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MEROMORPHIC FUNCTIONS OF LOWER ORDER LESS THAN ONE

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MEROMORPHIC FUNCTIONS OF LOWER ORDER LESS THAN ONE.

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C O N T E N T S

INTRODUCTION.

Polynomials have the following two properties:

1) A polynomial takes on every complex value the same number of times.

2) On large circles $|z| = r$ the absolute value of a polynomial $p(z)$ is large and

$$\lim_{r \to \infty} \frac{|p(re^{i\alpha})|}{|p(re^{i\beta})|} = 1$$

uniformly in α and β.

The example of the exponential function shows that neither of these two properties subsists for entire functions. These lectures discuss the problem of finding analogues for the properties 1) and 2) for entire and meromorphic functions of lower order. In sections 1 and 2 some auxiliary results are given. In 3-5 analogues of property 2) are discussed and in sections 6-8 analogues of property 1).

A knowledge of the fundamentals of Nevanlinna Theory is assumed, such as it can be found in W.K.Hayman's Meromorphic Functions, Chapters 1 and 2.

SECTION 1.

POLYA PEAKS.

Let $G(t)$ be a real-valued function of t defined in $t > t_0 \geq 0$. By a <u>sequence of Pólya peaks of order</u> σ we mean a sequence of positive numbers $\{r_n\}$, $r_n \to \infty$ as $n \to \infty$, such that there are two sequences $\{\epsilon_n\}$, $\{C_n\}$ where $\epsilon_n \to 0$ and $C_n \to \infty$ as $n \to \infty$, with the property that

$$G(t)t^{-\sigma} < G(r_n)r_n^{-\sigma}(1+\epsilon_n) \qquad \left(\frac{r_n}{C_n} < t < C_n r_n\right).$$

LEMMA. *If* $G(t)$ *is an increasing continuous function of* t *and if for every* $\epsilon > 0$

$$\text{(1.1)} \quad \liminf G(t)t^{-\sigma-\epsilon} = 0, \ \limsup G(t)t^{-\sigma+\epsilon} = \infty \qquad (t \to \infty)$$

then $G(t)$ *has sequences of Pólya peaks of order* σ.

PROOF. We shall construct an auxiliary function $\eta(t)$ such that

1. $\eta(t)$ is real-valued and continuous in $t > t_0$
2. $\eta(t) \to 0 \qquad (t \to \infty)$
3. $\eta'(t)$ exists except at isolated points and

$$\eta'(t) = o\left(\frac{1}{t \log t}\right)$$

4. $\phi(t) = G(t).t^{-\sigma+\eta(t)}$

satisfies

$$\underline{\lim}\ \phi(t) = 0, \qquad \overline{\lim}\ \phi(t) = \infty \qquad (t \to \infty)$$

Notice that 3. has the following consequence. There is a function $\epsilon(t)$, $0 < \epsilon(t)$, $\epsilon(t) \to 0$ $(t \to \infty)$ and a function $C(t)$, $C(t) \to \infty$ $(t \to \infty)$, such that

$$(1.2) \qquad \left| \frac{r^{\eta(r)}}{t^{\eta(t)}} - 1 \right| < \epsilon(t) \qquad \left(\frac{r}{C(r)} < t < r\,C(r) \right)$$

To prove the existence of an $\epsilon(t)$ so that (1.2) holds, we first notice that the assertion is equivalent to

$$(1.3) \qquad \left| \log \left(\frac{r^{\eta(r)}}{t^{\eta(t)}} \right) \right| < \epsilon_1(t) \qquad \left(\frac{r}{C(r)} < t < r\,C(r) \right)$$

where $\epsilon_1(t) \to 0$ as $t \to \infty$.

But

$$\left| \log \left(\frac{r^{\eta(r)}}{t^{\eta(t)}} \right) \right| = \left| \int_t^r \frac{d}{du} \left(\log u^{\eta(u)} \right) du \right|$$

$$< \int_t^r \left| \frac{d}{du} \left(\log u^{\eta(u)} \right) \right| |du|$$

$$< \int_t^r \frac{|\eta(u)|}{u} \; |du| + \int_t^r |\eta'(u)| \log u \; |du|$$

By 2.and 3 this inequality becomes

$$\left| \log \frac{r^{\eta(r)}}{t^{\eta(t)}} \right| < \int_t^r o\left(\frac{1}{u}\right) \; |du| = o\left(\left|\log \frac{r}{t}\right| \right)$$

It is now easy to find functions $\epsilon_1(t)$, $C(r)$ such that (1.3) holds, which implies (1.2) for a suitable $\epsilon(t)$.

The actual construction of $\eta(t)$ is as follows: Divide the segment $t > t_o$ of the t-axis into successive intervals $I_1, J_1, I_2, J_2, \ldots$.

In I_k, $\quad \eta'(t) = 0, \; \eta(t) = \dfrac{(-1)^k}{k}$

In J_k, $\quad \eta'(t) = \dfrac{(-1)^{k-1}}{k \; t \log t}$.

The end-points of the intervals are determined in succession by requiring that at the right-hand end-point t_k of I_k

$$\phi(t_k) \leq \frac{1}{k} \qquad (k \text{ odd}, \; k > 1)$$

$$\phi(t_k) \geq k \qquad (k \text{ even})$$

Since in I_k, $\eta(t) = \frac{(-1)^k}{k}$ such values of $\phi(t)$ are certainly possible by the hypothesis (1.1). The intervals J_k are chosen so that $\eta(t)$ varies from the value $\frac{(-1)^k}{k}$ to the value $\frac{(-1)^{k-1}}{k+1}$ in J_k. This is possible, because

$$\int^{\infty} \frac{dt}{t \log t} = \infty .$$

It is obvious that the function $\eta(t)$ constructed in this way satisfies all requirements.

Consider now the function $\phi(t) = G(t)t^{-\sigma+\eta(t)}$. Let $\tau = \tau(t)$ be the least value of x such that

$$\phi(\tau) = \sup_{t_0 \leq x \leq t} \phi(x).$$

Then $\tau(t) \leq t$ and, since $\phi(t) \to \infty$ $(t \to \infty)$, $\tau(t) \to \infty$ with t. Notice also that

$$(1.4) \qquad \underline{\lim} \frac{\tau(t)}{t} = 0 .$$

For, otherwise

$$(1.5) \qquad \tau(t) > At \qquad (t > t_1)$$

for some positive A. But then

$$\phi(t) = G(t)t^{-\sigma+\eta(t)} \geq G(t)t^{-\sigma+\eta(\tau)} \left(\frac{\tau}{t}\right)^{\sigma} \frac{t^{\eta(t)}}{\tau^{\eta(\tau)}}$$

$$\geq \phi(\tau)A^{\sigma} \cdot \frac{1}{2} \qquad (t > t_2)$$

using (1.2) and (1.5). But this contradicts $\underline{\lim}\ \phi(t) = 0$. Therefore (1.4) must hold.

Choose a sequence $\{t_n\}$ such that $t_n \to \infty$, $\frac{\tau(t_n)}{t_n} \to 0$. Put $r_n = \tau(t_n)$. Then

$$\phi(t) \leq \phi(r_n) \qquad (t_o < t < t_n).$$

Replacing $\phi(t)$ by $G(t)t^{-\sigma+\eta(t)}$ and remembering (1.2), the Lemma is established with

$$C_n = \min \left\{ C(r_n), \frac{t_n}{r_n} \right\}$$

and $\epsilon_n = \epsilon(r_n)$.

SECTION 2.

THE APPROXIMATION LEMMA

LEMMA. 2.1 (Approximation Lemma). *Let* $f(z)$ *be a meromorphic function and let* $f(0) = 1$. *Let* $a_1, a_2, \ldots$ *be the zeros and* $b_1, b_2, \ldots$ *be the poles of* $f(z)$. *If* q *is a non-negative integer and if*

$$0 \leq |z| = r < \tfrac{1}{2} R$$

then

$$\log|f(z)| = \mathcal{R}\left\{\gamma_0 + \gamma_1 z + \ldots + \gamma_q z^q\right\} + \sum_{|a|<R} \log\left|E\left(\frac{z}{a},q\right)\right| - \sum_{|b|<R} \log\left|E\left(\frac{z}{b},q\right)\right| + S_q(z,r)$$

where $E(u,q)$ *is the Weierstrass primary factor,*

$$E(u,q) = \begin{cases} 1-u & (q=0) \\ (1-u)\exp\left(u + \frac{u^2}{2} + \cdot + \frac{u^q}{q}\right) & (q>0), \end{cases}$$

$$|S_q(r,z)| < 16.\left(\frac{r}{R}\right)^{q+1} T(2R,f)$$

and

$$\gamma_0 = 0, \; \gamma_m = \frac{1}{\pi\rho^m} \int_{-\pi}^{\pi} \log|f(\rho e^{i\theta})| \, e^{-im\theta} d\theta \qquad (m \geq 1)$$

$(f(z) \neq 0 \;\text{ in }\; |z| \leq \rho)$.

PROOF. The Lemma is a consequence of the Poisson-Jensen formula in the form

$$\log f(z) = \sum_{|a| \leq R} \log \frac{R(a-z)}{R^2-\bar{a}z} - \sum_{|b| \leq R} \log \frac{R(b-z)}{R^2-\bar{b}z}$$

$$+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(Re^{i\theta})| \frac{Re^{i\theta}+z}{Re^{i\theta}-z} \, d\theta + i\,C.$$

valid in $|z| < R$ for suitable determinations of the logarithms. Differentiating $(q+1)$ times

$$\left(\frac{d}{dz}\right)^{q+1} \log f(z) = - \sum_{|a| \leq R} q!\,(a-z)^{-q-1} +$$

$$+ \sum_{|b| \leq R} q!\,(b-z)^{-q-1}$$

$$+ \sum_{|a| \leq R} q!(R^2-\bar{a}z)^{-q-1}\bar{a}^{q+1} - \sum_{|b| \leq R} q!\,\bar{b}^{q+1}(R^2-\bar{b}z)^{-q-1}$$

$$+ \frac{(q+1)!}{\pi} \int_{-\pi}^{\pi} \log|f(Re^{i\phi})| \frac{Re^{i\phi}}{(Re^{i\phi}-z)^{q+2}} \, d\phi$$

(2.1)
$$= - \sum_{|a| \leq R} q!(a-z)^{-q-1} + \sum_{|b| \leq R} q!\,(b-z)^{-q-1}$$
$$+ I(z) + U(z)$$

where

$$|I(z)| < q!\, R^{q+1}\, (R^2-Rr)^{-q-1} \left(n(R,o)+n(R,\infty)\right)$$

(2.2)
$$< q!\,(R-r)^{-q-1} \frac{1}{\log 2} \left(N(2R,o)+N(2R,\infty)\right)$$
$$< \frac{2}{\log 2}\, q!\,(R,r)^{-q-1}\, T(2R,f)$$

and

$$|U(z)| < 2(q+1)! \frac{R}{(R-r)^{q+2}} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(Re^{i\emptyset})\ d\emptyset$$
$$< \frac{2(q+1)!R}{(R-r)^{q+2}} \left\{ m(R,f)+m(R,\tfrac{1}{f}) \right\}$$

(2.3)
$$< \frac{4(q+1)!\,R}{(R-r)^{q+2}}\ T(R,f).$$

If we assume that there is no zero or pole of $f(z)$ on the straight-line segment with end-points 0 and $z = re^{i\theta}$, then we can obtain $\log f(z)$ by $(q+1)$ successive integrations from (2.1). After these integrations, the left-hand side of (2.1) becomes

$$\log f(z) - \sum_{m=0}^{q} \gamma_m z^m$$

where

$$\gamma_m = \frac{1}{m!} \left(\frac{d}{d\zeta}\right)^m \log f(\zeta) \Big|_{\zeta=0}$$

By (2.1) with $R = \rho$ so small that $f(z) \neq 0$ and $\neq \infty$ in $|z| \leq \rho$, $q = m-1$, $z = 0$

$$\gamma_m = \frac{1}{\pi\rho^m} \int_{-\pi}^{\pi} \log |f(\rho e^{i\phi})| e^{-im\phi} \, d\phi$$

The function $-\log E(\frac{t}{d},q)$ has the $(q+1)^{st}$ derivative $q!(d-t)^{-q-1}$ and its first q derivatives at the origin are equal to 0. Therefore $(q+1)$ successive integrations of

$$-\sum_{|a| \leq R} q!(a-z)^{-q-1} + \sum_{|b| \leq R} q!(b-z)^{-q-1}$$

yield

$$\sum_{|a| \leq R} \log E(\frac{z}{a},q) - \sum_{|b| \leq R} \log E(\frac{z}{b},q).$$

In the same way integration of error terms leads to new error terms. By (2.2) and (2.3) these are at most $-\frac{2}{\log 2} T(2R,f) \cdot \log E(\frac{r}{R},q)$ and $-4R \frac{d}{dr} \log E(\frac{r}{R},q+1)T(R,f)$.

For $r < \frac{1}{2} R$

$$- \log E(\tfrac{r}{R},q) = \sum_{q+1}^{\infty} \frac{1}{k}\left(\frac{r}{R}\right)^{k} < \frac{1}{q+1} \sum_{q+1}^{\infty}\left(\frac{r}{R}\right)^{k}$$

$$< \frac{1}{(q+1)} \cdot \left(\frac{r}{R}\right)^{q+1} \frac{1}{1-\frac{r}{R}} \leq 2.\left(\frac{r}{R}\right)^{q+1}$$

and

$$- R \frac{d}{dr} \log E(\tfrac{r}{R},q+1) = \sum_{q+2}^{\infty}\left(\frac{r}{R}\right)^{k-1} \leq 2.\left(\frac{r}{R}\right)^{q+1}.$$

The final result is

$$(2.4) \qquad \log f(z) = \sum_{0}^{q} \gamma_m z^m + \sum_{|a| \leq R} \log E(\tfrac{r}{a},q) - \sum_{|b| \leq R} \log E(\tfrac{r}{b},q) + S;$$

taking real parts

$$\log |f(z)| = \mathcal{R} \sum_{0}^{q} \gamma_m z^m + \sum_{|a| \leq R} \log |E(\tfrac{r}{a},q)|$$

$$- \sum_{|b| \leq R} \log |E(\tfrac{r}{b},q)| + S_1,$$

$$|S_1| \leq |S| < \frac{4}{\log 2}\left(\frac{r}{R}\right)^{q+1} T(2R,f) + 8\left(\frac{r}{R}\right)^{q+1} T(R,f)$$

$$< 16\left(\frac{r}{R}\right)^{q+1} T(2R,f).$$

A simple continuity argument allows us to drop the restriction that the straight line segment from 0 to z must be free of zeros and poles of $f(z)$.

As a first application of the approximation Lemma we shall prove an estimate needed in the sequel.

