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COMPLEX VARIABLE PROOFS OF TAUBERIAN THEOREMS

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COMPLEX VARIABLE PROOFS OF TAUBERIAN THEOREMS

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Introduction

Although the subject of Tauberian theorems is now a classical one, it has attracted new interest in the last decade since new complex variable methods were found to establish Tauberian theorems. The aim of these lectures is to give an account of this development.

To make the lectures self-contained, we shall give a brief introduction to summability methods in Chapter I, whereas in Chapter II we make the function theoretic preparations. Chapter III will contain Tauberian theorems with gap condition or σ-condition.

D. G.

C H A P T E R 1

I N T R O D U C T I O N T O S U M M A B I L I T Y.

§ 1. Summability methods, some examples.

The most important notion in analysis is that of convergence. Summability methods serve as a generalization of the classical notion of convergence. The problem is to associate a generalized limit to a convergent or divergent sequence.

The principle adopted in the summability methods is, if the series Σa_n is given, we apply a transformation V to the sequence of partial sums s_n of the series, denoted by $\sigma_n = V(s_n)$, or the transformation may be obtained as a function of a continuous variable ($\sigma(x) = V(s_n)$). Then we study the ordinary convergence of the newly obtained sequence or function. We say that the given series is summable by the method V if the transformed sequence or function tends to a limit.

DEFINITION. We say $\underline{V\text{-lim}}\ s_n = s$ if and only if $\sigma_n \to s$ as $n \to \infty$ (or $\sigma(x) \to s$ as $x \to \genfrac{}{}{0pt}{}{+\infty}{+1}$).

We give here only four summability methods, though other summability methods are there.

A. Cesáro method. Let an arbitrary series Σa_n be given, form $s_n (n = 0,1,2,\ldots.)$, the partial sums of the series, and define the transformation

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} s_k \quad (n = 0,1,2,\ldots.)$$

This is called the Cesáro means.

DEFINITION. We say the series Σa_n is Cesáro summable to s, $C - \Sigma a_n = s$, if and only if $\sigma_n \to s$ as $n \to \infty$

Examples: (1) $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (-1)^n$.

Here

$$s_n = \begin{cases} 1 \text{ for } n \text{ even.} \\ 0 \text{ for } n \text{ odd.} \end{cases}$$

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} s_k \to \frac{1}{2} \quad \text{as } n \to \infty .$$

Hence

$$C - \sum_{n=0}^{\infty} (-1)^n = \frac{1}{2} .$$

(2) Consider the geometric series $\sum_{n=0}^{\infty} z^n$.

We have for $z \neq 1$

$$s_n = \frac{1-z^{n+1}}{1-z},$$

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} \frac{1-z^{k+1}}{1-z}$$

$$= \frac{1}{1-z} - \frac{z}{1-z} \frac{1}{n+1} \sum_{k=0}^{n} z^k$$

$$= \frac{1}{1-z} - \frac{1}{n+1} \frac{z}{1-z} \frac{1-z^{n+1}}{1-z}.$$

The term $\frac{1-z^{n+1}}{1-z}$ is bounded for $|z| \leq 1$ and $z \neq 1$, and so $\sigma_n \to \frac{1}{1-z}$ as $n \to \infty$ for $|z| \leq 1, z \neq 1$.

<u>THEOREM</u>. $C - \Sigma a_n = s \implies a_n = o(n)$.

<u>PROOF</u>. We may assume without loss of generality that $s = 0$; (otherwise consider $\Sigma a_n'$ with $a_0' = a_0 - s$ and $a_n' = a_n (n > 0)$).

Hence

$$\sigma_n \to 0 \Longrightarrow \sum_{k=0}^{n} s_k = o(n)$$
$$\Longrightarrow s_n = o(n)$$
$$\Longrightarrow a_n = o(n). \|$$

COROLLARY. The C-method is useless for analytic continuation.

B. The Abel Method.

In this method, we transform the given series into a new series which is a function of a continuous variable. The transformation is defined as

$$\sigma(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (0 \leq x < 1).$$

DEFINITION. We say Σa_n is A-summable to s, or A-$\Sigma a_n = s$, if and only if

(i) $\sigma(x)$ converges for x in (0,1),

(ii) $\sigma(x) \to s$ for $x \to 1-$.

Examples: (1) $1+z+z^2+\ldots\ldots$

The Abel transformation

$$\sigma(x) = \sum_{n=0}^{\infty} z^n x^n$$

$$= \frac{1}{1-x\cancel{z}}, \qquad |z| \leq 1$$

$$\to \frac{1}{1-z} \text{ as } x \to 1 \text{ for } |z| \leq 1,\ z \neq 1.$$

Therefore A- $\Sigma z^n = \frac{1}{1-z}$ $(|z| \leq 1,\ z \neq 1)$.

(2) This method can be used in finding the solution of the Dirchlet Problem for $|z| \leq 1$. For the solution of the problem, expand the given function f in a Fourier series, i.e.

$$f(e^{i\varphi}) = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\varphi + b_k \sin k\varphi).$$

Form the Abel transform

$$\sigma(r) = u(r,\varphi) = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\varphi + b_k \sin k\varphi)r^k.$$

This converges for $0 \leq r < 1$. It is easy to show

(i) u is harmonic in the unit circle,

(ii) u is continuous in $|z| \leq 1$, and $u(1,\varphi) = f(e^{i\varphi})$, if the given function f was continuous on $|z| = 1$.

C. General Summability Method.

In general, for a given series Σa_n form the partial sums s_n, and the transformation is defined as

$$\sigma_n = \sum_{k=0}^{\infty} a_{n.k}\, s_k, \quad n = 0,1,2,\ldots..$$

where $V = (a_{nk})$ $(n = 0,1,2,\ldots;\ k = 0,1,2,\ldots.)$ is a weight matrix. This is sometimes also called a F.F. transformation.

DEFINITION. We say Σa_n is V-summable to s, or $V\text{-}\Sigma a_n = s$, if and only if

(i) σ_n exists for $n = 0,1,2,\ldots.$,

(ii) $\sigma_n \to s$ as $n \to \infty$.

The fundamental problem in these methods is to study the relation between the statements $\Sigma a_n = s$ and $V\text{-}\Sigma a_n = s$. The problem splits into two parts. The first part is when we could say V is a permanent method, that is to say when $\Sigma a_n = s$ always implies $V\text{-}\Sigma a_n = s$. The second part of the problem is just converse to the first part which we study as Tauberian theorems in Chapter III. As a solution to the first part, Toeplitz has given 3 conditions on V.

THEOREM. (Toeplitz) The method $V(=(a_{nk}))$ is permanent if and only if the following three conditions are satisfied.

(a) $\lim_n a_{nk} = 0$ for fixed $k = 0,1,2,\ldots$;

(b) $\sum_{k=0}^{\infty} a_{nk}$ is an absolutely convergent series and

$\lim_n \sum_k a_{nk} = 1$;

(c) $\sum_k |a_{nk}| \leq M$ for all $n = 0,1,2,\ldots$.

