MATSCIENCE REPORT 62

LECTURES ON

DESCRIPTION OF PARTICLES WITH ANY SPIN AND WITH INTERNAL SYMMETRY

R. H. GOOD, JR.
Visiting Professor, MATSCIENCE.

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DESCRIPTION OF PARTICLES WITH ANY SPIN AND WITH INTERNAL SYMMETRY

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Visiting Professor, MATSCIENCE.

* Institute for Atomic Research and Department of Physics, Iowa State University, Ames, Iowa, U.S.A.

I. Introduction

The problem considered here is that of describing free particles and antiparticles of definite nonzero mass m, spin $s = 0, \frac{1}{2}, 1, 3/2$ and with internal SU(2) or SU(3) symmetry.

Several approaches to this problem already exist: (a) Spinor-analysis equations of motion (Dirac, Fierz, and Pauli)[1,2,3] (b) Hyper-Dirac equations of motion (Bargmann, Wigner)[4] (c) A Schrodinger equation with Hamiltonian $\beta(m^2+p^2)^{1/2}$ (Foldy)[5] (d) The helicity formalism (Jacob and Wick)[6] (e) Proca type equations for integer spin[7] (f) Rarita-Schwinger type equations for half-integer spin.[7]

All these formulations are equivalent, although it may be a complicated affair to find the details of the connection between two of them. Bargmann and Wigner[4] have emphasized that a set of functions describing all possible states of a free particle form the basis of a representation of the Poincare group and have shown that, except for equivalences, there is only one representation for given finite m and s. Thus there is only one type of system to study, for given mass and spin, and the only question is how to describe it in a way that lends itself to application.

In the description reviewed here, the particle is described by a (wave function which is the basis of a (0,s) $\oplus$ (s,0) representation of the Lorentz group. One value of the description is that it permits all properties of the free particle to be discussed in a straightforward way, in parallel to the well-known discussions in Dirac theory. Detailed formulas for almost everything can be worked out. For example for any spin the plane wave solutions can be found and formulas for the various polarization[8] and position[9] operators are known. Another usefulness of the description is that it gives an easy way to build up interactions. To make phenomenological interactions with form factors you just combine the wave functions to make scalars. Some progress has been made in including effects of external electric and magnetic fields on particles with various spins by formulating the wave equation in such a way that there are no auxiliary conditions and then replacing p_μ by $p_\mu - eA_\mu$.

The utility of the (0,s) $\oplus$ (s,0) representation was suggested independently by Joos[11], Weinberg[12], and by Weaver, Hammer, and Good[13]. Mathews[14] gave a valuable discussion of the uniqueness of and general formulas for the Hamiltonian. Williams Draayer and Weber[15,16] also gave several general formulas and showed how to handle the special types of series that occur in this subject. The relationships between all the formulations was discussed by Sankaranarayanan and Good[8] for spin 1 and by Shay Song and Good[17] for spin 3/2. The quantization of the theory, has been studied by Weaver[18], Mathews[19], Mathews and Ramakrishnan[20], and Nelson and Good[21].

This present review follows Nelson and Good's work[21]. Some background material has been added in the earlier sections and some subjects are discussed more at length here. However the original paper contains more details and especially the treatment of SU_3 self-conjugate multiplets.

II. Transformation Rule for the Wave Function

The basic idea in this description is to set up the properties of the system neatly in the rest frame and then make the Lorentz transformation to the laboratory. The process is straightforward once a certain form for the wave function transformation is adopted.

Consider to begin with the representation of Lorentz transformations continuous with the identity by 2×2 unimodular matrices. It is known that

$$A^{\dagger} \sigma_{\alpha} A = a_{\alpha\beta} \sigma_{\beta} \qquad (1)$$

where A are the two-by-two unimodular matrices, σ_{α} are $(\underset{\sim}{\sigma}, i)$ where $\underset{\sim}{\sigma}$ are the three Pauli matrices, and $a_{\alpha\beta}$ are the Lorentz transformation coefficients

$$X_{\alpha}' = a_{\alpha\beta} X_{\beta} \quad , \quad a_{\alpha\beta}\, a_{\alpha\gamma} = \delta_{\beta\gamma}$$

Here X_4 is it (factors of c and $\hbar$ will be omitted). The matrix A can be written explicitly as

$$A = e^{\frac{1}{2} i \underset{\sim}{T} \cdot \underset{\sim}{\sigma}} \qquad (2)$$

where $\underset{\sim}{T}$ has three complex components corresponding to the six parameters of the continuous Lorentz group. To see some of this in detail consider the special case

$$\underset{\sim}{T} = i\,\hat{\underset{\sim}{v}}\ \operatorname{arctanh} v \tag{3}$$

where v is the length of a vector $\underset{\sim}{v}$ and $\hat{\underset{\sim}{v}}$ is $\underset{\sim}{v}/v$. Then A can be written as

$$A = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\tfrac{1}{2} i\, \underset{\sim}{T}\cdot\underset{\sim}{\sigma}\right)^n$$

$$= \sum_{\substack{\text{even}\\ n}} \frac{1}{n!}\left(\tfrac{1}{2} i T\right)^n + \frac{i\,\underset{\sim}{T}\cdot\underset{\sim}{\sigma}}{T} \sum_{\substack{\text{odd}\\ n}} \frac{1}{i\,n!}\left(\tfrac{1}{2} i T\right)^n$$

$$= \cos\tfrac{1}{2}T + \frac{i\,\underset{\sim}{T}\cdot\underset{\sim}{\sigma}}{T}\sin\tfrac{1}{2}T \tag{4}$$

where T is defined to be i arctanh v ; this is a consequence of $(\underset{\sim}{T}\cdot\underset{\sim}{\sigma})^2 = T^2$. The coefficients $a_{\alpha\beta}$ can be evaluated in a straightforward way, for example

$$a_{4\beta}\,\sigma_\beta = A^\dagger \sigma_4 A$$

$$= \left(\cos\tfrac{1}{2}T + i\,\hat{\underset{\sim}{T}}\cdot\underset{\sim}{\sigma}\sin\tfrac{1}{2}T\right)\sigma_4\left(\cos\tfrac{1}{2}T + i\,\hat{\underset{\sim}{T}}\cdot\underset{\sim}{\sigma}\sin\tfrac{1}{2}T\right)$$

$$= \left(\cos^2\tfrac{1}{2}T - \sin^2\tfrac{1}{2}T\right)\sigma_4 - 2\,\underset{\sim}{T}\cdot\underset{\sim}{\sigma}\sin\tfrac{1}{2}T\cos\tfrac{1}{2}T$$

$$= -\sin T\,\hat{\underset{\sim}{T}}\cdot\sigma + \cos T\,\sigma_4$$

$$= \frac{-iv}{\sqrt{1-v^2}}\,\hat{\underset{\sim}{v}}\cdot\underset{\sim}{\sigma} + \frac{1}{\sqrt{1-v^2}}\,\sigma_4$$

Thus one finds that

$$a_{4i} = \frac{-i v_i}{\sqrt{1-v^2}} , \quad a_{44} = \frac{1}{\sqrt{1-v^2}} \tag{5a}$$

The other results are

$$a_{ij} = \delta_{ij} - \hat{v}_i \hat{v}_j \left(1 - \frac{1}{\sqrt{1-v^2}}\right), \quad a_{i4} = \frac{i v_i}{\sqrt{1-v^2}} \tag{5b}$$

These are the coefficients for a pure Lorentz transformation, the primed axes having velocity $\underset{\sim}{v}$ relative to the unprimed. Thus $e^{\frac{1}{2} i \underset{\sim}{\tau} \cdot \underset{\sim}{\sigma}}$ with $\underset{\sim}{\tau}$ given by $i \hat{\underset{\sim}{v}}$ arctanh v are the unimodular matrices for pure Lorentz transformations. Similarly one finds that if $\underset{\sim}{\tau}$ is real the Lorentz transformation coefficients are

$$\begin{aligned} a_{ij} &= \delta_{ij} \cos \tau + \epsilon_{ijk} \hat{\tau}_k \sin \tau + \hat{\tau}_i \hat{\tau}_j (1-\cos\tau), \\ a_{i4} &= a_{4i} = 0 , \\ a_{44} &= 1 \end{aligned} \tag{6}$$

which is a space rotation in the right hand sense about the direction $\hat{\underset{\sim}{\tau}}$ through the angle τ. Any Lorentz transformation continuous with the identity can be produced by taking products of these two types. Hausdorff's theorem shows how such a product must simplify. The theorem says that

$$e^A e^B = e^{A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A,[A,B]] + \frac{1}{12}[[A,B],B] + \cdots} \tag{7}$$

where only higher and higher order commutators occur in the exponent. In a product like

$$e^{i\tau_A\cdot s}\, e^{i\tau_B\cdot s}$$

where $s = \frac{1}{2}\sigma$ are the spin $1/2$ matrices a commutator simplifies according to

$$[\tau_A\cdot s\,,\,\tau_B\cdot s] = \tau_{Ai}\,\tau_{Bj}\,[s_i\,,\,s_j]$$

$$= \tau_{Ai}\,\tau_{Bj}\,i\,\epsilon_{ijk}\,s_k$$

$$= i\,(\tau_A\times\tau_B)\cdot s$$

Thus the entire exponent simplifies somehow down to (<u>something</u>)$\cdot s$

and

$$e^{i\tau_A\cdot s}\, e^{i\tau_B\cdot s} = e^{i\tau_C\cdot s} \qquad (8)$$

It is clear then that the unimodular 2×2 matrices representing the continuous Lorentz group can be written as $e^{i\tau\cdot s}$ and that the complex three-vector τ is a parameter for labelling Lorentz transformations.