LEMMA. 2.2 Let $f(z)$ be meromorphic with $f(0) = 1$. Suppose $r < \frac{1}{2} R$. Then

$$T(r,f) < \sum_{|a|<R} \log\left(1+\frac{r}{|a|}\right) + \sum_{|b|<R} \log\left(1+\frac{r}{|b|}\right) + \frac{kr\,T(2R,f)}{R}.$$

PROOF. By the approximation lemma with $q = 0$

$$\log |f(re^{i\theta})| = \sum_{|a|<R} \log\left|1-\frac{z}{a}\right| - \sum_{|b|<R} \log\left|1-\frac{z}{b}\right| + S.$$

$$\log^+ |f(re^{i\theta})| \le \sum_{|a|<R} \log^+ \left|1-\frac{z}{a}\right| + \sum_{|b|<R} \log^+\left(\left|1-\frac{z}{b}\right|\right) + S,$$

where

$$|S| < 16\,\frac{r}{R}\,T(2R,f).$$

Therefore

$$(2.5) \qquad m(r,f) < \sum_{|a|<R} m(r,1-\tfrac{z}{a}) + \sum_{|b|<R} m\left(r, \frac{1}{1-\frac{z}{b}}\right) + S_1$$

$$|S_1| < 16\,\frac{r}{R}\,T(2R,f).$$

By Jensen's formula

$$m(r, \frac{1}{1-\frac{z}{b}}) = m(r,1-\tfrac{z}{b}) - \log^+ \frac{r}{|b|}$$

so that

$$(2.6) \qquad \sum_{|b|<R} m(r, \frac{1}{1-\frac{z}{b}}) = \sum_{|b|<R} m(r, 1-\tfrac{z}{b}) - N(r,f).$$

Since

$$\left|1 - \frac{re^{i\theta}}{d}\right| \leq 1 + \frac{r}{|d|} \qquad (d \neq 0)$$

$$(2.7) \qquad m(r,1-\tfrac{z}{d}) \leq \log(1 + \frac{r}{|d|})$$

The lemma now follows from (2.5), (2.6) and (2.7).

SECTION 3.

SLOWLY GROWING FUNCTIONS.

In this section we show that the relation

$$\log|f(re^{i\theta})| \sim \log M(r) \qquad (r \to \infty)$$

which is valid for polynomials has a very close analogue for entire functions with

$$T(r,f) = O\left((\log r)^2\right).$$

The theorem in this section is a special case of a result of W.K.Hayman who considered subharmonic functions [9].

An ϵ-set is a countable set of discs not containing $z = 0$ and subtending angles at the origin whose sum s is finite. The number s is called the extent of the ϵ-set.

Let E be an ϵ-set. Then

a) If L_θ is the ray $\arg z = \theta$, then $L_\theta \cap E$ is bounded for almost all θ.

For, E can be divided into a finite set of discs and an ϵ-set E_1 of extent $< \epsilon$. If $L_\theta \cap E$ is unbounded, then $L_\theta \cap E_1 \neq \emptyset$. But this means that θ is in a set of measure ϵ.

b) The set of r for which $|z| = r$ meets E is of finite logarithmic measure.

For, $|z| = r$ meets the disc $|z-z_0| < |z_0| \sin \delta$ which subtends 2δ at 0, if

$$r_1 = |z_0| \cdot (1-\sin \delta) < r < |z_0| (1+\sin \delta) = r_2.$$

This set of r has the logarithmic length

$$\int_{r_1}^{r_2} \frac{dt}{t} = \log \frac{1+\sin \delta}{1-\sin \delta} .$$

For $\delta < \frac{\pi}{4}$, $\log \frac{1+\sin \delta}{1-\sin \delta} < A\delta$, which proves our assertion.

THEOREM 3.1. <u>If</u> $f(z)$ <u>is a transcendental entire function with</u>

$$T(r,f) = O\left((\log r)^2\right)$$

<u>then</u>

$$\log |f(re^{i\theta})| \sim \log M(r,f) \sim T(r,f)$$

<u>as</u> $re^{i\theta} \to \infty$ <u>outside an</u> ϵ-<u>set</u>.

PROOF. The function $f(z)$ is of order 0 and therefore

$$f(z) = C z^s \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{a_\nu}\right)$$

Omission of the factor Cz^s changes $\log |f|$, $\log M(r,f)$ and $T(r,f)$ by a $O(\log r)$-term. It is therefore enough to consider the case $f(0) = 1$,

$$f(z) = \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{a_\nu}\right).$$

Then

$$\log |f(re^{i\theta})| \leq \int_0^{\infty} \log(1+\tfrac{r}{t})\, dn(t) = r \int_0^{\infty} \frac{n(t)\,dt}{t(t+r)} .$$

Now

$$n(t) \leq \frac{1}{\log t} \int_t^{t^2} \frac{n(u)}{u}\, du \leq \frac{1}{\log t} \; N(t^2) = O(\log t)$$

(3.1)

$$r \int_0^{\infty} \frac{n(t)}{t(t+r)}\, dt < \int_0^{r} \frac{n(t)}{t}\, dt + O\left(\int_r^{\infty} \frac{r \log t}{t^2}\, dt \right)$$

$$< N(r) + O(\log r).$$

Therefore

(3.2) $$\log |f(re^{i\theta})| < N(r) + O(\log r) \sim N(r).$$

In particular

$$\log M(r,f) < N(r) + O(\log r) \sim N(r).$$

On the other hand, by Jensen's formula

$$N(r) \leq \log M(r,f)$$

so that

$$\log M(r,f) \sim N(r). \tag{3.3}$$

We shall prove also that

$$\log M(r,f) - \log |f(re^{i\theta})| = o\big(N(r)\big). \tag{3.4}$$

uniformly in θ as $re^{i\theta} \to \infty$ outside an ϵ-set. The theorem follows at once from (3.3) and (3.4). Let $2^k \leq r < 2^{k+1}$ $(k \geq 1)$.

$$0 \leq \log M(r,f) - \log|f(re^{i\theta})| \leq \sum \log\left(1+\frac{r}{|a|}\right) - \sum \log\left|1-\frac{re^{i\theta}}{a}\right| = S$$

$$S \equiv \sum_{|a|<2^{k-1}} \log \frac{|a|+r}{|a-re^{i\theta}|} + \sum_{2^{k-1}\leq|a|<2^{k+2}} + \sum_{2^{k+2}\leq|a|}$$

$$= S_1+S_2+S_3.$$

S_1 is easily estimated. In S_1, since $\log \frac{1+x}{1-x} < 3x \ (0<x<\frac{1}{2})$

$$0 < \log \frac{|a|+r}{|a-re^{i\theta}|} < \log \frac{1+\frac{|a|}{r}}{1-\frac{|a|}{r}} < 3\,\frac{|a|}{r}$$

$$S_1 < \frac{3}{r}\int_0^{\frac{1}{2}r} t\, dn(t) < 3\int_0^{\frac{1}{2}r} dn(t) = O(\log r),$$

by (3.1). Since $f(z)$ is transcendental, $\log r = o\big(N(r)\big)$, so that

$$S_1 = o(N(r)).$$

Similarly

$$0 \leq S_3 \leq 3 \sum_{|a| \geq 2r} \frac{r}{|a|} \leq 3r \int_{2r}^{\infty} \frac{dn(t)}{t} < 3r \int_{2r}^{\infty} \frac{n(t)\,dt}{t^2}$$

$$\leq O\left(3r \int_{2r}^{\infty} \frac{\log t}{t^2}\,dt\right) = O\left(r \cdot \frac{\log r}{r}\right) = o(N(r)).$$

It remains to show

$$(3.5) \qquad \frac{S_2(re^{i\theta})}{N(r)} \to 0 \quad \text{as} \quad re^{i\theta} \to \infty \text{ outside an } \epsilon\text{-set.}$$

We need

CARTAN'S LEMMA. *If* $P(z) = \prod(z-\alpha_r)$ *is a polynomial of degree* q, *then*

$$\log |P(z)| > q \log h.$$

outside circles containing the α_r *the sum of whose radii is less than* $2\,eh$

LEMMA 3.1. *There is a constant* A *such that in* $2 < r < R$

$$(3.6) \qquad S_2(z) < A\lambda \log R \qquad (z = re^{i\theta})$$

outside an ϵ*-set of extent* $e^{-\lambda}$ *for all* $\lambda \geq 8$.

PROOF. Suppose $2^k \leq r < 2^{k+1} < 2R$. Let

$$\mu_k = n(2^{k+2}) - n(2^{k-1}).$$

Then by Cartan's Lemma

$$S_2(re^{i\theta}) = \sum_{2^{k-1} < |a| < 2^{k+2}} \log \frac{r+|a|}{|re^{i\theta}-a|}$$

$$\leq \mu_k \log(2^{k+2} + 2^{k+1}) - \log \left|\prod (re^{i\theta}-a)\right|$$

$$< \mu_k \log \frac{2^{k+3}}{h_k}$$

provided $re^{i\theta}$ is outside a set E_k of discs the sum of whose radii is at most $2e\, h_k$.

The choice

$$h_k = 2^{k+3}\, e^{-\frac{A\lambda \log R}{\mu_k}}$$

makes

(3.7) $$S_2(z) < A\lambda \log R \qquad (z \notin E_k,\ 2^k \leq |z| < 2^{k+1}).$$

Let α_k be the angle subtended by E_k. In view of (3.1) for sufficiently large A

$$\alpha_k < 2^{-k+3} \sum \rho < 2^{-k+4}\, e\, h_k$$

$$< 2^7 \cdot e^{-\frac{A\lambda \log R}{\mu_k}} < 2^7\, e^{-\lambda - \frac{1}{2}\frac{A \log R}{\mu_k}\lambda}$$

$$< \frac{2^8 \mu_k}{A \log R}\, e^{-\lambda}, \qquad \lambda \geq 8$$

since for $x > 8$, $e^{-x} < \frac{1}{x}$.

Every point of the plane is in at most 3 of the annuli $2^{k-1} < |\zeta| \leq 2^{k+2}$. Therefore

$$\mu_1 + \mu_2 \cdots + \mu_\ell \leq 3n(2^{\ell+2})$$

Summing (3.8) over all integers from 1 to $\ell = \left[\frac{\log R}{\log 2}\right] + 1$ shows that $E = \bigcup_{k=1}^{\ell} E_k$ is an ϵ-set of extent less than

$$e^{-\lambda} \frac{2^8\, 3n(2^{\ell+2})}{A \log R} < e^{-\lambda} \frac{2^8\, 3n(8R)}{A \log R} < e^{-\lambda}$$

if A is sufficiently large, by (3.1). Outside E, (3.6) holds, by (3.7). The Lemma is proved.

Let $a_1, a_2, \ldots$ be the zeros of $f(z)$ numbered so that $|a_1| \leq |a_2| \leq |a_3| \leq \cdots$. Suppose $p > 64$ is such that $2 < |a_p| = \rho < |a_{p+1}| = \rho'$. By Lemma 3.1, with

$$\lambda = p^{\frac{1}{2}}, \ R = \rho'^2$$

$$S_2(re^{i\theta}) < 2A\, p^{\frac{1}{2}} \log \rho' \qquad (\rho^2 \leq r < \rho'^2)$$

provided $re^{i\theta} \notin E_p$, where E_p is an ϵ-set of extent less than $e^{-p^{\frac{1}{2}}}$.

We must show that $S_2(re^{i\theta})$ is small compared with $N(r)$. There are three cases to distinguish.

1. $\rho' < 2\rho^2$. Then for $\rho^2 < r < \rho'^2$

$$N(r) > \int_{\rho}^{\rho^2} \frac{n(t)}{t}\, dt \geq p \log \rho > p \log \left(\frac{\rho'}{2}\right)^{\frac{1}{2}} > C\, p \log \rho' .$$

2. $\rho' > 2\rho^2$, $\rho^2 < r < \frac{1}{2}\rho'$. There are no zeros of $f(z)$ in $\rho < |\zeta| < \rho'$. But if $\rho^2 < r \leq \frac{1}{2}\rho'$, then $\rho < \frac{1}{2} r < 2r < \rho'$. Therefore there are no zeros contributing to $S_2(z)$.

$$S_2(z) = 0 = o\,(N(r)).$$

3. $\rho' > 2\rho^2$. $\frac{1}{2}\rho' \leq r \leq \rho'^2$.

$$N(r) > \int_{\rho}^{\frac{1}{2}\rho'} \frac{n(t)}{t}\, dt = p \log \frac{\rho'}{2\rho} > p \log \left(\frac{\rho'}{2}\right)^{\frac{1}{2}}$$

$$> C_p \log \rho' .$$

In all three cases

$$\left|\frac{S_2(re^{i\theta})}{N(r)}\right| < Cp^{-\frac{1}{2}} \qquad (|a_p|^2 \leq r < |a_{p+1}|^2)$$

outside an exceptional $\mathcal{E}$-set E_p of extent $\exp(-p^{\frac{1}{2}})$. Therefore (3.5) holds outside the $\mathcal{E}$-set $\bigcup E_p$.

This completes the proof of Theorem 3.1.

SECTION 4.

WIMAN'S THEOREM

A celebrated result, conjectured by both Littlewood and Lindelöf in 1908 and later proved independently by Wiman [20] and Valiron [19] is

THEOREM 4.1. (Wiman's Theorem). *If* $f(z)$ *is an entire function of order* $\lambda < 1$ *and if*

$$m^*(r,f) = \inf_{|z| = r} |f(z)|$$

then

$$\limsup_{r \to \infty} \frac{\log m^*(r,f)}{\log M(r,f)} \geq \operatorname{Cos} \pi\lambda .$$

This result was sharpened by Kjellberg to

THEOREM 4.2. *If* $f(z)$ *is an entire function of lower order* $\mu < 1$, *then*

$$\limsup_{r \to \infty} \frac{\log m^*(r,f)}{\log M(r,f)} \geq \operatorname{Cos} \pi\mu .$$

THEOREM. 4.2. will be an immediate consequence of the following Lemma due to Kjellberg [11] .

LEMMA 4.3. *Let* $f(z)$ *be an entire function*

(4.1) $\sigma = \liminf_{r \to \infty} r^{-\alpha} \log M(r,f)$, $\tau = \limsup_{r \to \infty} r^{-\alpha} \log M(r)$

<u>If</u> $0 < \alpha < 1$ <u>and</u>

$$\text{(4.2)} \qquad \log m^*(r,f) - \cos \pi\alpha \log M(r,f) \leq 0 \qquad (r > r_0)$$

<u>Then either</u> (i) $\sigma = \tau = \infty$ <u>or</u> (ii) $0 < \sigma, \tau < \infty$.

<u>Deduction of Theorem 4.2 from Lemma 4.3</u>.

If $\alpha > \mu$, then $\sigma = 0$, so that (4.2) cannot hold, i.e., for some arbitrary large r

$$\log m^*(r,f) - \cos \pi\alpha \log M(r,f) > 0 \qquad (\alpha > \mu)$$

Theorem 4.2 follows after division by $\log M(r,f)$ on letting α tend to μ.

The proof of Lemma 4.3 is based on an elementary, but highly ingenious Lemma of Denjoy (only the part relating to $h_1(r)$ will be required in the proof).