PROOF. We only give the proof of the sufficiency part. Assume $s_n \to s$ and V satisfies the conditions. Now we have to prove that $\sigma_n - s \to 0$ as $n \to \infty$.

$$\sigma_n - s = \sum_k a_{nk} s_k - s$$

$$= \sum_k a_{nk}(s_k - s) + s(\Sigma a_{nk} - 1), \ n = 1, 2, \ldots .$$

Let $\varepsilon > 0$ be given, then $|s_k - s| < \frac{\varepsilon}{M}$ $(k \geq n')$. We split the above sum into two parts and get

$$\sigma_n - s = \sum_{k<n'} a_{nk}(s_k - s) + (\sum_{k \geq n'} a_{nk}(s_k - s) + s(\Sigma a_{nk} - 1)$$

Since

1) $|\sum_{k<n'}| < \varepsilon$ for $n > N_1$ by condition (a),

2) $|\sum_{k \geq n'}| < \frac{\varepsilon}{M} M$ by condition (c),

3) $|s(\Sigma a_{nk} - 1)| < \varepsilon$ for $n > N_2$ by condition (b),

$$|\sigma_n - s| < 3\varepsilon \quad \text{for } n > \max(N_1, N_2)$$

and ε arbitrary, the result follows.|

<u>REMARK</u>. There is a similar theorem for a function to function transformation in place of our sequence to sequence transformation.

D. <u>The methods of Borel</u>. Given the series Σa_n, form the partial sums s_n and define

$$\sigma(x) = e^{-x} \sum_{n=0}^{\infty} \frac{s_n x^n}{n!} \quad (x > 0),$$

called the Borel transformation of $\{s_n\}$.

DEFINITION. We say Σa_n is <u>Borel summable</u> to s, or B- $\Sigma a_n = s$, if and only if

(i) $\sigma(x)$ exists for every $x > 0$
(i.e. $\sigma(x)$ is an entire function)

(ii) $\lim_{x\to\infty} \sigma(x) = s.$

To specify that the transformation is on s_n we write $B(x, s_n)$ instead of $\sigma(x)$ when necessary.

Given a series $a_0 + a_1 + a_2 \ldots\ldots.,$ form the shifted series $0 + a_0 + a_1 + a_2 + \ldots\ldots\ldots$ If we call the partial sums of the former series s_n, that of the latter S_n, we see that $S_n = s_{n-1}$. Now let us study the summability of these two series and the relation in the sense of Borel.

DEFINITION. We say $B' - \Sigma a_n = s$ if $B(x,S_n) \to s$ as $x \to \infty$.

Relation between B and B$'$

$$\int_0^x \sum_0^\infty \frac{s_n t^n}{n!} dt = \sum_0^\infty \frac{s_n x^{n+1}}{(n+1)!}$$

$$= \sum_0^\infty \frac{S_{n+1} x^{n+1}}{(n+1)!}$$

$$= \sum_0^\infty \frac{S_n x^n}{n!} \qquad \text{since } S_o = 0.$$

i.e.,

$$\int_0^x e^t B(t,s_n)dt = e^x B(x,S_n)$$

or

$$(*) \qquad B(x,S_n) = \int_0^x e^{t-x} B(t,s_n)dt.$$

The conclusion from B to B$'$ is

THEOREM. $B - \Sigma a_n = s \Longrightarrow B' - \Sigma a_n = s.$

PROOF. We have to check Toeplitz conditions for the integral transform (*).

a) is trivial.

b) and c): $$\int_0^x K(x,t)dt = e^{-x}(e^x - 1) \leq 1$$

and $\to 1$ as $x \to \infty$,

where

$$K(x,t) = \begin{cases} e^{t-x}, & 0 \leq t \leq x \\ 0, & t > x. \end{cases} \quad ||$$

REMARK. The converse is not true in general.

Example. Define $\{S_n\}$ by $B(x, S_n) = \frac{\sin x^2}{x}$

i.e.,

$$\Sigma \frac{S_n x^n}{n!} = e^x \frac{\sin x^2}{x}$$

a) Clearly we have $B' - \Sigma a_n = 0$.

b) $B(x, s_n)e^x = \frac{d}{dx} (B(x, S_n)e^x)$

$$= \frac{d}{dx} (\frac{\sin x^2}{x} e^x)$$

$$= e^x \left[\frac{\sin x^2}{x} + 2 \cos x^2 - \frac{\sin x^2}{x^2}\right]$$

$$B(x, s_n) = 2 \cos x^2 + o(1),$$

which implies that $\lim_{x \to \infty} B(x, s_n)$ does not exist.

Application to the geometric series Σz^n, z fixed.

$$s_n = \frac{1-z^{n+1}}{1-z} \quad (z \neq 1),$$

$$B(x, s_n(z)) = e^{-x} \sum_{0}^{\infty} \frac{\frac{1}{1-z} - \frac{z}{1-z} z^n}{n!} x^n .$$

$$= \frac{1}{1-z} - \frac{z}{1-z} e^{x(z-1)}.$$

The factor $e^{x(z-1)} \to 0$ as $x \to \infty$ if Re $z < 1$, So B- $\Sigma z^n = \frac{1}{1-z}$ for Re $z < 1$.

Here actually we get the analytic continuation of the geometric series outside the circle of convergence.

§ 2. Application of summability methods to Analytic continuation.

A. Analytic continuation by Euler's method. In this method, the transformation matrix (a_{nk}) is given by

$$a_{nk} = \begin{cases} \binom{n}{k} \frac{1}{2^n}, & k \leq n \\ 0, & k > n. \end{cases}$$

So, for a given series Σa_n, form the partial sums s_n, and the transformation will be

$$\sigma_n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} s_k .$$

It is easy to check that this transformation satisfies the Toeplitz conditions, so the method is permanent.

<u>DEFINITION</u>. We say E- $\lim s_n = s$ if and only if $\lim_n \sigma_n = s$.

Example. $\Sigma z^n = 1 + z + z^2 + \ldots\ldots$

Partial sums $s_n(z) = \dfrac{1-z^{n+1}}{1-z}$

$$\sigma_n = \frac{1}{2^n} \sum_0^n \binom{n}{k} s_k(z)$$

$$= \frac{1}{1-z} - \frac{z}{1-z} \frac{1}{2^n} \sum_0^n \binom{n}{k} z^k.$$

$$= \frac{1}{1-z} - \frac{z}{1-z} \frac{(1+z)^n}{2^n}$$

where

$$\frac{(1+z)^n}{2^n} \longrightarrow 0 \quad \text{as } n \to \infty \text{ for } |1+z| < 2.$$

Hence

$$E\text{-}\ \Sigma z^n = \frac{1}{1-z}$$

if z is in the circle $|1+z| < 2$.