Next consider the $(2s+1)^{st}$ square matrices $e^{i\boldsymbol{\tau}\cdot\mathbf{s}}$ for any spin s. These are in correspondence with the Lorentz transformations since they have the same parameters $\boldsymbol{\tau}$ as the two-by-two. They also form a representation of the Lorentz group since the calculation of a product $e^{i\boldsymbol{\tau}_A\cdot\mathbf{s}}\,e^{i\boldsymbol{\tau}_B\cdot\mathbf{s}}$ using Hausdorff's theorem depends only on the commutation rules; these are the same for all spins so the product $e^{i\boldsymbol{\tau}_C\cdot\mathbf{s}}$ must be the same for all spins. These matrices $e^{i\boldsymbol{\tau}\cdot\mathbf{s}}$ are in fact the (0,s) irrepresentation of the continuous Lorentz group with the parameters explicitly displayed.

What about the (s,0) representation? For any set of spin matrices $\mathbf{s}$ there is a unitary matrix C_s such that

$$C_s\,\mathbf{s} = -\mathbf{s}^{*}C_s \tag{9}$$

where the asterisk denotes complex conjugation. Mostly the standard representation for the spin matrices will be used, with s_3 diagonal and elements of s_1 real and positive. Then C_s is $e^{i\pi s_2}$; it is real and C_s^2 is $(-1)^{2s}$. The convention is to take $C_s = I$ for spin 0. If $e^{i\boldsymbol{\tau}\cdot\mathbf{s}}$ are matrices forming a representation then

$$(C_s e^{i\underset{\sim}{T}\cdot\underset{\sim}{s}} C_s^{-1})^* = (e^{-i\underset{\sim}{T}\cdot\underset{\sim}{s}^*})^*$$

$$= e^{i\underset{\sim}{T}^*\cdot\underset{\sim}{s}} .$$

also form a representation since

$$(C_s e^{i\underset{\sim}{T}_A\cdot\underset{\sim}{s}} C_s^{-1})^* (C_s e^{i\underset{\sim}{T}_B\cdot\underset{\sim}{s}} C_s^{-1})^*$$

$$= (C_s e^{i\underset{\sim}{T}_A\cdot\underset{\sim}{s}} e^{i\underset{\sim}{T}_B\cdot\underset{\sim}{s}} C_s^{-1})^*$$

These matrices $e^{i\underset{\sim}{T}^*\cdot\underset{\sim}{s}}$ are the (s,0) irrep.

For transformation between two Lorentz frames where the descriptions of an event are related by

$$X'_\alpha = a_{\alpha\beta} X_\beta ,$$

the (0,s) $\oplus$ (s,0) wave functions are related by

$$\psi'(x') = \begin{pmatrix} e^{i\underset{\sim}{T}\cdot\underset{\sim}{s}} & 0 \\ 0 & e^{i\underset{\sim}{T}^*\cdot\underset{\sim}{s}} \end{pmatrix} \qquad (11)$$

The parameters $\underset{\sim}{T}$ can be found from the coefficients $a_{\alpha\beta}$; in particular $\underset{\sim}{T}$ is $i\,\hat{\underset{\sim}{v}}$ artanh v for a pure Lorentz transformation, the primed axes having velocity $\underset{\sim}{v}$ relative to the unprimed.

III. Summary of Matrices Used.

For spin s the matrices are $2(2s+1)$-square and are defined as

$$\alpha_\mu = \frac{1}{s}\begin{pmatrix} s_\mu & 0 \\ 0 & -s_\mu \end{pmatrix}, \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \dot{s}_\mu = \begin{pmatrix} s_\mu & 0 \\ 0 & s_\mu \end{pmatrix}$$

$$\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & C_s \\ -C_s & 0 \end{pmatrix} \tag{12}$$

Some of the spin $1/2$ properties still apply in general. For example β anticommutes with α and γ_5 and α_μ and γ_5 commute. The C matrix is the charge conjugation matrix with the properties

$$\begin{gathered} C \alpha_\mu C^{-1} = \alpha^*, \quad C \beta C^{-1} = -\beta^*, \\ C s_\mu C^{-1} = -s^*, \quad C \gamma_5 C^{-1} = -\gamma_5^* \\ C^2 = (-1)^{2s+1} \end{gathered} \tag{13}$$

IV. Rest Frame State Functions.

One starts from the assumption that eigenstates of energy and momentum for particle and antiparticle exist. In the rest frame the Hamiltonian is identified as $m\beta$ and the polarization operator as $\beta \underset{\sim}{s}$ (discuss why $\beta \underset{\sim}{s}$ is used here in preference to just $\underset{\sim}{s}$ later). These commute and so eigenfunctions u_R can be found such that

$$\begin{aligned} \beta\, u_{R\epsilon k} &= \epsilon\, u_{R\epsilon k} \\ \beta\, \underset{\sim}{s} \cdot \underset{\sim}{\hat{e}}\, u_{R\epsilon k} &= k\, u_{R\epsilon k} \end{aligned} \qquad (14)$$

where $\epsilon = \pm 1$ for the particle/antiparticle, $\underset{\sim}{\hat{e}}$ is a unit vector in an arbitrary quantization direction, and k ranges from $-s$ to $+s$. Choose the nonmalization so that

$$u^{\dagger}_{R\epsilon k}\, u_{R\epsilon' k'} = \delta_{\epsilon\epsilon'}\, \delta_{kk'} \qquad (15)$$

This leaves many phase factors in the $u_{R\epsilon k}$ not yet determined. It is a big help in the later discussion of T, C,P questions to have these phases chosen intelligently. As a first point one notices that $(C\, u_{R\epsilon k})^*$ must be proportional to $u_{R-\epsilon k}$ because

$$\begin{aligned} \beta\,(C\, u_{R\epsilon k})^* &= (\beta^*\, C\, u_{R\epsilon k})^* \\ &= -(C\beta\, u_{R\epsilon k})^* \\ &= -\epsilon\,(C\, u_{R\epsilon k})^* \end{aligned}$$

$$\beta\, \underset{\sim}{s} \cdot \underset{\sim}{\hat{e}}\,(C\, u_{R\epsilon k})^* = k\,(C\, u_{R\epsilon k})^*.$$

Also one can step up and down the angular momentum quantum numbers in the usual way. These two considerations lead to choosing the phases such that

$$\left(C\,v_{R\epsilon k}\right)^{*} = \epsilon^{2s+1}\,v_{R,-\epsilon,k} \qquad (16a)$$

$$\beta\, s_{\pm}\, v_{R\epsilon k} = \sqrt{s(s+1)-k(k\pm\epsilon)}\; v_{R,\epsilon,k\pm\epsilon} \qquad (16b)$$

Here $s_{\pm}$ is $\underset{\sim}{s}\cdot\hat{\underset{\sim}{f}} \pm i\,\underset{\sim}{s}\cdot\hat{\underset{\sim}{g}}$, where $\hat{\underset{\sim}{e}}$, $\hat{\underset{\sim}{f}}$, and $\hat{\underset{\sim}{g}}$ form an orthogonal right hand set. To see that this is really all right, look at the equations first for $\epsilon = \pm 1$. Then Eq.(16b) is just the usual angular momentum phase choice.

$$s_{\pm}\, v_{R1k} = \sqrt{s(s+1)-k(k\pm 1)}\; v_{R1\,k\pm 1}$$

and the two equations for $\epsilon = +1$ serve to define all the $v_{R\epsilon k}$ once any one of them is specified. There the equations for $\epsilon = -1$ can be derived from those for $\epsilon = +1$. For example if

$$\left(C\,v_{R1k}\right)^{*} = v_{R-1k}$$

then

$$\begin{aligned}\left(C\,v_{R-1k}\right)^{*} &= \left(C\left(C\,v_{R1k}\right)^{*}\right)^{*}\\ &= C^{2}\,v_{R1k}\\ &= (-1)^{2s+1}\,v_{R1k}\end{aligned}$$

A similar proof can be made to justify Eq.(16b).

This leaves one phase factor still undetermined for the entire set of $v_{R\epsilon k}$. Another consideration is that γ_5 anticommutes with β and $\beta \underset{\sim}{s}$ so that $\gamma_5 v_{R\epsilon k}$ is proportional to $v_{R-\epsilon-k}$. By similar reasoning one can show that the final phase factor can be chosen so that

$$\gamma_5 v_{R\epsilon k} = (-1)^{s+\epsilon k+1} v_{R,-\epsilon,-k} \tag{17}$$

and that the $v_{R\epsilon k}$ are then completely specified except for a single over all plus or minus sign.

The wave function in the rest frame is then

$$\psi_{R\epsilon k} = v_{R\epsilon k}\, e^{-i\epsilon m t_R} \tag{18}$$

and the equation of motion is

$$m\beta\psi_{R\epsilon k} = i\,\frac{\partial\psi_{R\epsilon k}}{\partial t_R} \tag{19}$$

V. Laboratory Frame State Functions.

Let $\underset{\sim}{q}$ and E be the physical momentum and energy of the pargicle or antiparticle so that E is positive and $\underset{\sim}{q}$/E is the velocity. Then the wave function transformation is, from Eq.(11),

$$\psi_L = \begin{pmatrix} e^{\mathbf{s}\cdot\hat{\mathbf{q}}\,\text{arctanh}\, q/E} & 0 \\ 0 & e^{-\mathbf{s}\cdot\hat{\mathbf{q}}\,\text{arctanh}\, q/E} \end{pmatrix}$$

$$= e^{s\,\boldsymbol{\alpha}\cdot\hat{\mathbf{q}}\,\text{arctanh}\, q/E}\, v_{R\epsilon k}\, e^{i\epsilon q_\alpha x_\alpha} \tag{20}$$

where q_α is $(\mathbf{q}, iE)$ so that $q_\alpha x_\alpha = -mt_R$. As usual the symbol $\mathbf{p}$ will be used for $-i\boldsymbol{\nabla}$ and for the eigenvalue, here $\epsilon\mathbf{q}$. With factors appropriate for the normalization set up later, the laboratory system functions are

$$\psi_{\mathbf{p}\epsilon k} = \frac{1}{(2\pi)^{3/2}}\frac{m^s}{E^{1/2}}\, e^{s\epsilon\,\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\,\text{arctanh}\, p/E}\, v_{R\epsilon k}\, e^{i(\mathbf{p}\cdot\mathbf{x}-\epsilon Et)} \tag{21}$$

At any time t these form a complete set for describing 2(2s+1) component functions of $\mathbf{x}$. One can specialize to $\hat{\mathbf{e}} = \hat{\mathbf{q}}$ in which case these are helicity states.