LEMMA 4.4. <u>Let</u> $0 \leq \phi \leq \pi$, $0 < \alpha < 1$. <u>Put</u>

$$h_1(r) = \int_r^\infty \frac{\log|1+xe^{i\phi}| - \cos \alpha\phi \log(1+x)}{x^{1+\alpha}}$$

$$h_2(r) = \int_r^\infty x^{-\alpha-1} \left(\frac{\pi\alpha}{\sin \pi\alpha} \log^+ x - \log(1+x) \right) dx$$

$$h_3(r) = \int_r^\infty \frac{\log|1-x| - \pi\alpha \cot \pi\alpha \log^+ x}{x^{1+\alpha}} \, dx .$$

<u>Then it is possible to find constants</u> $A > 0$, $B > 0$ such that, for $j = 1,2,3$

$$A\,\frac{\log(1+r)}{r^{\alpha}} < h_j(r) < B\,\frac{\log(1+r)}{r^{\alpha}} \qquad (0 < r).$$

PROOF. The proof runs along the same lines for $j = 1,2,3$. The steps are

(1) $h_j(0) = h_j(\infty) = 0$

(2) $h_j(r) > 0$ $(0 < r < \infty)$, because there is an r_j such that $h'(r) > 0$ $(r < r_j)$, $h'(r) < 0$ $(r > r_j)$

(3) $0 < \underline{\lim}\,\frac{r^{\alpha}}{\log(1+r)}\,h_j(r) \leq \overline{\lim}\,\frac{r^{\alpha}}{\log(1+r)}\,h_j(r) < \infty$

as $r \to 0$ and as $r \to \infty$.

It follows then at once from the continuity of h_j and (3) that

$$0 < \inf \frac{r^{\alpha}}{\log(1+r)}\,h_j(r) \leq \sup \frac{r^{\alpha}}{\log(1+r)}\,h_j(r) < \infty \qquad (r > 0)$$

which proves the Lemma.

We give the details for $h(r) = h_1(r)$. The other cases are a little simpler. 1) Obviously $h_1(\infty) = 0$

$$h_1(0) = \mathcal{R}\int_0^{\infty} \frac{\log(1+xe^{i\phi}) - \cos\phi\,\log(1+x)}{x^{1+\alpha}}\,dx$$

where the branch of the logarithm with $0 \leq \arg(1+xe^{i\phi}) < 2\pi$ is chosen. By applying Cauchy's theorem to $z^{-1-\alpha}\log(1+z)$ in the sector $|z| < R$, $0 < \arg z < \phi$ and letting $R \to \infty$,

$$\int_0^\infty \frac{\log(1+xe^{i\phi})e^{-i\alpha\phi}}{x^{1+\alpha}}\,dx = \int_0^\infty \frac{\log(1+x)}{x^{1+\alpha}}\,dx .$$

Multiplying by $e^{i\alpha\phi}$ and taking real parts proves $h_1(0) = 0$.

(2) Let $u(r) = r^{1+\alpha} h'(r) = \operatorname{Cos}\alpha\phi \log(1+r)$

$$- \tfrac{1}{2}\log(1+r^2 + 2r\operatorname{Cos}\phi)$$

Then

$$u'(r) = \frac{d}{dr}(r^{1+\alpha} h'(r)) = \frac{\operatorname{Cos}\alpha\phi}{1+r} - \frac{r+\operatorname{Cos}\phi}{1+r^2 + 2r\operatorname{Cos}\phi}$$

$$= \frac{\operatorname{Cos}\alpha\phi - \operatorname{Cos}\phi + r\{2\operatorname{Cos}\phi\operatorname{Cos}\alpha\phi - 1 - \operatorname{Cos}\phi\} + r^2(\operatorname{Cos}\alpha\phi - 1)}{(1+r)(1+r^2 + 2r\operatorname{Cos}\phi)}$$

The numerator in the expression for $u'(r)$ is a quadratic the product of whose roots is

$$\frac{\operatorname{Cos}\alpha\phi - \operatorname{Cos}\phi}{\operatorname{Cos}\alpha\phi - 1} < 0$$

$$u'(0) = \operatorname{Cos}\alpha\phi - \operatorname{Cos}\phi > 0 \text{ and } u'(\infty) < 0.$$

Therefore $u'(r)$ has a single positive root $r = \rho$. For $r < \rho$, $u'(r) > 0$ and for $r > \rho$, $u'(r) < 0$. Therefore $u(r)$ increases from the value 0 at 0 to a maximum at $r = \rho$ and decreases then steadily. Since $u(\infty) = -\infty$, there is a unique value r_1 such that $u(r) > 0$ $(r < r_1)$ and $u(r) < 0$ $(r > r_1)$ i.e. $h'(r) > 0$, $(r < r_1)$ and $h'(r) < 0$ $(r > r_1)$.

3) As $r \to \infty$

$$\frac{r^{\alpha} h(r)}{\log(1+r)} = \frac{r^{\alpha}}{\log(1+r)} \int_r^{\infty} \frac{\log x(1-\cos\alpha\phi) + O\left(\frac{1}{x}\right)}{x^{1+\alpha}}\, dx$$

$$= \frac{r^{\alpha}}{\log(1+r)} \left\{ \frac{1-\cos\alpha\phi}{\alpha} \; \frac{\log r}{r^{\alpha}} - \int_r^{\infty} O(x^{-1-\alpha})\, dx \right\}$$

$$\to \frac{1-\cos\alpha\phi}{\alpha}$$

As $r \to 0$, by (1)

$$\frac{r^{\alpha} h(r)}{\log(1+r)} = \frac{r^{\alpha}}{\log(1+r)} \int_0^{r} \frac{\cos\alpha\phi \log(1+x) - \log|1+xe^{i\phi}|}{x^{1+\alpha}}\, dx$$

$$= \frac{r^{\alpha}}{\log(1+r)} \int_0^{r} \frac{(\cos\alpha\phi - \cos\phi)x + O(x^2)}{x^{1+\alpha}}\, dx$$

$$= \frac{r^{\alpha}}{\log(1+r)} \; \frac{\cos\alpha\phi - \cos\phi}{1-\alpha}\, r^{1-\alpha} + O\left(\frac{r^{\alpha}}{\log(1+r)}\right)$$

$$\to \frac{\cos\alpha\phi - \cos\phi}{1-\alpha}.$$

PROOF OF LEMMA 4.3. There is no loss of generality in assuming that $f(0) \neq 0$. For, otherwise $f(z) = C\, z^s g(z)$ $(s > 0)$ and

$$\log m^*(r,f) - \cos \pi\alpha \log M(r,f)$$

$$= \log m^*(r,g) - \cos \pi\alpha \log M(r,f) + (1-\cos \pi\alpha) \log |Cr^s|$$

so that $g(z)$ also satisfies the hypothesis of the Lemma. Obviously $\sigma = \sigma(f) = \sigma(g)$ and $\tau = \tau(f) = \tau(g)$.

By the approximation lemma applied to $\frac{f(z)}{f(0)}$, we obtain

$$\log |f(z)| = \sum_{|a| \leq R} \log\left|1-\frac{z}{a}\right| + \log |f(0)| + O\left(\frac{r}{R}\, T(2R,f)\right), \quad (|z| = r < \tfrac{1}{2} R)$$

(4.3)

$$= \sum_{|a| \leq R} \log\left|1-\frac{z}{a}\right| + O\left(1+\frac{r}{R}\, T(2R,f)\right)$$

To abbreviate, write S for $O\left(1+\frac{r}{R}\, T(2R,f)\right)$. It is easy to see that

$$\log |1+te^{i\phi}| = \log |1+te^{-i\phi}| \qquad (t > 0)$$

is monotonely decreasing function of ϕ in $0 \leq \phi \leq \pi$. Therefore by (4.3)

$$\log m^*(r,f) \geq \sum_{|a| \leq R} \log \left| 1 - \frac{r}{|a|} \right| + S \qquad (r \leq \tfrac{1}{2}R)$$

(4.4) $$\log M(r,f) \leq \sum_{|a| \leq R} \log \left(1 + \frac{r}{|a|} \right) + S \qquad (r \leq \tfrac{1}{2}R)$$

Also if ϕ is chosen so that $|f(re^{i\phi})| = m^*(r,f)$ we have

$$\log m^*(r,f) + \log M(r,f) \geq \log |f(re^{i\phi})| + \log |f(-re^{i\phi})|$$

$$= \sum_{|a| \leq R} \log \left| 1 - \frac{r}{a} e^{i\phi} \right| + \log \left| 1 + \frac{r}{a} e^{i\phi} \right| + S$$

$$\geq \sum_{|a| \leq R} \log \left| 1 - \frac{r^2}{|a|^2} \right| + S .$$

Hence, for $0 < \alpha < 1$,

$$\log m^*(r,f) - \cos \pi\alpha \log M(r,f)$$

$$= \log m^*(r,f) + \log M(r,f) - (1 + \cos \pi\alpha) \log M(r,f)$$

$$\geq \sum_{|a| \leq R} \left\{ \log \left| 1 - \frac{r}{|a|} \right| + \log \left| 1 + \frac{r}{|a|} \right| - (1 + \cos \pi\alpha) \cdot \log \left| 1 + \frac{r}{|a|} \right| \right\} + S$$

(4.5) $$\geq \sum_{|a| \leq R} \left\{ \log \left| 1 - \frac{r}{a} \right| - \cos \pi\alpha \cdot \log \left| 1 + \frac{r}{|a|} \right| \right\} + S$$

(The first step in this chain is necessary to take care of the case $\cos \pi\alpha < 0$).

Let $\frac{1}{2}R = \rho$. By (4.4) with the notation of Lemma 2

$$\int_r^\rho t^{-1-\alpha}\left\{\log m^*(t,f) - \cos \pi\alpha \log M(t,f)\right\} dt$$

$$\geq \sum_{|a| < R} \int_r^\rho t^{-1-\alpha}\left\{\log \left|1-\frac{t}{|a|}\right| - \cos \pi\alpha \log\left(1+\frac{t}{a}\right)\right\} dt$$

$$+ O\left(\rho^{-\alpha} T(4\rho,f) + r^{-\alpha}\right)$$

Abbreviating $O\left(\rho^{-\alpha} T(4\rho,f) + r^{-\alpha}\right)$ by S_1, we have, using Lemma 4.4,

$$\int_r^\rho t^{-1-\alpha}\left\{\log m^*(t,f) - \cos \pi\alpha \log M(t,f)\right\} dt$$

$$\geq \sum_{|a| \leq 2\rho} |a|^{-\alpha} \int_{\frac{r}{|a|}}^{\frac{\rho}{|a|}} u^{-1-\alpha}\left\{\log|1-u| - \cos \pi\alpha \log|1+u|\right\} du + S_1$$

$$\geq \sum_{|a| \leq 2\rho} |a|^{-\alpha}\left\{h_1\left(\frac{r}{|a|}\right) - h_1\left(\frac{\rho}{|a|}\right)\right\} + S_1$$

$$\geq A \sum_{|a| \leq 2\rho} r^{-\alpha} \log\left(1+\frac{r}{|a|}\right) - B \sum_{|a| \leq 2\rho} \rho^{-\alpha}\left(1+\frac{\rho}{|a|}\right) + S_1 \tag{4.6}$$

Now

$$\sum_{|a| \leq 2\rho} \log\left(1+\frac{\rho}{|a|}\right) = \sum_{|a| \leq 2\rho} \left\{\log \frac{3\rho}{|a|} + \log\left(1+\frac{|a|}{\rho}\right) - \log 3\right\}$$

(4.7)

$$\leq \sum_{|a| \leq 2\rho} \log \frac{3\rho}{|a|} \leq N(3\rho, \tfrac{1}{f}) \leq T(4\rho, f).$$

Also, by (4.4)

$$(4.8) \qquad \sum_{|a| \leq 2\rho} \log\left(1+\frac{r}{|a|}\right) \geq \log M(r,f) - O(\tfrac{r}{\rho}\, T(4\rho, f) \;.$$

Finally, using (4.7) and (4.8) in (4.6), for $r < \rho$,

$$\int_r^\rho t^{-1-\alpha}\left\{\log m^*(t,f) - \operatorname{Cos} \pi\alpha \log M(t,f)\right\} dt$$

$$(4.9) \qquad \geq Ar^{-\alpha} \log M(r,f) - B\rho^{-\alpha}\,(T(4\rho,f) + S_1.$$

Under the hypotheses of the Lemma the left hand side of (4.9) is ≤ 0 for $r > r_0$. But the right hand side can be made positive for some arbitrarily large r and ρ if either

$$\lim\inf\, (4\rho)^{-\alpha}\, T(4\rho,f) \leq \lim\inf\, \rho^{-\alpha} \log M(\rho,f) = \sigma = 0$$

or $\sigma < \infty$, $\tau = \lim\sup r^{-\alpha} \log M(r,f) = \infty$.

THEOREM 4.2 remains true in the limiting case $\mu = 1$.

THEOREM 4.2'. *If* $f(z)$ *is an entire function of lower order* 1, *then*

$$\limsup \frac{\log m^*(r,f)}{\log M(r,f)} \geq -1.$$

PROOF. Let $F(z^2) = f(z)\, f(-z)$. $F(\zeta)$ is an entire function of ζ and the lower order of $F(\zeta)$ is $\frac{1}{2}$. Therefore, by Theorem 4.2

$$\limsup \frac{\log m^*(r^2,F)}{\log M(r^2, F)} \geq 0$$

i.e.,

$$\log m^*(r^2,F) + \epsilon \log M(r^2,F) > 0 \qquad (r > r_o(\epsilon)). \tag{4.10}$$

But

$$\log m^*(r^2,F) \leq \log m^*(r,f) + \log M(r,f)$$

and

$$\log M(r^2,F) \leq 2 \operatorname{Log} M(r,f)$$

so that (4.10) implies

$$\log m^*(r,f) + \log M(r,f) + 2\epsilon \log M(r,f) \geq 0 \quad (r > r_o(\epsilon))$$

i.e.

$$\limsup \frac{\log m^*(r,f)}{\log M(r,f)} \geq -1.$$

This completes the proof of Theorem 4.2'.

Lemma 4.3 suggests that Theorem 4.2 could be replaced by the stronger statement

$$\log m^*(r,f) - \cos \pi\mu \log M(r,f) \geq 0$$

for some arbitrarily large r.