As we are interested in proving the Fabry gap theorem, which we state later, let us study the summability of a general power series $f(z) = \Sigma a_n z^n$ near $z = 1$. Let us write

$$a_n z^n = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi}\ \frac{z^n}{\xi^n}\,d\xi$$

where C is a contour around $z = 0$ and z fixed. Now

$$\sigma_n(z) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} s_k$$

$$= \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi}\left[\frac{1}{1-\frac{z}{\xi}} - \frac{\frac{z}{\xi}}{1-\frac{z}{\xi}}\left(\frac{1+\frac{z}{\xi}}{2}\right)^n\right]d\xi$$

$$\sigma_n(z) - f(z) = \frac{-1}{2\pi i}\int_C \frac{f(\xi)}{\xi}\ \frac{z}{\xi - z}\left(\frac{\xi+z}{2\xi}\right)^n d\xi\ ,$$

if C also surrounds the point z. The right hand side of the above tends to zero when $\left|\frac{\xi+z}{2\xi}\right| < 1$. It is easily seen that the quantity $\left|\frac{\xi+1}{2\xi}\right|$ will be less than 1 for all ξ outside the circle $|\xi-\frac{1}{3}| = \frac{2}{3}$. In other words $\left|\frac{\xi+1}{2\xi}\right| < 1$ if $|\xi-\frac{1}{3}| > \frac{2}{3}$. Now let us assume that f is regular in $|z-\frac{1}{3}| \leq \frac{2}{3}$ then f is regular in and on C, a concentric circle of radius greater than $\frac{2}{3}$. If $\xi \in C$, we have

$$\left|\frac{\xi+1}{2\xi}\right| \leq q' < 1 \Rightarrow \left|\frac{\xi+z}{2\xi}\right| \leq q < 1 \text{ for } z \in N(1),\ \xi \in C,$$

which in turn implies

$$\sigma_n(z) - f(z) = O(q^n) \text{ for } z \in N(1),\ n = 1,2,\ldots$$

We have therefore proved the following theorem.

THEOREM. *If f is regular in* $|z-\frac{1}{3}| \leq \frac{2}{3}$ *then for some neighbourhood* N(1) *of the point* z=1 *and for some* q < 1 *we have*

$$\sigma_n(z) - f(z) = O(q^n),\ z \in N(1).$$

Example. $f(z) = \frac{1}{2z^2+1}$. This example shows also that we may get the analytic continuation of f beyond its circle of convergence.

B. Turán's proof of the Fabry gap theorem.

Before stating the theorem let us formulate the Fabry gap condition.

Given a power series $f(z) = \Sigma a_n z^n$ with radius of convergence equal to 1. Assume many $a_n = o$ i.e., $a_n = o$ $(n \neq n_k)$. We say that the series satisfies Fabry's gap condition if $a_n = 0$ for $n \neq n_k$, with $\frac{n_k}{k} \to \infty$ as $k \to \infty$.

REMARK. $n_{k+1} - n_k \to \infty \Longrightarrow \frac{n_k}{k} \to \infty$.

PROOF. Write $n_{k+1} - n_k = \ell_k$, $k = 1,2,\ldots$, so,

$$\sum_{k=1}^{N} (n_{k+1} - n_k) = n_{N+1} - n_1 = \sum_{k=1}^{N} \ell_k .$$

Divide throughout by N, we have

$$\frac{n_{N+1} - n_1}{N} = \frac{1}{N} \sum_{k=1}^{N} \ell_k .$$

The r.h.s. tends to ∞ since $l_k \to \infty$ and so do the arithmetic means. Hence

$$\frac{n_N}{N} \longrightarrow \infty \quad \text{as } N \to \infty .$$

FABRY'S GAP THEOREM. Let $f(z) = \Sigma a_n z^n$ have radius of convergence equal to 1. Then Fabry gaps imply that f is not continuable beyond the unit circle.

To prove this we use Turán's fundamental lemma; which we state without proof.

LEMMA. Let P(z) be a polonomial with N terms and denote

$$M(\delta) = \max_{\substack{|z|=1 \\ |\arg z| \leq \delta}} |P(z)|$$

Then for some constant C.

$$M(\pi) \leq \left(\frac{c}{\delta}\right)^N M(\delta).$$

PROOF OF FABRY'S GAP THEOREM. Assume $f(z)$ is regular at $z = 1$, then from the theorem proved in part A we have

$$\sigma_{n_k}(z)-f(z) = \mathcal{O}(q^{n_k}) \text{ for some } q<1 \text{ and } z\in N(1)$$

where $\sigma_{n_k}(z)$ contains only the powers $n_1, n_2, \ldots n_k$.

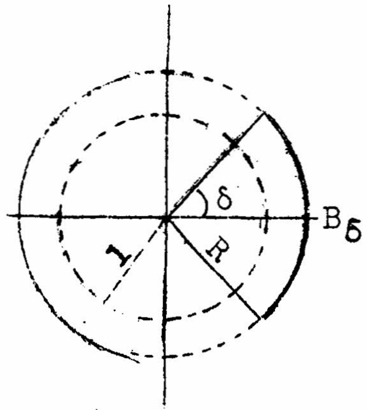
Now form

$$\sigma_{n_{k+1}}(z)-\sigma_{n_k}(z) = \mathcal{O}(q^{n_k}), \quad z\in N(1) \text{ and } z\in B_\delta \text{ (see Fig.)}.$$

The l.h.s. of the above is a polynomial of at most $k+1$ terms. By Turán's Lemma we have

$$\max_{|z| \leq R} \left|\sigma_{n_{k+1}}(z) - \sigma_{n_k}(z)\right| \leq \left(\frac{C}{\delta}\right)^{k+1} C' q^{n_k}$$

$$\leq C'' \left[\left(\frac{C}{\delta}\right)^{\frac{k}{n_k}} q\right]^{n_k} \qquad k=1,2,\ldots$$



As $\frac{k}{n_k} \to 0$ as $k \to \infty$, the term $\left(\frac{c}{\delta}\right)^{k/n_k} q \leq q' < 1$ for $k > k_0$.

Hence

$$\max_{|z| \leq R} \left| \sigma_{n_{k+1}}(z) - \sigma_{n_k}(z) \right| = O(q')^k, \quad k=1,2,\ldots$$

Hence $\{\sigma_{n_k}(z)\}$ converges uniformly to a function $F(z)$ for $|z| \leq R$. $F(z)$ being the uniform limit of analytic functions, is analytic in $|z| < R$. But $\sigma_{n_k}(z) \longrightarrow f(z)$ in the unit circle because the Euler method is a permanent method. Therefore $F(z) = f(z)$ in $|z| < 1$, or $F(z)$ is the analytic continuation of f to $|z| < R$. This contradicts the hypothesis, that the radius of convergence is 1.‖

CHAPTER II

SOME THEOREMS OF PHRAGMÉN-LINDELÖF TYPE

In this chapter we are going to prove some theorems of Phragmén-Lindelöf which constitute a generalization of the maximum principle, and we give some applications of the P.L. theorem. In the last section we give properties of Blaschke products for a half plane.