The general wave function can be found by summing all these possible states with arbitrary coefficients, say

$E^{-1/2} A_{\epsilon k}(\underset{\sim}{p})\, d\underset{\sim}{p}$ to get .

$$\psi(\underset{\sim}{X}, t) = \frac{m^{s}}{(2\pi)^{3/2}} \int \frac{d\underset{\sim}{p}}{E} \sum_{\epsilon k} A_{\epsilon k}(\underset{\sim}{p})$$

$$e^{s\epsilon\, \underset{\sim}{\alpha} \cdot \hat{\underset{\sim}{p}} \operatorname{arctanh} P/E}\, u_{R\epsilon k}\, e^{i(\underset{\sim}{p} \cdot \underset{\sim}{X} - \epsilon E t)} \qquad (22)$$

Here it is neat to break the exponential up into two parts. Put for short

$$\omega = \operatorname{arctanh} P/E$$

Then one can write, since $s\underset{\sim}{\alpha} = -\gamma_5 \underset{\sim}{s}$,

$$e^{s\epsilon\, \underset{\sim}{\alpha} \cdot \hat{\underset{\sim}{p}}\, \omega} = \sum_{\substack{\text{even}\\ n}} \frac{1}{n!} (s\epsilon\, \underset{\sim}{\alpha} \cdot \hat{\underset{\sim}{p}}\, \omega)^n + \sum_{\substack{\text{odd}\\ n}} (\text{same})$$

$$= \sum_{\substack{\text{even}\\ n}} \frac{1}{n!} (\underset{\sim}{s} \cdot \hat{\underset{\sim}{p}}\, \omega)^n - \gamma_5 \epsilon \sum_{\substack{\text{odd}\\ n}} \frac{1}{n!} (\underset{\sim}{s} \cdot \hat{\underset{\sim}{p}}\, \omega)^n$$

$$= \cosh(\underset{\sim}{s} \cdot \hat{\underset{\sim}{p}}\, \omega) - \gamma_5 \epsilon \sinh(\underset{\sim}{s} \cdot \underset{\sim}{p}\, \omega)$$

As the exponential occurs in Eq.(22), ϵ can be replaced by β acting on the $u_{R\epsilon k}$. Consequently the right hand side of Eq.(22) can be factored so that

$$\psi(\underset{\sim}{x},t) = \frac{m^s}{E^{1/2}} S\phi(\underset{\sim}{x},t). \tag{23}$$

where the operator S is given by

$$S = \cosh(\underset{\sim}{s}\cdot\hat{\underset{\sim}{p}}\,\omega) - \gamma_5\beta\sinh(\underset{\sim}{s}\cdot\hat{\underset{\sim}{p}}\,\omega) \tag{24}$$

and $\phi(\underset{\sim}{x},t)$ is then

$$\phi(\underset{\sim}{x},t) = \frac{1}{(2\pi)^{3/2}}\int\frac{d\underset{\sim}{p}}{E^{1/2}}\sum_{\epsilon k} A_{\epsilon k}(\underset{\sim}{p})\, u_{R\epsilon k}\, e^{i(\underset{\sim}{p}\cdot\underset{\sim}{x} - \epsilon E t)} \tag{25}$$

This is Foldy's wave function since it satisfies the equation

$$E\beta\,\phi = i\,\frac{\partial\phi}{\partial t} \tag{26}$$

Here E denotes the operator $(p^2 + m^2)^{1/2}$. Equation (23) is the generalization of the Foldy-Wouthuysen transformation[23] to all spins. For spin $1/2$ it simplifies to their formula

$$\psi = \frac{E + m - \beta\underset{\sim}{\alpha}\cdot\underset{\sim}{p}}{[2E(E+m)]^{1/2}}\,\phi \tag{27}$$

(This can be verified by combining Eqs. (24) and (30) below). The transformation is unitary for spin $\frac{1}{2}$ but not for higher spin.

Many of the quantities of interest are expressible in terms of hyperbolic functions of $\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}$, as shown by Williams, Draayer, and Weber. These can be simplified into polynomials of degree $2s$ in $\underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}$ because these matrices satisfy the characteristic equation

$$(\underset{\sim}{s} \cdot \underset{\sim}{\hat{p}} - s)(\underset{\sim}{s} \cdot \underset{\sim}{\hat{p}} - s + 1) \cdots (\underset{\sim}{s} \cdot \underset{\sim}{\hat{p}} + s) = 0$$

Weinberg[12], Williams, Draayer and Weber[15] and Weber and Williams[16] have discussed the problem of simplifying the hyperbolic functions to polynomials and give many detailed results. An elementary method that will always give the result in a specific case is to consider the operator $\underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}$ to be diagonalized and then solve for the polynomial coefficients. For example consider spin $1/2$ in which case

$$\cosh(\underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}\, \omega) = a + b\, \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}$$

where a and b areto be determined. Evaluating the equation at the two eigenvalues $\pm 1/2$ of $\underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}$ gives

$$\cosh\left(\tfrac{1}{2}\omega\right) = a + \tfrac{1}{2} b$$
$$\cosh\left(-\tfrac{1}{2}\omega\right) = a - \tfrac{1}{2} b$$

Then b is zero and

$$a = \cosh\left(\tfrac{1}{2}\omega\right)$$

The hyperbolic functions of $\frac{1}{2}\omega$ are given by

$$\cosh\left(\tfrac{1}{2}\omega\right) = \frac{E+m}{[2m(E+m)]^{1/2}} \tag{28}$$
$$\sinh\left(\tfrac{1}{2}\omega\right) = \frac{p}{[2m(E+m)]^{1/2}}$$

You can see this is true because these values satisfy $\cosh^2 - \sinh^2 = 1$ and they give

$$\cosh\omega = \cosh^2\left(\tfrac{1}{2}\omega\right) + \sinh^2\left(\tfrac{1}{2}\omega\right) = E/m \tag{29a}$$

$$\sinh\omega = 2\sinh\left(\tfrac{1}{2}\omega\right)\cosh\left(\tfrac{1}{2}\omega\right) = p/m \tag{29b}$$

so that

$$\tanh\omega = p/E$$

as required. The conclusion is that

$$\cosh\left(\underset{\sim}{s}\cdot\hat{\underset{\sim}{p}}\,\omega\right) = \frac{E+m}{[2m(E+m)]^{1/2}} \tag{30a}$$

and similarly one finds that

$$\sinh\left(\underset{\sim}{s}\cdot\hat{\underset{\sim}{p}}\,\omega\right) = \frac{2\underset{\sim}{s}\cdot\underset{\sim}{p}}{[2m(E+m)]^{1/2}} \tag{30b}$$

This process works in general because there are as many eigenvalues as coefficients in the polynomial.

The laboratory-frame operators for the time-development and polarization are found by transforming the rest-frame operators:

$$H = SE\beta S^{-1} \tag{31}$$

$$\underset{\sim}{\theta} = S\beta \underset{\sim}{s} S^{-1} \tag{32}$$

Note H^2 is E^2, $\underset{\sim}{\theta} \cdot \underset{\sim}{\theta} = s(s+1)$ and H and $\underset{\sim}{\theta}$ commute. The plane wave states can be written as

$$\psi_{\underset{\sim}{p}\epsilon k} = \frac{m^{s}}{(2\pi)^{3/2} E^{1/2}} S\, v_{R\epsilon k}\, e^{i(\underset{\sim}{p}\cdot\underset{\sim}{x} - \epsilon E t)} \tag{33}$$

so they are eigenstates

$$\frac{H}{E}\psi_{\underset{\sim}{p}\epsilon k} = \epsilon\,\psi_{\underset{\sim}{p}\epsilon k}, \tag{34}$$

$$\underset{\sim}{\theta}\cdot\hat{\underset{\sim}{e}}\,\psi_{\underset{\sim}{p}\epsilon k} = k\,\psi_{\underset{\sim}{p}\epsilon k} \tag{35}$$

It follows from Eq.(26) that

$$H\psi = i\frac{\partial\psi}{\partial t} \tag{36}$$

so H is the Hamiltonian for the particle and antiparticle.

You can get a detailed expression for by first verifying that

$$S^{-1} = \operatorname{sech}(2\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}) S^{\dagger} \tag{37}$$

this way:

$$\operatorname{sech}(2\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}})\left[\cosh(\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}) + \gamma_5 \beta \sinh(\omega \underset{\sim}{s} \cdot \underset{\sim}{p})\right]$$

$$\left[\cosh(\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}) - \gamma_5 \beta \sinh(\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}})\right]$$

$$= \operatorname{sech}(2\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}})\left[\cosh^2(\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}) + \sinh^2(\omega \underset{\sim}{s} \cdot \underset{\sim}{p})\right]$$

$$= \operatorname{sech}(2\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}) \cosh(2\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}})$$

$$= 1$$

The general formula for the Hamiltonian then becomes

$$H = \left[\cosh(\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}) - \gamma_5 \beta \sinh(\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}})\right] E\beta$$

$$\operatorname{sech}(2\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}})\left[\cosh(\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}) + \gamma_5 \beta \sinh(\omega \underset{\sim}{s} \cdot \underset{\sim}{p})\right]$$

$$= \operatorname{sech}(2\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}})\left[\cosh^2(\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}) E\beta - \right.$$

$$\left. - \sinh^2(\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}) E\beta - 2\gamma_5 E \sinh(\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}) \cosh(\omega \underset{\sim}{s} \cdot \underset{\sim}{p})\right]$$

$$= \operatorname{sech}(2\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}) E\beta - \tanh(2\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}) E\gamma_5 \tag{38}$$

The Hamiltonian is Hermitian. Some of the detailed results that come from evaluating the hyperbolic functions are

Spin s	Hamiltonian H
0	$E\beta$
$1/2$	$\underset{\sim}{\alpha}\cdot\underset{\sim}{p} + \beta$
1	$\frac{(2E^2-m^2)\beta + 2E\underset{\sim}{\alpha}\cdot\underset{\sim}{p} - 2(\underset{\sim}{\alpha}\cdot\underset{\sim}{p})^2\beta_E}{2E^2-m^2}$

Onee H is known there is a general formula for $\underset{\sim}{\theta}$ [8]

$$\underset{\sim}{\theta} = \frac{i s\,\underset{\sim}{\alpha}\times\underset{\sim}{p} + \underset{\sim}{s}H}{m} - \frac{\underset{\sim}{s}\cdot\underset{\sim}{p}\,H\,\underset{\sim}{p}}{mE(E+m)}$$

For spin $1/2$ this operator is the known three-vector polarization operator[24],

$$\underset{\sim}{\theta} = \underset{\sim}{s}\cdot\underset{\sim}{p}\,\frac{H}{E}\,\underset{\sim}{p} + \underset{\sim}{p}\times(\beta\,\underset{\sim}{s}\times\underset{\sim}{p})$$

The usual spin-up and spin-down functions of Dirac theory are eigenfunctions of θ_z .

