MATSCIENCE REPORT 61

GROUP THEORY AND UNITARY SYMMETRY

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THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS 20. (INDIA)

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Group Theory and Unitary Symmetry

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1. Elementary Particles

Particles are characterized by definite electric charge (in units of the charge of an electron), mass, spin and life time.

TABLE

Most recent Table on Particles and Resonances

Interaction between particles.

It is possible to classify the interactions among particles into the following groups.

1) Electromagnetic Interaction:- This is perhaps the most well understood type of interaction. This is the interaction between electric charges. The law governing this is the well-known coulomb's law. The interaction is characterized by a coupling parameter $\frac{e^2}{4\pi} \approx \frac{1}{137}$ (in units of $\hbar = c = 1$)

2) Gravitational Interaction:- This is the interaction between particles virtue of their massiveness. The law governing such an interaction is the well known Newton's law or the Einstein's equations. The strength is typically characterised by a coupling $\frac{K^2}{4\pi} \approx 10^{-39}$. The typical size of a nucleon

being 1 Fermi (1 Fermi = 10^{-13} cm), the gravitational interaction between particles is very small and will not be considered in our discussions. However, one is not clear whether if one goes into substructures like quarks, the masses steadily increasing with decreasing distance, this gravitational interaction could be neglected. But this is an open question.

3. Strong Interactions

This is the interaction between the mesons and baryons. which exhibits fantastic symmetry properties and respects all conservation laws. The strength of strong interaction is typically characterized by $\frac{g^2}{4\pi} \sim 1$ (137 times stronger than electromagnetic interaction!). Various methods have been tried to understand this type of interaction. Certainly, the study of 'Symmetry' has helped us to understand the pattern and symmetries of this interaction. But the dynamics of it is far from clear.

4. Weak Interactions

This type of interaction is perhaps the least understood by physicists. It has the maximum complex and is a good source of light to the actual dynamics. The coupling constant describing this kind of interaction is not dimensionless. For convenience, it is chosen to be

$$g \sim 1.01 \times 10^{-5} \, m_p^{-2}$$

Whether or not such an interaction is mediated by Intermediate bosons is still an open question while what is clear from experiments is that if such a boson exists, it must have a fantastically heave mass.

This classification gives one of the most important orientation in the study of Elementary Particles. The deep and difficult question of the origin of such a classification is still an open problem. But much light has been thrown by the study of 'symmetries', and conservation laws. The strong interactions offer the maximum symmetry while weak and electromagnetic interactions violate many conservation laws.

REVIEW OF THE DIRAC EQUATION

Empirical facts:- If one sees the list of particles in Table I, it is obvious that each particles is attributed a definite electric charge, mass and spin. Among the spin $1/2$ particles, we find two different groups, one with light mass (called the leptons), the other group fairly massive (called the baryons). In order that the universe is stable, the attribute of a baryon number is bestowed on each particle. Its value is one for baryons and zero for leptons. If one postulates its conservation, we can prevent protons decaying to leptons. Our main interest is to explain the systematics of Table I by theoretical means.

Charge Independence:- The proton and the neutron have identical properties except for the electric charge. As for as strong interactions are concerned (if one neglects the relatively very weak electromagnetic interactions), they behave like identical particles. The same is true for the three pions as well. An empirical relation $Q = \frac{1}{2} + I_z$, where Q is the electric charge and I_z is the projection on the Z-axis of a spinlike object I (called the isotopic spin) was observed to start with. The multiplicity of a group of particles differing only in charge is found to be (2I+1). But to make such a formula work for the 'nucleons' (a name for protons and neutron), it was then modified as $Q = I_z + \frac{N}{2}$ where N is the nucleon number.

But with the discovery of the K-mesons (which carries no baryon number), the above empirical formula had to be modified as

$$Q = I_z + N/2 + S/2$$

where S is called the 'strangeness'. At present, all the known physical particles obey such a relation. This empirical relation is known is Gell-Mann-Nishijima relation.

The interesting feature is the experimental observation that though the neutron and proton differ in electric charge, the p-p forces = p-n forces = n-n forces. This fact comes from the binding energy considerations in nuclear physics from the study of mirror nuclei. Of course, small electromagnetic corrections are understood. Such an experimental observation is called the 'charge-independence' of strong interactions.

Invariance of a Theory under Lorentz transformations. LORENTZ GROUP. The idea is that Physics is unchanged if goes from a given frame to its Lorentz transformed frame.

DISCRETE TRANSFORMATION

CPT and their

Implications and Reactions

Charge conjugation:- Definition: Charge conjugation is defined as particle to antiparticle conjugation without changing the spin or momentum. In other words, the conjugation C is defined as commuting with all connected Pcincare transformations. It leaves momentum p and spin s invariant.

Invariance under charge conjugaation requires that

Probability transition $A \to B$

$=$ Probability transition $A^c \to B^c$

where

A^c is obtained from A by changing all particles to their antiparticles without changing their energy momentum and polarisation.

Applications and similar discussion on P and T (see Sridhar's notes)

<u>G-Conjugation</u>:- <u>Definition</u>: This is defined an extended conjugation (called by Michel as isoparity),

$$G = C\, e^{i\pi T_2} = CR$$

where C is the usual charge conjugation operation and R is a rotation around the second isospin axis through an angle π. The operation is defined such that the self-conjugate boson field is an eigenstate of this operator (In this case the pions are eigenstates of this operator.) A system of n pions is an eigenstate of G with an eigenvalue $(-1)^n$. The assumption of G-conservation in reactions lead to severe selection rules.

A close observation of Table I reveals that any theory has to explain the multiplet structure of the particles. The strong interactions are charge independent and conserve the hypercharge ($Y = B + S$) . We look for some generalization of charge-independence. In the case of isospin, all members of an isomultiplet have the strong interaction properties and differ only in their electric charge properties. Similarly one looks for a generalization of this concept such that members of a larger multiplet have the same 'super-strong' interaction properties, but differ in their 'medium-strong' interaction properties which respect only charge-independence and also differ in their electric charge properties.

II. Mathematical Preliminaries

1. Group Theory:-

An abstract group is a set G of elements which possesses a binary law of composition, such that

(i) for any two elements $a, b \in G$, $a.b$ is an element of G. Here the dot denotes the binary law of composition called sometimes the 'group multiplication'.

(ii) if a, b, c are in G,

$$a.(b.c) = (a.b).c$$

This is known as the associative law for the binary operation.

(iii) there exists an element I in G such that for any $a \in G$, $a.I = I.a = a$. The element I is called the identity element of G.

(iv) for each element a in G, there exists an element $a^{-1} \in G$ such that $a.a^{-1} = a^{-1}.a = I$. a^{-1} is called the inverse of a. For any $a, b \in G$, if $a.b = b.a$, then G is said to be abelian.

A topological space is a set for which the following conditions are satisfied.

(i) The intersection of any finite number of open subsets is open,

(ii) The union of any number of open subsets is open,

(iii) The empty subset and the whole space are open.

(iv) To each pair of distinct points, there are open sets containing them which do not intersect.

A set G of elements is a topological group if

(i) G is an abstract group,

(ii) G is a topological space

(iii) (a) If a and b are elements of G, then for each open set W of a.b, there exists open subsets U,V containing a,b respectively such that $U.V. \subset W$ i.e. for any $x \in U$, $y \in V$, $x.y \in W$,

(b) If a is in G, then for every open set V containing a^{-1} there exist open sets U containing a such that $U^{-1} \subset V$ (i.e. for any $x \in U$, x^{-1} is in V).

A topological group G is called a Lie group, if the following conditions are satisfied:

(i) A coordinate system can be introduced in G. By this we mean that to every r-tuple $(S^1, \ldots\ldots, S^r)$ in an open set containing (0,0,...0) of a r-dimensional enclidean space, we can associate an element S in an open set U of G containing I in a one-one bicontinuous manner such that the r-tuple (0,0,...0) corresponds to I. $(S^1 \ldots\ldots S^r)$ are called the coordinates of S. Let W be a sufficiently small open set containing I and contained in U so that for any $s,t \in W$, $s.t = u \in W$. Then

$$u^\alpha = f^\alpha (s^1, \ldots s^r, t^1, \ldots\ldots t^r) \tag{1}$$

$$\alpha = 1, \ldots\ldots, r$$

where u^α denotes the α^{th} coordinate of u.

(ii) The function f^α is analytic (differentiable an arbitrary number of times) in the 2r parameters $s^1, \ldots s^r, t^1, \ldots t^r$ The number r is called the dimension of G.

2. Structure Constants:-

Let G be a Lie Group. As I = (0, .. , 0) and uI = Iu = u, from (1) we get

$$f^\alpha (s^1, \ldots, s^r, 0, 0, \ldots 0) = s^\alpha$$

$$f^\alpha (0, 0, \ldots 0, t^1, \ldots t^r) = t^\alpha \tag{2}$$

In view of (ii) in the definition of a Lie Group, we can develop f^α as a Taylor's series which we give below with summation convention as.

$$\begin{aligned} f^\alpha (s^1, \ldots s^r, t^1, \ldots t^r) &= s^\alpha + t^\alpha + a^\alpha_{\beta\gamma} s^\beta t^\gamma \\ &+ g^\alpha_{\beta\gamma\delta} s^\beta s^\gamma t^\delta + h^\alpha_{\beta\gamma\delta} s^\beta t^\gamma t^\delta \\ &+ \ldots\ldots \end{aligned} \tag{3}$$

If $s\tilde{s} = I$, then from (3) we can manage to get the α^{th} coordinate $\tilde{s}^\alpha$ in terms of the coordinates of s as

$$\tilde{s}^\alpha = -s^\alpha + a^\alpha_{\beta\gamma} s^\beta s^\gamma + \ldots\ldots \tag{4}$$

If s and t are elements of G, consider the element q(s,t) = s t s^{-1} t^{-1} (called the commutator of s and t). Then from (3) and (4) we obtain

$$q^{\alpha}(s,t) = C^{\alpha}_{\beta\gamma} s^{\beta} t^{\gamma} + \ldots\ldots \tag{5}$$

The constants $C^{\alpha}_{\beta\gamma}$ for α, β, γ varying from 1, ... , r are called the structure constants of G. From (3) and (5) we deduce that $C^{\alpha}_{\beta\gamma} = a^{\alpha}_{\beta\gamma} - a^{\alpha}_{\gamma\beta}$. Hence

$$C^{\alpha}_{\beta\gamma} = - C^{\alpha}_{\gamma\beta} \tag{6}$$

From the associate law for G, we can deduce another condition on the r^3 structure constants of G. If s, t, u are in G, as (s.t). u = s. (t.u),

$$f^{\alpha}\left[f^{\alpha}(s,t), u\right] = f^{\alpha}\left[s, f^{\alpha}(t,u)\right] \quad \text{cf. (1)}$$

Substituting (3) for the elements of third order in the equation (which is identically fulfilled in the first and second order), we have for the coefficient of $s^{\beta} t^{\gamma} u^{\delta}$, the relation

$$a^{\alpha}_{\sigma\delta} a^{\sigma}_{\beta\gamma} - a^{\alpha}_{\beta\sigma} a^{\sigma}_{\gamma\delta} = h^{\alpha}_{\beta\gamma\delta} + h^{\alpha}_{\beta\delta\gamma} - g^{\alpha}_{\beta\gamma\delta} - g^{\alpha}_{\gamma\beta\delta} \tag{7}$$

As $h^{\alpha}_{\beta\gamma\delta} = - h^{\alpha}_{\gamma\beta\delta}$ etc., the right hand side vanishes when summed with respect to α while the left hand side in view of (6) becomes

$$C^{\alpha}_{\beta\sigma} C^{\sigma}_{\gamma\delta} + C^{\alpha}_{\gamma\sigma} C^{\sigma}_{\delta\beta} + C^{\alpha}_{\delta\sigma} C^{\sigma}_{\beta\gamma} = 0 \tag{8}$$

N.B. The structure constants depend on the coordinate system chosen for the euclidean space.

3. Lie Algebra:-

Let $\mathfrak{g}$ be the r-dimensional vector space in which the following operation of composition of vectors is defined:

(i) to every pair ξ, η of vectors, there corresponds a vector $\omega = [\xi, \eta]$ called the Lie bracket of ξ, η,

(ii) $[\xi_1, c_1\eta_1 + c_2\eta_2] = c_1[\xi_1, \eta_1] + c_2[\xi_1, \eta_2]$

when c_1, c_2 are constants and '+' is vector addition,

(iii) $[\xi, \eta] + [\eta, \xi] = 0$

(iv) $[[\xi, \eta], \theta] + [[\eta, \theta], \xi] + [[\theta, \xi], \eta] = 0$

for any triple ξ, η, θ. (This property is known as the Jacobi identity). Then $\mathfrak{g}$ is called a Lie Algebra.

Let $\{e_\alpha\}$, $\alpha = 1, \ldots, r$ be a basis for $\mathfrak{g}$.

Then $[e_\beta, e_\gamma] = c^\alpha_{\beta\gamma} e_\alpha$ (written with summation convention) for a suitable choice of r^3 constants $c^\alpha_{\beta\gamma}$, α, β, γ, varying from 1 to r. $\{c^\alpha_{\beta\gamma}\}$ are called the structure constants of $\mathfrak{g}$ (These obvious obviously depend on the choice of the basis).

From (iii) and (iv) it follows that

$$\text{(a)} \qquad c^\alpha_{\beta\gamma} = -c^\alpha_{\gamma\beta}$$

(b) $\quad C^{\alpha}_{\beta\sigma} C^{\sigma}_{\gamma\delta} + C^{\alpha}_{\gamma\sigma} C^{\sigma}_{\delta\beta} + C^{\alpha}_{\delta\sigma} C^{\sigma}_{\beta\gamma} = 0$

Conversely, if $\{ C^{\alpha}_{\beta\gamma} \}$ are r^3 constants satisfying (a) and (b), then by defining $[e_\beta, e_\gamma] = c^{\alpha}_{\beta\gamma} e_\alpha$ and extending by linearity the bracket operation for any two vectors of a r-diml. vector space for which $\{ e_\alpha \}$ is a basis, we get a Lie algebra structure $\mathfrak{g}$ for which $\{ C^{\alpha}_{\beta\gamma} \}$ are the structure constants. Thus a Lie Algebra is completely specified by its structure constants.

4. <u>The Lie Algebra of a Lie Group.</u>

A collection $((s(\tau))$ of elements in a Lie group G depending continuously on a real parameter τ varying on a real interval such that $s(o) = I$ is called a <u>curve</u> in G.

We shall say that the curve $s(\tau)$ has a <u>tangent</u> if the derivatives $\xi^\alpha = \left. \frac{d s^\alpha(\tau)}{d\tau} \right|_{\tau=0}$ exist. The r-vector whose components are ξ^α, $\alpha = 1, \ldots r$ is called the <u>tangent vector</u> of the curve in question.

Thus we <u>associate with a r-dimensional Lie group</u> G, <u>r-dimensional vector space</u> $\mathfrak{g}$ composed of all tangents to the curves in G. This association can be shown to be independent in a natural manner of the choice of a basis for the euclidean space which gives the coordinates of elements of G.

If ξ, η are tangents to the curves $s(\tau)$, $t(\tau)$ respectively and if $u(\tau)$ is the curve such that

$u(\tau) = s(\tau)\, t(\tau)$, the tangent θ to $u(\tau)$ is given by $\theta = \xi + \eta$.

