 3.3.15 *For each $e \in \mathfrak{D}_\lambda$, $C_{w_{0,\lambda}}T_e = \sum_{d\in\mathfrak{D}_\lambda, d\leq e} g_{e,d}C_{w_{0,\lambda}d}$ where $g_{e,e} = 1$ and $g_{e,d} = 0$ if $d \not\leq e$.*

PROOF: We once again use the description of the set $\{w|\ w \leq_{\mathrm{R}} w_{0,\lambda}\}$ as the collection of elements $w_{0,\lambda}d$ such that $d \in \mathfrak{D}_\lambda$.

Proceed by induction on the $l(e)$, $e \in \mathfrak{D}_\lambda$. When $e = 1$ the claim is obvious. So, let $l(e) \geq 1$. Write $e = ds$ such that $d \in \mathfrak{D}_\lambda$ and $l(ds) = l(d) + 1$. Using the expression for $C_{w_{0,\lambda}}T_d$ assumed by the induction hypothesis we get that

$$C_{w_{0,\lambda}}T_e = C_{w_{0,\lambda}}T_dT_s = C_{w_{0,\lambda}d} + \sum_{d'\in\mathfrak{D}_\lambda, d'<d} g_{d,d'}C_{w_{0,\lambda}d'}T_s$$

By induction hypothesis it is clear that the only term in the right-hand side of the above relation that contributes to the coefficient of $C_{w_{0,\lambda}ds}$ is $C_{w_{0,\lambda}d}T_s$. So, inserting the expression $C_{w_{0,\lambda}ds} + C_{w_{0,\lambda}d} + \sum_{zs<z<w_{0,\lambda}d} \mu(z, w_{0,\lambda}d)C_z$ in place of $C_{w_{0,\lambda}d}T_s$ (see Equation (2.6)) in the right-hand side of the above equation we conclude that the coefficient of $C_{w_{0,\lambda}e}$ in $C_{w_{0,\lambda}}T_e$ is $g_{e,e} = g_{d,d} = 1$. The rest of the claim is obvious by length considerations. $\square$

3.3.5 Cell modules

It follows from the definition of the pre-order $\leq_{\mathrm{L}}$ that the A-span $\langle C_y|y \leq_{\mathrm{L}} w\rangle_A$ of $\{C_y|y \leq_{\mathrm{L}} w\}$, for w in $\mathfrak{S}_n$ fixed, is a left ideal of $\mathcal{H}$; so is $\langle C_y|y <_{\mathrm{L}} w\rangle_A$. The quotient $L(w) := \langle C_y|y \leq_{\mathrm{L}} w\rangle_A / \langle C_y|y <_{\mathrm{L}} w\rangle_A$ is called the *left cell module* associated to w. It is a left $\mathcal{H}$-module. Right cell modules $R(w)$ and two sided cell modules are defined similarly. They are right modules and bimodules respectively. From the way these modules are defined, it is clear that the images of the elements C_y, $y \sim_{\mathrm{L}} w$ (resp., C_y, $y \sim_{\mathrm{R}} w$) form a basis for $L(w)$ (resp., $R(w)$), which is called the *C-basis* of the cell module. For notational convenience we shall continue to use C_y for the image of C_y, $y \sim_{\mathrm{L}} w$ in $L(w)$. Similarly in $R(w)$.

Remark 3.3.16 Alternatively, one may choose to use the C'-basis of $\mathcal{H}$ to define the cell modules. For $w \in \mathfrak{S}_n$, if we denote by $L(w)^\diamond$ the left C'-cell module defined analogously as above. Then by (2.7), it is immediate that $L(w)^\diamond$ is just the image of $L(w)$ under the map induced by the involution j. Similarly the image of $R(w)$ under j is the right C'-cell module, denoted as $R(w)^\diamond$. It should be noted at this point that the involution j is not an

A-algebra homomorphism, so it does not induce an $\mathcal{H}$-isomorphism between the $R(w)$ and $R(w)^\diamond$. However, in due course we describe an isomorphism between $R(w)$ and $R(w_0w)^\diamond$ after extending scalars to $\mathbb{Q}(v)$ (see Lemma 3.4.1 below).

Let y and w be permutations of the same RSK-shape λ. The left cell modules $L(y)$ and $L(w)$ are then $\mathcal{H}$-isomorphic. In fact, the association $C_{(P,Q(y))} \leftrightarrow C_{(P,Q(w))}$ gives an isomorphism: see [KL79, §5], [Gec06, Corollary 5.8]. The right cell modules $R(y)$ and $R(w)$ are similarly isomorphic, and we sometimes write $R(\lambda)$ for $R(y) \simeq R(w)$. Analogous statements hold also for the left/right C'-cell modules and we write $R(\lambda)^\diamond$ for $R(y)^\diamond \simeq R(w)^\diamond$.

When a homomorphism from A to a commutative ring k is specified, such notation as $R(w)_k$ and $R(\lambda)_k$ make sense: see §2.2, Notation 3.3.4.

The 'T-basis' of $R(\lambda')$ and its relationship to the C-basis

We have already seen that the right $\mathcal{H}$-module $C_{w_{0,\lambda}}\mathcal{H}$ has two bases: the 'C-basis' and 'T-basis'. And in Lemma 3.3.15 we saw that after a suitable reordering they are in unitriangular relationship with each other. We use this to define the T-basis for $R(\lambda')$

The elements $w \leq_{\mathrm{R}} w_{0,\lambda}$ are precisely $w_{0,\lambda}d$, $d \in \mathfrak{D}_\lambda$ (Proposition 2.3.18 (1)). Let d_1, $\ldots$, d_M be the elements of $\mathfrak{D}_\lambda$ ordered so that $i \leq j$ if $d_i \leq d_j$ in the Bruhat order. By Lemma 3.3.15 and its proof, the two bases above are related by a uni-triangular matrix with respect to an ordering as above :

$$\begin{pmatrix} C_{w_{0,\lambda}}T_{d_1} \\ \vdots \\ C_{w_{0,\lambda}}T_{d_M} \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ \star & & 1 \end{pmatrix} \begin{pmatrix} C_{w_{0,\lambda}d_1} \\ \vdots \\ C_{w_{0,\lambda}d_M} \end{pmatrix}$$

Let us now read this equation in the quotient $R(w_{0,\lambda})$ of $C_{w_{0,\lambda}}\mathcal{H}$. Let $d_{i_1}, \ldots, d_{i_m}$ with $1 \leq i_1 < \ldots < i_m \leq M$ be such that they are all the prefixes of w_λ—see Lemma 2.3.18 (2)—so that $w_{0,\lambda}d_{i_1}, \ldots, w_{0,\lambda}d_{i_m}$ are all the elements right equivalent to $w_{0,\lambda}$. Writing $e_1, \ldots,$ e_m in place of $d_{i_1}, \ldots, d_{i_m}$, and noting that $C_{w_{0,\lambda}d_j}$ vanishes in $R(\lambda')$ unless $w_{0,\lambda}d_j \sim_{\mathrm{R}} w_{0,\lambda}$, we have:

$$\begin{pmatrix} C_{w_{0,\lambda}}T_{e_1} \\ \vdots \\ C_{w_{0,\lambda}}T_{e_m} \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ \star & & 1 \end{pmatrix} \begin{pmatrix} C_{w_{0,\lambda}e_1} \\ \vdots \\ C_{w_{0,\lambda}e_m} \end{pmatrix}$$

We conclude that

Proposition 3.3.17 *The elements $C_{w_{0,\lambda}}T_{e_1}$, $\ldots$, $C_{w_{0,\lambda}}T_{e_m}$ where $e_1, \ldots, e_m$ are all the prefixes of w_λ form an A-basis for $R(\lambda')$. Further, if the e_i's are ordered such that $i \leq j$ if $e_i \leq e_j$ (where $\leq$ denotes the Bruhat order) then it is in uni-triangular relationship with the C-basis $C_{w_{0,\lambda}e_1}$, $\ldots$, $C_{w_{0,\lambda}e_m}$ of $R(\lambda')$. It is called the T-*basis *of $R(\lambda')$.* □

Irreducibility and other properties of the cell modules

Let k be a field such that $\mathcal{H}_k$ is semisimple. The proof of Theorem 3.3.18 below follows [KL79, §5].

Theorem 3.3.18 ([KL79, §5]) *Assume that $\mathcal{H}_k := \mathcal{H} \otimes_A k$ is semisimple. Then*

1. $\mathrm{End}_{\mathcal{H}_k} R(\lambda)_k = k$ *and* $R(\lambda)_k$ *is absolutely irreducible, for all* $\lambda \vdash n$.
2. $R(\lambda)_k \not\simeq R(\mu)_k$ *for partitions* $\lambda \neq \mu$ *of* n.
3. $R(\lambda)_k$, $\lambda \vdash n$, *are a complete set of simple* $\mathcal{H}_k$*-modules.*
4. $\mathcal{H}_k \simeq \prod_{\lambda \vdash n} \mathrm{End}_k R(\lambda)_k$.

Proof: By Proposition 3.1.1, it is enough to exhibit a filtration of the right regular representation of $\mathcal{H}_k$ in which the quotients are precisely $R(\lambda)_k^{\oplus d(\lambda)}$, $\lambda \vdash n$, each occurring once. We will in fact exhibit a decreasing filtration $\mathfrak{F} = \{F_i\}$ by right ideals (in fact, two sided ideals) of $\mathcal{H}$ in which the quotients F_i/F_{i+1} are precisely $R(\lambda)^{\oplus d(\lambda)}$, $\lambda \vdash n$, each occurring once. Since $R(\lambda)$ are free A-modules, it will follow that $\mathfrak{F} \otimes_A k$ is a filtration of $\mathcal{H}_k$ whose quotients are $R(\lambda)_k^{\oplus d(\lambda)}$, and the proof will be done.

Let $\succeq$ be a total order on partitions of n that refines the dominance partial order $\trianglerighteq$. Let $\lambda_1 \succ \lambda_2 \succ \ldots$ be the full list of partitions arranged in decreasing order with respect to $\succeq$. Set $F_i := \langle C_w | \text{RSK-shape}(w) \preceq \lambda_i \rangle_A$. It is enough to prove the following:

1. The F_i are right ideals in $\mathcal{H}$ (they are in fact two sided ideals).
2. $F_i/F_{i+1} \simeq R(\lambda_i)^{\oplus d(\lambda_i)}$.

It follows from the definition in §2.2.2 of the relation $\leq_{\mathrm{LR}}$ that, for any fixed permutation w, $\langle C_x | x \leq_{\mathrm{LR}} w \rangle_A$ is a two sided ideal of $\mathcal{H}$. But $x \leq_{\mathrm{LR}} w$ if and only if $\text{RSK-shape}(x) \trianglelefteq \text{RSK-shape}(w)$, by the characterization in Proposition 2.3.5. Thus, $\langle C_x | \text{RSK-shape}(x) \trianglelefteq \lambda \rangle_A$ is a two sided ideal, and F_i being equal to the sum $\sum_{j \geq i} \langle C_x | \text{RSK-shape}(x) \trianglelefteq \lambda_j \rangle_A$ of two sided ideals is a two sided ideal. This proves (1).

To prove (2), let S_1, S_2, … be the distinct right cells contained in the two sided cell corresponding to shape λ_i. It follows from the assertions in Proposition 2.3.7 that there are $d(\lambda_i)$ of them and the cardinality of each is $d(\lambda_i)$. Fix a permutation w of shape λ_i. Consider the right cell module $R(w)$, which by definition is the quotient of the right ideal $\langle C_x | x \leq_{\mathrm{R}} w \rangle_A$ by the right ideal $\langle C_x | x \lneq_{\mathrm{R}} w \rangle_A$. If $x \leq_{\mathrm{R}} w$ then evidently $x \leq_{\mathrm{LR}} w$ and (by Proposition 2.3.5) $\text{RSK-shape}(x) \trianglelefteq \lambda_i$, so $\text{RSK-shape}(x) \preceq \lambda_i$. Thus we have a map induced by the inclusion: $\langle C_x | x \leq_{\mathrm{R}} w \rangle_A \to F_i/F_{i+1}$.

We claim that the above map descends to an injective map from the quotient $R(\lambda_i)$. It descends because $x \lneq_{\mathrm{R}} w$ implies $x \lneq_{\mathrm{LR}} w$: if $x \sim_{\mathrm{LR}} w$, then $x \sim_{\mathrm{R}} w$ by Proposition 2.3.6. To prove that the map from $R(\lambda_i)$ is an injection, let $\sum_{x \leq_{\mathrm{LR}} w} a_x C_x$ belong to F_{i+1} with $a_x \in A$. Suppose that $a_x \neq 0$ for some fixed x. Then, since the C_y form an A-basis of $\mathcal{H}$, we conclude that RSK-shape$(x) \preceq \lambda_{i+1}$, so RSK-shape$(x) \neq \lambda_i$, and (by Proposition 2.3.5) $x \lneq_{\mathrm{LR}} w$. But this means $x \not\sim_R w$, so $x \lneq_{\mathrm{R}} w$, and thus the image in $R(\lambda_i)$ of $\sum_{x \leq_{\mathrm{LR}} w} a_x C_x$ vanishes.

The image of $R(w)$ in F_i/F_{i+1} is spanned by the classes $\overline{C_x}$, $x \sim_{\mathrm{R}} w$. Choosing w_1 in S_1, w_2 in S_2, ... we see that the images of $R(w_1)$, $R(w_2)$, ... in F_i/F_{i+1} form a direct sum (for the C_x are an A-basis of $\mathcal{H}$). The $R(w_j)$ are all isomorphic to $R(\lambda)$ (see §3.3.5). This completes the proof of (2) and also of the theorem. □

3.4 McDonough - Pallikaros Isomorphism

The aim of this section is to prove that the Specht module S^λ and the right cell module $R(\lambda)$ corresponding to a partition $\lambda \vdash n$, are isomorphic ([MP05, Theorem 3.5]). We follow the approach in [MP05], which involves proving the existence of an isomorphism after specializing to $F = \mathbb{Q}(v)$, and using this we then prove the required isomorphism over $\mathcal{H}$. For the first step, we will need the following lemma:

Lemma 3.4.1 *For $w \in \mathfrak{S}_n$, $R(w)_F$ and $R(w_0 w)^\diamond_F$ are isomorphic as $\mathcal{H}_F$-modules.*

Proof: Let $\mathfrak{C}$ denote the right cell of w and $\mathfrak{C}'$ denote the right cell of $w_0 w$. Then from Lemma 2.2.12 we notice that $w_0\mathfrak{C}$ equals $\mathfrak{C}'$. Let β, β' be the representations of $\mathcal{H}$ on $R(w)$ and $R(w_0 w)^\diamond$ respectively. From the relation (2.6) and Lemma 2.2.8, we get

$$C_x T_s = \begin{cases} C_x + C_{xs} + \sum_{ys<y<x} \mu(y,x) C_y, & \text{if } xs > x \\ -v^{-1} C_x, & \text{if } xs < x \end{cases}$$

Applying j to the above relations we get the corresponding relations for the C'-basis. Also, note that $\mu(x,y) = \mu(w_0 y, w_0 x)$ for all $x, y \in W$ (see Remark 2.2.13). With these observations and the fact that $y < x$ if and only $w_0 x < w_0 y$ (Lemma 2.2.12) it can be deduced that, after a suitable re-ordering of the bases of $R(w)$ and $R(w_0 w)^\diamond$, the matrices of $\beta(T_s)$ (with respect to the C-basis of $R(w)$), and $\beta'(T_s)$ (with respect to the C'-basis of $R(w_0 w)^\diamond$) are transposes of each other. From this it follows that $\beta(T_w) = \beta'(T_{w^{-1}})^t$ for all $w \in \mathfrak{S}_n$. Hence, if χ_β and $\chi_{\beta'}$ are the characters of β and β' respectively then $\chi_\beta(T_w) = \chi_{\beta'}(T_{w^{-1}})$ for all $w \in \mathfrak{S}_n$. Extending scalars to $\mathbb{Q}$ via $v \mapsto 1$, we get that $\chi^1_\beta(w) = \chi^1_{\beta'}(w^{-1})$ where χ^1_β (resp. $\chi^1_{\beta'}$) denotes the character of the representation β (resp. β') of $\mathcal{H}_\mathbb{Q}$ ($\cong \mathbb{Q}\mathfrak{S}_n$) obtained by specializing. Since w^{-1} is conjugate to w in $\mathfrak{S}_n$ we in fact have, $\chi^1_\beta(w) = \chi^1_{\beta'}(w)$ for all $w \in \mathfrak{S}_n$. Thus, $R(w)_\mathbb{Q}$ and $R(w_0 w)^\diamond_\mathbb{Q}$ are $\mathcal{H}_\mathbb{Q}$-isomorphic.

Let $F = \mathbb{Q}(v)$. By Theorem 3.3.18 we know that the collection $R(\lambda)_F$, $\lambda \vdash n$, are all the non-isomorphic irreducibles for $\mathcal{H}_F$, and by an analogous argument, so is also the collection $R(\lambda)^\diamond_F$, $\lambda \vdash n$. Suppose that $R(w)_F \ncong R(w_0 w)^\diamond_F$ then we get that for some $\mu \neq \lambda' =$ RSK-shape$(w_0 w)$, $R(\mu)^\diamond_F$ has to be isomorphic to $R(\lambda)_F$ (observe that RSK-shape(w)=λ). In particular, this would mean that the characters associated to the representations $R(\mu)_F^\diamond$ and $R(\lambda)_F$ have to be equal. However, these characters evaluated on $\mathcal{H}$ ($\subset \mathcal{H}_F$) take values in A and in fact coincide with the characters of $R(\mu)^\diamond$ and $R(\lambda)$, respectively. This means that on specializing $\mathcal{H}$ to $\mathbb{Q}$ the characters associated to the representations $R(\mu)^\diamond$ and $R(\lambda)$ are equal. On the other hand, from the previous paragraph we have $R(\lambda)_\mathbb{Q} \cong R(\lambda')_\mathbb{Q}^\diamond$. So we have $R(\mu)_\mathbb{Q}^\diamond \cong R(\lambda')_\mathbb{Q}^\diamond$, thereby leading to incompatibility in the number of irreducibles $R(\lambda)_\mathbb{Q}^\diamond$ since $\mu \neq \lambda'$ (Theorem 3.3.18 for right C'-cell modules). Therefore, we conclude that $R(w)_F \cong R(w_0 w)_F^\diamond$. □

Now we are ready to prove the first step. The idea is to show that the $R(\lambda)_F$ is a composition factor for both $x_\lambda \mathcal{H}_F$ and $y_\lambda \mathcal{H}_F$. Then, showing that S_F^λ is the only composition factor common to these modules we deduce the required isomorphism. The details are outlined in the following proposition.