This is not the case. There are entire functions of order μ, $0 < \mu < 1$, such that

$$\log m^*(r,f) - \cos \pi\mu \log M(r,f) \leq 0, \qquad r > r_0. \tag{4.2}$$

A simple example is

$$f(z) = \frac{\sin \sqrt{z}}{\sqrt{z}} \qquad (\mu = \tfrac{1}{2})$$

Kjellberg has investigated the class of functions satisfying (4.2) and proved that all such functions are of regular growth. We shall prove Kjellberg's result in the form

THEOREM 4.3. *If* $0 < \alpha < 1$ and

$$\log m^*(r,f) - \cos \pi\alpha \log M(r,f) < K, \qquad r > r_0 \tag{4.11}$$

then

$$\lim_{r \to \infty} r^{-\alpha} \log M(r,f) \quad (\leq \infty) \text{ exists}.$$

PROOF. For suitable choice of the constant C. $Cf(z)$ satisfies the hypothesis of Lemma 4.3. Therefore either

$$r^{-\alpha} \log M(r,f) \to \infty$$

or

$$r^{-\alpha} \log M(r,f) < K \tag{4.12}$$

(K denotes a positive constant, not necessarily the same at each occurrence). It remains to prove the theorem under the hypothesis (4.12). We need

LEMMA 4.5. *Let*

$$f(z) = \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{a_\nu}\right) \qquad 0 < |a_1| \leq |a_2| \leq \cdots$$

be an entire function of order $\lambda \leq \alpha < 1$. *Then*

$$\tau(f) = \limsup r^{-\alpha} \log M(r,f) < \infty$$

implies

$$\tau(f_1) = \limsup r^{-\alpha} \log M(r,f_1) < \infty$$

where

$$f_1(z) = \prod_{\nu=1}^{\infty} \left(1 + \frac{z}{|a_\nu|}\right)$$

PROOF.

$$\log M(\rho,f_1) = \log f_1(\rho) = \sum_{|a| \le \rho} \log\left(1+\frac{\rho}{|a|}\right)$$

$$+ \sum_{|a|>\rho} \log\left(1+\frac{\rho}{|a|}\right)$$

$$= \sum_{|a| \le \rho} \left\{ \log \frac{2\rho}{|a|} - \log 2 + \log\left(1+\frac{|a|}{\rho}\right)\right\}$$

$$+ \int_{\rho}^{\infty} \log\left(1+\frac{\rho}{u}\right) dn(u,\tfrac{1}{f})$$

$$\le N(2\rho,\tfrac{1}{f}) + \sum_{|a|<\rho} (-\log 2 + \log 2) + \rho \int_{\rho}^{\infty} \frac{n(u,\frac{1}{f})}{u(u+\rho)}\, du$$

$$\le N(2\rho,\tfrac{1}{f}) + \rho \int_{\rho}^{\infty} \frac{N(u,\frac{1}{f})}{(u+\rho)^2}\, du.$$

Since

$$N(t,\tfrac{1}{f}) \le T(t,f) \le \log M(t,f) \le (\tau(f)+\epsilon)t^{\alpha} \qquad (t > t_1(\epsilon))$$

$$\log M(\rho,f_1) \le \log M(2\rho,f) + \rho(\tau(f)+\epsilon) \int_{\rho}^{\infty} \frac{u^{\alpha}}{(u+\rho)^2}\, du$$

$$\le \log M(2\rho,f) + \rho^{\alpha}(\tau(f)+\epsilon) \int_{1}^{\infty} \frac{x^{\alpha}dx}{(1+x)^2}.$$

The lemma follows on dividing by ρ^{α} and letting $\rho \to \infty$.

PROOF OF THEOREM 4.9. We may suppose without loss of generality that $f(0) = 1$. Otherwise

$$f(z) = C\, z^{s}\, g(z)\,, \quad g(o) = 1 \qquad C \neq 0,\ s > o$$

$\log m^*(r,f) - \operatorname{Cos} \pi\alpha \log M(r,f)$.

$$= \log m^*(r,g) - \operatorname{Cos} \pi\alpha \log M(r,g) + (1 - \operatorname{Cos} \pi\alpha) \log |C\, r^{s}|$$

Therefore g also satisfies the hypothesis of the theorem and it is obviously sufficient to prove the theorem for $g(z)$.

By (4.12), $f(z)$ is of order $\leq \alpha < 1$, so that

$$f(z) = \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{a_{\nu}}\right)$$

Let

$$f(z) = \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{|a_{\nu}|}\right)$$

By (4.12) and Lemma 4.5,

$$\limsup \rho^{-\alpha}\, T(\rho,F) \leq \limsup \rho^{-\alpha} \log M(\rho,F) < \infty .$$

By (4.9) (applied to $F(z)$), (4.12) and Lemma 4.5

$$(4.13) \qquad \int_{r}^{\rho} t^{-1-\alpha} \left\{ \log m^*(t,F) - \operatorname{Cos} \pi\alpha \log M(t,F) \right\} dt > -K.$$

But

$$\log m^*(t,F) - \text{Cos}\ \pi\alpha \log M(t,F) - K < 0$$

so that

$$\int^{\rho} t^{-1-\alpha} \left\{ \log m^*(t,F) - \text{Cos}\ \pi\alpha \log M(t,F) - K \right\} dt$$

either tends to $-\infty$ or to a finite limit as $\rho \to \infty$. Therefore (4.13) implies that

$$\int_a^{\infty} t^{-1-\alpha} \left\{ \log m^*(t,F) - \text{Cos}\ \pi\alpha \log M(t,F) - K \right\} dt \qquad (a > o)$$

exists. Since

$$\int_a^{\infty} t^{-1-\alpha}\, dt < \infty$$

this implies that

$$\int_a^{\infty} h(t)dt = \int_a^{\infty} t^{-1-\alpha} \left\{ \log m^*(t,F) - \text{Cos}\ \pi\alpha \log M(t,F) \right\} dt$$

exists and since

$$t^{-1-\alpha} \left\{ \log m^*(t,F) - \text{Cos}\ \pi\alpha \log M(t,F) \right\}^{+} = \max\ (o,h(t)) < Kt^{-1-\alpha} \in L(a,\infty)$$

the same is true of $h^-(t) = \min(o, h(t))$ and so

$$t^{-1-\alpha}|\log m^*(t,F) - \cos \pi\alpha \log M(t,F)| = h^+(t) - h^-(t) \in L(a,\infty).$$

(4.14)

Next we derive an integral formula for $r^{-\alpha} \log M(r,F)$.
Consider

$$e^{\pi i\alpha} \int_C \log F(z) z^{-1-\alpha} \frac{(e^{-\pi i} z)^{2\alpha} - r^{2\alpha}}{z^2 - r^2}$$

where C is the boundary of the semicircle

$$|z| < R, \quad 0 < \arg z < \pi \, .$$

with indentations at 0, r and the points $-|a_r|$. By Cauchy's theorem the value of the integral is 0. On the other hand letting $z = t$, on the positive real axis, $z = e^{\pi i} t$ on the negative real axis, the contribution of the real axis to the integral is

$$-\int_0^R \log|F(-t)| t^{-1-\alpha} \frac{t^{2\alpha} - r^{2\alpha}}{t^2 - r^2} dt +$$

$$+\int_0^R \log|F(t)| t^{-1-\alpha} \frac{e^{-\pi i\alpha} t^{2\alpha} - e^{\pi i\alpha} r^{2\alpha}}{t^2 - r^2} dt$$

where the second integral is to be understood as a principal value at $t = r$.

The indentation at the pole $z = r$ of the integrand gives a contribution

$$-\pi i \log F(r) r^{-1-\alpha} \frac{(e^{-\pi i\alpha} - e^{\pi i\alpha}) r^{2\alpha}}{2r}$$

$$= -\pi r^{\alpha-2} \log F(r) \sin \pi\alpha.$$

The other indentations give no contribution when they are allowed to shrink to points. The big semicircle gives a contribution of the order of magnitude

$$\sup_{0 \le \theta \le \pi} |\log F(re^{i\theta})| R^{\alpha-2}$$

$$= \sup_{0 \le \theta \le \pi} \left|\log|F(re^{i\theta})| + i \arg F(re^{i\theta})\right| R^{\alpha-2}.$$

Since F is of order $\le \alpha < 1$,

$$\log |F(re^{i\theta})| R^{\alpha-2} \to 0 \qquad (R \to \infty)$$

The formula (2.4) whose real part yields the approximation lemma with $q = 0$ also shows that

$$|\arg F(Re^{i\theta})| < \pi\, n(2R,\tfrac{1}{F}) + O(T(4R,F)) = O(T(4R,F))$$

and so

$$|\arg F(re^{i\theta})| R^{\alpha-2} \to 0 \qquad (R \to \infty).$$

Collecting all these pieces of information together, taking real parts and noting that $F(-t) = m^*(t,F)$, $F(t) = M(t,F)$, we obtain

$$r^{-\alpha}\log F(r)$$

$$= -\frac{1}{\pi \sin \pi\alpha} \int_0^{\infty} \left\{ \log m^*(t,F) - \cos \pi\alpha \log M(t,F) \right\} t^{-\alpha-1} \frac{\left(\frac{t}{r}\right)^{2\alpha} - 1}{\left(\frac{t}{r}\right)^{2} - 1} \, dt \, .$$

Since

$$0 < \frac{u^{2\alpha} - 1}{u^2 - 1} < K \qquad (u > 0)$$

the absolute value of the integrand is dominated by the integrable function

$$Kt^{-1-\alpha} \left| \log m^*(t,F) - \cos \pi\alpha \log M(t,F) \right| \qquad (0 < t < \infty).$$

We may therefore perform the limisint transition $r \to \infty$ under the sign of integration, by Lebesgue's Theorem of dominated convergence. This yields

$$\lim_{r \to \infty} r^{-\alpha} \log M(r,F) =$$

$$\frac{1}{\pi \sin \pi\alpha} \int_0^{\infty} t^{-1-\alpha} dt \left\{ \log m^*(t,F) - \cos \pi\alpha \log M(t,F) \right\} .$$

In particular this limit exists and is finite.

By the inequalities (see reasoning leading to (4.5))

$$\log M(r,f) \le \log M(r,F)$$
$$\log m^*(r,f) + \log M(r,f) \ge \log m^*(r,F) + \log M(r,F)$$

$$(4.15) \qquad K \ge \log m^*(r,f) - \cos \pi\alpha \log M(r,f)$$
$$\ge \log m^*(r,F) - \cos \pi\alpha \log M(r,F)$$

$$0 \le (1+\cos \pi\alpha)\, t^{-1-\alpha} \left\{ \log M(t,F) - \log M(t,f) \right\}$$
$$(4.16) \qquad \le t^{-1-\alpha} \Big\{ \log m^*(t,f) + \log M(t,f) - \log m^*(t,f) - \log M(t,F)$$
$$+ (1+\cos \pi\alpha) \left[\log M(t,F) - \log M(t,f) \right] \Big\} .$$

By (4.14) and (4.15) the right hand side of (4.16) is in $L(0,\infty)$ (no trouble at 0 since $f(0) = 1$). Therefore

$$(4.17) \qquad \int_0^\infty t^{-1-\alpha} \left\{ \log M(t,F) - \log M(t,f) \right\} dt < \infty .$$

Suppose now that

$$\underline{\lim}\, r^{-\alpha} \log M(r,f) < \lim r^{-\alpha} \log M(r,F) = b$$

Then we can find a constant c, $0 < c < b$ and arbitrarily large ρ such that

$$\rho^{-\alpha} \log M(\rho,f) < c$$

and so, since $\log M(r,f)$ is an increasing function of r

$$t^{-\alpha} \log M(t,f) < \left(\frac{\rho}{t}\right)^{\alpha} c \qquad (t < \rho) .$$

On the other hand, for all large t

$$t^{-\alpha} \log M(t,F) > \frac{b+c}{2} .$$

Therefore

$$\int_{\rho_1}^{\rho} t^{-1-\alpha} \left\{ \log M(t,F) - \log M(t,f) \right\} dt > \int_{\rho_1}^{\rho} t^{-1} \left\{ \frac{b+c}{2} - \left(\frac{\rho}{t}\right)^{\alpha} c \right\} dt$$

If ρ_1 is chosen so that $\left(\frac{\rho}{\rho_1}\right)^{\alpha} c = \frac{b+2c}{3}$ i.e.,

$\rho_1 = \left(\frac{3c}{b+2c}\right)^{\frac{1}{\alpha}} \rho = k\rho$, then the integrand $\left(\frac{b+c}{2} - \frac{\rho^{\alpha}}{t^{\alpha}} c \right) t^{-1}$

is greater than $\frac{b-c}{6t}$ so that

$$\int_{k\rho}^{\rho} t^{-1-\alpha} \left\{ \log M(t,F) - \log M(t,f) \right\} dt > \frac{b-c}{6} \log \frac{1}{k} .$$

But this contradicts (4.17). Therefore

$$\underline{\lim}\, r^{-\alpha} \log M(r,f) = \lim r^{-\alpha} \log M(r,F) \geq \overline{\lim}\, r^{-\alpha} \log M(r,f) ,$$

i.e. $$\underline{\lim}\, r^{-\alpha} \log M(r,f) = \overline{\lim}\, r^{-\alpha} \log M(r,f)$$

and the theorem is proved.

Theorem 4.3 has been generalized by M.Essén. He replaced the hypothesis that (4.2) holds by the hypothesis

$$I(r) = \int_0^r t^{-\alpha-1} \left\{ \log m^*(t,f) - \cos \pi\alpha \log M(t,f) \right\} dt < K$$

and proved. If $r^{-\alpha} \log M(r,f)$ has a finite upper bound, then $I(r)$ is bounded below and $\lim\ r^{-\alpha} \log M(r,f)$ and $\lim I(r)$ exist or fail to exist together.

Problem. Essen's method is different from ours. Does the method of proof of Theorem 4.3 also work for Essen's theorem ?

SECTION 5.

ANALOGUE OF WIMAN'S THEOREM FOR MEROMORPHIC FUNCTIONS.

A number of analogues of Wiman's theorem for meromorphic functions are known. Most of these are due to A.A. Goldberg and I.V.Ostrovski ([8] , [9]).

THEOREM 5.1. <u>If</u> $f(z)$ <u>is a meromorphic function of lower order</u> $\mu < 1$ <u>and if</u> $\mu < \alpha < 1$, <u>then for some arbitrary large values of</u> r.

$$(5.1) \qquad \frac{\pi\alpha}{\operatorname{Sin} \pi\alpha} N(r,\tfrac{1}{f}) - \log M(r,f) - \frac{\pi\alpha \operatorname{Cos} \pi\alpha}{\operatorname{Sin} \pi\alpha} N(r,f) > 0.$$

<u>If</u> $\mu < \alpha < \frac{1}{2}$, <u>then for some arbitrarily large</u> r

$$(5.2) \qquad \log m^*(r,f) - \cos \pi\alpha \log M(r,f) + \pi\alpha \sin \pi\alpha\, N(r,f) > 0$$

A consequence of this theorem is

THEOREM 5.2. <u>If</u> $f(z)$ <u>is a meromorphic function of lower order</u> $\mu < \frac{1}{2}$, <u>then for arbitrarily large</u> r <u>and</u> $\epsilon > 0$,

$$(5.3) \qquad \frac{\log^+ m^*(r,f)}{T(r,f)} \geq \frac{\pi\mu}{\sin \pi\mu} \left\{ \cos \pi\mu - 1+\delta(\infty,f)-\epsilon \right\}$$

COROLLARY. _If_ $f(z)$ _is of lower order_ $\mu < \frac{1}{2}$ _and if_ $\delta(\infty, f) > 1 - \cos \pi\mu$, _then_ ∞ _is the only deficient value of_ $f(z)$.