With the same notations as in previous para let $q(\tau)$ be the curve such that

$$q(\tau) = s(\tau)\, t(\tau)\, (s(\tau))^{-1} (t(\tau))^{-1}$$

Introducing the parameter $\sqrt{\tau}$ (the positive root of τ) let θ be the tangent vector of $q(\tau)$. We define a bracket operation on $\mathfrak{g}$ thus: For ξ, η in $\mathfrak{g}$ define $[\xi, \eta] = \theta$, θ defined as above.

We can check up that the bracket operation defined on $\mathfrak{g}$ satisfies the conditions of a Lie algebra. Further the structure constants of the Lie algebra are the same as those of the Lie group G in corresponding coordinates. $\mathfrak{g}$ <u>is called the Lie algebra of the Lie group G</u>.

We quote Lie's Fundamental Theorem:

To every Lie group there corresponds a Lie algebra of the same dimension, conversely, every lie algebra determines uniquely up to 'local isomorphism' a Lie group (cf. Pontrjagin).

5. <u>Special classes of Lie Groups and Algebras:</u>

In sec. 1, we defined an abelian group. <u>For an abelian Lie group</u>, in view of (6) in § 2, <u>all structure constants are zero</u>.

<u>A commutative Lie algebra</u> is one where the bracket of any two elements is 0. <u>The structure constants of a commutation Lie algebra are zero</u>.

A subgroup of a group G is a non-void subset which with respect to the induced operation constitutes a group. An invariant subgroup S is a subgroup such that whatever be x in G, $x \cdot S\, x^{-1}$ is contained in S.

A subring of a Lie algebra $\mathfrak{g}$ is a subgroup of $\mathfrak{g}$ with respect to vector addition which is closed for bracket operation. An ideal J of $\mathfrak{g}$ is a subring of such that whatever α in $\mathfrak{g}$ and ξ in J, $[\alpha, \xi]$ is in J.

By a simple group, we mean a group which does not contain any invariant subgroup other than the whole group and the identity element, considered as subgroups. A group is semi-simple if it does not contain any abelian invariant subgroup.

The Lie $\mathfrak{g}$ algebra of a simple (respectively semi-simple) Lie group is itself said to be simple (semi-simple).

6. Some properties of Semi-simple Lie Algebras.

Let $\mathfrak{g}$ be a Lie algebra with structure constants $\{C^{\alpha}_{\beta\gamma}\}$ α, β, γ ranging from 1 to r. We define the symmetric tensor g_{ik} thus:

$$g_{ik} = g_{ki} = C^{\alpha}_{i\beta}\, C^{\beta}_{k\alpha} \qquad (1)$$

Theorem 1: (Cartan's criterion); The necessary and sufficient condition that a Lie algebra $\mathfrak{g}$ be semi-simple is that $\det |g_{ik}| \neq 0$.

By a Linear Lie algebra, we mean a Lie algebra the elements of which are linear operators acting on a vector space.

<u>Lemma</u>: Let $\mathfrak{g}$ be a Lie algebra and A an element of $\mathfrak{g}$. Define for each X in $\mathfrak{g}$, $A(X)$ as the element $[A, X]$ of $\mathfrak{g}$. Then $A(X)$ is a linear operator on the vector space underlying $\mathfrak{g}$. (The operator $A(X)$ is 0 if and only if A commutes with each element of $\mathfrak{g}$).

With usual addition of operators and defining the bracket of two operators defined in the above lemma as the operator defined by the bracket of the elements determining them, we can prove that such operators constitute a (linear) Lie algebra, As a semi-simple Lie algebra cannot contain a commutative ideal, it follows that such a Lie algebra is identical with the Lie algebra constituted by the operators $A(X)$. Hence we have

<u>Theorem 2</u>. Every semi-simple Lie algebra is a linear Lie algebra.

For linear operators we have the usual product operator which is an associative operation. Thus in any semi-simple Lie algebra we have an associative product denoted by $\cdot$

We now define the <u>CasimirForm</u> of a semi-simple Lie algebra $\mathfrak{g}$. Let g^{ik} be normalized cofactors of det $|g_{ik}|$ (cf. (1) above), i.e. $g^{i\alpha} g_{\alpha k} = \delta^i_k$. As $\det |g_{ik}| \neq 0$ $g^{i\alpha}$ exist.

<u>The quadratic form</u>,

$$F = g^{\alpha\beta} e_\alpha \cdot e_\beta \tag{2}$$

(where $e_\alpha \cdot e_\beta$ is the associative product referred to above) is called the Casimir Form of g.

Theorem 3. (Casimir). The casimir form F of a semi-simple Lie algebra commutes with every element (equivalently, $[F, e_r] = 0$.

The following theorem relates semi-simple Lie algebras to simple ones.

Theorem 4. (Cartan) Every semi-simple Lie algebra is a 'direct sum' of all its simple ideals.

7. The Standard Form of a Semi-simple Lie Algebra.

Let g be a Lie algebra of dimension r. Consider the eigen value problem of the operator $A(X)$ defined in the Lemma in Sec.6. i.e. $A(X) = [A, X] = \rho X$. If the secular equation of the operator has r distinct roots, then we have r linearly independent eigen vectors which can be used as a basis for the vector space underlying g. If, however, the secular equation has degenerate roots, r linearly independent vectors may not exist. Hence, a coordinate system for g cannot be arrived at by the above method. But for semi-simple Lie algebras we have the following

<u>THEOREM</u> (Cartan): For a semi-simple Lie algebra g if we choose A so that the secular equation of $A(x)$ has the maximum number of distinct roots (which we can), the only degenerate root is $\rho = 0$ and if ℓ is the multiplicity of the root, there exist corresponding to this root, ℓ linearly independent eigenvectors any two of which commute.

The number ℓ is called <u>rank</u> of g.

We shall choose as basis the ℓ linearly independent eigenvectors (say) $H_1, \ldots, H_\ell$ corresponding to the degenerate root. $\rho = 0$ together with the $(r-\ell)$ linearly independent eigenvectors $E_\alpha, E_\beta \cdots \cdots$ corresponding to the distinct roots $\alpha, \beta, \ldots,$

The commutational relations for $H_1, \ldots \ldots H_\ell$; $E_\alpha, E_\beta, \ldots$ can be obtained to be

$$[H_i, H_j] = 0, \tag{1}$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha, \tag{2}$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \text{ if } (\alpha+\beta) \text{ is not a vanishing root,} \tag{3}$$

$$[E_\alpha, E_{-\alpha}] = \alpha^i H_i . \tag{4}$$

The structure constants are then,

$$C^\tau_{ij} = 0, \quad C^\tau_{i\alpha} = \alpha_i \delta^\tau_\alpha, \quad C^{\alpha+\beta}_{\alpha\beta} = N_{\alpha\beta},$$

$$C^\tau_{\alpha\beta} = 0 \quad \text{if } \tau \neq \alpha+\beta .$$

Further,

$$[A, H_i] = 0, \tag{5}$$

$$[A, E_\alpha] = \alpha E_\alpha \tag{6}$$

As A is an eigenvector of $[A, x] = \rho x$,

$$A = \lambda^i H_i \tag{7}$$

From (6), (7) and (2), it follows that

$$\alpha = \lambda^i \alpha_i \tag{8}$$

8. <u>The Concept of Root</u>:-

The form (8) of sec.7 is called a <u>root</u> of the semi-simple Lie algebra $\mathfrak{g}$. It can be thought of as a vector in a l-dimensional vector space.

A root is said to be positive if its first nonvanishing component is positive (in an arbitrary basis). A root is called <u>simple</u> (sometimes the terminology primitive or elementary is also used in the literature) if it is a positive root and in addition cannot be decomposed into the sum of two positive roots.

<u>Theorem</u>: (1) For a simple group of rank l there exist l simple roots and they are all linearly independent (We shall call the set of simple roots the π-system).

(2) Any non-simple root can be expressed as a linear combination of the simple roots $\sum_{\alpha \in \Pi} R_i \alpha_i$ where R_i are <u>all</u> positive or <u>all</u> negative integers

(3) If α is a root, then $-\alpha$ is also a root for any simple group.

(4) If α and β are two roots then

$$2 \frac{(\alpha\beta)}{(\alpha\alpha)} = \text{integer}$$

and $\beta - \frac{2(\alpha\beta)}{(\alpha\alpha)}\alpha$ is also a root. Here $(\alpha\beta)$ denotes their scalar product. If φ is the angle between α and β, then from Th. (4) above follows that

$$\cos^2 \varphi = \frac{1}{4} mn ,$$

and

$$\frac{|\alpha|^2}{|\beta|^2} = \frac{m}{n} .$$

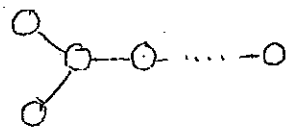
Here m and n are integers. This would mean that the angle φ can assume only certain values (implying thereby some kind of a quantization of the angle.) In particular, this is true for the simple roots. The allowed angles are 90^0, 120^0, 135^0 and 150^0 and the ratio between their lengths become

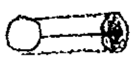
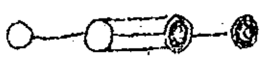
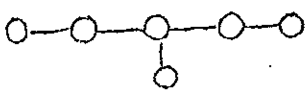
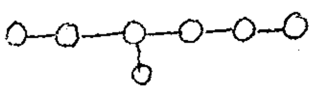
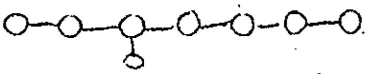
$$\frac{|\alpha|^2}{|\beta|^2} = \begin{array}{ll} 1 & \text{if } \varphi = 120^0 \\ 2 & \text{if } \varphi = 135^0 \\ 3 & \text{if } \varphi = 150^0 \end{array}$$

If $= 90^0$, then the ratio of lengths is undetermined.

Dynkin Diagrams:-

The geometrical properties of the simple roots in the π-system characterize in a unique manner the corresponding infinitesimal Lie groups. Therefore, it is most convenient to incorporate them in a schematic diagram. These diagrams (the so called Schouten-Dynkin diagrams) are drawn in fig.

	Classical Groups	N = number of parameters
A_ℓ		$\ell^2 + 2\ell$
B_ℓ		$2\ell^2 + \ell$
C_ℓ		$2\ell^2 + \ell$
D_ℓ $(\ell > 2)$		$2\ell^2 - \ell$

	Exceptional group	
G_2		14
F_4		52
E_6		78
E_7		133
E_8		248

Cartan's solution of all possible simple Lie groups.

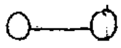
We have seen earlier that the lengths of the simple roots of a given simple Lie group can assume only certain values. This together with the fact that the angles are restricted can be symbolically described by associating with each simple root a small circle. For roots of greater length, the circle is marked in black. If the angle between the two consecutive simple roots is equal to 120°, 135°, or 150° the corresponding circles are joined by simple, double or triple lines respectively. If the angle is 90°, the circles are not joined. For a group of rank ℓ, there are ℓ simple roots and therefore ℓ circles (black or white). Now the language of the diagrams is clear. In terms of these diagrams, simple Lie groups can be classified as classical and exceptional groups.

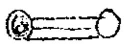
Classical Groups:-

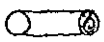
The realization of A_ℓ is the group of unitary, unimodular matrices in complex space of $(\ell+1)$ dimensions $SU(\ell+1)$. The realization of B_ℓ and D_ℓ are the real orthogonal groups in $(2\ell+1)$ and 2ℓ dimensions respectively. The realization of C_ℓ is the group of unitary matrices in complex 2ℓ dimensions satisfying the condition $U^T J U = J$ where J is a non-singular antisymmetric matrix. In other words, the realization of C_ℓ is the symplectic group in complex 2ℓ dimensions.

Some simple examples of root structures:-

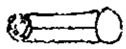
For $\ell = 1$, there is just one simple root +1. The π-space is just the single object $\{1\}$. For $\ell = 2$, the π-space is two-dimensional.

A_2:  Two simple roots of equal length and the angle between them is 120°.

B_2: 

C_2: 

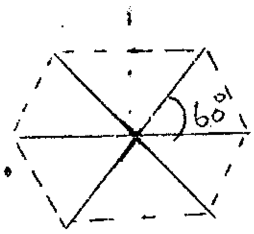
Two simple roots. Their length ratio is $\sqrt{2}$. The angle between them is 135°.

G_2:  Two simple roots with the length ratio $\sqrt{3}$, and angle 150°.

D_2:  is <u>semisimple</u> $D_2 \approx A_1 \times A_1$

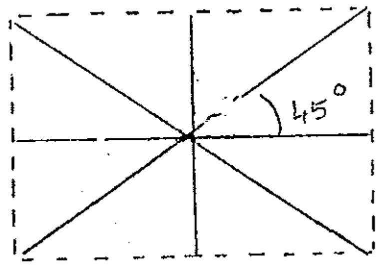
It follows that from the Dynkin diagrams we can read off immediately the rank of the group, the length of the simple roots and their mutual angles.

It should be kept in mind that not all the roots are simple. If the order of the group is N (denoting the total number of elements), ℓ of the elements commute among themselves (ℓ fold degeneracy) Out of the rest (N-ℓ) elements, each gives rise to a root vector. However, since both α and $-\alpha$ are roots, the distinct roots are only $\frac{N-\ell}{2}$ in number. Out of these ℓ, we have seen, are simple. Therefore, there are $\frac{N-3\ell}{2}$ non-simple roots. The entire root diagram could be constructed (the root diagram is two dimensional when $\ell = 2$ for example). The root diagrams for A_2, B_2 C_2 and G_2 are shown in the fig.



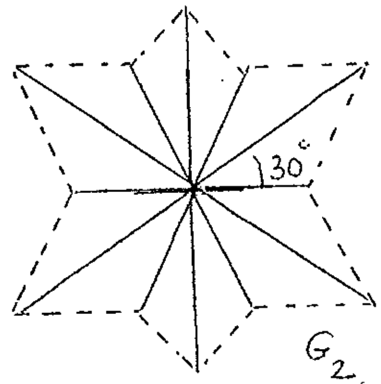
A_2

$N = 8$



B_2

$N = 10$



G_2

$N = 14$

In general the entire root diagram is obtained in the following way.

<u>Classical Groups</u>:- A_ℓ The collection of $\ell(\ell+1)$ differences $\{(e_i - e_j)\}$, $i, j = 1 \ldots (\ell+1)$ of $(\ell+1)$ unit vectors yields all the roots. The dimension of the algebra is $(\ell+1)^2 - 1$.

B_ℓ :- The roots are obtained from $\{\pm e_i\}$, $\{\pm e_i \pm e_k\}$
$i, k = 1 \ldots \ell$

The dimension of the algebra is $\ell(2\ell+1)$.

C_ℓ :- The collection $\{\pm 2e_i\}$, $\{\pm e_i \pm e_j\}$ yields the roots of C_ℓ. $i, j = 1 \ldots \ell$

D_ℓ: The collection $\{\pm e_i \pm e_k\}$, $i, k = 1 \ldots, \ell > 2$ yields all the roots. There ~~ale~~ $2\ell(\ell-1)$ of them and the dimension of the algebra is $\ell(2\ell-1)$.

<u>Exceptional Groups</u>:= G_2 : The collection $\{\pm\ e_i - e_j\}$ and $\pm\{e_i - 2e_j + e_k\}$, $i, j, k = 1, 2, 3$ yields all the roots. The order of the group is 14.

F_4: The diagram of B_4 with 16 more vectors $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$ (Total 48 vectors and dimension is 52.)