§1. The Main Theorem.

The classical maximum principle says that if a function f is analytic inside a domain D, continuous on the boundary ∂D and $|f(z)| \leq M$, $\xi \in \partial D$, then $|f(z)| \leq M$ for all $z \in D$. On the other hand a function analytic inside a domain D and continuous as $z \to \xi \in \partial D$ except for one boundary point $\xi_0 \in \partial D$ with $|f(z)| \leq M$ for all $\xi \in \partial D$, $\xi \neq \xi_0$ will imply $|f(z)| \leq M$ for all $z \in D$, provided some growth condition is imposed on f near the exceptional point ξ_0. This is our generalization of the maximum principle. We are concerned with the most important case when $\xi_0 = \infty$ and D is an angle.

A. The Theorem of Phragmén-Lindelöf.

THEOREM. Let f be regular in $|\arg z| < \alpha\pi$ $(0 < \alpha \leq 1)$, continuous at finite boundary points, and $|f(re^{\pm i\alpha\pi})| \leq M$. Further assume $|f(z)| = O(e^{r^\beta})$ $(r = |z|,\ z \in \text{angle})$. Then $|f(z)| \leq M$ for all z in the angle if $\alpha\beta < 1/2$.

In particular,

1) When $\beta = 1$, it implies that if f is of exponential type and the angle is $< \pi$, $|f(z)| \leq M$ on the sides of the angle implies $|f(z)| \leq M$ inside the angle.

2) When $\alpha = 1$, $\beta < 1/2$, f is an entire function of order $< 1/2$. By the theorem it cannot have a radial asymptotic path except when f is constant.

PROOF. It is sufficient to prove the case $\alpha = 1/2$ (otherwise substitute $\omega = z^{\frac{1}{2\alpha}}$). Now consider the auxiliary function

$$(*) \qquad F(z) = f(z)e^{-\varepsilon z^\gamma}, \quad \beta < \gamma < 1.$$

for a given $\varepsilon > 0$ and γ fixed. The function F is regular in Re $z > 0$ and continuous for Re $z \geq 0$. On the imaginary axis, we have

$$|F(iy)| \leq Me^{-\varepsilon r^{\gamma}\cos\gamma\varphi}.$$

Since $\cos\gamma\varphi > 0$ for $\varphi = \pi/2$, $e^{-\varepsilon r^{\gamma}\cos\gamma\varphi}$ is less than 1. Hence

$$|F(iy)| \leq M.$$

For any z, $|z| = r \geq r_0$ it follows by hypothesis

$$|F(z)| \leq Ce^{r^{\beta}}e^{-\varepsilon r^{\gamma}\cos\gamma\varphi}.$$

Since $\cos\gamma\varphi \geq d > 0$, the right hand side of the above tends to zero as r tends to ∞. By applying the classical maximum principle we have

$$|F(z)| \leq M$$

inside a semicircle of radius $r(> r_0(\varepsilon, \gamma))$. Since our r is arbitrary, the above inequality is true for Re $z \geq 0$. Therefore from (*)

$$|f(z)| \leq Me^{\varepsilon r^{\gamma}\cos\gamma\varphi}$$

for $z = re^{i\varphi}$, Re $z \geq 0$. Now for a fixed z and γ, letting $\varepsilon \to 0$, the result follows.‖

B. Refinement for functions of exponential type.

Let us assume that f is regular in the right half plane and is of exponential type. If f is bounded by M on the imaginary axis, it need not be bounded by M in Re $z > 0$. However, this is true under an additional hypothesis.

THEOREM. *Let f be of exponential type in* Re $z \geq 0$ *and* $\overline{\lim_{x \to +\infty}} \frac{\log |f(x)|}{x} \leq 0$. *If* $|f(z)| \leq M$ *for* $z = iy$ *then* $|f(z)| \leq M$ *in* Re $z > 0$.

PROOF. Consider the function

$$F(z) = f(z)e^{-\varepsilon z} \quad \text{for any } \varepsilon > 0.$$

This F is of exponential type in Re $z \geq 0$ and tends to zero on the real axis as $x \to \infty$. So F has a maximum N(say) on the real axis at some point x_0. On the imaginary axis $|F(z)| \leq M$. Now by the Phragmén-Lindelöf theorem

$$|F(z)| \leq \max(M,N)$$

for $0 \leq \arg z \leq \pi/2$. Arguing similarly, the same inequality holds for $-\pi/2 \leq \arg z \leq 0$ also. Hence

$$|F(z)| \leq \max(M, N), \operatorname{Re} z > 0.$$

We now claim that $N \leq M$. Suppose $N > M$. Take a circle round the point x_0. F has the maximum inside the circle, which is possible only if F is a constant, which is a contradiction.

Now it follows that

$$|F(z)| \leq M, \operatorname{Re} z > 0$$

or

$$|f(z)| \leq Me^{\varepsilon x}, \operatorname{Re} z \geq 0.$$

For z fixed, letting $\varepsilon \to 0$, the result follows.||

COROLLARY. *If f is of exponential type in* $\operatorname{Re} z \geq 0$, $\overline{\lim_{x \to +\infty}} \frac{\log|f(x)|}{x} \leq C$ *and* $|f(iy)| \leq M$, *then* $|f(z)| \leq Me^{Cx}$ *for* $\operatorname{Re} z \geq 0$.

PROOF. Apply the theorem to the function

$$F(x) = f(z)e^{-Cz}.||$$

§2. Applications of the Phragmén-Lindelöf theorem.

A. Theorems of Montel.

THEOREM. *If f is regular and bounded in* $0 \leq \arg z \leq \pi/2$ *and* $f(x) \to 0$ *as* $x \to \infty$, *then*

$$f(z) \Longrightarrow^{*} 0 \text{ as } z \to \infty$$

in $0 \leq \arg z \leq \frac{\pi}{2} - \delta$ $(\delta > 0)$.