The invariant integral between any two free-particle wave functions $\psi^{(\ell)}$ and $\psi^{(n)}$ can be expressed in various forms, all evidently related by formulas given above:

$$\begin{aligned}
(\psi^{(\ell)}, \psi^{(n)}) &= \int d\underset{\sim}{x}\;\phi^{(\ell)\dagger}\phi^{(n)} \\
&= m^{-2s}\int d\underset{\sim}{x}\,\psi^{(\ell)\dagger}E(S^{-1})^\dagger S^{-1}\psi^{(n)} \\
&= m^{-2s}\int d\underset{\sim}{x}\,\psi^{(\ell)\dagger}E\,\mathrm{sech}(2\omega\underset{\sim}{s}\cdot\hat{\underset{\sim}{p}})\psi^{(n)} \\
&= m^{-2s}\int d\underset{\sim}{x}\,\psi^{(\ell)\dagger}\tfrac{1}{2}[H,\beta]_+\psi^{(n)} \\
&= -\tfrac{1}{2}i m^{-2s}\int d\underset{\sim}{x}\left(\frac{\partial\bar{\psi}^{(\ell)}}{\partial t}\psi^{(n)} - \bar{\psi}^{(\ell)}\frac{\partial\psi^{(n)}}{\partial t}\right), \qquad (39)
\end{aligned}$$

where $\bar{\psi}$ is $\psi^{+}\beta$. The first expression is Foldy's form[5] and one sees from it that the inner product is positive definite. The integral is Lorentz-invariant in the sense that, if you evaluate it in two different reference frames where the functions are related by Eq.(11), the same number results

$$\left(\psi^{(\ell)\prime}, \psi^{(n)\prime}\right) = \left(\psi^{(\ell)}, \psi^{(n)}\right)$$

To see how this comes about one starts from the fact that each function is of the form given in Eq.(22) so, since the space and time dependence is of the form $e^{i(\underset{\sim}{p}\cdot\underset{\sim}{x} - \epsilon E t)}$, satisfies the Klein-Gordon equation

$$\frac{\partial^2 \bar{\psi}^{(\ell)}}{\partial x_\alpha \partial x_\alpha} = m^2 \bar{\psi}^{(\ell)} ,$$

$$\frac{\partial^2 \psi^{(n)}}{\partial x_\alpha \partial x_\alpha} = m^2 \psi^{(n)} .$$

The first equation can be multiplied by $\psi^{(n)}$ the second by $\bar{\psi}^{(\ell)}$ and the two subtracted. This leads to

$$\frac{\partial}{\partial x_\alpha}\left(\frac{\partial \bar{\psi}^{(\ell)}}{\partial x_\alpha}\psi^{(n)} - \bar{\psi}^{(\ell)}\frac{\partial \psi^{(n)}}{\partial x_\alpha}\right) = 0$$

Consequently the current defined by

$$j_\alpha^{(\ell, n)} = i\frac{1}{2} m^{-2s}\left(\frac{\partial \bar{\psi}^{(\ell)}}{\partial x_\alpha}\psi^{(n)} - \bar{\psi}^{(\ell)}\frac{\partial \psi^{(n)}}{\partial x_\alpha}\right)$$

has zero divergence. Furthermore this is a Lorentz four-vector because $\bar{\psi}^{(\ell)}\psi^{(n)}$ is a scalar:

$$\bar{\psi}^{(\ell)'}\psi^{(n)'} = \psi^{+(\ell)'}\beta\,\psi^{(n)'}$$

$$= \psi^{+(\ell)}\begin{pmatrix} e^{-i\underset{\sim}{\tau}^{*}\cdot\underset{\sim}{a}} & 0 \\ 0 & e^{-i\underset{\sim}{\tau}\cdot\underset{\sim}{a}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{i\underset{\sim}{\tau}\cdot\underset{\sim}{a}} & 0 \\ 0 & e^{i\underset{\sim}{\tau}^{*}\cdot\underset{\sim}{a}} \end{pmatrix}$$

$$= \psi^{+(\ell)}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\psi^{(n)}$$

$$= \bar{\psi}^{(\ell)}\psi^{(n)}$$

The integral over all space of the fourth component of a divergenceless four-vector is a Lorentz scalar. Thus $-i\int j_4^{(\ell,n)}d\underset{\sim}{X}$ is an invariant and this coincides with the last expression in Eq.(39). The factors in the functions $\psi_{\underset{\sim}{p}\epsilon k}$ of Eq.(21) have been chosen so that the normalization is

$$(\psi_{\underset{\sim}{p}\epsilon k},\psi_{\underset{\sim}{p}'\epsilon' k'}) = \delta_{\epsilon\epsilon'}\,\delta_{kk'}\,\delta(\underset{\sim}{p}-\underset{\sim}{p}') \tag{40}$$

As well as the function ψ it is important to consider the function

$$\tilde{\psi}(\underset{\sim}{X},t) = \left[\tfrac{1}{2}(1-\gamma_5)+\tfrac{1}{2}(1+\gamma_5)\epsilon\right]\psi(x,t) \tag{41}$$

where ϵ here means $E^{-1}\partial/\partial t$. Since $\left[\frac{1}{2}(1\pm\gamma_5)\right]^2$ is $\left[\frac{1}{2}(1\pm\gamma_5)\right]$ and ϵ^2 applied to these functions is 1, the operator in square brackets in Eq.(40) has an inverse, itself.

Consequently $\tilde{\psi}$ has as much information in it about the system as ψ has and serves as an equivalent description. For spin zero, the components of $\tilde{\psi}$ are equal and are the ordinary Klein-Gordon function. Thus if ψ_{KG} satisfies

$$E^2 \psi_{KG} = \left(i \frac{\partial}{\partial t}\right)^2 \psi_{KG}$$

and we write

$$\tilde{\psi} = \begin{pmatrix} \psi_{KG} \\ \psi_{KG} \end{pmatrix}$$

then

$$\psi = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} \psi_{KG} \\ \psi_{KG} \end{pmatrix} = \begin{pmatrix} \psi_{KG} \\ \epsilon \psi_{KG} \end{pmatrix}$$

and

$$\begin{aligned} E\beta\psi &= E \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_{KG} \\ \epsilon \psi_{KG} \end{pmatrix} \\ &= \begin{pmatrix} \epsilon E \psi_{KG} \\ E \psi_{KG} \end{pmatrix} \\ &= i \frac{\partial}{\partial t} \psi \end{aligned}$$

which is the Hamiltonian wave equation for spin zero. For spin $1/2$, ψ is the usual Dirac wave function. For spin 1, the components of $\tilde{\psi}$ are closely related to the Proca field components [8]. For spin $\frac{3}{2}$, certain derivatives of ψ are the Rarita-Schwinger components.[17] The wave function used by Joos[11] and Weinberg[12] is of the ψ type for half-integer spin and the $\tilde{\psi}$ type for integer spin.

For space reflection

$$x_i' = -x_i \; , \; t' = t$$

the wave function transformation rule is chosen to be

$$\psi'(x') = \eta_p \beta \psi(x) \tag{42}$$

and for time reflection

$$x_i' = x_i \; , \; t' = -t$$

it is chosen to be

$$\psi'(x') = \eta_T [C \gamma_5 \beta \psi(x)]^* \tag{43}$$

Here the η's are phase-factors with absolute value of unity. With some calculation one can show that the transformations of Eqs.(11), (42) and (43) leave the theory cvoariant in the sense that the same equations, for example $H\psi = i\partial\psi/\partial t$ apply in the primed system as in the unprimed system.

There are two possible ways to define the charge conjugation in such a way that the system is covariant:

$$\psi^{c1}(x) = \eta_{c1} [C \psi(x)]^* \tag{43}$$

$$\psi^{c2}(x) = \eta_{c2} [C (H/E) \psi(x)]^* \tag{44}$$

To see that the first type, for example, leaves the system covariant, you start from

$$H\psi = i\,\partial\psi/\partial t \;,$$

operate with C and take the complex conjugate. This leads to

$$(CHC^{-1})^{*}(C\psi)^{*} = -i\,\partial(C\psi)^{*}/\partial t$$

From eqs. (13) and (38) one finds that

$$(CHC^{-1})^{*} = -H$$

so that

$$H\psi^{C1} = i\,\partial\psi^{C1}/\partial t$$

as required. Evidently multiplication by H/E also leaves the system covariant so Eq.(44) gives a reasonable charge conjugation operation also. The difference between the two is in their period. For the first type one finds

$$\begin{aligned}(\psi^{C1})^{C1} &= \eta_{C1}\left[C\eta_{C1}(C\psi)^{*}\right]^{*}\\ &= C^{2}\psi\\ &= (-1)^{2s+1}\psi\end{aligned}$$

where as for the second type

$$
\begin{aligned}
(\psi^{c2})^{c2} &= \eta_{c2}\left[C\,(H/E)\,\eta_{c2}\left(C\,(H/E)\psi\right)^*\right]^* \\
&= \left(C(H/E)C^{-1}\right)^* C^* C\,(H/E)\,\psi \\
&= -(H/E)(-1)^{2s+1}(H/E)\psi \\
&= (-1)^{2s}\psi
\end{aligned}
$$

One chooses to use type 1 for half-integer spin and type 2 for integer spin so as to have period two for the operation in any case.