Proposition 3.4.2 *Let $\lambda \vdash n$ and $w_{0,\lambda}$ be the longest element in W_λ. Then the right cell module corresponding to $w_{0,\lambda}$, $R(w_{0,\lambda})_F$, is isomorphic to $S_F^{\lambda'}$ as an $\mathcal{H}_F$-module.*

Proof: We have seen in Lemma 2.3.18(4) that $w = w_0 w_{0,\lambda} w_\lambda$ is a prefix of every element in the right cell containing it. Using this fact, an inductive argument on $l(y)$ where $y \sim_\mathrm{R} w$ shows that $C'_y \in C'_w \mathcal{H}$ for all $y \sim_\mathrm{R} w$. We thus note that $C'_w \mathcal{H}_F$ has a composition factor isomorphic to $R(w)_F^\diamond$. Once again in Lemma 2.3.18(3), we have seen that $w \sim_\mathrm{L} w_{0,\lambda'}$ so that $C'_w \in \mathcal{H} C'_{w_{0,\lambda'}}$ (by analogue of Proposition 3.3.14 for $\leq_\mathrm{L}$), which immediately produces for us an element $h \in \mathcal{H}$ such that multiplication by h on the left gives a surjection from $x_{\lambda'} \mathcal{H}_F$ to $C'_w \mathcal{H}_F$. Thus, $R(w)_F^\diamond$ becomes a composition factor also of $x_{\lambda'} \mathcal{H}_F$. On the other hand, by Proposition 3.3.14, the module $R(w_{0,\lambda})$ is a composition factor of $y_\lambda \mathcal{H}_F$. But as was seen in Lemma 2.3.18(2), $w_0 w = w_{0,\lambda} w_\lambda \sim_\mathrm{R} w_{0,\lambda}$. So we have by the definition of the right cell module that $R(w_{0,\lambda})_F = R(w_0 w)_F$, which by Lemma 3.4.1 above, is isomorphic to $R(w)_F^\diamond$. Thus $R(w_{0,\lambda})_F$ is a common composition factor of both $x_{\lambda'} \mathcal{H}_F$ and $y_\lambda \mathcal{H}_F$. However, using Lemma 3.3.1 we can deduce that the only factor common to these two modules is the Specht module $S_F^{\lambda'}$. Indeed, if any irreducible module is common to $x_{\lambda'} \mathcal{H}_F$ and $y_\lambda \mathcal{H}_F$ then it should occur in the product $x_{\lambda'} \mathcal{H}_F y_\lambda \mathcal{H}_F$, which by Lemma 3.3.1 is equal to $x_{\lambda'} T_{w_\lambda} y_\lambda \mathcal{H}_F = S_F^{\lambda'}$. Thus, $R(w_{0,\lambda})_F$ is isomorphic to $S_F^{\lambda'}$ (note that $R(w_{0,\lambda})_F$ and $S_F^{\lambda'}$ are irreducible; see Theorem 3.3.18, Corollary 3.3.10). □

We can now establish the isomorphism, which we shall refer to as the "MP-isomorphism" between S^λ and $R(\lambda)$.

Proposition 3.4.3 ([MP05, Theorem 3.5]) *Let $\lambda \vdash n$. Then $S^\lambda \cong_\mathcal{H} R(\lambda)$.*

Proof: Let $N_{w_{0,\lambda'}}$ be the module $\langle C_y | y \leq_\mathrm{R} w_{0,\lambda'} \rangle_A$ and $\hat{N}_{w_{0,\lambda'}}$ denote its submodule $\langle C_y | y <_\mathrm{R} w_{0,\lambda'} \rangle_A$. We have seen already in Proposition 3.3.14 that $N_{w_{0,\lambda'}}$ is the same as the right ideal $C_{w_{0,\lambda'}} \mathcal{H}$. Define a map

$$\begin{array}{rccc} \theta: & N_{w_{0,\lambda'}} & \longrightarrow & S^\lambda \\ & m & \mapsto & x_\lambda T_{w_\lambda} m \end{array} \tag{3.7}$$

Using Proposition 3.3.13 we can immediately deduce the surjectivity of the above map. We now claim that the ker $\theta = \hat{N}_{w_{0,\lambda'}}$. First of all, suppose that $\hat{N}_{w_{0,\lambda'}} \nsubseteq$ ker θ, then extending scalars to $F = \mathbb{Q}(v)$ we still have $\hat{N}_{w_{0,\lambda'}}{}_F \nsubseteq$ ker θ_F. This in turn implies that θ_F is non-zero on $\hat{N}_{w_{0,\lambda'}}{}_F$. So by the irreducibility of S_F^λ (Corollary 3.3.10) we deduce that S_F^λ is a composition factor of $\hat{N}_{w_{0,\lambda'}}{}_F$. This contradicts the fact that the multiplicity of $R(\lambda)_F$ in $N_{w_{0,\lambda'}}{}_F = y_\lambda \mathcal{H}_F$ is 1 (a fact that can be verified as in the last part of the proof of Proposition 3.3.18). We thus have $\hat{N}_{w_{0,\lambda'}} \subseteq$ ker θ. Counting the rank over A of the quotient $N_{w_{0,\lambda'}}/\hat{N}_{w_{0,\lambda'}}$ $(= R(w_{0,\lambda'}))$, we then deduce that the map induced on the quotient is an isomorphism. As the RSK-shape$(w_{0,\lambda'})$ is λ we have the required isomorphism. $\square$

3.5 Interplay between S^λ and $S^{\lambda'}$

The aim of this section is to display the identical behaviour of S^λ and $S^{\lambda'}$ with respect to irreducibility, over a field k. In other words, S_k^λ is irreducible if and only if $S_k^{\lambda'}$ is so ([DJ87, Theorem 3.5]; [Mur95, Theorem 5.2]). This is achieved by giving an isomorphism between S^λ and the dual of an A-isomorphic copy of $S^{\lambda'}$ which reflects the behaviour of $S^{\lambda'}$ in terms of irreducibility (Proposition 3.5.3).

It would be convenient to use the following two notations, in order to describe the isomorphism that we seek:

Notation 3.5.1 For a (right) $\mathcal{H}$-module M, denote by $M^\dagger$ the $\mathcal{H}$-module whose underlying $\mathbb{Z}$-module is M but with $\mathcal{H}$-action given by $m.h{:=}mj(h)$ where j is the involution on $\mathcal{H}$ given by $\sum_w a_w T_w \mapsto \sum_w \epsilon_w \overline{a_w} T_w$.

Notation 3.5.2 For a (left) $\mathcal{H}$-module M, we denote by M^* the (right) $\mathcal{H}$-module whose underlying A-module is the same as that of M and with the action of $\mathcal{H}$ being given by $mh := h^* m$ where $m \in M$, $h \in \mathcal{H}$ and $h \mapsto h^*$ is the involutive anti-automorphism given by $T_w \mapsto T_{w^{-1}}$.

In particular, for a (right) $\mathcal{H}$-module M, the dual $M^{\text{dual}} := \text{Hom}_A(M, A)$ is naturally a left $\mathcal{H}$-module: $(m)(h\phi) := (mh)\phi$, for $\phi \in M^{\text{dual}}$, $m \in M,\ h \in \mathcal{H}$. In the notation given above, M^{dual^*} is a right $\mathcal{H}$-module: $(m)(\phi h) := (mh^*)\phi$, for $\phi \in M^{\text{dual}}$, $m \in M,\ h \in \mathcal{H}$.

With the above notation, we have

Proposition 3.5.3 *There is an $\mathcal{H}$-isomorphism, $(S^{\lambda'^\dagger})^{\text{dual}\,*} \cong S^\lambda$. In particular, for a field k, S_k^λ is irreducible if and only if $S_k^{\lambda'}$ is so.*

To prove the above isomorphism, we proceed as follows:

Let $N_{w_{0,\lambda'}}$ be the (right) $\mathcal{H}$-module $\langle C_y | y \leq_{\text{R}} w_{0,\lambda'} \rangle_A$, which is the same as the monomial module $y_{\lambda'}\mathcal{H}$ (see Proposition 3.3.14; Eq. (3.3)).

We define a A-bilinear form on $N_{w_{0,\lambda'}}$ in the same way as was done for M^λ (§3.3.1): on the basis $\{y_{\lambda'} T_d \mid d \in \mathfrak{D}_{\lambda'}\}$ (see Proposition 3.3.13) of $N_{w_{0,\lambda'}}$ the form is defined by setting

$\langle y_{\lambda'}T_d, y_{\lambda'}T_e\rangle$ equal to 1 or 0 accordingly as the elements d, e of $\mathfrak{D}_{\lambda'}$ are equal or not. The form is evidently symmetric. Also, as was done in §3.3.1 for $n_1,\ n_2 \in N_{w_{0,\lambda'}}$,

$$\langle n_1 h, n_2\rangle = \langle n_1, n_2 h^*\rangle \tag{3.8}$$

where $*$ is the A-linear, anti-automorphism of the algebra $\mathcal{H}$ given by $T_w \mapsto T_{w^{-1}}$.

Since $N_{w_{0,\lambda'}}$ is A-free and the bilinear form $\langle,\rangle$ on $N_{w_{0,\lambda'}}$ is non-degenerate, we have an isomorphism $\alpha : N_{w_{0,\lambda'}} \to {N_{w_{0,\lambda'}}}^{\mathrm{dual}^*}$ induced by $\langle,\rangle$. By (3.8), α is in fact an $\mathcal{H}$-isomorphism.

Let $\tilde{S}^{\lambda'}$ be the $\mathcal{H}$-submodule of $N_{w_{0,\lambda'}}$ given by $\tilde{z}_{\lambda'}\mathcal{H}$ where $\tilde{z}_{\lambda'} := y_{\lambda'}T_{w_{\lambda'}}x_\lambda$. Note that $j(z_{\lambda'}) = \tilde{z}_{\lambda'}$ (by the definition of the involution j, and Equations (3.3), (2.7)).

With the notation as described above, it follows that *the involution j of $\mathcal{H}$ induces an isomorphism between ${S^{\lambda'}}^{\dagger}$ and the $\mathcal{H}$-module $\tilde{S}^{\lambda'}$*. It can hence be easily seen that $\tilde{S}^{\lambda'}$ has similar properties as the Specht module $S^{\lambda'}$. Listed below are some of the properties that we would need:

1. (see, Theorem 3.3.7) The set $\{\tilde{z}_{\lambda'}T_d |\ t^{\lambda'}w_{\lambda'}d$ is a standard tableau$\}$ forms an A-basis for $\tilde{S}^{\lambda'}$: apply the involution j to the standard basis of $S^{\lambda'}$ as given by Theorem 3.3.7.

2. (see, Theorem 3.3.9) Let U be a submodule of $N_{w_{0,\lambda'}}$, we define $U^{\perp} := \{n \in N_{w_{0,\lambda'}} |\ \langle n, u\rangle = 0$ for all $u \in U\}$. Let F be a field, $a \in F$ be invertible. Let $\mathcal{H}_F$ be the specialization of $\mathcal{H}$ via $v \mapsto a$. Then $\tilde{S}^{\lambda'}_F \subset U$ or $U \subset \tilde{S}^{\lambda'\perp}_F$. (see also [DJ87, §3.1(vi)]).

From Lemma 3.3.8 it follows that $\tilde{S}^{\lambda'}$ has an A-complement in $N_{w_{0,\lambda'}}$. So, the inclusion $\tilde{S}^{\lambda'} \subset N_{w_{0,\lambda'}}$ yields a surjection from ${N_{w_{0,\lambda'}}}^{\mathrm{dual}^*} \to (\tilde{S}^{\lambda'})^{\mathrm{dual}^*}$. Composing this with the isomorphism $\alpha : N_{w_{0,\lambda'}} \to {N_{w_{0,\lambda'}}}^{\mathrm{dual}^*}$, we get a $\mathcal{H}$-module homomorphism from $N_{w_{0,\lambda'}}$ onto $(\tilde{S}^{\lambda'})^{\mathrm{dual}^*}$, whose kernel is obviously $\tilde{S}^{\lambda'\perp}$. Thus,

$$N_{w_{0,\lambda'}}/\tilde{S}^{\lambda'\perp} \xrightarrow{\sim} (\tilde{S}^{\lambda'})^{\mathrm{dual}^*} \tag{3.9}$$

Proposition 3.5.4 ([DJ87, Theorem 3.5]) *The kernel of the map θ defined in (3.7) is $\tilde{S}^{\lambda'\perp}$. So, we have an $\mathcal{H}$-isomorphism, $(\tilde{S}^{\lambda'})^{\mathrm{dual}^*} \cong S^\lambda$.*

Proof: By definition of θ, $\tilde{z}_{\lambda'} \mapsto x_\lambda T_{w_\lambda}\tilde{z}_{\lambda'}$. Let

$$a = \langle x_\lambda T_{w_\lambda} y_{\lambda'} T_{w_{\lambda'}} x_\lambda, x_\lambda\rangle = \langle x_\lambda T_{w_\lambda} y_{\lambda'} T_{w_{\lambda'}}, x_\lambda x_\lambda\rangle = f_\lambda \langle x_\lambda T_{w_\lambda} y_{\lambda'} T_{w_{\lambda'}}, x_\lambda\rangle$$

where $0 \neq f_\lambda \in A$ such that $x_\lambda^2 = f_\lambda x_\lambda$ (using Eq.(2.9)). By Lemma 3.3.8 we therefore get that $0 \neq a \in A$. Hence $\tilde{z}_{\lambda'} \notin \ker\theta$ and so, $\tilde{S}^{\lambda'} \not\subseteq \ker\theta$. Then, by extending scalars to $F = \mathbb{Q}(v)$ and using property (2) we have that $\ker\theta_F \subset \tilde{S}^{\lambda'\perp}_F$. This implies that $\ker\theta \subset \tilde{S}^{\lambda'\perp}$. Comparing dimensions we obtain our conclusion. □

Combining the above proposition with the observation that ${S^{\lambda}}^{\dagger}$ is $\mathcal{H}$-isomorphic (via j) to $\tilde{S}^\lambda$, proves Proposition 3.5.3. □

Chapter 4

RSK bases for certain quotients of the group ring

In this chapter, we look first at the tabloid representation of $\mathfrak{S}_n$. Fix a partition λ of n. As seen earlier (§3.2), $\mathbb{C}\mathcal{T}_\lambda$ defines a representation of $\mathfrak{S}_n$. Let $\rho_\lambda : \mathbb{C}\mathfrak{S}_n \to \mathrm{End}_{\mathbb{C}}\mathbb{C}\mathcal{T}_\lambda$ be the defining map. We seek a basis for the image of $\mathbb{C}\mathfrak{S}_n$ under ρ_λ or equivalently, for the quotient of $\mathbb{C}\mathfrak{S}_n$ by the kernel of ρ_λ. This is discussed in §4.2.1 where we present the *RSK basis* for this quotient space. The question in consideration may well be posed over any field (not necessarily $\mathbb{C}$). This is addressed through the remainder of §4.2. Also discussed in §4.3 is a Hecke analogue of §4.2.2 with a proof.

In §4.4, we shift our focus to a representation of $\mathfrak{S}_n$ which is of invariant theoretic interest, namely the space $V^{\otimes n}$ where V is a finite dimensional vector space over $\mathbb{C}$. Using a classical result in invariant theory, we then notice in Proposition 4.4.3 that the image of $\mathbb{C}\mathfrak{S}_n$ in the endomorphism ring of $V^{\otimes n}$ is identified with the quotient $\mathbb{C}\mathfrak{S}_n/\ker \rho_\lambda$ for a particular λ, where ρ_λ is the map discussed in the previous paragraph. Thus, we obtain a basis for the image of $\mathbb{C}\mathfrak{S}_n$ in $\mathrm{End}_{\mathbb{C}}V^{\otimes n}$, using results in §4.2.1 for the tabloid representation over $\mathbb{C}$.

The image of $\mathbb{C}\mathfrak{S}_n$ in $\mathrm{End}_{\mathbb{C}}V^{\otimes n}$ has a certain invariant theoretic appeal which is explained in greater detail in §4.4. Inspired by this appeal, we extend the result discussed above which presents a basis for the image of $k\mathfrak{S}_n$ in $\mathrm{End}_k V^{\otimes n}$ when $k = \mathbb{C}$, to the case when k is an arbitrary commutative ring with unity, satisfying some mild conditions. This is done in Theorem 4.4.1. The rest of §4.4 is devoted to presenting bases for certain other rings of invariants through sections §4.4.1, §4.4.2, §4.4.3, as a consequence of Theorem 4.4.1.

We begin this chapter with a couple of observations which play a key role in the approach we take to produce bases for the endomorphism rings under consideration.

4.1 Key observations

The first observation that we make in this section is a simple one which will enable us to move from the C-basis to T-basis of certain quotients by two-sided ideals of the Hecke algebra. The second observation is crucial for many of the results that we discuss here. It partially justifies our choice of the cell modules over the Specht modules to realize the irreducible representations of $\mathfrak{S}_n$ and its Hecke algebra.

4.1.1 Moving from C-basis to T-basis of certain quotients

From Theorem2.2.6(2), we get $T_w \equiv C_w \bmod \langle T_x | x < w \rangle_A$. From this in turn we get, by induction on the Bruhat-Chevalley order, the following: for a subset $\mathfrak{S}$ of $\mathfrak{S}_n$, the (images of) elements T_w, $w \in \mathfrak{S}_n \setminus \mathfrak{S}$, form a basis for the A-module $\mathcal{H}/\langle C_x | x \in \mathfrak{S} \rangle_A$. The same thing holds also in specializations $\mathcal{H}_k$ of $\mathcal{H}$ (§2.2): the (images of) elements T_w, $w \in \mathfrak{S}_n \setminus \mathfrak{S}$, form a basis for the k-module $\mathcal{H}_k/\langle C_x | x \in \mathfrak{S} \rangle_k$.

4.1.2 Images of the C-basis elements in End$R(\lambda)$

The image of C_y in End$R(\lambda)$ vanishes unless $\lambda \trianglelefteq$ RSK-shape(y), for, if C_z occurs with non-zero coefficient in $C_x C_y$ (when expressed as an A-linear combinations of the C-basis), where RSK-shape$(x) = \lambda$, and $\lambda \ntrianglelefteq$ RSK-shape(y), then $z \leq_{\mathrm{L}} y$ (by definition), so RSK-shape$(z) \trianglelefteq$ RSK-shape(y) (Proposition 2.3.5), which means that RSK-shape$(z) \neq \lambda$, so $z \not\sim_R x$ (Proposition 2.3.7).