PROOF. Let us assume first

$$|f(z)| \leq 1,\ 0 \leq \arg z \leq \frac{\pi}{2}$$

and

$$|f(x)| \leq \varepsilon \quad (x \geq 0,\ 0 < \varepsilon < 1).$$

Consider the auxiliary function

$$F(z) = f(z)\,\frac{z^{-\frac{2}{\pi} i \log \varepsilon}}{\varepsilon}.$$

---

* Read 'uniformly tends to'.

this function F is regular for $0 \leq \arg z \leq \pi/2$ and is bounded by 1 on real and imaginary axis. Hence by the Phragmén-Lindelöf theorem

$$|F(z)| \leq 1, \quad 0 \leq \arg z \leq \frac{\pi}{2}$$

or

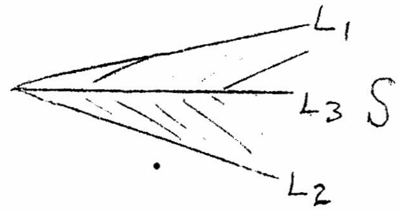
$$|f(z)| \leq \varepsilon \left| e^{\frac{2}{\pi} i \log \varepsilon \cdot \log z} \right|, \quad 0 \leq \arg z \leq \frac{\pi}{2}$$

so that

$$\begin{aligned} |f(z)| &\leq \varepsilon\, e^{-\frac{2}{\pi} \log \varepsilon \arg z} \\ &\leq \varepsilon\, e^{-\frac{2}{\pi} \log \varepsilon \left(\frac{\pi}{2} - \delta\right)}, \quad 0 \leq \arg z \leq \frac{\pi}{2} - \delta \\ &= e^{\frac{2}{\pi} \delta \log \varepsilon} \\ &= \varepsilon^{\frac{2\delta}{\pi}}. \end{aligned}$$

The general case follows easily from these considerations, because $|f(x)| < \varepsilon$ $(x \geq x_0)$ and therefore $|f(z)| \leq \varepsilon^{2\delta/\pi}$ if $0 \leq \arg (z-x_0) \leq \frac{\pi}{2} - \delta$.||

COROLLARY. *If f is regular and bounded in an angle S, and $f(z) \to a$ as $z \to \infty$ on one side of S and $f(z) \to b$ as $z \to \infty$ on the other side of S, then $a = b$ and $f(z) \to a$ uniformly in S as $z \to \infty$.*



PROOF. Let $f \to a$ on L_1 and $f \to b$ on L_2 as $z \to \infty$ ~~or f — a → 0~~ or $f - a \to 0$ and $f - b \to 0$ as $z \to \infty$. By our theorem, both $f - a$ and $f - b$ tend to zero on L_3. Hence $a = b$ and $f \Longrightarrow a$ as $z \to \infty$ in S. ||

B. The f-f' theorem.

Given a function f which tends to s as $x \to \infty$, in general we cannot conclude that $f'(x) \to 0$ as $x \to \infty$. For example $f(x) = \dfrac{\sin x^2}{x} \to 0$ as $x \to 0$. But $f'(x) = 2 \cos x^2 + o(1)$ does not tend to zero as $x \to 0$. This will be true provided the given function does not grow too rapidly.

THEOREM. *If f is of exponential type in* $\operatorname{Re} z \geq 0$ *and* $f(x) \to 0$ *as* $x \to \infty$ *then* $f'(x) \to 0$ *as* $x \to \infty$.

PROOF. Choose a $K' > K$ where $|f(z)| = (e^{K|z|})$ in $\operatorname{Re} z \geq 0$ for some K. Consider the function

$$F(z) = f(z)e^{iK'z}$$

Now

$$F(x) \to 0 \quad \text{as } x \to \infty$$

and

$$F(iy) \to 0 \quad \text{as } y \to \infty$$

because of our choice of K'. Since F is of exponential type in the first quadrant, by the Phragmén-Lindelöf theorem we get, that F is bounded in $0 \leq \arg z \leq \pi/2$. So from Montel's theorem we have

$$|F(z)| < \varepsilon, \ 0 \leq \arg z \leq \pi/2 \ \text{ and } |z| \geq R(K', \varepsilon).$$

Analogously we have

$$|F(z)| < \varepsilon, \ -\pi/2 \leq \arg z \leq 0, \ |z| \geq R'(K', \varepsilon)$$

or

$$|f(z)| \leq \varepsilon\, e^{K'|y|}, \ \operatorname{Re} z \geq 0, \ |z| \geq \max\,(R, R').$$

Therefore $f \to 0$ uniformly as $z \to \infty$ in every strip of constant width around the x-axis. Now take a circle C around a point x, with radius r, then

$$f'(x) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi - x)^2}\,d\xi .$$

Taking the absolute values we have

$$|f'(x)| \leq \frac{1}{2\pi}\int_C \frac{|f(\xi)|}{|(\xi-x)^2|}|d\xi|$$

$$\leq \frac{o(1)2\pi}{2\pi r^2}$$

$$= o(1).$$

Hence the result follows.||

REMARKS.

1) R.P. Boas in his paper (M.Z.68) has proved that exponential type is the best possible condition on f.

2) Lang (M.Z.69) has proved that

$$\text{if } \frac{f(x+h)-f(x)}{h} \to 0 (x \to +\infty,\ h > 0 \text{ fixed}),$$

then $f'(x) \to 0 (x \to \infty)$ if f is of type $< \frac{2\pi}{h}$. This contains the theorem proved above.

3) When the positive real axis is replaced by a Jordan arc C which goes to ∞,

a) if the order of $f \leq 1/2$, Gaier and Delange have proved that the f-f' theorem holds.

b) if the order of $f = 1$ (exponential type), the theorem does not hold in general, but Jaenisch (1964) has proved that in every δ neighbourhood of C there exists a Jordan arc C' such that

$$\lim_{\substack{z \in C' \\ z \to \infty}} f'(z) = 0$$

§ 3. Blaschke products for a half plane.

A. Definition, simple properties and an example.

DEFINITION. Let $\{\lambda_k\}$ be a strictly increasing sequence of positive real numbers such that $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty$. Then the Blaschke Product is defined as

$$B(z) = \prod_{k=1}^{\infty} \frac{\lambda_k - z}{\lambda_k + z} = \prod_{k=1}^{\infty} \left(1 - \frac{2z}{\lambda_k + z}\right)$$

Properties of B(z):

1) $B(z)$ is analytic in z-plane for $z \neq -\lambda_k$;

2) $B(z)$ has simple poles at $z = -\lambda_k$;

3) $B(z)$ has simple zeros at $z = \lambda_k$;

4) Since

$$\left|\frac{\lambda_k - z}{\lambda_k - z}\right| \text{ is } \begin{cases} < 1 & \operatorname{Re} z > 0 \\ = 1 & \operatorname{Re} z = 0 \\ > 1 & \operatorname{Re} z < 0 \end{cases}$$

$$|B(z)| \text{ is } \begin{cases} < 1 & \operatorname{Re} z > 0 \\ = 1 & \operatorname{Re} z = 0 \\ > 1 & \operatorname{Re} z < 0 \end{cases}$$

5) $B(-z) = \dfrac{1}{B(z)}$

Example. When $\lambda_k = k^2$, to study $B(x)$ $(x > 0)$

$$B(x) = \frac{\prod_{k=1}^{\infty}\left(1-\frac{x}{k^2}\right)}{\prod_{k=1}^{\infty}\left(1-\frac{x}{k^2}\right)}$$

$$= \frac{\frac{\sin \pi\sqrt{x}}{\pi\sqrt{x}}}{\frac{\sinh \pi\sqrt{x}}{\pi\sqrt{x}}}$$

$$= \frac{\sin \pi\sqrt{x}}{\sinh \pi\sqrt{x}}$$

$$|B(x)| = \mathcal{O}(e^{-\pi\sqrt{x}}) \quad (x \to +\infty)$$

3. Blaschke Products for Hadamard sequences.

DEFINITION. A sequence $\{\lambda_k\}$ will be called an Hadamand sequence if $\frac{\lambda_{k+1}}{\lambda_k} \geq q > 1$ $(k = 1,2,\ldots)$.