A question is what operators in the first-quantized theory are to be identified with physically observable quantities. A consistent point of view, which is adopted here, is that the function $\psi_{\underset{\sim}{p}\,\epsilon k}$ actually describes a particle or an anti-particle. (This is not the hole-theory point of view in which $\psi_{\underset{\sim}{p}\,-1\,k}$ describes a state which, for a vacuum, is filled. There the antiparticle is the absense of a particle from the state $\psi_{\underset{\sim}{p}\,-1\,k}$). The operators for the physically observable quantities are assigned to be: energy E, momentum $\underset{\sim}{P}(H/E)$, angular momentum ($\underset{\sim}{x}\times\underset{\sim}{p}+\underset{\sim}{s}$) (H/E), polarization $\underset{\sim}{\theta}$. This means that the energy is positive definite and it agrees with the values you get by thinking through the hole theory point of view. A property of all these operators, say Ω , is that

$$\Omega = C^{-1}\Omega^* C \tag{45}$$

The polarization operator in the rest frame was chosen at Eq.(14) to be $\beta \underset{\sim}{s}$ rather than $\underset{\sim}{s}$ so that Eq.(45) finally would apply to $\underset{\sim}{\Theta}$. The proof of Eq.(45) for all these operators follows readily from the fact that

$$C^{-1} S^{*} C = S$$

which gives, since H/E is $S\beta S^{-1}$, that

$$C^{-1}(H/E)^{*} C = -(H/E) \tag{46}$$

To understand the significance of Eq.(45), consider the matrix element of some operator Ω between two states $\psi^{(\ell)}$ and $\psi^{(n)}$

$$(\psi^{(\ell)}, \Omega\psi^{(n)}) = m^{-2s} \int d\underset{\sim}{x}\, \psi^{(\ell)\dagger} E \operatorname{sech}(2\omega \underset{\sim}{s}\cdot\hat{\underset{\sim}{p}})\Omega\psi^{(n)}$$

By taking the complex conjugate and inserting factors of one finds that

$$\begin{aligned}(\psi^{(\ell)}, \Omega\psi^{(n)})^{*} &= m^{-2s} \int d\underset{\sim}{x}\, (C\psi^{(\ell)*})^{T} E \operatorname{sech}(2\omega \underset{\sim}{s}\cdot\hat{\underset{\sim}{p}}) \\ &\qquad C\Omega^{*}C^{-1}(C\psi^{(n)*}) \\ &= (\psi^{(\ell)c1}, C\Omega^{*}C^{-1}\psi^{(n)c1})\end{aligned} \tag{47}$$

(47)

One can also argue that H/E is Hermitian with respect to this inner product,

$$(\psi^{(\ell)}, (H/E)\,\psi^{(n)}) = m^{-2s}\int d\underset{\sim}{x}\,\psi^{(\ell)\dagger} E (S^{-1})^{\dagger} S^{-1} (H/E)\,\psi^{(n)}$$

$$= m^{-2s}\int d\underset{\sim}{x}\,\psi^{(\ell)\dagger} E (S^{-1})^{\dagger} S^{-1} S \beta S^{-1} \psi^{(n)}$$

$$= m^{-2s}\int d\underset{\sim}{x}\,\psi^{(\ell)\dagger} E (\beta S^{-1})^{\dagger} S^{-1} \psi^{(n)}$$

$$= m^{-2s}\int d\underset{\sim}{x}\,\psi^{(\ell)\dagger} E [S^{-1}(H/E)]^{\dagger} S^{-1} \psi^{(n)}$$

$$= ((H/E)\,\psi^{(\ell)}, \psi^{(n)})$$

so that Eq.(46) also leads to

$$(\psi^{(\ell)}, \Omega\,\psi^{(n)})^{*} = (\psi^{(\ell)c1}, (H/E)^{2} C\Omega^{*}C^{-1} (H/E)^{2} \psi^{(n)c1})^{*}$$

$$= (\psi^{(\ell)c2}, (H/E) C\,\Omega^{*} C^{-1} (H/E)\,\psi^{(n)c2})^{*}$$

$$= (\psi^{(\ell)c2}, C\Omega^{*}C^{-1}\psi^{(n)c2})^{*} \tag{48}$$

where, in the final step, the fact that (H/E) commutes with Ω for all those operators considered here, was used. The point of Eqs.(47) and (48) is that, as an alternate to calculating matrix elements of operators Ω between functions, one can calculate the conjugate of matrix elements between operators $C\Omega^{*}C^{-1}$ and the charge-conjugate functions. The significance of Eq.(45) is that Ω and $C\Omega^{*}C^{-1}$ are the same operator so that the identification of operators with physical quantities is the same whether particles or antiparticles are preferred, in the sense of being described by positive-frequency functions. The equation of motion is of course the same since if

$$H\psi = i\,\partial\psi/\partial t$$

then from Eq.(46)

$$H\psi^{c1} = i\,\partial\psi^{c1}/\partial t$$

and similar for ψ^{c2}. Thus the theory is covariant by charge conjugation, both for the equations of motion and for the physical assignments.

VI. Field Operators.

In the usual way one starts from the normalized plane wave functions $\psi_{\underset{\sim}{p}\epsilon k}$, as given by Eqs. (21) or (33), and defines the field operator by

$$\psi(\underset{\sim}{x},t)=\int d\underset{\sim}{p}\sum_{k}\left[a_{1k}(\underset{\sim}{p})\psi_{\underset{\sim}{p}1k}+a^{*}_{-1k}(\underset{\sim}{p})\psi_{\underset{\sim}{p}-1k}\right] \qquad (49)$$

where $a_{\underset{\sim}{p}\epsilon k}$ and $a^{*}_{\underset{\sim}{p}\epsilon k}$ are the destruction and creation operators for the single particle states with labels $\underset{\sim}{p}, \epsilon, k$. The operators are postulated to satisfy the rules

$$\left[a_{\epsilon k}(\underset{\sim}{p_1}), a_{\delta l}(\underset{\sim}{p_2})\right]_{\pm}=0 \qquad (50)$$

$$\left[a_{\epsilon k}(\underset{\sim}{p_1}), a^{*}_{\delta l}(\underset{\sim}{p_2})\right]_{\pm}=\delta_{kl}\,\delta_{\epsilon\delta}\,\delta(\underset{\sim}{p_1}-\underset{\sim}{p_2})$$

for fermions/bosons. Here and below the asterisk denotes Hermitian conjugation in the Fock space. Also it is postulated that any operator $a_{\epsilon k}(\underset{\sim}{p})$ applied to the vacuum $|0\rangle$ gives zero and that all physical states are produced by applying the operators $a^{*}_{\epsilon k}(\underset{\sim}{p})$ in any number to the vacuum. It can be shown that this quantization is Lorentz covariant[21].

Starting from this definition, one can calculate the commutation rules for the field operators straightforwardly. Let $\psi_\alpha(x_1)$ denote the α' th component of $\psi_\alpha(\underset{\sim}{x}_1, t)$. Then one finds, considering any spin and either type of statistics,

$$\left[\psi_\alpha(x_1), \psi_\beta^*(x_2)\right]_\pm$$

$$= \left[\int d\underset{\sim}{p}_1 \sum_k \left\{ a_{+1k}(\underset{\sim}{p}_1)\left(\psi_{\underset{\sim}{p}_1+1k}(x_1)\right)_\alpha + a_{-1k}^*(\underset{\sim}{p}_1)\left(\psi_{\underset{\sim}{p}_1-1k}(x_1)\right)_\alpha\right\},\right.$$

$$\left.\int d\underset{\sim}{p}_2 \sum_\ell a_{+1\ell}^*(\underset{\sim}{p}_2)\left(\psi_{\underset{\sim}{p}_2+1\ell}(x_2)\right)_\beta^* + a_{-1\ell}(\underset{\sim}{p}_2)\left(\psi_{\underset{\sim}{p}_2-1\ell}(x_2)\right)_\beta^*\right\}\Big]_\pm$$

$$= \int d\underset{\sim}{p}_1 \int d\underset{\sim}{p}_2 \sum_{k,\ell}\left\{\left(\psi_{\underset{\sim}{p}_1+1k}(x_1)\right)_\alpha \left(\psi_{\underset{\sim}{p}_2+1\ell}(x_2)\right)_\beta^* \delta_{k\ell}\,\delta(\underset{\sim}{p}_1-\underset{\sim}{p}_2)\right.$$

$$\left.\pm \left(\psi_{p_1-1k}(x_1)\right)_\alpha \left(\psi_{\underset{\sim}{p}_2-1\ell}(x_2)\right)_\beta^* \delta_{k\ell}\,\delta(\underset{\sim}{p}_1-\underset{\sim}{p}_2)\right\}$$

$$= \int d\underset{\sim}{p} \sum_k \left\{\left(\psi_{\underset{\sim}{p}+1k}(x_1)\right)_\alpha \left(\psi_{\underset{\sim}{p}+1k}(x_2)\right)_\beta^* \pm \left(\psi_{\underset{\sim}{p}-1k}(x_1)\right)_\alpha \left(\psi_{p-1k}(x_2)\right)_\beta^*\right\}$$