4.2 Tabloid representations of $\mathfrak{S}_n$

Let k be an arbitrary field. Let λ be a partition of n and $\rho_\lambda : k\mathfrak{S}_n \to \mathrm{End}_k k\mathcal{T}_\lambda$ be the map defining the tabloid representation as described in §3.2. This is a morphism of k-algebras. Thus, obtaining a basis for the image of ρ_λ can be reduced to obtaining one for the quotient $k\mathfrak{S}_n/\ker\,\rho_\lambda$. As the set of permutations in $\mathfrak{S}_n$ gives a generating set for the quotient $k\mathfrak{S}_n/\ker\,\rho_\lambda$, we hope to find a suitable subset of permutations that actually form a basis for it.

We deal with this issue assuming initially that $k = \mathbb{C}$ (§4.2.1) where the decomposition, into irreducibles, of $\mathbb{C}\mathcal{T}_\lambda$ as in Proposition 3.2.2 is known. In this case, the observations made in §4.1.2, §4.1.1 above leads us almost immediately to a basis of $\mathbb{C}\mathfrak{S}_n/\ker\,\rho_\lambda$ consisting of permutations. Using this, we then prove the analogous statement with the base ring taken to be $\mathbb{Z}$. Finally owing to the fact that a field of characteristic 0 is flat over $\mathbb{Z}$, we readily have the result even over such a field. In the case when the characteristic of k is positive, the statement fails to be true, in general. Section 4.2.3, illustrates this by an example.

4.2.1 Results over $\mathbb{C}$

We first consider the case when $k = \mathbb{C}$. We have the *RSK-basis* for the tabloid representation given by the following theorem:

Theorem 4.2.1 *Permutations with* RSK-shape*s μ such that $\mu \trianglerighteq \lambda$ form a $\mathbb{C}$ basis of $\mathbb{C}\mathfrak{S}_n$ modulo the kernel of $\rho_\lambda : \mathbb{C}\mathfrak{S}_n \to \mathrm{End}_{\mathbb{C}}\mathbb{C}\mathcal{T}_\lambda$.*

PROOF: The tabloid representation of $\mathfrak{S}_n$ (defined over $\mathbb{C}$) has a decomposition into irreducibles given by $\oplus_{\mu \trianglerighteq \lambda}(\mathcal{S}^\mu_{\mathbb{C}})^{m(\mu)}$, $m(\mu) > 0$ (see Proposition 3.2.2). Since the multiplicities $m(\mu)$ in the decomposition are positive, the kernel of ρ_λ is the same as that of the map $\rho'_\lambda : \mathbb{C}\mathfrak{S}_n \to \mathrm{End}_{\mathbb{C}}(\oplus_{\mu \trianglerighteq \lambda}\mathcal{S}^\mu_{\mathbb{C}})$. The image of ρ'_λ is clearly contained in $\oplus_{\mu \trianglerighteq \lambda}\mathrm{End}_{\mathbb{C}}\mathcal{S}^\mu_{\mathbb{C}}$. Since the $\mathcal{S}^\mu_{\mathbb{C}}$ are non-isomorphic for distinct μ and are irreducible,[1] it follows from a density argument (see for example [Bou73, Chapter 8, §4, No. 3, Corollaire 2]) that ρ'_λ maps onto $\oplus_{\mu \trianglerighteq \lambda}\mathrm{End}_{\mathbb{C}}\mathcal{S}^\mu_{\mathbb{C}}$. Since $\dim \mathcal{S}^\mu_{\mathbb{C}} = d(\mu)$, where $d(\mu)$ is the number of standard tableaux of shape μ, and the $\mathcal{S}^\mu_{\mathbb{C}}$ as μ varies over all partitions of n are a complete set of irreducible representations,[2] we obtain, by counting dimensions:

$$\dim \ker \rho'_\lambda = \dim \mathbb{C}\mathfrak{S}_n - \dim\left(\oplus_{\mu \trianglerighteq \lambda}\mathrm{End}_{\mathbb{C}}\mathcal{S}^\mu_{\mathbb{C}}\right) = \sum_{\mu \vdash n} d(\mu)^2 - \sum_{\mu \trianglerighteq \lambda} d(\mu)^2 = \sum_{\mu \ntrianglerighteq \lambda} d(\mu)^2$$

Now consider $\mathbb{C}\mathfrak{S}_n$ as the specialization of the Hecke algebra $\mathcal{H}$ as follows (§2.2): $\mathbb{C}\mathfrak{S}_n \simeq \mathcal{H} \otimes_A \mathbb{C}$, where $\mathbb{C}$ is an A-algebra via the map $A \to \mathbb{C}$ defined by $v \mapsto 1$. By the observation §4.1.2, the images $C_w \otimes 1$ in $\mathcal{H} \otimes_A \mathbb{C} \simeq \mathbb{C}\mathfrak{S}_n$ of the Kazhdan-Lusztig basis elements C_w of $\mathcal{H}$ belong to the kernel of ρ'_λ if RSK-shape$(w) \ntrianglerighteq \lambda$. The number of such w being equal to $\sum_{\mu \ntrianglerighteq \lambda} d(\mu)^2$, which as observed above equals $\dim \ker \rho'_\lambda$, we conclude that

$$\ker \rho_\lambda = \ker \rho'_\lambda = \langle C_w \otimes 1 | \text{RSK-shape}(w) \ntrianglerighteq \lambda \rangle_{\mathbb{C}}. \tag{4.1}$$

By observation §4.1.1, the images of $T_w \otimes 1$, RSK-shape$(w) \trianglerighteq \lambda$, form a basis for $\mathcal{H} \otimes_A \mathbb{C} / \langle C_x \otimes 1 | \text{RSK-shape}(w) \ntrianglerighteq \lambda \rangle_{\mathbb{C}} \simeq \mathbb{C}\mathfrak{S}_n / \ker \rho'_\lambda$. But the image in $\mathbb{C}\mathfrak{S}_n$ of $T_w \otimes 1$ is the permutation w. This completes the proof of Theorem 4.2.1. □

4.2.2 Results over $\mathbb{Z}$

We now prove Theorem 4.2.1 with $\mathbb{Z}$ coefficients in place of $\mathbb{C}$ coefficients.

Let $\rho_{\lambda,\mathbb{Z}}$ be the map $\mathbb{Z}\mathfrak{S}_n \to \mathrm{End}_{\mathbb{Z}}\mathbb{Z}\mathcal{T}_\lambda$ defining the tabloid representation. We claim that Eq. (4.1) holds over $\mathbb{Z}$:

$$\ker \rho_{\lambda,\mathbb{Z}} = \langle C_w \otimes 1 | \text{RSK-shape}(w) \ntrianglerighteq \lambda \rangle_{\mathbb{Z}} \tag{4.2}$$

[1] It follows from the isomorphism in (Proposition 3.4.3) and the corresponding fact for cell modules proved in §3.3.5.

[2] Same comment as in footnote 1 applies to both assertions.

Once this is proved, the rest of the argument is the same as in the complex case: namely, use observation §4.1.1.

We first show the containment $\supseteq$. We have $(C_w \otimes 1)\mathbb{C}\mathcal{T}_\lambda = (C_w \otimes 1)\mathbb{Z}\mathcal{T}_\lambda \otimes_{\mathbb{Z}} \mathbb{C}$ (by flatness of $\mathbb{C}$ over $\mathbb{Z}$). Since $(C_w \otimes 1)\mathbb{Z}\mathcal{T}_\lambda$ is a submodule of the free module $\mathbb{Z}\mathcal{T}_\lambda$, it is free. By Eq. (4.1), $(C_w \otimes 1)\mathbb{C}\mathcal{T}_\lambda = 0$ if RSK-shape$(w) \ntrianglerighteq \lambda$, so $\supseteq$ holds.

To show the other containment, set $\mathfrak{m} = \langle C_w \otimes 1 | \text{RSK-shape}(w) \ntrianglerighteq \lambda\rangle_{\mathbb{Z}}$, and consider $\ker \rho_{\lambda,\mathbb{Z}}/\mathfrak{m}$. Since $\mathbb{Z}\mathfrak{S}_n/\mathfrak{m}$ is free, so is its submodule $\ker \rho_{\lambda,\mathbb{Z}}/\mathfrak{m}$, and we have

$$\frac{\ker \rho_{\lambda,\mathbb{Z}}}{\mathfrak{m}} \otimes_{\mathbb{Z}} \mathbb{C} = \frac{\ker \rho_{\lambda,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}}{\mathfrak{m} \otimes_{\mathbb{Z}} \mathbb{C}} = \frac{\ker \rho_{\lambda,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}}{\langle C_w \otimes 1 | \text{RSK-shape}(w) \ntrianglerighteq \lambda\rangle_{\mathbb{C}}}.$$

By the flatness of $\mathbb{C}$ over $\mathbb{Z}$, we have $\ker \rho_{\lambda,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \ker \rho_\lambda$. The last term in the above display vanishes by Eq. (4.1), and so $\subseteq$ holds (since $\ker \rho_{\lambda,\mathbb{Z}}/\mathfrak{m}$ is free). The proof of Theorem 4.2.1 over $\mathbb{Z}$ is complete.

Remark 4.2.2 Let k be a field of characteristic 0 and $\rho_{\lambda,k}$ the map $k\mathfrak{S}_n \to \mathrm{End}_k k\mathcal{T}_\lambda$ defining the representation on tabloids of shape λ. The analogue of Eqs. (4.1) and (4.2) holds over k, since, by the flatness of k over $\mathbb{Z}$, we have $\ker \rho_{\lambda,k} = \ker \rho_{\lambda,\mathbb{Z}} \otimes_{\mathbb{Z}} k$. Now one can use observation §4.1.1 as in the earlier cases to finish the proof of Theorem 4.2.1 even over k.

4.2.3 Failure over fields of positive characteristic

Theorem 4.2.1 does not hold in general over a field k of positive characteristic. We give an example of a non-trivial linear combination of permutations of RSK-shape dominating λ that acts trivially on the tabloid representation $k\mathcal{T}_\lambda$. Let k be a field of characteristic 2. Let $n = 4$ and $\lambda = (2, 2)$. Let us denote a permutation in $\mathfrak{S}_4$ by writing down in sequence the images under it of 1 through 4: e.g., 1243 denotes the permutation σ defined by $1\sigma = 1$, $2\sigma = 2$, $3\sigma = 4$, and $4\sigma = 3$. It is readily seen that the eight permutations in the display below are all of shape $(3, 1)$.

$$2134, \quad 2341, \quad 2314, \quad 1342, \quad 3124, \quad 1243, \quad 4123, \quad 1423.$$

Notice that shape $(3, 1) \trianglerighteq (2, 2)$ and that the sum of the above eight permutations acts trivially on $k\mathcal{T}_\lambda$.

4.3 Remarks on the Hecke Analogue of §4.2.2

The tabloid module having an analogue in the setup of the Hecke algebra of $\mathfrak{S}_n$, we are led to asking whether Theorem 4.2.1, with $\mathbb{Z}$-coefficients in place of $\mathbb{C}$-coefficients, has a natural $\mathcal{H}$-analogue. We state and prove this here.

Theorem 4.3.1 *The elements T_w, RSK-shape$(w) \trianglerighteq \lambda$, form a basis for $\mathcal{H}$ modulo the kernel of the map $\mathcal{H} \to \mathrm{End}_{\mathcal{H}} M^\lambda$ defining M^λ as a right $\mathcal{H}$-module.*

PROOF: We will show that the C_w, RSK-shape$(w) \ntrianglerighteq \lambda$, form a basis for the kernel. By the observation in §4.1.1, this will suffice. The longest step in the proof is to show that such C_w belong to the kernel. Assuming for the moment this to be the case, let us finish the rest of the proof. Suppose that the kernel is strictly larger than $\langle C_w | \text{RSK-shape}(w) \ntrianglerighteq \lambda \rangle_A$. Then there exists a non-trivial linear combination of C_w, RSK-shape$(w) \trianglerighteq \lambda$, in the kernel. Since M^λ is torsion-free (it is free over A), we may assume that not all coefficients vanish at $v = 1$. But then such a linear combination would not vanish in $\mathcal{H}_\mathbb{C}$ (where $\mathcal{H}_\mathbb{C} := \mathcal{H} \otimes_A \mathbb{C} = \mathbb{C}\mathfrak{S}_n$, $\mathbb{C}$ being the the specialization via $v \mapsto 1$), contradicting Theorem 4.2.1, and we're done.

We now show that the C_w, RSK-shape$(w) \ntrianglerighteq \lambda$, annihilate $M^\lambda = x_\lambda \mathcal{H}$. Let $\mu =$ RSK-shape(w) and assume $\mu \ntrianglerighteq \lambda$. Proceed by induction on the domination order $\trianglelefteq$, and assume that C_y kills M^λ for RSK-shape$(y) \triangleleft \mu$. First suppose that w is the longest element of shape μ, that is, $w = w_{0,\mu'}$—the base case of the induction is also proved by the argument in this case. By Remark 3.3.2, $x_\lambda \mathcal{H} y_{\mu'} = 0$; but $y_{\mu'}$ equals $C_{w_{0,\mu'}}$ (see (3.3) above) up to a factor of sign and a power of v. Thus $M^\lambda C_{w_{0,\mu'}} = 0$.

Next suppose that $w \leq_\mathrm{R} w_{0,\mu'}$. Then, by Proposition 3.3.14, C_w belongs to $C_{w_{0,\mu'}}\mathcal{H}$, so that $M^\lambda C_w \subseteq M^\lambda C_{w_{0,\mu'}} \mathcal{H} = 0$. If $w \leq_\mathrm{L} w_{0,\mu'}$, then (again by Proposition 3.3.14, left-sided version) C_w belongs to $\mathcal{H} C_{w_{0,\mu'}}$, so that $M^\lambda C_w \subseteq M^\lambda \mathcal{H} C_{w_{0,\mu'}} = M^\lambda C_{w_{0,\mu'}} = 0$.

Now suppose that w of RSK-shape μ is not left or right equivalent to $w_{0,\mu'}$. The association $C(P, Q(w)) \leftrightarrow C(P, t_\mu)$ gives an $\mathcal{H}$-isomorphism between the left cell modules $L(w)$ and $L(P, t_\mu)$ (§3.3.5), and as seen in the previous paragraph $x_\lambda \mathcal{H} C(P, t_\mu) = 0$ since (P, t_μ) is left equivalent to $w_{0,\mu'} = (t_\mu, t_\mu)$ (Remark 2.3.9). These two together imply that $x_\lambda \mathcal{H} C_w = 0$ in the quotient $L(w)$. Equivalently, $x_\lambda \mathcal{H} C_w \subseteq \langle C_y | y \lneq_\mathrm{L} w \rangle_A$.

By Eq. (2.10), $(\sum_{x \in W_\lambda} v_x^{-2}) x_\lambda \mathcal{H} C_w = x_\lambda^2 \mathcal{H} C_w \subseteq x_\lambda \langle C_y | y \lneq_\mathrm{L} w \rangle_A$. On the other hand, by Proposition 2.3.6 and Proposition 2.3.5, the y appearing on the right hand side of the last containment are such that RSK-shape$(y) \triangleleft \mu$. By induction $x_\lambda \mathcal{H} C_y = 0$ for such y. Thus $(\sum_{x \in W_\lambda} v_x^{-2}) x_\lambda \mathcal{H} C_w = 0$. But $\sum_{x \in W_\lambda} v_x^{-2}$ being a non-zero scalar, and M^λ being torsion-free A-module (it is a free A-module) we conclude that $x_\lambda \mathcal{H} C_w = 0$. This finishes the proof that the C_w, RSK-shape$(w) \ntrianglerighteq \lambda$ belong to the kernel. □

4.4 Certain rings of invariants

Let k denote a commutative ring with unity. Let V be a free k-module of finite rank d over k. Then there is a natural action of the group of automorphism of V, denoted as $GL(V)$, on the k-module V.

Let $n \in \mathbb{N}$ be fixed. Consider the symmetric group $\mathfrak{S}_n$ acting on the space of n-tensors $V^{\otimes n}$ by permuting the factors. The action of $\sigma \in \mathfrak{S}_n$ on pure tensors is given as:

$$(v_1 \otimes \cdots \otimes v_n)\sigma := v_{1\sigma} \otimes \cdots \otimes v_{n\sigma}$$

This action commutes with the natural diagonal action of $GL(V)$ on $V^{\otimes n}$. This implies that the map $\phi_n : k\mathfrak{S}_n \to \mathrm{End}_k V^{\otimes n}$ defining the action of $\mathfrak{S}_n$ on $V^{\otimes n}$ has image lying inside $\mathrm{End}_{GL(V)} V^{\otimes n}$ – the space of $GL(V)$-invariant endomorphism on $V^{\otimes n}$. The following fundamental theorem (see [dCP76, Theorems 4.1, 4.2]) in classical invariant theory states that this map is a surjection onto $\mathrm{End}_{GL(V)} V^{\otimes n}$:

> Assume that no non-zero polynomial of degree n with coefficients in k vanishes on k. (This holds for example when k is an infinite field, no matter what n is.) Then the k-algebra map ϕ_n maps onto $\mathrm{End}_{GL(V)} V^{\otimes n}$ and its kernel is the two-sided ideal $J(n,d)$ – the two-sided ideal generated by the element $y_d := \sum_{\tau \in \mathfrak{S}_{d+1}} \epsilon(\tau)\tau$, where $\mathfrak{S}_{d+1}$ is the subgroup of $\mathfrak{S}_n$ consisting of the permutations that fix point-wise the elements $d+2, \ldots, n$; when $n \leq d$, $J(n,d)$ is defined to be 0.

With the assumption on k as in the above statement, the quotient $k\mathfrak{S}_n / J(n,d)$ can hence be identified with the space of $GL(V)$-invariant endomorphism on $V^{\otimes n}$. By this identification, describing a basis for the quotient will provide us also with a description of a basis for $\mathrm{End}_{GL(V)} V^{\otimes n}$. A basis for the quotient is indeed given by the next theorem.

Theorem 4.4.1 *Let k be a commutative ring with unity. For n, $d \in \mathbb{N}$, let $J(n,d)$ be the two-sided ideal defined as above. Then the permutations σ of $\mathfrak{S}_n$ such that the sequence 1σ, $\ldots$, $n\sigma$ has no decreasing subsequence of length more than d form a basis for $k\mathfrak{S}_n/J(n,d)$.*

The main ingredient in the proof is Lemma 4.4.2 below. It is a two-sided analogue of Proposition 3.3.14.