PROOF OF COROLLARY. $m^*(r,f) \to \infty$ as $r \to \infty$ through a suitable sequence by Theorem 5.2. Therefore $\underline{\lim}\, m(r, \frac{1}{f-c}) = 0$ for every complex c; $\delta(c,f) = 0$.

Remark. Theorem 5.2 does not give any information about the values of r for which (5.3) holds. The proof actually yields the following additional information.

Let ρ_m be a sequence of Polya peaks of order μ of $T(r,f)$. There is a constant $C = C(\epsilon)$ such that (5.3) holds for an r in $(\rho_m, C\rho_m)$ for all large m.

Deduction of Theorem 5.2 from Theorem 5.1. There is nothing to prove if $\delta(\infty) < 1 - \cos \pi\mu$. Suppose now that

$$\delta(\infty, f) > 1 - \cos \pi\mu.$$

Choose c such that

$$N\left(r, \frac{1}{f-c}\right) \sim T(r,f).$$

Apply (5.1) to $\frac{1}{f-c}$, noting that

$$N(r, f-c) = N(r,f) < (1 - \delta(\infty) + \epsilon_1)\, T(r,f), \qquad r > r_0(\epsilon_1)$$

$$M(r, \frac{1}{f-c}) = \frac{1}{m^*(r, f-c)}.$$

For $\mu < \alpha < 1$, $r > r_o(\epsilon_1)$, we have

$$\frac{\pi\alpha}{\sin \pi\alpha}(1-\delta(\infty)+\epsilon_1)\ T(r,f)+\log m^*(r,f-c)-\pi\alpha \cot \pi\alpha\ T(r,f)(1-\epsilon_1) > 0.$$

Rearranging

$$\log m^*(r,f-c) > \frac{\pi\alpha}{\sin \pi\alpha}\left(\delta(\infty)-1+\cos \pi\alpha-2\epsilon_1\right)T(r,f)$$

and so

$$\begin{aligned}\log^+ m^*(r,f) &\geq \log^+ m^*(r,f-c) + \log^+ |c| + \log 2 \\ &> \frac{\pi\alpha}{\sin \pi\alpha}(\delta(\infty)-1+\cos \pi\alpha-2\epsilon_1)\ T(r,f)- \\ &\qquad - \log^+ |c| - \log 2\end{aligned}$$

Since $T(r,f) \to \infty$ with r this yields

$$\log^+ m^*(r,f) > \frac{\pi\alpha}{\sin \pi\alpha}(\delta(\infty)-1+\cos \pi\alpha-3\epsilon_1)\ T(r,f), \quad (r>r_o(\epsilon_1)).$$

By choosing ϵ_1 sufficiently small and taking α close to μ we can make

$$\frac{\pi\alpha}{\sin \pi\alpha}(\delta(\infty)-1+\cos \pi\alpha - 3\epsilon_1) > \frac{\pi\mu}{\sin \pi\mu}(\delta(\infty)-1+\cos \pi\mu - \epsilon)$$

and the theorem is proved.

<u>PROOF OF THEOREM 5.1.</u> The proof is along the same lines as the proof of Wiman's theorem. We only sketch the details. By the approximation Lemma

$$\log\left|f(re^{i\theta})\right| = \sum_{|a|<R} \log\left|1-\frac{z}{a}\right| - \sum_{|b|<R} \log\left|1-\frac{z}{b}\right| + $$

$$+ O\left(\frac{r}{R}\, T(2R,f)\right)$$

valid for $r < \frac{1}{2}R$. Therefore, by the monotonicity of $|1-te^{i\emptyset}|$ in $0 \le \emptyset \le \pi$.

$$\log M(r,f) - \frac{\pi\alpha}{\sin\pi\alpha}\, N(r,\tfrac{1}{f}) + \frac{\pi\alpha\cos\pi\alpha}{\sin\pi\alpha}\, N(r,f)$$

$$\le \sum_{|a|<R} \left\{\log\left|1+\frac{r}{|a|}\right| - \frac{\pi\alpha}{\sin\pi\alpha}\log^{+}\frac{r}{|a|}\right\}$$

$$+ \sum_{|b|<R} \left\{\frac{\pi\alpha\cos\pi\alpha}{\sin\pi\alpha}\log^{+}\frac{r}{|b|} - \log\left|1-\frac{r}{|b|}\right|\right\} +$$

$$+ O\left(\frac{r}{R}\, T(2R,f)\right) .$$

Multiply by $r^{-1-\alpha}$ and integrate from r to $\rho = \frac{1}{2}R$. Then

$$\int_r^{\rho} t^{-1-\alpha}\left\{\log M(t,f) - \frac{\pi\alpha}{\sin\pi\alpha}\, N(t,\tfrac{1}{f}) + \frac{\pi\alpha\cos\pi\alpha}{\sin\pi\alpha}\, N(t,f)\right\} dt$$

$$\leq - \sum_{|a| \leq 2\rho} \int_r^\rho t^{-1-\alpha} \left\{ \frac{\pi\alpha}{\sin \pi\alpha} \log^+ \frac{t}{|a|} - \log \left(1 + \frac{t}{|a|}\right) \right\} dt$$

$$(5.4) \quad - \sum_{|b| \leq 2\rho} \int_r^\rho t^{-1-\alpha} \left\{ \log \left|1 - \frac{t}{|b|}\right| - \frac{\pi\alpha \cos \pi\alpha}{\sin \pi\alpha} \log^+ \frac{t}{|b|} \right\} dt$$

$$+ \quad O(\rho^{-\alpha}\, T(4\rho, f)).$$

Using Lemma 4.4,

$$\int_r^\rho t^{-1-\alpha} \left\{ \frac{\pi\alpha}{\sin \pi\alpha} \log^+ \frac{t}{|a|} - \log \left(1 + \frac{t}{|a|}\right) \right\} dt$$

$$= |a|^{-\alpha} \left(h_2\left(\frac{r}{|a|}\right) - h_2\left(\frac{\rho}{|a|}\right) \right)$$

$$> Ar^{-\alpha} \log \left(1 + \frac{r}{|a|}\right) - B\rho^{-\alpha} \log \left(1 + \frac{\rho}{|a|}\right)$$

and similarly, employing $h_3(r)$,

$$\int_r^\rho t^{-1-\alpha} \left\{ \log \left|1 - \frac{t}{|b|}\right| - \frac{\pi\alpha \cos \pi\alpha}{\sin \pi\alpha} \log^+ \frac{t}{|b|} \right\} dt$$

$$> Ar^{-\alpha} \log \left(1 + \frac{r}{|b|}\right) - B\rho^{-\alpha} \log \left(1 + \frac{\rho}{|b|}\right).$$

By means of these estimates, (5.4) becomes

$$\int_r^\rho t^{-1-\alpha} \left\{ \log M(t,f) - \frac{\pi\alpha}{\sin \pi\alpha} N(t, \tfrac{1}{f}) + \frac{\pi\alpha \cos \pi\alpha}{\sin \pi\alpha} N(t,f) \right\} dt$$

$$< -Ar^{-\alpha} \sum \left\{ \log \left(1 + \frac{r}{|a|}\right) + \log \left(1 + \frac{r}{|b|}\right) \right\}$$

$$+ B\rho^{-\alpha} \sum \left\{ \log \left(1 + \frac{\rho}{|a|}\right) + \log \left(1 + \frac{\rho}{|b|}\right) \right\}$$

$$+ O\left(\frac{T(4\rho, f)}{\rho^{\alpha}}\right)$$

$$< -Ar^{-\alpha} \left\{ \sum \log \left(1 + \frac{r}{|a|}\right) + \sum \log \left(1 + \frac{r}{|b|}\right) \right\}$$

$$+ O\left(\frac{T(4\rho, f)}{\rho^{\alpha}}\right). \tag{5.5}$$

In the last line we have used the estimate (4.7).

If $\alpha > \mu$, then the right hand side of (5.5) will be negative for any assigned r and some $\rho > r$. Therefore the integrand on the left-hand side must be negative for arbitrarily large values of t.

The proof of the second assertion proceeds along the same lines

$$\log m^*(r,f) - \cos \pi\alpha \log M(r,f) + \pi\alpha \sin \pi\alpha \, N(r,f)$$

$$\geq \sum_{|a| \leq R} \left\{ \log \left|1 - \frac{r}{|a|}\right| - \cos \pi\alpha \log\left[1 + \frac{r}{|a|}\right] \right\}$$

$$- \sum_{|b| \leq R} \left\{ \log \left(1 + \frac{r}{|b|}\right) - \cos \pi\alpha \log \left|1 - \frac{r}{|b|}\right| - \pi\alpha \sin \pi\alpha \log^{+} \frac{r}{|b|} \right\}$$

$$+ O\left(\frac{r}{R} T(2R, f)\right)$$

and so, since $\sin \pi\alpha = \dfrac{1-\cos^2 \pi\alpha}{\sin \pi\alpha}$,

$$\int_r^\rho t^{-1-\alpha}\left\{\log m^*(t,f) - \cos \pi\alpha \log M(t,f)+\pi\alpha \sin \pi\alpha\, N(t,f)\right\} dt$$

$$\geq \sum_{|a|\leq 2\rho} |a|^{-\alpha}\left\{ h_1\left(\frac{r}{|a|}\right) - h_1\left(\frac{\rho}{|a|}\right)\right\} + \cos \pi\alpha \sum_{|b|\leq 2\rho}\left\{|b|^{-\alpha}\right.$$

$$\left.\left(h_3\left(\frac{r}{|b|}\right) - h_3\left(\frac{\rho}{|b|}\right)\right)\right\}$$

$$+ \sum_{|b|\leq 2\rho} |b|^{-\alpha}\left(h_2\left(\frac{r}{|a|}\right) - h_2\left(\frac{\rho}{|a|}\right)\right) + O\left(\rho^{-\alpha} T(4\rho,f)\right) .$$

This implies exactly in the same way as before that the left hand side is greater than

$$r^{-\alpha} A_1\left\{ \sum_{|a|\leq 2\rho} \log\left(1 + \frac{r}{|a|}\right) + \sum_{|b|\leq 2\rho} \log\left(1 + \frac{r}{|b|}\right)\right\} -$$

$$- B_1 \frac{T(4\rho,f)}{\rho^\alpha}$$

which can be made positive for any r by suitable choice of ρ .

SECTION 6.

DEFICIENT VALUES OF MEROMORPHIC FUNCTIONS OF LOWER ORDER LESS THAN ONE.

An Important Formula.

Let $f(z)$ be a transcendental meromorphic function of order λ $(\leq \infty)$ and lower order $\mu < 1$, $f(0) = 1$. Let $a_1, a_2, \ldots$ be the zeros and $b_1, b_2, \ldots$ be the poles of $f(z)$.

We denote by $\eta(r)$ a non-decreasing positive continuous function such that

$$\frac{\eta(r)}{T(r,f)} \to 0 \qquad (r \to \infty)$$

and

$$E(r,c,\eta(r)) = E(r,\tfrac{1}{f-c},\eta(r)) = \left\{\theta \,\middle|\, 0 \leq \theta < 2\pi,\ \log|f(re^{i\theta})-c| < -\eta(r)\right\},$$

$$(c \neq \infty)$$

$$E(r,f,\eta(r)) = E(r,\infty,\eta(r)) = \left\{\theta \,\middle|\, 0 \leq \theta < 2\pi,\ \log|f(re^{i\theta})| > \eta(r)\right\}$$

The Lebesgue measure of $E(r,\infty,\eta(r))$ we denote by

$$\text{mes}\left\{E(r,\infty,\eta(r))\right\} = 2\gamma(r,\infty) = 2\gamma(r) = 2\gamma.$$

so that

$$0 \leq \gamma \leq \pi.$$

By the definition of $m(r,f)$ and the approximation Lemma we have for $R > 2r$

$$m(r,f) \leq \frac{1}{2\pi} \int_{E(r,\infty,\eta)} \log |f(re^{i\theta})| \, d\theta + \eta(r)$$

$$\leq \sum_{|a| < R} \frac{1}{2\pi} \int_{E(r,\infty,\eta)} \log \left| 1 - \frac{re^{i\theta}}{a} \right| d\theta$$

$$(6.1) \qquad - \sum_{|b| < R} \frac{1}{2\pi} \int_{E(r,\infty,\eta)} \log \left| 1 - \frac{re^{i\theta}}{b} \right| d\theta$$

$$+ O\left(\eta(r)\right) + O\left(\frac{r}{R} T(2R,f)\right) .$$

Using again the remark that

$$\log |1 + te^{-i\theta}| = \log |1 + te^{i\theta}| \qquad (t > 0)$$

is a decreasing function of θ in $0 \leq \theta \leq \pi$, we see that

$$\frac{1}{2\pi} \int_E \log \left| 1 - \frac{re^{i\theta}}{a} \right| d\theta \leq \frac{1}{2\pi} \int_{-\gamma}^{\gamma} \log \left| 1 + \frac{r}{|a|} e^{i\theta} \right| d\theta$$

and

$$- \frac{1}{2\pi} \int_E \log \left| 1 - \frac{re^{i\theta}}{b} \right| d\theta \leq - \frac{1}{2\pi} \int_{-\pi}^{-\pi+\gamma} \log \left| 1 + \frac{r}{|b|} e^{i\theta} \right| d\theta -$$

$$- \frac{1}{2\pi} \int_{\pi-\gamma}^{\pi} \log \left| 1 + \frac{r}{|b|} e^{i\theta} \right| d\theta$$

$$\leq - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left|1 + \frac{r}{|b|} e^{i\theta}\right| d\theta + \frac{1}{2\pi} \int_{-(\pi-\gamma)}^{\pi-\gamma} \log \left|1 + \frac{r}{|b|} e^{i\theta}\right| d\theta .$$

By Jensen's formula the first integral on the right hand side is equal to $- \log^{+} \frac{r}{|b|}$. Hence, finally, by (6.1)

$$m(r,f) \leq -N(r,f) + \sum_{|a| < R} \frac{1}{2\pi} \int_{-\gamma}^{\gamma} \log \left|1 + \frac{r}{|a|} e^{i\theta}\right| d\theta$$

$$+ \sum_{|b| < R} \frac{1}{2\pi} \int_{-(\pi-\gamma)}^{\pi-\gamma} \log \left|1 + \frac{r}{|b|} e^{i\theta}\right| d\theta$$

$$+ O\left(\frac{r}{R} T(2R,f) + \eta(r)\right)$$

(6.2) $$T(r,f) \leq \sum_{|a| < R} \frac{1}{2\pi} \int_{-\gamma}^{\gamma} \log \left|1 + \frac{r}{|a|} e^{i\theta}\right| d\theta$$

$$+ \sum_{|b| < R} \frac{1}{2\pi} \int_{-(\pi-\gamma)}^{\pi-\gamma} \log \left|1 + \frac{r}{|b|} e^{i\theta}\right| d\theta$$

$$+ O\left(\frac{r}{R} T(2R,f) + \eta(r)\right) .$$

Let $0 \leq \theta < \pi$. Then

$$\log \left| 1 + \frac{re^{i\theta}}{|a|} \right| = \mathcal{R}\left\{\log \left(1 + \frac{re^{i\theta}}{a}\right)\right\}$$

$$\sum_{|a|<R} \log \left| 1 + \frac{re^{i\theta}}{|a|} \right| = \mathcal{R} \int_0^R \log\left(1 + \frac{re^{i\theta}}{|u|}\right) \, dn(u,\tfrac{1}{f}) .$$