E_6: The diagram A_5, the vectors $\pm\sqrt{2}\, e_7$ and $\frac{1}{2}(\pm e_1 \pm \cdots \cdots \pm e_6) \pm e_7/\sqrt{2}$

Constitute the root diagram of E_6, where we take four positive and four negative in the first fraction. The total number of vectors are 72 and the dimension is 78.

E_7:- The diagram A_7 and the vectors $\frac{1}{2}(\pm e_1 \pm e_2 \pm \cdots \pm e_8)$

where we take four positive and four negative signs constitutes the root diagram of E_7. The number of vectors is 126 and the dimension is 133.

E_8: The diagram D_8 and the vectors $\frac{1}{2}(\pm e_1 \pm \cdots\cdots \pm e_8)$ with each sign occurring an even number of times forms the root diagram of E_8. There are 240 vectors and the dimension of the algebra is 248

III. Representation of Lie Group and Lie Algebras

Let G be a Lie group. If to each element of G, we can associate a linear operator R(g) of a certain n dimensional vector space V such that if $g_1 \cdot g_2 = g_3 \in G$, then $R(g_1) R(g_2) = R(g_3)$ and the association $g \longrightarrow R(g)$ is further continuous, then R is a n-dimensional representation of G.

Let $\mathfrak{g}$ be the Lie algebra. If to each element ξ of $\mathfrak{g}$ we can associate an operator $A(\xi)$ acting on V such that

$$A(\xi + \eta) = A(\xi) + A(\eta)$$

$$A(c\xi) = c\, A(\xi)$$

$$A([\xi, \eta]) = [A(\xi), A(\eta)]$$

then A is said to be a n-dimensional representation of $\mathfrak{g}$.

Theorem 1:- Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then any representation of G is a representation of $\mathfrak{g}$ and vice versa.

Theorem 2: The commutation relation of the Lie algebra (hence that of the Lie group) is true for any representation. Two representation $A_1(\xi)$ and $A_2(\xi)$ are said to be equivalent, if there exists a nonsingular operator U such that

$$U A_1(\xi) U^{-1} = A_2(\xi)$$

for any ξ.

A representation $\xi \to A(\xi)$ is <u>reducible</u>, if the operators $A(\xi)$ acting on the vector space V leave a proper sub-space of V invariant.

If a representation $A(\xi)$ is reducible, then, it could be brought, by equivalence, to the standard matrix form

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

A representation which could not be brought to this form by equivalence is called an <u>irreducible representation</u>.

A representation $\xi \to A(\xi)$ is <u>decomposable</u> if the operators $A(\xi)$ leave two mutually orthogonal subspaces which together span the whole space V. If a representation A is decompose then there is an equivalent representation in which A could be brought to the form $\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$.

<u>Theorem 3</u>: Every representation of a compact Lie group (see chavalley 'Lie groups' for defipition) is finite dimensional and is equivalent to a unitary representation. Thus R(g) takes the form

$$R(g) = \exp \ i \epsilon^{\alpha} X_{\alpha}$$

where ϵ^{α}'s are real and X_{α} is hermitian.

<u>Theorem 4</u>: For a unitary group, if a representation is reducible, then it is fully reducible to the form

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

The concept of Weight:

Consider a n-dimensional matrix representation of a semi-simple Lie algebra $\mathfrak{g}$. The representation is completely specified by r-matrices (r being the dimension of $\mathfrak{g}$) D_ρ $\rho = 1, \ldots r$ which satisfy the equation

$$[D_\rho, D_\sigma] = C_{\rho\sigma}^{\lambda} D_\lambda$$

where $C_{\rho\sigma}^{\lambda}$ are the structure constants of $\mathfrak{g}$. Let us express the representation with respect to the standard Cartan form. Let $H_1', \ldots H_\ell', E_\alpha', \ldots E_\gamma'$ be the matrices in the representation corresponding to the basis $H_1, \ldots H_\ell, E_\alpha, \ldots E_\gamma$ of $\mathfrak{g}$. Let u be the simultaneous eigenvector of the diagonal matrices $H_1', \ldots H_\ell'$ so that

$$H_i' u = m_i u$$

Then the l-components $(m_1, \ldots m_\ell)$ can be thought of as the components of a ℓ-dimensional vector m which is called the weight vector. It should be noticed that while the root vectors characterize the infinitesimal Lie group), the weight vectors characterize the representation.

Theorem 1: Every representation has at least one weight (see Racah's Princeton notes for proof).

Theorem 2: A vector u of weight m which is a linear combination of vector u_k of weights m_k, $m_k \neq m$ for each k, must vanish. (The corresponding theorem in matrices is that the eigenvectors corresponding to two distinct eigenvalues of a hermitian matrix are orthogonal)

Theorem 3: There exists at most n linearly independent weights corresponding to a representation.

Theorem 4: If u is a vector of weight m, then $E_\alpha u'$ is an eigenvector with weight $(m+\alpha)$.

Theorem 5: If a representation is irreducible, then all the H_i's (we drop the primes for convenience and these denote the matrix representation) can be simultaneously diagonalized.

Theorem 6: If m is a weight and α is a root then

$$2\frac{(m\alpha)}{(\alpha\alpha)} = \text{integer}$$

and $m - \frac{2(m\alpha)}{(\alpha\alpha)}\alpha$ is a weight

(Note: There is no theorem analogus to that of the roots that if m_1 and m_2 are weights, then

$$2\frac{(m_1 m_2)}{(m_1 m_1)} \quad \text{is an integer)}$$

Theorem: The set of all weights is invariant under the Weyl group S of transformations generated by reflections with respect to the hyperplanes passing through the origin and perpendicular to the roots.

Definitions: A weight is said to be positive, if its first non-vanishing component (in an arbitrary basis) is positive. One weight is said to be higher than the other, if their difference is positive. Thus weights are equivalent if they are connected by a transformation belonging to S

A weight higher than all its equivalents is said to be dominant. A weight is called simple if it belongs to only one eigenvector. The highest among the dominant weights is called the highest weight.

Theorem: An irreducible representation is uniquely characterized by its highest weight which is simple.

Theorem: Two irreducible representations are equivalent if their highest weights are equal.

Theorem: For a semi-simple Lie algebra of rank ℓ, there are weights (called fundamental dominant weights) such that any dominant weight is a non-negative integral linear combination of them.

Theorem: There are ℓ fundamental irreducible representations $A_1, \ldots A_\ell$, which have the fundamental weights as their highest weights. The dimension of the representation with highest weight Λ is given by

$$d = \prod_{\alpha \in \Sigma_+} 1 + \frac{(\Lambda \alpha)}{(g \alpha)}$$

where

$$g = \frac{1}{2} \sum_{\beta \in \Sigma_+} \beta$$

Σ_+ is the system of all positive roots.

III. Review of Unitary Groups

The set of all (n x n) non-singular matrices with complex entries form a group under matrix multiplication. An interesting subgroup is the set of unitary (n x n) matrices. This group is called U(n). If g is a typical element, then it can always be diagonalized. Then, any unitary matrix. g can be written as

$$g = \exp i \epsilon H ,$$

where ϵ's are real and H is a hermitian matrix. If in addition g is such that det g = +1, then it follows for infinitesimal ϵ

$$\det g \approx 1 + i \epsilon \, Tr H + O(\epsilon^2) ,$$

so that

$$Tr \; H = 0$$

Now an arbitrary hermitian matrix is given in terms of n real diagonal elements (which are necessarily real for a hermitian matrix) and $\frac{n(n-1)}{2}$ complex elements above the main diagonal. Thus, it has n^2 independent real parameters. If in addition its trace is zero, it depends only on (n-1) real diagonal elements and therefore only on (n^2-1) real parameters. These (nxn) unitary unimodular matrices form the group SU(n) under matrix multiplication.

The topological properties of this group are:

(1) <u>SU(n) is compact</u>: By this we mean that if we are given an infinite sequence of elements $g_1, \dots, g_i, \dots$, we can always extract a sub-sequence which converges to an element of the group. As we had seen in an earlier section that an immediate consequence of this topological property is that

(a) the irreducible representations are finite dimensional and are equivalent to unitary representations.

(b) if any representation is reducible, it is fully reducible and so

(c) Any representation can be split into a direct sum of irreducible representations.

(2) <u>SU(n) is a Lie Group</u>: This means that the functions defined in the parameter space are analytic. It should be emphasized, that for a Lie group, the number of elements can be infinite. The only condition is that the number of parameters (called <u>coordinates</u> earlier) are <u>finite</u>.

(3) <u>SU(n) is a simply connected group</u>: By 'connected', we mean that for a given element g, we can always find a <u>continuous function</u> $g(t)$, $0 \le t \le 1$, such that $g(0) = I$ = identity element of the group and $g(1) = g$. In a simply connected group two such 'paths' $f(t)$, $g(t)$ leading from I to g can be continuously transformed from one another. The Lie algebra of $SU(\ell+1)$, (ℓ = rank of the group = n-1) is called A_ℓ in Cartan's notation (see the discussion in our earlier chapters).

The root diagram consists of the collection $\{(e_i - e_j)\}$, $i, j = 1 \ldots (\ell+1)$ of unit vectors e_i in a (l+1) dimensional space. The dimension of the algebra is $(\ell+1)^2 - 1$. We have also seen that there are l independent simple roots and the Dynkin diagram is $\circ\!\!-\!\!\circ\!\!-\!\!\circ \cdots -\!\!\circ$

Weight Diagrams: Since the root space is (l+1) dimensional it is convenient to describe the weights also as vectors in a (l+1) dimensional space, but because they will not then automatically lie in the l-space with normal

$$\frac{1}{(\ell+1)^{\frac{1}{2}}} \quad (1, 1, \ldots 1)$$

we impose on the weights the condition

$$\sum_{i=1}^{\ell+1} m_i = 0$$

Then they also be lie in the same l-dimensional space as the roots.

The Weyl group in this case is isomorphic to the permutation group in (l+1) dimension. By this we mean that the elements of the Weyl group permute the (l+1) components of a given weight and yield its equivalents. The condition that $2\frac{(m\alpha)}{(\alpha\alpha)}$ = integer shows that

$$m_i = \frac{\text{integer}}{(\ell+1)} \quad \text{and} \quad m_i - m_j = \text{integer}$$

An immediate consequence is that the highest weight of any representation should satisfy the condition

$$m_1 \geq m_2 \geq \cdots \geq m_{\ell+1}$$

and of course

$$\sum_{i=1}^{\ell+1} m_i = 0$$

and so

$$m_1 \geq 0$$

$$m_{\ell+1} \leq 0$$

To find the representation of the algebra A_ℓ, we proceed as follows. We first build the so-called the 'self-representation', then construct out of that 1-fundamental representations and finally use them to build any representation. Consider the set of all $(\ell+1) \times (\ell+1)$ unitary matrices. We choose as a basis for their algebra, the matrices

$$\overline{H}_i = |i\rangle\langle i|$$

$$E_\alpha = |i\rangle\langle j| \qquad j > i$$

$$E_{-\alpha} = |i\rangle\langle j| \qquad j < i$$

where $|i\rangle$ stands for a set of orthonormal vectors in a (1+1) space. Then

$$\left[\overline{H}_i , E_\alpha\right] = \left(\delta_{ij} - \delta_{ik}\right) E_k$$

The roots are then just $(e_i - e_j)$. Thus the self representation or realization of A_l is the group of unitary matrices. To get the realization for the group SU(l+1), we proceed as follows: Define

$$H_i = \overline{H}_i - \frac{1}{(l+1)} I$$

where I is $(l+1)$ dimensional unit matrix. The matrices H_i are traceless and

$$\sum_{i=1}^{l+1} H_i = 0 ,$$

so that the are not all linearly independent. Thus take the form

$$H_1 = \frac{1}{l+1} \begin{bmatrix} l & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & \ddots & \\ & & & & -1 \end{bmatrix},$$

$$\vdots$$

$$H_{l+1} = \frac{1}{l+1} \begin{bmatrix} -1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & l \end{bmatrix}.$$

Their simultaneous eigenvectors are the vectors

$$u_1 = (1, 0, 0 \cdots 0)$$
$$u_2 = (0, 1, 0, \cdots 0)$$
$$\vdots$$
$$u_{\ell+1} = (0, 0, \cdots 1)$$

and their corresponding weights are

$$\frac{1}{\ell+1} \quad (\ell, -1, -1, \cdots -1)$$
$$\frac{1}{\ell+1} \quad (-1, \ell, \cdots\cdots -1)$$
$$\vdots$$
$$\frac{1}{\ell+1} \quad (-1, -1, \cdots -1, \ell)$$

The highest weight here is obviously

$$\frac{1}{\ell+1} \quad (\ell, -1, -1, \cdots, -1)$$

We now proceed to construct the other fundamental representations: If we call the above self-representation as D_1, then the other fundamental representations are:

Representation	Highest weight
D_1	$m^{(1)} = \frac{1}{\ell+1}(\ell, -1, \cdots, -1)$
$D_2 = (D_1 \times D_1)_{A \cdot S}$	$m^{(2)} = \frac{1}{\ell+1}(\ell-1, \ell-1, 2, \cdots -2)$
$D_3 = (D_1 \times D_1 \times D_1)_{CAS}$	$m^{(3)} = \frac{1}{\ell+1}(\ell-2, \ell-2, \ell-2, -3, -3, \cdots -3)$
$\vdots$	
$D_\ell = \underbrace{(D_1 \times D_1 \cdots \times D_1)}_{\longleftarrow \ \ell \ \longrightarrow}{}_{CAS}$	$m^{(\ell)} = \frac{1}{\ell+1}(1, 1 \cdots, -\ell)$

A.S = Anti symmetric ; CAS = Completely Anti Symmetric

The highest weight of D_2 is obtained by adding the highest and the next to it of D_1 and that of D_3 by adding the first, second and their weights of D_1. It remains for us to show that these are the ℓ-fundamental representations. Let

$$\begin{aligned} \lambda_1 &= (m_1 - m_2) \\ \lambda_2 &= (m_2 - m_3) \\ \lambda_\ell &= (m_\ell - m_{\ell+1}) \end{aligned} \qquad \begin{aligned} &\lambda_i \geq 0 \\ &\lambda_i = \text{integer} \end{aligned}$$

and λ_i = integers, then we see that

$$m = \sum_{i=1}^{\ell} \lambda_i \, m^{(i)}$$

Hence, if we form the direct product

$$D = \underbrace{D_1 \times D_1 \times \cdots \times D_1}_{\lambda_1} \times \underbrace{D_2 \times D_2 \times \cdots \times D_2}_{\lambda_2} \times \cdots \times \underbrace{D_\ell \times D_\ell \times \cdots \times D_\ell}_{\lambda_\ell}$$

$$= D_m \dot{+} \cdots$$

the leading irreducible representation.

D_m occurring in the reduction will have the highest weight in D with highest weight m. In this way, we can construct the irreducible representation D_m with any given highest weight m out of the ℓ-fundamental representations.

Description of the irreducible representations of SU(n)

In this we shall describe in detail the description of an irreducible representation (I.R.) of SU(n), in terms of tensors and Young Tableau. While our discussion will be quite general for SU(n), we shall illustrate them for the group SU(3), in view of the later applications.