Lemma 4.4.2 *Let $\zeta(d)$ denote the partition $(d+1, 1, \ldots, 1)$ of n. The two-sided ideal $C_{w_{0,\zeta(d)}}$ is a free A-submodule of $\mathcal{H}$ with basis C_x,* RSK-shape(x) *has more than d rows (or, equivalently,* RSK-shape$(x) \trianglelefteq \zeta(d)'$*).*

Proof: Since $w_{0,\zeta(d)}$ has shape $\zeta(d)'$ (see Remark 2.3.9), it follows from the combinatorial description of $\leq_{\mathrm{LR}}$ in Proposition 2.3.5 that $x \leq_{\mathrm{LR}} w_{0,\zeta(d)}$ if and only if RSK-shape$(x) \trianglelefteq \zeta(d)'$. So it is clear from the definition of the relation $\leq_{\mathrm{LR}}$ (§2.2.2) that the two-sided ideal $\mathcal{H}C_{w_{0,\zeta(d)}}\mathcal{H}$ is contained in $\langle C_x | \text{RSK-shape}(x) \trianglelefteq \zeta(d)' \rangle_A$. To show the reverse containment, we first observe that if $x = w_{0,\mu'}$, the longest element of its shape then C_x belongs to the right ideal $C_{w_{0,\zeta(d)}}\mathcal{H}$ whenever $\mu = \text{RSK-shape}(x) \trianglelefteq \zeta(d)'$: it is enough, by Proposition 3.3.14, to show that $x \leq_{\mathrm{R}} w_{0,\zeta(d)}$; on the other hand, by Lemma 2.3.18(1), $x \leq_{\mathrm{R}} w_{0,\zeta(d)}$ is equivalent to $x(1) > x(2) > \ldots > x(d+1)$, which clearly holds for the elements x that we are considering.

Now suppose that x is a general element of RSK-shape $\mu \trianglelefteq \zeta(d)'$. Proceed by induction on the domination order of μ. The base case is proved by the argument in the previous paragraph. Assume $C_y \in \mathcal{H}C_{w_{0,\lambda}}\mathcal{H}$ for y such that RSK-shape$(y) \triangleleft \mu$. Let $x \leftrightarrow (P,Q)$ under RSK-correspondence. Then, on the one hand, the association $C(P,Q) \leftrightarrow C(t_\mu, Q)$

gives an $\mathcal{H}$-isomorphism between the right cell modules $R(x)$ and $R(v)$, where v is the permutation corresponding under RSK to (t_μ, Q) (§3.3.5); on the other, since $v \leftrightarrow (t_\mu, Q)$ is right equivalent to $w_{0,\mu'} \leftrightarrow (t_\mu, t_\mu)$, there exists, by Proposition 3.3.14, an element h in $\mathcal{H}$ such that $C_v = C_{w_{0,\mu'}}h$; so that, by the definition of right cell modules and the isomorphism between $R(w_{0,\mu'})$ and $R(u)$ where $u \leftrightarrow (P, t_\mu)$ under RSK,

$$C_x \equiv C_u h \pmod{\langle C_y | y \lneq_{\mathrm{R}} u \rangle_A}.$$

Now, $y \lneq_{\mathrm{R}} u$ implies, by Propositions 2.3.6, 2.3.5, RSK-shape$(y) \lhd \mu \unlhd \zeta(d)'$; and, by the induction hypothesis, $C_y \in \mathcal{H}C_{w_{0,\zeta(d)}}\mathcal{H}$. As to $C(P, t_\mu)$, being left equivalent to $C(t_\mu, t_\mu)$, it belongs, once again by Proposition 3.3.14, to the left ideal $\mathcal{H}C_{w_{0,\mu'}}$, which as shown in the previous paragraph is contained in $\mathcal{H}C_{w_{0,\zeta(d)}}\mathcal{H}$. Thus $C_x = C(P, Q) \in \mathcal{H}C_{w_{0,\zeta(d)}}\mathcal{H}$, and we are done. □

Proof of Theorem 4.4.1 given Lemma 4.4.2 As seen in §2.2, $k\mathfrak{S}_n$ is the specialization of the Hecke algebra $\mathcal{H}$: $k\mathfrak{S}_n \simeq \mathcal{H}_k := \mathcal{H} \otimes_A k$, where k is an A-algebra via the natural ring homomorphism $A \to k$ defined by $v \mapsto 1$. Under the map $\mathcal{H} \to \mathcal{H} \otimes_A k$ given by $x \mapsto x \otimes 1$, the image of $C_{w_{0,\zeta(d)}}$ is $C_{w_{0,\zeta(d)}} \otimes 1 = y_d$, by Eq. (3.3). Denoting by $\tilde{J}$ the two-sided ideal of $\mathcal{H}$ generated by $C_{w_{0,\zeta(d)}}$, we thus have $\mathcal{H}/\tilde{J} \otimes_A k \simeq k\mathfrak{S}_n/J(n,d)$.

On the other hand, combining Lemma 4.4.2 with the observation in §4.1.1, we see that $\mathcal{H}/\tilde{J}$ is a free A-module with basis T_x, as x varies over permutations of whose RSK-shapes have at most d rows. The image of T_x in $k\mathfrak{S}_n/J(n,d)$ being the residue class of the corresponding permutation x, the theorem is proved following the easy observation that the permutations σ of $\mathfrak{S}_n$ such that RSK-shape(σ) has atmost d rows is just the set of permutations σ of $\mathfrak{S}_n$ such that the sequence $1\sigma, \ldots, n\sigma$ has no decreasing subsequence of length more than d. □

An alternative proof of Theorem 4.4.1 in the special case of $k = \mathbb{C}$

In order to place this discussion in its proper perspective, we use Theorem 4.2.1 for tabloid representations to arrive at a different proof of Theorem 4.4.1 in the case when $k = \mathbb{C}$. We do this by establishing that the kernel of the map ρ_λ as in Theorem 4.2.1 for a suitable λ turns out to be the ideal $J(n,d)$ as defined in the beginning of this section.

Given positive integers n and d it is easy to see that there exists a unique partition $\lambda(n,d) \vdash n$ that has at most d parts and is smallest in the dominance order among those with at most d parts. For example, $\lambda(8,2) = (4,4)$. Then we have,

Proposition 4.4.3 *Consider the linear representation of $\mathfrak{S}_n$ on the free vector space $\mathbb{C}\mathcal{T}_{\lambda(n,d)}$ generated by tabloids of shape $\lambda(n,d)$ (defined above). The ideal $J(n,d)$ (as described above with coefficients in $\mathbb{C}$) is the kernel of the $\mathbb{C}$-algebra map $\mathbb{C}\mathfrak{S}_n \to \mathrm{End}_{\mathbb{C}}\mathbb{C}\mathcal{T}_{\lambda(n,d)}$ defining this representation.*

Proof: On the one hand, it is easily seen that the generator y_d of the two sided ideal $J(n,d)$

(defined above) belongs to $\ker \rho_{\lambda(n,d)}$. Indeed, given a tabloid $\{T\}$ of shape $\lambda(n,d)$, there evidently exist integers a and b, with $1 \leq a, b \leq d+1$, that appear in the same row of T. This implies that the transposition (a,b) fixes $\{T\}$. Writing $\mathfrak{S}_{d+1}$ as a disjoint union $S \cup S(a,b)$ (for a suitable choice of a subset S), we have $y_d\{T\} = \sum_{\sigma \in \mathfrak{S}_{d+1}} \epsilon(\sigma)\sigma\{T\} = \sum_{\sigma \in S} \epsilon(\sigma)(\sigma - \sigma(a,b))\{T\} = 0$.

On the other hand, as computed in the proof of Theorem 4.2.1 above, $\ker \rho_{\lambda(n,d)}$ as a $\mathbb{C}$-vector space has dimension $\sum_{\mu \not\trianglerighteq \lambda(n,d)} d(\mu)^2$. It suffices therefore to show that $J(n,d)$ too has this same dimension. Since $y_d = C_{w_{0,\zeta(d)}}|_{v=1}$ (see Equation (2.8)), it follows from Lemma 4.4.2 below, that the ideal $J(n,d)$ has dimension $\sum_{\mu \trianglelefteq \zeta(d)'} d(\mu)^2$, where $\zeta(d)$ is the partition of $(d+1, 1, \ldots, 1)$ of n and $\zeta(d)'$ denotes its transpose. But $\mu \not\trianglerighteq \lambda(n,d)$ if and only if μ has more than d rows if and only if $\mu \trianglelefteq \zeta(d)'$. $\square$

In view of the above proposition and Theorem 4.2.1, we immediately arrive at the following description of a basis for the above quotient:

Theorem 4.4.4 *Let $k = \mathbb{C}$. Then those permutations σ of $\mathfrak{S}_n$ such that* RSK-shape(σ) *has atmost d rows form a basis for $k\mathfrak{S}_n/J(n,d)$.* $\square$

It is easy to see from the definition of the RSK-correspondence that the permutations described above are precisely those permutations σ in $\mathfrak{S}_n$ such that the sequence $1\sigma, \ldots, n\sigma$ has no decreasing subsequence of length more than d, thus arriving at Theorem 4.4.1 when $k = \mathbb{C}$.

4.4.1 Multilinear invariants

As earlier, let k denote a commutative ring with unity and V be a free k-module of finite rank d. Let n be a fixed positive integer. Consider the space of multilinear functions on the n-fold product $(\mathrm{End}_k V)^{\times n}$. There is a natural action of $GL(V)$ on this space induced from the action of $GL(V)$ on $(\mathrm{End}_k V)^{\times n}$ by simultaneous conjugation. By the universal property of tensor product, one can identify the space of multilinear functions on $(\mathrm{End}_k V)^{\times n}$ with the space of linear functions on $(\mathrm{End}_k V)^{\otimes n}$. Also, since this identification is compatible with the $GL(V)$ actions naturally induced on both these spaces, we can restrict this identification to the level of their respective subspaces of $GL(V)$-invariant functions. In other words, the space of multilinear invariant functions on $(\mathrm{End}_k V)^{\times n}$ is identified with the linear invariant functions on $(\mathrm{End}_k V)^{\otimes n}$. We can take this identification one step further via the $GL(V)$-equivariant isomorphism $V^* \otimes V \cong \mathrm{End}_k V$ given by $(\alpha \otimes u)(v) := \alpha(v)u$. We have the following $GL(V)$-isomorphisms (and thus, for the k-duals):

$$(\mathrm{End}_k V)^{\otimes n} \cong (V^* \otimes V)^{\otimes n} \cong (V^*)^{\otimes n} \otimes V^{\otimes n}$$

On the other hand, we have the identification $(V^* \otimes V)^* \cong \mathrm{End}_k V$ given by the usual $GL(V)$-equivariant pairing $(A, \alpha \otimes u) \mapsto \alpha(Au)$ leading us to the $GL(V)$-isomorphism

$$((V^*)^{\otimes n} \otimes V^{\otimes n})^* \cong \mathrm{End}_k(V^{\otimes n})$$

Thus, the space of multilinear invariant functions on $(\mathrm{End}_k V)^{\times n}$ is naturally identified with the space $\mathrm{End}_k(V^{\otimes n})^{GL(V)}$. We have seen in the beginning of §4.4 that (with the hypothesis on k as required) the latter space, $\mathrm{End}_k(V^{\otimes n})^{GL(V)}$, occurs as the image of the map ϕ_n mentioned there. As was also observed there, the image of the map ϕ_n is isomorphic to the quotient $k\mathfrak{S}_n/J(n,d)$ for which the permutations σ of $\mathfrak{S}_n$ such that the sequence $1\sigma, \ldots, n\sigma$ has no decreasing subsequence of length more than d form a basis (Theorem 4.4.1). So, going through ϕ_n followed by the above isomorphism, we observe that the image of $\phi_n(\sigma)$ in $((V^*)^{\otimes n} \otimes V^{\otimes n})^*$ is the $GL(V)$-invariant linear function, call it ψ_σ, given by

$$\alpha_1 \otimes \cdots \otimes \alpha_n \otimes u_1 \otimes \cdots \otimes u_n \mapsto (\alpha_1 \otimes \cdots \otimes \alpha_n)(u_{1\sigma} \otimes \cdots \otimes u_{n\sigma})$$

Note that, if we denote by $\alpha \otimes v$ also the matrix of the associated endomorphism under the identification of $V^* \otimes V$ with $\mathrm{End}_k V$ then $\alpha \otimes v \;.\; \beta \otimes u = \alpha \otimes \beta(v) u$ and $\mathrm{Trace}(\alpha \otimes v) = \alpha(v)$. Now it can be easily seen that the linear invariant ψ_σ on $(V^*)^{\otimes n} \otimes V^{\otimes n}$ composed with the isomorphism $(\mathrm{End}_k V)^{\otimes n} \cong (V^*)^{\otimes n} \otimes V^{\otimes n}$, described above, produces the linear $GL(V)$-invariant function on $(\mathrm{End}_k V)^{\otimes n}$ given by

$$A_1 \otimes \cdots \otimes A_n \mapsto \mathrm{Trace}(A_{i_1} A_{i_2} \cdots) \cdots \mathrm{Trace}(A_{i_p} A_{i_{p+1}} \cdots)$$

where $\sigma = (i_1, i_2, \ldots) \ldots (i_p, i_{p+1}, \ldots)$ and A_i denotes the matrix of the endomorphism $\alpha_i \otimes u_i \in \mathrm{End}_k V$, with respect to a fixed basis of V. Since endomorphisms of the form $\alpha \otimes u$ span $\mathrm{End}_k V$, we have seen that

Theorem 4.4.5 *Let k be a commutative ring with unity. Assume that no non-zero polynomial, in one variable, of degree n with coefficients in k vanishes identically on k. Let T_σ be the multilinear function on $(\mathrm{End}_k V)^{\times n}$ defined by:*

$$(A_1, \ldots, A_n) \mapsto \mathit{Trace}(A_{i_1} A_{i_2} \cdots) \cdots \mathit{Trace}(A_{i_p} A_{i_{p+1}} \cdots)$$

where $\sigma = (i_1, i_2, \ldots) \ldots (i_p, i_{p+1}, \ldots)$. Then the functions T_σ where σ varies over permutations in $\mathfrak{S}_n$ having no decreasing subsequence of length more than d, form a basis for the space of multilinear invariants on $(\mathrm{End}_k V)^{\times n}$. □

4.4.2 A monomial basis for the tensor algebra

Let T denote the tensor algebra $\oplus_{i \geq 0} (\mathrm{End}_k V)^{\otimes i}$. This is a graded algebra with $(\mathrm{End}_k V)^{\otimes i}$ as the i-th graded k-subspace and the multiplication on its homogeneous elements is given by $u.v := u \otimes v$.

The group $GL(V)$ acts on $(\mathrm{End}_k V)^{\otimes n}$ for each n and hence on T. This action preserves the algebra structure of T, so the ring of $GL(V)$-invariants is in fact a sub-algebra. So

$$T^{GL(V)} \cong \oplus_n ((\mathrm{End}_k V)^{\otimes n})^{GL(V)} \cong \oplus_n (\mathrm{End}_{GL(V)} V^{\otimes n})$$

By the classical theorem quoted in the beginning of §4.4, the map ϕ_n is an isomorphism between the quotient $k\mathfrak{S}_n/J(n,d)$ and $\mathrm{End}_{GL(V)}V^{\otimes n}$, with the condition on k as required there. Theorem 4.4.1 therefore gives a basis of $\mathrm{End}_{GL(V)}V^{\otimes n}$ for each n. By just taking the disjoint union of these bases, for $n \geq 0$, we obtain a basis for $T^{GL(V)}$, denote it as $\mathcal{B}$.

Theorem 4.4.6 *The basis $\mathcal{B}$ as described above is monomial, i.e. closed under products.*

PROOF: We obtain a description of the k-algebra $T^{GL(V)}$ as follows: Consider the space $\mathfrak{S} := \oplus_{n\geq 0} k\mathfrak{S}_n$ with the following multiplication: for π in $\mathfrak{S}_m$ and σ in $\mathfrak{S}_n$, $\pi \cdot \sigma$ is the permutation in $\mathfrak{S}_{m+n}$ that, as a self-map of $[m+n]$, is given by

$$\pi \cdot \sigma(i) := \begin{cases} \pi(i) & \text{if } i \leq m \\ \sigma(i-m)+m & \text{if } i \geq m+1 \end{cases}$$

For each n, consider the subspace $\mathfrak{P}_n$ of $k\mathfrak{S}_n$ spanned by permutations that have no decreasing subsequence of length more than d (equivalently, the k-span of permutation with RSK-shape having atmost d rows). The direct sum $\mathfrak{P} := \oplus_{n\geq 0}\mathfrak{P}_n$ is a sub-algebra of $\mathfrak{S}$.

The restriction to $\mathfrak{P}_n$ of the canonical map $k\mathfrak{S}_n \to k\mathfrak{S}_n/J(n,d)$ is a vector space isomorphism (Theorem 4.4.1). Let Θ_n be the isomorphism

$$\Theta_n\colon\ \ k\mathfrak{S}_n/J(n,d) \simeq \mathrm{End}_{GL(V)}(V^{\otimes n})$$

induced by ϕ_n. Thus $\oplus_{n\geq 0}\Theta_n$ is a vector space isomorphism of the algebra $\mathfrak{P}$ onto $T^{GL(V)}$. It is evidently also an algebra isomorphism. In particular, $\mathcal{B}$ is closed under products as required. □

4.4.3 Rings of polynomial invariants

In this subsection, k denotes a field of characteristic 0. We first recall the notion of *polynomial invariants* and then describe the well-known process of obtaining them from suitable multilinear invariants. We then use this to obtain a generating set for the polynomial invariants of $(\mathrm{End}_k V)^{\times n}$, as a consequence of results mentioned in the earlier sections.

Let W be a vector space over k of dimension m. A function $f : W \to k$ is called *polynomial* if it is given by a polynomial in the co-ordinates with respect to a basis of W. Let $k[W]$ denote the set of polynomial functions on W, which forms a ring. A polynomial function is called *homogeneous of degree d* if $f(tw) = t^d w$ for all $t \in k, w \in W$. Every polynomial function is in a unique way the sum of homogeneous functions, called its *homogeneous components*. Thus $k[W] = \oplus k[W]_d$ where $k[W]_d$ is the set of polynomial functions that are homogeneous of degree d.

