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Proceedings of the Conference on

CLIFFORD ALGEBRA, ITS GENERALIZATION AND APPLICATIONS

(30th January -- 1st February 1971)

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Edited by
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PREFACE

A CHARMED ENTRANCE

It is considered a worn-out cliche to emphasise that Pauli and Dirac matrices are the representations of a Clifford algebra of anticommuting elements. However, the exact mathematical transition from Pauli to Dirac matrices has never been studied in rigorous detail. Recently, by a series of fortuitous circumstances it was demonstrated at MATSCIENCE that the transition from Pauli to Dirac matrices revealed a charmed entrance into a fascinating structure called L-matrices with a definite relationship between the number of elements and the dimensions of the matrix representations of a Clifford algebra. Even more surprising was the possibility of entering the unexplored domain of ω-commutation relations where ω is the general root of unity.

We later found that these extensions have claimed the attention of mathematicians some years ago but their contributions had not gained general currency since they were published in journals not easily accessible. Nono and Morinaga in Japan had arrived at the Generalized Clifford Algebra based on ω-commutation relations

$$AB = \omega BA$$

and this was independently studied by Yamazaki followed ten years later by A.O.Morris. It was therefore found worthwhile to arrange a conference when the creators of the mathematical structure could meet those who had for the first time applied these ideas to elementary particle physics. They met at Ootacamund, that lovely hill resort of colourful South India, where sunsets swathe the verdant valleys and misty mountains with the hues of heaven and forest pines sway to the 'gentle wind whose breath can teach the wilds to love tranquillity'.

It is our earnest hope that at a time when the theoretical physicist of this Gell-Mannic era is satiate with the triumphs of unitary symmetry in the mood

"unarm, my friends - the long days' work is done"

we are able to discern the dawn of a new algebra which may further enlighten elementary particle theory in the years to come.

Alladi Ramakrishnan

(Alladi Ramakrishnan)

CONTENTS

GENERALIZED CLIFFORD ALGEBRA AND LINEARIZATIONS OF A PARTIAL DIFFERENTIAL EQUATION: $g^{i_1 i_2 \cdots i_m} \nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_m} \psi = c^m \psi$

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About twenty years ago, the purpose of our work was to generalize the theory of spinors as a whole, and to find its applications for physics. As well known, the theory of spinors is established on the combined basic concepts: Clifford algebra with (-1)-commutation rule, orthogonal group (or Lorentz group), linearization of quadratic form: $(x_i \gamma_i)^2 = \gamma_0 \sum_{i=1}^{n} (x^i)^2$, and linearization of quadratic wave equation: $\Box\psi = (cm)^2\psi$. First of all, our task was to construct an associative algebra generalizing the ordinary Clifford algebra, and to investigate some properties related to the algebra, and the results obtained at that time were published in a paper [1]. Thereafter, the theory of ω-Clifford algebra with ω-commutation rule, being a special one of our generalized Clifford algebra, were well cultivated independently by many authors: Professor K. Yamazaki [2] and A.O. Morris [3], and members of Madras school [4,5,6]. It is a great pleasure and also surprise for me to see that this mathematical theory is now being applied beautifully to the theory of elementary particles by Madras school under the direction of Professor A. Ramakrishnan [7,8,9,10].

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In Part I, we shall state the outlines of our paper [1] and, in Part II, generalizing some results of Professor K.Morinaga on the linearization of wave equation $\Box\psi = mc\psi$ [11], we shall have some results on the linearizations of a partial differential equation: $g^{i_1 i_2 \cdots i_m} \nabla_{i_1}\nabla_{i_2}\cdots\nabla_{i_m}\psi = c^m \psi$ [12].

I. Generalized Clifford Algebra.

1. Generalized Clifford Algebra:

We shall define a generalized Clifford algebra (briefly G.C.algebra), by extending the concept of the ordinary Clifford algebra.

Let G be the direct product of n groups $G_1, G_2, \ldots, G_n$, and let a symbol e_A correspond to each element $A \equiv (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $(\alpha_i \in G_i)$ of G. We define the following linear associative algebra ϑ with the basic elements e_A, $A \in G$ on a field of characteristic zero.

i) ϑ is a left and right linear space on a field K.

ii) There is a mapping $a \to a^A$ in K, such that

$$e_A \cdot a = a^A \cdot e_A \quad \text{for any } e_A, A \in G \text{ and } a \in K.$$

iii) There is defined an associative multiplication in ϑ, such that, for any two basic elements e_A and e_B:

$$A \equiv (\alpha_1, \alpha_2, \ldots, \alpha_n) \text{ and } B \equiv (\beta_1, \beta_2, \ldots, \beta_n),$$

$$e_A e_B = \zeta(A,B), \quad AB \equiv (\alpha_1\beta_1, \alpha_2\beta_2, \ldots, \alpha_n\beta_n),$$

where

$$\zeta(A,B) = \prod_{k>i}^{*} \rho(\alpha_k,\beta_i)^{(\alpha_1\beta_1,\alpha_2\beta_2,\dots,\alpha_{i-1}\beta_{i-1},\alpha_i,\dots,\alpha_{k-1})}$$

$$\equiv \rho(\alpha_n,\beta_1)^{(\alpha_1,\alpha_2,\dots,\alpha_{n-1})} \cdot \rho(\alpha_{n-1},\beta_1)^{(\alpha_1,\dots,\alpha_{n-2})} \cdots \rho(\alpha_2,\beta_1)^{(\alpha_1)}$$

$$\cdot \rho(\alpha_n,\beta_2)^{(\alpha_1\beta_1,\alpha_2,\dots,\alpha_{n-1})} \cdot \rho(\alpha_{n-1},\beta_2)^{(\alpha_1\beta_1,\alpha_2,\dots,\alpha_{n-2})} \quad \rho(\alpha_3,\beta_2)^{(\alpha_1\beta_1,\alpha_2)}$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\cdot \rho(\alpha_n,\beta_{n-1})^{(\alpha_1\beta_1,\alpha_2\beta_2,\dots,\alpha_{n-2}\beta_{n-2},\alpha_{n-1})} ,$$

in terms $\rho(\alpha_k,\beta_i) \in K$, $(k>i)$, and

$$\rho(\alpha_k,\varepsilon_i) = \rho(\varepsilon_k,\beta_i) = 1 \qquad (k>i),$$

for the unit element ε_i of G_i and the unit element 1 of K.

iv) For any basic elements e_A, e_B and any element $a \in K$

$$e_A \cdot (a\, e_B) = (e_A\, a) \cdot e_B .$$

Such a linear associative algebra ϑ shall be called a generalized Clifford algebra (G.C. algebra), and its elements generalized Clifford numbers. And moreover we shall call $G_1,\dots,G_n$ the basic groups of ϑ, and $\rho(\alpha_i,\beta_j)$ the structure numbers of ϑ.

Here $\rho\overset{(\alpha_1\beta_1,\ldots,\alpha_{i-1}\beta_{i-1},\alpha_i,\cdots,\alpha_{k-1})}{(\alpha_k,\beta_i)}$, lacking the upper indices $\alpha_k,\ldots,\alpha_n,\beta_i,\cdots,\beta_n$ in (1.3), means $\rho^A(\alpha_k,\beta_i)$ for $A\equiv(\alpha_1\beta_1,\alpha_2\beta_2,\ldots,\alpha_{i-1}\beta_{i-1},\ldots,\alpha_{k-1},\varepsilon_k,\ldots,\varepsilon_n)$.

We have a theorem derived directly from this definition.

Theorem 1. Let G_i and G_j $(i>j)$ be the two basic groups of a G.C. algebra ϑ, and let $\rho(\alpha_i,\beta_j)$ be the structure numbers of ϑ. Then we have the relations

$$\rho(\alpha_i,\overset{1}{\beta}_j\overset{2}{\beta}_j)=\rho(\alpha_i,\overset{1}{\beta}_j)\rho^{\overset{1}{\beta}_j}(\alpha_i,\overset{2}{\beta}_j)\text{ for any }\alpha_i\in G_i;\ \overset{1}{\beta}_j,\overset{2}{\beta}_j\in G_j,$$

$$\rho(\overset{1}{\alpha}_i\overset{2}{\alpha}_i,\beta_j)=\rho^{(\overset{1}{\alpha}_i)}(\overset{2}{\alpha}_i,\beta_j)\,\rho(\overset{1}{\alpha}_i,\beta_j)\text{ for any }\overset{1}{\alpha}_i,\overset{2}{\alpha}_i\in G_i;\ \beta_j\in G_j.$$

The set $N_j(\alpha_i)\equiv\{\beta_j;\beta_j\in G_j,\ \rho(\alpha_i,\beta_j)=1\}$ is a subgroup of G_j. The necessary and sufficient condition for $N_j(\alpha_i)$ to be a normal subgroup of G_j is that $\rho^{(\beta_j)}(\alpha_i,\theta_j)=\rho(\alpha_i,\theta_j)$ for any $\beta_j\in N_j(\alpha_i)$, $\theta_j\in G_j$. The necessary and sufficient condition for $N_j(\alpha_i)$ to contain the commutator group Q_j of G_j is that $\rho(\alpha_i,\overset{1}{\beta}_j\overset{2}{\beta}_j)=\rho(\alpha_i,\overset{2}{\beta}_j\overset{1}{\beta}_j)$ for any $\overset{1}{\beta}_j,\overset{2}{\beta}_j\in G_j$.

As for $N_i'(\beta_j)=\{\alpha_i,\alpha_i\in G_i,\ \rho(\alpha_i,\beta_j)=1\}$, the similar results are obtained.

Here $\rho^{(\beta_j)}(\cdot,\cdot)$ means $\rho^B(\cdot,\cdot)$ for $B\equiv(\varepsilon_1,\ldots,\varepsilon_{j-1},\beta_j,\varepsilon_{j+1},\ldots,\varepsilon_n)$.

2. G.C. algebra reducible to G.C. algebra each whose group is commutative.

Let ϑ' be a G.C. algebra which has the basic groups G_i' $(i=1,2,\ldots,n)$ and has the structure numbers $\rho'(\alpha_i',\beta_j')$.

Then we shall define that the G.C. algebra ϑ is homomorphic to ϑ', when the basic groups G_i are homomorphic onto G_i' respectively, and in the homomorphisms: $\alpha_i \to \alpha_i'$ it holds $\rho(\alpha_i, \beta_j) = \rho'(\alpha_i', \beta_j')$. By the correspondence from the basic element e_A, $A \in G \equiv G_1 \times G_2 \times \cdots \times G_n$ to $e'_{A'}$,

$$A' \in G' \equiv G_1' \times G_2' \times \cdots \times G_n' ,$$

the G.C. algebra ϑ is homomorphic to ϑ' in the ordinary sense. In this terminology we have

Theorem 2. Let G.C. algebra ϑ be defined by the basic groups G_i and the structure numbers $\rho(\alpha_i, \beta_j)$, $(i > j$, $i, j = 1, 2, \ldots, n)$. Then the necessary and sufficient condition in order that the G.C. algebra ϑ is homomorphic to a G.C. algebra ϑ' each of whose basic groups is commutative is that

$$\rho(\alpha_i, \overset{1}{\beta}_j \overset{2}{\beta}_j) = \rho(\alpha_i, \overset{2}{\beta}_j \overset{1}{\beta}_j) \quad \text{for any} \quad \alpha_i \in G_i,\ \overset{1}{\beta}_j, \overset{2}{\beta}_j \in G_j,$$
$$\rho(\overset{1}{\alpha}_i \overset{2}{\alpha}_i, \beta_j) = \rho(\overset{2}{\alpha}_i \overset{1}{\alpha}_i, \beta_j) \quad \text{for any} \quad \overset{1}{\alpha}_i, \overset{2}{\alpha}_i \in G_i,\ \beta_j \in G_j.$$

3. G.C. algebra whose basic elements are commutative with its structure numbers

Now we shall consider the G.C. algebra whose basic elements e_A, $A \in G$ are commutative with its structure numbers $\rho(\alpha_i, \beta_j)$, $(i > j$, $i, j = 1, 2, \ldots, n)$. In this case we have

<u>Theorem 3</u>. Let e_A be a symbol corresponding to an element $A \in G \equiv G_1 \times G_2 \times \cdots \times G_n$, and let

$$\Lambda \equiv \{ \rho(\alpha_i, \beta_j) ; \alpha_i \in G_i , \beta_j \in G_j , i > j , i, j = 1, 2, \ldots, n \}$$

be a set of numbers which are commutative with e_A, $A \in G$. Then the necessary and sufficient condition that there is a G.C. algebra having G_i as the basic groups and Λ as the set of the structure numbers is that

1. Λ is a commutative set,

and

2. $\rho(\alpha_i, \overset{1}{\beta}_j \overset{2}{\beta}_j) = \rho(\alpha_i, \overset{1}{\beta}_j)\, \rho(\alpha_i, \overset{2}{\beta}_j)$ for any $\alpha_i \in G_i$, $\overset{1}{\beta}_j, \overset{2}{\beta}_j \in G_j$,

$\rho(\overset{1}{\alpha}_i \overset{2}{\alpha}_i, \beta_j) = \rho(\overset{1}{\alpha}_i, \beta_j)\, \rho(\overset{2}{\alpha}_i, \beta_j)$ for any $\overset{1}{\alpha}_i, \overset{2}{\alpha}_i \in G_i$, $\beta_j \in G_j$.

As we shall see from this theorem, in the G.C. algebra whose basic elements are commutative with its structure numbers, it is evident that

$$\rho(\alpha_i, \overset{1}{\beta}_j \overset{2}{\beta}_j) = \rho(\alpha_i, \overset{2}{\beta}_j \overset{1}{\beta}_j) \quad \text{for any } \alpha_i \in G_i ,\ \overset{1}{\beta}_j, \overset{2}{\beta}_j \in G_j ,$$

$$\rho(\overset{1}{\alpha}_i \overset{2}{\alpha}_i, \beta_j) = \rho(\overset{2}{\alpha}_i \overset{1}{\alpha}_i, \beta_j) \quad \text{for any } \overset{1}{\alpha}_i, \overset{2}{\alpha}_i \in G_i ,\ \beta_j \in G_j .$$

So by Theorem 2, such a G.C. algebra ϑ is homomorphic to the G.C. algebra ϑ' each of whose basic groups is commutative. In particular, let us suppose that the basic groups $G_1, G_2, \cdots, G_n$ are the cyclic groups of order $m_1, m_2, \ldots, m_n$ respectively, i.e. $G_i \in [\, \varepsilon_i, \alpha_i, \ldots, \alpha_i^{m_i - 1} \,]$, $\alpha_i^{m_i} = \varepsilon_i$. By Theorem 3, the set $\Lambda \equiv \{ \rho(\alpha_i^{\lambda_i}, \alpha_j^{\lambda_j}) ; \alpha_i \in G_i ,$ $\lambda_i = 0, 1, \ldots, m_i - 1 ,\ i, j = 1, 2, \ldots, n \}$ of the structure numbers of the G.C. algebra with the basic groups $G_1, \cdots, G_n$ is commutative and is determined by the conditions:

$$\rho(\alpha_i, \alpha_j^{\lambda+\mu}) = \rho(\alpha_i, \alpha_j^{\lambda})\rho(\alpha_i, \alpha_j^{\mu}),$$
$$\rho(\alpha_i^{\lambda+\mu}, \alpha_j) = \rho(\alpha_i^{\lambda}, \alpha_j)\rho(\alpha_i^{\mu}, \alpha_j), \quad (i > j)$$

from which we have

$$\rho(\alpha_i^{\lambda_i}, \alpha_j^{\mu_j}) = (\rho(\alpha_i^{\lambda_i}, \alpha_j))^{\mu_j} = (\rho(\alpha_i, \alpha_j))^{\lambda_i\mu_j},$$
$$(\mu_j = 0, 1, \dots, m_j - 1)$$

since $\alpha_i^{m_i} = \varepsilon_i$, we have

$$\rho(\alpha_i, \alpha_j)^{m_i} = \rho(\alpha_i^{m_i}, \alpha_j) = \rho(\varepsilon_i, \lambda_j) = 1,$$

and

$$\rho(\alpha_i, \alpha_j)^{m_j} = \rho(\alpha_i, \alpha_j^{m_j}) = \rho(\alpha_i, \varepsilon_j) = 1,$$

and therefore $\rho(\alpha_i, \alpha_j)$ must be a primitive ℓ_{ij}-th root ω_{ij} of unity, where ℓ_{ij} is a factor of m_i and m_j.

In this case we have

Theorem 4. For the G.C. algebra with the basic groups $G_i = [\varepsilon_i, \alpha_i, \dots, \alpha_i^{m_i-1}]$, $\alpha_i^{m_i} = \varepsilon_i$, the structure numbers $\rho(\alpha_i^{\lambda_i}, \alpha_j^{\mu_j})$ are equal to $\omega_{ij}^{\lambda_i\mu_j}$ where ω_{ij} is a primitive ℓ_{ij}-th root of unity and ℓ_{ij} is a factor of m_i and m_j.

4. Linearization of $\sum_{i=1}^{n}(x^i)^m$

Now, in particular, we shall consider the G.C. algebra whose basic groups, are the same cyclic group of order m: $[\varepsilon, \alpha, \alpha^2, \dots \alpha^{n-1}]$, $\alpha^m = \varepsilon$, and whose basic elements are commutative with its structure numbers. Suppose that the field K is commutative and contains a primitive m-th root of unity. If we take $m = 2$ and K as the field of real or complex numbers, then we obtain the theory of ordinary spinors. As a special case of Theorem 4 we have

Theorem 4'. For the G.C. algebra with the basic groups $G_i \equiv [\varepsilon, \alpha, \alpha^2, \dots, \alpha^{m-1}]$, $\alpha^m = \varepsilon$, the structure numbers $\rho(\alpha^\lambda, \alpha^\mu)$ are equal to $\omega^{\lambda\mu}$ where ω is any m-th root of unity in K.

So if we write $\overset{\lambda}{e}_i \equiv e_{(\varepsilon, \dots, \varepsilon, \underset{(i)}{\alpha^\lambda}, \varepsilon, \dots, \varepsilon)}$, then we have

$$\overset{\lambda}{e}_i \overset{\mu}{e}_i = \omega^{\lambda\mu} \overset{\mu}{e}_j \overset{\lambda}{e}_i , \quad (i > j),$$

in particular,

$$\overset{\lambda}{e}_i \overset{\lambda}{e}_j = \omega^{\lambda^2} \overset{\lambda}{e}_j \overset{\lambda}{e}_i , \quad (i > j).$$

We can prove the following theorem.

Theorem 5. Let $\overset{\lambda}{e}_i$ be the basic element $e_{(\varepsilon, \dots, \varepsilon, \underset{(i)}{\alpha^\lambda}, \varepsilon, \dots, \varepsilon)}$ of the G.C. algebra whose basic groups are the same cyclic group of order m and whose structure numbers are $\rho(\alpha^\lambda, \alpha^\mu) = \omega^{\lambda\mu}$ (ω is a primitive m-th root of unity in K). Then we have an identity

$$\left(\sum_{i=1}^{n} x^i \overset{\lambda}{e}_i \right)^\ell = e_0 \sum_{i=1}^{n} (x^i)^\ell , \quad (e_0 \equiv e_{(\varepsilon, \varepsilon, \dots, \varepsilon)}),$$

for $\ell = m/(\lambda, m)$, if and only if $(m/(\lambda, m), (\lambda, m)) = 1$.

5. Structure of the G.C. algebra associated with the linearization of $\sum_{i=1}^{n}(x^i)^m$

Here we write p_o and p_i for e_o and e_i' in 4 respectively. So we have

$$p_i p_j = \omega p_j p_i , \quad (i>j),$$

where ω is a primitive m-th root of unity. The G.C. algebra generated by p_o and p_i ($i = 1, 2, \ldots, n$) will be called the G.C. algebra associated with the linearization: $(\sum_{i=1}^{n} x^i p_i)^m = p_o \sum_{i=1}^{n}(x^i)^m$. This G.C. algebra might be called ω-Clifford algebra, (ω.C. algebra), emphasising ω-commutation rule.