$$= \frac{m^{2s}}{(2\pi)^3}\int \frac{d\underset{\sim}{p}}{E}\sum_k\left\{\left(S\,v_{R+1k}\right)_\alpha \left(S\,v_{R+1k}\right)_\beta^* e^{-iE(t_1-t_2)}\right.$$

$$\left.\pm \left(S\,v_{R-1k}\right)_\alpha \left(S\,v_{R-1k}\right)_\beta^* e^{iE(t_1-t_2)}\right\} e^{i\underset{\sim}{p}\cdot(\underset{\sim}{x}_1-\underset{\sim}{x}_2)}$$

where Eq.(33) for $\psi_{\underset{\sim}{p}\epsilon k}$ was used in the last step. The completeness of the functions $\nu_{R\epsilon k}$ can be used to evaluate the k sums this way:

$$\sum_k (S\,\nu_{R\pm 1 k})_\alpha (S\,\nu_{R\pm 1 k})^*_\beta$$

$$= \frac{1}{2}\sum_{k\epsilon}\big(S(1\pm\beta)\,\nu_{R\epsilon k}\big)_\alpha \big(S\,\nu_{R\epsilon k}\big)^*_\beta$$

$$= \frac{1}{2}\left[S(1\pm\beta)\right]_{\alpha\rho} S^*_{\beta\sigma}\sum_{k\epsilon}(\nu_{R\epsilon k})_\rho (\nu_{R\epsilon k})^*_\sigma$$

$$= \frac{1}{2}\left[S(1\pm\beta)S^+\right]_{\alpha\beta}$$

Let the indices α, β be suppressed so the equations now read

$$\left[\psi(x_1),\psi^*(x_2)\right]_\pm$$

$$= \frac{m^{2s}}{(2\pi)^3}\int\frac{d\underset{\sim}{p}}{2E}\Big[S(1+\beta)S^+ e^{-iE(t_1-t_2)}$$

$$\pm\, S(1-\beta)S^+ e^{iE(t_1-t_2)}\Big]e^{i\underset{\sim}{p}\cdot(\underset{\sim}{x}_1-\underset{\sim}{x}_2)}$$

Here S is a function of $\underset{\sim}{p}$, given in Eq.(24), and $\underset{\sim}{p}$ is considered to be real in getting S^+. Let $S(\underset{\sim}{p})$ inside the integral be replaced by $S_1 = S(-i\underset{\sim}{\nabla}_1)$ outside the integral. With some rearrangement of terms one finds

$$[\psi(x_1), \psi^*(x_2)]_\pm$$

$$= \frac{m^{2s}}{(2\pi)^3} S_1 S_1^+ \int \frac{d\underset{\sim}{p}}{E} \frac{e^{-iE(t_1-t_2)} \pm e^{iE(t_1-t_2)}}{2} \times e^{i\underset{\sim}{p}\cdot(\underset{\sim}{x_1}-\underset{\sim}{x_2})}$$

$$+ \frac{m^{2s}}{(2\pi)^3} S_1 \beta S_1^+ \int \frac{d\underset{\sim}{p}}{E} \frac{e^{-iE(t_1-t_2)} \mp e^{iE(t_1-t_2)}}{2} \times e^{i\underset{\sim}{p}\cdot(\underset{\sim}{x_1}-\underset{\sim}{x_2})}$$

These integrals are expressible by the usual invariant function

$$\Delta(x)$$

$$= \frac{-i}{(2\pi)^3} \int \frac{d\underset{\sim}{p}}{2E} \left(e^{ip_\mu x_\mu} - e^{-ip_\mu x_\mu}\right)$$

(51)

where p_4 is $\sqrt{p^2+m^2}$. Thus, replacing $\underset{\sim}{p}$ by $-\underset{\sim}{p}$ in the second term gives

$$i\Delta(x) = \frac{1}{(2\pi)^3} \int \frac{d\underset{\sim}{p}}{E} \frac{e^{-iEt} - e^{iEt}}{2} e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}},$$

$$i\hat{\epsilon}\Delta = \frac{1}{(2\pi)^3} \int \frac{d\underset{\sim}{p}}{E} \frac{e^{-iEt} + e^{iEt}}{2} e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}},$$

where $\hat{\epsilon}$ is $E^{-1}(i\,\partial/\partial t)$. This gives

$$[\psi(x_1), \psi^*(x_2)]_\pm$$
$$= i m^{2s} \{\epsilon_1\}_F [S_1 S_1^+ + S_1 \beta S_1^+ \hat{\epsilon}_1] \Delta(x_1 - x_2)$$

where $\{\epsilon_1\}_F$ denotes ϵ_1 for fermions, unity for bosons. Finally Eq.( ) gives

$$S_1 S_1^+ = \cosh(2\omega_1 \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}_1)$$

and Eq.(24) gives

$$S\beta S^+ = [\cosh(\underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}\,\omega) - \gamma_5 \beta \sinh(\underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}\,\omega)]$$
$$\beta [\cosh(\underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}\,\omega) - \beta\gamma_5 \sinh(\underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}\,\omega)]$$
$$= \beta[\cosh^2(\underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}\,\omega) - \sinh^2(\underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}\,\omega)]$$
$$- 2\gamma_5 \sinh(\underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}\,\omega)\cosh(\underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}\,\omega)$$
$$= \beta - \gamma_5 \sinh(2\underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}\,\omega)$$

All the commutators and anticommutators are given by the formula

$$[\psi(x_1), \psi^*(x_2)]_\pm$$
$$= i m^{2s} \{\epsilon_1\}_F \left[\cosh(2 \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}_1 \omega_1) + \beta \hat{\epsilon}_1 - \gamma_5 \sinh(2 \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}_1 \omega_1) \hat{\epsilon}_1 \right] \Delta (x_1 - x_2) \qquad (52)$$

It is obvious from the commutation or anticommutation rules for the $a_{\epsilon k}(\underset{\sim}{p})$ that $[\psi(x_1), \psi(x_2)]_\pm = 0$.
The equal time commutation rules can be found from these by using the values

$$\Delta(\underset{\sim}{x}, 0) = 0, \quad \hat{\epsilon} E \Delta(\underset{\sim}{x}, 0) = -i\delta(\underset{\sim}{x})$$

One expects the field operators to commute or anticommute for spacelike separations. The functions $\Delta(x)$ and $\hat{\epsilon} E \Delta(x)$ are zero when x is spacelike, the functions $\hat{\epsilon}\Delta$ and $E\Delta$ are not. From the study of the hyperbolic functions[12,16] it is known that for integer spin $\cosh(2\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}})$ and $E^{-1}\sinh(2\omega \underset{\sim}{s} \cdot \underset{\sim}{p})$ are polynomials in $\underset{\sim}{s} \cdot \underset{\sim}{p}$ and p^2, for half-integer spin $E^{-1}\cosh(2\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}})$ and $\sinh(2\omega \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}})$ are such polynomials. Consequently, for Fermi statistics and half-integer spin,

$$[\psi(x_1), \psi^*(x_2)]_+$$
$$= i m^{2s} \left[E_1^{-1} \cosh(2\omega_1 \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}_1) \epsilon E_1 \Delta + \beta - \gamma_5 \sinh(2\omega_1 \underset{\sim}{s} \cdot \underset{\sim}{\hat{p}}_1) \Delta \right]$$

and the right-hand-side is zero for space-like separations. For half-integer spin and Fermi statistics the system is causal in this sense but in no other case does a zero occur on the right. Mathews[19] considers this to be a crticism of the theory for integer spins and sets up different plane wave states with different Hamiltonians not defined in the rest frame. However, Nelson and Good make the point that the theory is all right if only a different field operator is used to build interactions with. The operator $\tilde{\psi}$ can be defined by

$$\tilde{\psi}(\underset{\sim}{x}, t) = \left[\tfrac{1}{2}(1-\gamma_5) + \tfrac{1}{2}(1+\gamma_5)\epsilon\right]\psi(\underset{\sim}{x}, t) \qquad (53)$$

the same as in the first quantized theory, Eq.(41). Equation (49) then gives

$$\begin{aligned}\tilde{\psi}&(\underset{\sim}{x}, t)\\ &= \int d\underset{\sim}{p} \sum_k \left[a_{1k}(\underset{\sim}{p})\psi_{\underset{\sim}{p}1k}\right.\\ &\qquad\left. - a^*_{-1k}(\underset{\sim}{p})\gamma_5 \psi_{\underset{\sim}{p}\,-1k}\right]\end{aligned} \qquad (54)$$

Commutation and anticommutation rules for $\tilde{\psi}$ can be calculated straightforwardly the same way as for ψ. The results are

$$\begin{aligned}&\left[\tilde{\psi}(x_1), \tilde{\psi}(x_2)\right]_{\pm} = 0\\ &\left[\tilde{\psi}(x_1), \tilde{\psi}^*(x_2)\right]_{\pm}\\ &= i m^{2s} \{\hat{\epsilon}_1\}_F \left[\cosh(2\omega_1 \underset{\sim}{s}\cdot\hat{\underset{\sim}{p}}_1) + \beta\right.\\ &\qquad\left. - \gamma_5 \sinh(2\omega_1 \underset{\sim}{s}\cdot\hat{\underset{\sim}{p}}_1)\hat{\epsilon}_1\right]\Delta(x_1 - x_2)\end{aligned} \qquad (55)$$

Here if integer-spin bosons are considered the right-hand-side is zero for space-like separations and in no other case. The point is that the function ψ is to be used to make interactions with half-integer spin, the function $\tilde{\psi}$ with integer spin and then the usual spin-statistics relation applies as well. In what follows. Ψ is used to denote ψ for half-integer spin fermions and $\tilde{\psi}$ for integer spin bosons. Equations ( ) and ( ) combine to make

$$[\Psi(x_1), \Psi(x_2)]_\pm = 0 ,$$

$$[\Psi(x_1), \Psi^*(x_2)]_\pm = i m^2 \left[\{\hat{\epsilon}_1\}_F \cosh(2\omega_1 \underset{\sim}{s} \cdot \hat{\underset{\sim}{p}}_1)\right.$$

$$\left. + \beta - \{\hat{\epsilon}_1\}_B \gamma_5 \sinh(2\omega_1 \underset{\sim}{s} \cdot \hat{\underset{\sim}{p}}_1)\right] \Delta(x_1 - x_2) \qquad (56)$$

Weinberg[12] gives the commutation rules in terms of the covariantly defined matrices. Equations (49) and (54) are written together as

$$\Psi = \int d\underset{\sim}{p} \sum_k \left[a_{1k}(\underset{\sim}{p}) \psi_{\underset{\sim}{p} 1k}\right.$$

$$\left. + \{-\gamma_5\} a^*_{-1k}(\underset{\sim}{p}) \psi_{\underset{\sim}{p} -1k}\right] \qquad (57)$$

VII. Self Conjugate Particles.

The charge conjugation operator $\mathscr{C}$ is defined to be a unitary operator in the Fock space with the properties

$$\mathscr{C}|0\rangle = |0\rangle$$

$$\mathscr{C}\psi(\underset{\sim}{x},t)\mathscr{C}^{-1} = [\mathscr{C}\{H/E\}_B\psi(\underset{\sim}{x},t)]^* \tag{58}$$

From the discussion of the charge conjugation in the first quantized theory it is clear that the $\mathscr{C}$ -transform has period two.