By two integrations by parts

$$\sum_{|a|<R} \log \left| 1 + \frac{re^{i\theta}}{|a|} \right| = \mathcal{R} \int_0^R \frac{n(u,\frac{1}{f})\, re^{i\theta}}{u(u+re^{i\theta})} \, du + \mathcal{O}\left(\frac{r}{R}\, n(R,\tfrac{1}{f})\right)$$

$$= \mathcal{R} \int_0^R \frac{N(u,\frac{1}{f})\, re^{i\theta}}{(u+re^{i\theta})^2} \, du + $$

$$+ \mathcal{O}\left(\frac{r}{R}\left[N(R,\tfrac{1}{f}) + n(R,\tfrac{1}{f})\right]\right) .$$

The error term can be replaced by $\mathcal{O}\left(\frac{r}{R}\, T(2R,f)\right)$ in the usual way. If $0 < \gamma < \pi$, then an integration with respect to θ from θ to γ, yields

$$\sum_{|a|<R} \frac{1}{2\pi} \int_{-\gamma}^{\gamma} \log \left| 1 + \frac{re^{i\theta}}{|a|} \right| d\theta = \sum_{|a|<R} \frac{1}{\pi} \int_0^{\gamma} \log \left| 1 + \frac{re^{i\theta}}{|a|} \right| d\theta$$

$$= \mathcal{R}\, \frac{1}{\pi} \int_0^R \frac{N(u,\frac{1}{f})}{i} \, du \int_0^{\gamma} \left\{ \frac{d}{d\theta} \left[\frac{-1}{u+re^{i\theta}} \right] \right\} d\theta + \mathcal{O}\left(\frac{r}{R}\, T(2R,f)\right)$$

$$= \mathcal{R}\,\frac{1}{\pi}\int_0^R \frac{N(u,\frac{1}{f})}{i}\left\{\frac{1}{u+r} - \frac{1}{u+re^{i\gamma}}\right\} du + O\left(\frac{r}{R}\,T(2R,f)\right)$$

$$= -\,\mathcal{R}\,\frac{1}{\pi i}\int_0^R \frac{N(u,\frac{1}{f})}{u+re^{i\gamma}}\, du + O\left(\frac{r}{R}\,T(2R,f)\right)$$

$$= -\,\mathcal{R}\,\frac{1}{\pi i}\int_0^R \frac{N(u,\frac{1}{f})(u-re^{i\gamma})}{u^2+r^2+2ur\cos\gamma}\, du + O\left(\frac{r}{R}\,T(2R,f)\right)$$

$$= \frac{1}{\pi}\int_0^R \frac{N(u,\frac{1}{f})\,r\sin\gamma}{u^2+r^2+2ur\cos\gamma}\, du + O\left(\frac{r}{R}\,T(2R,f)\right) .$$

Using this value in (6.2) and the analogous formula for the sum over the contribution from the poles we arrive finally at the fundamental formula: If $0 < \gamma = \gamma\,(r,\infty,\eta) < \pi$, then

$$T(r,f) \le \frac{1}{\pi}\int_0^R \frac{r\sin\gamma}{u^2+r^2+2ur\cos\gamma}\, N(u,\tfrac{1}{f})\,du$$

$$+\,\frac{1}{\pi}\int_0^R \frac{r\sin\gamma}{u^2+r^2-2ur\cos\gamma}\, N(u,f)\,du$$

(6.3)

$$+\; O\left(\frac{r}{R}\,T(2R,f) + \eta(r)\right) \qquad (r > r_0).$$

We note that (6.3) remains valid without essential change if the assumption $f(0) = 1$ is dropped, since in this case

$$f(z) = cz^{s} F(z), \qquad F(0) = 1$$

$$T(r,f) = T(r,F) + (\log r)$$

$$N(r,f) = N(r,F) + n(o,f) \log r \geq N(r,F) \qquad (r \geq 1)$$

$$N(r,\tfrac{1}{f}) = N(r,\tfrac{1}{F}) + n(o,\tfrac{1}{f}) \log r \geq N(r,\tfrac{1}{F}) \quad (r \geq 1)$$

so that by an application of (6.3) to $F(z)$ we obtain

$$T(r,f) \leq \frac{1}{\pi} \int_0^R \frac{r \sin \gamma}{u^2+r^2+2tr \cos \gamma} N(t,\tfrac{1}{f})\,dt + \int_0^R \frac{r \sin (\pi-\gamma) N(t,f)\,dt}{t^2+r^2+2tr \cos(\pi-\gamma)}$$

$$(6.4) \qquad + \; O\left(\frac{r}{R}\, T(2R,f) + \eta(r) + \log r\right) .$$

In exactly the same way one can prove the following extension.

Let $0 < \rho < 1$. Let

$$E_\rho(r,f) = E_\rho(r,\infty) = \left\{ \theta \,\middle|\, |f(re^{i\theta})| > e^{\rho T(r,f)}, \; |\theta| < \pi \right\}$$

$$2\gamma_\rho = 2\gamma_\rho(r,c) = |E_\rho(r,f)| .$$

Then, by following the method of proof of (6.4) but integrating over E_ρ and noting

$$\frac{1}{2\pi}\int_{E_\rho(r,\infty)} \log|f(re^{i\theta})|\,d\theta > m(r,f) - \frac{\rho}{2\pi}(2\pi - 2\gamma_\rho(r))T(r,f),$$

$$T(r,f)\left(1-\rho\left(1-\frac{\gamma_\rho}{\pi}\right)\right) \leq \frac{1}{\pi}\int_0^R \frac{r\sin\gamma_\rho}{u^2+r^2+2ur\cos\gamma_\rho} N(u,\tfrac{1}{f})\,du$$

$$(6.5) \qquad + \frac{1}{\pi}\int_0^R \frac{r\sin\gamma_\rho}{u^2+r^2-2ur\cos\gamma_\rho} N(u,f)\,du$$

$$+ O\left(\frac{r}{R}T(2R,f) + \log r\right)$$

where

$$0 < \gamma_\rho = \gamma_\rho(r,f) < \pi.$$

As an application of (6.3) we prove

LEMMA 6.1. _Let_ $f(z)$ _be a meromorphic function of lower order_ $\mu < 1$ _and order_ $\lambda(\leq \infty)$. _Let_ β _be a limit of_ $\gamma(r,c,\eta(r))$ _as_ $r \to \infty$ _through a sequence of Polya-peaks of_ $T(r)$ _of order_ ρ _where_ ρ _satisfies_ $\mu \leq \rho \leq \lambda$ _and_ $\rho < 1$. _If_

$$u = 1-\delta(d,f),\ v = 1-\delta(c,f) \qquad (d \neq c)$$

then

$$(6.6) \qquad \sin \pi\rho \leq u \sin \beta\rho + v \sin (\pi-\beta)\rho \qquad \rho > o$$

$$\pi \leq u\beta + v(\pi-\beta) \qquad \rho = o,$$

If β_ρ *is a limit point of* $\gamma_\rho (r,c,f)$, *then*

$$(6.6') \qquad (1-\rho (1 - \frac{\beta\rho}{\pi}) \sin \pi\rho \leq u \sin \beta_\rho\rho + v \sin (\pi-\beta_\rho)\rho \quad (\rho>o)$$

$$\pi\left(1 - \rho(1 - \frac{\beta_\rho}{\pi})\right) \leq u\beta_\rho + v (\pi-\beta_\rho) \qquad (\rho = o).$$

PROOF. We give only the proof of (6.6), the proof of (6.6') is along the same lines starting from (6.5) in the place of (6.4). We may suppose without loss of generality $d = o$, $c = \infty$, by considering a linear transform of $f(z)$, if necessary.

Suppose first that there is a sequence of Polya-peaks of order ρ, $\{r_m\}$ such that $o< \gamma (r_m,f,\eta) < \pi$, $\gamma(r_m,f,\eta) \to \beta$.

By the definition of Polya peaks we may suppose that

$$(6.7) \qquad T(t,f) < T(r_m,f) (1+\epsilon_m) (t/r_m)^\rho$$

$$(r_m'' \leq t \leq r_m' \; ; \; r_m'/r_m > m, \; r_m/r_m'' > m)$$

where $\epsilon_m \to o$ as $m \to \infty$.

By the definition of deficiency

$$N(t,\tfrac{1}{f}) < (u + \eta_m)\, T(t,f) \qquad r''_m \leq t \tag{6.8}$$

$$N(t,f) < (v + \eta_m)\, T(t,f) \qquad r''_m \leq t \tag{6.9}$$

$$\eta_m \to 0 \quad \text{as} \quad m \to \infty .$$

Also $N(t,f) \leq T(r''_m, f)$; $N(t,\frac{1}{f}) \leq T(r''_m,f)$ $\quad (r''_m > t)$.

With the estimates (6.7), (6.8) and (6.9) one obtains from (6.6) with $R = \frac{1}{2} r'_m$ ($> 2r_m$ for $m > m_o$),

$$T(r_m,f) \leq (u+\eta_m)(1+\epsilon_m)\, \frac{1}{\pi} \int_{r''_m}^{r'_m} T(r_m,f)\left(\frac{u}{r_m}\right)^{\rho} \frac{r_m \sin \gamma_m}{u^2+r_m^2+2ur_m\cos \gamma_m}\, du$$

$$+ (v+\eta_m)(1+\epsilon_m)\, \frac{1}{\pi} \int_{r''_m}^{r'_m} T(r_m,f) \left(\frac{u}{r_m}\right)^{\rho} \frac{\sin \gamma_m}{u^2+r_m^2-2ur_m \cos \gamma_m}\, du$$

$$+ \int_0^{r''_m} T(r''_m) \frac{r_m \sin \gamma_m}{u^2+r_m^2+2ur_m \cos \gamma_m}\, du$$

$$+ \int_0^{r''_m} T(r''_m,f) \frac{r_m \sin \gamma_m}{u^2+r^2-2ur_m \cos \gamma_m}\, du$$

$$+ O\left\{\left(\frac{r_m}{r'_m}\right)^{1-\rho} T(r_m,f)\right\} + o\left(T(r_m,f)\right) .$$

Or, using (6.7) again on the right hand side and putting $u/r_m = x$, we obtain

$$T(r_m,f) \leq \left[(u+\eta_m)(1+\epsilon_m)\int_0^\infty x^\rho P(x,\gamma_m)\,dx\right.$$

$$+ (v+\eta_m)(1+\epsilon_m)\int_0^\infty x^\rho P(x,\pi-\gamma_m)\,dx$$

$$\left.+ O\left[\left\{\left(\frac{r''_m}{r_m}\right)^\rho \int_0^{r''_m} \frac{1}{r_m}\,du + \left(\frac{r_m}{r'_m}\right)^{1-\rho} + o(1)\right\} T(r_m,f)\right]\right]$$

where

$$P(x,\gamma) = \frac{1}{\pi}\,\frac{\sin\gamma}{x^2+1+2x\cos\gamma} .$$

The error term is $o\,(T(r_m,f))$.

By a contour integration $\left(\text{or from } \int_0^\infty \frac{x^{\rho-1}}{x+e^{i\gamma}}\,dx\right)$, we have

$$\int_0^\infty x^\rho \, P(x,\Upsilon)\,dx = \frac{\sin \Upsilon\rho}{\sin \pi\rho} \qquad 0 < \rho < 1$$

$$\int_0^\infty P(x,\Upsilon)\,dx = \Upsilon/\pi \qquad \rho = 0$$

Therefore, dividing (6.10) by $T(r_m,f)$ and letting $m \to \infty$

$$1 \leq u \frac{\sin \beta\rho}{\sin \pi\rho} + \frac{\sin (\pi-\beta)\rho}{\sin \pi\rho} v \qquad \rho > 0$$

which is (6.6) for $\rho > 0$. Similarly for $\rho = 0$.

If $\Upsilon(r_m,f) = 0$ for all large m, then $m(r_m,f) = 0$ and so $N(r_m,f) = T(r_m,f) + o(1)$, so that $\delta(\infty,f) = 0$.

$$1 \leq v$$

which is (6.6) for $\beta = 0$. Similarly (6.6) is true if $\Upsilon(r_m,f) = \pi$ for all large m.

SECTION 7.

THE SPREAD OF A DEFICIENT VALUE.

Let $f(z)$ be a meromorphic function of lower order $\mu < 1$. Let $\beta(c)$ be the geatest lower bound of all

$$\underline{\lim}\ \gamma(r_n', c, \eta(r))$$

where $r_n' \to \infty$ through a sequence of Pólya-peaks of order μ. Then

$$\beta(c) = \lim \gamma(r_n, c, \eta(r))$$

where $\{r_n\}$ is a suitable sequence of Pólya peaks of order μ. We shall call $\beta(c)$ the <u>spread</u> of the value c. We shall prove that for a deficient value the spread is bounded below by a positive constant depending only on μ and $\delta(c,f)$.

If we choose the function $\eta(r)$ entering into the definition of $E(r,c,\eta(r))$ as a function tending to ∞, then it is clear that for any finite set of c_j's the sets $E(r,c_j,\eta(r))$ are non-intersecting for $r > r_o$ so that

$$0 \leq \sum \gamma\ (r, c_j, \eta(r)) \leq \pi \qquad (r > r_o)$$

and so, letting $r \to \infty$ in a suitable manner.

$$(7.1) \qquad 0 \leq \sum \beta(c_j) \leq \pi$$

By limiting transition this remains true if the summation is extended over all deficient values, finite or infinite in number.

THEOREM 7.1. If $f(z)$ is meromorphic of lower order $\mu < 1$ and if for some c

$$v = 1-\delta(c,f) \leq \cos \pi\mu$$

then $\mu \leq \frac{1}{2}$ and c is the only deficient value of $f(z)$ and for any sequence of Pólya-peaks of order μ

$$\gamma(r_m; c, \eta(r)) \to \pi = \beta(c).$$

Remark. This theorem is closely related to the corollary of Theorem 5.2, but the information is slightly different.