Here our aim is to study the behaviour of $B(z)$ formed with an Hadamard sequence.

CLAIM. The behaviour of $B(z)$ in the plane is 'essentially determined' by one factor.

Choose a $\rho > 0$ (fixed) such that $\frac{1+\rho}{1-\rho} < q$ and take any $z \in C$ with $|z| = r$. Then there is atmost one λ_k in $(r(1-\rho), r(1+\rho))$; otherwise there would be λ_k, λ_{k+1} such that $\frac{\lambda_{k+1}}{\lambda_k} \leq \frac{r(1+\rho)}{r(1-\rho)} < q$ which contradicts our choice of ρ.

Now for a given z, we write $B(z)$ as product of three factors,

$$B(z) = \prod_1 \prod_2 \prod_3$$

where

$$\prod_1 = \prod_{\lambda_k \leq r(1-\rho)},$$

$\prod_2$ consists of atmost one factor

and

$$\prod_3 = \prod_{\lambda_k \geq r(1+\rho)}$$

Estimate of $\overline{|\,|}_1$

$$\left|\frac{\lambda_k - z}{\lambda_k + z}\right| \le \frac{\lambda_k + r}{r - \lambda_k} = 1 + \frac{2}{\frac{r}{\lambda_k} - 1}, \quad \lambda_k \le r(1-\rho)$$

since $1 \le \frac{r(1-\rho)}{\lambda_k}$, we have

$$\left|\frac{\lambda_k - z}{\lambda_k + z}\right| \le 1 + \frac{2}{\frac{r}{\lambda_k} - \frac{r}{\lambda_k}(1-\rho)}$$

$$= 1 + \frac{2}{\rho}\,\frac{\lambda_k}{r}$$

$$\le e^{C_1 \frac{\lambda_k}{r}} .$$

Hence

$$\left|\overline{|\,|}_1\right| < e^{\frac{C_1}{r}\left[\lambda_1 + \lambda_2 + \ldots + \lambda_K\right]}$$

$$= e^{C_1 \frac{\lambda_K}{r}\left[1 + \frac{\lambda_{K-1}}{\lambda_K} + \ldots + \frac{\lambda_1}{\lambda_K}\right]}$$

Now since $\frac{\lambda_{K-1}}{\lambda_K} \leq \frac{1}{q}$, then $\frac{\lambda_{K-2}}{\lambda_K} \leq \frac{1}{q^2}$ and so on, and writing

$$C_1\left[1 + \frac{1}{q} + \frac{1}{q^2} + \ldots\right] = C_2$$

we have

$$\left|\prod_1\right| < e^{C_2 \frac{\lambda_K}{r}} \leq e^{C_3} \text{ since } \lambda_K \leq r(1-\rho) \cdot$$

where C_3 depend on q only.

<u>Estimate of</u> $\prod_3$

$$\left|\frac{\lambda_k - z}{\lambda_k - z}\right| \leq \frac{\lambda_k + r}{\lambda_k - r} = \frac{1 + \frac{r}{\lambda_k}}{1 - \frac{r}{\lambda_k}}$$

$$\leq e^{C_4 \frac{r}{\lambda_k}}$$

$$\left|\prod_3\right| < e^{C_4 r\left[\frac{1}{\lambda_{K'+1}} + \frac{1}{\lambda_{K'+2}} + \ldots\right]}$$

$$= e^{C_4 \frac{r}{\lambda_{K'+1}}\left[1 + \frac{\lambda_{K'+1}}{\lambda_{K'+2}} + \ldots\ldots\right]}.$$

Writing

$$C_5 = C_4 \left[1 + \frac{1}{q} + \frac{1}{q^2} + \dots\right]$$

we obtain

$$\left|\prod{}_3\right| \leq e^{C_6}.$$

Thus

$$\left|\prod{}_1 \prod{}_3\right| \leq e^{C_3 + C_6} = C \quad \text{on } |z| = r.$$

Now replacing z by -z and using the property (5) of the Blaschke products, we have

$$\left|\prod{}_1 \prod{}_3\right| \geq \frac{1}{C} \quad \text{on } |z| = r.$$

Hence

$$B(z) = \prod{}_1 \prod{}_2 \prod{}_3$$

where $\prod_1 \cdot \prod_3$ is bounded from above and below on $|z| = r$ and $\prod_2$ contains atmost one factor; the corresponding λ_k lies in $(r(1-\rho), r(1+\rho))$ and the constant depends on q only.

Consequences.

1. Suppose $|z|$ stays away from λ_k i.e. $z \in \bigcup_{k=1}^{\infty} R_k$ where R_k is the annulus, $\{\lambda_k(1+2\rho) \leq |z| \leq \lambda_{k+1}(1-2\rho)\}$.

Now we assume $\rho < \frac{1}{2}$ and $\frac{1+2\rho}{1-2\rho} \leq q$ (which implies $\frac{1+\rho}{1-\rho} \leq q$). It can be easily checked that there exists no K such that

$$|z|(1-\rho) < \lambda_K < |z|(1+\rho).$$

Hence, the Blaschke product $B(z)$ is bounded above by C and below by $1/C$, as $\prod_2$ contains no factor.

2) When z stays away from the poles, i.e. whem $|z+\lambda_k| \geq C > 0$ $(k = 1,2,3\ldots)$, then

$$\begin{aligned}
|B(z)| &\leq \mathcal{O}(1)\left|\frac{z-\lambda_K}{z+\lambda_K}\right| \\
&\leq \mathcal{O}(1)\frac{|z-\lambda_K|}{C} \\
&\leq \mathcal{O}(1)\,(|z| + |\lambda_K|) \\
&= \mathcal{O}(|z|)
\end{aligned}$$

since $\lambda_K \leq |z|(1+\rho)$.

3) When z is on a ray emanating from $z = 0$, with $\arg z = \varphi \neq \pi$

$$|B(z)| = \mathcal{O}(1)\left|\frac{\lambda_k - z}{\lambda_k + \bar{z}}\right|$$

$$= \mathcal{O}(1)\left|\frac{1-\frac{z}{\lambda_k}}{1+\frac{\bar{z}}{\lambda_k}}\right|$$

$$= \mathcal{O}(1)\left|\frac{1-\xi}{1+\xi}\right|$$

where ξ will be on the same ray. Hence

$$|B(z)| \leq C(\varphi_0),$$

in $|\arg z| \leq \varphi_0 < \pi$.