Let $W_1, \ldots, W_r$ be finite dimensional vector spaces such that $W = \oplus_i W_i$. Then, a function $f \in k[W]$ is said to be *multi-homogeneous of degree $\underline{h} = (h_1, \ldots, h_r)$* if $f(t_1v_1, \ldots, t_rv_r) = t_1^{h_1} \cdots t_r^{h_r} f(v_1, \ldots, v_r)$ for all $t_1, \ldots, t_r \in k$ and $v_i \in W_i$. We have a decomposition given by $k[W] = \oplus_{\underline{h}\in\mathbb{N}^r} k[W]_{\underline{h}}$, where $k[W]_{\underline{h}}$ is the set of polynomial functions that are multi-homogeneous of degree $\underline{h}$.

The group $GL(W)$ of automorphisms of W, acts on $k[W]$ where the action is induced from the left action on W. If we consider W as a linear representation of a group G *i.e.,* there is a homomorphism $\rho : G \to GL(W)$, then G acts on W and hence on $k[W]$ via ρ. We define a *polynomial G-invariant* to be a polynomial function which is constant on the G-orbits of W. Denote by $k[W]^G$ the set of polynomial G-invariants. It is a subring of $k[W]$. Also, since the $GL(W)$-action on $k[W]$ preserves the degree of a homogeneous polynomial, we have $k[W]^G = \oplus_{d\in\mathbb{N}} k[W]^G_d$. Similarly, when $W = \oplus_i W_i$ we have $k[W]^G = \oplus_{\underline{h}\in\mathbb{N}^r} k[W]^G_{\underline{h}}$.

Polarisation, Restitution and their generalizations

Polarisation: Let $f \in k[W]$ be a homogeneous function of degree d. Let $v_1, \dots v_d$ be arbitrary d vectors in W. For $t_1, \dots, t_d \in k$, we obtain an expression for $f(t_1v_1 + \cdots + t_dv_d)$ of the form

$$f(t_1v_1 + \cdots + t_dv_d) = \sum_{s_1+\cdots+s_d=d} t_1^{s_1} \dots\ t_d^{s_d} f_{s_1,\dots,s_d}(v_1, \dots, v_d) \tag{4.3}$$

where $f_{s_1,\dots,s_d}$ is a multi-homogeneous function on $W^{\oplus d}$ of degree $(s_1, \dots, s_d)$. Then the *polarisation* of f, denoted as $\mathcal{P}f$, is the multilinear function $f_{1,\dots,1}$ in the above expression.
Restitution: This is the inverse operator (upto a scalar) to polarisation, by which we obtain a homogeneous polynomial from a multilinear function. Let F be multilinear function on $W^{\oplus d}$. The *restitution* of F is the homogeneous polynomial of degree d in $k[W]$ defined by $\mathcal{R}F(u) := F(u, \dots, u)$.

Remark 4.4.7 The operators

$$\mathcal{P} : k[W]_d \to k[W^{\oplus d}]_{(1,\dots,1)} \quad \text{and} \quad \mathcal{R} : k[W^{\oplus d}]_{(1,\dots,1)} \to k[W]_d$$

are $GL(W)$-equivariant. In fact, if we consider W as a linear representation of a group G, then these operators are G-equivariant under the G-action induced on $k[W]$.

Proposition 4.4.8 *Assume char $k = 0$ and let W be a finite dimensional representation of a group G. Then every homogeneous polynomial G-invariant $f \in k[W]^G$ of degree d is obtained by the restitution of a multilinear G-invariant on $W^{\oplus d}$.*

Proof: This follows from Remark 4.4.7 and the observation that $\mathcal{R}\mathcal{P}f = d!f$. Indeed, by setting $v_i = v$ for all $i = 1, \dots, d$, in (4.3) and comparing it with the relation,

$$f((\sum_i t_i)v) = (\sum_i t_i)^d f(v) = (t_1^d + \cdots + d!t_1 \cdots t_d)f(v)$$

we get that f is the restitution of $\frac{1}{d!}\mathcal{P}f$, which is a G-multilinear invariant whenever f is a homogeneous G-invariant. □

Generalizing to several copies: For the application we have in mind, we try to slightly generalize the above notions.

Let $f \in k[W^{\oplus r}]$ be a multi-homogeneous polynomial of degree $\underline{h} = (h_1, \dots, h_r)$. Let $\mathcal{P}_i$ denote the polarisation with respect to the i-th variable. Then the *polarisation* of f is defined as

$$\mathcal{P}f := \mathcal{P}_r\mathcal{P}_{r-1}\cdots\mathcal{P}_1 f \in k[W^{\oplus h_1} \oplus \cdots \oplus W^{\oplus h_r}]$$

It is a multilinear polynomial since it is linear in each variable, by construction. As earlier, we define the *restitution* $\mathcal{R}F$ of a multilinear function $F \in k[W^{\oplus h_1} \oplus \cdots \oplus W^{\oplus h_r}]$ by

$$\mathcal{R}F(v_1, \dots, v_r) = F(\underbrace{v_1, \dots, v_1}_{h_1}, \cdots, \underbrace{v_r, \dots, v_r}_{h_r})$$

If we consider the $GL(W)$-action on W^d acting diagonally then the operators $\mathcal{P}$, $\mathcal{R}$ are $GL(W)$-equivariant. And as before, we have the following

Proposition 4.4.9 *Assume char $k = 0$. Let W be a finite dimensional representation of a group G. Let $n \geq 0$ be fixed. Then for each $\underline{h} = (h_1, \dots, h_r) \in \mathbb{N}^r$ such that $h_1 + \cdots + h_r = n$ the restitution of a multilinear invariant on $W^{h_1} \oplus \cdots \oplus W^{h_r}$ gives a homogeneous polynomial invariant on $W^{\oplus r}$ of degree n. Further, as $\underline{h}$ varies, these elements linearly span $k[W^{\oplus r}]^G_n$.*

Proof: The first part of the statement just follows from the definition of the operator $\mathcal{R}$ along with the observation that a multi-homogeneous invariant of degree $(h_1, \dots, h_r)$ is homogeneous of degree $h_1 + \cdots + h_r = n$. For obtaining the second part, we notice that if g is a multi-homogeneous function of degree $\underline{h} = (h_1, \dots, h_r)$ then $\mathcal{R}\mathcal{P}g = h_1! \cdots h_r! g$. Now use the decomposition of $f \in k[W^r]_n$ as a sum of multi-homogeneous functions given by

$$f(v_1, \dots, v_r) = \sum_{s_1 + \cdots + s_d = n} f_{s_1, \dots, s_d}(v_1, \dots, v_d)$$

□

Invariants of $n \times n$ matrices

With all the preliminaries in place, we finally come to the aim of this subsection which is to describe a basis for ring of polynomial invariants of $(\mathrm{End}_k V)^{\times n}$.

For a fixed integer $m \geq 0$, given a permutation σ of m elements and a map ν of $[m]$ to $[n]$ (we use the notation $[m]$ to denote $\{1, \dots, m\}$), consider the function $f(\sigma, \nu)$ defined on $(\mathrm{End}_k V)^{\times n}$ as follows: writing σ as a product $(i_1 i_2 \cdots) \cdots (i_{p+1} i_{p+2} \cdots)$ of disjoint cycles,

$$f(\sigma, \nu)(A_1, \dots, A_n) := \mathrm{Trace}(A_{\nu(i_1)} A_{\nu(i_2)} \cdots) \cdots \mathrm{Trace}(A_{\nu(i_{p+1})} A_{\nu(i_{p+2})} \cdots)$$

The $f(\sigma, \nu)$ are clearly $GL(V)$-invariant polynomials of degree m. We obtain,

Theorem 4.4.10 *For $m \in \mathbb{Z}^{\geq 0}$ fixed, the invariant functions $f(\sigma, \nu)$, where σ varies over permutations of $\{1, \ldots, m\}$ that do not have any decreasing subsequence of length exceeding d; ν varies over all maps from $[m]$ to $[n]$, form a k-linear spanning set for the ring of GL(V)-invariant polynomial functions on $(\mathrm{End}_k V)^{\times n}$ of degree m.*

Proof: By Proposition 4.4.9, we know that the restitution, with respect to various possible multi-degrees $\underline{h} = (h_1, \ldots, h_r) \in \mathbb{N}^r$ such that $h_1 + \cdots + h_r = n$, of multilinear invariants on $(\mathrm{End}_k V)^{\times m}$ gives a spanning set for the ring of polynomial invariants on $(\mathrm{End}_k V)^{\times n}$. However, Theorem 4.4.5 gives a sub-collection of multilinear invariants $\{T_\sigma \mid \sigma \in \mathfrak{S}_m;\ \sigma$ *has no decreasing subsequence of length exceeding* $d\}$ that forms a basis for the ring of multilinear invariant functions on $(\mathrm{End}_k V)^{\times m}$. Since $\mathcal{R}$ is a linear operator, the images of these T_σ's under the restitution $\mathcal{R}$ would suffice to span the ring of polynomial invariants. The statement now follows by observing that the the restitution of T_σ with respect to a multi-degree $(h_1, \ldots, h_r)$ whose sum is m, gives the map

$$(A_1, \ldots, A_r) \mapsto T_\sigma(\underbrace{A_1, \ldots, A_1}_{h_1}, \cdots, \underbrace{A_r, \ldots, A_r}_{h_r})$$

which is just $f(\sigma, \nu)$ for a suitable ν. $\square$

Picture invariants

For non-negative integers t and b, set $V_b^t := V^{*\otimes b} \otimes V^{\otimes t}$. Given non-negative integers t_i, b_i, for $i = 1, \ldots, s$, we wish to describe a generating set for the ring of polynomial $\mathrm{GL}_k(V)$-invariant functions on the space $W = V_{b_1}^{t_1} \times \cdots \times V_{b_s}^{t_s}$ of several tensors.

In [DKS03, §3], the notion of a 'picture invariant' is introduced, generalizing the functions $f(\sigma, \nu)$ defined above. Picture invariants span the space of invariant polynomial functions ([DKS03, Proposition 7]) on W.

We recall from [DKS03] the definition of picture invariants. Choose a basis $v_1, \ldots, v_d$ for V and let $v^1, \ldots, v^d$ be the dual basis of V^*. Let $T_{l_1 \cdots l_t}^{u_1 \cdots u_b}$ be the co-ordinate function on V_b^t that is 1 on the basis element $v_{u_1} \otimes \cdots \otimes v_{u_t} \otimes v^{l_1} \otimes \cdots \otimes v^{l_b} \in V_b^t$ and 0 on the other basis elements. The ring of polynomial functions on the space V_b^t can be identified with $k[T_{l_1 \cdots l_t}^{u_1 \cdots u_b}]$ in d^{t+b} variables. More generally, for W (as defined above) the co-ordinate ring can be identified with the polynomial ring $k[T(i)_{l_1 \cdots l_{t_i}}^{u_1 \cdots u_{b_i}}]$ in $\sum_{i=1}^s d^{t_i + b_i}$ variables. For non-negative integers $m_1, \ldots, m_s$ such that $\sum_{i=1}^s m_i t_i = \sum_{i=1}^s m_i b_i = N$ and $\sigma \in \mathfrak{S}_N$, by the associated *picture invariant* on W, we mean the following element of $k[W]$:

$$\sum_{(r_1, \ldots, r_N) \in [d]^N} \prod_{i=1}^{s} \left(\prod_{j=1}^{m_i} T(i)_{r_{(\sum_{p<i} m_p b_p + (j-1)b_i + 1)}, \cdots r_{(\sum_{p<i} m_p b_p + j b_i)}}^{r_{\sigma(\sum_{p<i} m_p b_p + (j-1)b_i + 1)}, \cdots, r_{\sigma(\sum_{p<i} m_p b_p + j b_i)}} \right)$$

Example 4.4.11 Let $W = V_2^1 \oplus V_1^2 \oplus V_0^1$ and $m_1 = 2,\ m_2 = 1,\ m_3 = 1$. The picture invariant corresponding to the permutation $(123)(45)$ is $\sum T(1)_{r_1 r_2}^{r_2} T(1)_{r_3 r_4}^{r_3} T(2)_{r_5}^{r_1 r_5} T(3)^{r_4}$.

Note that $k[W]$ can be identified with

$$Sym_k(W^*) = \oplus_{j\geq 0} Sym_k^j(W^*) = \oplus_{j\geq 0} \oplus_{\{(m_1,\cdots,m_s):\sum m_i = j\}} \otimes_{i=1}^s Sym_k^{m_i}((V_{b_i}^{t_i})^*).$$

The natural map from $\otimes_{i=1}^k((V_{b_i}^{t_i})^*)^{\otimes m_i} \to \otimes_{i=1}^s Sym_k^{m_i}((V_{b_i}^{t_i})^*)$ leads to a surjection from the space of $GL(V)$-invariants V_M^N to the space $(\otimes_{i=1}^s Sym_k^{m_i}((V_{b_i}^{t_i})^*))^{GL(V)}$, where $M = \sum m_i t_i$, $N = \sum m_i b_i$. This followed by the observation that non-zero $GL(V)$-invariants exist only if $N = M$, and in that case, $V_N^N \cong \mathrm{End}(V^{\otimes N})$ enables us to use Theorem 4.4.1. So we get that the space of $GL(V)$-invariants of V_N^N is spanned by the elements $\sum_{(r_1,\ldots,r_N)} v_{r_1} \otimes \cdots \otimes v_{r_N} \otimes v^{r_{\sigma(1)}} \otimes \cdots \otimes v^{r_{\sigma(N)}}$ as σ varies over $\mathfrak{S}_N$ with no decreasing sub-sequence of length exceeding d and hence, we also obtain a spanning set for $(\otimes_{i=1}^s Sym_k^{m_i}((V_{b_i}^{t_i})^*))^{GL(V)}$ (by the surjection above). Chasing through the above isomorphisms, it can be seen that these elements are precisely the picture invariants. Thus,

Theorem 4.4.12 (Compare [DKS03, Proposition 7]) *Only those picture invariants with underlying permutations having no decreasing sub-sequences of length exceeding d suffice to span as a k-vector space the ring of $GL_k(V)$-invariant polynomial functions on the space $V_{b_1}^{t_1} \times \cdots \times V_{b_s}^{t_s}$ of several tensors.* □

Chapter 5

Cell modules: RSK basis, irreducibility

We have seen in the previous chapter that the image of $\mathbb{C}\mathfrak{S}_n$ in the endomorphism ring of the tabloid representation permits a basis consisting of permutations. Over $\mathbb{C}$, as the Specht module (associated to a partition of n) is an irreducible module for $\mathbb{C}\mathfrak{S}_n$ a more natural question to ask is whether the endomorphism ring of the Specht module permits such a basis.

By the RSK-correspondence we know that the number of permutations of RSK-shape λ is the same as the dimension of $\mathrm{End}_{\mathbb{C}} S^\lambda$, so an obvious choice for a basis as above would be the collection of all permutations of RSK-shape λ. However, this choice fails to form a basis in general as is illustrated by the example in §5.1.1. Failing to find a suitable choice of permutations that form a basis for $\mathrm{End}_{\mathbb{C}} S^\lambda$, we look for possible candidates in the group algebra $\mathbb{C}\mathfrak{S}_n$.

Going through the proof of Theorem 4.2.1, one immediately notices that the Kazhdan-Lusztig basis elements parametrized by the permutations of RSK-shape λ suggest themselves to form a basis that we seek. In Theorem 5.1.1 we make the precise statement and prove it using the same ideas as in Theorem 4.2.1. More generally, we prove in Theorem 5.1.2 that for any finite-dimensional representation U of $\mathbb{C}\mathfrak{S}_n$, we obtain a basis for the image of $\mathbb{C}\mathfrak{S}_n$ in the endomorphism ring $\mathrm{End}_{\mathbb{C}}(U)$. All these statements are presented and proved in the set-up of the Hecke algebra $\mathcal{H}_k$ and its right cell modules $R(\lambda)_k$, where k is a field over which $\mathcal{H}_k$ is semisimple. Recalling that the cell module $R(\lambda)$ is isomorphic to S^λ (§3.4), we note that the same statements are true also for the Specht modules of $\mathcal{H}_k$.

The proof of Theorem 5.1.1 relies on the semisimplicity of $\mathcal{H}_k$ which is not true in general. As was indicated also in the introduction, we deal with the case when $\mathcal{H}_k$ is not semisimple by a head-on approach via the matrix $\mathcal{G}(\lambda)$ defined in §5.2.1. The matrix $\mathcal{G}(\lambda)$ encodes the action of $\mathcal{H}$ on $R(\lambda)$ in a systematic way, described in §5.2.2. The determinant of this matrix turns out to be related to the determinant of a bilinear form defined on $R(\lambda)$. This is done in §5.2.3. This also turns out to be the key step in arriving at a bases for $\mathrm{End}R(\lambda)_k$ in the non-semisimple case giving analogues of Theorem 5.1.1 under certain additional conditions. In §5.1.3, these analogues are formulated but their proofs are deferred to §5.4, after all the ground work required for it is done.

It is easy to see that if the module $R(\lambda)_k$ is not irreducible then a basis for $\mathrm{End}R(\lambda)_k$ as in Theorem 5.1.1 is not possible. We pursue this idea to obtain a criterion for the irreducibility of $R(\lambda)_k$ in terms of the determinant of $\mathcal{G}(\lambda)$, stated precisely in Theorem 5.3.1.

5.1 RSK Bases for End$R(\lambda)_k$

Let k be an arbitrary field and $a \in k$ be invertible. Let $\mathcal{H}_k$ denote the specialization of $\mathcal{H}$ via $v \mapsto a$ (refer §2.2).

In this section, we focus on presenting a basis for the endomorphism ring of the right cell module $R(\lambda)_k$ corresponding to a partition $\lambda \vdash n$. The case when $\mathcal{H}_k$ is semisimple is treated differently from the case when it is not. In the semisimple case we use the Wedderburn theory that we had discussed in §3.3.5. The non-semisimple case is more complicated and the result is proved with a hypothesis on the partition λ, namely that it is e-regular (see §5.1.3).

Before we proceed further, it is good to note that under the isomorphism given in Proposition 3.4.3, all the discussion in this section extends verbatim to the endomorphism ring of the corresponding Specht module.

5.1.1 An illustrative example

The purpose of this example is to show that images in $\mathrm{End}_{\mathbb{C}}R(\lambda)_{\mathbb{C}}$ of permutations of RSK-shape λ do not in general form a basis of $\mathrm{End}_{\mathbb{C}}R(\lambda)_{\mathbb{C}}$.