Now we shall investigate the center and ideal of W.C. algebra, in each case of the followings:

Case (I) : n is even,

Case (II_1) : n is odd and m is also odd,

Case (II_2) : n is odd, $n \equiv 1$ (mod. 4) and m is even,

Case (II_3) : n is odd, $n \equiv 3$ (mod. 4), m is even, and there exists an element k in K such $\omega = k^2$,

Case (II_4) : n is odd, $n \equiv 3$ (mod. 4), m is even $m = 2m_o$ there does not exist an element k in K such that $\omega = k^2$

We have the following results:

Theorem 6. If n is even (i.e. Case (I)), then the center $\mathcal{Z}$ of ϑ is $\{p_o\}$ and the ideals of ϑ are only $\{o\}$ and the whole algebra ϑ . If n is odd, then the center $\mathcal{Z}$ of ϑ is $\{ \overset{o}{P}, \overset{1}{P}, \overset{2}{P}, \ldots, \overset{m-1}{P} \}$, where $\overset{\mu}{P} = p_1^{\mu} p_2^{-\mu} p_3^{\mu} \cdots p_{n-1}^{\mu} p_n^{-\mu}$.

And in the cases (II_1), (II_2) and (II_3), $\mathfrak{V}$ is decomposed into the direct sum $\sum_{\lambda=0}^{m-1} \oplus \mathfrak{V}\overset{\lambda}{Z}$ of the simple ideals of $\mathfrak{V}$ where

$$\overset{\lambda}{Z} = \frac{1}{m}\sum_{\mu=0}^{m-1} \omega^{\lambda\mu - \frac{1}{4}(n-1)(\mu-1)} \cdot \overset{\mu}{P} \qquad \text{for the cases } (II_1) \text{ and } (II_2),$$

$$\overset{\lambda}{Z} = \frac{1}{m}\sum_{\mu=0}^{n-1} \omega^{\lambda\mu - \frac{1}{4}(n-1)\mu^2} \cdot \overset{\mu}{P} \qquad \text{for the case } (II_3);$$

moreover, as for any ideal $\mathfrak{A}$ of $\mathfrak{V}$, $\mathfrak{A} \cap \mathfrak{v}$ is an ideal $\sum_{\mathfrak{A}\overset{\lambda}{Z}\neq 0} \oplus \{\overset{\lambda}{Z}\}$ of $\mathfrak{v}$ and $\mathfrak{A} = \sum_{\mathfrak{A}\overset{\alpha}{Z}\neq 0} \oplus \mathfrak{V}\overset{\alpha}{Z}$. Similarly, in the case (II_4) we have $\mathfrak{V} = \sum_{\alpha=0}^{m_0-1} \oplus \mathfrak{V}\mathfrak{q}_\alpha$ $\quad (\mathfrak{V}\mathfrak{q}_\alpha = \mathfrak{V}\overset{\alpha}{U})$

$$\mathfrak{A} \cap \mathfrak{v} = \sum_{\mathfrak{A}\overset{\alpha}{U}\neq 0} \oplus \mathfrak{q}_\alpha \quad \text{and} \quad \mathfrak{A} = \sum_{\mathfrak{A}\cdot\overset{\alpha}{U}\neq 0} \oplus \mathfrak{V}\mathfrak{q}_\alpha \quad \text{where} \quad \mathfrak{q}_\alpha = \{\overset{\alpha}{U}, \overset{m_0+\alpha}{U}\},$$

$$\overset{\alpha}{U} = \frac{2}{m}\sum_{\substack{\mu=0 \\ (\text{even})}}^{m-2} \omega^{\alpha\mu - \frac{1}{4}(n-1)\mu^2} \cdot \overset{\mu}{P},$$

and

$$\overset{m_0+\alpha}{U} = \frac{2}{m}\sum_{\substack{\nu=1 \\ (\text{odd})}}^{m-1} \omega^{\alpha\nu - \frac{1}{4}(n-1)\nu^2 + \frac{1}{2}} \cdot \overset{\nu}{P}.$$

Moreover, $\mathfrak{V}\overset{\lambda}{Z}$ $(\lambda = 0, 1, \cdots, m-1)$ are mutually isomorphic and have m^{n-1} linearly independent elements respectively, and $\mathfrak{V}\overset{\alpha}{\mathfrak{q}}$ $(\alpha = 0, 1, \ldots, m_0-1)$ are mutually isomorphic and have $2m^{n-1}$ linearly independent elements respectively.

Here $\{\mathfrak{q}_1, \mathfrak{q}_2, \cdots, \mathfrak{q}_s\}$ denotes the linear space over K spanned by $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_s$.

6. Matric representation of the G.C. algebra associated with the linearization of $\sum_{i=1}^{n}(x^i)^m$.

We can investigate the representation of the G.C. algebra by means of the structure of $\mathfrak{V}$-discussed in 5, and the general theory of the representations of algebra. But, in this section we shall consider directly the actual representation of the G.C. algebra $\mathfrak{V}$. We must suppose that K contains a primitive m-th root ω of unity, and the square root $\omega^{1/2}$ of ω.

We have the following results.

Theorem 7. Let ω be a primitive m-th root of unity if K contains $\omega^{1/2}$ then the general system of matrices P_i $(i=1,2,\dots,n)$ such that

$$P_i P_j = \omega P_j P_i \ (i>j) \quad \text{and} \quad P_i^m = E$$

are written as follows:

(I) If n is even: $n = 2r$, then

$$P_{2s-1} = W^{-1}[\underbrace{\underbrace{\Omega_3 \times \cdots \times \Omega_3}_{s} \times \Omega_1 \times E_m \times \cdots \times E_m}_{r} \times E_l]W,$$

$$P_{2s} = W^{-1}[\underbrace{\underbrace{\Omega_3 \times \cdots \times \Omega_3}_{s} \times \Omega_2 \times E_m \times \cdots \times E_m}_{r} \times E_l]W$$

$$(s = 1,2,\dots,r)$$

where W is an arbitrary regular matrix of order $m^r l$, $l = 1,2,\dots$.

(II) If n is odd: $n = 2r+1$, then

$$P_{2s-1} = W^{-1}[\underbrace{\underbrace{\Omega_3\times\cdots\times\Omega_3}_{s}\times\Omega_1\times E_m\times\cdots\times E_m}_{r}\times E_\ell]W,$$

$$P_{2s} = W^{-1}[\underbrace{\underbrace{\Omega_3\times\cdots\times\Omega_3}_{s}\times\Omega_2\times E_m\times\cdots\times E_m}_{r}\times E_\ell]W,$$

$$P_n = P_{2r+1} = W^{-1}[\underbrace{\Omega_3\times\cdots\times\Omega_3}_{r}\times R\,]W,$$

$$(s = 1, 2, \dots, r),$$

where R is any matrix of order ℓ such that $R^m = E_\ell$, we may take $R = \begin{bmatrix} \omega^{\alpha_1} & & 0 \\ & \omega^{\alpha_2} & \\ 0 & & \ddots\ \omega^{\alpha_\ell} \end{bmatrix}$, ($\alpha_1,\dots,\alpha_\ell$: any integers, $\ell = 1, 2, \cdots$) and W is an arbitrary regular matrix of order $m^r\ell$. Here

$$\Omega_1 = \begin{bmatrix} 1 & & & \\ & \omega & & 0 \\ & & \ddots & \\ 0 & & & \omega^{m-1} \end{bmatrix} ; \qquad \Omega_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \vdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and $\Omega_3 = \omega^{\frac{m-1}{2}}\,\Omega_1^{-1}\,\Omega_2$.

Now if we make correspond the matrix $P_1^{\lambda_1} P_2^{\lambda_2} \cdots P_n^{\lambda_n}$, P_i being the matrices in Theorem 7, to the basic element $p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_n^{\lambda_n}$ of the G.C. algebra ϑ , then we obtain the general representation of ϑ' . Then we have the following theorem.

Theorem 8. The general representation of the G.C. algebra ϑ associated with the linearization of $\sum_{i=1}^{n}(x^i)^m$ is generated by the system of matrices P_i ($i = 1, 2, \ldots, n$) in Theorem 7. If n is even: $n = 2r$, then the faithful representation of minimal order yields the complete matric algebra in m^r dimensions, and therefore the representation is irreducible. If n is odd: $n = 2r+1$, the faithful representation of minimal order is the direct sum of m complete matric algebras in m^r dimensions, and therefore it is completely reducible. The above results were obtained independently by Professor A.O.Morris [3]. By Theorem 7, there exist only three matrices Ω_1, Ω_2 and Ω_3 in the lowest dimension m, satisfying the two generalized Clifford conditions, from which the general system of matrices P_i ($i = 1, 2, \ldots, n$) can be constructed. The method of construction seems to be essentially the same as the σ- operation by Professor Ramakrishnan [4].

7. Linear transformation leaving invariant $\sum_{i=1}^{n}(x^i)^m$, $(m>2)$.

In this last section, we shall determine the linear transformation

$$x'^i = \sum_{j=1}^{n} h^i_j \, x^j$$

leaving invariant $\sum_{i=1}^{n}(x^i)^m$, $(m > 2)$. The result obtained is the following:

Theorem 9. The set of all linear transformations leaving invariant $\sum_{i=1}^{n}(x^i)^m$, $(m>2)$ is a finite group of order $m^n \cdot n!$ These linear transformations are written as follows:

$$x^i = \omega^{\lambda_i} x^{j(i)}$$

where $j(i)$, $i=1,2,\cdots,n$ is any permutation of $1,2,\ldots,n$, ω is a primitive m-th root of unity, and λ_i is any integer mod. m.

II. Linearization of a partial differential equation:

$$g^{i_1 i_2 \cdots i_m} \nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_m} \psi = c^m \psi .$$

1. Preliminaries

The two following theorems were proved by Professor K.Morinaga [11].

Theorem I. A partial differential equation

$$\alpha^i \nabla_i \psi = mc \psi$$

is a linearization of the wave equation

$$\Box \psi = (mc)^2 \psi ,$$

if and only if

$$\alpha^{(i} \alpha^{j)} = g^{ij} E_N ,$$

i.e., α^i are matrices of Dirac type.

Theorem II. A partial differential equation

$$\alpha^i \nabla_i \psi = mc \psi , \quad (mc \neq 0)$$

is a linearization of the wave equation

$$\Box\psi = (mc)^2\psi$$

if and only if there exists an integer $p\ (\geq 0)$ such that

$$\alpha^{(i_1}\alpha^{i_2}\cdots\alpha^{i_{p+2})} = g^{(i_1 i_2}\alpha^{i_3}\cdots\alpha^{i_{p+2})}.$$

The condition for p = 1 becomes to:

$$\alpha^{(i_1}\alpha^{i_2}\alpha^{i_3)} = g^{(i_1 i_2}\alpha^{i_3)},$$

which is satisfied by each of the following conditions:

$$\alpha^{(i_1}\alpha^{i_2)} = g^{i_1 i_2},$$

and

$$\alpha^{(i_1}\alpha^{|i_2|}\alpha^{i_3)} = g^{(i_1|i_2|}\alpha^{i_3)}.$$

The latter is the Duffin-Kemmer relation.

We shall generalize these results for a partial differential equation:

$$g^{i_1 i_2 \cdots i_m}\nabla_{i_1}\nabla_{i_2}\cdots\nabla_{i_m}\psi = c^m\psi.$$

Let x^i be n variables , $\nabla_i = \dfrac{\partial}{\partial x_i}$ $(i = 1, 2, \cdots, n)$, α^i N x N matrices, and ψ an N-vector, then a partial differential equation:

$$\alpha^i\nabla_i\psi = c\psi \tag{1}$$

is called a linearization of a given partial differential equation:

$$g^{i_1 i_2\cdots i_m}\nabla_{i_1}\nabla_{i_2}\cdots\nabla_{i_m}\psi = c^m\psi, \tag{2}$$

if and only if any solution of (1) is always a solution of (2).
Here $g^{i_1 i_2 \cdots i_m}$ is symmetric with respect to $i_1, i_2, \ldots, i_m$.
If we take

$$g^{i_1 i_2 \cdots i_m} = \delta^{i_1 i_2 \cdots i_m} = \begin{cases} 1 & \text{for } i_1 = i_2 = \cdots = i_m, \\ 0 & \text{otherwise}, \end{cases}$$

then we have

$$g^{i_1 i_2 \cdots i_m} \nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_m} \psi = \sum_{i=1}^{n} \frac{\partial^m}{\partial x^{i_m}} \psi .$$

Let $v = \alpha^i v_i$, then we define $\|v\|$ as follows:

$$\|v\| = (g^{i_1 i_2 \cdots i_m} v_{i_1} v_{i_2} \cdots v_{i_m})^{1/m},$$

since

$$\begin{aligned} v^m &= (\alpha^i v_i)^m = \alpha^{i_1} \alpha^{i_2} \cdots \alpha^{i_m} v_{i_1} v_{i_2} \cdots v_{i_m} \\ &= \alpha^{i_1} \alpha^{i_2} \cdots \alpha^{i_m} v_{(i_1} v_{i_2} \cdots v_{i_m)} \\ &= \alpha^{(i_1} \alpha^{i_2} \cdots \alpha^{i_m)} v_{i_1} v_{i_2} \cdots v_{i_m} , \end{aligned}$$

We see that

$$v^m = \|v\|^m E_N , \tag{3}$$

if and only if

$$\alpha^{(i_1} \alpha^{i_2} \cdots \alpha^{i_m)} = g^{i_1 i_2 \cdots i_m} E_N .$$

Here ( )means the symmetric part, E_N is the unit matrix of order N. From (3) it follows:

$$(\det v)^m = \|v\|^{mN},$$

so that $\det v = 0$ is equivalent to $\|v\| = 0$.

Remark. If we assume that, for a primitive m-th root ω of unity,

$$\alpha^i\alpha^j = \omega\, \alpha^j\alpha^i \ , \quad (i>j)$$

$$(\alpha^i)^m = E_N \ ,$$

then we have

$$\alpha^{(i_1}\alpha^{i_2}\cdots\alpha^{i_m)} = \delta^{i_1 i_2\cdots i_m} E_N \ .$$

similarly we see that

$$v^{m+p} = \alpha^{(i_1}\alpha^{i_2}\cdots\alpha^{i_m}\alpha^{j_1}\cdots\alpha^{j_p)} v_{i_1}v_{i_2}\cdots v_{i_m}v_{j_1}\cdots v_{j_p},$$

and

$$\|v\|^m v^p = (g^{i_1 i_2\cdots i_m} v_{i_1}v_{i_2}\cdots v_{i_m})(\alpha^{j_1}\alpha^{j_2}\cdots\alpha^{j_p} v_{j_1}v_{j_2}\cdots v_{j_p})$$

$$= g^{i_1 i_2\cdots i_m}\alpha^{j_1}\cdots\alpha^{j_p}\, v_{(i_1}v_{i_2}\cdots v_{i_m}v_{j_1}\cdots v_{j_p)}$$

$$= g^{(i_1 i_2\cdots i_m}\alpha^{j_1}\alpha^{j_2}\cdots\alpha^{j_p)} v_{i_1}v_{i_2}\cdots v_{i_m}v_{j_1}\cdots v_{j_p}\ .$$

so we have that

$$v^{m+p} = \|v\|^m v^p , \qquad (4)$$

if and only if

$$\alpha^{(i_1}\alpha^{i_2}\ldots\alpha^{i_m}\alpha^{j_1}\ldots\alpha^{j_p)} = g^{(i_1 i_2 \ldots i_m}\alpha^{j_1}\alpha^{j_2}\ldots\alpha^{j_p)} . \qquad (5)$$

If there exists an integer $p(\geqq 0)$ such that

$$v^{m+p} = \|v\|^m v^p ,$$

then a partial differential equation:

$$\alpha^i \nabla_i \Psi = c\,\Psi \quad (c \neq 0) \qquad (1)$$

is a linearization of the partial differential equation:

$$g^{i_1 i_2 \ldots i_m} \nabla_{i_1}\nabla_{i_2}\cdots\nabla_{i_m}\Psi = c^m \Psi . \qquad (2)$$

In fact, operating $(\alpha^i \nabla_i)^{p+m-1}$ on the both sides of (1), we have

$$\alpha^{i_1}\alpha^{i_2}\ldots\alpha^{i_{p+m}} \nabla_{i_1}\nabla_{i_2}\cdots\nabla_{i_m}\Psi = c(\alpha^i\nabla_i)^{p+m-1}\Psi , \qquad (6)$$

since clearly it holds

$$\nabla_{i_1}\nabla_{i_2}\cdots\nabla_{i_{p+m}} = \nabla_{(i_1}\nabla_{i_2}\cdots\nabla_{i_{p+m})} ,$$

from (6) it follows

$$\alpha^{(i_1}\alpha^{i_2}\ldots\alpha^{i_{p+m})}\nabla_{i_1}\nabla_{i_2}\cdots\nabla_{i_{p+m}}\Psi = c^{p+m}\Psi .$$

By using (5) we have

$$g^{(i_1 i_2 \cdots i_m} \alpha^{i_{m+1}} \cdots \alpha^{i_{m+p})} \nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_m} \nabla_{i_{m+1}} \cdots \nabla_{i_{m+p}} \Psi = c^{m+p} \Psi ,$$

i.e.,

$$g^{i_1 i_2 \cdots i_m} \nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_m} (\alpha^i \nabla_i)^p \Psi = c^{m+p} \Psi .$$

By using (1) again we have

$$c^p g^{i_1 i_2 \cdots i_m} \nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_m} \Psi = c^{m+p} \Psi ,$$

so that, since $c \neq 0$, it holds

$$g^{i_1 i_2 \cdots i_m} \nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_m} \Psi = c^m \Psi .$$

Thus, (1) is a linearization of (2).

2. Main Theorems

Theorem 1. A partial differential equation

$$\alpha^i \nabla_i \Psi = c \Psi \qquad (1)$$

is a linearization of the partial differential equation:

$$g^{i_1 i_2 \cdots i_m} \nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_m} \Psi = c^m \Psi , \qquad (2)$$

if and only if

$$v^m = (\alpha^i v_i)^m = \| v \|^m \cdot E_N \quad \text{for any } v_i ,$$

i.e.

$$\alpha^{(i_1} \alpha^{i_2} \cdots \alpha^{i_m)} = g^{i_1 i_2 \cdots i_m} E_N . \qquad (3)$$

<u>Proof</u>: Let $\mathring{v}_i$ be an arbitrary vector such that

$$g^{i_1 i_2 \cdots i_m} \mathring{v}_{i_1} \mathring{v}_{i_2} \cdots \mathring{v}_{i_m} = 1,$$

and then we transform $\alpha^i \mathring{v}_i$ into the Jordan canonical form:

$$U(\alpha^i \mathring{v}_i)U^{-1} = \sum \oplus \begin{bmatrix} \lambda & \mu & \cdots & 0 \\ & \lambda & \ddots & \\ & & \ddots & \mu \\ 0 & & & \lambda \end{bmatrix}, \quad (\mu = 0 \text{ or } 1).$$

Under the assumption that (1) is a linearization of (2), we can prove that

$$U(\alpha^i \mathring{v}_i)U^{-1} = \begin{bmatrix} \omega^{n_1} & & & 0 \\ & \omega^{n_2} & & \\ & & \ddots & \\ 0 & & & \omega^{n_N} \end{bmatrix},$$

where ω is an m-th root of unity. Therefore we see that

$$(U(\alpha^i \mathring{v}_i)U^{-1})^m = E_N,$$

i.e.,

$$(\alpha^i \mathring{v}_i)^m = E_N.$$

For an arbitrary v_i, if we take

$$\mathring{v}_i = \frac{v_i}{\| v \|},$$

then clearly $\| \mathring{v}_i \| = 1$, so that we have

$$\left(\alpha^i \frac{v_i}{\| v \|}\right)^m = E_N,$$

which is equivalent to:

$$v^m = (\alpha^i v_i)^m = \|v\|^m E_N ,$$

i.e.,

$$\alpha^{(i_1}\alpha^{i_2}\cdots\alpha^{i_m)} = g^{i_1 i_2\cdots i_m} E_N . \qquad (3)$$

Conversely, if α^i satisfy condition (3), then by operating $(\alpha^i \nabla_i)^{m-1}$ on the both sides of (1) we have

$$\alpha^{i_1}\alpha^{i_2}\cdots\alpha^{i_m}\nabla_{i_1}\nabla_{i_2}\cdots\nabla_{i_m}\psi = c(\alpha^i\nabla_i)^{m-1}\psi$$

i.e.

$$\alpha^{(i_1}\alpha^{i_2}\cdots\alpha^{i_m)}\nabla_{i_1}\nabla_{i_2}\cdots\nabla_{i_m}\psi = c^m\psi ,$$

by means of (3) we have

$$g^{i_1 i_2\cdots i_m}\nabla_{i_1}\nabla_{i_2}\cdots\nabla_{i_m}\psi = c^m\psi .$$

Therefore, (1) is a linearization of (2).