What effect does the $\mathscr{C}$ -transform have on the individual operators $a_{\epsilon k}(\underset{\sim}{p})$? This is found by combining Eqs. (49) and (58):

$$\begin{aligned} \int d\underset{\sim}{p} \sum_k [\mathscr{C} a_{1k}(\underset{\sim}{p})\mathscr{C}^{-1}\psi_{\underset{\sim}{p}1k} &+ \mathscr{C} a^*_{-1k}(\underset{\sim}{p})\mathscr{C}^{-1}\psi_{\underset{\sim}{p}-1k}] \\ = \int d\underset{\sim}{p} \sum_k [a^*_{1k}(\underset{\sim}{p})(\mathscr{C}\psi_{\underset{\sim}{p}1k})^* &+ (-1)_B\, a_{-1k}(\underset{\sim}{p})(\mathscr{C}\psi_{\underset{\sim}{p}-1k})^* \end{aligned} \tag{59}$$

Equations (16a), (33) and the fact that

$$\mathscr{C}S(\underset{\sim}{p}) = S^*(-\underset{\sim}{p})\mathscr{C}$$

combine to simplify the functions on the right:

$$(\mathscr{C}\psi_{p\epsilon k})^* = \left[\mathscr{C}\frac{m^s}{(2\pi)^{3/2}E^{1/2}} S\,v_{R\epsilon k}\, e^{i(p\cdot x-\epsilon Et)}\right]^*$$

$$= \left[S^*(-p)\frac{m^s}{(2\pi)^{3/2}E^{1/2}}\mathscr{C}v_{R\epsilon k}\,e^{i(p\cdot x-\epsilon Et)}\right]^*$$

$$= \frac{S(-p)\,m^s}{(2\pi)^{3/2}E^{1/2}}\,\epsilon^{2s+1}\,v_{R-\epsilon k}\,e^{-i(p\cdot x-\epsilon Et)}$$

$$= \epsilon^{2s+1}\,\psi_{-p,-\epsilon,k}$$

(60)

Consequently the right hand side of Eq.(59) is

$$\int dp\sum_k\left[a^*_{1k}(p)\,\psi_{-p,-1k}+a_{-1k}(p)\,\psi_{-p,1,k}\right]$$

$$=\int dp\sum_k\left[a^*_{1k}(-p)\,\psi_{p-1k}+a_{-1k}(-p)\,\psi_{p1k}\right]$$

The functions $\psi_{p\epsilon k}$ form a complete set so Eq.(59) gives

$$\mathscr{C}a_{\epsilon k}(p)\,\mathscr{C}^{-1}=a_{-\epsilon k}(-p)$$

(61)

Bearing in mind that $\epsilon \underset{\sim}{p}$ is the physical momentum, one sees that the $\mathcal{C}$ -transform carries the particle operator into the antiparticle operator with the same polarization and physical momentum. This is just as it should be and it justifies the choice of Eq.(58) as the definition of charge conjugation.

What is the result of applying the $\mathcal{C}$ -transform to the operator Ψ ? The answer is

$$\mathcal{C}\Psi\mathcal{C}^{-1} = (\mathcal{C}\{\gamma_5\}_B \Psi)^* \tag{62}$$

as is easily verified. For half-integer spin fermions Ψ is ψ and this agrees with Eq.(58). For integer spin bosons Eqs.(54) and (61) give, for the left hand side,

$$\mathcal{C}\tilde{\psi}\mathcal{C}^{-1} = \int d\underset{\sim}{p} \sum_k \left[a_{-1k}(-\underset{\sim}{p})\psi_{\underset{\sim}{p}1k} - a^*_{1k}(-\underset{\sim}{p})\gamma_5\psi_{\underset{\sim}{p}-1k}\right]$$

Also Eqs.(54) and (60) give for the right hand side

$$(\mathcal{C}\gamma_5\tilde{\psi})^* = \int d\underset{\sim}{p} \sum_k \left[a_{1k}(\underset{\sim}{p})(-\gamma_5^*)\mathcal{C}\psi_{\underset{\sim}{p}1k} - a^*_{-1k}(\underset{\sim}{p})\mathcal{C}\psi_{\underset{\sim}{p}-1k}\right]$$

$$= \int d\underset{\sim}{p} \sum_k \left[-\gamma_5 a^*_{1k}(\underset{\sim}{p})\psi_{-p-1k} + a_{-1k}(\underset{\sim}{p})\psi_{-p1k}\right]$$

so Eq.(62) verifies all right.

The fact that the $\mathcal{C}$ -transform has period two permits self-conjugate operators to be set up. Let $\Psi_{(\rho)}$, where ρ is ± 1 , be defined by

$$\Psi_{(\rho)} = \frac{1}{2}\left(\Psi + \rho\, \mathcal{C}\Psi\mathcal{C}^{-1}\right) \tag{63}$$

Then this has the property

$$\mathcal{C}\Psi_{(\rho)}\mathcal{C}^{-1} = \frac{1}{2}\left(\mathcal{C}\Psi\mathcal{C}^{-1} + \rho\, \mathcal{C}^{2}\Psi\mathcal{C}^{-2}\right)$$

$$= \rho\frac{1}{2}\left(\Psi + \rho\, \mathcal{C}\Psi\mathcal{C}^{-1}\right)$$

$$= \rho\, \Psi_{(\rho)} \tag{64}$$

See how these self-conjugate operators are expressed in terms of the creation and destruction operators.by combining Eqs.(57) and (63):

$$\Psi_{(\rho)} = \int d\underset{\sim}{p} \sum_{k}\left[\frac{a_{1k}(\underset{\sim}{p}) + \rho\, a_{-1k}(-\underset{\sim}{p})}{2}\, \psi_{\underset{\sim}{p}1k} + \frac{a^{*}_{-1k}(\underset{\sim}{p}) + \rho\, a^{*}_{1k}(-\underset{\sim}{p})}{2}\{-\gamma_5\}_{B}\, \psi_{p-1k}\right]$$

Replace $\underset{\sim}{p}$ by $-\underset{\sim}{p}$ in the second term and write this as

$$\Psi_{(\rho)} = \frac{1}{\sqrt{2}}\int d\underset{\sim}{p} \sum_{k}\Big[b_{\rho k}(\underset{\sim}{p})\,\psi_{\underset{\sim}{p}1k} + \rho\, b^{*}_{\rho k}(\underset{\sim}{p})\{-\gamma_5\}_{B}\,\psi_{-\underset{\sim}{p}-1k}\Big] \tag{65}$$

where

$$b_{\rho k}(\underset{\sim}{p}) = \frac{a_{1k}(\underset{\sim}{p}) + \rho\, a_{-1k}(-\underset{\sim}{p})}{\sqrt{2}} \tag{66}$$

The two fields $\Psi_{(\rho)}$ thus each have their own operators $b_{\rho k}(\underset{\sim}{p})$. The commutation rules of the b's are found from those of the a's to be

$$\left[b_{\rho k}(\underset{\sim}{p}_1), b_{\sigma \ell}(\underset{\sim}{p}_2)\right]_{\pm} = 0$$
$$\left[b_{\rho k}(\underset{\sim}{p}_1), b^{*}_{\sigma \ell}(\underset{\sim}{p}_2)\right]_{\pm} = \delta_{k\ell}\, \delta_{\rho\sigma}\, \delta(\underset{\sim}{p}_1 - \underset{\sim}{p}_2) \tag{67}$$

This means for one thing that the $b_{1k}(\underset{\sim}{p}_1)$ anticommute/commute with the $b_{-1\ell}(\underset{\sim}{p}_2)$ so the fields $\Psi_{(+1)}$ and $\Psi_{(-1)}$ are correspondingly uncoupled.