Consider any mixed tensor

$$T^{\alpha\beta\ldots\gamma}_{ij\ldots k} \qquad \left.\begin{matrix}\alpha,\beta,\gamma\\ i,j,k\end{matrix}\right\} = 1\ldots n$$

with the property under unitary, unimodular transformation

$$\bar{A}^{\alpha\beta\ldots\gamma}_{ij\ldots k} = a^{\alpha\lambda}\, a^{\beta\mu}\ldots a^{\gamma\nu}\, (a^{-1})_{\ell i}\, (a^{-1})_{mj} \ldots (a^{-1})_{nk}\; T^{\lambda\mu\ldots\nu}_{\ell m\ldots n}$$

with the a's satisfying the unitary unimodular condition. The following numerical tensors can be easily verified to be invariant under such transformations

$$\bar{\delta}^{i}_{j} = \sum_{k,l} a_{ik} (a^{-1})_{lj} \; \delta^{k}_{l}$$

$$= \sum_{l} a_{il} (a^{-1})_{lj} = \delta^{i}_{j}$$

$$\bar{\epsilon}^{ijk} = \sum_{l,m,n} a_{il} \; a_{jm} \; a_{kn} \; \epsilon^{lmn}$$

$$= \det a \;\; \epsilon^{ijk}$$

$$= \epsilon^{ijk} \quad \text{since} \quad \det a = 1$$

A general mixed tensor with upper and lower indices can be written as

$$A^{\alpha\beta\ldots\delta}_{ij\ldots l}$$

Define tensors

$$B^{\beta\ldots\delta}_{j\ldots l} = \delta^{i}_{\alpha} \; A^{\alpha\beta\ldots\delta}_{ij\ldots l}$$

B behaves like a tensor with (p-1) upper and (q-1) lower indices.

$$C^{\gamma\ldots\delta}_{\mu i j \ldots l} = \epsilon_{\mu\alpha\beta} A^{\alpha\beta\ldots\delta}_{ij\ldots l}$$

C behaveslike a tensor with $(p-2)$ upper indices and $(q+1)$ lower indices.

$$D^{m\alpha\beta\ldots\delta}_{k\ldots l} = \epsilon^{mij} A^{\alpha\beta\ldots\delta}_{ij\ldots l}$$

is a tensor with $(p+1)$ upper and $(q-2)$ lower indices. In other words, what we have been able to do was to <u>reduce</u> the mixed tensor A to tensors B, C and D with the help of the invariant numerical tensors δ^i_j, $\epsilon_{\alpha\beta\gamma}$ and $\epsilon^{\alpha\beta\gamma}$. A will be <u>Ir-reducible</u> if the action of these numerical tensors is zero on A. i.e. we <u>cannot form</u> non-zero B, C, D. In order that A is irreducible B = o, C = o, D = o. We see that

$$B = 0 \quad \text{if} \quad A^{i\beta\ldots\gamma}_{ij\ldots k} = 0$$

A is traceless in each pair of upper and lower indices.

C = o when A is symmetric in the upper indices

D = o, when A is symmetric in the lower indices.

Thus, a mixed tensor with p upper and of lower indices will be irreducible if

(1) it is totally symmetric in the p upper indices

(2) it is totally symmetric in the lower indices

and (3) it is traceless

A traceless tensor symmetric in all p upper indices and symmetric in all q lower indices has dimension (the number of independent components) in C_n given by

$$N = \frac{(p+n-2)!\ (q+n-2)!}{p!\ q!\ (n-1)!\ (n-2)!}\ (p+q+n-1)$$

$$= \left\{ \frac{(p+n+1)!}{p!\ (n-1)!}\ \frac{(q+n-1)!}{q!\ (n-1)!} - \frac{(p+n-2)!}{(p-1)!\ (n-1)!}\ \frac{(q+n-2)!}{(q-1)!\ (n-1)!} \right\}$$

<u>Example:</u> In the case of SU(3), n = 3,

$$N = \frac{(p+1)!\,(q+1)!}{p!\;q!\;2!}\,(p+q+2)$$

$$= \frac{1}{2}\,(p+1)\,(q+1)\,(p+q+2)$$

$$= (1+p)\,(1+q)\,\left(1+\frac{p+q}{2}\right)$$

<u>Description of an I.R. with Young Tablean</u>

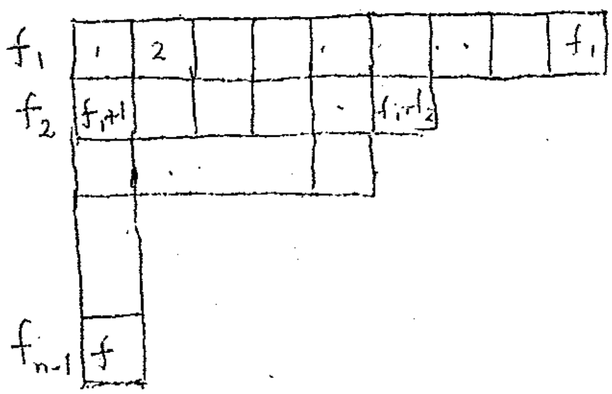
A standard Young Tablean is an array of f boxes with f_1 boxes in the first row, f_2 boxes in the second row, and f_{n-1} boxes in the (n-1)-th row, where $f_1, f_2, \ldots, f_{n-1}$ satisfy the relations

$$f_1 \geq f_2 \geq \cdots \geq f_{n-1}$$

and

$$\sum_{i=1}^{n-1} f_i = f$$

A tablean is usually drawn as follows:



To this tablean corresponds the following symmetry operations

(1) symmetrize completely with respect to the first f_1 indices, the following f_2 indices and so on getting a tensor

$$T'_{\{i_1 \ldots i_{f_1}\}, \{i_{f_1+1}, \ldots i_{f_1+f_2}\}, \ldots}$$

(2) Then antisymmetrize the tensor T' with respect to the indices $\{i_1, i_{f_1+1}, \ldots\}$ .. the indices $\{i_2, i_{f_1+2}, \ldots\}$, $\{i_3, i_{f_1+3}, \ldots\}$ and so on. The resulting tensor T'' form the basis of an invariant subspace which generates an irreducible representation of SU(n). In a different notation, we can write

$$T'' = Y T$$

where

$$Y = \sum \delta_q \; q \; p$$

is the Young symmetrizer. The sum in Y is carried over all permutations p of integers in the same row, and all permutations q of integers in the same column and δ_q is the signature of the permutation $\delta_q = +1$ for q even and $= -1$ for q odd. The tableau has no more than (n-1) rows. This follows from the fact that it is impossible to antisymmetrize more than n indices each index running from $1 \ldots n$ (${}^nC_r$ for $r > n$ has no meaning), and we restrict ourselves to transformations with determinant 1 so that a column with n boxes is equivalent to 1. Thus, there is one-one correspondence between a regular Young Tableau of no more than (n-1) rows and the I.R.'s of SU(n).

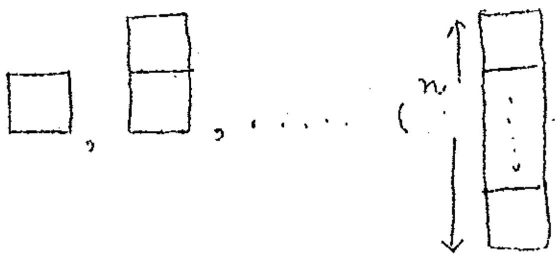
A tableau with zero box (dot) corresponds to the identity representation. The tableau with one box corresponds to the self-representation. Among the other interesting representation we note the following:

(1) The representations with one row only i.e. $f_1 = f$ These correspond to totally symmetric tensors. The dimension of the representation of a tensor with f indices totally symmetry (allowing repetitions) can be computed as

$$N = \binom{n+f-1}{f} = \frac{(n+f-1)!}{f!\,(n-1)!}$$

There are infinite number of such representations.

(2) The representations with one column. Excluding the identity representation, there are (n-1) such representations.



These correspond to totally antisymmetric tensors. If t is the length of the column then the dimension of this representation is given by.

$$N = \binom{n}{t} = \frac{n!}{t!\,(n-t)!}$$

We shall give later the dimension of an arbitrary Young Tableau. For unitary representation, by contragradient a given representation, we mean its complex conjugate. For a given tablean of SU(n) corresponding to a representation A, we get the contragradient a given representation, we mean its complex conjugate. For a given tableau of SU(n) corresponding to a representation A, we get the contragradient representation by the following process.

(1) Draw the initial Young Tableau.

(2) Complete the drawing to obtain a rectangle of horizontal dimension f_1 and vertical dimension n

(3) The complementary part is the desired Young Tableau if one rotates it by π. The procedure can be seen to be equivalent to saying that if $f_1' \geq \ldots \geq f_{n-1}'$ are the rows of the Young Tableau corresponding to the contragradient representation, then

$$f_1' = f_1$$
$$f_2' = f_1 - f_{n-1}$$
$$\vdots$$
$$f_p' = f_1 - f_{n-p+1}$$

A representation is self contragradient if

$$f_p = f_p'$$

so that

$$f_p + f_{n-p+1} = f_1$$
$$p = 1, \ldots, n$$

A representation and its contragradient have the same dimension.

Thus an I.R. of SU(n) can be described by

(1) the components of its highest weight $(\lambda_1, \ldots \lambda_\ell)$ where $\lambda_1 \ldots \lambda_\ell$ are integers. This mean that the given I.R. has been obtained from

$$\underbrace{D^1 \times D^1 \times \ldots \times D^1}_{\lambda_1} \times \underbrace{D^2 \times D^2 \times \ldots \times D^2}_{\lambda_2} \times \ldots \times \underbrace{D^\ell \times D^\ell \times \ldots \times D^\ell}_{\lambda_\ell}$$

where $D^1, \ldots D^\ell$ are the fundamental representations.

(2) By a traceless tensor, symmetric in lower and symmetric in upper indices.

(3) By a standard Young Tableau.

For the special case of SU(3) for which there are only two fundamental representations.

The number of upper indices = λ_1

and the number of lower indices = λ_2

where (λ_1, λ_2) denotes the highest weight. In this case the connection between the f's of Young Tableau and λ's is

$$\lambda_1 = f_1 - f_2$$

$$\vdots$$

$$\lambda_\ell = f_{\ell-1} - f_\ell$$

<u>The Character</u>.

If D, with matrices $D(s)$, is a representation of some group G, then one calls the system of numbers

$$\chi(s) = \operatorname{tr} D(s)$$

the character of the reoresentation. Since the trace is unaltered by equivalent transformations, <u>equivalent representations have the same character</u>. The following formula gives the character of the reoresentation belonging to the Young Tableau $(f_1, f_2, \ldots\ldots, f_{n-1})$ as a symmetric function of the eigen-values $(\epsilon_1, \epsilon_2, \cdots \epsilon_{n})$ in the diagonal form of a general element g of

$$\chi_{f_1 \cdots f_n} = \frac{\begin{vmatrix} \epsilon_1^{f_1+n-1} & \epsilon_1^{f_2+n-2} & \cdots & \epsilon_1^{0} \\ \epsilon_2^{f_1+n-1} & \cdots & & \epsilon_2^{0} \\ \vdots & & & \vdots \\ \epsilon_n^{f_1+n-1} & \cdots & & \epsilon_n^{0} \end{vmatrix}}{\begin{vmatrix} \epsilon_1^{n-1} & \epsilon_1^{n-2} & \cdots & \epsilon_1^{0} \\ \vdots & & & \vdots \\ \epsilon_n^{n-1} & \epsilon_n^{n-2} & \cdots & \epsilon_n^{0} \end{vmatrix}}$$

From this formula one gets the dimension N by letting $\epsilon_1, \ldots \epsilon_n \to 1$ i.e. N is the character of the Identity representation. The limitting problem $\epsilon_1 \ldots \epsilon_n \longrightarrow 1$ should be carefully carried as otherwise $\chi_{f_1 \ldots f_n}$ will become indeterminate. The characters distinguish the IR's. This is true in general, since the characters of different IR's are orthogonal to each other.

First we set

$$\ell_1 = f_1 + n - 1$$
$$\ell_2 = f_2 + n - 1$$
$$\ldots$$
$$\ell_n = f_n + n - 1$$

In order to take proper limit, we relax the condition

$\epsilon_1 \ldots \epsilon_n = 1$ and choose

$$\epsilon_1 = (\epsilon)^{n-1} \quad \ldots \quad \epsilon_n = \epsilon^0 \qquad \epsilon_i = (\epsilon)^{n-i}$$

With $\epsilon \to e^{i\phi}$ and $\phi \to 0$, we have

$$N = \lim_{\phi \to 0} \frac{\begin{vmatrix} (\epsilon^{\ell_1})^{n-1} & (\epsilon^{\ell_2})^{n-1} & \ldots & (\epsilon^{\ell_n})^{n-1} \\ (\epsilon^{\ell_1})^{n-2} & (\epsilon^{\ell_2})^{n-2} & \ldots & (\epsilon^{\ell_n})^{n-2} \\ \vdots & & & \vdots \\ (\epsilon^{\ell_1})^{0} & (\epsilon^{\ell_2})^{0} & \ldots & (\epsilon^{\ell_n})^{0} \end{vmatrix}}{\begin{vmatrix} (\epsilon^{n-1})^{n-1} & (\epsilon^{n-2})^{n-1} & \ldots & (\epsilon^{0})^{n-1} \\ \ldots & & & \\ \vdots & & & \\ (\epsilon^{n-1})^{0} & (\epsilon^{n-2})^{0} & \ldots & (\epsilon^{0})^{0} \end{vmatrix}}$$

Using the result

$$\Delta(x_1 \quad x_n) = \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ & \cdot & & \\ & \cdot & & \\ x_1^0 & x_2^0 & \cdots & x_n^0 \end{vmatrix}$$

$$= \prod_{i<j} (x_i - x_j)$$

and taking into account

$$\left(\epsilon^{l_i} - \epsilon^{l_j}\right) = e^{i\phi(l_i - l_j)}$$

$$\underset{\phi \to 0}{\approx} \quad i\phi \quad (l_i - l_j)$$

we obtain

$$N = \frac{\Delta(l_1, l_2, \quad l_n = 0)}{\Delta(n-1, n-2, \quad 0)}$$

where

$$\Delta(n-1, n-2, \cdots 0) = (n-1)!(n-2)! \quad \cdots \quad 1!$$

The character is very useful when we discuss the direct products of representations as we shall see later.

Direct Products of Representations

Tensor Method: We shall limit ourselves to the discussion of SU(3) while the method is quite general. We have seen earlier that an I.R. is described by a traceless tensor, symmetric in upper indices and symmetric in lower indices. The basic functions for the representation $D(p,q)$ are the components of the traceless, symmetric tensor $T^{\nu_1 \dots \nu_q}_{\mu_1 \dots \mu_p}$. By using tensor multiplication and contraction, the basic functions for product representations may be obtained and reduced. We shall give many examples to illustrate the method.

$D(0,1) \otimes D(1,0)$: The basis for $D(1,0)$ is ϕ_ν while for D(0,1) is ψ^μ. The product $\phi_\nu \psi^\mu$ is reducible since trace is invariant under unitary transformation. We write

$$\psi^\mu \phi_\nu = \left\{ \psi^\mu \phi_\nu - \frac{1}{3} \delta^\mu_\nu \psi^\lambda \phi_\lambda \right\} + \frac{1}{3} \delta^\mu_\nu \psi^\lambda \phi_\lambda$$

$$\mu, \nu, \lambda = 1, 2, 3$$

where the tensor in the bracket provides the basis of the representation $D(1,1)$ and the second term corresponds to the trace. Symbolically, we express this reduction as

$$D(0,1) \otimes D(1,0) = D(1,1) \oplus D(0,0)$$

or in terms of their dimensions,

$$\bar{3} \otimes 3 = 8 \oplus 1$$

(ii) $D(1,0) \otimes D(1,0)$:- The reducible product now is $\phi_\mu \psi_\nu$. The trace has now no meaning and we split this into its symmetric and antisymmetric parts.

$$\phi_\nu \psi_\mu = \frac{1}{2}\left(\phi_\nu \psi_\mu + \phi_\mu \psi_\nu\right) + \frac{1}{2}\left(\phi_\nu \psi_\mu - \phi_\mu \psi_\nu\right)$$

$$= S_{\nu\mu} + A_{\nu\mu}$$

$S_{\nu\mu}$ is already symmetric and it corresponds to the representation $D(2,0)$. However, the antisymmetric part $A_{\nu\mu}$ is equivalent to a tensor with just one index as

$$\epsilon^{\nu\mu\lambda} A_{\nu\mu} \sim T^\lambda$$

Thus the reduction is then

$$D(1,0) \otimes D(1,0) = D(2,0) \oplus D(0,1)$$

or symbolically

$$3 \otimes 3 = 6 \oplus \bar{3}$$

where the bar on 3 denotes that it transforms as a contragradient vector.