Theorem 2. A partial differential equation

$$\alpha^i\nabla_i\psi = c\,\psi \quad (c \neq 0) \qquad (1')$$

is a linearization of the partial differential equation:

$$g^{i_1 i_2\cdots i_m}\nabla_{i_1}\nabla_{i_2}\cdots\nabla_{i_m}\psi = c^m\psi , \qquad (2')$$

if and only if there exists an integer $p\ (\geqq 0)$ such that

$$v^{m+p} = \|v\|^m v^p ,$$

i.e.

$$\alpha^{(i_1}\alpha^{i_2}\cdots\alpha^{i_m}\alpha^{j_1}\cdots\alpha^{j_p)} = g^{(i_1 i_2\cdots i_m}\alpha^{j_1}\cdots\alpha^{j_p)} . \qquad (4)$$

<u>Proof</u>: Let $\mathring{v}_i$ be an arbitrary vector such that

$$g^{i_1 i_2 \cdots i_m} \mathring{v}_{i_1} \mathring{v}_{i_2} \cdots \mathring{v}_{i_m} = 1 ,$$

and then we transform $\alpha^i \mathring{v}_i$ into the Jordan canonical form:

$$X = U(\alpha^i \mathring{v}_i)U^{-1} = \sum \oplus \begin{bmatrix} \lambda & \mu & & 0 \\ & \lambda & \ddots & \\ & & \ddots & \mu \\ 0 & & & \lambda \end{bmatrix} , \ (\mu = 0 \text{ or } 1).$$

Under the assumption that (1') is a linearization of (2'), we can prove that

$$X = U(\alpha^i \mathring{v}_i)U^{-1} = \begin{bmatrix} \omega^{n_1} & & & 0 \\ & \omega^{n_2} & & \\ & & \ddots & \\ 0 & & & \omega^{n_r} \end{bmatrix} \oplus \sum \oplus \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ & & 0 \ddots & \\ 0 & & & \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \end{bmatrix}$$

Let p_0 be the maximal value of orders of blocks belonging to zero roots, then we have

$$(X^m - E_N)X^{p_0} = O_N ,$$

so that $$((\alpha^i \mathring{v}_i)^m - E_N)(\alpha^i \mathring{v}_i)^{p_0} = O_N .$$

Let p be the maximal value of p_0 for changing $\mathring{v}_i$ under the restriction $\| \mathring{v} \|^m = 1$, then clearly we have

$$((\alpha^i \mathring{v}_i)^m - E_N)(\alpha^i \mathring{v}_i)^p = O_N .$$

For any v_i , if we take $\mathring{v}_i = \dfrac{v_i}{\|v\|}$, then we know $\|\mathring{v}_i\|^m = 1$, and hence it holds

$$((\alpha^i v_i)^m - \|v\|^m E_N)(\alpha^i v_i)^p = O_N ,$$

that is,

$$v^{m+p} = \|v\|^m v^p .$$

The converse was already proved in 1.

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STRUCTURE OF ALGEBRA EXTENSION OF FINITE ABELIAN GROUP

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Introduction

During the preparations of [2] and [3], the author has not been aware of [1]. Now it should be referred and its logical connection with [2], [3] should be clarified. The concept of the 'generalized Clifford algebra' given in [1] may be regarded as a special kind of 'ring extension' given in [2] when the basic field is commutative (Cf. 2).

An algebra extension is a ring extension whose elements commute with the elements of the basic field. As a special case, the structure and representation of 'usual' generalized Clifford algebra which is written C_n^m were studied in [1] originally. [2] and [3] contain some clarifications of the structure of more general algebra extensions.

All concepts and theories given in this lecture will be taken only from [2], [3], except some illustrations. For the reader's convenience, some prerequisites will be added.

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1. Prerequisites

To avoid possible confusion, we shall recall some elementary concepts from the standard algebra.

By a monoid, we mean a set M with an associative binary operation

$$M \times M \rightarrow M \quad ((x, y) \mapsto xy)$$

such that there exists the neutral element 1. A group is a monoid whose elements x have their inverses x^{-1}.

Note. If the binary operation is written additively: $(x,y) \mapsto x+y$, the neutral element is written 0 and the inverse of x is written $-x$.

Let M, M' be monoids. A (monoid) morphism from M to M' is a mapping $h : M \rightarrow M'$ satisfying the conditions

$$h(xy) = h(x)h(y) \qquad (x,y \in M),$$

$$h(1_M) = 1_{M'},$$

where 1_M and $1_{M'}$ mean the neutral elements of M and M' respectively.

Note. If M' is a group, the first condition implies the second one.

Let M_1, M_2 be monoids. A pairing (= bimorphism) from M_1, M_2 to M' is a mapping $p : M_1 \times M_2 \rightarrow M'$ such that, for arbitrarily fixed elements $a_1 \in M_1$, $a_2 \in M_2$, the partial mappings

$$p(\cdot, a_2) : M_1 \rightarrow M' \qquad (x_1 \mapsto p(x_1, a_2))$$

$$p(a_1, \cdot) : M_2 \rightarrow M' \qquad (x_2 \mapsto p(a_1, x_2))$$

are morphisms.

By a *ring* we mean a set R with two binary operations - addition and multiplication - such that it is an abelian group under the addition, it is a monoid under the multiplication, and the multiplication

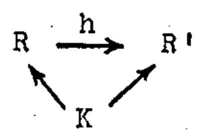
$$R \times R \longrightarrow R \ ((x, y) \longmapsto xy)$$

is a pairing under the addition. A *field* is a ring K such that $K^* = K - \{0\}$ forms an abelian group under the multiplication, we shall fix a field K.

Let R, R' be rings. A (*ring*) *morphism* from R to R' is a mapping $h : R \longrightarrow R'$ which is a monoid morphism under the addition and multiplication.

By a *ring over* K, we mean a ring R with a ring morphism $K \longrightarrow R$. If R is non-trivial, namely $R \neq \{0\}$, K may be regarded to be contained in R. An *algebra over* K is a ring over K whose elements commute with the elements of K.

Let R, R' be rings over K. A (*ring*) *morphism over* K from R to R' is a (ring) morphism $h : R \longrightarrow R'$ making the following diagram commutative:



Which means $h(\lambda a) = \lambda h(a)$, $h(a\lambda) = h(a)\lambda$ $(\lambda \in K,\ a \in R)$; an algebra morphism means always a ring morphism over K.

A ring R over K is regarded as a two-sided vector space over K whose scalor multiplications are naturally defined. A subset of R which is closed under the addition and scalar multiplication, is called a two-sided K-subspace. If R is an algebra over K, the word 'two-sided' may be replaced.

For a bijective (monoid or ring) morphism, its inverse mapping is also a morphism. Such a morphism is called a (monoid or ring) isomorphism. Let M be a monoid or ring. An isomorphism from M to itself is called an automorphism of M.

Let G be a (multiplicative) group acting on an abelian (multiplicative) group Ω as automorphism, we denote by σ the action. A mapping

$$f : G \times G \longrightarrow \Omega$$

is called a 2-cocycle of G in Ω, if it satisfies

$$(\sigma(x)f(y, z))\, f(xy, z)^{-1} f(x\, yz)\, f(x, y)^{-1} = 1 \qquad (x,y,z \in G).$$

All cocycles form an abelian group $Z^2_\sigma(G,\Omega)$.

For any mapping

$$g : G \longrightarrow \Omega,$$

the mapping $f : G \times G \longrightarrow \Omega$ defined by

$$f(x, y) = (\sigma(x)g(y))\, g(xy)^{-1} g(x) \qquad (x,y \in G)$$

is a 2-cocyle, such a 2-cocyle is called a 2-coboundary.

All 2-coboundaries form a subgroup $B^2_\sigma(G,\Omega)$ of $Z^2_\sigma(G,\Omega)$.

The quotient group $H^2_\sigma(G,\Omega) = Z^2_\sigma(G,\Omega)/B^2_\sigma(G,\Omega)$ is called the 2-cohomology group of G in Ω, its element is called a 2-cohomology class.

For the trivial action σ, $H^2_\sigma(G,\Omega)$ will be denoted simply by $H^2(G,\Omega)$.

2. The concept of ring and algebra extension

DEFINITION. Let K be a field and G a (multiplicative) group. A ring extension of G over K is a ring R over K with a direct sum decomposition

$$R = \sum_{x \in G} R_x$$

of R into a family of two-sided K-subspaces R_x, indexed by G, such that

1) $R_x = K e_x = e_x K$ for some $e_x \neq 0$ ($x \in G$);

2) $R_x R_y = R_{xy}$ ($x, y \in G$).

An algebra extension of G over K is a ring extension of G over K which is an algebra over K.

Note. For an algebra R over K, 1) is equivalent to the following:

1') Every R_x is of one dimension over K.

Two ring extensions R, R' of G over K are said to be isomorphic if there exists a ring isomorphism $h : R \to R'$ over K satisfying

$$h(R_x) = R'_x \qquad (x \in G).$$

Let a ring extension R of G over K be given. Then R_1 is identified with K, and G acts naturally on K as ring automorphisms we denote it by σ_R:

$$(\sigma_R(x))\ (\lambda) = e_x\ \lambda e_x^{-1} \quad (\lambda \in K,\ x \in G),$$

where the right side depends only on λ, x. σ_R may be considered as an action of G on the multiplicative group K^*.

Note. If σ_R is faithful, R is nothing but a so-called 'crossed product' of G and K where G is regarded as the Galois group of K over the fixed subfield. If and only if σ_R is trivial, R is an algebra extension.

Taking non-zero elements $e_x \in R_x$ $(x \in G)$, we define a mapping $f : G \times G \to K^*$ by the equalities

$$e_x\ e_y = f(x, y)\ e_{xy} \qquad (x, y \in G).$$

Then f is a 2-cocycle of G in K* whose cohomology class $c_R \in H^2_{\sigma_R}(G, K^*)$ depends only on R. In this way, we have a one-to-one correspondence between

all isomorphic classes of ring extensions of G over K

and

all 2-cohomology classes in $H^2_\sigma(G,K^*)$. where σ runs over all actions of G on the field K.

In particular, we have a one-to-one correspondence between

all isomorphic classes of algebra extensions of G over K

and

all 2-cohomology classes in $H^2(G,K^*)$.

Now suppose that G is decomposed into the direct product of normal subgroups $G_1,\dots,G_n$. Then we have the natural morphisms

$$H^2_\sigma(G,K^*) \longrightarrow H^2_{\sigma_i}(G_i, K^*) \qquad (c \mapsto c_i) \qquad (1 \leqq i \leqq n),$$

where σ_i is the action of G_i on K induced by σ. If a ring extension R of G over K satisfies the condition

$$(c_R)_i = 1 \qquad (1 \leqq i \leqq n),$$

which means $R_i = \sum_{x \in G_i} R_x$ is essentially the group algebra of G_i over K, then R is almost nothing but the 'generalized Clifford algebra' given in [1]*.

3. <u>Some basic considerations</u>

Hereafter we shall always assume the following

ASSUMPTION. G <u>is a finite abelian group with exponent</u> e (= the greatest order of elements) <u>and</u> K <u>is a field containing a primitive</u> e-th <u>root of unity</u>.

$H^2_{ab}(G,K^*)$ denotes the subgroup of $H^2(G,K^*)$ consisting of all cohomology classes containing 'abelian cocyle' f

$$f(x,y) = f(y,x) \qquad (x, y \in G).$$

Let Ω be an arbitrary abelian group. $P_{a.s.}(G,\Omega)$ denotes

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*The author would like to call such a ring extension with direct product decomposition 'of M-N type'. However, exactly saying, this is not the same as given in [1], because of the difference of formulations. To get the concept given in [1], we should add a little more restrictions.

the group consisting of all 'anti-symmetric' pairings

$\varphi : G \times G \longrightarrow \Omega$:

$$\varphi(x,y) = \varphi(y,x)^{-1} \quad (x, y \in G).$$

For any $c \in H^2(G,\Omega)$, $\varphi_c \in P_{a.s.}(G,\Omega)$ is defined by

$$\varphi_c(x,y) = f(x,y)\, f(y,x)^{-1} \qquad (x,y \in G)$$

where f belongs to c, φ_c depends only on c.

LEMMA 1. <u>We have the following natural splitting exact</u> sequence.

$$1 \longrightarrow H^2_{ab}(G,\Omega) \longrightarrow H^2(G,\Omega) \longrightarrow P_{a.s.}(G,\Omega) \longrightarrow 1$$

$$(c \longmapsto \varphi_c)$$

<u>Proof.</u> See [2] p. 160.

<u>Note.</u> Let $\underline{Z}_m$ denote the cyclic group of order m. For $G = \underline{Z}_{m_1} \times \cdots \times \underline{Z}_{m_n}$, $H^2_{ab}(G,\Omega)$ has the following structure.

$$\cong \Omega/\Omega^{m_1} \times \cdots \times \Omega/\Omega^{m_n}$$

where Ω^m is consisting of all m-th powers of elements in Ω. (see [2] p. 159). In particular, $H^2{}_{ab}(G,\underline{C}^*)$ is trivial, where $\underline{C}$ is the field of complex numbers. Hence we have

COROLLARY. $H^2(G,\underline{C}^*) \cong P_{a.s.}(G,\underline{C}^*)$.

$\mathrm{Hom}(G,\Omega)$ denotes the group consisting of all morphisms from G to Ω. Let Ω be the quotient group $\overline{K}^*/K^*$ where $\overline{K}$ is the algebraic closure of K. Then to any $\psi \in \mathrm{Hom}(G,\Omega)$, taking a section $\lambda : G \to \bar{K}^*$ of ψ and 2-cocycle

$$f(x, y) = \lambda(y)\ \lambda(xy)^{-1}\ \lambda(x),$$

we assign a cohomology class $c \in H^2_{ab}(G,K^*)$ containing f, c depends only on ψ, which will be denoted by ψ_c.

LEMMA 2. <u>We have the following isomorphism</u>

$$H^2_{ab}(G,K^*) \cong \mathrm{Hom}(G,K^*/K^*).$$
$$(c \mapsto \psi_c)$$

<u>Proof</u>. See [3] pp. 41-42.

Now, for any $c \in H^2(G,K^*)$, we define two subgroups N_c and M_c of G as follows. N_c denotes the annihilator of φ_c, namely the subgroup consisting of all $y \in G$ such that

$$\varphi_c(x, y) = 1 \quad \text{for all} \quad x \in G.$$

Applying Lemma 1 to the group N_c and K^*, we see the restriction c' of c to N_c belongs to $H^2_{ab}(N_c,K^*)$. So, by the correspondence in Lemma 2, we have $\psi_{c'}$) $\mathrm{Hom}(N_{c'},K^*/K^*)$. M_c denotes the kernel of $\psi_{c'}$, this is a subgroup of $N_c \subset G$.

4. <u>Structure theorem</u>.

Any semi-simple algebra A over a field K is decomposed into the direct sum of all minimal two-sided ideals which are simple algebras:

$$A = S_1 + \dots + S_\mu.$$

Note. μ depends only on A and may be written μ_A. This is equal to the number of all non-similar irreducible representations of A over K.

According to this decomposition, we have the following expressions

$$1 = \varepsilon_1 + \cdots + \varepsilon_\mu \quad (\varepsilon_i \in S_i).$$

Then we define two subalgebras B(A) and C(A) of A as follows. B(A) denotes the subalgebra generated by $\varepsilon_1, \ldots, \varepsilon_\mu$: $B(A) = K\varepsilon_1 + \ldots\ldots + K\varepsilon_\mu$. C(A) denotes the center of A which is the sum of centres of $S_i (1 \leqq i \leqq \mu)$:

$$C(A) = C(S_1) + \ldots + C(S_\mu).$$

THEOREM 1. Let G be a finite abelian group of exponent e and K a field containing a primitive e-th root of unity. Then any algebra extension A of G over K has the following structure:

1) A is a semi-simple algebra with mutually isomorphic simple components.

2) $B(A) = \sum_{x \in M_c} A_x$.

(A corresponds to $c \in H^2(G, K^*)$.)

3) $C(A) = \sum_{x \in N_c} A_x$.

Proof. See [2] pp. 175, 179-182 and [3] pp. 44-45.

<u>Remark</u>. 1) follows from the fact that the group $\hat{G} = \mathrm{Hom}(G,K^*)$ acts on A and induces some transitive permutations on all simple components. [3] contains a simplified proof for this fact which means that there exists only one non-similar irreducible 'projective' representation of G over K corresponding to a given cohomology class c, M_c is the kernel of this representation.

Moreover the centres $C(S_i)$ are isomorphic Kummer fields over K whose automorphisms are induced by G. See [2] pp.180-182 in detail.

<u>Note</u>. If K is the field $\underline{C}$ of complex numbers, we have $M_c = N_c$ and

$$A \cong M_d(\underline{C}) \oplus \cdots \oplus M_d(\underline{C}) \qquad (\mu_A \text{ copies})$$

where $M_d(\underline{C})$ is the total matric algebra over K of degree d. We note that $\mu_A d^2$ is equal to the order of G.

5. <u>Illustrations</u>-1

Taking some simple cases, let us illustrate the theory given in the sections 3,4. In this section, we assume that

$$K = \underline{C} \quad \text{(the field of complex numbers)}$$

$$G = \underline{Z}_m \times \cdots \times \underline{Z}_m \quad \text{(n copies)}.$$

Then the anti-symmetric pairings $\varphi \in P_{a.s.}(G,K^*)$ are given by the tables

φ	s_1	s_2	....	s_n
s_1	ω_{11}	ω_{12}		ω_{1n}
s_2	ω_{21}	ω_{22}		ω_{2n}
⋮				
s_n	ω_{n1}	ω_{n2}		ω_{nn}

where the s_i generate G and the $\omega_{ij} \in K$ satisfy the conditions

$$\omega_{ij}{}^m = 1, \quad \omega_{ij} = \omega_{ji}^{-1} \qquad (1 \leqq i, j \leqq n).$$

We have such $m^{n(n-1)/2}$ tables. Hence, by Corollary to Lemma 1, there are $m^{n(n-1)/2}$ non-isomorphic algebra extensions of G over K.

Note. The above number is not in the sense of algebra but 'algebra extension'. The number of non-isomorphic algebra extensions in the sense of 'algebra' can be also calculated using Theorem 2 in the next section. (Cf.1.)

For each anti-symmetric pairing φ , we take the pairing f given by the table

f	s_1	s_2	...	s_n
s_1	1	1		1
s_2	ω_{21}	1		1
⋮				
s_n	ω_{n1}	ω_{n2}		1

where all entries in the upper half triangle are 1.

Note. Any pairing $f : G \times G \rightarrow K^*$ is a 2-cocycle of G in K^*.

We see that the cohomology class c containing f determines $\varphi = \varphi_c$. Using the above f, the corresponding algebra extension A is given by the generators $e_1, \ldots, e_n$ with the fundamental relations

$$e_i^{\,m} = 1, \; e_i \, e_j = \omega_{ij} \, e_j \, e_i \; (i > j).$$

($e_i = e_{s_i} \in A_{s_i}$ $(1 \leq i \leq n)$ according to the previous notation).

Now let us find the structure of A. The subgroups M_c and N_c of G are the same and calculated as follows:

Let ω be a primitive m-th root of unity and α_{ij} integers such that

$$\omega_{ij} = \omega^{\alpha_{ij}} \quad (1 \leq i, j \leq n).$$

Since N_c is the annihilator of φ_c, we have

$$M_c = N_c = \{ s_1^{\beta_1} \ldots s_n^{\beta_n} ; \sum_{j=1}^{n} \alpha_{ij} \beta_j \quad 0 (\text{mod. } m) \; (1 \leq i \leq n) \} .$$

Note. This subgroup is reduced to the unit group which means A is simple, if and only if the determinant of the matrix (α_{ij}) is coprime with m.

Example 1. ('usual' generalised Clifford algebras). The algebra C_n^m may be defined by the generators $e_1, \ldots, e_n$ with the relations

$$e_i^{\,m} = 1 \; (1 \leq i \leq n) \quad e_i \, e_j = \omega \, e_j \, e_i \quad (i > j),$$

where ω is a primitive m-th root of unity.

In this case, we have

$$\alpha_{ij} = \begin{cases} 1 & (i > j) \\ 0 & (i = j) \\ -1 & (i < j). \end{cases}$$

If n is even, then det $(\alpha_{ij}) = 1$ and C_n^m is simple.