C. Blaschke products for other sequences.

In this part we restrict ourselves to sequences $\{\lambda_k\}$ satisfying

(*) $$\lambda_{k+1} - \lambda_k \geq \theta\lambda_k^{\sigma}, \; k = 1,2,\ldots.$$

where $\theta > 0$ and $0 < \sigma < 1$. For such a sequence we claim that

$$1) \sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty$$

$$(**) \qquad 2) \quad \lambda_{k+1}^{\tau} - \lambda_k^{\tau} > C = C(\theta, \sigma),$$

for $\tau = 1-\sigma$, $k=1,2,\ldots.$

PROOF. 1) The condition (*) implies

$$\lambda_{k+1} - \lambda_k \geq C > 0 \text{ and } \lambda_k \geq C'k .$$

Now substituting back in (*) we obtain

$$\lambda_{k+1} - \lambda_k \geq C''k^{\sigma} .$$

Summing the above over k from 1 to m-1, we get

$$\lambda_m - \lambda_1 \geq C''' \sum_{k=1}^{m} k^{\sigma} > C''m^{\sigma+1}$$

which implies 1).

2) From the condition (*)

$$\lambda_{k+1}^{\tau} \geq \lambda_k^{\tau}\left[1+\theta\lambda_k^{\sigma-1}\right], \quad \tau=1-\sigma$$

$$\geq \lambda_k^{\tau}\left[1+\frac{\tau\theta}{2}\lambda_k^{\sigma-1}\right]^{\tau} \text{for } k > k_0(\theta,\sigma)$$

which implies 2).

Now we want to study the growth of the derivative of the Blaschke product $B(z)$ at $z = \lambda_m$, $m = 1, 2, \ldots$ We have

$$|B'(\lambda_m)| = \frac{1}{2\lambda_m} \prod_{k\neq m} \frac{\lambda_k-\lambda_m}{\lambda_k+\lambda_m} = \frac{1}{2\lambda_m}\cdot\frac{1}{p_m}$$

where $p_m = \prod_{k\neq m} \frac{\lambda_k+\lambda_m}{\lambda_k-\lambda_m}$

<u>THEOREM</u>. If $\{\lambda_k\}$ <u>satisfies</u> (*), then

$$0 < \log p_m < A\lambda_m^{\tau}, \quad \tau = 1-\sigma, \quad 0<\sigma<1$$

<u>for some constant A</u>.

<u>PROOF</u>. Let us fix m, then

$$\log p_m = \sum_{k<m} \log \frac{1+\frac{\lambda_k}{\lambda_m}}{1-\frac{\lambda_k}{\lambda_m}} + \sum_{k>m} \log\frac{\frac{\lambda_k}{\lambda_m}+1}{\frac{\lambda_k}{\lambda_m}-1}.$$

Denoting $\nu = \frac{1}{\tau}$ (>1) and $\left(\frac{\lambda_k}{\lambda_m}\right)^{\tau} = x_k^{(m)} = x_k$, for $k = 1,2,\ldots,$, the distance between two consecutive x_k's is

$$x_{k+1} - x_k = \lambda_m^{-\tau}(\lambda_{k+1}^{\tau} - \lambda_k^{\tau}) \geq C\lambda_m^{-\tau}, \quad k=1,2,\ldots$$

because of (**). Hence we obtain

$$\lambda_m^{-\tau} C \log p_m \leq \sum_{k<m} \log\left|\frac{1+x_k^{\nu}}{1-x_k^{\nu}}\right| (x_{k+1} - x_k) + \sum_{k>m} \log\left|\frac{1+x_k^{\nu}}{1-x_k^{\nu}}\right| \cdot (x_k - x_{k-1}).$$

The right hand side of the above inequality represents the lower Riemann sum of the function $h(x) = \log\left|\frac{1+x^{\nu}}{1-x^{\nu}}\right|$. Therefore we have

$$\lambda_m^{-\tau} C \log p_m < \int h(x)dx = \text{constant}$$

or

$$\log p_m < A\lambda_m^{+\tau}.$$

CHAPTER III

COMPLEX VARIABLE PROOFS OF TAUBERIAN THEOREMS

As indicated before, in this chapter we deal with the inverse theorems. We conclude from certain transformations back to the given sequence or function. This is done usually under an additional hypothesis which is called the "Tauberian condition." Two types of Tauberian conditions are prominent: (1) growth condition on a_n and (2) gap condition i.e., many a_n vanish. The term 'Tauberian' comes from a theorem due to Tauber (1897) which states: If $\Sigma a_n x^n \to s (x \to 1-0)$ and $a_n = o\left(\frac{1}{n}\right)$ then $\Sigma a_n = s$.

§ 1. Equivalence of the Borel Methods.

THEOREM. (Gaier, 1953). *The methods* B *and* B' *are equivalent if* $a_n = \mathcal{O}(K^n)$ *for some* $K < \infty$.

PROOF. We have to prove only $B'\text{-}\Sigma a_n = s \Rightarrow B - \Sigma a_n = s$

From Ch. I, § 1, D, we have

$$B(x, S_n) = \int_0^x e^{t-x} B(t, s_n) dt .$$

If $a_n = \mathcal{O}(K^n)$, then $S_n = \mathcal{O}((2K)^n)$. Therefore

$$\left| \Sigma \frac{S_n x^n}{n!} \right| \leq \mathcal{O}(1) \; \Sigma \frac{(2K|x|)^n}{n!}$$

$$= \mathcal{O}(1) e^{2K|x|} .$$

So, from above it follows that the Borel transformation $B(x, S_n)$ is an entire function of exponential type. Moreover, it has a limit s for $x \to \infty$. Therefore its derivative tends to zero as $x \to +\infty$ (by the f - f' theorem, Ch. II, § 2, C) i.e.,

$$\frac{d}{dx}\left[B(x,S_n)\right] = e^{-x} \cdot e^x B(x,s_n) + \int^x e^t B(t,s_n)dt(-e^{-x})$$

$$= B(x,s_n) - B(x,S_n) \to 0, \text{ as } x \to +\infty.$$

Thus the result follows. ||

REMARK. Garten has proved the theorem for $a_n = \mathcal{O}(n^k)$ and Karamata has proved it for $a_n = \mathcal{O}(e^{n^\rho})$, $\rho < 1/3$. The condition $a_n = \mathcal{O}(K^n)$ is best possible, by using Boas' result on the f - f' theorem.

§ 2. The Classical High Indices Theorem.

In this section, we want to make the inverse conclusion for the Abel method when many a_n's vanish in the given series. Such theorems were known for some time, with real variable proofs. Recently Gaier (1965) and Halasz (1967) have attacked these theorems with complex variable methods.

Before coming to the theorem, we mention some properties of Dirichlet series. A Dirichlet series is a series of the form

$$\Sigma a_n e^{-\lambda_n z}$$

where $\{\lambda_n\}$ is an increasing sequence of positive numbers with $\lambda_n \to \infty$. This series

1) converges in a half plane, i.e., if it converges at a point z_0, then it converges for all z with $\mathrm{Re}\, z > \mathrm{Re}\, z_0$.