(iii) $\underline{D(1,0) \otimes D(1,1)}$: The reducible tensor in this case is $T^{\mu}_{\nu} \phi_{\lambda}$. This, we write as

$$T^{\mu}_{\nu} \phi_{\lambda} = \left[\frac{1}{2} \left(T^{\mu}_{\nu} \phi_{\lambda} + T^{\mu}_{\lambda} \phi_{\nu} \right) - \frac{1}{8} \left(\delta^{\mu}_{\lambda} T^{\alpha}_{\nu} \phi_{\alpha} + \delta^{\mu}_{\nu} T^{\alpha}_{\lambda} \phi_{\alpha} \right) \right]$$

$$+ \left[\frac{1}{2} \left(T^{\mu}_{\nu} \phi_{\lambda} - T^{\mu}_{\lambda} \phi_{\nu} \right) - \frac{1}{4} \left(\delta^{\mu}_{\lambda} T^{\alpha}_{\nu} \phi_{\alpha} - \delta^{\mu}_{\nu} T^{\alpha}_{\lambda} \phi_{\alpha} \right) \right]$$

$$+ \left[\frac{3}{8} \delta^{\mu}_{\lambda} T^{\alpha}_{\nu} \phi_{\alpha} - \frac{1}{8} \delta^{\mu}_{\nu} T^{\alpha}_{\lambda} \phi_{\alpha} \right]$$

The quantity in the first bracket is symmetric in the lower indices and also traceless and so it forms the basis for D(2,1). The quantity in the second bracket is antisymmetric in the lower indices and so it is equivalent to a tensor with an upper index. This along with the other upper index makes it the basis for the I.R. $D(0,2)$. The quantity in the third bracket corresponds to $D(1,0)$. Then

$$D(1,0) \otimes D(1,1) = D(2,1) \oplus D(0,2) + D(1,0)$$

or symbolically.

$$3 \otimes 8 = 15 \oplus \bar{6} \oplus 3 .$$

Let us now take the more difficult case $D(1,1) \otimes D(1,1)$. The reducible tensor in this case is $T^{\mu}_{\nu} \; S^{\alpha}_{\beta}$. Of the possible contractions we notice, that if the two upper indices are contracted with the two lower indices are contracted with the two lower indices we get the trace. For notational convenience, we introduce the following tensors after Okübo,

$$\bar{Q}^{\mu\alpha}_{\nu\beta} = T^{\mu}_{\nu} \; S^{\alpha}_{\beta} + T^{\mu}_{\beta} \; S^{\alpha}_{\nu} + T^{\alpha}_{\nu} \; S^{\mu}_{\beta} + T^{\alpha}_{\beta} \; S^{\mu}_{\nu}$$

with the obvious property

$$\bar{Q}^{\mu\alpha}_{\nu\beta} = \bar{Q}^{\alpha\mu}_{\nu\beta} = \bar{Q}^{\mu\alpha}_{\beta\nu} = \bar{Q}^{\alpha\mu}_{\beta\nu}$$

(symmetry in both upper and lower indices)

$$\bar{R}^{\mu\alpha}_{\nu\beta} = T^{\mu}_{\nu} \; S^{\alpha}_{\beta} + T^{\mu}_{\beta} \; S^{\alpha}_{\nu} - T^{\alpha}_{\nu} \; S^{\mu}_{\beta} - T^{\alpha}_{\beta} \; S^{\mu}_{\nu}$$

This obeys the property

$$\bar{R}^{\mu\alpha}{}_{\nu\beta} = \bar{R}^{\mu\alpha}{}_{\beta\nu} = -\bar{R}^{\alpha\mu}{}_{\nu\beta} = -\bar{R}^{\alpha\mu}{}_{\beta\nu}$$

(symmetry in the lower indices and antisymmetry in the upper indices)

$$\bar{P}^{\mu\alpha}{}_{\nu\beta} = T^{\mu}_{\nu} S^{\alpha}_{\beta} + T^{\alpha}_{\nu} S^{\mu}_{\beta} - T^{\mu}_{\beta} S^{\alpha}_{\nu} - T^{\alpha}_{\beta} S^{\mu}_{\nu}$$

with the property

$$\bar{P}^{\mu\alpha}{}_{\nu\beta} = \bar{P}^{\alpha\mu}{}_{\nu\beta} = -\bar{P}^{\alpha\mu}{}_{\beta\nu} = -\bar{P}^{\mu\alpha}{}_{\beta\nu}$$

(symmetry in the upper indices and antisymmetry in the lower indices)

and lastly

$$\bar{X}^{\mu\alpha}{}_{\nu\beta} = T^{\mu}_{\nu} S^{\alpha}_{\beta} - T^{\alpha}_{\nu} S^{\mu}_{\beta} - T^{\mu}_{\beta} S^{\alpha}_{\nu} + T^{\alpha}_{\beta} S^{\mu}_{\nu}$$

with the property

$$\bar{X}^{\mu\alpha}{}_{\nu\beta} = -\bar{X}^{\alpha\mu}{}_{\nu\beta} = -\bar{X}^{\mu\alpha}{}_{\beta\nu} = \bar{X}^{\alpha\mu}{}_{\beta\nu}$$

However, these tensors are not traceless. We find the corresponding traceless tensors as

$$Q^{\mu\alpha}_{\nu\beta} = \bar{Q}^{\mu\alpha}_{\nu\beta} - \frac{1}{5}\left[\delta^{\mu}_{\nu}\, \bar{Q}^{\lambda\alpha}_{\lambda\beta} + \delta^{\mu}_{\beta}\, \bar{Q}^{\lambda\alpha}_{\nu\lambda} + \delta^{\alpha}_{\nu}\, \bar{Q}^{\mu\lambda}_{\lambda\beta} + \delta^{\alpha}_{\beta}\, \bar{Q}^{\mu\lambda}_{\nu\lambda} \right] + \frac{1}{20}\left[\delta^{\mu}_{\nu}\, \delta^{\alpha}_{\beta} + \delta^{\mu}_{\beta}\, \delta^{\alpha}_{\nu} \right] \bar{Q}^{\lambda\rho}_{\lambda\rho}$$

The upper indices of Q are symmetric and so are the lower indices and it is traceless. It then forms the basis for the I.R. D(2,2) and the dimension of this representation is 27. From $\bar{R}$, we form the traceless tensor,

$$R^{\mu\alpha}_{\nu\beta} = \bar{R}^{\mu\alpha}_{\nu\beta} - \frac{1}{3}\left\{ \delta^{\mu}_{\nu}\, \bar{R}^{\lambda\alpha}_{\lambda\beta} + \delta^{\mu}_{\beta}\, \bar{R}^{\lambda\alpha}_{\nu\lambda} + \delta^{\alpha}_{\nu}\, \bar{R}^{\mu\lambda}_{\lambda\beta} + \delta^{\alpha}_{\beta}\, \bar{R}^{\mu\lambda}_{\nu\lambda} \right\}$$

Then R is symmetric in the lower indices and antisymmetric in the upper two indices. The antisymmetry in the two upper indices can be removed using the $\in$ tensor which adds one more lower index and thus R forms the basis for the I.R. D(3,0). Similarly, we find that from $\overline{P}$ we can construct the following traceless tensors P which forms the basis for the I.R. of D(0,3)

$$
\begin{aligned}
P^{\mu\alpha}_{\nu\beta} = {} & \overline{P}^{\mu\alpha}_{\nu\beta} \\
& - \frac{1}{3}\Big\{ \delta^{\mu}_{\nu}\, \overline{P}^{\lambda\alpha}_{\lambda\beta} + \delta^{\mu}_{\beta}\, \overline{P}^{\lambda\alpha}_{\nu\lambda} \\
& \quad + \delta^{\alpha}_{\nu}\, \overline{P}^{\mu\lambda}_{\lambda\beta} + \delta^{\alpha}_{\beta}\, \overline{P}^{\mu\lambda}_{\nu\lambda} \Big\}
\end{aligned}
$$

The tracelss tensor for $\overline{X}^{\mu\alpha}_{\nu\beta}$ is

$$
\begin{aligned}
X^{\mu\alpha}_{\nu\beta} = {} & \overline{X}^{\mu\alpha}_{\nu\beta} - \Big[\delta^{\mu}_{\nu}\, \overline{X}^{\lambda\alpha}_{\lambda\beta} + \delta^{\mu}_{\beta}\, \overline{X}^{\lambda\alpha}_{\nu\lambda} \\
& \quad + \delta^{\alpha}_{\nu}\, \overline{X}^{\mu\lambda}_{\lambda\beta} + \delta^{\alpha}_{\beta}\, \overline{X}^{\mu\lambda}_{\nu\lambda} \Big] \\
& + \frac{1}{2}\Big[\delta^{\mu}_{\nu}\, \delta^{\alpha}_{\beta} - \delta^{\mu}_{\beta}\, \delta^{\alpha}_{\nu} \Big] \overline{X}^{\lambda\rho}_{\lambda\rho}
\end{aligned}
$$

We now note that since X is antisymmetric in the upper indices, and also in the lower indices, by using the ϵ tensor we can show that this is equivalent to a tensor with one upper index and one lower index f^{λ}_{ρ} and since X is traceless, we can also show that f^{λ}_{ρ} is traceless. Thus the complete reduction of $T^{\mu}_{\nu}\, S^{\alpha}_{\beta}$ into its irreducible constituents can be written as

$$T^{\mu}_{\nu}\, S^{\alpha}_{\beta} = \frac{1}{4}\left[\bar{Q}^{\mu\alpha}_{\nu\beta} + \bar{R}^{\mu\alpha}_{\nu\beta} + \bar{P}^{\mu\alpha}_{\nu\beta} + \bar{X}^{\mu\alpha}_{\nu\beta}\right]$$

$$= \frac{1}{4}\left[Q^{\mu\alpha}_{\nu\beta} + R^{\mu\alpha}_{\nu\beta} + P^{\mu\alpha}_{\nu\beta}\right]$$

$$+ \Big\{ -\frac{1}{5}\,\delta^{\mu}_{\nu}\left[(ST)^{\alpha}_{\beta} + (TS)^{\alpha}_{\beta}\right] - \frac{1}{5}\,\delta^{\alpha}_{\beta}\left[(ST)^{\mu}_{\nu} + (TS)^{\mu}_{\nu}\right] + \frac{3}{10}\,\delta^{\mu}_{\beta}\left[(ST)^{\alpha}_{\nu} + (TS)^{\alpha}_{\nu}\right] + \frac{3}{10}\,\delta^{\alpha}_{\nu}\left[(ST)^{\mu}_{\beta} + (TS)^{\mu}_{\beta}\right]\Big\}$$

$$+ \frac{1}{6}\Big\{ \delta^{\mu}_{\beta}\left[(ST)^{\alpha}_{\nu} - (TS)^{\alpha}_{\nu}\right] + \delta^{\alpha}_{\nu}\left[(ST)^{\mu}_{\beta} - (TS)^{\mu}_{\beta}\right]\Big\}$$

$$- \frac{1}{40}\langle TS\rangle \left\{ 11\,\delta^{\mu}_{\beta}\,\delta^{\alpha}_{\nu} - 9\,\delta^{\mu}_{\nu}\,\delta^{\alpha}_{\beta}\right\}$$

where

$$(TS)^{\alpha}_{\beta} = T^{\alpha}_{\lambda} \; S^{\lambda}_{\beta}$$

$$\langle TS \rangle = T^{\rho}_{\lambda} \; S^{\lambda}_{\rho}$$

The quantity $\left\{ (ST)^{\alpha}_{\beta} + (TS)^{\alpha}_{\beta} \right\}$, corresponds to the symmetric octet representatiom (symmetric under the exchange $T \leftrightarrow S$) and $\left\{ (ST)^{\alpha}_{\beta} - (TS)^{\alpha}_{\beta} \right\}$ corresponds to the antisymmetric octet representation. These seems to be no operation within the group which could distinguish these two octet representations. The complete reduction now is

$$\begin{aligned} D(1,1) \otimes D(1,1) = D(2,2) &\oplus D(3,0) \oplus D(0,3) \\ &\oplus D_s(1,1) \oplus D_a(1,1) \\ &\oplus D(0,0) \end{aligned}$$

or symbolically.

$$8 \otimes 8 = 27 \oplus 10 \oplus \overline{10} \oplus 8_s \oplus 8_a \oplus 1$$

We here notice two important differences between SU(2) and SU(3). In SU(2), the tensor ϵ^{ij} makes ϕ_i and ϕ^j equivalent while these are inequivalent in the case of SU(3). Secondly in the reduction above we found that the octet representation occurs twice in the case of SU(3). This feature is totally absent in the case of SU(2), and we call this property of SU(2) as simple reducibility.

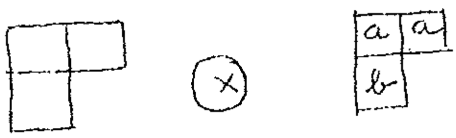
These two differences offer most of the troubles when we want to find the generalized Cleblish-Gordan coefficients. This we shall discuss later.

The Method of reduction using Young Tableau.

Without going to the proof, we shall describe the method used in actual computation. Suppose we are given two I.R's (Recall an earlier theorem that a standard Young Tableau characterizes an I.R. uniquely). We follow the following procedure.

Step 1:- Draw the Young Tableau corresponding to these I.R.'s and label one of them with indices such that all the boxes in the same row are given the same index. In practice, it is wiser to label the Tableau which has the least number of boxes.

Example: $D(1,1) \otimes D(1,1)$ of $SU(3)$



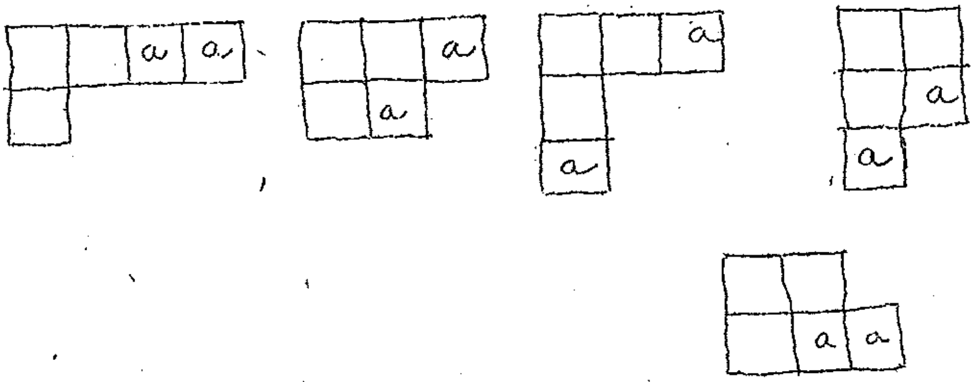
If the representation is labelled as $D(\lambda_1 \ldots \lambda_\ell)$ where $(\lambda_1 \ldots \lambda_\ell)$ are the components of the highest weight then

$$\lambda_1 = f_1 - f_2, \ \ldots \ \lambda_\ell = f_\ell - f_{\ell+1}$$

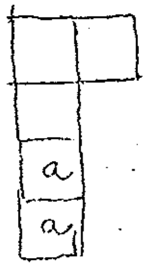
where $(f_1 \ldots f_{\ell+1})$ denote the number of bixes in the first row, $\ldots, (\ell+1)$th row of the corresponding Young Tableau. So we see for SU(3) $\mathcal{D}(\lambda_1=1, \lambda_2=1)$ corresponds to the Young Tableau with $f_1 = \lambda_1 + \lambda_2 = 2$ and $f_2 = 1 = \lambda_2$. This correspondence is understood in all our discussions.