If n is odd, then solving the equation

$$\left.\begin{array}{l} -\beta_2 - \beta_3 - \cdots\cdots - \beta_n \equiv 0 \\ \beta_1 \quad - \beta_3 - \cdots\cdots - \beta_n \equiv 0 \\ \beta_1 + \beta_2 \quad - \cdots\cdots - \beta_n \equiv 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \beta_1 + \beta_2 + \beta_3 + \cdots\cdots + \beta_{n-1} \equiv 0 \end{array}\right\} \pmod{m}$$

we see that $M_c = N_c$ is generated by the single element

$$s_1 s_2^{-1} s_3 s_4^{-1} \cdots ,$$

whose order is m. Hence we have

$$C_n^m \cong \begin{cases} M_{m^\nu}(\underline{C}) & (n = 2\nu) \\ M_{m^\nu}(\underline{C}) \oplus \cdots \oplus M_{m^\nu}(\underline{C}) & (n = 2\nu + 1) \ (\text{m copies}) \end{cases}$$

Example 2. The algebra generated by $e_1, \ldots, e_{2\nu}$ with the relations

$$e_i^m = 1 \ (1 \leq i \leq 2\nu), \ e_i e_j = \begin{cases} \omega e_j e_i & (i = j+1 \text{ is even}) \\ e_j e_i & (\text{otherwise}) \end{cases}$$

is isomorphic to $M_{m^\nu}(\underline{C})$. The proof is left to the readers as an exercise.

6. Existence theorem.

According to the notation in Theorem 1, we shall determine all possible subgroups M_c and N_c of G.

THEOREM 2. Let G be a finite abelian group of exponent e and K a field containing a primitive e-th root of unity. Then for given two subgroups M and N of G there exists an algebra extension A such that $M_c = M$ and $N_c = N$, if and only if the following hold:

1) The quotient group G/N is of symmetric type.
2) The quotient group $\bar{K}^*/K^*$ contains a subgroup isomorphic to N/M, where N is supposed to contain M.

(Let c correspond to A).

Proof. See [3] pp. 49-50.

Note. A group H is called 'of symmetric type' if it is isomorphic to the direct product of two isomorphic groups: $H = H' \times H'$. If K is the field $\underline{C}$ of complex numbers, then the condition 2) is trivial and we have:

COROLLARY. Let G be a finite abelian group and d, μ natural numbers. Then there exists an algebra extension A isomorphic to

$$M_d(\underline{C}) \oplus \cdots\cdots \oplus M_d(\underline{C}) \qquad (\mu \text{ copies}),$$

if and only if G contains a subgroup N such that

1) G/N is of symmetric type
2) The order of N is equation to μ
3) The order of G is equal to μd^2.

7. Illustrations-2

Taking the same group as in the section 5, let us illustrate the theory given in the section 6. Namely we assume

$$G = \underline{Z}_m \times \cdots \times \underline{Z}_m \quad \text{(n copies)}.$$

Example 1. As already shown, any algebra extension A of G over the field $\underline{C}$ has the following structure:

$$A = M_d(\underline{C}) \oplus \cdots \oplus M_d(\underline{C}) \qquad (\mu \text{ copies})$$

with the trivial condition

$$m^n = \mu d^2.$$

Now applying Corollary to Theorem 2, we can know what numbers d, μ are really possible:

There exists an algebra extension A of $G = \underline{Z}_m \times \cdots \times \underline{Z}_m$ (n copies) over $\underline{C}$ such that

$$A \cong M_d(\underline{C}) \oplus \cdots \oplus M_d(\underline{C}) \qquad (\mu \text{ copies})$$

if and only if d, μ satisfy the conditions:

1) $m^n = \mu d^2$.
2) If n is odd, then m devides μ.

Proof. Let N be any subgroup of G. Then there are n positive integers $m_1, \dots, m_n$ such that

$$G/N \cong Z_{m_1} \times \cdots \times Z_{m_n} \text{ and } m_i \text{ devides } m_{i+1} \ (1 \leqq i < n),$$

since G/N has n generators. Considering the orders of its elements, the m_i must be divisors of m. Conversely, for any divisors m_i of m, there clearly exists such a subgroup N. On the other hand, we note that, for a finite abelian group of order h, any divisor of h is possible as an order of some subgroup.

Hence, by Corollary to Theorem 2, we easily see the existence of an algebra extension of given type under the conditions 1), 2). Let us see the converse. For an even number n, the condition 2) is trivial. For an odd number n, if G/N is of symmetric type then $m_1 = 1$ which implies that m devides μ, namely we have 2).

Example 2. We take $K = \underline{Q}$ (the field of rational numbers) and m = 2 for the simplicity. Then we can determine all 'commutative'. algebra extensions of $G = \underline{Z}_2 \times \cdots \times \underline{Z}_2$ over $\underline{Q}$:

$$A \cong L \oplus \cdots \oplus L \ (\mu \text{ copies}),$$

where L is a Kummer field over Q. By Theorem 2, we easily see that any divisor μ of 2^n is really possible. We note that any Kummer field with a Galois group homomorphic to G is possible as L. Such a fact holds also for the general case. (See [2] pp. 190-192.)

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PROJECTIVE REPRESENTATIONS OF FINITE GROUPS

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Notation

Let G be a finite group, K an algebraically closed field of characteristic $p \nmid |G|$ (for example the field of complex numbers) $K^* = K - \{0\}$, V a finite dimensional vector space over K with $(V:K) = n$, GL(V) the group of non-singular linear transformations on V(or the group of invertible $n \times n$ matrices over K).

1. Factor Sets

Definition 1.1. A mapping $\alpha : G \times G \longrightarrow K^*$ is called a factor set of G in K if

$$\alpha(x,y)\,\alpha(xy,z) = \alpha(x,yz)\,\alpha(y,z), \text{ for all } x,y,z \in G \qquad (1.1)$$

and

$$\alpha(x,e) = \alpha(e,x) = \alpha(e,e) = 1, \text{ for all } x \in G, \qquad (1.2)$$

where e denotes the identity element of G. Two factor sets α and β of G in K are said to be equivalent if there exists a mapping $\mu : G \longrightarrow K^*$ such that $\mu(e) = 1$ and

$$\beta(x,y) = \frac{\mu(x)\mu(y)}{\mu(xy)}\,\alpha(x,y) \qquad \text{for all } x,y \in G.$$

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Factor sets satisfy various elementary properties which are obtained in the following series of lemmas. Let $M(G,K^*)$ denote the set of all factor sets of G in K.

Lemma 1.2. $M(G,K^*)$ is a multiplicative abelian group.

Proof. Exercise.

Lemma 1.3. If $\alpha \in M(G,K^*)$, then there exists $\beta \in M(G,K^*)$ such that β is equivalent to α and

$$\beta(x, x^{-1}) = 1 \qquad \text{for all } x \in G. \qquad (1.3)$$

Proof. Define

$$\mu(x) = \frac{1}{\sqrt{\alpha(x, x^{-1})}} \qquad \text{for all } x \in G.$$

Then, the factor set β defined by

$$\beta(x, y) = \frac{\mu(x)\mu(y)}{\mu(xy)}\alpha(x, y)$$

has the required property, since

$$\beta(x, x^{-1}) = \left\{ \frac{\alpha(e, e^{-1})}{\alpha(x, x^{-1})\alpha(x^{-1}, x)} \right\}^{\frac{1}{2}} \alpha(x, x^{-1}) = 1 .$$

Lemma 1.4. (Schur [49]) If $\alpha \in M(G,K^*)$, then there exists $\beta \in M(G,K^*)$ such that β is equivalent to α and

$$\beta(x, y)^{|G|} = 1 \qquad \text{for all } x, y \in G. \qquad (1.4)$$

Proof. Define

$$\mu(x) = \prod_{g \in G} \alpha(x, g) ,$$

then

$$\frac{\mu(x)\mu(y)}{\mu(x,y)} = \frac{\prod_{g\in G}\alpha(x,g)\prod_{g\in G}\alpha(y,g)}{\prod_{g\in G}\alpha(xy,g)} = \alpha(x,y)^{|G|},$$

using (1.1). If we now let

$$\nu(x) = \mu(x)^{-\frac{1}{|G|}},$$

then the factor set β defined by

$$\beta(x,y) = \frac{\nu(x)\nu(y)}{\nu(xy)}\alpha(x,y) \quad \text{for all } x,y \in G$$

has the required property.

Definition 1.5. A factor set $\alpha \in M(G,K^*)$ is called a normalized factor set if

$$\alpha(x,x^{-1}) = 1 \qquad \text{for all } x \in G. \qquad (1.5)$$

Lemma 1.6. If $\alpha \in M(G,k^*)$ is a normalized factor set, then

$$\alpha^{-1}(x,y) = \alpha(y^{-1},x^{-1}) \qquad \text{for all } x,y \in G. \qquad (1.6)$$

Proof. By(1.1)

$$\alpha(x,y)\,\alpha(xy,y^{-1}) = \alpha(x,e)\alpha(y,y^{-1}) = 1,$$

and

$$\alpha(xy,y^{-1})\alpha(x,x^{-1}) = \alpha((xy),(xy)^{-1})\alpha(y^{-1},x^{-1}).$$

That is, we have

$$\alpha(x,y)^{-1} = \alpha(xy, y^{-1}) = \alpha(y^{-1}, x^{-1}) ,$$

as required.

Definition 1.7. An element $s \in G$ is called a α-regular element if

$$\alpha(s,x) = \alpha(x,s) \qquad \text{for all} \quad x \in C_G(s) = \{x \in G \mid xs = sx\} .$$

($C_G(s)$ is called the centralizer of s in G), where $\alpha \in M(G,K^*)$.

We shall consider in particular for each $\alpha \in M(G,K^*)$ and for each α-regular element $s \in G$

$$f_\alpha(x,s) = \alpha(x,s)\alpha^{-1}(xsx^{-1}, x) \qquad \text{for all } x \in G. \tag{1.7}$$

We note in particular that $f_\alpha(x,s)=1$ if $x \in C_G(s)$. Further, if α is a normalized factor set

$$f_\alpha(x,s) = \alpha(x,s)\,\alpha(xs, x^{-1}) . \tag{1.8}$$

Then, the following can be proved.

Lemma 1.8. If $\alpha \in M(G,K^*)$ is a normalized factor set and $s \in G$ is a α-regular element, then

(i) $f_\alpha(x,s) = f_\alpha^{-1}(x,s^{-1})$ for all $x \in G$ (1.9)

(ii) if $xsx^{-1} = ysy^{-1}$ with $x,y \in G$, then

$$f_\alpha(x,s) = f_\alpha(y,s) . \tag{1.10}$$

Proof. (i) $f_\alpha(x,s)f_\alpha(x,s^{-1}) = \alpha(x,s)\alpha(xs,x^{-1})\alpha(x,s^{-1})\alpha(xs^{-1},x^{-1})$
$= \alpha(x,s)\alpha(xs,x^{-1})\alpha(x,s^{-1}x^{-1})\alpha(s^{-1},x^{-1})$
$=1$ by Lemma 1.5.

(ii) $f_\alpha(x,s)\, f_\alpha^{-1}(y,s) = \alpha(x,s)\alpha(xs,x^{-1})\alpha(xsx^{-1},y)\alpha^{-1}(y,s)$

$$= \alpha(x,s)\alpha(xs,x^{-1}y)\alpha(x^{-1},y)\,\alpha^{-1}(y,s)$$

$$= \alpha(x,sx^{-1}y)\,\alpha(s,x^{-1}y)\,\alpha(x^{-1},ys)\alpha^{-1}(x^{-1}y,s)$$

But $xsx^{-1} = ysy^{-1}$ and $x^{-1}y \in C_G(s)$ and since s is α-regular

$$\alpha(s,x^{-1}y)\,\alpha^{-1}(x^{-1}y,s) = 1$$

Thus

$$f_\alpha(x,s)\, f_\alpha^{-1}(y,s) = \alpha(x,sx^{-1}y)\,\alpha(x^{-1},ys)$$
$$= \alpha(x,x^{-1}ys)\,\alpha(x^{-1},ys)$$
$$= 1$$

These results are now used to prove

<u>Theorem 1.9</u>. If $\alpha \in M(G,K^*)$ is normalized and s is a α-regular element of G, then

(i) s^{-1} is a α-regular element,

(ii) every element conjugate to s is α-regular.

Proof. (i) It is easily verified that $C_G(s) = C_G(s^{-1})$. Further

$$f_\alpha(x,s) = 1 \qquad \text{for all } x \in C_G(s)$$

and by Lemma 1.8 (i)

$$f_\alpha(x,s^{-1}) = 1 \qquad \text{for all } x \in C_G(s^{-1}),$$

and so s^{-1} is a α-regular element.

(ii) Again, it is easily verified that $C_G(tst^{-1}) = t\,C_G(s)$

We must show that

$$f_\alpha(y, tst^{-1}) = 1 \qquad \text{for all } y \in C_G(tst^{-1})$$

But

$$f_\alpha(y, tst^{-1}) = \alpha(y, tst^{-1})\,\alpha^{-1}(tst^{-1}, y)$$

Since $y \in C_G(tst^{-1})$, it follows that

$$(yt)s(yt)^{-1} = tst^{-1} ,$$

and thus by Lemma 1.8(ii)

$$f_\alpha(yt, s) = f_\alpha(t, s)$$

and

$$\begin{aligned} f_\alpha(yt, s) &= \alpha(yt, s)\,\alpha(yts, t^{-1}y^{-1}) \\ &= \alpha(yt, s)\,\alpha(yts, t^{-1})\,\alpha(tst^{-1}y, y^{-1})\,\alpha^{-1}(t^{-1}, y^{-1}) \\ &= \alpha(y, ts)\,\alpha(yts, t^{-1})\,\alpha(tst^{-1}y, y^{-1})\,\alpha(t, s) \\ &= \alpha(y, tst^{-1})\,\alpha(tst^{-1}y, y^{-1})\,\alpha(ts, t^{-1})\,\alpha(t, s) \\ &= f_\alpha(y, tst^{-1})\, f_\alpha(t, s) . \end{aligned}$$

That is

$$f_\alpha(y, tst^{-1}) = 1 \quad \text{for all } y \in C_G(tst^{-1})$$

As a result of this theorem, we have

Definition 1.10. A class of conjugate elements of G is called a α-regular class if all its elements are α-regular. (In the physical literature such a class has been called a ray class (Harter [22]).

We note in particular that if a class $\mathcal{K}$ of G is α -regular, then the class $\mathcal{K}^*$ of inverses of elements of $\mathcal{K}$ is α -regular.

Before proceeding to our main theorem, we note the following

Lemma 1.11. If an element $s \in G$ is α -regular, then it is β - regular for all $\beta \in M(G,K^*)$ which are equivalent to α .

Proof. Exercise.

Theorem 1.12. (Conlon [10]). If $\alpha \in M(G,K^*)$, then there exists a $\gamma \in M(G,K^*)$ equivalent to α such that both the following conditions are satisfied

$$\text{(i) } \gamma(x, x^{-1}) = 1 \qquad \text{for all } x \in G \qquad (1.11)$$

$$\text{(ii) } f_\gamma(x, s) = 1 \quad \text{for all } \gamma\text{-regular } s \in G \text{ and for all } x \in G. \qquad (1.12)$$

Proof. If $\alpha \in M(G,K^*)$, then by Lemma 1.3 there exists a $\beta \in M(G,K^*)$ equivalent to α which satisfies (1.11).

Let $\{a_1, \ldots, a_n\}$ be an arbitrary β -regular class of G, then $a_i = x_i a_1 x_i^{-1}$ where $G = \bigcup_{i=1}^{n} C_G(a_1) x_i$. Now, define

$$\mu(a_i) = f_\beta(x_i, a_1), \quad (i = 1, \ldots, n).$$

Similarly, define $\mu(a)$ for all β -regular elements $a \in G$. If a is not β -regular, let

$$\mu(a) = 1.$$

Then by (1.10), μ is well-defined and $\mu(a_1) = 1$ (in particular $\mu(e) = 1$). Now, let

$$\gamma(x,y) = \frac{\mu(x)\mu(y)}{\mu(xy)} \beta(x,y) \qquad \text{for all } x, y \in G.$$

Then

$$\gamma(a_i, a_i^{-1}) = \frac{\mu(a_i)\,\mu(a_i^{-1})}{\mu(e)}\,\beta(a_i, a_i^{-1})$$

$$= f_\beta(x_i, a_1)\, f_\beta(x_i, a_1^{-1})$$

$$= f_\beta(x_i, a_1)\, f_\beta^{-1}(x_i, a_1)$$

$$= 1$$

by (1.9) and so γ satisfies (1.11). In addition

$$f_\gamma(x, a_i) = \gamma(x, a_i)\,\gamma^{-1}(x a_i x^{-1}, x)$$

$$= \frac{\mu(a_i)}{\mu(x a_i x^{-1})}\,\beta(x, a_i)\,\beta^{-1}(x a_i x^{-1}, x)$$

$$= \beta(x_i, a_1)\,\beta(x, a_i x_i)\,\beta(x^{-1}, x x_i)\,\beta^{-1}(x x_i, a_1)$$

$$= \beta(x_i, a_1)\,\beta(x, x_i a_1)\,\beta(x^{-1}, x x_i)\,\beta^{-1}(x x_i, a_1)$$

$$= \beta(x_i, x_i)\,\beta(x^{-1}, x x_i)$$

$$= 1$$

and so γ also satisfies (1.12).

Remark. Unfortunately, the factor set γ defined in the above theorem does not in general simultaneously satisfy (1.4). Conlon [10] has in fact proved that

$$\gamma(x, y)^{2|G|} = 1 \qquad \text{for all } x, y \in G.$$

A factor set $\alpha \in M(G, K^*)$ which satisfies (1.11) and (1.12) is called a c-normalised factor set.

2. Projective Representations and Twisted Group Algebras

Definition 2.1 A mapping $T:G \longrightarrow GL(V)$ is called a projective representation with representation space V and factor set α if

$$T(x)T(y) = \alpha(x,y)T(xy) \text{ for all } x,y \in G \qquad (2.1)$$

$$T(e) = I_V , \qquad (2.2)$$

where $\alpha(x,y) \in K$. (Alternatively, it may be considered as a mapping $T:G \longrightarrow GL(n,K)$ satisfying (2.1) and $T(e)=I_n$, when it is referred to as a projective representation of degree n with factor set α . These two definitions are taken to be equivalent).

It is easily verified that the $\alpha : G \times G \rightarrow K^*$ is a factor set of G in K, that is, $\alpha \in M(G,K^*)$. If $\alpha(x,y)=1$, for all $x,y \in G$, then T is an ordinary representation of G, which we shall call a linear representation.

Definition 2.2. If S and T are projective representations of G with representation spaces V_1 and V_2 respectively and factor sets α and β respectively, then S and T are projectively equivalent if there exists an isomorphism $P:V_1 \longrightarrow V_2$ and a mapping $\mu : G \longrightarrow K^*$ with $\mu(e)=1$ such that

$$PS(x) = \mu(x)T(x)P \quad \text{for all } x \in G. \qquad (2.3)$$

If $\mu(x)=1$, for all $x \in G$ then S and T are said to be linearly equivalent.

Lemma 2.3. The factor sets of projectively equivalent projective representations are equivalent. Linearly equivalent projective representations have the same factor set.

Proof. Exercise.

Lemma 2.4. For any $\alpha \in M(G,K^*)$, there exists a projective representation T of G with factor set α.

Proof. Let V be a vector space with K-basis $\{\nu(x) \mid x \in G\}$

Define $T: G \longrightarrow GL(V)$ by

$$T(x)\nu(y) = \alpha(x,y)\nu(xy) \qquad \text{for all } x,y \in G.$$

Then we have

$$\begin{aligned} T(x)T(y)\nu(z) &= T(x)\alpha(y,z)\nu(yz) \\ &= \alpha(x,yz)\alpha(y,z)\nu(xyz) \\ &= \alpha(x,y)\alpha(xy,z)\nu(xyz) \\ &= \alpha(x,y)T(xy)\nu(z) \end{aligned}$$

for all $x,y,z \in G$. That is

$$T(x)T(y) = \alpha(x,y)T(xy) \text{ for all } x,y \in G.$$

Further,

$$\begin{aligned} T(e)\nu(x) &= \alpha(e,x)\nu(x) \\ &= \nu(x), \end{aligned}$$

that is

$$T(e) = I_V.$$

Thus, T is the required projective representation with factor set α.

Definition 2.5. Conlon [10], Yamazaki [60], Tazawa [55].