Let $n = 4$ and $\lambda = (2,2)$. Then $R(\lambda)_{\mathbb{C}}$ is the unique 2 dimensional complex irreducible representation of $\mathfrak{S}_4$. Consider the action of $\mathfrak{S}_4$ on partitions of $\{1,2,3,4\}$ into two sets of two elements each. There being three such partitions, we get a map $\mathfrak{S}_4 \to \mathfrak{S}_3$, which is surjective and has kernel $\{\text{identity}, (12)(34), (13)(24), (14)(23)\}$. Pulling back the 2-dimensional complex irreducible representation of $\mathfrak{S}_3$ via the above map, we get an irreducible 2-dimensional representation of $\mathfrak{S}_4$, which therefore has to be $R(\lambda)_{\mathbb{C}}$. Thus the elements in the kernel of $\mathfrak{S}_4 \to \mathfrak{S}_3$ are also in the kernel of $\mathfrak{S}_4 \to \mathrm{End}_{\mathbb{C}}R(\lambda)_{\mathbb{C}}$. In $\mathfrak{S}_4$, the permutations of RSK-shape λ are (13)(24), (1342), (1243) and (12)(34). The first and last of these are in the kernel of $\mathfrak{S}_4 \to \mathrm{End}_{\mathbb{C}}R(\lambda)_{\mathbb{C}}$ so they are in fact mapped to the same element in $\mathrm{End}_{\mathbb{C}}R(\lambda)_{\mathbb{C}}$.

5.1.2 The case of semisimple $\mathcal{H}_k$

In this subsection, we assume that $\mathcal{H}_k$ is semisimple. From [DJ86, Theorem 4.3] we recall that $\mathcal{H}_k$ is semisimple except precisely when

- either $a^2 = 1$ and the characteristic of $k \leq n$

- or $a^2 \neq 1$ is a primitive r^{th} root of unity for some $2 \leq r \leq n$.

Theorem 5.1.1 *For λ a partition of n, the images in $\mathrm{End}_k R(\lambda)_k$ of the Kazhdan-Lusztig basis elements C_x, $\text{RSK-shape}(x) = \lambda$ form a basis (for $\mathrm{End}_k R(\lambda)_k$).*

PROOF: By Theorem 3.3.18(4), $\mathcal{H}_k \simeq \oplus_{\lambda \vdash n} \mathrm{End} R(\lambda)_k$. The projections to $\mathrm{End} R(\nu)_k$ of C_x, $\text{RSK-shape}(x) \trianglelefteq \lambda$, vanish if $\nu \not\trianglelefteq \lambda$ (§4.1.2). Therefore the projections of the same elements to $\oplus_{\mu \trianglelefteq \lambda} \mathrm{End} R(\mu)_k$ form a basis: note that the number of such elements equals $\sum_{\mu \trianglelefteq \lambda} \dim \mathrm{End}_k R(\mu)_k$. Again by §4.1.2, the projections of C_x, $\text{RSK-shape}(x) \triangleleft \lambda$, vanish in $\mathrm{End} R(\lambda)_k$. This implies that the projections of C_x, $\text{RSK-shape}(x) = \lambda$, in $\mathrm{End} R(\lambda)_k$ form a spanning set. Since the number of such C_x equals $\dim \mathrm{End} R(\lambda)_k$, the theorem follows.□

Theorem 5.1.2 *Let U be a finite dimensional representation of $\mathcal{H}_k$ and $\mathcal{S}$ the subset of partitions λ of n such that $R(\lambda)_k$ appears in a decomposition of U into irreducibles. Then the images in $\mathrm{End}_k U$ of C_x, $x \in \mathfrak{S}_n$ such that $\text{RSK-shape}(x) \in \mathcal{S}$ form a basis for the image of $\mathcal{H}_k$ (under the map $\mathcal{H}_k \to \mathrm{End}_k U$ defining U).*

PROOF: It is enough to prove the assertion assuming $U = \oplus_{\lambda \in \mathcal{S}} R(\lambda)_k$. The image of $\mathcal{H}_k$ in $\mathrm{End}_k U$ is $\oplus_{\lambda \in \mathcal{S}} \mathrm{End}_k R(\lambda)_k$ (Theorem 3.3.18 (1), density theorem, and [Bou73, Corollaire 2, page 39]). Proceed by induction on the cardinality of $\mathcal{S}$. It is enough to show that the relevant images in $\mathrm{End}_k U$ are linearly independent, for their number equals the dimension of $\oplus_{\lambda \in \mathcal{S}} \mathrm{End}_k R(\lambda)_k$. Suppose that a linear combination of the images vanishes. Choose $\lambda \in \mathcal{S}$ such that there is no $\mu \in \mathcal{S}$ with $\lambda \triangleleft \mu$. Projections to $\mathrm{End}_k R(\lambda)_k$ of all C_x, $\lambda \neq \text{RSK-shape}(x) \in \mathcal{S}$, vanish (§4.1.2). So projecting the linear combination to $\mathrm{End}_k R(\lambda)_k$ and using Theorem 5.1.1, we conclude that the coefficients of C_x, $\text{RSK-shape}(x) = \lambda$, are all zero. The induction hypothesis applied to $\mathcal{S} \setminus \{\lambda\}$ now finishes the proof. □

5.1.3 The case of arbitrary $\mathcal{H}_k$

The results in this section are true for arbitrary $\mathcal{H}_k$.

Let $a \in k$ be such that $\mathcal{H}_k$ is the specialization of $\mathcal{H}$ via $v \mapsto a$. We denote by e the smallest positive integer such that $1 + a^2 + \cdots + a^{2(e-1)} = 0$; if there is no such integer, then $e = \infty$. Recall that, a shape λ is called *e-regular* if the number of rows in it of any given length is less than e.

Theorem 5.1.3 *For an e-regular shape λ such that $R(\lambda)_k$ is irreducible, the Kazhdan-Lusztig basis elements C_w, w of RSK-shape λ, thought of as operators on $R(\lambda)_k$ form a basis for $\mathrm{End}_k R(\lambda)_k$.*

(Proof deferred until §5.4).

Theorem 5.1.4 *Suppose that λ', the transpose shape of λ, is e-regular and that $R(\lambda)_k$ is irreducible. Then the elements C'_w, $\text{RSK-shape}(w) = \lambda'$, as operators on $R(\lambda)_k$ form a basis for $\mathrm{End}_k R(\lambda)_k$.*

(Proof deferred until §5.4)

Theorem 5.1.5 *Let $\mathcal{S}$ be the set of e-regular shapes λ such that the Specht module $R(\lambda)_k$ is irreducible. Let U be a finite dimensional semisimple $\mathcal{H}_k$-module, every irreducible component of which is of the form $R(\lambda)_k$, $\lambda \in \mathcal{S}$. Let $\mathfrak{T}$ be the subset of $\mathcal{S}$ consisting of those shapes λ such that $R(\lambda)_k$ appears as a component of U. Then the images in* $\mathrm{End}_k U$ *of C_x, $x \in \mathfrak{S}_n$ such that* RSK-shape(x) *belongs to $\mathfrak{T}$, form a basis for the image of $\mathcal{H}_k$ in* $\mathrm{End}_k U$ *(under the map $\mathcal{H}_k \to \mathrm{End}_k U$ defining U).*

PROOF: The proof is similar to that of Theorem 5.1.2. □

Remark 5.1.6 Note that Theorems 5.1.3, 5.1.5 are generalizations of Theorems 5.1.1, 5.1.2 respectively to the case of arbitrary $\mathcal{H}_k$.

5.2 The matrix $\mathcal{G}(\lambda)$ and its determinant

Let λ be a fixed partition of n. Our goal in this section is to study the action of the elements C_w, RSK-shape$(w) = \lambda$, on the right cell module $R(\lambda)$. The motivation for doing this is to prove Theorem 5.1.3.

We observe in §5.2.2 that all information about the action can conveniently be gathered together into a matrix $\mathbb{G}(\lambda)$ which breaks up nicely into blocks of the same size (Proposition 5.2.3). The non-zero blocks all lie along the diagonal and are all equal to a certain matrix $\mathcal{G}(\lambda)$ defined in §5.2.1. This matrix encodes the multiplication table modulo lower cells of the C_w of RSK-shape λ. In §5.2.3, we show that this relates to the matrix of a bilinear form on $R(\lambda)$.

5.2.1 Definition of the matrix $\mathcal{G}(\lambda)$

Let $P_1, \ldots, P_m$ be the complete list of standard tableaux of shape λ.

Lemma 5.2.1 *For i, j, k, $l \in \{1, \ldots, m\}$ we have,*

$$C(P_i, P_j) \cdot C(P_k, P_l) = g_j^k C(P_i, P_l) \mod \langle C_y | \text{RSK-shape}(y) \lhd \lambda,\ y \lneq_{\mathrm{L}} (P_k, P_l),\ y \lneq_{\mathrm{R}} (P_i, P_j) \rangle_A \quad (5.1)$$

PROOF: To prove the above relation, consider the expression of the left hand side as a linear combination of the C-basis elements. For any C_y occurring with non-zero coefficient in this expression, we have $y \leq_{\mathrm{R}} (P_i, P_j)$ and $y \leq_{\mathrm{L}} (P_k, P_l)$, by the definition of the pre-orders (§2.2.2). By Proposition 2.3.5, RSK-shape$(y) \unlhd \lambda$; and if RSK-shape$(y) \neq \lambda$, then $y \lneq_{\mathrm{R}} (P_i, P_j)$ and $y \lneq_{\mathrm{L}} (P_k, P_l)$. If RSK-shape$(y) = \lambda$, then, by Proposition 2.3.7, $y \sim_{\mathrm{LR}} (P_k, P_l)$; by Proposition 2.3.6, $y \sim_{\mathrm{L}} (P_k, P_l)$; so, the Q-symbol of y is P_l; and, analogously, the P-symbol of y is P_i. That g_j^k depends only on the indices i and l follows from the description of the $\mathcal{H}$-isomorphisms between one sided cells of the same RSK-shape as recalled in §3.3.5, and (5.1) is proved. $\square$

Definition 5.2.2 *With g_j^k, $1 \leq j,k \leq m$ given by the above lemma, we define the matrix $\mathcal{G}(\lambda) := (g_j^k)_{1\leq j,k\leq m}$*

5.2.2 Relating the matrix $\mathcal{G}(\lambda)$ to the action on $R(\lambda)$

Enumerate as $P_1, \ldots, P_m$ all the standard Young tableaux of shape λ. Let us write $C(k,l)$ for the C-basis element $C(P_k, P_l)$. Consider the ordered basis $C(1,1)$, $C(1,2)$, $\ldots$, $C(1,m)$ of $R(\lambda)$. Denote by $\mathfrak{e}_i^j$ the element of End$R(\lambda)$ that sends $C(1,i)$ to $C(1,j)$ and kills the other basis elements. Any element End$R(\lambda)$ can be written uniquely as $\sum \alpha_i^j \mathfrak{e}_i^j$, for some $\alpha_i^j \in A$. Arrange the coefficients as a row matrix like this:

$$\left(\begin{array}{ccccccccccccccc} \alpha_1^1 & \alpha_2^1 & \ldots & \alpha_m^1 & | & \alpha_1^2 & \alpha_2^2 & \ldots & \alpha_m^2 & | & \ldots & | & \alpha_1^m & \alpha_2^m & \ldots & \alpha_m^m \end{array} \right)$$

Now consider such row matrices for $\rho_\lambda(C(k,l))$, where $\rho_\lambda : \mathcal{H} \to \mathrm{End}R(\lambda)$ is the map defining the representation of $\mathcal{H}$ on $R(\lambda)$. Arrange them one below the other, the first row corresponding to the value $(1,1)$ of (k,l), the second to $(2,1)$, $\ldots$, the m^{th} row to $(m,1)$, the $(m+1)^{\text{th}}$ row to $(1,2)$, $\ldots$, and the last to (m,m). We thus get a matrix—denote it $\mathbb{G}(\lambda)$—of size $d(\lambda)^2 \times d(\lambda)^2$, where $d(\lambda) := \dim R(\lambda)$.

Let us compute $\mathbb{G}(\lambda)$ in the light of (5.1). Setting $\alpha_i^j(k,l) := \alpha_i^j(\rho_\lambda(C(k,l)))$, we have (mind the abuse of notation: this equation holds in $R(\lambda)$, not in $\mathcal{H}$):

$$C(1,i)\rho_\lambda C(k,l) = \sum_j \alpha_i^j(k,l) C(1,j).$$

The left hand side is just $C(1,i).C(k,l)$, so applying (5.1) to it and reading the result as an equation in $R(\lambda)$, we see that it equals $g_i^k C(1,l)$. Thus

$$\alpha_i^j(k,l) = \begin{cases} g_i^k & \text{if } j = l \\ 0 & \text{otherwise} \end{cases}$$

which means the following:

Proposition 5.2.3 *The matrix $\mathbb{G}(\lambda)$ (defined earlier in this section) is of block diagonal form, with uniform block size $d(\lambda) \times d(\lambda)$, and each diagonal block equal to the matrix $\mathcal{G}(\lambda) = (g_i^k)$ of §5.2.1, where the row index is k and the column index i.* $\square$

5.2.3 The Dipper-James bilinear form on $R(\lambda)$

Pulling back via the isomorphism θ of §3.4 (see (3.7) for definition of θ) the restriction to S^λ of the bilinear form on M^λ defined in §3.3.1, we get a bilinear form on $R(\lambda)$ (which we continue to denote by $\langle , \rangle$). Let us compute the matrix of this form with respect to the basis $C(1,1), \ldots, C(1,m)$, where, as in §5.2.2, $P_1, \ldots, P_m$ is an enumeration of all standard tableaux of shape λ, and $C(k,l)$ is short hand notation for $C(P_k, P_l)$. We further assume that $P_1 = t_\lambda$, so that the right cell with P-symbol P_1 is the one containing $w_{0,\lambda'}$ (which under RSK corresponds to the pair (P_1, P_1)—see Remark 2.3.9

Proposition 5.2.4 *The (i,j)-th entry in the matrix of the bilinear form $\langle , \rangle$ on $R(\lambda)$ is given by,*

$$\langle C(1,i), C(1,j)\rangle = \epsilon_{w_{0,\lambda'}} v_{w_{0,\lambda'}} v_{w_\lambda}^2 g_j^i$$

Proof: The explanations for the steps in the following calculation appear below:

$$\begin{aligned}\langle C(1,i), C(1,j)\rangle &= \langle C(1,i)\theta, C(1,j)\theta\rangle = \langle x_\lambda T_{w_\lambda} v_{w_\lambda} C(1,i), x_\lambda T_{w_\lambda} v_{w_\lambda} C(1,j)\rangle \\ &= v_{w_\lambda}^2 \langle x_\lambda T_{w_\lambda}, x_\lambda T_{w_\lambda} C(1,j) C(1,i)^*\rangle = v_{w_\lambda}^2 \langle x_\lambda T_{w_\lambda}, x_\lambda T_{w_\lambda} C(1,j) C(i,1)\rangle \\ &= v_{w_\lambda}^2 \langle x_\lambda T_{w_\lambda}, x_\lambda T_{w_\lambda} g_j^i C(1,1)\rangle = v_{w_\lambda}^2 g_j^i \langle x_\lambda T_{w_\lambda}, x_\lambda T_{w_\lambda} \epsilon_{w_{0,\lambda'}} v_{w_{0,\lambda'}} y_{\lambda'}\rangle \\ &= \epsilon_{w_{0,\lambda'}} v_{w_{0,\lambda'}} v_{w_\lambda}^2 g_j^i \sum_{w \in W_{\lambda'}} \epsilon_w v_w^{-1} \langle x_\lambda T_{w_\lambda}, x_\lambda T_{w_\lambda} T_w\rangle = \epsilon_{w_{0,\lambda'}} v_{w_{0,\lambda'}} v_{w_\lambda}^2 g_j^i\end{aligned}$$

The first equality follows from definition of the form on $R(\lambda)$; the second from the definition of θ; the third from (3.5); the fourth from the relation $C_w^* = C_{w^{-1}}$ (by definition, $T_w^* = T_{w^{-1}}$, so the characterisation $C_w \equiv T_w \mod \mathcal{H}_{>0}$ implies that $C_w^* = C_{w^{-1}}$). For the fifth, substitute for $C(1,j)C(i,1)$ using (5.1) and observe that the 'smaller terms' on the right hand side belong to the kernel of θ (§3.4). The sixth follows by substituting for $C(1,1) = C_{w_{0,\lambda'}}$ from (3.3); the seventh from the definition of $y_{\lambda'}$; and the final equality by combining the definition of the form and Remark 2.3.14: observe that $T_{w_\lambda} T_w = T_{w_\lambda w}$ since $l(w_\lambda) + l(w) = l(w_\lambda w)$ (similar to Proposition 2.3.10(3)) and that $w_\lambda w$ belongs to $\mathfrak{D}_\lambda$.

□

Having come thus far, we immediately get a formula relating the determinant of the form $\langle , \rangle$ with the determinant of the matrix $\mathcal{G}(\lambda)$.

Corollary 5.2.5 *The determinant of the matrix of the form $\langle , \rangle$ on $R(\lambda)$ with respect to the basis $C(1,1), \ldots, C(1,m)$ equals*

$$\epsilon_{w_{0,\lambda'}}^{d(\lambda)} v_{w_{0,\lambda'}}^{d(\lambda)} v_{w_\lambda}^{2d(\lambda)} \det \mathcal{G}(\lambda) \tag{5.2}$$

□

The above corollary plays a key role for plugging the seemingly extraneous condition of e-regularity in Theorem 5.1.3 (see Proposition 5.4.1).