Step 2:- Attach the indices in the first row on the Young Tableau of the other representation such that only standard tableau are retained and that two of these indices do not occur in the same column.

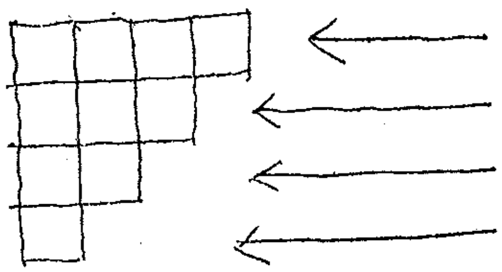
Ex:- In the example given above we see the allowed tableau are



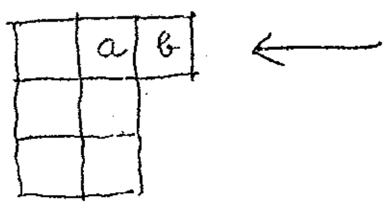
Notice we have omitted the non-standard tableau and also the one in which two a's occur in the same column



Then in these resulting tableau attach the indices in the second row such that we again retain only the standard tableau and with no two indices of the second row occurring in the same column. One more important caution is that when we go in the direction indicated below

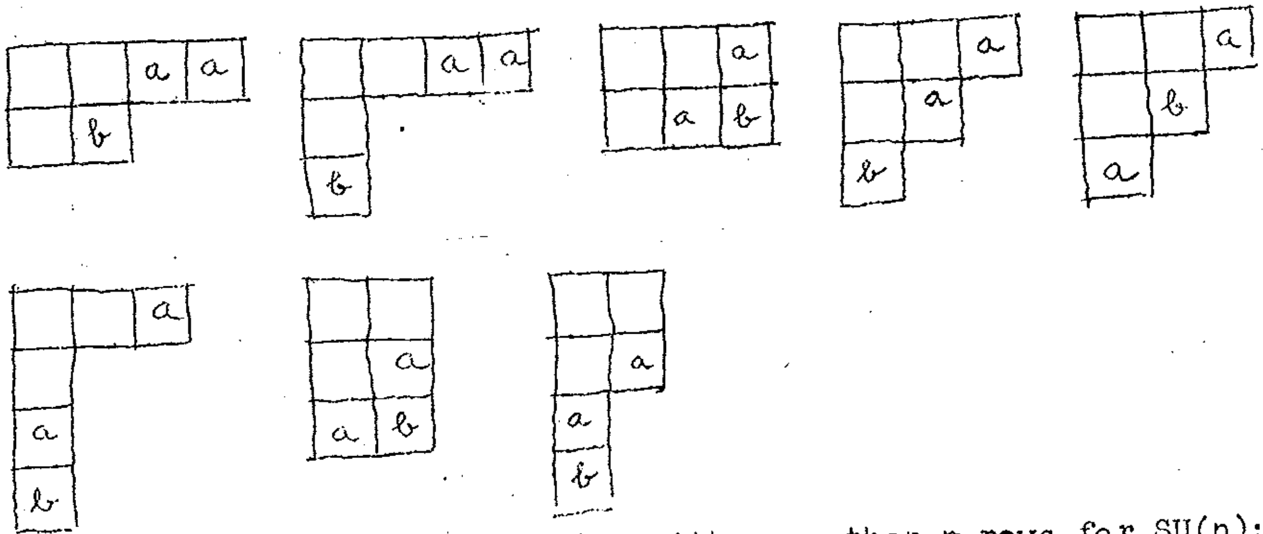


of the resulting tableau, at no stage, there should be more b's than a's, more c's than b's and so on. An example of a tableau which is not allowed is

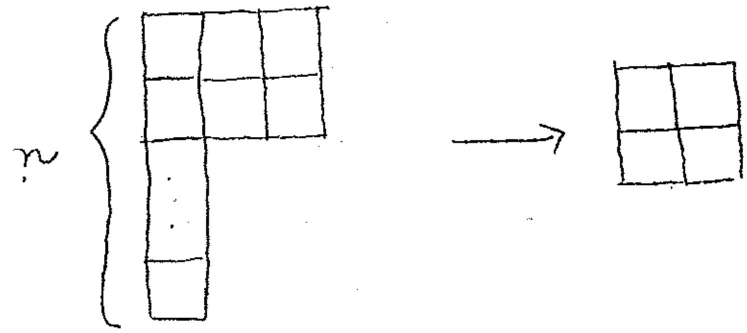


since when one goes in the direction indicated, b is seen first before a. This is a very important rule.

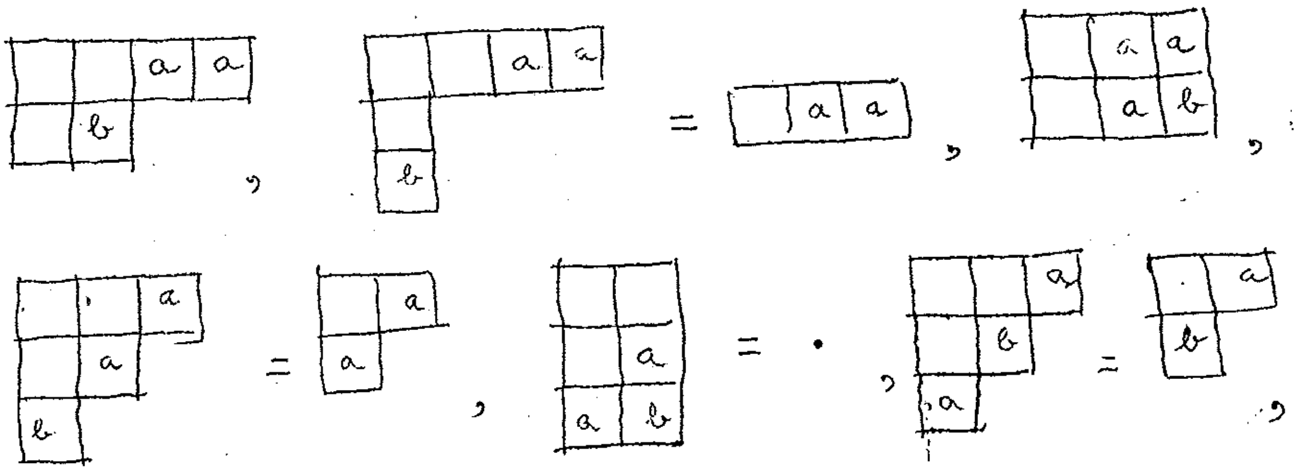
In exactly the same way the indices in the third row and so on are attached. In the example we gave, we get the following



Step III. Omit all the tableau with more than n rows for SU(n): Omit in a given tableau, column of length n. Retain only the rest. This means

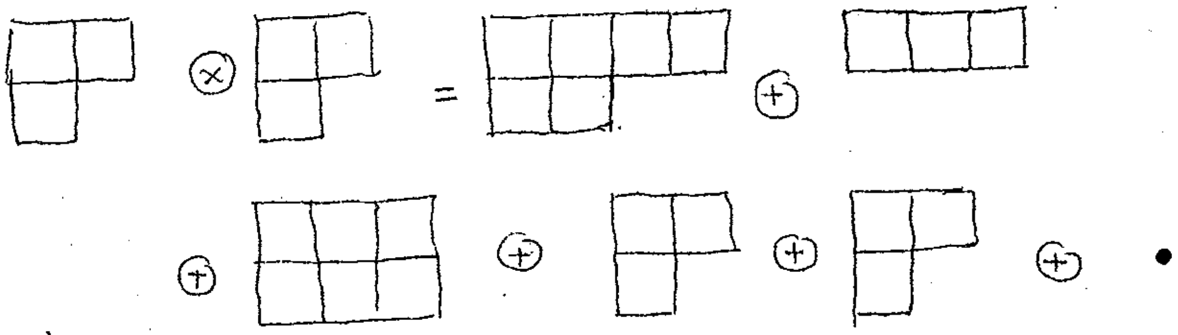


In the example above for SU(3), the only tableau retained are



and Notice that we have omitted tableau with more than three rows

so that



Step IV: Labelling:- Now read the resulting tableau in the highest weight notation. i.e. the number of columns with one box λ_1, the number of columns with two boxes λ_2 and soon. Thus in our example

$$D(1,1) \otimes D(1,1) = D(2,2) \oplus D(3,0) \oplus D(0,3)$$
$$\oplus D(1,1) \oplus D(1,1) \oplus D(0,0)$$

Step V: Use the formula for the dimension of the representation

$$D(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$$

$$N = (1+\lambda_1) \cdots\cdots (1+\lambda_\ell)$$
$$\left(1+\frac{\lambda_1+\lambda_2}{2}\right) \quad \left(1+\frac{\lambda_2+\lambda_3}{2}\right) \cdots\cdots \left(1+\frac{\lambda_{\ell-1}+\lambda_\ell}{2}\right)$$
$$\left(1+\frac{\lambda_1+\lambda_2+\lambda_3}{3}\right) \left(1+\frac{\lambda_2+\lambda_3+\lambda_4}{3}\right) \cdots \left(1+\frac{\lambda_{\ell-2}+\lambda_{\ell-1}+\lambda_\ell}{3}\right)$$
$$\cdots\cdots$$
$$\left(1+\frac{\lambda_1+\cdots+\lambda_\ell}{\ell}\right)$$

In our example SU(3)

$$N = (1+\lambda_1)\ (1+\lambda_2)\ \left(1+\frac{\lambda_1+\lambda_2}{2}\right)$$

so that

$$8 \otimes 8 = 27 \oplus 10 \oplus \overline{10} \oplus 8 \oplus 8 \oplus 1$$

The method of Young Tableau is very practical when one goes to higher rank group.

Method of Characters:

Here again, let us demonstrate the method using some examples in SU(3). For a triplet representation (the matrices are 3 x 3)

$$\chi_3(\epsilon_1, \epsilon_2, \epsilon_3) = \epsilon_1 + \epsilon_2 + \epsilon_3$$

where ϵ_1, ϵ_2 and ϵ_3 are the components in the diagonal form of the element α. For the representation $\overline{3}$

$$\chi_{\overline{3}}(\alpha) = \epsilon_1^* + \epsilon_2^* + \epsilon_3^*$$
$$= \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} + \frac{1}{\epsilon_3}$$

since $\epsilon^* \epsilon = 1$ for $|\epsilon| = 1$

Consider next the octet representation $D(1,1)$, it could be computed as

$$\chi_8(\epsilon_1, \epsilon_2, \epsilon_3) = \frac{\epsilon_1}{\epsilon_2} + \frac{\epsilon_1}{\epsilon_3} + \frac{\epsilon_2}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_3} + \frac{\epsilon_3}{\epsilon_1} + \frac{\epsilon_3}{\epsilon_2} + 2$$

In terms of the characters (which are just numbers), the direct product operation becomes ordinary multiplication and direct sum reduces to ordinary sum. As an example consider the product $3 \otimes \bar{3}$.

$$\chi_3 \chi_{\bar{3}} = (\epsilon_1 + \epsilon_2 + \epsilon_3)(\epsilon_1^* + \epsilon_2^* + \epsilon_3^*)$$

$$= \epsilon_1 \epsilon_1^* + \epsilon_2 \epsilon_2^* + \epsilon_3 \epsilon_3^*$$

$$+ \frac{\epsilon_1}{\epsilon_2} + \frac{\epsilon_1}{\epsilon_3} + \frac{\epsilon_2}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_3} + \frac{\epsilon_3}{\epsilon_1} + \frac{\epsilon_3}{\epsilon_2}$$

$$= \left\{ 2 + \frac{\epsilon_1}{\epsilon_2} + \frac{\epsilon_1}{\epsilon_3} + \frac{\epsilon_2}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_3} + \frac{\epsilon_3}{\epsilon_1} + \frac{\epsilon_3}{\epsilon_2} \right.$$

$$+ 1$$

$$= \chi_8 + \chi_1$$

so that

$$3 \otimes \bar{3} = 8 \oplus 1$$

We have given earlier Weyl's character formula to compute the character of any representation.

Invariants of SU(ℓ+1):

We have earlier defined Casimir forms. A Casimir operators are functions of generators $f(x_1 \ldots x_{n^2-1})$ which commute with all the generators of the algebra

$$\left[f_i (X_1 \ldots X_{n^2-1}) \, , \, X_\alpha \right] = 0 \qquad \text{for all } \alpha .$$

It has been shown by many people (Okubo, Biedenharn, Umezawa. M and Gruber and Raifeartaigh) that there are ℓ independent Casimir operators for a group of rank ℓ. For the group $SU(\ell+1)$ the explicit construction of these operators is as follows: If $\overset{S}{X}_\alpha, \ldots \overset{S}{X}_{n^2-1}$ are the generators of the group SU(n) in the self-representation then

$$I_\ell = Tr \left(\sum_\alpha X^\alpha \; \overset{S}{X}_\alpha \right)^\ell$$

where the trace is taken with respect to the representation $\overset{S}{X}_\alpha$. For the special case of SU(3), there are two Casimir operators which are constructed as follows. The eight matrices in the self representation are given by

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_4 = E_1^\dagger$$

$$E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_5 = E_2^\dagger$$

$$E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_6 = E_3^\dagger$$

or

$$E_{\pm 1} = \frac{E_1 \pm i E_2}{\sqrt{2}}, \quad E_{\pm 2} = \frac{E_3 \pm i E_4}{\sqrt{2}}$$

and

$$E_{\pm 3} = \frac{E_5 \pm i E_6}{\sqrt{2}}$$

The Casimir operators are

$$F^2 = \mathrm{Tr} \begin{pmatrix} H_1 + H_2 & E_1 & E_2 \\ E_4 & H_1 - H_2 & E_3 \\ E_5 & E_6 & -2H_2 \end{pmatrix}^2$$

and

$$G^3 = \mathrm{Tr} \begin{pmatrix} H_1 + H_2 & E_1 & E_2 \\ E_4 & H_1 - H_2 & E_3 \\ E_5 & E_6 & -2H_2 \end{pmatrix}^3$$

Since $E_4 = E_1^{\dagger}$, $E_5 = E_2^{\dagger}$ and $E_6 = E_3^{\dagger}$, the eigen-values of these operators can be easily computed. It can also be verified that these are the only two independent 'invariants'. This we shall demonstrate for the case of SU(3) when we discuss the '<u>mass formula</u>'.