Let $\alpha \in M(G,K)$ and

$$(KG)_\alpha = \left\{ \sum_{x \in G} \xi_x \nu(x) \,\middle|\, \xi_x \in K \right\},$$

where $\{\nu(x) \mid x \in G\}$ is a set of elements in one-one correspondence with the elements of G. Then, addition on $(KG)_\alpha$ is defined by

$$\sum_{x\in G} \xi_x \nu(x) + \sum_{x\in G} \eta_x \nu(x) = \sum_{x\in G} (\xi_x + \eta_x) \nu(x) ,$$

scalar multiplication of elements of $(KG)_\alpha$ by elements of K by

$$\xi \left(\sum_{x\in G} \xi_x \nu(x) \right) = \sum_{x\in G} \xi \xi_x \nu(x) ,$$

and multiplication on $(KG)_\alpha$ by

$$\left(\sum_{x\in G} \zeta_x \nu(x) \right) \left(\sum_{x\in G} \eta_x \nu(x) \right) = \sum_{x,y\in G} \zeta_x \eta_y \alpha(x,y) \nu(xy)$$

Then, the following lemma is easily proved .

Lemma 2.6. $(KG)_\alpha$ is an associative algebra with identity $\nu(e)$, called the twisted group algebra associated to $\alpha \in M(G,K^*)$.

This is a natural generalization of the familiar group algebra which plays such an important role in the ordinary representation theory of G(see e.g. [11]). Our aim now is to show that all of the usual theorems on group algebras can be generalized to twisted group algebras.

Lemma 2.7. There is a one-one correspondence between the projective representations of G with factor set α and the representations of $(KG)_\alpha$. Furthermore, there is a one-one correspondence between the representations of $(KG)_\alpha$ and finite dimensional left $(KG)_\alpha$ -modules.

Proof. This is left as an exercise, the proof being identical with that for the corresponding result for ordinary representations (See e.g. [11] , p.44-48).

Thus, the problem of determining all the projective representations of G with factor set α is equivalent to the determination of all left $(KG)_\alpha$ -modules V which are finite dimensional as vector spaces. If V is a left $(KG)_\alpha$ -module, then define $T: G \to GL(V)$ by

$$T(x)\,V = \nu(x)\,V \qquad \text{for all } x \in G, v \in V.$$

Then $T(x) \in GL(V)$, since V is a left $(KG)_\alpha$ -module, i.e.

$$T(x)(\alpha v_1 + v_2) = \nu(x)(\alpha v_1 + v_2) = \alpha\,\nu(x)\,v_1 + \nu(x)\,v_2$$
$$= \alpha\,T(x)\,v_1 + T(x)\,v_2,$$

and $T(x)v = 0 \Rightarrow \nu(x)v = 0$ which is impossible. Further

$$\begin{aligned} T(x)T(y)v &= \nu(x)\nu(y)v \\ &= \alpha(x,y)\nu(xy)v \\ &= \alpha(x,y)T(x,y)v, \end{aligned}$$
$$\text{for all } x, y, \in G,\ v \in V,$$

and $T(e)v = \nu(e)v = v$. This representation constructed from the left $(KG)_\alpha$ -module V is called the projective representation <u>afforded</u> by V. In particular $(KG)_\alpha$ itself may be regarded as a left $(KG)_\alpha$ module. The projective representation afforded by $(KG)_\alpha$ is called the <u>left regular projective representation with factor set</u> α of G. That is $\{\nu(g_i) \mid i = 1, \dots, |G|\}$ may be regarded as a K-basis for the vector space $(KG)_\alpha$, if g is an arbitrary element of G, define

$$\begin{aligned} R(x)\nu(g_i) &= \nu(x)\nu(g) \qquad \text{for all } g \in G \\ &= \alpha(x, g_i)\,\nu(xg_i) = \alpha(xg_i) \end{aligned}$$

and

$$R(x)\,R(y)\,\nu(g) = \alpha(x,y)\,R(x,y)\,\nu(g) \quad \text{for all } g \in G$$

and so

$$R(x)\,R(y) = \alpha(x,y)\,R(xy)$$

The matrix representation $R(x)$ is the matrix whose only non-zero element in the i'th column is $\alpha(x, g_i)$ which appears in the j'th row if $x g_i = g_j$. For the sake of completeness, we include the following definitions.

Definition 2.8. A left $(KG)_\alpha$ -module V is irreducible if its only submodules are the trivial submodules $\{o\}$ and V. A left $(KG)_\alpha$ -module V is a completely reducible module if for every submodule V_1 of V there exists a submodule V_2 of V such that

$$V = V_1 \oplus V_2 .$$

The following generalization of Maschke's Theorem is now proved

Theorem 2.9. Every left $(KG)_\alpha$ -module V is completely reducible.

Proof. Let V_1 be a non-trivial submodule of V. Then V_1 is a subspace of V and there exists a subspace W of V such that

$$V = V_1 \oplus W.$$

Thus, there exists an $E \in \mathrm{Hom}_K(V, V_1)$ such that if

$$v = v_1 + \omega \quad , v \in V, \; v_1 \in V_1 , \; \omega \in W ,$$

then $Ev = v_1$. Let $F: V \longrightarrow V$ be defined by

$$Fv = \frac{1}{|G|} \sum_{x \in G} \nu(x)\, E\, \nu(x)^{-1} v \qquad \text{for all } v \in V,$$

(which is meaningful since $|G| \neq o$ in K). Then $F \in \mathrm{Hom}_{(KG)_\alpha}(V,V)$, since for all $v \in V$,

$$F\nu(y)v = \frac{1}{|G|}\sum_{x\in G} \nu(x) E \nu(x)^{-1}\nu(y)v$$

$$= \frac{1}{|G|}\nu(y)\sum_{z\in G} \frac{\alpha(z^{-1}y^{-1}, y)\alpha(z, z^{-1})}{\alpha(yz, (yz)^{-1})\alpha(y, z)} \nu(z) E \nu(z)^{-1} v$$

and by (1.1)

$$\alpha(z^{-1}y^{-1}, y)\,\alpha(z^{-1}, z) = \alpha(z^{-1}y^{-1}, yz)\,\alpha(y, z),$$

that is

$$F\nu(y)v = \nu(y)Fv \qquad \text{for all } y \in G, v \in V.$$

Furthermore, $FV \subseteq V_1$ since

$$\frac{1}{|G|}\sum_{x\in G} \nu(x) E \nu(x)^{-1} v \in \frac{1}{|G|}\sum_{x\in G} \nu(x) E V \in V_1 ,$$

since V_1 is a submodule of V. Also if $v_1 \in V_1$,

$$Fv_1 = \frac{1}{|G|}\sum_{x\in G} \nu(x) E \nu(x)^{-1} v_1$$

$$= \frac{1}{|G|}\sum \nu(x)\nu(x^{-1}) v_1$$

$$= v_1 ,$$

since V_1 is a submodule of V and $EV_1 = V_1$. Thus $FV=V_1$. Now, let $V_2 = \{v - Fv \mid v \in V\}$, then V_2 is a submodule of V and

$$V = V_1 \oplus V_2 .$$

That is, V is completely reducible.

As a consequence of this theorem, using well-known results (see e.g. [11]), we have

Corollary 2.10. $(KG)_\alpha$ is a semi-simple algebra.

Corollary 2.11. If $T:G \longrightarrow GL(n,K)$. is a projective representation of G with factor set α , then T is linearly equivalent to a projective representation S(with factor set α) such that

$$S(x) = \begin{bmatrix} S_1(x) & & \\ & \ddots & \\ & & S_n(x) \end{bmatrix}$$

where $S_i(x)(i=1,\ldots,n)$ are irreducible projective representations with factor set α .

Corollary 2.12. ([11] , p.186) $(KG)_\alpha \cong M_{n_1}(K) + \ldots + M_{n_s}(K)$, where $M_{n_i}(K)$ $(i=1,\ldots,s)$ is the full matrix algebra of $n_i \times n_i$ matrices over K. Further, there are s distinct irreducible inequivalent $(KG)_\alpha$ -modules $M_1,\ldots,M_s$. (In fact, $Hom_K(M_i,M_i) \cong M_{n_i}(K)$ and $(M_i : K)=n_i$, each $M_{n_i}(K)$ is isomorphic to a direct sum of n_i copies of M_i . This in turn implies that the irreducible projective representation of degree n_i afforded by M_i appears n_i times as an irreducible component of the regular projective representation with factor set α afforded by $(KG)_\alpha$).

We now consider $Z = Z((KG)_\alpha)$ the centre of $(KG)_\alpha$, that is

$$Z = \{ b \in (KG)_\alpha \mid ba = ab \quad \text{for all } a \in (KG)_\alpha \}$$

It is easily verified (see [11] , p.187) that Z is itself an algebra over K and that

$$Z \cong Z(M_{n_1}(K)) \oplus \cdots \oplus Z(M_{n_s}(K)) .$$

Further $Z(M_{n_i}(K)) \cong K$ (i.e. the only matrices which commute with all matrices in $M_{n_i}(K)$ are scalar multiples of the identity matrix). Thus (Z:K) = s, the number of inequivalent irreducible projective representations with factor set α = (Z:K).

The following theorem gives another method for computing the dimension of Z over K.

Theorem 2.13. Let $\alpha \in M(G,K^*)$ be a c-normalized factor set. Let $\mathcal{K}_1, \ldots, \mathcal{K}_t$ be the α-regular classes of G and let

$$K_i = \sum_{g \in \mathcal{K}_i} \nu(g) \qquad (i = 1, \ldots, t)$$

Then $\{K_1, \ldots, K_t\}$ is a K-basis for $Z = Z((KG)_\alpha)$.

Proof. If $x \in G$, then

$$\nu(x)^{-1} K_j \nu(x) = \sum_{g \in \mathcal{K}_j} \nu(x)^{-1} \nu(g) \nu(x)$$

$$= \sum_{g \in \mathcal{K}_j} \nu(x^{-1} g x) = K_j ,$$

and so K_j (j=1,...,t) $\in$ Z. The set $\{K_1, \ldots, K_t\}$ is clearly linearly independent over K since each consists of a sum of disjoint sets of group elements. Let $y = \sum_{g \in G} \zeta_g \nu(g) \in Z$. We first show that if $\zeta_g \neq 0$, then g is a α-regular element. That is, we must show that if $x \in C_G(g)$, then $f_\alpha(x,g) = 1$. Since $y \in Z$, $\nu(x)^{-1} y \nu(x) = y$ for all $x \in G$, or

$$\zeta_g \nu(x)^{-1}\nu(g)\nu(x) + \sum_{g'\neq g} \zeta_{g'} \nu(x)^{-1}\nu(g')\nu(x) = \zeta_g \nu(g) + \sum_{g'\neq g} \zeta_{g'}\nu(g').$$

This means that

$$\zeta_g f_\alpha(x,g)\nu(g) + \sum_{g'\neq g} \zeta_{g'} f_\alpha(x,g')\nu(x^{-1}g'x) = \zeta_g \nu(g) + \sum_{g'\neq g} \zeta_{g'}\nu(g')$$

and as $\zeta_g \neq 0$ it follows that $f_\alpha(x,g) = 1$ since $x^{-1}g'x = g$ would contradict $g' \neq g$. Thus g is a α-regular element. Hence, if $y = \sum_{g\in G} \zeta_g \nu(g) \in Z$, the summation is over α-regular elements of G. Further now

$$\nu(x)^{-1} y \nu(x) = y,$$

implies that

$$\sum_{g\in G} \zeta_g f_\alpha(x,g)\nu(x^{-1}gx) = \sum_{g\in G} \zeta_g \nu(g)$$

and since $f_\alpha(x,g) = 1$ for all α-regular $g \in G$, and $x \in G$, then

$$\zeta_g = \zeta_{xgx^{-1}}.$$

Thus y is a linear combination of $\{K_1, \ldots, K_t\}$ as required, and the proof is complete.

It now follows directly that

Corollary 2.14. The number of non-equivalent distinct irreducible projective representations of G with factor set α is equal to the number of α-regular classes of G.

Let P_1 and P_2 be projective representations of G with factor sets α and β respectively and representation spaces V_1 and V_2 respectively. Then, define

$$(P_1 \otimes P_2)(x)(v_1 \otimes v_2) = (P_1(x) \otimes P_2(x))(v_1 \otimes v_2) \text{ for all } x \in G, \; v_1 \in V_1, v_2 \in V_2,$$

then

$$(P_1 \otimes P_2)(x)(P_1 \otimes P_2)(y) = \alpha(x,y)\beta(x,y)(P_1 \otimes P_2)(xy) \quad \text{for all } x, y \in G .$$

That is, $P_1 \otimes P_2$ (the direct product of P_1 and P_2) is a projective representation of G with factor set $\alpha\beta$ where $\alpha\beta(x,y) = \alpha(x,y)\beta(x,y)$ and representation space $V_1 \otimes V_2$. (This indicates an error in Rudra's [46] treatment of the direct product of projective representations. In his paper an incorrect construction for obtaining the direct product of two projective representations with factor set α to give a projective representation with factor set α is given. The above shows that the resulting projective representation will have factor set α^2. Rudra's construction is only valid for projective representations projectively equivalent to an ordinary representation).

<u>Remark</u>. The following theorem due to Clifford [9] (see also [11], p.351) shows that projective representations arise naturally in the study of ordinary representations.

<u>Theorem 2.15</u>. If H is a normal subgroup of G and V is an irreducible (KG)-module, then V restricted to H is a direct sum of S isomorphic irreducible KH-modules. If T is the representation of G afforded by V, then

$$T(x) = P_1(x) \otimes P_2(x) \qquad \text{for all } x \in G$$

where P_1 and P_2 are irreducible projective representations of G and P_1 is a projective representation of G/H of degree s.

For further work of this type, see Tucker [56] , [57] , Conlon [10] and Mackey [31] .

3. Schur Multiplier and Representation Groups

If $\alpha, \beta \in M(G, K^*)$, we have called α and β equivalent factor sets if there exists $\mu : G \longrightarrow K^*$ such that

$$\beta(x,y) = \frac{\mu(x)\mu(y)}{\mu(xy)} \alpha(x,y)$$

This is easily verified to be an equivalence relation on $M(G,K^*)$. Let $\{\alpha\}$ denote the equivalence class which contains $\alpha \in M(G,K^*)$. Let $H^2(G,K^*)$ denote the set of equivalence classes of factor sets. If $\{\alpha\}$, $\{\beta\}$, we define multiplication on $H^2(G,K^*)$ by

$$\{\alpha\}\{\beta\} = \{\alpha\beta\}$$

Then, we obtain the following

Theorem 3.1. $H^2(G,K^*)$ is a finite abelian group. The order of every element of $H^2(G,K^*)$ is a factor of the order of G.

Proof. We have already proved in Lemma 1.4 that if $\alpha \in M(G,K^*)$, and we define $\mu : G \rightarrow K^*$ by $\mu(x) = \prod_{g \in G} \alpha(x,g)$, then

$$\alpha(x,y)^{|G|} = \frac{\mu(x)\mu(y)}{\mu(xy)} ,$$

or in other words

$$\{\alpha\}^{|G|} = \{1\}$$

which proves the second statement. Furthermore, in Lemma 1.4, we showed that there exists a factor set β equivalent to α which is a $|G|$-th root of unity. Thus, every equivalence class $\{\alpha\}$ contains a representative $\{\beta\}$ which is a $|G|$-th root of unity. Thus, there are at most a finite number of equivalence classes of factor sets and the first statement is proved.

Definition 3.2. $H^2(G,K^*)$ is called the Schur Multiplier of G in K.

Before proceeding to give further crucial results on Schur Multipliers, we demonstrate its importance in the theory of projective representations of G.

A pair (G^*,π), where G^* is a group and π a homomorphism of G^* onto G is called a group extension of G. (G^*,π) is called a central group extension if the kernel A of π is contained in the centre of G^*. Thus $\frac{G^*}{A} \cong G$ and $A \subseteq Z(G^*)$. Let $\{\omega(x) \mid x \in G\}$ be a set of coset representatives of A in G^* which are in one-one correspondence with the elements of G. Then, we have

$$a(x,y) = \omega(x)\omega(y)\omega(xy)^{-1} \in A \qquad \text{for all } x,y \in G$$

Let T be a linear representation of G^* with representation space V such that

$$T(a(x,y)) = \alpha(x,y)I_V ,$$

where $\alpha(x,y) \in K^*$. Let T be an irreducible linear representation of G^*, then since $A \subseteq Z(G^*)$ it follows by Schur's lemma that

$T(a) = \alpha(a)I_V$, for all $a \in A$. In particular, let $T(a(x,y)) = \alpha(x,y)$. If we define $P: G \longrightarrow GL(V)$ by $P(x) = T(\omega(x))$ for all $x \in G$, then

$$\begin{aligned} P(x)P(y) &= T(\omega(x))T(\omega(y)) = T(a(x,y)\omega(xy)) \\ &= \alpha(x,y)P(xy) \quad \text{for all } x,y \in G, \end{aligned}$$

that is P is a projective representation of G with factor set α. We say that the projective representation P is <u>linearized</u> by the linear representation T of G*.

<u>Definition 3.3</u>. A <u>representation group</u> G* of G is a finite group G* of lowest possible order which is a central group extension of G such that every projective representation of G is equivalent to a projective representation of G which is linearized by a linear representation of G*.

The following theorem due to Schur [49] proves the existence of a representation group. The proof given here is due to Asano and Shoda [4].

<u>Theorem 3.4</u>. G has at least one representation group G* of order $|H^2(G,K^*)||G|$. Furthermore, the kernel A of the extension is isomorphic to $H^2(G,K^*)$.

Proof. $H^2(G,K^*)$ is a finite abelian group and so is a direct product of cyclic groups

$$H^2(G,K^*) = (\{\alpha^{(1)}\}) \times \cdots \times (\{\alpha^{(n)}\}),$$

where $(\{\alpha^{(i)}\})$ denotes the cyclic group generated by $\alpha^{(i)}$ of order e_i (say) $(i=1,\ldots,n)$. By theorem 3.1, $\{\alpha^{(i)}\}$ may be chosen as an

e_i'th root of unity. Let ζ_i denote a primitive e_i'th root of unity, then

$$\alpha^{(i)}(x,y) = \zeta_i^{a^{(i)}_{x,y}} \qquad \text{where} \quad 0 \le a^{i}_{(x,y)} \le e_i - 1$$

By(1.1) it follows that

$$a^{(i)}_{x,y} + a^{(i)}_{xy,z} \equiv a^{(i)}_{x,yz} + a^{(i)}_{y,z} \pmod{e_i} \qquad (3.1)$$

Then if $\{\alpha\} \in H^2(G,K^*)$, then α is equivalent to a factor set such that

$$\begin{aligned} \beta(x,y) &= (\alpha^{(1)}(x,y))^{l_1}(\alpha^{(2)}(x,y))^{l_2}\cdots(\alpha^{(r)}(x,y))^{l_r} \qquad 0 \le l_i \le e_i' \\ &= (\zeta_1^{l_1})^{a^{(r)}_{(x,y)}}\cdots(\zeta_r^{l_r})^{a^{(r)}_{(x,y)}} \end{aligned} \qquad (3.2)$$

Let A be an arbitrary finite abelian group such that $A \cong H^2(G,K^*)$ and let $a_i \in A$ correspond to the element $\alpha^{(i)} \in H^2(G,K^*)$. Let

$$a(x,y) = a_1^{a^{(1)}_{(x,y)}} \cdots a_r^{a^{(r)}_{(x,y)}} ,$$

then it follows from (3.1) that

$$a(x,y)\,a(xy,z) = a(x,yz)\,a(y,z) \quad , \; x,y,z \in G .$$

Let $\chi \in \mathrm{Hom}(A,K^*)$; define

$$\psi_\chi(x,y) = \chi(a(x,y)) = \chi(a_1)^{a^{(1)}_{(x,y)}}\cdots\chi(a_r)^{a^{(r)}_{(x,y)}} ,$$

then $\psi_\chi \in H^2(G,K^*)$ and $\chi(a_i)$ is an e_i'th root of unity.

Then as χ runs through the characters of A, then by (3.2) the ψ_χ run through all the factor systems of $H^2(G,K^*)$.

Let $G^* = \{(x,a) \mid x \in G, a \in A\}$ and define

$$(x,a)(y,b) = (xy,\ a(x,y)ab)$$

Then, it is easily verified that G* is a group such that $\{(1,a) \mid a \in A\} \cong A$ is contained in Z(G*). Further, let $\nu(x)=(x,1)$ then $\{\nu(x) \mid x \in G\}$ is a set of coset representation of A in G* and

$$\nu(x)\nu(y) = (x,1)(y,1) = (xy,\ a(x,y)) = (1,a(x,y))(xy,1)$$

it follows that $\frac{G^*}{A} \cong G$ and G* is a central extension of G.