Also Eq. (67) says that the b's are single-particle destruction operators. To describe this situation in more detail: the state $a^{*}_{1k}(\underset{\sim}{p})|0\rangle$ is a particle with physical momentum $\underset{\sim}{p}$ and polarization k ; the state $a^{*}_{-1k}(-\underset{\sim}{p})|0\rangle$ is an antiparticle with physical momentum $\underset{\sim}{p}$ and polarization k ; consequently $b^{*}_{\rho k}(\underset{\sim}{p})\,|0\rangle$ is a self-conjugate object with physical momentum $\underset{\sim}{p}$ and polarization k . Also it follows from Eqs.(61) and (66) that

$$\mathcal{C}\, b_{\rho k}(\underset{\sim}{p})\, \mathcal{C}^{-1} = \rho\, b_{\rho k}(\underset{\sim}{p}) \tag{68}$$

so the state $b^*_{\rho k}(\underset{\sim}{k})|0\rangle$ is actually an eigenstate of the operator $\mathcal{C}$:

$$\mathcal{C}\, b^*_{\rho k}(\underset{\sim}{k})\,|0\rangle = \rho\, b^*_{\rho k}(\underset{\sim}{k})\,\mathcal{C}\,|0\rangle$$
$$= \rho\, b^*_{\rho k}(\underset{\sim}{k})\,|0\rangle$$

Another question is whether the self-conjugate fields $\Psi_{(\rho)}$ are causal in the sense of commuting or anticommuting for space-like separations. This can be quickly settled from the properties already established. It is only necessary to consider $\left[\Psi_{(\rho)}(x_1), \Psi^*_{(\rho)}(x_2)\right]_\pm$ because can be deduced from it by using Eq.(64) in the form

$$\rho\,\Psi_{(\rho)} = \left(\mathcal{C}\{\gamma_5\}_B\,\Psi_{(\rho)}\right)^*$$

First one observes from Eq.(63) that

$$\left[\Psi_{(\rho)}(x_1), \Psi^*_{\rho}(x_2)\right]_\pm = \frac{1}{4}\Big[\Psi(x_1) + \rho\left(\mathcal{C}\{\gamma_5\}_B\Psi(x_1)\right)^*,$$
$$\Psi^*(x_2) + \rho\left(\mathcal{C}\{\gamma_5\}_B\Psi(x_2)\right)\Big]$$
$$= \frac{1}{4}\left[\Psi(x_1), \Psi^*(x_2)\right]$$
$$+\frac{1}{4}\left[\left(\mathcal{C}\{\gamma_5\}_B\Psi(x_1)\right)^*, \mathcal{C}\{\gamma_5\}_B\Psi(x_2)\right]$$

which is the same for both values of ρ. Second, one writes Ψ as $\Psi_{(+1)} + \Psi_{(-1)}$ and uses the fact that the two parts anticommute/commute to get

$$[\Psi(x_1), \Psi^*(x_2)]_\pm = [\Psi_{(+1)}(x_1), \Psi^*_{(+1)}(x_2)]_\pm$$

$$+ [\Psi_{(-1)}(x_1), \Psi^*_{(-1)}(x_2)]_\pm$$

Finally, since the two terms on the right are equal, they must each equal half the left hand side

$$[\Psi_{(\rho)}(x_1), \Psi^*_{(\rho)}(x_2)]_\pm$$

$$= \frac{1}{2}[\Psi(x_1), \Psi^*(x_2)]_\pm \qquad (69)$$

The function on the right was given in Eq. (56). The conclusion is that the self-conjugate fields have causal anticommutation or commutation rules.

VIII. Self Conjugate Isospin Multiplets.

For a particle-antiparticle system with isotopic spin t there are wave function operators $\psi_\mu(\underset{\sim}{x}, t)$ where μ is the isospin component label, ranging from -t to +t. In terms of the complete set of functions $\psi_{\underset{\sim}{p}\epsilon k}$ the operators are

$$\Psi_\mu = \int d\underset{\sim}{p} \sum_k \Big[a_{1k\mu}(\underset{\sim}{p}) \psi_{\underset{\sim}{p}1k}$$

$$+ \{-\gamma_5\}_B \, a^*_{-1k-\mu}(\underset{\sim}{p}) \psi_{\underset{\sim}{p}-1k} \Big] \qquad (70)$$

The subscripts are put on the a's in agreement with the usual ideas about isospin; the antiparticle has opposite isospin component to the particle. The a's with the same μ index have the same commutation rules as before; those with different μ indices anticommute/commute.

The isospin operators are defined by

$$\mathcal{T}_i = -\frac{1}{2} i m^{-2s} \int d\underset{\sim}{x} \sum_{\mu\mu'} :\Big[\frac{\partial \bar{\Psi}_\mu}{\partial t} (T_i)_{\mu\mu'} \Psi_{\mu'} - \bar{\Psi}_\mu (T_i)_{\mu\mu'} \frac{\partial \Psi_{\mu'}}{\partial t} \Big]: \qquad (71)$$

where $(T_i)_{\mu\mu'}$ are the (2t+1)-square angular momentum matrices in the standard representation. This form is suggested by the invariant integral defined in Eq. (39). It makes the isospin operators $\mathcal{T}_i$ be Lorentz scalars. The dots indicate the normal product of the operators: when written out in terms of the a's, every term involving an a^* and an a is organized so that the a^* is on the left, anticommutators for fermions and commutators for bosons being neglected in the process. As a result of this normal ordering, $\mathcal{T}_i$ in terms of the a's is found to be

$$\mathcal{T}_i = \sum_{\mu\mu'} (T_i)_{\mu\mu'} \int d\underset{\sim}{p} \sum_{k} \Big[a^*_{1k\mu}(\underset{\sim}{p})\, a_{1k\mu'}(\underset{\sim}{p}) - a^*_{-1k-\mu'}(\underset{\sim}{p})\, a_{-1k-\mu}(\underset{\sim}{p}) \Big]$$

so that $\mathcal{T}_i \,|\,0\rangle = 0$, and the vacuum is invariant to isospin rotations, as it should be.

It is straightforward to calculate the properties of the $\mathcal{T}_i$ from the above definitions. All the correct properties can be verified. The operators have the correct algebra

$$[\mathcal{T}_i , \mathcal{T}_j] = i \epsilon_{ijk} \mathcal{T}_k \tag{72}$$

they are the generators of isospace rotations of the field operators

$$[\Psi_\mu , \mathcal{T}_i] = \sum_{\mu'} (T_i)_{\mu\mu'} \Psi_{\mu'} , \tag{73}$$

and they combine the right way with $\mathcal{C}$

$$\begin{aligned} \mathcal{C}\, \mathcal{T}_2\, \mathcal{C}^{-1} &= \mathcal{T}_2 \\ \mathcal{C}\, \mathcal{T}_{1 \text{ or } 3}\, \mathcal{C}^{-1} &= - \mathcal{T}_{1 \text{ or } 3} \end{aligned} \tag{74}$$

(The charge conjugation operation is now defined by

$$\mathcal{C}\, \Psi_\mu\, \mathcal{C}^{-1} = (\mathcal{C} \{\gamma_5\} \Psi_{\bar{\mu}})^* . \;)$$

An interesting question is whether isospin multiplets that are in some sense self-conjugate can be set up. This question was first raised by Carruthers[25] who remarked that only for integer isospin can there be a self-conjugate system of ~~scalar~~ bosons. New isospin multiplets can be made if the conjugation operation commutes with $\mathcal{T}_i$. As is well known, the G-parity

$$\mathcal{G} = \mathcal{C}\, e^{i\pi \mathcal{T}_2} \tag{75}$$

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$$\mathcal{C}\,\mathcal{T}_{1\ \mathrm{or}\ 3}\,\mathcal{C}^{-1} = -\mathcal{T}_{1\ \mathrm{or}\ 3} \tag{74}$$

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$$\mathcal{G} = \mathcal{C}\,e^{i\pi\mathcal{T}_2} \tag{75}$$

has this property. The operator $e^{i\pi\tau_2}$ rotates through angle $(-\pi)$ about the 2 axis in isospace so

$$e^{+i\pi\tau_2}\,\tau_2\,e^{-i\pi\tau_2} = \tau_2\ ;$$

$$e^{+i\pi\tau_2}\,\tau_{1\,\mathrm{or}\,3}\,e^{-i\pi\tau_2} = -\tau_{1\,\mathrm{or}\,3} \tag{76}$$

Equations (74) and (76) together imply that

$$[\mathcal{G}, \tau_i] = 0 \tag{77}$$

From this it follows that the fields

$$\Psi_{(\rho)\mu} = \frac{1}{2}\left(\Psi_\mu + \rho\,\mathcal{G}\,\Psi_\mu\,\mathcal{G}^{-1}\right) \tag{78}$$

where ρ is +1 or -1, separately satisfy Eq. (73)

$$\left[\Psi_{(\rho)\mu}, \tau_i\right] = \sum_{\mu'}(T_i)_{\mu\mu'}\Psi_{(\rho)\mu'} \tag{79}$$

and so each forms an isospin t multiplet. The remaining question is when the $\mathcal{G}$-transform has period two so that the multiplet is self-conjugate in the sense that

$$\mathcal{G}\,\Psi_{(\rho)\mu}\,\mathcal{G}^{-1} = \rho\,\Psi_{(\rho)\mu} \tag{80}$$

In the standard representation $\left(e^{-i\pi T_2}\right)_{\mu\mu'}$ is $(-1)^{t-\mu}\delta_{\mu,-\mu'}$ so that

$$e^{i\pi T_2}\Psi_\mu e^{-i\pi T_2} = \sum_{\mu'}\left(e^{-i\pi\cdot T_2}\right)_{\mu\mu'}\Psi_{\mu'}$$
$$= (-1)^{t-\mu}\Psi_{-\mu}$$

The square of this transform is then

$$e^{2i\pi T_2}\Psi_\mu e^{-i2\pi T_2} = (-1)^{2t}\Psi_\mu \; .$$

This just expresses the fact that a minus sign comes in when a half-integer spin system is rotated through 2π. The $\mathcal{G}$-transform still has period two so

$$\mathcal{G}^2\Psi_\mu\mathcal{G}^{-1} = (-1)^{2t}\Psi_\mu \; .$$

Thus for integer isospin Eq. (80) holds and the self-conjugate isospin multiplet exists. The rest of the discussion goes through the same as for the self-conjugate fields in the previous section, with modifications to take care of the μ index. The field commutators are found to be

$$\left[\Psi_{(\rho)\mu}(x_1), \Psi^*_{(\sigma)\nu}(x_2)\right]_\pm$$
$$= \frac{1}{2}\delta_{\rho\sigma}\delta_{\mu\nu}\left[\Psi(x_1), \Psi^*(x_2)\right]_\pm$$

where the right hand side is known from Eq. (56). Thus the self-conjugate fields have causal commutation rules. These ideas have been extended to SU_3 multiplets in reference 21.

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