5.3 A criterion for irreducibility of $R(\lambda)_k$

Let $a \in k$ be invertible. We consider the specialization $\mathcal{H}_k$ via $v \mapsto a$. Since the dimension of $\mathrm{End}_k R(\lambda)_k$ is $d(\lambda)^2$ we observe that if the Kazhdan-Lusztig basis elements C_w, w of

RSK-shape λ, thought of as operators on $R(\lambda)_k$, are linearly independent then they in fact form a basis for $\mathrm{End}_k R(\lambda)_k$. In such a case, the module $R(\lambda)_k$ has to be irreducible. In view of this observation, we have the following:

Theorem 5.3.1 *If* $\det \mathcal{G}(\lambda)|_{v=a}$ *does not vanish in* k*, then* $R(\lambda)_k$ *is irreducible.*

PROOF: Suppose that $\det \mathcal{G}(\lambda)|_{v=a}$ does not vanish in k. Then, by Proposition 5.2.3, the matrix $\mathbb{G}(\lambda)$ is invertible (in k, after specializing to $v = a$). Thus the elements C_w, w of RSK-shape λ, are linearly independent (and so form a basis) as operators on $R(\lambda)_k$. In particular, $R(\lambda)_k$ is irreducible, and the assertion is proved. □

5.4 Proofs of Theorems 5.1.3, 5.1.4

Proposition 5.4.1 *If* λ *is* e*-regular, the bilinear form* $\langle , \rangle_k$ *on* $R(\lambda)_k$ *is non-zero.*

PROOF: This can be readily seen by using Proposition 3.3.11: for e_1, $e_2 \in S_k^\lambda$ given by Proposition 3.3.11, we have by the definition of the bilinear form on $R(\lambda)_k$ that $\langle \theta(e_1), \theta(e_2) \rangle = \langle e_1, e_2 \rangle \neq 0$, as required. □

Proof of Theorem 5.1.3: By (3.5), the radical of the form $\langle , \rangle_k$ on $R(\lambda)_k$ is a $\mathcal{H}_k$-submodule. Since $R(\lambda)_k$ is assumed irreducible, the form is either identically zero or non-degenerate. But, as shown in Proposition 5.4.1 above, it is non-zero under the assumption of e-regularity of λ. Thus its matrix with respect to any basis of $R(\lambda)_k$ has non-zero determinant. By (5.2), $\det \mathcal{G}(\lambda)|_{v=a}$ is such a determinant (up to a sign and power of a), so it is non-zero. It now follows from Proposition 5.2.3 that the operators C_w, w of RSK-shape λ, form a basis for $\mathrm{End} R(\lambda)_k$. □

Proof of Theorem 5.1.4: By Theorem 5.1.3, the C_w, RSK-shape$(w) = \lambda'$, as operators on $R(\lambda')_k$ form a basis for $\mathrm{End} R(\lambda')_k$ (note that $R(\lambda')_k$ is irreducible by Proposition 3.5.3; recall $R(\lambda') \cong S^{\lambda'}$). By Proposition 3.5.3 we have an isomorphism $R(\lambda)_k \simeq (R(\lambda')_k^\dagger)^{\mathrm{dual}\ *}$ the C_w, RSK-shape$(w) = \lambda'$, as operators on $(R(\lambda')_k^\dagger)^{\mathrm{dual}\ *}$ form a basis. By our notations (refer §3.5), we know that the C_w action on the latter space is given by $j(C_w^*) = \epsilon_w C'_{w^{-1}}$ (since $C_w^* = C_{w^{-1}}$, Remark 2.2.7(i)). The result now follows by observing that the RSK-shapes of w and w^{-1} are the same. □

5.4.1 Condition of e-regularity in Theorem 5.1.3

The condition of e-regularity assumed in Theorem 5.1.3 is necessary. In other words, for an arbitrary shape λ it is possible that the module $R(\lambda)_k$ is irreducible, however, the Kazhdan-Lusztig basis elements C_w, w of RSK-shape λ, thought of as operators on $R(\lambda)_k$, do not form a basis for $\mathrm{End}_k R(\lambda)_k$. We illustrate this in the following example.

Let $\mathcal{H}$ be the Hecke algebra corresponding to $\mathfrak{S}_4$. Let $\lambda = (2, 2)$, k a field and $\mathcal{H}_k$ the specialization given by $v \mapsto 1$ (see §2.2, p.11). Consider the right cell module $R(\lambda)_k$. For simplicity of calculations we work with the Specht module associated to λ instead of $R(\lambda)_k$

both of which are isomorphic (Proposition 3.4.3). Recall from §3.2 that the standard basis of the Specht module is given by

$$\mathfrak{e}_T := \sum \epsilon(\sigma)\{T\sigma\}$$

where T is a standard tableau of shape λ; the sum is taken over permutations σ in the column stabiliser of T; and $\{T\sigma\}$ denotes the tabloid corresponding to the tableau $T\sigma$. Hence the following two elements form the standard basis for S_k^λ:

$$e_1 = \begin{array}{c}\hline 1\ \ 2\\ \hline 3\ \ 4\\ \hline\end{array} - \begin{array}{c}\hline 2\ \ 3\\ \hline 1\ \ 4\\ \hline\end{array} - \begin{array}{c}\hline 1\ \ 4\\ \hline 2\ \ 3\\ \hline\end{array} + \begin{array}{c}\hline 3\ \ 4\\ \hline 1\ \ 2\\ \hline\end{array} \qquad e_2 = \begin{array}{c}\hline 1\ \ 3\\ \hline 2\ \ 4\\ \hline\end{array} - \begin{array}{c}\hline 2\ \ 3\\ \hline 1\ \ 4\\ \hline\end{array} - \begin{array}{c}\hline 1\ \ 4\\ \hline 2\ \ 3\\ \hline\end{array} + \begin{array}{c}\hline 2\ \ 4\\ \hline 1\ \ 3\\ \hline\end{array}$$

With respect to the above basis, the elements $T_{(1,2)}$, $T_{(2,3)}$, $T_{(3,4)}$ in $\mathcal{H}_k \cong k\mathfrak{S}_4$ act on S_k^λ as

$$\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

If $\mathrm{char}(k)$ is 2 then $e = 2$. Note that in this case λ is not e-regular (nor is λ' e-regular). Also, by the above calculation it can be easily seen that when the characteristic of k is 2 the Specht module S_k^λ is irreducible.

The permutations w such that RSK-shape$(w) = (2,2)$ are 3142, 2413, 2143, 3412 (where 3142 denotes the permutation given by $1 \mapsto 3,\ 2 \mapsto 1,\ 3 \mapsto 4,\ 4 \mapsto 2$). Using the notation $T(i_1, \ldots, i_j) := T_{(i_1,i_1+1)} \cdots T_{(i_j,i_j+1)}$, the KL-basis elements $C_w \in \mathcal{H}_k$ corresponding to the above four permutations are given by:

$$\begin{aligned}
C_{2132} &= -T(1) - T(3) + T(1,2) + T(1,3) + T(2,1) + T(2,3) + T(3,2) + \\
&\quad - T(1,2,1) - T(1,3,2) - T(2,1,3) - T(2,3,2) + T(2,1,3,2), \\
C_{2413} &= -\mathrm{id} + T(1) + T(2) + T(3) - T(1,3) - T(2,1) - T(2,3) + T(2,1,3), \\
C_{3142} &= -\mathrm{id} + T(1) + T(2) + T(3) - T(1,2) - T(1,3) - T(3,2) + T(1,3,2), \\
C_{2143} &= \mathrm{id} - T(1) - T(3) + T(1,3).
\end{aligned}$$

By the earlier calculations, we note that $T_{(1,2)}$ $(= T(1))$ and $T_{(3,4)}$ $(= T(3))$ act identically on S_k^λ. Also, $T(i)^2 = id$ and $T(1,2,1) = T(2,1,2)$ in $k\mathfrak{S}_4$. Hence it can be easily verified that the above C_w's acts trivially on S_k^λ (since char $k = 2$). $\square$

Chapter 6

A FORMULA FOR det $\mathcal{G}(\lambda)$ AND AN APPLICATION

We begin this chapter by investigating more closely the matrix $\mathcal{G}(\lambda)$ introduced in §5.2.1. This matrix was seen, in §5.2.3, to be related to the matrix of the Dipper-James bilinear form on $R(\lambda)$ (w.r.t the C-basis). In §6.1, we describe the relation between the latter matrix and the matrix of the bilinear form on the Specht module S^λ which was defined in §3.3.2 (w.r.t the standard basis) and thus naturally leading us to an explicit relation between the matrix $\mathcal{G}(\lambda)$ and the matrix of the bilinear form on S^λ (w.r.t the standard basis).

The main result of this chapter is a combinatorial formula for the determinant of $\mathcal{G}(\lambda)$ deduced from the above relation. As an application of this formula we give, in §6.3, a new proof of the Carter conjecture ([Jam78]) regarding irreducibility of the Specht module. The formula is stated in §6.2, while we work through its proof only in §6.4, assuming some known results recalled in §6.4.1.

6.1 Relating det $\mathcal{G}(\lambda)$ and the Gram determinant

The *Gram matrix* of S^λ is the matrix of the restriction to the Specht module S^λ of the bilinear form $\langle , \rangle$ on M^λ (defined in §3.3.1), with respect to its 'standard basis' as given in Theorem 3.3.7. The determinant of this matrix is called the *Gram determinant*, denoted as $\det(\lambda)$.

Our goal in this section is to relate $\mathcal{G}(\lambda)$ to the Gram matrix. Towards this, let us compute the image under the map θ of the T-basis elements of $R(\lambda)$. Given a prefix e of $w_{\lambda'}$, we have

$$\begin{aligned}
C_{w_{0,\lambda'}} T_e \theta &= v_{w_\lambda} x_\lambda T_{w_\lambda} C_{w_{0,\lambda'}} T_e && \text{(by the definition of } \theta \text{ in (3.7))}\\
&= \epsilon_{w_{0,\lambda'}} v_{w_{0,\lambda'}} \left(v_{w_\lambda} x_\lambda T_{w_\lambda} y_{\lambda'}\right) T_e && \text{(by (3.3))}\\
&= \epsilon_{w_{0,\lambda'}} v_{w_{0,\lambda'}} z_\lambda T_e && \text{(by the definition of } z_\lambda \text{ in §3.3)}\\
&= \epsilon_{w_{0,\lambda'}} v_{w_{0,\lambda'}} v_e^{-1} (v_e z_\lambda T_e)
\end{aligned}$$

From the above calculation, we have for prefixes e, e' of w_λ:

$$\langle C_{w_{0,\lambda'}} T_e\theta, C_{w_{0,\lambda'}} T_{e'}\theta\rangle = v^2_{w_{0,\lambda'}} v_e^{-1} v_{e'}^{-1} \langle v_e z_\lambda T_e, v_{e'} z_\lambda T_{e'}\rangle$$

Note that $v_e z_\lambda T_e$ is a standard basis element of S^λ. Hence the right hand side of the above relation is $v^2_{w_{0,\lambda'}} v_e^{-1} v_{e'}^{-1}$ times the (e, e')-th entry of the Gram matrix. On the other hand, we had defined the bilinear form on $R(\lambda)$ in such a way that,

$$\langle C_{w_{0,\lambda'}} T_e\theta, C_{w_{0,\lambda'}} T_{e'}\theta\rangle = \langle C_{w_{0,\lambda'}} T_e, C_{w_{0,\lambda'}} T_{e'}\rangle$$

Thus we conclude that the determinant of the matrix of the bilinear form $\langle,\rangle$ on $R(\lambda)$ with respect to the T-basis equals $v^{2d(\lambda)}_{w_{0,\lambda'}} (\prod_e v_e)^{-2} \det(\lambda)$. Combining this with Proposition 3.3.17, which relates the C-basis of $R(\lambda)$ with its T-basis, and Corollary 5.2.5, we get the following:

Proposition 6.1.1 *The determinant of the matrix $\mathcal{G}(\lambda)$ is given by,*

$$\det \mathcal{G}(\lambda) = (\epsilon_{w_{0,\lambda'}} v_{w_{0,\lambda'}} v^{-2}_{w_{\lambda'}})^{d(\lambda)} (\prod_e v_e)^{-2} \det(\lambda) \tag{6.1}$$

where the product is taken over all prefixes e of $w_{\lambda'}$.

6.2 Hook Formula for the determinant of $\mathcal{G}(\lambda)$

We now give a formula for the determinant of the matrix $\mathcal{G}(\lambda)$. We set the following notation:

- $[\lambda]$:= the set of nodes in the Young diagram of shape λ;
- h_{ab}:= hook length of the node $(a, b) \in [\lambda]$ (see §2.3.1).
- For a positive integer m,

$$[m]_v := v^{1-m} + v^{3-m} + \cdots + v^{m-3} + v^{m-1}$$
$$[m]_q := 1 + v^2 + v^4 + \cdots + v^{2(m-1)}$$

Assuming λ has r rows, we can associate to λ a decreasing sequence—called the *β-sequence*—of positive integers, the hook lengths of the nodes in the first column of λ. The shape can be recovered from the sequence, so the association gives a bijection between shapes and decreasing sequences of positive integers. Given such a sequence $\beta_1 > \ldots > \beta_r$, write $d(\beta_1, \ldots, \beta_r)$ for the number $d(\lambda)$ of standard tableaux of shape λ (§2.3.1). Extend the definition of $d(\beta_1, \ldots, \beta_r)$ to an arbitrary sequence of $\beta_1, \ldots, \beta_r$ of non-negative integers at most one of which is zero as follows: if the integers are not all distinct, then it is 0; if the integers are all distinct and positive, then it is $\operatorname{sign}(w)\, d(\beta_{w(1)}, \ldots, \beta_{w(r)})$ where w is the permutation of the symmetric group $\mathfrak{S}_r$ such that $\beta_{w(1)} > \ldots > \beta_{w(r)}$; if the integers are distinct and one of them—say β_k—is zero, then it is $d(\beta_1 - 1, \beta_2 - 1, \ldots, \beta_{k-1} - 1, \beta_{k+1} - 1, \ldots, \beta_r - 1)$, which is defined by induction on r.

Theorem 6.2.1 (Hook Formula) *For a partition λ of n,*

$$\det \mathcal{G}(\lambda) = \epsilon_{w_{o,\lambda'}}^{d(\lambda)} \prod \left(\frac{[h_{ac}]_v}{[h_{bc}]_v} \right)^{d(\beta_1,\ldots,\beta_a + h_{bc},\ldots,\beta_b - h_{bc},\ldots,\beta_r)} \tag{6.2}$$

with notation as above, where $\beta_1 > \ldots > \beta_r$ is the β-sequence of λ and the product runs over $\{(a,b,c) | (a,c),(b,c) \in [\lambda]$ and $a < b\}$.

The proof of the above theorem is deferred until §6.4.

6.3 A new proof for Carter's conjecture

In this section, as an immediate corollary of the above theorem, we present a new proof of a conjecture of Carter (proved in [Jam78]).

Let p denote the smallest positive integer such that $p = 0$ in k; if no such integer exists, then $p = \infty$. For an integer h, define $\nu_p(h)$ as the largest power of p (possibly 0) that divides h in case p is positive, and as 0 otherwise. Let e be the smallest positive integer such that $1 + a^2 + \cdots + a^{2(e-1)} = 0$; if there is no such integer, then $e = \infty$. For an integer h, define

$$\nu_{e,p}(h) := \begin{cases} 0 & \text{if } e = \infty \text{ or } e \nmid h \\ 1 + \nu_p(h/e) & \text{otherwise} \end{cases}$$

The (e,p)-*power diagram* of shape λ is the filling up of the nodes of the shape λ by the $\nu_{e,p}$'s of the respective hook lengths.

Observe that $e = p$ if $a = 1$. Then the (e,p)-power diagram is just the p-power diagram of Carter [Jam78].

Corollary 6.3.1 [Jam78, JM97] *If the (e,p)-power diagram of λ has either no column or no row containing different numbers, then S_k^λ is irreducible.*

Proof: It is enough to do the case when no column of the (e,p)-power diagram has different numbers: if the condition is met on rows and not on columns, we can pass to λ' and use the observation (Proposition 3.5.3) that S_k^λ is irreducible if and only if $S_k^{\lambda'}$ is.

So assume that in every column of the (e,p)-power diagram the numbers are all the same. We claim that each of the factors $[h_{ac}]_v / [h_{bc}]_v$ on the right hand side of (6.2) makes sense as an element of k and is non-zero. Combining the claim with Theorems 6.2.1 and 5.3.1 yields the assertion.

To prove the claim, we need the following elementary observations, where h denotes a positive integer:

- $[h]_v$ vanishes in k if and only if e is finite and divides h.

- if e is finite and divides h, then $[h]_v = ([h/e]_v)\,|_{v=v^e}\,[e]_v$.
- $a^{2e} = 1$ if e is finite.

If either $e = \infty$ or e does not divide any of the hook lengths in shape λ, then the claim follows from the the first of the above observations. So now suppose that e is finite and divides either h_{ac} or h_{bc}. By our hypothesis, e then divides both h_{ac} and h_{bc}; moreover both h_{ac}/e and h_{bc}/e are divisible by p to the same extent. Using the second and third observations above, we conclude that the image in k of $[h_{ac}]_v/[h_{bc}]_v$ is the same as that of the rational number h_{ac}/h_{bc} (written in reduced form), and so is non-zero. $\square$

6.4 Proof of the hook formula

In this section we establish the formula (6.2) given in §6.2. Throughout this section, we let F denote the field $\mathbb{Q}(v)$, the field of fractions of A.

6.4.1 Preliminaries

We begin by recalling some preliminary results from [DJ87], [JM97], that we will use.

An orthogonal basis for the Specht module[1]

Fix a partition $\lambda \vdash n$. Theorem 3.3.7 presents the standard basis for Specht module S_F^λ. As S_F^λ is equipped with a bilinear form (§3.2) we may also look for a basis that is orthogonal with respect to this bilinear form. We are interested in such an orthogonal basis as a means of simplifying the computation of the determinant of the Dipper-James bilinear form on S_F^λ.

Following the exposition as in [DJ87] (recall difference in notations §2.2) for constructing this basis for S_F^λ, we first define the *Jucys-Murphy elements* of $\mathcal{H}$ as follows: $L_1 := T_1$ and

$$L_k := v^{-1}(T_{(k-1,k)} + T_{(k-2,k)} + \cdots + T_{(1,k)}) \text{ for } k = 2, \ldots, n.$$

Let $\{d \in \mathfrak{S}_n \mid d \text{ is a prefix of } w_{\lambda'}\} = \{d_1, \ldots, d_m\}$ $(m = \dim_A S^\lambda)$ ordered in such a way that $i < j$, if $l(d_i) < l(d_j)$. Then $e_i := v_{d_i} z_\lambda T_{d_i}$, $i = 1, \ldots, m$ is the standard basis of S_F^λ.

For notational convenience we set $q := v^2$. For a Young diagram of shape λ, we define the *residue of the* (i,j)*-node* to be $1+q+\cdots+q^{(j-i-1)}$, if $j \geq i$ and $-q^{-1}-q^{-2}-\cdots-q^{-(i-j)}$ if $i > j$. For $1 \leq k \leq n$, define $r_i(k)$ to be the residue of the node containing k in $S_i := t^\lambda w_\lambda d_i$.