Let $T:G \longrightarrow GL(V)$ be an arbitrary projective representation of G with factor set α. Then, by the above considerations, there exists a linear character ψ of A such that

$$\psi(a(x,y)) = \alpha(x,y).$$

Define $T^*:G^* \longrightarrow GL(V)$ by

$$T^*(v(x)a) = T(x)\psi(a),$$

then

T* is a linear representation of G*, that is, T is a linearizable representation. Thus G* is a central extension of G such that every projective representation of G is linearized in G* and $|G^*|=|H^2(G,K^*)||G|$. But Schur [45] also proved that the order of a representation group is $\geqslant |H^2(G,K^*)||G|$ and so G* is a representation group of G.

In [49], Schur also proved that the Representation Group of G* is a characterised by the following properties:- G* contains a subgroup A such that

$$\text{I} \quad A \subseteq G^{*'} \cap Z(G^*)$$

$$\text{II} \quad \frac{G^*}{A} \cong G$$

$$\text{III} \quad |A| = |H^2(G, K^*)|$$

The representation group is not in general unique. Schur [50] obtained an upper bound for the number of non-isomorphic representation groups.

<u>Theorem 3.5</u>. The number of non-isomorphic representation groups of G is less than or equal to

$$\prod_{i,j} (\varepsilon_i, \eta_j).$$

where $\{\varepsilon_i\}$ and $\{\eta_j\}$ are the invariants of $\frac{G}{G'}$ and $H^2(G, K^*)$ respectively. If G is a perfect group (i.e. G'=G), the equality sign holds. If $\left(\left|\frac{G}{G'}\right|, H^2(G,K^*)\right) = 1$, then G has only one representation group.

For an alternative elementary proof of this theorem, see Asano and Shoda [4].

The proof of Theorem 3.4 is a purely existence proof and is of no value in the construction of a representation group. Schur [50] also gave a proof for the existence of a representation group which at the same time gives a general method for the construction of the representation group and Schur multiplier.

Theorem 3.6. Let F be a free group in n generators such that $F/R \cong G$. Let $H = \frac{F}{[R,F]}$ and $N = \frac{R}{[R,F]}$. Then N is a finitely generated abelian group contained in the centre of H. Let N_1 be the torsion group of N, then $N_1 \doteq \frac{R \cap F'}{[R,F]} \cong H^2(G,K^*)$ and rank N=n. If $N = N_1 \times N_2$, then $\frac{H}{N_2}$ is a representation group of G.

For a modern proof of this theorem, see Huppert [24]. Other results were obtained by Schur and others which are useful in determining the Schur multiplier. These results, together with others involving projective representations are now presented without proof.

Theorem 3.7. (Schur [50]). The degree of every irreducible projective representation of G is a divisor of $|G|$.

For a proof of this theorem see for example [11].

Theorem 3.8. (Schur [50]). The only primes that occur in $|H^2(G,K^*)|$ are those which occur to at least the second power in $|G|$. Thus, groups of square free order have trivial Schur multiplier.

Theorem 3.9. (Schur [50]). Let S be a subgroup of G of index n and $M^{(n)}$ denote the elements of $H^2(G,K^*)$ of order relatively prime to n. Then $M^{(n)}$ is isomorphic to a subgroup of $H^2(S,K^*)$.

Theorem 3.10. (Schur [50]). Let S be a p-Sylow subgroup of G. Then a p-Sylow subgroup of $H^2(G,K^*)$ is isomorphic to a subgroup of $H^2(S,K^*)$. For an alternative proof and an improvement of this result see Kochendorffer [29].

This result is especially useful in proving that the Schur multiplier of a group is trivial.

Theorem 3.11. (Schur [51]). If G has n generators and r defining relations, then $H^2(G,K^*)$ has (r-n) generators. Thus if $r-n \leq 0$,

$H^2(G,K^*)$ is trivial.

Definition 3.12. Let G and H be arbitrary groups, then define

$$P(G,H;K^*) = \{ f : G \times H \to K^* \mid f(g_1g_2,h) = f(g_1,h)f(g_2,h),\ f(g,h_1h_2) = f(g,h_1)f(g,h_2) \text{ for all } g,g_1,g_2 \in G,\ h,h_1,h_2 \in H \}$$

If $f_1, f_2 \in P(G,H;K^*)$ define

$$f_1f_2\,(g,h) = f_1(g,h)\,f_2(g,h).$$

Then $P(G,H;K^*)$ is an abelian group. This may be used to determine the Schur Multiplier of a direct product of finite groups.

Theorem 3.13. If G and H are finite groups, then

$$H^2(G \times H, K^*) \cong H^2(G,K^*) \times H^2(H,K^*) \times P(G,H;K^*)$$

This may be extended by induction to prove that if $G = G_1 \times G_2 \times \cdots \times G_r$, then

$$H^2(G,K^*) = \prod_{i=1}^{r} H^2(G_i,K^*) \times \prod_{1 \le i \le j \le r} P(G_i,G_j;K^*)$$

4. Projective Characters

Throughout this section we shall assume that the factor set $M(G,K^*)$ is c-normalised, that is

(i) $\alpha(x, x^{-1}) = 1$, for all $x \in G$

(ii) $\alpha(x,s)\alpha^{-1}(xsx^{-1},x) = \alpha(x,s)\alpha(xs,x^{-1})$

$= 1$ for all α-regular s and for

Definition 4.1. Let T be a projective representation of G with factor set α . Define

$$\zeta(x) = \text{Trace } T(x) \qquad \text{for all } x \in G$$

Then ζ is called a projective character of G.

Lemma 4.2. ζ is a class function on G.

Proof. We shall prove that

$$\zeta(xsx^{-1}) = \zeta(s) \qquad \text{for all } \alpha\text{-regular } s \in G \text{ and for all } x \in G$$

and

$$\zeta(s) = 0 \quad \text{if } s \text{ is not } \alpha\text{-regular.}$$

It is easily verified that for arbitrary $x \in G$ and α-regular $s \in G$

$$T(x)T(s)T(x)^{-1} = \alpha(x,s)\,\alpha^{-1}(xsx^{-1}, x)\,T(xsx^{-1})$$
$$= T(xsx^{-1}) \quad ,$$

since α satisfies (ii) above. Thus, since equivalent matrices have the same traces, the first result follows.

If s is not α-regular, there exists $x \in C_G(s)$ such that $\alpha(x,s) \neq \alpha(s,x)$. Since $xsx^{-1} = s$,

$$T(s) = \alpha(x,s)\,\alpha^{-1}(s,x)\,T(s)$$

and taking traces implies that

$$\zeta(s) = \alpha(x,s)\,\alpha^{-1}(s,x)\,\zeta(s)$$

which means that $\zeta(s) = 0$.

Lemma 4.3. Linearly equivalent projective representations have the same character.

Proof. Exercise.

Projective characters also satisfy the usual orthogonality relations.

Let $T_1, \ldots, T_t$ form a complete set of irreducible projective representations with factor set α with degrees $f_1, \ldots, f_t$ and characters $\zeta_1, \ldots, \zeta_t$ respectively. Let $C = [\gamma_{rs}]$ be an arbitrary $f_i \times f_j$ matrix and

$$P = \sum_{x \in G} Z_i(x)\, C\, Z_j(x^{-1}) .$$

Then, it follows that

$$Z_i(y) P = \sum_{x \in G} \alpha(y, x)\, Z_i(yx)\, C\, Z_j(x^{-1})$$

$$= \left\{ \sum_{x' \in G} \alpha(y, y^{-1}x')\,\alpha^{-1}(x'^{-1}, y)\, Z_i(x')\, C\, Z_j(x'^{-1}) \right\} Z_j(y)$$

$$= P\, Z_j(y)$$

if we assume that α is normalized (i.e. $\alpha(x, x^{-1}) = 1$ for all $x \in G$). Thus, we may apply Schur's lemma and obtain that $P=0$ if $i \neq j$ and $P = \lambda I_{f_i}$ if $i=j$. For fixed l, m, put $\gamma_{lm} = p$, and $\gamma_{rs} = 0$ if $(r, s) \neq (l, m)$, then if $Z_i(x) = [a^{(i)}_{rs}(x)]$,

$$\sum_{x \in G} a^{(i)}_{rl}(x)\, a^{(j)}_{sm}(x^{-1}) = 0$$

and

$$\sum_{x\in G} a_{rl}^{(i)}(x)\, a_{ms}^{(i)}(x^{-1}) = \lambda_{lm}\,\delta_{rs} ,$$

for all (r,l,m,s). Put s=r and sum over s, then

$$\sum_{x\in G} \sum_{s=1}^{f_i} a_{rl}^{(i)}(x)\, a_{mr}^{(i)}(x^{-1}) = f_i \lambda_{lm}$$

and so

$$\sum_{x\in G} a_{rl}^{(i)}(x)\, a_{ms}^{(i)}(x^{-1}) = \frac{|G|}{f_i}\,\delta_{rs}\,\delta_{lm} .$$

Now, put l=r and m=s and sum over r and s which gives

$$\sum_{x\in G} \zeta_i(x)\,\zeta_j(x^{-1}) = \delta_{ij}\,|G| , \qquad (4.1)$$

which is the first orthogonality relation for projective characters. Let $\mathcal{K}_1, \ldots, \mathcal{K}_t$ denote the α -regular classes of G; since projective characters are class functions, define

$$\zeta_i^{\rho} = \zeta_i(x) \qquad \text{for any } x \in \mathcal{K}_\rho$$

By Theorem 1.9, the class containing the inverses of the elements of $\mathcal{K}_\rho$ is also α -regular; denote this class as $\mathcal{K}_{\rho^*}$. Then (4.1) can be rewritten as

$$\sum_{\rho=1}^{t} h_\rho\,\zeta_i^{\rho}\,\zeta_j^{\rho^*} = \delta_{ij}\,|G|$$

where h_ρ is the number of elements in the class K_ρ. From this, the second orthogonality relation

$$\sum_{i=1}^{t} \zeta_i^{\rho} \zeta_i^{\sigma *} = \frac{|G|}{h_\rho} \delta_{\rho\sigma} \tag{4.2}$$

is obtained in the usual way. Over the complex field,

$$\zeta_j^{\rho *} = \overline{\zeta_j^{\rho}}$$

As a consequence, we can prove

Theorem 4.4. Two projective representations T_1 and T_2 with factor set α are linearly equivalent if and only if they have the same projective character.

Proof. Exercise.

Let H be a subgroup of G and let V be a left $(KG)_\alpha$ -module, then V may be regarded as a left $(KH)_\alpha$ -module by restricting the domain of operators to the subalgebra $(KH)_\alpha$; in this case, we denote the module by V_H. If V affords the projective representation T and V_H the projective representation T_H, then

$$T_H(h) = T(h) \qquad \text{for all } h \in H.$$

Let ζ denote the projective character of T and ζ_H the projective character of T_H. If ζ is an irreducible character then in general ζ_H is a reducible character of H.

We now consider the inverse problem; let W be a left $(KH)_\alpha$ -module. Then

$$(KG)_\alpha \otimes_{KH_\alpha} W = W^G$$

$$\zeta^G(g) = \sum_{i=1}^{t} \mathring{\zeta}(g_i^{-1} g g_i) = \frac{1}{|H|} \sum_{t \in G} \mathring{\zeta}(t^{-1} g t),$$

where $\mathring{\zeta}(x) = \zeta(x)$ if $x \in H$

$= 0$ if $x \notin H$.

This conforms with the usual formula (see [11] , p.266) for obtaining a class function on G from a class function on H. Therefore, all the usual induced character theorems for linear characters are valid for projective characters without further proof. For example,

Theorem 4.6. (Frobenius Reciprocity Theorem). Let $T_1, \ldots, T_t$ be a complete set of inequivalent projective representations of G with factor set α and $S_1, \ldots, S_s$ a complete set of inequivalent projective representations of H with factor set α . Let the character of T_i be ϕ_i (i=1,...,t) and the projective character of S_i be ζ_i (i=1,...,s) then if

$$\phi_{iH} = \sum_{j=1}^{s} a_{ij} \zeta_j \quad , \ (i = 1, \ldots, t) \ ,$$

then

$$\zeta_j^G = \sum_{i=1}^{t} a_{ij} \ \phi_i \quad (j = 1, \ldots, s) .$$

5. <u>Schur Multipliers and Projective Representation of Particular Groups.</u>

We shall concentrate on finite groups which are important in physics. Namely, cyclic groups, dihedral groups and symmetric groups.

<u>Example 1.</u> <u>Cyclic groups $\mathcal{K}_n$.</u>

Let $\mathcal{K}_n$ be a cyclic group of order n generated by x. Then $x^n = e$ and by Theorem 3.11 it follows immediately that $H^2(\mathcal{K}_n, K^*) = E$. Alternatively, if we let T be a projective representation of $\mathcal{K}_n$ of degree m and factor set α then

$$T(x)^n = \mu(x) I_m ,$$

where $\mu(x) = \alpha(x,x)\alpha(x^2,x)\dots\alpha(x^{n-1},x)$. Define

$$T'(x) = \frac{1}{(\mu(x))^{\frac{1}{n}}} T(x) , \qquad \text{for all } x \in G,$$

then

$$(T'(x))^n = I_m ,$$

and so T' is a linear representation of $\mathcal{K}_n$ which is projectively equivalent to T. That is, every projective representation of $\mathcal{K}_n$ is projectively equivalent to a linear representation of $\mathcal{K}_n$.

<u>Example 2.</u> <u>Dihedral Groups D_{2n}</u>

$$D_{2n} = \{ x, y \mid x^n = e,\ y^2 = e,\ y^{-1} x y = x^{n-1} \}$$

If n is odd, D_{2n} has only cyclic Sylow subgroups and thus $H^2(D_{2n}, K^*) = E$. If n is even, then Schur showed that D_{4n} is a

$$T(S_{2i}) = \frac{\sqrt{-1}}{\sqrt{2}} \Big(\epsilon \otimes \cdots \otimes \underset{i-1}{\epsilon} \otimes (\underset{i}{\epsilon} \otimes \underset{i+1}{\sigma} + \underset{i}{\rho} \otimes \underset{i+1}{\tau}) \otimes \underset{i+2}{\tau} \otimes \cdots \otimes \underset{\nu}{\tau} \Big)$$
$$(i = 1, \ldots, \nu - 1)$$

and

$$T(S_{2i+1}) = \frac{\sqrt{-1}}{\sqrt{2}} \Big(\epsilon \otimes \cdots \otimes \underset{i-1}{\epsilon} \otimes (\underset{i}{\rho \otimes \sigma}) \otimes \tau \otimes \cdots \otimes \underset{\nu}{\tau} \Big)$$
$$(i = 1, \ldots, \nu)$$

then the relations (5.4) are easily verified. For a systematic construction of projective (spin) representations using Clifford Algebras, see Morris [36] . Schur [51] has completely determined the irreducible projective representations of the symmetric and alternating groups. Projective character tables for small values of n are available in Morris [36] .

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$$P(xy) = P(x)P(y).$$

Thus, it follows that

$$P(xy)P(x) = P(y),$$

$$P(xy)P(x)\{P(xy)\}^{-1} = P(y)\{P(xy)\}^{-1}$$

$$= P(y)\{P(x)P(y)\}^{-1}$$

$$= P(x)^{-1}.$$

and raising this to the nth power gives $\lambda^2 = 1$ and so $\lambda = \pm 1$. If $\lambda = 1$ then P is a linear representation of D_{2n} and for a non-trivial projective representation λ must be equal to -1. If n is odd, let $P'(x) = -P(x)$, $P'(y) = P(y)$ and

$$P'(xy) = -P(xy) = -P(x)P(y) = P'(x)P'(y)$$

and then

$$(P'(x))^n = (P'(y))^2 = (P'(xy))^2 = I_m.$$

and so P' is a linear representation. That is, if n is odd, D_{2n} has no non-trivial projective representation. If n is even, no further such simplification is possible and furthermore, if we put

$$P(x) = \begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix} \quad \text{and} \quad P(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$, where ω is a

primitive 2nth root of unity, then

$[P(x)]^n = -I_2$, $P(y)^2 = I_2$ and $P(xy)^2 = I_2$. P is not equivalent to a linear representation as can be verified by comparing characters.

Example 3. Symmetric groups.

Let S_n be the symmetric group of degree n. Then S_n is generated by the transpositions $S_i = (i, i+1)$ $(i=1,\ldots,n-1)$ with the defining relations

$$S_i^2 = e \qquad (i=1,\ldots,n-1)$$

$$(S_i, S_{i+1})^3 = e \qquad (i=1,\ldots,n-2)\ .$$

$$S_i S_j = S_j S_i \qquad (i+1 < j)$$

Schur [51] proves that S_n has two representation groups Γ_n and Γ_n' of order $2(n!)$, where Γ_n is generated by f, t_i $(i=1,\ldots,n-1)$ with defining relations

$$f^2 = e\ ,\quad t_i^2 = f\ ,\ (t_i t_{i+1})^3 = f\ ,\ t_i t_j = f t_j t_i\ ,$$

and Γ_n' is generated by f, t_i' $(i=1,\ldots,n-1)$ with defining relations

$$f^2 = e\ ,\quad t_i'^2 = e\ ,\ (t_i' t_{i+1}')^3 = e\ ,\ t_i' t_j' = f t_j' t_i'\ .$$

The Schur Multiplier $M^2(S_n, K^*) \cong Z_2$.

Again in this case, we shall obtain the projective representations directly. Let P be a projective representation of S_n of degree m. Then

$$P(S_i)^2 = \alpha_i I_m \quad (i=1,\ldots,n-1), \tag{5.1}$$

$$(P(S_i)P(S_{i+1}))^3 = \beta_i I_m\ (i=1,\ldots,n-2), \tag{5.2}$$

$$P(S_i)P(S_j) = \gamma_{ij}\ P(S_j)P(S_i) \quad (i<j+1)\ . \tag{5.3}$$

From (5.3) it follows that

$$P(s_i)\,P(s_j)\,P(s_i)^{-1} = \gamma_{ij}\,P(s_j),$$

and squaring both sides gives

$$\gamma_{ij}^2 = 1 .$$

Let $s_i = (i,i+1)$ and $s_j = (j,j+1)$, where $j > i+1$. Then $i, i+1, j, j+1$ are four distinct elements. Let $s_i' = (i', i'+1)$, $s_j' = (j', j'+1)$ and $t \in S_n$ be defined by

$$t = \begin{pmatrix} \dots & i & i+1 & \dots & j & j+1 & \dots \\ \dots & i' & i'+1 & \dots & j' & j'+1 & \dots \end{pmatrix}$$

then

$$t^{-1} s_i t = s_i' \quad \text{and} \quad t^{-1} s_j t = s_j' \quad \text{and so}$$

$$P(t)^{-1} P(s_i) P(t) = \lambda P(s_i') \text{ and } P(t)^{-1} P(s_j) P(t) = \mu\, P(s_j'),$$

where $\lambda, \mu \in K^*$. From (5.3) it follows that

$$P(t)^{-1} P(s_i) P(s_j) P(t) = \gamma_{ij}\, P(t)^{-1} P(s_j) P(s_i) P(t),$$

and

$$\lambda\mu\, P(s_i') P(s_j') = \lambda\mu\, \gamma_{ij}\, P(s_j')\, P(s_i') .$$

Thus $\gamma_{ij} = \gamma_{i'j'}$ for all $i, i', j, j' \in \{1, 2, \dots, n\}$. Let $\gamma_{ij} = c$, where $c = \pm 1$. From (5.2) it follows that

$$T(s_i)\,T(s_{i+1})\,T(s_i) = \beta_i\, T(s_{i+1})^{-1} T(s_i)^{-1} T(s_{i+1})^{-1}$$

and squaring both sides gives

$$\beta_i^2 = \alpha_i^3 \, \alpha_{i+1}^3 \, .$$

The α_i $(i = 1, \ldots, n-1)$ in (5.1) can be chosen arbitrarily. Let $\alpha_i = c$ $(i=1,\ldots,n-1)$, then $\beta_i = \pm 1$ $(i=1,\ldots,n-2)$. Then define

$$T'(S_i) = c\,\beta_1 \cdots \beta_{i-1}\, T(S_i) \quad \text{if } i \text{ is even},$$

$$= \beta_1 \cdots \beta_{i-1}\, T(S_i) \quad \text{if } i \text{ is odd}.$$

Then, an elementary calculation shows that

$$T'(S_i)^2 = c\, I_m \, , \; [T'(S_i)T'(S_{i+1})]^3 = c\, I_m \, ,$$
$$T'(S_i)\, T'(S_j) = c\, T'(S_j)\, T'(S_j) \, . \qquad (5.4)$$

That is, every projective representation is projectively equivalent to a projective representation satisfying (5.4). If $c=1$, then T' is a linear representation. A non-linear projective representation, if it exists will satisfy (5.4) with $c=-1$. We now give a construction for such a projective representation.