PROOF OF THEOREM 7.1. We give the proof for $\mu > 0$. For $\mu = 0$, it proceeds along the same lines, using the second formula of (6.6). If β is a limit point of $\{\gamma(r_m, c, \eta(r))\}$, then by (6.6) with $\rho = \mu$, with

$$u = 1-\delta(d,f), \qquad v = 1-\delta(c,f)$$

$$\begin{aligned} \sin \pi\mu &\leq u \sin \beta\mu + v \sin (\pi-\beta)\mu \\ &\leq \sin \beta\mu + \cos \pi\mu \sin (\pi-\beta)\mu \\ &\leq \sin ((\pi-(\pi-\beta))\mu) + \cos \pi\mu \sin (\pi-\beta)\mu \\ &\qquad\qquad = \sin \pi\mu \cos(\pi-\beta)\mu. \end{aligned}$$

Hence $\cos(\pi-\beta)\mu = 1$; $\beta = \pi$. Returning to (6.6)

$$\sin \pi\mu \leq u \sin \pi\mu \;;\; u = 1 \text{ and } \delta(d,f) = 0.$$

COROLLARY. _A function of lower order_ 0 _has at most one deficient value_.

THEOREM 7.2. _If_ $f(z)$ _is meromorphic of lower order_ $\mu < 1$ _and if_

$$\cos \pi\mu < v = 1-\delta(c,f) < 1$$

then

I. _For any_ $d \neq c$ _and_ $u = 1-\delta(d,f)$

(7.2) $$u^2+v^2-2uv \cos \pi\mu \geq \sin^2 \pi\mu.$$

II. _If_ β_1 _is any limit point of a sequence_ $\{\gamma(r_m,c,\eta(r))\}$ _and_ β_2 _any limit point of a sequence_ $\{\gamma(r_m,d,\eta(r))\}$, _where_ $r_m \to \infty$ _through a sequence, of Pólya-peaks, then_

$$\pi-\beta_1-\beta_2 < \frac{2}{\mu} \operatorname{arc\,cos} \frac{\sin \pi\mu}{\Delta}$$

III. _With_ β_1 _as in_ II

$$\frac{2}{\mu} \operatorname{arc\,tan} \frac{u-v \cos \mu\pi - \sqrt{\Delta^2-\sin^2\mu\pi}}{(1+v) \sin \pi\mu} \leq \beta_1$$

$$\leq \frac{2}{\mu} \text{ arc tan } \frac{u-v \cos \pi\mu + \sqrt{\Delta^2 - \sin^2 \mu\pi}}{(1+v) \sin \pi\mu}$$

IV. $\beta_1 > \pi - \frac{2}{\mu} \text{ arc tan } \frac{v - \cos \pi\mu}{\sin \pi\mu}$

<u>where</u> $\Delta^2 = u^2 + v^2 - 2uv \cos \pi\mu$.

<u>PROOF</u>. Under the hypotheses of the Theorem $\cos \pi\mu < 1$, $\mu > 0$. By (6.6) with $\rho = \mu$.

$$\sin \pi\mu \leq u \sin \beta_1\mu + v \sin (\pi - \beta_1)\mu$$

(7.3) $$\sin \pi\mu \leq (u - v \cos \pi\mu) \sin \beta_1\mu + v \sin \pi\mu \cos \beta_1\mu.$$

Define η and Δ by

$$\Delta = (u - v \cos \pi\mu)^2 + v^2 \sin^2 \pi\mu = u^2 + v^2 - 2uv \cos \pi\mu$$

$$\sin \eta = \frac{u - v \cos \pi\mu}{\Delta}, \quad \cos \eta = \frac{v \sin \pi\mu}{\Delta}.$$

Then (7.3) becomes

(7.4) $$\sin \pi\mu \leq \Delta \operatorname{Cos} (\beta\mu - \eta).$$

Since $\cos (\beta\mu - \eta) \leq 1$, this proves (7.2).

To prove III, put $x = \tan(\frac{1}{2}\beta, \mu)$. Then (7.3) becomes

$$J = (1+v) \sin \pi\mu \, x^2 - 2(u - v \cos \pi\mu)x + (1-v) \sin \pi\mu \leq 0 .$$

This inequality is true, if x lies between the two non-negative roots

$$\frac{u-v\cos\pi\mu \pm\sqrt{(u-v\cos\pi\mu)^2-(1-v^2)\sin^2\pi\mu}}{(1+v)\sin\pi\mu} =$$

$$= \frac{u-v\cos\pi\mu \pm \sqrt{\Delta^2-\sin^2\pi\mu}}{(1+v)\sin\pi\mu} .$$

Assertion III follows immediately.

Proof of II. We note next that

$$(7.5) \qquad u-v\cos\pi\mu > 0$$

For, by Theorem 7.1 $v < 1$ implies $u > \cos\pi\mu$, so that (7.5) is true for $\cos\pi\mu \geq 0$. If $\cos\pi\mu < 0$, then (7.5) could only fail to be true, if $u = v = 0$, but this is impossible by Theorem 7.1. By (7.5),

$$0 < \eta < \tfrac{\pi}{2}.$$

If χ is defined by

$$0 < \chi < \tfrac{\pi}{2}, \cos\chi = \frac{\sin\pi\mu}{\Delta}$$

then (7.4) can be rephrased

$$\text{Cos}\,\chi < \text{Cos}(\beta_1\mu - \eta)$$

and since $-\frac{\pi}{2} < \beta_1\mu - \eta < \pi$, this implies

$$|\beta_1\mu - \eta| < \chi$$

$$(7.6) \qquad \eta - \chi < \beta_1\mu < \eta + \chi$$

since $\cos\eta < \cos\chi$, the lower bound for $\beta_1\mu$ is positive.

By interchanging c and d and calling β_2 any limit point of $\Upsilon(r_m, d, \eta(r))$, we have

$$\sin \pi\mu \leq u \sin(\pi - \beta_2)\mu + v \sin \beta_2\mu$$

which is (7.3) with β_1 replaced by $\pi-\beta_2$. Since (7.6) was a consequence of (7.3) and $v < 1$, we have now

$$\eta - \chi < (\pi-\beta_2)\mu < \eta + \chi$$

and so

$$(\pi-\beta_2-\beta_1)\mu < \eta_1 + \chi - (\eta_1 - \chi) = 2\chi = 2 \text{ arc cos } \frac{\sin \pi\mu}{\Delta} .$$

PROOF of IV. It is clear from (7.3) that the least possible value of β_1 satisfying (7.3) for a given value of v will be obtained for $u = 1$. Interchanging u and v and β_1 and $\pi-\beta_1$ in (7.3) we obtain for $u = 1$ from III

$$\pi-\beta_1 \leq \frac{2}{\mu} \text{ arc tan } \frac{v - \cos\pi\mu + \sqrt{1+v^2-2v\cos\pi\mu - \sin^2\pi\mu}}{2\sin\pi\mu}$$

$$\leq \frac{2}{\mu} \text{ arc tan } \frac{v-\cos\pi\mu}{\sin\pi\mu}$$

This is IV.

<u>Geometrical interpretation of Theorem 7.1 and 7.2. I</u>.

Interpret (u,v) where

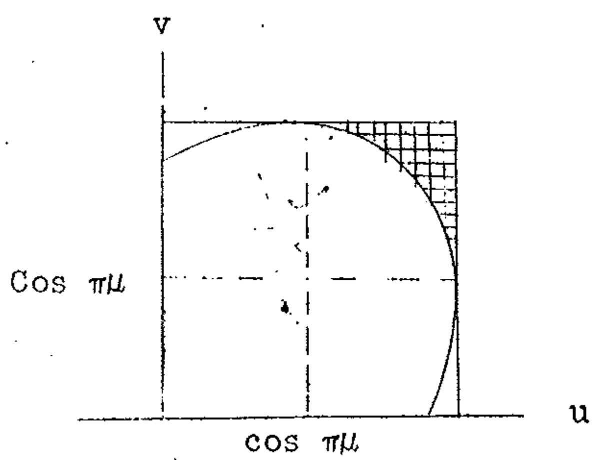
$$u = 1-\delta(d,f) \qquad v = 1-\delta(c,f)$$

as a point in the plane with Cartesian coordinates u,v.

The obvious inequalities $o \leq u \leq 1$, $0 \leq v \leq 1$ say that (u,v) lies in the unit square in the first quadrant. Theorem 7.1 says that any point with $v < \cos \pi$ must be on the line $u = 1$ (This statement is vacuous for $\mu > \frac{1}{2}$). The inequality (7.2) restricts (u,v) to the outside of the ellipse

$$u^2 + v^2 - 2uv \cos \pi\mu = \sin^2 \pi\mu.$$

This ellipse touches the lines $u = 1$, $v = 1$ at the points $(1, \cos \pi\mu)$, $(\text{Cos } \pi\mu, 1)$. The point (u,v) therefore lies in the region drawn in with heavy lines.



Simple examples show there are functions of order μ for which (u,v) has any assigned position in the shaded area.

Next we obtain an estimate for the spread of a deficient value for functions of arbitrary, finite lower order.

THEOREM 7.3. _Let_ $f(z)$ _be a meromorphic function of lower order_ $\mu < \infty$. _Let_ q _be an integer such that_ $q > \mu$. _If_

$$1 - \cos \frac{\pi\mu}{q} \leq \delta(c,f) \tag{7.7}$$

then the spread of c _with respect to_ f _satisfies_

$$\beta(c) > \frac{\pi}{q} .$$

_If_

$$1 - \cos \frac{\pi\mu}{q} - \delta(c,f) = \rho > 0 \tag{7.8}$$

then

$$\beta(c) > \frac{\pi}{q} - \frac{2}{\mu} \cos^{-1} \frac{u - \cos \pi\mu}{\sin \pi\mu} = \frac{\pi}{q} - \frac{2}{\mu} \tan^{-1} \frac{\rho}{\sin \pi\mu}$$

PROOF. Let $\omega = e^{2\pi i/q}$. Replacing f by $f-c$ or $\frac{1}{f}$ (if $c = \infty$), we may suppose $c = 0$. Let $\rho_1, \rho_2, \ldots$ be the zeros of $f(z)$. Since the set of σ such that $T(r,f) \sim N(r,\frac{1}{f-\sigma})$ is non-countable, we can find such a σ different from all the numbers of the form $f(\omega^k \rho_\ell)$ $(k = 0, 1, 2, \ldots q-1, \ell = 1, 2, \ldots)$. Consider

$$\phi(z^q) = \frac{\prod_{i=0}^{q-1} f(\omega^i z)}{\prod_{j=0}^{q-1} f(\omega^j z-\sigma)}$$

$\phi(z)$ is meromorphic. Since, by the choice of σ, no zero of the denominator is cancelled by a zero of the numerator

$$N(r, \frac{1}{\phi(z^q)}) = q\ N(r, \frac{1}{f})$$

$$N(r, \phi(z^q)) = q\ N(r, \frac{1}{f-\sigma}) \sim q\ T(r,f).$$

On the other hand

$$N(r, \phi(z^q)) < T(r, \phi(z^q)) < \sum T(r, \frac{f(\omega^j z)}{f(\omega^j z)-\sigma}) = qT(r,f) + O(1).$$

Therefore

$$T(r, \phi(z^q)) \sim q\ T(r,f).$$

Now

$$T(r, \phi(z^q)) = T(r^q, \phi(z))$$

$$N(r, \phi(z^q)) = N(r^q, \phi(z)).$$

Therefore

$$N(r, \frac{1}{\phi(z)}) = N(r^{\frac{1}{q}}, \frac{1}{\phi(z^q)}) = q\, N(r^{\frac{1}{q}}, \frac{1}{f(z)})$$

$$T(r, \phi) = T(r^{\frac{1}{q}}, \phi) \sim q\, T(r^{\frac{1}{q}}, f).$$

Therefore

$$\delta(0, \phi) = \delta(0, f)$$

and the lower order of ϕ is $\mu(\phi) = \frac{\mu}{q}$.

If $\{r_m\}$ is a sequence of Pólya peaks of order μ of f, then $\{r_m^q\}$ is a sequence of Pólya peaks of order $\frac{\mu}{q}$ of ϕ.

Under the hypothesis (7.7), $\phi(z)$ satisfies the hypothesis of Theorem 7.1.

Since $r_m^q\, e^{iq\theta}$ runs once round the whole circumference $|\zeta| = r_m^q$ as θ varies from $-\frac{\pi}{q}$ to $\frac{\pi}{q}$, the conclusion of Theorem 7.1 is expressed by

$$\text{meas}\left\{\theta \,\middle|\, \log |\phi(r_m^q\, e^{iq\theta})| < -\eta(r_m);\ |\theta| \leq \frac{\pi}{q}\right\} > \frac{2\pi}{q} - \epsilon$$

$$(m > m_0).$$

By considering separately the cases $|f| < 2\sigma$ and $\sigma \leq \frac{1}{2}|f|$ it is easily seen that

(7.9) $$\frac{|f|}{|f - \sigma|} \geq \min \left\{ \frac{|f|}{3\sigma}, \frac{2}{3} \right\}$$

If $\eta(r_m) > 9 \log \frac{3}{2}$, the inequality

$$\log |\phi(r_m^q e^{iq\theta})| < -\eta(r_m),$$

then (7.9) implies

$$\min_j \log |f(r_m^q e^{i\theta} \omega^j)| \leq -\frac{q(r_m)}{q} \log 3\sigma$$

$$\leq -\frac{\eta(r_m)}{2q} = -K\eta(r_m),$$

if $\eta(r_m) > 2 \log 3\sigma$.

Therefore at leat one $\log |f(r_m e^{i\theta} \omega^j)| < -K\eta(r_m)$ for θ in a set F of measure $\frac{2\pi}{q} - \epsilon$ contained in $|\theta| \leq \frac{\pi}{q}$. But this is equivalent to saying that the set of θ in $|\theta| \leq \pi$ for which $\log |f(r_m e^{i\theta})| < -K\eta(r_m)$ is of measure $\geq \frac{2\pi}{q} - \epsilon$. This proves the first assertion.

To prove the second assertion one applies Theorem 7.2 IV to $\phi(\zeta)$ instead of Theorem 7.1.

Examples show that the estimate $\beta(c) \geq \frac{\pi}{q}$ is best possible if

$$\delta(c,f) = 1-\cos\frac{\pi\mu}{q} \qquad (q = 1,2,\ldots).$$

This has suggested

Edrei's Conjecture. *For every meromorphic function of lower order* $\mu < \infty$

$$\beta(c) \geq \min\left\{\pi, \frac{1}{\mu}\arccos(1-\delta(c,f))\right\}. \tag{7.10}$$

Support for the conjecture comes

a) from the truth of Theorem 7.3

b) from the fact that it agrees with the conclusion of Theorem 7.1.

c) from the possibility of deducing most of Theorem 7.2 from it. By (7.1), for any two values c,d $(c \neq d)$

$$0 \leq \beta(c) + \beta(d) \leq \pi \quad . \tag{7.11}$$

If $1-\delta(c,f) = v$ and $1-\delta(d,f) = u > \cos\pi\mu$, then (7.11) and (7.10) lead to

$$0 < \beta(c) < \pi-\beta(d) < \pi,$$

$$v \geq \cos(\beta(c)\mu) \geq \cos(\pi-\beta(d))\,\mu \geq u\cos\pi\mu + \sqrt{1-u^2}\,\sin\pi\mu$$

so that

$$(v-u \cos \pi\mu)^2 \geq (1-u^2) \sin^2 \pi\mu.$$

This is (7.2).