We define operators $E_i \in \mathcal{H}_F$ $(1 \leq i \leq m)$ via the Murphy operators as follows:

$$E_i := \prod_{k=1}^{n} \prod_{\substack{j=1 \\ r_i(k) \neq r_j(k)}}^{m} \frac{L_k - r_j(k)}{r_i(k) - r_j(k)}$$

[1] A more detailed exposition can be found at `http://www.imsc.res.in/~preena/Appendix.pdf`

Define elements $f_i \in S_F^\lambda$ by setting $f_i := e_i E_i^m$ for $i = 1, \ldots, m$.

In the next lemma, we list out the properties satisfied by the elements $f_i \in S_F^\lambda$. That they form an orthogonal basis will follow from these properties.

Lemma 6.4.1 [DJ87, Lemma 4.6] *Let $1 \le i \le m$. Then,*

1. *for $1 \le j \le i-1$, $e_j E_i^m = 0$;*
2. *for $1 \le m \le n$, $f_i L_m = r_i(m) f_i$;*
3. *if $1 \le j \le d$, $i \ne j$ then $\langle f_i, f_j \rangle = 0$;*
4. *$\langle f_i, e_i \rangle = \langle f_i, f_i \rangle$ and $\langle f_i, e_i \rangle = 0$ for $1 \le j \le i-1$.*
5. *$f_i = e_i +$ a linear combination of e_j, $j < i$.* □

We do not include the proofs[1] of the above properties which can be deduced from certain *Garnir relations* satisfied by z_λ.

A combinatorial result

We now recall (without proof) a combinatorial result from [JM97].

Let $\lambda \vdash n$ be a fixed partition. Following the notation as in [JM97, §2.16, p247] we define $\Delta_\mu(\lambda)$ in the special case of $\mu = 1^n$. In this case, $\mathcal{T}_0(\lambda, \mu)$ denotes the set of standard tableaux of shape λ. For each $s \in \mathcal{T}_0(\lambda, \mu)$ and $(i, j) \in [\lambda]$, let

$$\Gamma_s(i,j) = \{(k,l) \in [\lambda] : l < j,\ s_{kl} < s_{ij},\ \text{and } s_{k'l} > s_{ij} \text{ for all } k' > k\}$$

where s_{kl} denotes the entry in the (k,l)-th node of s. Note, $\Gamma_s(i,j)$ is the set of nodes (k,l) such that the l-th column of s lies to the left of the j-th column and s_{kl} is the largest entry in the l-th column which is less that s_{ij}. Set

$$\gamma_s = \prod_{(i,j)\in[\lambda]} \prod_{(k,l)\in\Gamma_s(i,j)} \frac{[j-i+k-l+1]_q}{[j-i+k-l]_q}.$$

The contribution to γ_s from the node $(k,l) \in \Gamma_s(i,j)$ is precisely $[\rho+1]_q/[\rho]_q$ where ρ is the *axial distance* from (k,l) to (i,j) defined to be $(k-i)+(j-l)$ if $k \ge i$ and $j \ge l$.

Finally, we define $\Delta_\mu(\lambda) = \prod_{s\in\mathcal{T}_0(\lambda,\mu)} \gamma_s$. We have the following combinatorial result from [JM97] (also see [Mat99, Theorem 5.2.7]):

Lemma 6.4.2 ([JM97], Corollary 2.30) *With notations as above and let $\alpha_1 > \alpha_2 > \ldots > \alpha_r \ge 0$ be the β-numbers for λ'. Then*

$$\Delta_\mu(\lambda) = \prod \left(\frac{[h_{ab}]_q}{[h_{ac}]_q}\right)^{d(\alpha_1,\ldots,\alpha_b+h_{ac},\ldots,\alpha_c-h_{ac},\alpha_r)}$$

where this product is over $\{a, b, c \mid b < c, (a,b) \text{ and } (a,c) \in [\lambda]\}$. □

6.4.2 Computing the Gram determinant det(λ)

Recall notations as in §6.4.1. Let $d_1, \ldots, d_m$ be the prefixes of $w_{\lambda'}$ ordered so that $i < j$ if $l(d_i) < l(d_j)$. Let $e_i := v_{d_i} z_\lambda T_{d_i}$, $1 \leq i \leq m$ be the standard basis of S^λ (see Theorem 3.3.7). Let f_i, $1 \leq i \leq m$ be the orthogonal basis of S_F^λ as defined in §6.4.1 (see also [DJ87, Theorem 4.7]). The basis f_i is in unitriangular relationship with the e_i (Lemma 6.4.1). Thus we have, $\det(\lambda) = \prod_{i=1}^m \langle f_i, f_i \rangle$.

We would like to give a more explicit formula for the Gram determinant. For this we set some notation. Let $S_1, \ldots, S_m$ be an enumeration of all the standard tableaux of shape λ such that $S_i = t_\lambda d_i$. For i, u such that $1 \leq i \leq m$, $1 \leq u \leq n$, let S_i^u denote the standard tableau obtained from S_i by deleting all nodes with entries exceeding u; set $\gamma_{ui} := \prod_{j=1}^{a-1} [h_{jb}]_q / [h_{jb} - 1]_q$ where (a, b) is the position of the node in S_i^u containing u, h_{jb} is the hook length in S_i^u of the node in position (j, b), and $[s]_q := 1 + v^2 + v^4 + \cdots + v^{2(s-1)}$ for a positive integer s.

With this notation, we get

Lemma 6.4.3 (compare [DJ87, Lemma 4.10]) *For $1 \leq i \leq n$, $\langle f_i, f_i \rangle = v^{2r_i} \prod_{u=1}^n \gamma_{ui}$ where the exponents are given by,*

$$\begin{aligned} r_1 &= l(w_\lambda) - l(w_{0,\lambda'}) \text{ and} \\ r_i &= r_j + 1 \text{ where } 1 \leq j < i \text{ such that } e_i = v e_j T_{(k-1,k)}. \end{aligned}$$

For outlining a proof of the above lemma we will need to recall a result from [DJ87, Theorem 4.9]. The proof of the following result involves properties of the basis f_i listed in Lemma 6.4.1 and some combinatorial observations regarding residues (see §6.4.1 for definition). Being fairly computational we do not include it here.

Theorem 6.4.4 ([DJ87, Theorem 4.9(ii)]) *Let $\lambda \vdash n$, $m = \dim S_F^\lambda$. Let $1 \leq i \leq m$ and $2 \leq k \leq n$. Denote by ρ the axial distance between the nodes occupied by k and $k-1$ in t_i, where the axial distance between the nodes (a, b) and (a', b') such that $a \geq a'$, $b' \geq b$ is defined to be $(a - a') + (b' - b)$. Let $q = v^2$. If $t_i(k-1, k) = t_j$ for some j then*

$$f_i T_{(k,k-1)} = \begin{cases} \frac{q^\rho}{[\rho]_q} f_i + f_j & \text{if } i < j, \\ -\frac{1}{[\rho]_q} f_i + \frac{q[\rho-1]_q[\rho+1]_q}{[\rho]_q^2} f_j & \text{if } i > j \end{cases} \tag{6.3}$$

□

Proof of Lemma 6.4.3: Proceed by induction on i.

<u>$i = 1$</u>: In this case, we have $f_1 = e_1 = z_\lambda$. Substituting for z_λ and in turn for y_λ from their definitions in §3.3, we get

$$\begin{aligned} \langle f_1, f_1 \rangle &= \langle z_\lambda, z_\lambda \rangle \\ &= \langle v_{w_\lambda} x_\lambda T_{w_\lambda} y_{\lambda'}, v_{w_\lambda} x_\lambda T_{w_\lambda} y_{\lambda'} \rangle \\ &= v_{w_\lambda}^2 \sum_{u,u' \in W_{\lambda'}} \epsilon_u \epsilon_{u'} v_u^{-1} v_{u'}^{-1} \langle x_\lambda T_{w_\lambda} T_u, x_\lambda T_{w_\lambda} T_{u'} \rangle \end{aligned}$$

Using in order (3.5), Lemma 2.2.4(4), Remark 2.3.14 and the definition of $\langle , \rangle$, we get:

$$\begin{aligned}
\langle x_\lambda T_{w_\lambda} T_u, x_\lambda T_{w_\lambda} T_{u'}\rangle &= \langle x_\lambda T_{w_\lambda}, x_\lambda T_{w_\lambda} T_{u'} T_{u^{-1}}\rangle \\
&= \langle x_\lambda T_{w_\lambda}, x_\lambda T_{w_\lambda} (T_{u'u^{-1}} + \textstyle\sum_{w\in W_{\lambda'}; w>u'u^{-1}} c_w T_w)\rangle \\
&= \langle x_\lambda T_{w_\lambda}, x_\lambda T_{w_\lambda u' u^{-1}} + \textstyle\sum_{w\in W_{\lambda'}; w>u'u^{-1}} c_w T_{w_\lambda w}\rangle \\
&= \begin{cases} 1 & \text{if } u = u' \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

so that

$$\langle f_1, f_1\rangle = v_{w_\lambda}^2 \sum_{u\in W_{\lambda'}} v_u^{-2} = v_{w_\lambda}^2 v_{w_{0,\lambda'}}^{-2} \sum_{u\in W_{\lambda'}} v_{w_{0,\lambda'}}^2 v_u^{-2} = v_{w_\lambda}^2 v_{w_{0,\lambda'}}^{-2} \sum_{u\in W_{\lambda'}} v_u^2$$

Routine calculations show:

$$\sum_{u\in \mathfrak{S}_n} v_u^2 = [n]_q^! \qquad \text{and} \qquad \sum_{u\in W_{\lambda'}} v_u^2 = [\lambda_1']_q^! \cdots [\lambda_r']_q^!$$

where $[n]_q^! := [n]_q[n-1]_q \cdots [1]_q$ and $\lambda' = (\lambda_1', \ldots, \lambda_r')$. Finally, a pleasant verification, given the fact that $S_1 = t_\lambda$, shows:

$$[\lambda_1']_q^! \cdots [\lambda_r']_q^! = \prod_{u=1}^{n} \gamma_{u1}$$

$\underline{i > 1}$: For $i > 1$ we have, $e_i = e_j T_{(m-1,m)}$ for some $2 \le m \le n$ and $1 \le j < i$. Using this and the properties of f_i listed in Lemma 6.4.1, we get the following equalities:

$$\begin{aligned}
\langle f_i, f_i\rangle &= \langle f_i, e_i\rangle = \langle f_i, e_j T_{(m-1,m)}\rangle = \langle f_i T_{(m-1,m)}, e_j\rangle \\
&= \langle c_i f_i + c_j f_j, e_j\rangle \qquad \text{(by Equation (6.3))} \\
&= c_j \langle f_j, f_j\rangle,
\end{aligned}$$

where $c_j = \frac{q[\rho-1]_q[\rho+1]_q}{[\rho]_q^2}$, ρ is the axial distance between m and $m-1$ in t_i.

Note that the length of the hook joining $m-1$ and m in S_i^m (as well as S_j^m) is $\rho+1$. Since $S_i = S_j(m-1, m)$ is standard and $l(w_\lambda d_i) < l(w_\lambda d_j)$ an argument as in Lemma 2.3.13 shows that the row-index of m is strictly bigger than that of $m-1$ in S_i. With this observation, it can be easily seen that for $1 \le u \le n$, $u \ne m-1,\ m$ we have $\gamma_{ui} = \gamma_{uj}$ and $\gamma_{m-1,i} = \gamma_{mj}$. Further, the term γ_{mi} differs from $\gamma_{m-1,j}$ only in one factor — the former has a factor $[h]_q/[h-1]_q$, while the latter has in its place $[h-1]_q/[h-2]_q$ (owing to the node containing m being removed in $S_j^m - 1$). This immediately leads us to the relation $\gamma_{mi} = q^{-1} c_j \gamma_{k-1,j}$. Putting together the conclusion of the last paragraph with the relations we just obtained along with the induction hypothesis, we arrive at the required statement for the index i in the lemma. Therefore, by induction we are done. □

Putting together Lemma 6.4.3 and the observation made in the beginning of this subsection about the relation between the standard basis e_i and the orthogonal basis f_i, we arrive at the following formula (compare[DJ87, Theorem 4.11]):

Lemma 6.4.5 *The determinant of the Gram matrix of the bilinear form $\langle , \rangle$ on S_F^λ is given by,*

$$\det(\lambda) = v^{2r} \prod_{i=1}^{m} \prod_{u=1}^{n} \gamma_{ui} \tag{6.4}$$

where, the integer $r = d(\lambda)\big(l(w_\lambda) - l(w_{0,\lambda'})\big) + \sum_{i=1}^{m} l(d_i)$. □

Finally, in the next lemma, we use the combinatorial result Lemma 6.4.2 to describe the product on the right side of Equation (6.4) in a simpler way.

Lemma 6.4.6 *For a partition λ and with notations as described above,*

$$\prod_{i=1}^{m} \prod_{u=1}^{n} \gamma_{um} = \prod \left(\frac{[h_{ac}]_q}{[h_{bc}]_q} \right)^{d(\beta_1,\ldots,\beta_a+h_{bc},\ldots,\beta_b-h_{bc},\ldots,\beta_r)} \tag{6.5}$$

where $\beta_1 > \ldots > \beta_r$ is the β-sequence of λ and the product runs over $\{(a,b,c)|(a,c),(b,c) \in [\lambda]$ and $a < b\}$.

Proof: The relation follows by observing that both sides of Equation (6.5) equals $\Delta_{(1^n)}(\lambda')$. We shall explain this in detail below:

By definition, $\mathcal{T}_0(\lambda', (1^n))$ is the collection of standard tableaux of shape λ'. For each $s \in \mathcal{T}_0(\lambda', (1^n))$ the set

$$\Gamma_s(i,j) = \{(k,l) \in [\lambda'] : l < j, s_{kl} < s_{ij}, \text{ and } s_{k'l} > s_{ij} \text{ for all } k' > k\}$$

is just the set of nodes (k,l) that appear in the last row of s^u (refer to notation in §6.4.1) where $u = s_{ij}$ and so the value $j - l + k - i + 1$ is the hook length of the node (i,l) in s^u . In particular also for $s = S_r'$– the transpose of S_r, where S_r is a standard tableau of shape λ. On the other hand the value $j - l + k - i + 1$ is also the hook length of the node (k,j) in S_r^u which we denote as h_{kj}. With this observation, we get

$$\prod_{(k,l)\in\Gamma_s(i,j)} \frac{[j-i+k-l+1]_q}{[j-i+k-l]_q} = \prod_{k:1\le k<j} \frac{[h_{kj}]_q}{[h_{kj}-1]_q}.$$

The right-hand side of this equation is just γ_{um} where $u = s_{ij} = S_r(j,i)$, the entry in the (j,i)-th node of S_r. This along with the definition of γ_s leads us to the relation, $\gamma_s = \prod_{u=1}^{n} \gamma_{um}$. Thus, by the definition of $\Delta_{(1^n)}(\lambda')$ we get that it equals the left-hand side of (6.5) *i.e.*,

$$\Delta_{(1^n)}(\lambda') = \prod_{s\in\mathcal{T}_0(\lambda',(1^n))} \gamma_s = \prod_{m=1}^{d} \prod_{u=1}^{n} \gamma_{um}$$

That the right-hand side of (6.5) is $\Delta_{(1^n)}(\lambda')$ follows immediately from the relation in Lemma 6.4.2 and re-indexing the product on the right-hand side of that relation over nodes of $[\lambda]$ rather than that of $[\lambda']$. $\square$

Thus, Lemma 6.4.5 and Lemma 6.4.6 together allows us to conclude that

Lemma 6.4.7 *With notations as above, we have ,*

$$\det(\lambda) = v^{2r} \prod \left(\frac{[h_{ac}]_q}{[h_{bc}]_q} \right)^{d(\beta_1,\ldots,\beta_a+h_{bc},\ldots,\beta_b-h_{bc},\ldots,\beta_r)} \tag{6.6}$$

where $\beta_1 > \ldots > \beta_r$ *is the* β*-sequence of* λ*,* $r = d(\lambda)\left(l(w_\lambda) - l(w_{0,\lambda'})\right) + \sum_{i=1}^m l(d_i)$ *and the product runs over* $\{(a,b,c)|(a,c),(b,c) \in [\lambda]$ *and* $a < b\}$*.* $\square$

6.4.3 Proof of Theorem 6.2.1

Both sides of Equation (6.2) are elements of A. To prove they are equal, we may pass to the quotient field $F := \mathbb{Q}(v)$ of A.

Combining Equations (6.1) and (6.6), we get

$$\epsilon_{w_{0,\lambda'}}^{d(\lambda)} v_{w_{0,\lambda'}}^{d(\lambda)} \det \mathcal{G}(\lambda) = \prod \left(\frac{[h_{ac}]_q}{[h_{bc}]_q} \right)^{d(\beta_1,\ldots,\beta_a+h_{bc},\ldots,\beta_b-h_{bc},\ldots,\beta_r)} \tag{6.7}$$

The left hand side is an element of A. As to the right hand side, it is regular with value 1 at $v = 0$, since the same is true for $[s]_q$ for every positive integer s. Thus both sides of the equation belong to $1 + v\mathbb{Z}[v]$ and

$$\det \mathcal{G}(\lambda) = \epsilon_{w_{0,\lambda'}}^{d(\lambda)} v_{w_{0,\lambda'}}^{-d(\lambda)} + \text{higher degree terms.}$$

The 'bar-invariance' of the C-basis elements (Theorem 2.2.6(1)) means that:

$$\overline{g_j^k} = g_j^k \text{ for } g_j^k \text{ as in (5.1) and so also } \overline{\det \mathcal{G}(\lambda)} = \det \mathcal{G}(\lambda).$$

Thus $\det \mathcal{G}(\lambda)$ has the form:

$$\epsilon_{w_{0,\lambda'}}^{d(\lambda)} v_{w_{0,\lambda'}}^{-d(\lambda)} + \cdots + \epsilon_{w_{0,\lambda'}}^{d(\lambda)} v_{w_{0,\lambda'}}^{d(\lambda)}$$

the terms represented by $\cdots$ being of v-degree strictly between $-d(\lambda)l(w_{0,\lambda'})$ and $d(\lambda)l(w_{0,\lambda'})$. Equating the v-degrees on both sides of (6.7) gives

$$d(\lambda)l(w_{0,\lambda'}) = \sum d(\beta_1,\ldots,\beta_a+h_{bc},\ldots,\beta_b-h_{bc},\ldots,\beta_r)\left(h_{ac}-h_{bc}\right).$$

Using this and substituting $v^{h_{ac}}[h_{ac}]_v$, $v^{h_{bc}}[h_{bc}]_v$, respectively for $[h_{ac}]_q$, $[h_{bc}]_q$ into (6.7), we arrive at the theorem. $\square$

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