Let

$$\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then it is easily verified that

$$\rho^2 = \sigma^2 = \tau^2 = \epsilon \, , \quad \rho\sigma = -\sigma\rho \, , \quad \rho\tau = -\tau\rho \, , \quad \sigma\tau = -\tau\sigma \, ,$$

Further, if $n = 2\nu$ or $n = 2\nu + 1$, set representation group of D_{2n} and $H^2(D_{2n}, K^*) \cong \mathbb{Z}_2$. Alternatively, the problem may be considered directly as in Frucht [18] . Let P be a projective representation of degree m with factor set α of D_{2n}. Then

$$(P(x))^n = \lambda I_m, \ (P(y))^2 = \mu I_m \text{ and } (P(xy))^2 = \nu I_m$$

Clearly, P may be replaced by a projectively, equivalent set projective representation P' such that

$$P'(x^r y^s) = P'(x)^r P'(y)^s \qquad 0 \leq r \leq n-1, \ 0 \leq s \leq 1.$$

(that is, put $P'(x^r y^s) = \alpha(x,x) \ldots \alpha(x^{r-1},x)\alpha(x^r,y^s)P(x^r y^s)$.)
In particular

$$P'(xy) = P'(x)\, P'(y).$$

Furthermore, let

$$P''(y) = \frac{1}{\sqrt{\mu}} P'(y) \quad \text{and} \quad P''(xy) = \frac{1}{\sqrt{\nu}} P'(xy),$$

then

$$(P''(y))^2 = I_m \quad \text{and} \quad (P''(xy))^2 = I_m.$$

Thus, we may assume that our factor set has been chosen such that

$$P(x)^n = \lambda I_m, \ P(y)^2 = P(xy)^2 = I_m \quad \text{and}$$

$$P(x^r y^s) = P(x)^r P(y)^s \quad \text{where } r = 0, \ldots, n-1; \ s = 0, 1 \text{ and}$$

in particular

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GENERALIZED CLIFFORD ALGEBRA AND ITS APPLICATIONS

OR

A NEW APPROACH TO INTERNAL QUANTUM NUMBERS

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The confluence of relativity and quantum mechanics was achieved when Dirac just wrote down his famous equation in 1928. This achievement was made possible since he was able to construct four mutually anti-commuting matrices so that the Hamiltonian was consistent with the quadratic relativistic relation between energy momento and mass. Earlier, in non-relativistic quantum mechanics three mutually anti-commuting matrices were found sufficient to include the concept of intrinsic spin. It was immediately noticed by Dirac that the quantum mechanical concept of spin was also imbedded in his Hamiltonian. In the years of uninterrupted triumph that followed the birth of relativistic quantum mechanics, the study of the mathematical significance of the transition from Pauli to Dirac matrices was considered quite academic and therefore ignored. But it was obvious that it was still a live and unsolved problem since immediately after Dirac's formulation. Pauli attempted such a study and as late as 1956, Feynman himself raised the question of the relationship between spin and relativity in his famous Caltech lectures even after the total triumph of his graphical formalism in electrodynamics.

We therefore set as our objective the understanding of the mathematical procedure of obtaining Dirac matrices from the basis Pauli set. We thought it was just the right time now to take it up since the spirit of the hour demanded a re-examination of the whole structure from the point of view of mathematical rigour and logical precision. To our strange surprise we found that the procedure which Dirac used was of such general significance that it could be extended into a grammar of anti-commuting matrices, the ramifications of which give us a better insight into various branches of theoretical physics -- relativity, complementarity, propagator formalism and the fundamental concepts of spin and mass of elementary particles. Even more surprising was the possibility of enlarging the concept of anti-commutation to ω-commutation where ω is a general root of unity.

This work was presented in a series of papers most of which were published in the 'Journal of Mathematical Analysis and Applications'[1]. We shall now present a rapid survey of this work at this conference*.

1. The Generalized Clifford Algebra.

Our starting point in the fundamental cyclic $n \times n$ matrix obtained by shifting in the $n \times n$ unit matrix k^{th} row to the $(k-1)^{th}$ row, $k = 2,3,\ldots,n$ and the first to the n-th row

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i.e.

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad (1)$$

where C operates on an $n \times n$ matrix to the right, it just does this shifting operation.

We immediately note that

$$C^n = I \qquad (2)$$

i.e. C is one of the n-th roots of the unit matrix of dimension n. The eigenvalues of the matrix C are the n roots of unity:

$$1, \omega, \omega^2, \; . \; . \; . \; . \; . \; . \; . \; . \; \omega^{n-1}$$

being a primitive n-th root of unity. The corresponding eigenvectors can be placed together as columns forming a matrix which we call at the U-matrix. Hence

$$U^{-1} CU = B$$

where B is a diagonal matrix.

$$B = \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{n-1} \end{bmatrix} \qquad (3)$$

and

$$U = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & & (\omega^{n-1})^{n-1} \end{bmatrix} \qquad (3a)$$

We immediately observe that B and C possess an interesting commutation relation:

$$GB = \omega BC \qquad (4)$$

This we shall call the ω-commutation relation which is the basis of the Generalized Clifford Algebra. It is an inexplicable fact in the history of matrix theory that this commutation relation has not been noticed till very recently.

We next observe that any $n \times n$ matrix A can be expressed as a linear combination of the n^2 independent matrices each of the form $B^k C^\ell$; $k,\ell = 0,1,2,\ldots \pi-1$, i.e.

$$A = \sum_{k,\ell=0}^{n-1} a_{k\ell} B^k C^\ell . \qquad (5)$$

Among these n^2 matrices there is a matrix CB such that there matfices (C,CB,B) have a mutual ω-commutation relation. Denoting these as Σ_x, Σ_y, Σ_z we find they constitute a generalization of the anticommuting Pauli matrices to the basis set of ω-commuting matfices. It is to be noted that the Σ_x, Σ_y, Σ_z

obey an ordered commutation relation.

$$
\begin{aligned}
\Sigma_x \Sigma_y &= \omega \Sigma_y \Sigma_x \\
\Sigma_y \Sigma_z &= \omega \Sigma_z \Sigma_y \\
\Sigma_x \Sigma_z &= \omega \Sigma_z \Sigma_x
\end{aligned} \qquad (6)
$$

the order being irrelevant for the case $\omega = -1$.

2. Generalized Gell-Mann-Nishijima relation.

The matrices $B, B^2, \ldots, B^{n-1}$ commute as they are powers of the matrix B. Calling the eigenvalues of $B, B^2, \ldots, B^{n-1}$ as $\eta_1, \eta_2, \ldots \eta_{n-1}$ let us define a set of 'quantum numbers' $s_1, s_2, \ldots s_n$ as the following linear combinations of the

$$
\begin{aligned}
s_1 &= \frac{1}{n}\{\eta_1 + \eta_2 + \cdots + \eta_{n-1}\} \\
s_2 &= \frac{1}{n}\{\omega^{n-1}\eta_1 + (\omega^{n-1})^2\eta_2 + \cdots + (\omega^{n-1})^{n-1}\eta_{n-1}\} \\
&\vdots \\
s_n &= \frac{1}{n}\{\omega\eta_1 + \omega^2\eta_2 + \cdots + \omega^{n-1}\eta_{n-1}\}
\end{aligned}
$$

These can be written as a vector-matrix equation as

$$
\vec{s} = \frac{1}{n} \mathfrak{S} \vec{\eta} \qquad (7)
$$

where

$$
\vec{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \quad ; \quad \vec{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix} \qquad (8)
$$

and

$$\mathcal{S} = U \qquad (9)$$

Since the first set in η is set equal to zero, the choice of the first column in $\mathcal{S}$ is irrelevant, $\mathcal{S}$ is the well-known Sylvester matrix. We now notice that in the case of the quark, we identify the quantum numbers s_1, s_2 and s_3 as

$$\begin{aligned} s_1 &= Q \\ s_2 &= Y - Q \\ s_3 &= -Y \end{aligned} \qquad (9a)$$

where Q and Y are the charge and hypercharge respectively. Moreover, the commuting matrices in Gell-Mann's formulation of SU(3) can be identified as a linear combination of the commuting matrices B, B^2 in the Generalized Clifford Algebra (G.C.A.) for $\omega^3 = 1$. The shift matrices of SU(3) algebra can also be expressed as linear combinations of products of B and C. In other words, we assert now that the commuting generators of Lie algebra are the linear combinations of the commuting elements of the G.C.A. The same is true in the case of the shift operators.

We now proceed to define eigenvalues $I_{k\ell}$ which are the differences between the s_k and s_ℓ , $(k \neq \ell)$

$$I_{k\ell} = s_k - s_\ell \quad ; \quad k, \ell = 1, 2, \ldots, n \qquad (10)$$

For $n = 3$, we can identify I_{12}, I_{23} and I_{31} as the z-components of I spin, U spin and V spin in the language of SU(3).

We now also recognise that the equation (10) for $n = 3$ as the Gell-Mann-Nishijima relation i.e.

$$I_{12} = 2\,I_z = s_1 - s_2 = 2\,Q - Y$$

$$\text{or} \quad Q = I_z + \frac{Y}{2} \;.$$

But eq.(10) can be applied to the SU(n) quarks, and thus constitutes the generalized Gell-Mann-Nishijima relations. What we have described now is the correspondence between the quantum numbers of the quarks in the language of the Lie and Clifford algebras.

If we consider physical particles as composed of quarks with $\eta^i(j)$ denoting the j-th quantum number of the i-th particle then we can define the quantum number of the physical particle η_j as

$$\eta(j) = \sum_i \eta^i(j) \qquad (11)$$

In a similar way, the composite s quantum number of a physical particle can be defined. The vector matrix relation given by eq. (8), still holds between $s(j)$ and $\eta(j)$.

3. Dirac procedure for Generalized Clifford Algebra.

The three Pauli matrices are the lowest dimensional matrices which obey anti-commutation relation. It has been shown by the author that to obtain a greater number of mutually anti-commuting matrices, we must increase the dimension of the matrices, more precisely there are (2m+1) mutually anticommuting matrices of dimension 2^m. Denoting them by $\mathcal{X}_1, \mathcal{X}_2, \cdots, \mathcal{X}_{2m+1}$ and their linear combination as

$$L_{2m+1} = \sum_{i=1}^{2m+1} \lambda_i \mathcal{X}_i \tag{12}$$

where the λ_i's are all real or all imaginary parameters we notice that

$$L^2_{2m+1} = \Lambda^2_m I = (\lambda^2_1 + \lambda^2_2 + \cdots + \lambda^2_{2m+1}) I \, . \tag{13}$$

Setting $m = 2$, $\lambda_1 = p_x$, $\lambda_2 = p_y$, $\lambda_3 = p_z$, $\lambda_4 = 0$, $\lambda_5 = m$ the rest mass and $\Lambda_2 = E$, energy, and $\Lambda_1 =$ helicity $\cdot\, |\vec{p}|$ we obtain the familiar features of the Dirac Hamiltonian.

A similar result holds even for the ω-commuting matrices It can be shown that there are $(2m+1)$ ω-commuting matrices of dimension n^m where $\omega^n = 1$. It follows therefore that

$$(L_{2m+1})^n = \Big(\sum_{i=1}^{2m+1} \lambda_i \mathcal{X}_i\Big)^n = (\Lambda_m)^n I \tag{14}$$

with

$$(\Lambda_m)^n = \lambda^n_1 + \lambda^n_2 + \cdots + \lambda^n_{2m+1}$$

This constitutes a generalization of the Dirac procedure to ω-commuting matrices and it is considered worthwhile to unravel the physical meaning of the sequence of eigenvalues.

$$\Lambda_m(1), \;\; \Lambda_m(2), \; \ldots, \;\; \Lambda_m(n)$$

where $\Lambda_m(i) = \omega^i \Lambda_m(1)$, ω being a primitive n-th root of unity and $i = 1,2,..n$.

The question now may be raised whey do we need to invoke the G.C.A. in elementary particle physics. My answer is: the structure is too fundamental to be unnoticed, too consistent to be ignored and much too pretty to be without consequence.

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GENERALIZED SPINOR STRUCTURE

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We present the principal results of [10] and [11] on some grading classes associated to generalised Clifford algebras. We apply these results to construct some special bases of unitary Lie algebras (see also [12] for SU_3) and to define the generalized spinor structures. One establishes a prolognation theorem of structures [6] associated to space-times, by means of metric of Sasakian type [13] , [15] .

By an algebra over a field F we understand an associative algebra with unity 1 and of finite dimension over F. The field F is identified with F.1.

Let G be a finite Abelian group, $\{e_a\}_{a\in G}$ a basis of the algebra A over F and $\theta: G\times G \to F$. If the structure of A is given by

(1) $$e_a e_b = \theta(a,b) e_{a+b} \qquad (a,b \in G),$$

then the triplet $\mathcal{G} = (G, \{e_a\}, \theta)$ is a <u>maximal</u> G-<u>grading</u> (or <u>maximal grading</u>) of A. There results immediately that e_0 $(0 \in G)$ and 1 are collinear and we always suppose $e_0 = 1$. The equality $(e_a e_{-a})e_a = e_a(e_{-a}e_a)$ implies $\theta(a,-a) = \theta(-a,a)$.

The maximal grading $\mathcal{G}$ is <u>normed</u> if, for any $a,b \in G$, we have

(2) $$\theta(a,b)\,\theta(b,a) = 1 \ .$$

For such a grading

(2') $$\theta(a,-a) = \theta(-a,a) = \varepsilon_a = \pm 1$$

for any $a \in G$ $(\varepsilon_0 = 1)$.

Let M be the set of all maximal G-gradings of A. Every system $k = \{k_a\}_{a \in G}$ $(k_0 = 1)$ of non zero scalars defines an operator $\bar{k}$ of M by

(1')
$$\bar{k}(\mathcal{G}) = (G, \{k_a e_a\}, \theta'), \quad \theta'(a,b) = \frac{k_a k_b}{k_{a+b}} \theta(a,b) .$$

Let M' be the set of all maximal G'-gradings of A (G is isomorphic to G'). Every isomorphism $h : G' \to G$ defines an operator $\bar{h} : M \to M'$ by

(1'')
$$\bar{h}(\mathcal{G}) = (G', \{e_{h(a)}\}, \theta'), \quad \theta'(a,b) = \theta(h(a),h(b)) .$$

Let $\mathcal{G} = (G, \{e_a\}, \theta)$ and $\mathcal{G}' = (G', \{e_b'\}, \theta')$ be two maximal gradings of A and A' respectively. If $G = G'$ and $\theta = \theta'$, then G and G' are <u>isomorphic</u>. If there exists an operator $\bar{h} \circ \bar{k}$ such that $\bar{h} \circ \bar{k}(\mathcal{G})$ and $\mathcal{G}'$ are isomorphic, then G and G' are <u>equivalent</u> $(\mathcal{G} \sim \mathcal{G}')$. In addition, if $k_a = \pm 1$ $(a \in G)$, then G and G' are ε-equivalent $(\mathcal{G} \overset{\varepsilon}{\sim} \mathcal{G}')$. If $\mathcal{G} \sim \mathcal{G}'$, then the algebras A and A' are isomorphic.

It is easily proved that $\sim$ (resp. $\overset{\varepsilon}{\sim}$) is an equivalence relation in the set of all maximal gradings (resp. normed maximal gradings) of the algebras over a field F. Assuming $\mathcal{G} \overset{\varepsilon}{\sim} \mathcal{G}'$, it follows that $\mathcal{G}'$ is normed if and only if $\mathcal{G}$ is normed.

Let $\mathcal{G}_i = (G_i, \{e^i_{a_i}\}, \theta_i)$ be a maximal grading of the algebras A_i over F $(i = 1,2)$. Then $A_1 \otimes_F A_2$ has the maximal grading (the <u>tensor product</u> of $\mathcal{G}_1$ with $\mathcal{G}_2$)

$$G_1 \otimes G_2 = (G \times G_2, \{e^1_{a_1} \otimes e^2_{a_2}\}, \theta) ,$$

where

$$\theta\,((a_1,a_2),(b_1,b_2)) = \theta_1(a_1,b_1)\,\theta_2(a_2,b_2).$$

If G_1 and G_2 are normed, then $G_1 \otimes G_2$ is normed.

Let $M_n(F)$ be the algebra of all $n \times n$ matrices over the field F. We suppose that F contains a primitive n-th root ω of the unity. Then, from the representation theory of generalized Clifford algebras [7] , [8] , there results that $M_n(F)$ is the polynomial algebra generated over F by the matrices e_1, e_2 subject to the relations

$$e_1^n = e_2^n = 1 , \quad e_2 e_1 = \omega e_1 e_2 . \tag{3}$$

Let Z_n be a cyclic group of period n and α_1, α_2 two generators of $Z_n \times Z_n$. The elements of $Z_n \times Z_n$ have the form $a = a_1\alpha_1 + a_2\alpha_2$, $b = b_1\alpha_1 + b_2\alpha_2$, ...

The matrices

$$e_a = e_1^{a_1} e_2^{a_2} \qquad (a \in Z_n \times Z_n) \tag{3'}$$

define a basis of $M_n(F)$ and

$$e_a e_b = \omega^{a_2 b_1} e_{a+b} \qquad (a, b \in Z_n \times Z_n). \tag{3''}$$

Hence $\{e_a\}$ is the basis of a $Z_n \times Z_n$ -maximal grading A_n of $M_n(F)$ and e_1, e_2 are called the generators of A_n . It is easily proved that A_n is independent up to an equivalence of e_1, e_2 and ω .

If $\zeta = \sqrt{\omega} \in F$ we define an operator k given by the scalars k_a $(a \in Z_n \times Z_n)$ as follows. If is odd, then we take

$$\zeta = \omega^{(p+1)/2}, \; k_a = \zeta^{a_1 a_2} .$$

If is even we associate to every $a \in Z_n \times Z_n$ the supplementary component a_3 given by $a_1 + a_2 + a_3 \equiv 0 \pmod{n}$ and we take

$$k_a = \zeta^{\sum_{s>t} a_s a_t + a_3^2} \qquad (0 \leq a_s < n \; ; \; s,t = 1,2,3) .$$

In both cases we have :

$$(4) \qquad k_a^2 = \omega^{a_1 a_2} \; , \; k_a = k_{-a} .$$

From relations (1'), (3'') and (4) there results that the maximal grading $A_n^0 = \bar{k}(A_n)$ is normed and the coefficients ε_a defined by (2') are all equal to 1. The matrices e_1, e_2 are also called the generators of A_n^0. For n even we consider the operator $\bar{k}'$ given by the scalars

$$k_a' = \zeta^{a_1} \qquad (a \in Z_n \times Z_n) , \quad 0 \leq a_1 < n .$$

Then the algebra $M_n(F)$ has also normed maximal grading $A_n^1 = \bar{k}'(A_n^0)$ for which

$$(5) \qquad \varepsilon_a = \begin{cases} 1 & \text{if } a_1 \equiv 0 \pmod{n} \\ -1 & \text{if } a_1 \not\equiv 0 \pmod{n} . \end{cases}$$

The matrices $e_1' = \zeta e_1$ and $e_2' = e_2$ are called the generators of A_n^1. The normed maximal gradings A_n^0 and A_n^1 are independent up to an ε-equivalence of e_1, e_2, ω and ζ.

It is known that every Abelian group G_n with n elements $(n > 1)$ has a representation of the form

(6) $$G_n \approx Z_{q_1} \times \cdots \times Z_{q_r} \qquad (q_1 \cdots q_r = n) ,$$

where

(6') $$q_i = p_i^{m_i} \qquad (p_i \text{ prime} ; \ i = 1, 2, \ldots, r).$$

The numbers r and q_i are invariants of G_n .

It is clearly that the algebra $M_n(F)$ $(\omega \in F)$ has the maximal $G_n \times G_n$ - grading

(7) $$G_n = A_{q_1} \otimes A_{q_2} \otimes \cdots \otimes A_{q_r} .$$

Two maximal gradings (7) are equivalent if and only if their grading groups are isomorphic.

THEOREM 1. Let G be a maximal G-grading of a simple algebra A over F. If the field F has characteristic zero and contains the roots of all binomial equations over F, then $G \approx G_n \times G_n$, $A \approx M_n(F)$ and $G \sim G_n$.

The proof is given in [11] where a more general class of maximal gradings, which are also characterised in terms of generalized Clifford algebras, is constructed.