<u>Deductions about</u> $\sum \delta_k$ <u>from Edrei's Conjecture</u>.

Let $f(z)$ be meromorphic with $\mu \leq \frac{1}{2}$. If one $\delta_k > 1-\cos \pi\mu$, then c_k is the only deficient value by Theorem 7.1. If $f(z)$ has more than one deficient value then

$$(7.12) \qquad o < \beta_k < \pi, \qquad \sum \beta_k \leq \pi$$

and $1-\delta_k \geq \cos \beta_k\mu$. Therefore an upper bound for $\sum \delta_k$ is given by

$$\text{least upper bound } \sum (1-\cos \beta_k\mu)$$

subject to (7.12). If there are two β's, β_1 and β_2 say, then $\sum (1-\cos \beta_k\mu)$ can be increased by replacing β_1 by $\beta_1+\beta_2$, β_2 by 0, since

$$2 - \cos \beta_1\mu - \cos \beta_2\mu = 2 - 2 \cos \frac{\beta_1+\beta_2}{2} \mu \cos \frac{\beta_1-\beta_2}{2} \mu$$

$$\leq 2 - 2 \cos \frac{\beta_1+\beta_2}{2} \mu \cos \frac{\beta_1+\beta_2}{2} \mu =$$

$$= 1 - \cos \frac{\beta_1+\beta_2}{2} \mu,$$

because $\mu|\beta_1-\beta_2| < \mu(\beta_1+\beta_2) < \frac{\pi}{2}$. Therefore $\max \sum (1-\cos \beta_k\mu)$ subject to $0 \leq \beta_k \leq \pi$, $\sum \beta_k \leq \pi$ is obtained for $\beta_1 = \pi$, all other $\beta = 0$ and the value of the maximum is $1-\cos \pi\mu$.

This suggests

THEOREM 7.4. *If* $f(z)$ *is a meromorphic function of lower order* $\mu < \frac{1}{2}$, *then*

$$\sum_c \delta(c,f) \leq 1 \quad \textit{if } f \textit{ has only one deficient value}$$

$$\sum_c \delta(c,f) \leq 1-\cos \pi\mu \quad \textit{if } f \textit{ has more than one deficient value.}$$

If $\frac{1}{2} < \mu < 1$, *then*

$$1 - \delta_k > \cos^+ \beta_k\mu = \max (0, \cos \beta_k\mu)$$

and $\sum \delta_k$ *is majorized by*

$$\max \sum (1-\cos^+ \beta_k\mu)$$

subject to $o < \beta_k < \pi$, $\sum \beta_k = \pi$.

If one $\beta_k > \frac{\pi}{2\mu}$, the sum can be increased by reducing to $\frac{\pi}{2\mu}$ and introducing a new term $1-\cos \beta$ with $\beta\mu+\frac{\pi}{2} = \beta_k\mu$. If two β's, β_1 and β_2 say, satisfy $\beta_j < \frac{\pi}{2\mu}$, $j = 1,2$, then $\sum (1-\cos^+ \beta_k\mu)$ can be increased by replacing β_1 by $\min \left\{ \beta_1+\beta_2, \frac{\pi}{2\mu} \right\} = \beta_1'$ and β_2 by $\max \left\{ o, \beta_1+\beta_2 - \frac{\pi}{2\mu} \right\} = \beta_2'$.

$$2 - \cos \beta_1\mu - \cos \beta_2\mu = 2 - 2 \cos \frac{\beta_1+\beta_2}{2}\, \mu \cos \frac{\beta_1-\beta_2}{2}\, \mu$$

$$< 2 - \cos \beta_1'\mu - \cos \beta_2'\mu$$

$$= 2 - 2 \cos \frac{\beta_1+\beta_2}{2}\, \mu \cos \frac{\beta_1'-\beta_2'}{2}\, \mu$$

$$\left(0 < \frac{\beta_1+\beta_2}{2}\, \mu = \frac{\beta_1'+\beta_2'}{2}\, \mu < \frac{\pi}{2} ;\quad \beta_1'-\beta_2' > |\beta_1-\beta_2| \right).$$

Therefore the upper bound is attained for all nonzero β's equal to $\frac{\pi}{2\mu}$ with at most one exception which is less than $\frac{\pi}{2\mu}$ i.e. $\beta_1 = \frac{\pi}{2\mu}$, $\beta_2 = \pi - \frac{\pi}{2\mu}$.

We can therefore make the guess

THEOREM 7.5. *If* $f(z)$ *is a meromorphic function of lower order* μ, $\frac{1}{2} < \mu < 1$, *then*

$$\sum \delta_k < 1 + 1 - \cos \left(\pi\mu - \frac{\pi}{2}\right) = 2 - \sin \pi\mu.$$

SECTION 8.

<u>PROOF OF THEOREM 7.4</u>. Edrei has managed to prove Theorem 7.4 independently of his conjecture [4].

THEOREM 8.1. *If* $f(z)$ *is of lower order* $\mu < \mu_o$ *and if* $f(z)$ *has at least two deficient values then*

$$\sum_c \delta(c,f) \leq 1-\cos \pi\mu.$$

Here μ_o is the least positive root of

$$\frac{\sin \pi\mu_o}{\pi\mu_o} = \frac{4}{9} \qquad (.64 < \mu_o < \tfrac{2}{3}).$$

<u>Remark.</u> For $\mu \leq \frac{1}{2}$ this bound is best possible. Note the interesting effect of the hypothesis of at least 2 deficient values. It reduces the maximum of $\sum \delta$ from 1 (attained by any entire function) to $1-\cos \pi\mu$.

PROOF. Arrange the deficient values in a sequence $c_1, c_2, \ldots$ such that $(\delta(c_n,f) = \delta_n)$

$$\delta_1 \geq \delta_2 \geq \delta_3 \geq \cdots$$

We must distinguish 2 cases:

I. $\delta_3 \geq 1-\cos \frac{\pi\mu}{3}$ II. $\delta_3 < 1 - \cos \frac{\pi\mu}{3}$.

PROOF IN THE CASE I. We have $\delta_j > 1-\cos\frac{\pi\mu}{3}$ $(j = 1,2,3)$ and so by Theorem 7.3

$$\beta(c_j) \geq \frac{\pi}{3} \qquad j = 1,2,3$$

since $\sum \beta(c_j) \leq \pi$, this means

$$\beta(c_j) = \frac{\pi}{3} \quad \text{and} \quad \sum_1^3 \beta(c_j) = \pi.$$

Therefore c_1, c_2, c_3 are the only deficient values and by Lemma 6.1 with $u = 1-\delta(c_{j_1})$, $v = 1-\delta(c_{j_2})$ $(j_1 \neq j_2,$ $1 \leq j_1, j_2 \leq 3)$ we have

$$\sin \pi\mu \leq u \sin\frac{\pi\mu}{3} + v \sin\frac{2\pi\mu}{3}.$$

Also, interchanging the role of c_1 and c_2,

$$\sin \pi\mu \leq u \sin\frac{2\pi\mu}{3} + v \sin\frac{\pi\mu}{3}.$$

Adding the two inequalities

$$2 \sin \pi\mu \leq (u+v)\left(\sin\frac{\pi\mu}{3} + \sin\frac{2\pi\mu}{3}\right)$$

$$= 2(u+v) \sin\frac{\pi\mu}{2} \cos\frac{\pi\mu}{6}$$

$$\delta_{j_1} + \delta_{j_2} = 2-u-v \leq 2 - \frac{\sin \pi\mu}{\sin\frac{\pi\mu}{2}\cos\frac{\pi\mu}{6}}$$

Write down for all choices j_1, j_2 and add

$$2(\delta_1+\delta_2+\delta_3) \leq 6 - \frac{3 \sin \pi\mu}{\sin \frac{\pi\mu}{2} \cos \frac{\pi\mu}{6}} = 6 \left[1 - \frac{\cos \frac{\pi\mu}{2}}{\cos \frac{\pi\mu}{6}} \right]$$

since $\cos \frac{\pi\mu}{6} < 1$,

$$\delta_1+\delta_2+\delta_3 \leq 3(1-\cos \frac{\pi\mu}{2}).$$

Now

$$1-\cos 2x - 3(1-\cos x) = (1-\cos x)(2 \cos x-1).$$

Therefore

$$3(1-\cos \frac{\mu\pi}{2}) \leq 1-\cos \pi\mu$$

if

$$\frac{1}{2} \leq \cos \frac{\pi\mu}{2} \leq 1 \quad \text{i.e.} \quad 0 \leq \mu < \frac{2}{3}.$$

Since μ_o is less than $\frac{2}{3}$, the assertion is proved in Case I.

<u>PROOF IN CASE II</u>. For $j \geq 3$, we can find an integer $q_j \geq 3$ such that

$$1-\cos \frac{\pi\mu}{q_j+1} \leq \delta_j < 1-\cos \frac{\pi\mu}{q_j} .$$

Then, by Theorem 7.3,

$$\beta(c_j) \geq \frac{\pi}{q_j+1} .$$

If $u = 1-\delta_1$, $v = 1-\delta_2$ then, by Theorem 7.2 II,

$$(8.1) \qquad \pi \sum_{j \geq 2} \frac{1}{q_j+1} \leq \sum_{j \geq 3} \beta(c_j) \leq \pi-\beta(c_1)-\beta(c_2)$$

$$\leq \frac{2}{\mu} \cos^{-1}\left(\frac{\sin \pi\mu}{\Delta} \right),$$

where

$$\Delta^2 = u^2+v^2-2u\, v \cos \pi\mu.$$

Also, since $1-\cos x \leq \frac{1}{2} x^2$,

$$(8.2) \qquad \sum_{j=3}^{\infty} \delta_j \leq \sum_{j=3}^{\infty} \left(1-\cos \frac{\pi\mu}{q_j}\right) \leq \frac{1}{2} \pi^2 \mu^2 \sum_{j=3}^{\infty} \frac{1}{q_j^2} .$$

But

$$(8.3) \qquad \sum_{3}^{\infty} \frac{1}{q_j^2} = \sum_{3}^{\infty} \frac{q_j+1}{q_j^2} \frac{1}{q_j+1} < \frac{4}{9} \sum \frac{1}{q_j+1} .$$

Combining (8.1),(8.2) and (8.3)

$$\sum_{j \geq 3} \delta_j \leq \frac{4}{9} \pi\mu \cos^{-1}\left(\frac{\sin \pi\mu}{\Delta} \right)$$

$$\frac{4\pi\mu}{9} \cos^{-1}\left(\frac{\sin \pi\mu}{\Delta}\right) = \frac{4\pi\mu}{9} \text{ arc tan } \frac{\sqrt{\Delta^2-\sin^2 \pi\mu}}{\sin \pi\mu}$$

$$< \frac{4}{9} \frac{\pi\mu}{\sin \pi\mu} \sqrt{\Delta^2 - \sin^2 \pi\mu}.$$

Therefore

$$\sum_{j\geq 1} \delta_j = \delta_1+\delta_2 + \sum_{j\geq 3} \delta_j \leq 2-u-v + \frac{4}{9} \frac{\pi\mu}{\sin \pi\mu} \sqrt{\Delta^2-\sin^2 \pi\mu} .$$

For given Δ and u,v subject to $0 < u, v \leq 1$, $\min(u+v)$ is attained if $u = 1$, $1+v^2-2v \cos \pi\mu = \Delta^2$ (Look at ellipse-diagram) i.e. $u = 1$, $v = \cos \pi\mu + \sqrt{\Delta^2-\sin^2 \pi\mu}$ (remember $v > \cos \pi\mu$). Hence

$$\sum_{j\geq 1} \delta_j \leq 2-(1+\cos \pi\mu + \sqrt{\Delta^2-\sin^2 \pi\mu}) + \frac{4\ \pi\mu}{9 \sin \pi\mu} \sqrt{\Delta^2-\sin^2 \pi\mu}$$

$$= 1-\cos \pi\mu - \sqrt{\Delta^2-\sin^2 \pi\mu} \left\{1 - \frac{4\ \pi\mu}{9 \sin \pi\mu}\right\}$$

$$\leq 1-\cos \pi\mu$$

if $1 - \frac{4\ \pi\mu}{9 \sin \pi\mu} \geq 0$ i.e. $\frac{\sin \pi\mu}{\pi\mu} \geq \frac{4}{9}$, $\quad \mu > \mu_o$.

Theorem 7.5 has not yet been proved, it is, however possible to prove

THEOREM 8.2. Let $f(z)$ be an entire function of lower order μ, $\frac{1}{2} < \mu < 1$ and of finite order. Then

$$\delta(c,f) \leq 2-\sin \pi\mu.$$

PROOF. There is nothing to prove if $\mu = 1$. For $\mu < 1$ the proof is based on the well known results of Nevanlinna Theory. As $r \to \infty$, for any n distinct complex numbers

$$(8.4) \qquad N(r,\tfrac{1}{f'}) + \sum_{1}^{n} m(r, \tfrac{1}{f-c_k}) \leq T(r,f') + O(\log r\, T(r))$$

$$(8.5) \qquad T(r,f') \leq T(r,f) + O(\log r\, T(r))$$

If the order of $f(z)$ is infinite, then r has to avoid an exceptional set of finite measure as it tends to ∞. By (8.5)

$$\underline{\lim} \frac{m(r,\frac{1}{f-c_k})}{T(r,f)} \geq \delta(c_k,f) = \underline{\lim} \frac{m(r,\frac{1}{f-c_k})}{T(r,f)} .$$

Therefore by (8.4)

$$1 \geq \overline{\lim} \frac{N(r,\frac{1}{f'})}{T(r,f')} + \underline{\lim} \sum \frac{m(r,\frac{1}{f-c_k})}{T(r,f')}$$

$$\geq \overline{\lim} \frac{N(r,\frac{1}{f'})}{T(r,f')} + \sum \underline{\lim} \frac{m(r,\frac{1}{f-c_k})}{T(r,f')}$$

$$1 \geq 1-\delta(o,f') + \sum \delta(c_k,f).$$

and so

$$\sum_{k=1}^{n} \delta(c_k, f) \leq \delta(o, f').$$

It follows that $\sum_{c} \delta(c,f)$ (summation over all deficient values) satisfies

$$\sum_{c} \delta(c,f) = \delta(\infty, f') + \sum_{c \neq \infty} \delta(c,f) \leq 1+\delta(o,f'). \tag{8.6}$$

By Theorem 7.2 I applied to f' with

$$u = 1-\delta(o,f'), \qquad v = 1-\delta(\infty, f') = 0$$

we obtain

$$\sin^2 \pi\mu \leq u^2 \; ; \quad u \geq \sin \pi\mu$$

$$\delta(o,f') \leq 1-\sin \pi\mu$$

and the Theorem follows from (8.6).

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