GENERATING FUNCTIONS OF CLASSICAL GROUPS AND EVALUATION OF PARTITION FUNCTIONS

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INTRODUCTION

The Clebsch-Gordan (C.G.) programme of classical groups suffers from two major difficulties. Unlike the rotation group in three dimensions for which the C.G. programme is well known, many other classical groups do not possess the properties of simple reducibility and the equivalence of an irreducible representation (I.R.) and its conjugate. Here, we mean by the lack of simple reducibility, the multiple occurrance of an I.R. in the product of two I.R's. This multiplicity is called the external multiplicity [1]. However, many relations have been worked out [2], [3], which relate this external multiplicity to the multiple occurrance of a given weight in an I.R. This multiple occurrance of a given weight in an I.R., a feature not shared by the I.R's of O(3), is called the internal multiplicity structure.

At present the internal multiplicity structure can be worked out using Kostant's formula [4]. There exist, however, many other methods (for instance, the recursion method of Fraudenthal [5]), although in practice, Kostant's formula is the most useful. Kostant's formula involves the partition function of expressing a non-negative integral linear combination of positive roots in terms of a non-negative integral linear combination of primitive roots. These partition functions have been known so far only for rank two and three groups [6].

Recently we have developed a method [7] of obtaining the partition functions for $A_\ell (\sim SU(\ell+1))$ using the generating functions. In this, we set up recursion relations for the partition functions, which are then used in conjunction with Kostant's formula to compute the internal multiplicities. Of course, the calculation gets more and more involved as one goes to large ℓ. However, the method is precise.

In this paper, we work out the generating functions for A_ℓ, B_ℓ, C_ℓ, D_ℓ and G_2. The calculations for the other exceptional groups F_4, E_6, E_7 and E_8 will be published elsewhere. We also obtain recursion relations for the internal multiplicity.

In Section 2, the general discussion of Kostant's formula is given. We discuss the cases of $A_\ell \sim SU(\ell+1)$, $B_\ell \sim O(2\ell+1)$, $C_\ell \sim (Sp_{2\ell})$, $D_\ell \sim O(2\ell)$ and G_2 in sections (3)-(7). The discussion includes the Weyl group, the structure of positive and primitive (simple) roots and the Diophantine equations. Explicit formulae are obtained and possible recursion relations for the partition functions are given. In Sec.(8), the connection between internal and external multiplicity structures is discussed. In Sec.(9), the conclusions are given. Many of the properties of the classical groups (structure of positive and primitive roots and so on) are contained in many places. We have taken them from the papers of Dynkin [8].

2. Kostant's formula

The inner multiplicity $M^{m}(m')$ of a weight m' belonging to the irreducible representation D(m) of highest weight m is given by Kostant's formula [4] which is

$$M^{m}(m') = \sum_{S \in W} \delta_S \; P\left[S(m+R_o) - (m'+R_o) \right] , \qquad (2.1)$$

where W is the Weyl group and R_o is half the sum of positive roots $\delta_S = \pm 1$ according as whether the reflection is even or odd respectively. P(M) is the partition function for the weight M. This is the number of ways the weight M can be written as a sum over all the positive roots

$$M = \sum_{i=1}^{n} a_i \varphi_i \quad , \qquad (2.2)$$

with different non-negative integers a_i . On the other hand, Antoine and Speiser [9] have shown that the vector

$$S(m+R_o) - (m'+R_o)$$

can be expressed for a fixed $S \in W$ uniquely in terms of the primitive roots as

$$S(m+R_o) - (m'+R_o) = \sum_{i=1}^{l} k_i \cdot \beta_i \, , \qquad (2.3)$$

ℓ being the rank of the group. From (2.2) and (2.3), it is clear that P(M) is the number of ways we can write

$$\sum_{i=1}^{\ell} k_i \cdot \beta_i = \sum_{\mu=1}^{n} a_\mu \varphi_\mu \tag{2.4}$$

$$k_i \geq 0, \quad a_\mu \geq 0$$

k_i and a_μ are integers

for given k_i. It can be shown that $P(k_1, \ldots k_\ell)$ is the multiplicity $\overline{M}(\gamma)$ of a vector γ of $\frac{1}{\Delta}$ [9] where the $\frac{1}{\Delta}$ is related to the character by Weyl's formula

$$\chi^m(\xi) = \frac{X(m+R_0)}{\Delta} \tag{2.5}$$

$$\Delta = X(R_0)$$

$X(m+R_0)$ is the alternating elementary sum

$$X(m+R_0) = \sum_{S \in W} \delta_S \exp\left[S(m+R_0), \xi\right] \tag{2.6}$$

where ξ are the coordinates of the toroid (the group parameters). Hence (2.1) can be written as

$$M^m(m') = \sum_{S \in W} \delta_S \quad \overline{M}(k_1^S, \ldots k_\ell^S).$$

If we can calculate the partition function $\overline{M}(k_1^s, \ldots k_\ell^s)$, then $M^m(m')$ can be computed in principle. In the following few sections, we shall explicitly calculate $\overline{M}(k_1^s, \ldots k_\ell^s)$ for the various classical groups.

3. $A_\ell (\sim SU(\ell+1))$

The roots of this algebra are given by $e_i - e_j$, $i, j = 1, \ldots (\ell+1)$.. The e_i form an orthogonal basis in $(\ell+1)$ dimensional space in which the roots and weights are defined. There are $\ell(\ell+1)$ roots. The $\frac{1}{2}\ell(\ell+1)$ positive roots are then obtained as $e_i - e_j$ $(i<j)$. The primitive (simple) roots in this case are $\beta_i = e_i - e_{i+1}$, $i = 1, \ldots \ell$.

Equation (2.4) then can be written as

$$k_i = C_{i\mu} a_\mu \qquad \begin{array}{l} i = 1, \ldots \ell \\ \mu = 1, \ldots, \frac{1}{2}\ell(\ell+1) \end{array} \tag{3.1}$$

where $\underset{\sim}{C}$ is the $\left(\frac{1}{2}\ell(\ell+1) \times \ell\right)$ dimensional rectangular matrix

$$C_{i\mu} = \begin{array}{c} \mu \longrightarrow 1, \ldots, \frac{1}{2}\ell(\ell+1) \\ i = 1, \ldots \ell \downarrow \left(\begin{array}{ccc|ccc|ccc|c|c} 1 & 0 \cdots & 0 & 1 & 0 \cdots & 0 & 1 & 0 \cdots & 0 & \cdots & 1 \\ 0 & 1 \cdots & 0 & 1 & 1 \cdots & 0 & 1 & 1 \cdots & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 1 \cdots & 0 & 1 & 1 \cdots & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \cdots & 0 & \cdot & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots & 0 & & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & & 1 \end{array} \right) \end{array} \tag{3.2}$$

It can easily be seen that only for the case of $\ell = 2$, the matrix C is a non-singular square matrix so that there is a unique solution i.e. $\overline{M}(k_1) = 1$. However, in general C is a rectangular matrix and so given the vector k and the matrix C, the number of a's is trivially infinite and it is only because we have the restriction that the elements of the matrix C are non-negative integers the very question of the number of solutions (number of a's, the components of the vector a are again non-negative integers) makes a meaning after all[1]. We recognise that the number of solution of Eq.(2.4) is given by the coefficient of $x_1^{k_1} x_2^{k_2} \ldots\ldots x_\ell^{k_\ell}$ of the generating functions. To solve the diophantine equations (3.1), (actually we mean finding the number of solutions for given k and C) we now use the method of generating functions [7]. Let $f(x_1, \ldots\ldots x_\ell)$

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1) I am grateful to Professor Ramakrishnan for focussing my attention to this general problem. There is a discussion about such a matrix equation in the book on 'Linear Differential Operators' by Cornelus, Lanczas, D.Van Nostrand Company Limited (London) (1961), p.115. However, the general problem of finding the number of solutions seems to remain open, although the generating function method we have developed in principle gives a solution to this problem.

be the generating function defined by

$$f_\ell(x_1,\dots,x_\ell) = \prod_{i=1}^{\frac{1}{2}\ell(\ell+1)} \frac{1}{\left(1 - x_1^{c_{1i}} x_2^{c_{2i}} \dots x_\ell^{c_{\ell i}}\right)} \tag{3.3}$$

$x_1, \dots, x_\ell$ are chosen arbitrary parameters with modulus less than one. $\overline{M}(k_1,k_2,\dots,k_\ell)$ is now given by the coefficient of $x_1^{k_1}\dots x_\ell^{k_\ell}$ in $f_\ell(x_1,\dots,x_\ell)$. This can be checked by actually expanding $f_\ell(x_1,\dots x_\ell)$ in power series. Since the matrix C is known, we can write the following important relation

$$f_\ell(x_1,\dots,x_\ell) = \left\{\prod_{i=1}^{\ell}\left(1 - \prod_{j=\ell-i+1}^{\ell} x_j\right)\right\}^{-1} f_{\ell-1}(x_1,\dots x_{\ell-1}) \tag{3.4}$$

Now, we can expand (3.4) in power series. $\overline{M}(k_1,\dots,k_\ell)$ is the coefficient of $x_1^{k_1}x_2^{k_2}\dots x_\ell^{k_\ell}$ in (3.4). If $\overline{M}(k_1,\dots k_{\ell-1})$ the coefficient of $x_1^{k_1},\dots\dots x_{\ell-1}^{k_{\ell-1}}$ in $f_{\ell-1}(x_1,\dots,x_{\ell-1})$ then it is easily seen that

$$\overline{M}(k_1,\dots,k_\ell) = \sum_{r_{\ell-1}=0}^{\infty}\dots\sum_{r_2=0}^{\infty}\sum_{r_1=0}^{\infty}\overline{M}(k_1-r_1;\,k_2-r_1-r_2;\,\dots\, k_{\ell-1}-r_1-r_2-\dots-r_{\ell-1}) \tag{3.5}$$

with

$$0 \le r_1 \le k_1\,; \quad 0 \le r_1+r_2 \le k_2\,; \;\dots\dots$$

$$0 \le r_1+r_2+\dots+r_{\ell-1} \le k_{\ell-1}\,.$$

and $$r_1 + r_2 + \dots + r_\ell = k_\ell ,$$

so that $$0 \leq r_1 + r_2 + \dots + r_{\ell-1} \leq \min(k_\ell, k_{\ell-1}) .$$

Define a new set of variables

$$i_1 = r_1 ; \quad i_2 = r_1 + r_2 ; \quad \dots \quad i_{\ell-1} = r_1 + r_2 + \dots + r_{\ell-1} , \qquad (3.6)$$

then

$$\bar{M}'(k_1, \dots k_\ell) = \sum_{i_{\ell-1} = i_{\ell-2}}^{\min(k_{\ell-1}, k_\ell)} \sum_{i_{\ell-2} = i_{\ell-3}}^{k_{\ell-2}} \dots \sum_{i_2 = i_1}^{k_2} \sum_{i_1 = 0}^{k_1}$$

$$\bar{M}(k_1 - i_1 ; k_2 - i_2 ; \dots ; k_{\ell-1} - i_{\ell-1}) . \qquad (3.7)$$

Eq.(3.7) is exactly the recursion relation we want since it facilitates the computation of the partition function for any A_ℓ (ℓ arbitrary) in terms of the simple partition function for A_2, viz.

$$\bar{M}(k_1, k_2) = \sum_{0}^{\min(k_1, k_2)} 1 = 1 + \min(k_1, k_2) \qquad (3.8)$$

which has been obtained earlier [10]. The weight space is again (ℓ+1) dimensional with the condition on the components of a weight m,

$$\sum_{i=1}^{\ell+1} m_i = 0 .$$

Using Weyl's theorems, it can be proved that the components are (integer)/(ℓ +1). The Weyl group in this case permutes the components of m and is of order (ℓ +1)! The dominant weights satisfy

$$m_1 \geq m_2 \geq \cdots\cdots \geq m_{\ell+1} , \quad \sum_{i=1}^{\ell+1} m_i = 0 . \qquad (3.9)$$

These properties of the dominant weight will be used in picking up the non-vanishing contributions to $M^m(m')$

4. B_ℓ ($\sim O_{2\ell+1}$)

The roots of this algebra are $\pm (e_i \pm e_j)$, $\pm e_i (i=1,\ldots \ell)$ There are $2\ell^2$ of them. The ℓ^2 positive roots may be obtained as $e_i - e_j$, $e_i + e_j$ and e_i $(i<j)$. The simple roots in this case are given by $\beta_{i-1} = e_{i-1} - e_i$, $\beta_\ell = e_\ell$.

Equation (24) then takes the form

$$k_i = C_{i\mu} \, a_\mu , \qquad i = 1, \ldots \ell , \quad \mu = 1, \ldots \ell^2 . \qquad (4.1)$$

where C is the $(\ell^2 \times \ell)$ dimensional rectangular matrix

$$
\begin{array}{cc}
 & \mu \rightarrow 1, \ldots\ldots \ell^2 \\
C_{i\mu} = \underset{i=1,\ldots\ell}{\downarrow} & \left(\begin{array}{c|cccc|cccc|cc|c}
 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdot & 0 & & & 0 \\
 & 1 & 1 & \cdots & 2 & 1 & 1 & \cdot & 1 & \vdots & \vdots & 0 \\
 & 1 & 1 & \cdot & 2 & 1 & 1 & \cdot & 2 & \vdots & \vdots & 0 \\
C^{A_\ell} & \vdots & \vdots & & & \vdots & \vdots & & \vdots & & & \vdots \\
\xleftrightarrow{\qquad\quad} & \vdots & \vdots & & & \vdots & \vdots & & \vdots & & & \vdots \\
\frac{1}{2}\ell(\ell+1) & 1 & 1 & & 2 & 1 & 1 & & 2 & & & 0 \\
 & 1 & 2 & & 2 & 1 & 2 & & 2 & & & 1 \\
 & 2 & 2 & & 2 & 2 & 2 & & 2 & & & 2
\end{array} \right)
\end{array} \tag{4.2}
$$

The generating function in this case is

$$
f_\ell^{B_\ell}(x_1, \ldots x_\ell) = \prod_{i=1}^{\ell^2} \frac{1}{\left(1 - x_1^{C_{1i}} x_2^{C_{2i}} \cdots x_\ell^{C_{\ell i}}\right)} \tag{4.3}
$$

It can be easily checked that unlike the case of A_ℓ , there is no simple recursion relation between $f_\ell^{B_\ell}$ and $f_\ell^{B_{\ell-1}}$. However, the following very interesting relation can be obtained, which of course is obvious from the structure of the C-matrix eq.(4.2)

$$
f_\ell^{B_\ell}(x_1, \ldots x_\ell) = \frac{f_\ell^{A_\ell}(x_1, \ldots x_\ell)}{\prod_{i=2}^{\ell} \prod_{j=0}^{\ell-i} \left(1 - \prod_{k=i-1}^{\ell} x_k \prod_{r=\ell-j}^{\ell} x_r\right)} \tag{4.4}
$$

It is therefore clear that for large values of ℓ the recursion relation Eq.(4.4) is not simple. For ℓ =2, Eq.(4.4) reads as

$$f_2^{B_2}(x_1,x_2) = \frac{f_2^{A_2}(x_1,x_2)}{(1-x_1x_2^2)} , \tag{4.5}$$

so that the recursion relation for $\overline{M}$ is

$$\overline{M}^{B_2}(k_1,k_2) = \sum_i \overline{M}^{A_2}(k_1-i\,;\,k_2-2i) \tag{4.6}$$

which is the relation obtained by Gruber and Zaccaria earlier [11].