The algebra $M_n(F)$ $(\zeta \in F)$ has the normed maximal $G_n \times G_n$- grading

(7') $$G_n^{\circ} = A_{q_1}^{\circ} \otimes A_{q_2}^{\circ} \otimes \cdots \otimes A_{q_r}^{\circ} \qquad (G_n^{\circ} \sim G_n) .$$

Suppose that in (6), (6') we have

$$p_1 = p_2 = \cdots = p_s = 2, \quad p_i \neq 2 \quad \text{for } i > s$$

and that the sequence $q_1, q_2, \ldots, q_s$ contains $t = t(G_n)$ different terms $q_1' > q_2' > \ldots > q_t'$. If s_j terms are equal to q_j' $(j = 1, \ldots, t)$, we may write

$$q_1 = \cdots = q_{s_1} = q_1' > q_{s_1+1} = \cdots = q_{s_1+s_2} = q_2' > \cdots >$$

$$q_{s-s_t+1} = \cdots = q_s = q_t' \quad (s_1 + s_2 + \cdots + s_t = s).$$

The invariant $t(G_n)$ is zero if and only if n is odd.

The algebra $M_n(F)$ (n even, $\xi \in F$) has the normed maximal $G_n \times G_n$ -grading

(8) $$G_n^j = A_{q_1}^{x_1} \otimes \cdots \otimes A_{q_s}^{x_s} \otimes A_{q_{s+1}}^{0} \otimes \cdots \otimes A_{q_r}^{0} \quad (G_n^j \sim G_n ; j = 1, \ldots, t),$$

where

(8') $$x_i = \begin{cases} 1 & \text{if } i = s_1 + \cdots + s_{j-1} + 1 \\ 0 & \text{otherwise} . \end{cases}$$

Hence the relations (7'), (8), (8') associate to every group G_n the normed maximal gradings G_n^j, $0 \leq j \leq t(G_n)$. The gradings G_n^j and $G_n^{j'}$ are $\mathcal{E}$-equivalent if and only if their grading groups are isomorphic and $j = j'$.

The results established in [10] together with Theorem 1 imply

THEOREM 2. Let G be a normed maximal G-grading of the simple algebra A over the field F considered in Theorem 1. Then $G \approx G_n \times G_n$, $A \approx M_n(F)$ and $G \overset{\mathcal{E}}{\sim} G_n^j$, $0 \leq j \leq t(G_n)$.

Let $\mathcal{G} = (G, \{E_a\}, \theta)$ be a normed maximal grading of the algebra $M_n(F)$. If the characteristic of F is not a divisior of n, the following relation of Pauli type [10] holds

(9) $$\sum_{a\in G} E^{i}_{ja} E^{ka}_{1} = n\, \delta^{i}_{1} \delta^{k}_{j} \qquad (E^{a} = \varepsilon_a E_{-a}) ,$$

where E^{i}_{ja} and E^{ia}_{j} are the elements of i-th line and j-th row of matrices E_a and E^a respectively, or

(9') $$\sum_{a\in G} E_a \times E^a = n\, \mathrm{tr} X \text{ (X arbitrary in } M_n(F)) .$$

From (9) it results that $\{E_a\}_{a\neq 0}$ is a basis of $M_n(F)'$ (subspace of $M_n(F)$ of all matrices of null trace). Hence

(9'') $$\mathrm{tr}(E^a E_b) = n\, \delta^{a}_{b} \quad (a,b \text{ arbitrary in } G).$$

Every pair (n,p) of integers $(0 \leq p \leq [n/2])$ defines an antiinvolution $X \to X^{+}$ of the algebra $M_n(\mathbf{C})$ ($\mathbf{C}$ complex field) by:

(10) $$X^{+} = H_p X^{*} H_p , \quad H_p = \begin{pmatrix} I_{n-p} & 0 \\ 0 & -I_p \end{pmatrix} \quad (I_p \text{ unity of } M_p(\mathbf{C})),$$

where X^{*} is the adjoint of X.

We consider the real vector spaces

$$\mathfrak{J}(n,p) = \{X \in M_n(\mathbf{C}) : X = X^{+}\} , \quad \mathfrak{J}(n,p)' = \mathfrak{J}(n,p) \cap M_n(\mathbf{C})'$$

and the real Lie groups

$$U(n,p) = \{X \in SL(n,\mathbf{C}) : XX^{+} = \pm 1\} , \quad U_0(n,p) = \{X \in SL(n,\mathbf{C}) : XX^{+} = 1\} .$$

The space $\mathfrak{J}(n,p)$ has a natural structure of real Jordan algebra. We have $U(n,0) = U_0(n,0) = SU_n$. If n is odd, then $U(n,p) = U_0(n,p)$.

The map $X \to \frac{1}{n}\,\mathrm{tr}\,X^2$ defines a ~~xx~~ real nondegenerated quadratic form Q_p (resp. Q_p') on $\mathcal{J}(n,p)$ (resp. $\mathcal{J}(n,p)'$). According to the relation $\mathcal{J}(n,p) = \mathcal{J}(n,p)' \oplus R\cdot 1$, we have $Q_p = Q_p' \oplus 1$, where $1 : x \to x^2$ $(x \in R)$. The index of Q_p' is equal to $(n-2p)^2-1$. The forms Q_0 and Q_0' are positive defined.

Let $\lambda(X)$ be the inner automorphism of $M_n(C)$ given by an arbitrary element $X \in SL(n,C)$ and let λ be the $n:1$ representation $X \to \lambda(X)$. The group $\lambda(SL(n,C))$ is the group of all automorphisms of $M_n(C)$. It is easily proved that $\mathcal{J}(n,p)$ is invariant to $\lambda(X)$ if and only if $X \in U(n,p)$. We denote by ρ_p and ρ_p' the $n:1$ representations of $U(n,p)$ induced by λ on $\mathcal{J}(n,p)$ and $\mathcal{J}(n,p)'$ respectively. We have $\rho_p = \rho_p' \oplus 1$, where $1 : X \to I_n$ and ρ_p' is equivalent with the adjoint representation of $U(n,p)$. On the other hand, $\rho_p(U(n,p)) \subset SO(Q_p)$ and $L(n,p) = \rho_p'(U(n,p)) \subset SO(Q_p')$.

We suppose that the normed maximal grading $G = (G, \{E_a\}, \theta)$ of $M_n(C)$ verifies, for some p , the conditions

(11) $\quad E_a^+ = E_a$

(a arbitrary in G) .

(11 ') $\quad E_a^* = \varepsilon_a E_{-a}$

These conditions are invariant to every operator $\bar{h} \circ \bar{k}$ where $k_a = \pm 1$ and h is an arbitrary isomorphism of G . On the other hand, we can suppose [10] that the generators e_1, e_2 (resp. e_1', e_2') of the normed maximal **grading** A_n^0 (resp. A_n^1) verify the relations

$$e_i^* = e_i^{-1}, \; e_i^+ = e_i \quad \left(n-2p = \frac{3+(-1)^n}{2}\right), \quad e_i^{\prime +} = e_i^{\prime} \quad (p = n/2).$$

Then, from (3), (3'), (4) there results that A_n^0 and A_n^1 verify (11), (11'). But according to (5), for A_n^1 we have $E_a^2 = -1$ $(a = \frac{n}{2}\alpha_1)$. Hence for even n, $U(n, n/2) \neq U_0(n, n/2)$. We also obtain

THEOREM 3. Every normed maximal grading G of $M_n(C)$ verifies, up to an isomorphism, conditions (11) and (11'). The integer $p = p(G)$ in (10) is uniquely determined by G and namely if $G \overset{\mathcal{E}}{\sim} G_n^j$, then $p = n/2$ for $j \neq 0$ and $n-2p$ is the number of the elements of period 2 (zero, too) of G_n for $j = 0$.

Therefore $\{E_a\}_{a \neq 0}$ and $\{iE_a\}_{a \neq 0}$ $(i^2 = -1)$ are bases of the space $\mathcal{T}(n,p)'$ and of the unitary Lie algebra $U(n,p)'$ respectively*). We consider a total ordering relation $<$ in G and we denote by $G' = \{a \in G : -a < a, a \neq 0\}$. Then

$$\{E_a + E^a, \; i(E_a - E^a)\}_{a \in G'}, \text{ and } \{E_a - E^a, \; i(E_a + E^a)\}_{a \in G'}, \quad (E^a = \varepsilon_a E_{-a})$$

give two bases of the spaces $\mathcal{T}(n,0)'$ and SU_n' respectively**). But we write the elements of $\mathcal{T}(n,0)'$ under the form $X = \sum_{a \neq 0}' X^a E_a$, where $\overline{X}^a = \varepsilon_a X^{-a}$ $(a \in G)$.

The quadratic form Q_p and the isomorphism $d\rho_p$ are given in the basis $\{E_a\}$ by

$$Q_p(X) = \sum_{a \in G}' \varepsilon_a X^a X^{-a}, \quad X \in \mathcal{T}(n,p) \tag{12}$$

$$d\rho_p(X)_a^b E_b = [X, E_a], \quad X \in U(n,p)'. \tag{13}$$

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*) If L is a Lie group, we denote by L' its Lie algebra.

**) In [12] this result for SU_3' is obtained, using the basis of the maximal grading A_3.

Using relations (9), (9'') and (13), we obtain the explicit form for $d\rho'_p$ and $d\rho'^{-1}_p$:

(13') $d\rho'_p(X)_a^{\,b} = \frac{1}{n}\, tr(X\,[E_a, E^b])$, $d\rho'^{-1}_p(T) = \frac{1}{n^2} \sum_{a,b \neq 0} T_a^{\,b} E^a E_b$

which hold also for $d\rho'_0$ and $d\rho'^{-1}_0$.

We consider the quadratic form Q on $V = R^4$ given by :

(14) $$Q(x) = -(x^1)^2 - (x^2)^2 - (x^3)^2 + (x^4)^2$$

and the following representation of $O(Q)$:

(15) $$\mu = \bigoplus_{r=0}^{4} \Lambda^r Id \ , \ Id: X \to X \ , \ \Lambda^0 Id : X \to 1 \ (X \in O(Q))$$

having the exterior algebra ΛV as representation space.

Choosing a system of generators $\alpha_1, \dots, \alpha_4$ of the group Z_2^4, we write its elements under the form $a = \sum a_i \alpha_i$, $b = \sum b_i \alpha_i, \dots$ where a_i, b_i are equal to 0 or 1 ($i = 1, \dots, 4$).

Let Δ br the set given by the zero element of Z_2^4 and all sequences $(i_1, \dots, i_r)$ where $1 \leq i_1 < \dots < i_r \leq 4$, $r = 1, \dots, 4$. The xxx map $\delta: \alpha_{i_1} + \dots + \alpha_{i_r} \to (i_1, \dots, i_r)$ establishes a 1:1 correspondence $Z_2^4 \to \Delta$.

Every orthonormal basis $\{e_i\}$ of V verifies in the Clifford algebra C(Q) the relations

(16) $$e_i e_j + e_j e_i = 2\eta_i \delta_{ij} \qquad (\eta_1 = \eta_2 = \eta_3 = -\eta_4 = -1)$$

and defines a basis $\{e_a\}_{a \in Z_2^4}$ of $C(Q)$, where

$e_a = e_{i_1} \cdots e_{i_r}$ with $\delta(a) = (i_1, \ldots, i_r)$, $e_0 = 1$.

There is a vector space isomorphism $\Lambda V \to C(Q)$ such that

$e_{i_1} \wedge \ldots \wedge e_{i_r} \to e_a$ for any orthonormal basis of V. Using this isomorphism, we suppose that the representation μ acts also on $C(Q)$.

Let h_o be the following automorphism of Z_2^4

$$\alpha_1 \to \alpha_1 + \alpha_2 + \alpha_3, \ \alpha_2 \to \alpha_1 + \alpha_2 + \alpha_4, \ \alpha_3 \to \alpha_1, \ \alpha_4 \to \alpha_2$$

and $G_0 = h_o(A_2^1 \otimes A_2^0) = (Z_2^4, \{E_a\}, \theta)$. The elements $e_i = E_{\alpha_i}$ verify the relations (16) and we can suppose that e_i is an orthonormal basis of V. From Theorem 3 there results that (11) holds for n = 4, p = 2 and according to (16) we have:

(17) $$E_a = n_a e_{i_1} \cdots e_{i_r}, \quad \varepsilon_a = \theta(a, -a) = \eta_{i_1} \cdots \eta_{i_r}$$

where

(17') $$n_a = \begin{cases} 1 & \text{if } r = 1 \text{ or } r = 4 \\ \sqrt{-1} & \text{if } r = 2 \text{ or } r = 3 \end{cases}, \quad \delta(a) = (i_1, \ldots, i_r).$$

Therefore we can suppose that μ acts also on the space $\mathcal{T}(4,2)$ and we denote by μ' the representation induced by μ on $\mathcal{T}(4,2)'$

Any $A \in O(Q)$ defines an orthonormal basis $e_i' = Ae_i$ of V and a map

$$T : E_a \to E_a' = n_a e_{i_1}' \cdots e_{i_r}', \quad T \in L(4,2).$$

Since $E'_a = \mu'(A) E_a$, there results that $\mu'(A) = T$ and we denote by S the subgroup of U(4.2) given by $\rho_2(S) = \mu(O(Q))$. The map $\sigma : X \to A$, $X \in \rho_2^{-1}(\mu(A))$ is a 4 : 1 representation of the group S on O(Q) and we have:

$$\rho_2(X) = \mu(\sigma(X)), \quad \rho'_2(X) = \mu'(\sigma(X)) \quad \text{(X arbitrary in S)} \tag{18}$$

Hence ρ_2 induces on the space V the representation σ and we can write:

$$d\sigma(X)_i^j = \frac{1}{4} tr(X [e_i, e^j]), \; d\sigma^{-1}(A) = \frac{1}{4} A_i^j e^i e_j \quad (e^i = \eta_i e_i). \tag{19}$$

Therefore σ^{-1} is locally isomorphic to the spinor representation of the Lorenth group O(Q). Similarly, a representation of the group O(4,R) can be obtained, by means of the normed maximal grading $\bar{h}_o(A_2^o \otimes A_2^o)$. From (13') and (19) there results that the representation $d\rho'^{-1}_p$ has, in the basis $\{E_a\}$, the expression as the spinor representation of O(Q)'.

Let M be a space-time which is, or is not, orientable and let $P = P(M, O(Q))$ be the principal fibre bundle of orhtonormal frames over M. A spinor structure of M [6] is a principal fibre bundle homomorphism $f : \Sigma(M, S) \to P(M, O(Q))$ [5] with the corresponding structure group homorphism $\sigma : S \to O(Q)$ such that the induced map on M is the identity map*).

* For every principal fibre bundle we suppose that the structure group acts freely on the total space. Hence the above map f is a 4 : 1 map.

The existence conditions of spinor structures defined by replacing $\mathfrak{S}$ by the natural $2:1$ representation Pin $\mathfrak{a} \to O(\mathfrak{a})$ are given in [2] and [3] . The same problem for the covering map of the connected Lorentz group $SL(2, \mathbb{C}) \to L_4$ is considered in [4] .

Let G be a normed maximal grading of $M_n(\mathbb{C})$ and let M be a Riemannian manifold of dimension $n^2 - 1$ and index $(n-2p)^2 - 1$, such that the principal fibre bundle of orthonormal frames over M has a reduction P to $L(n,p) = \rho_p'(U(n,p)) \subset O(\mathfrak{a}_p')$. If $p \neq 0$ we suppose that $p = p(G)$. *A generalised spinor structure* of M associated to G is a principal fibre bundle homomorphism $f: \sum(M, U(n,p)) \to P(M, L(n,p))$ with the corresponding structure group homomorphism $\rho_p': U(n,p) \to L(n,p)$ such that the induced map on M is the identity map.

We can study the spinor and spin-tensor fields both for generalized spinor structures and usual spinor structures. The relations (13') and (19) imply similar expressions of the spinor ~~structure~~ covariant derivatives in both cases.

THEOREM 4. *Let* $f: \sum \to P$ *be a spinor structure of the space-time M. There exists a Riemannian manifold M' of dimension 15 and index -1 a projection* π *of M' onto M and a generalized spinor structure* $f': \sum' \to P'$ *of M' associated to the normed maximal grading* G_0 ($\sum'$ *reducible to S) such that the transition functions* $\psi_{\alpha\beta}$ *and* $\psi'_{\alpha\beta}$ *of the principal fibre bundles* $\sum$ *and* $\sum'$ *respectively verify the condition*

(20) $$\psi'_{\alpha\beta} = \psi_{\alpha\beta} \circ \pi .$$

Proof. Let V be a n-dimensional vector space and let B be a symmetric nondegenerated bilinear form on V. There exists a unique symmetric nondegenerated bilinear form B_r on $\otimes^r V$ such that:

$$B_r(X_1\otimes\cdots\otimes X_r, Y_1\otimes\cdots\otimes V_r) = B(X_1,Y_1)\cdots B(X_r,Y_r)$$

for all $X_1, \ldots, X_r, Y_1, \ldots, Y_r \in V$. We denote by B_r' the restriction of $\frac{1}{r!} B_r$ to $\Lambda^r V$. If for the basis $\{e_i\}$ we have $B(e_i, e_j) = k_i \delta_{ij}$ then

$$B_r'(e_{i_1}\wedge\cdots\wedge e_{i_r}, e_{j_1}\wedge\cdots e_{j_r}) = k_{i_1}\cdots k_{i_r}\delta_{i_1 j_1}\cdots\delta_{i_r j_r}.$$

We associate to any point x of the space-time M the vector space $F_x = \bigoplus_{r=2}^{4} \Lambda^r T_x$, where $T_x = T_x(M)$ is the tangent space of M at x. The metric g of M induces on each space F_x the bilinear form $g' = g_2' \oplus g_3' + g_4'$.

Let M' be the vector bundle over M of all pairs (x,y) where $x \in M$, $y \in F_x$ and let $\pi : M' \to M$ be the natural projection $(x,y) \to x$. The Riemannian connection of M associates to each point $u \in M'$ the horizontal space H_u and the vertical space $V_u = T_u(F_x)$ with $x = \pi(u)$ such that $T_u(M') = H_u \oplus V_u$. Let h be the isomorphism which carries each vector $X \in T_x(M)$ in its horizontal lift $hX \in H_u$ and let v be the translation $F_x \to V_u$. The manifold M' has Saskian metric of index -1

$$G = g\circ(h^{-1}\times h^{-1}) \oplus g'\circ(v^{-1}\times v^{-1}). \tag{21}$$

Let $\{U_\alpha\}$ be an open covering of M such that on every U_α there exists a field $\{\theta_i\}$ of orthonormal frames with respect to g. If $\{\theta_i'\}$ is the field of orthonormal frames on U_β, then the transition functions of P are given by:

$$\theta_i'(x) = \gamma_{\alpha\beta}(x)_i^j\, \theta_j(x) \,, \quad x \in U_\alpha \cap U_\beta \tag{22}$$

and we can suppose that

$$\gamma_{\alpha\beta} = \sigma \circ \psi_{\alpha\beta} \,. \tag{23}$$

From (12), (17), (17') and (22) there follows that the field of frames $\{\gamma_a\}$ $(a \in Z_2^4 \,,\ a \neq 0)$ on $V_\alpha = \pi^{-1}(U_\alpha)$ given by :

$$\gamma_a = \begin{cases} h \circ \tilde{\theta}_i \quad (\tilde{\theta}_i = \theta_i \circ \pi) & \text{if } \delta(a) = i \\ v\,(\tilde{\theta}_{i_1} \wedge \cdots \tilde{\theta}_{i_r}) & \text{if } \delta(a) = (i_1, \ldots, i_r), r \geq 2 \end{cases}$$

and the similar field $\{\gamma_a'\}$ on V_β are orthonormal with respect to the metric (21) and we have:

$$\gamma_a'(u) = \mu'(\varphi_{\alpha\beta}(x))_a^b\, \gamma_b(u) \qquad (u \in V_\alpha \cap V_\beta \,,\ x \in \pi(u)\,. \tag{22'}$$

Hence the principal fibre bundle of orthonormal frames over M' has a reduction P' to L(4,2) whose transition functions are

$$\varphi'_{\alpha\beta} = \mu' \circ \varphi_{\alpha\beta} \circ \pi \,. \tag{23'}$$

Relations (18), (23) and (23') imply

$$\varphi'_{\alpha\beta} = \rho'_2 \circ \Psi_{\alpha\beta} \circ \pi \quad . \tag{23"}$$

We consider Σ' , the principal fibre bundle whose transition functions are given by (20) and the homomorphism $f': \Sigma' \to P'$ which preserves the conditions $\varphi'_{\alpha\beta} = \rho'_2 \circ \varphi_{\alpha\beta}$. Q.E.D.

A similar theorem for the normed maximal grading $\overline{h}_o(A^o_2 \otimes A^o_2)$ and Riemannian manifolds of dimension 4 and index zero holds.

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