The weight space is ℓ dimensional and the components may be integers or half integers. The Weyl group in this case consists of all possible permutation of the components of m together with all possible changes of sign and is therefore of order $2^\ell\,\ell!$. The dominant weights satisfy

$$m_1 \geqslant m_2 \geqslant \ldots\ldots \geqslant m_\ell \geqslant 0 . \tag{4.7}$$

5. C_ℓ ($\sim Sp(2\ell)$)

The roots of this algebra are $\pm(e_i \pm e_j)$, $\pm 2e_i$ $(i=1,\ldots \ell)$. It should be stressed that the factor 2 in the second class of roots is very important and makes this algebra different from B_ℓ .

There are $2\ell^2$ roots. The ℓ^2 positive roots are given by $e_i - e_j$, $e_i + e_j$, $2e_i$ $(i<j)$. The simple roots in this case are $\beta_{i-1} = e_{i-1} - e_i$ $(i = 1, \ldots\ldots \ell)$, $\beta_\ell = 2e_\ell$.

Eq.(2.4) is then

$$\kappa_i = C_{i\mu}\, a_\mu \qquad \begin{array}{l} i = 1, \ldots \ell \\ \mu = 1, \ldots \ell^2 \end{array} \tag{5.1}$$

where C is the $(\ell^2 \times \ell)$ dimensional rectangular matrix

$$C_{i\mu} = \begin{array}{cc} & \mu \to 1, \ldots \ell^2 \\ \downarrow i = 1, \ldots \ell & \left(\begin{array}{c|ccc|ccc|ccc} C^{A_\ell} & 1 & 1 \cdots & 2 & 0 & \cdot & 0 & \cdot & \cdot & 0 \\ & 1 & 1 \cdots & 2 & 1 & \cdot & 2 & \cdot & \cdot & 0 \\ \xleftrightarrow{\frac{1}{2}\ell(\ell+1)} & 1 & 1 & 2 & 1 & \cdot & 2 & \cdot & \cdot & 0 \\ & \vdots & \vdots & & & & & & & \vdots \\ & 1 & 2 & 2 & 1 & & 2 & & & 0 \\ & 2 & 2 & 2 & 2 & & 2 & & & 2 \\ & 1 & 1 & 1 & 1 & & 1 & & & 1 \end{array} \right) \end{array} \tag{5.2}$$

The generating function is of the same type of $f_\ell^{B_\ell}(x_1, \ldots x_\ell)$, but the elements of C are different in view of Eq.(5.2). Again in this case, there is no simple recursion relation between $f_\ell^{C_\ell}$

and $f_{\ell-1}^{C_{\ell-1}}$. However, the following relation can be easily verified.

$$f_{\ell}^{C_\ell}(x_1,\dots x_\ell) = \frac{f_\ell^{A_\ell}(x_1,\dots x_\ell)}{\prod_{i=1}^{\ell}\prod_{j=1}^{\ell-i}\left(1-\prod_{k=i}^{\ell} x_k \prod_{r=\ell-j}^{\ell-1} x_r\right)} \tag{5.3}$$

For the special case of ℓ =2, the above relation reads as

$$f_2^{C_2}(x_1,x_2) = \frac{f_2^{A_2}(x_1,x_2)}{(1-x_1^2x_2)} \tag{5.4}$$

so that the same relation (4.6) is derived with $k_1 \longleftrightarrow k_2$

$$\overline{M}^{C_2}(k_1,k_2) = \sum_i \overline{M}(k_1-2i\,;\,k_2-i) \tag{5.5}$$

This is not surprising because of the known isomorphism between C_2 and B_2.

The weight space is again ℓ-dimensional and the components of the weight one integers. The Weyl group is the same as that for B_ℓ and is of order $2^\ell\,\ell!$. This consists of all the permutations of the components of the weight and all changes in sign. The dominant weight satisfies

$$m_1 \geq m_2 \geq \dots \geq m_\ell \geq 0 \tag{5.6}$$

6. $D_\ell (\sim O(2\ell))$

The roots are given by $\pm(e_i \pm e_j)$ $i,j = 1,\ldots \ell$ and there are $2(\ell^2 - \ell)$ of them. The $\ell(\ell-1)$ positive roots are then $e_i + e_j$ and $e_i - e_j$ $(i<j)$. The simple roots are

$\beta_{i-1} = e_{i-1} - e_i$, $(i=1,\ldots\ell)$ and $\beta_\ell = e_{\ell-1} + e_\ell$ Eq.(2.4)

is then

$$k_i = C_{i\mu} a_\mu \qquad i = 1,\ldots \ell \qquad \mu = 1,\ldots \ell(\ell-1) \tag{6.1}$$

where C is the $\ell(\ell-1) \times \ell$ dimensional rectangular matrix

$$C_{i\mu} = \begin{array}{c} \\ \downarrow \\ i = 1,\ldots,\ell \end{array} \begin{array}{c} \mu \longrightarrow 1,\ldots \ell(\ell-1) \\ \left(\begin{array}{c|cccc|cccc|ccc|cc} & 1 & 0 & \cdot & 0 & 1 & 1 & \cdot & 1 & 0 & \cdots & 0 & \cdot & 0 \\ & 1 & 1 & \cdot & 0 & 1 & 1 & \cdot & 2 & 1 & \cdots & 1 & \cdot & 0 \\ & 1 & 1 & \cdot & 0 & 1 & 1 & \cdot & 2 & 1 & & 2 & \cdot & 0 \\ C^{A'_\ell} & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ \underset{\frac{1}{2}\ell(\ell+1)-1}{\longleftrightarrow} & & & & & & & & & & & & & \\ & 1 & 1 & & 0 & 1 & 2 & & 2 & 1 & & 2 & & 1 \\ & 1 & 1 & & 1 & 2 & 2 & & 2 & 2 & & 2 & & 2 \\ & 0 & 0 & & 0 & 1 & 1 & & 1 & 1 & & 1 & & 1 \\ & 1 & 1 & & 1 & 1 & 1 & & 1 & 1 & & 1 & & 1 \end{array} \right) \end{array} \tag{6.2}$$

where $C^{A_\ell'}$ denotes the matrix C^{A_ℓ} with the column $(0,0,\ldots 0,1,1)$ missing. In this case also, there is the following recursion relation

$$f_\ell^{D_\ell}(x_1,\ldots x_\ell) = \frac{f_\ell^{A_\ell}(x_1,\ldots x_\ell)\,[1-x_{\ell-1}x_\ell]}{\left\{\prod_{k=2}^{\ell-2}\left(1-\prod_{r=\ell-k}^{\ell-2} x_r x_\ell\right)\right\}\cdot\left\{\prod_{r=0}^{\ell-4}\prod_{k=0}^{\ell-r-4}\left(1-\prod_{s=r+1}^{\ell} x_s \prod_{t=\ell-k-2-r}^{\ell-2} x_t\right)\right\}} \tag{6.3}$$

For $\ell=2$, the above relation gives

$$f_2^{D_2}(x_1,x_2) = f_2^{A_2}(x_1,x_2)\,[1-x_1x_2] = \frac{1}{(1-x_1)(1-x_2)} \tag{6.4}$$

and so $\overline{M}(k_1,k_2)=1$ for all k_1, k_2. This of course is a known result. For $\ell=3$, this yields

$$f_3^{D_3}(x_1,x_2,x_3) = \frac{f_3^{A_3}(x_1,x_2,x_3)\,[1-x_2x_3]}{(1-x_1x_3)} \tag{6.5}$$

so that

$$\overline{M}(k_1,k_2,k_3) = \sum_{i=0}^{\min(k_1,k_3)}\Big[\overline{M}^{A_3}(k_1-i;\,k_2;\,k_3-i) - \overline{M}^{A_3}(k_1-i;\,k_2-1;\,k_3-i-1)\Big] \tag{6.6}$$

The weight space is ℓ dimensional. The components of the weight must be integers or half-integers. The Weyl group in this case consists of all permutations of the components of the weight (corresponding to the reflection perpendicular to the roots $e_i - e_j$) and all changes of sign <u>in pairs</u> (corresponding to the reflection perpendicular to the roots $e_i + e_j$, and is of order $2^{\ell-1}\,\ell!$.

The condition for a weight to be dominant is

$$m_1 \geq m_2 \geq \ldots \geq m_{\ell-1} \geq |m_\ell|$$

7. G_2.

The roots for this exceptional group are $\pm(e_i - e_j)$, $\pm e_i$ $\quad i, j = 1,2,3$; $e_3 = -(e_1+e_2)$. The six positive roots are $(e_1 - e_2), (e_1 - e_3), (e_2 - e_3), e_1, e_2$ and $-e_3 = (e_1 + e_2)$.

The simple roots are $\beta_1 = e_1 - e_2$ and $\beta_2 = e_2$. Eq.(2.4) then becomes

$$k_i = C_{i\mu}\, a_\mu \qquad i = 1,2 \qquad \mu = 1,2,\ldots 6. \tag{7.1}$$

where the (6 x 2) rectangular matrix C is

$$C_{i\mu} = \begin{array}{c c} & \mu = 1, \ldots 6 \\ i=1,2 & \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 3 & 3 \end{pmatrix} \end{array} \tag{7.2}$$

The generating function is then

$$f^{G_2}(x_1, x_2) = (1-x_1)^{-1}(1-x_2)^{-1}(1-x_1x_2)^{-1}(1-x_1x_2^2)^{-1} \cdot (1-x_1x_2^3)^{-1}(1-x_1^2x_2^3)^{-1} \tag{7.3}$$

and so one immediately sees the following relations

$$\begin{aligned} f^{G_2}(x_1,x_2) &= \frac{f_2^{A_2}(x_1,x_2)}{(1-x_1x_2^2)(1-x_1x_2^3)(1-x_1^2x_2^3)} \\ &= \frac{f_2^{B_2}(x_1,x_2)}{(1-x_1x_2^3)(1-x_1^2x_2^3)} \end{aligned} \tag{7.4}$$

It follows therefore [11], [12]

$$\overline{M}^{G_2}(k_1,k_2) = \sum_{i,j,k} \overline{M}(k_1-i-j-2k;\, k_2-2i-3j-3k) \tag{7.5}$$

The above sum has been explicitly carried out in ref. [12] for various inequalities of k_1 and k_2. From (7.4) it also follows that

$$\overline{M}^{G_2}(k_1,k_2) = \sum_{i,j} \overline{M}^{B_2}(k_1-i-2j;\, k_2-3i-3j) \tag{7.6}$$

The weight space in this case is again three dimensional like A_2 with the components of a weight satisfying

$$m_1 + m_2 + m_3 = 0$$

The components of the weights are integers. The Weyl group is of order 12 and consists of the six permutations of (m_1, m_2, m_3) corresponding to the reflection perpendicular to the roots $(e_1 - e_2)$, $(e_2 - e_3)$, $(e_1 - e_3)$ and six permutations with a total change in sign corresponding to the roots e_i. The dominant weight satisfies

$$m_1 \geq m_2 \geq m_3 \ ; \quad m_1 \geq 0 \ ; \ m_2 \leq 0 \ ; \ m_3 \leq 0 \tag{7.7}$$

8. External Multiplicity

In the case of rotation groups in three dimensions, an IR is characterised by the eigenvalue j of the single Casimir operator J^2, which is integral or half integral. One is then familiar with the C.G. series

$$D^{j_1} \otimes D^{j_2} = \sum_{j=j_1+j_2}^{|j_1-j_2|} \oplus \, D^{j} \tag{8.1}$$

where D^j denotes an I.R. with the highest weight j. If $j_1 > j_2$, (in which case we shall say that the representation D^{j_1} dominates D^{j_2}), the right hand side of (8.1) can be interpreted as those I.R's whose highest weights are obtained by adding to the highest weight of the dominant I.R. i.e. D^{j_1} , all the weights of the I.R. D^{j_2} (from j_2 to $-j_2$). This is the main constent of Biedenharn's theorem [2]. The conditions for one I.R. to dominate another I.R. have been worked out [1]. The general idea follows from the two equivalent formulae for the character

$$\chi^{m}(\xi) = \sum_{m' \in D(m)} M^{m}(m') \exp i(m', \xi) \qquad (8.2)$$

where the $\chi^{m}(\xi)$ is the character of an I.R. with the highest weight m and ξ are the group parameters. The other formula is

$$\chi^{m}(\xi) = \frac{X(m+R_0)}{X(R_0)} \qquad (8.3)$$

where

$$X(m+R_0) = \sum_{S \in W} \delta_S \exp i\left(S(m+R_0), \xi\right)$$

Suppose, we are interested in the product of I.R.'s $D(\Lambda_1)$ and $D(\Lambda_2)$ with Λ_1 and Λ_2 as their highest weights respectively.

Then

$$\chi(\Lambda_1)\,\chi(\Lambda_2) = \frac{\sum\limits_{S\in W} \delta_S \exp i\left[S(\Lambda_1+R_0),\xi\right]}{\sum\limits_{S\in W} \delta_S \exp i\left[S(R_0),\xi\right]}$$

$$\cdot \sum_{m'\in D(\Lambda_2)} M^{\Lambda_2}(m') \exp i(m',\xi) , \tag{8.4}$$

where we have used Eq.(8.2) for $\chi(\Lambda_2)$ and (8.3) for $\chi(\Lambda_1)$

Eq.(8.4) can now be regrouped to be written as

$$\chi(\Lambda_1)\,\chi(\Lambda_2) = \sum_{\substack{S\in W \\ m'\in D(\Lambda_2)}} \frac{\delta_S M^{\Lambda_2}(m') \exp i\left[S(\Lambda_1+m'+R_0),\xi\right]}{\sum\limits_{S\in W} \delta_S \exp i\left[S(R_0),\xi\right]} \tag{8.5}$$

where we have used the property

$$S(P) + S(Q) = S(P+Q) \tag{8.6}$$

Eq.(8.5) can now be interpreted as follows. In the product $D(\Lambda_1) \times D(\Lambda_2)$ where $D(\Lambda_1)$ dominates $D(\Lambda_2)$, only these IR's with the highest weight Λ_1+m' occur $m' \in D(\Lambda_2)$ in the reduction. These IR's occur with the multiplicity $M^{\Lambda_2}(m')$ i.e. multiplicity of the weight m' in the IR with highest weight Λ_2. The condition of dominance of one I.R. over the other is needed to the make (Λ_1+m') dominant. These have been more general formulae of G.Racah and D.Speiser [3] which do not involve the condition that one I.R. dominates the other. For our purpose, Eq.(8.5) is quite sufficient. Thus, we realize that the external multiplicity is very closely related to the internal multiplicity structure.

9. CONCLUSION

We have constructed generating function for the various classical groups. A_ℓ , B_ℓ , C_ℓ , D_ℓ and G_2 . These are then used to set up recursion relations for the partition function which enter Kostant's formula for the inner multiplicity structure. The essential idea of the whole analysis is the realization that the number of solutions of the matrix equation $k = C\alpha$ (for given k and c) where the matrix C is in general a rectangular matrix with non-negative integer coefficients and the components of the vectors k and a are again non-negative integers is given by the coefficient of $x_1^{k_1} \ldots x_\ell^{k_\ell}$ of the generating function. In many cases the explicit evaluation of the number of solutions is not possible and so we have set up recursion relations. While in the case of A_ℓ , the recursion relation is between the partition functions of A_ℓ and $A_{\ell-1}$, in the cases of B_ℓ , C_ℓ and D_ℓ the recursion relations for their partition functions are among these and of A_ℓ . For G(2), there are two recursion relations one with A_2 and the other with B_2. We have also discussed the connection between the internal and external multiplicity structures.

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