2) converges absolutely in a half plane, i.e., if it converges absolutely at a point z_0, then it converges absolutely for all z with $\mathrm{Re}\, z \geq \mathrm{Re}\, z_0$.

3) will have the same half plane as its domain of convergence and domain of absolute convergence if the sequence $\{\lambda_n\}$ satisfies the condition $\lambda_{n+1} - \lambda_n \geq c(> 0)$ for all n.

See, for example, the book by Titchmarsh, Chapter IX.

Furthermore, we shall need a Blaschke product. Let $\{\lambda_n\}$ be an Hadamand sequence (Ch. II, § 3, B). Define

$$B(z) = \prod_{n=1}^{\infty} \frac{\sqrt{\lambda_n} - \sqrt{z}}{\sqrt{\lambda_n} + \sqrt{z}}, \quad z \in D$$

where D is the complex plane slit along the negative axis. This function will be regular in D, have simple zeros at $z = \lambda_n$, and be of absolute value < 1 in D, whereas $|B(z)| = 1$ on the boundary of D. Observe that the sequence $\{\sqrt{\lambda}_n\}$ is again an Hadamard sequence. Now for a z such that $|z| = R = \sqrt{q}\,\lambda_n$, it is easy to verify that $\sqrt{z}$ lies in the n^{th} annulus

$$\sqrt{\lambda}_n(1+\rho) < \sqrt{|z|} < \sqrt{\lambda_{n+1}}(1-\rho)$$

for sufficiently small ρ. Hence by Ch. II, § 3, B, we have on $|z| = \sqrt{q}\lambda_n$

$$\frac{1}{C} \leq |B(z)| \leq C,$$

where C depends on $\{\lambda_n\}$.

<u>THEOREM</u>. (Hardy-Littlewood, 1926). Let

$$f(x) = \Sigma a_n e^{-\lambda_n x}$$

<u>be convergent for</u> $x > o$ <u>where</u> $\{\lambda_n\}$ <u>is an Hadamard sequence with</u> $\lambda_1 > 0$. <u>Then</u>

a) $\left| \sum_{n=1}^{N} a_n \right| \leq C \sup_{x>o} |f(x)|$, <u>for all</u> N.

<u>where</u> C <u>depends only on</u> $\{\lambda_n\}$,

b) $\lim_{x \to 0+} f(x) = 0 \Longrightarrow \sum_{n=1}^{\infty} a_n = 0.$

<u>PROOF</u>. Let $|f(x)| \leq A$, $(x > o)$. Let us assume at first $\Sigma \frac{|a_n|}{\lambda_n} < \infty$. Consider the Laplace transform

$$(*) \qquad F(z) = \int_o^\infty f(t)e^{tz}dt.$$

This is a regular function in Re $z < 0$. We study some of its properties.

1) On the negative real axis, $z = -x$, $x > o$ we have

$$|F(-x)| \leq A \int_o^\infty e^{-tx}dt = \frac{A}{x}.$$

2) From (*), we have

$$F(z) = \int_0^\infty \sum_1^\infty a_n e^{-t\lambda_n} e^{tz} dt, \ \text{Re } z < 0$$

$$= \sum_1^\infty a_n \int_0^\infty e^{t(z-\lambda_n)} dt$$

$$= \sum_1^\infty \frac{-a_n}{z-\lambda_n}$$

where inverting the order of integration and summation is justified because of our additional assumption that $\Sigma \frac{|a_n|}{\lambda_n} < \infty$. The series on the right is convergent for $z \neq \lambda_n$. Hence it is the analytic continuation of F to the whole plane. Moreover, it is a meromorphic function in the whole plane with simple poles at $z = \lambda_n$. The residues of F are $-a_n$ at $z = \lambda_n$.

3) Growth of F outside $|z-\lambda_n| = 1$, $n = 1,2,\ldots$.

From the above we have

$$|F(z)| \leq \Sigma \frac{|a_n|}{|z-\lambda_n|} .$$

We split the summation on the right hand side into three parts:

$$|F(z)| \leq \sum_{\lambda_n < \frac{|z|}{2}} {}_1 + \sum_{\frac{|z|}{2} \leq \lambda_n \leq 2|z|} {}_2 + \sum_{\lambda_n > 2|z|} {}_3 .$$

Since, for $\lambda_n < \frac{|z|}{2}$ we have $|z - \lambda_n| \geq \lambda_n$, for $\frac{|z|}{2} \leq \lambda_n \leq 2|z|$ we have $2|z||z-\lambda_n| \geq \lambda_n$, we could estimate $|F(z)|$ by

$$|F(z)| \leq \Sigma_1 \frac{|a_n|}{\lambda_n} + \Sigma_2 |a_n| \frac{2|z|}{\lambda_n} + \Sigma_3 \frac{2|a_n|}{\lambda_n}$$

$$\leq (2+2|z|) \cdot \sum_1^\infty \frac{|a_n|}{\lambda_n} = O(|z|).$$

This shows that outside the circles F does not grow fast.

Now we construct a regular function in the slit plane D in order to apply the Phragmén-Lindelöf theorem. Consider

$$H(z) = zF(z)B(z)$$

where B(z) is the Blaschke product defined in the beginning of this section. Then H(z) is regular in D and $|H(z)| \leq A$ for $z = -x$ $(x>o)$. Moreover $|H(z)| = O(1)|z|^2$ in D. Now by Phragmén-Lindelöf theorem we obtain

$$|H(z)| \leq A \text{ for all } z \in D$$

or

$$|F(z)| \leq \frac{A}{|z|B(z)|} \leq \frac{A}{|z|} \cdot C = \frac{CA}{R}$$

on $|z| = R = \sqrt{q}\,\lambda_N$, where C depends on $\{\lambda_n\}$.

By the residue theorem, applied to F, we have

$$\left|\sum_{n=1}^{N}(-a_n)\right| \leq \left|\frac{1}{2\pi i}\right| \int_{|z|=R} |F(z)||dz| \leq \frac{CA}{R}\cdot\frac{2\pi R}{\pi} = CA.$$

The proof of a) will be complete if we remove our additional assumption $\Sigma \frac{|a_n|}{\lambda_n} < \infty$. For that, consider

$$f(x+\delta) = \sum_1^\infty a_n e^{-\lambda_n \delta} e^{-\lambda_n x} \qquad (\delta > 0, \text{ fixed})$$

From the hypothesis, we have

$$|f(x+\delta)| \leq A, \quad (x > o).$$

Now using the property 3) of the Dirichlet series defined earlier, we obtain

$$\sum_1^\infty \frac{|a_n| e^{-\lambda_n \delta}}{\lambda_n} \leq \frac{1}{\lambda_1} \sum_1^\infty |a_n| e^{-\lambda_n \delta} < \infty.$$

Now treating a_n's as $a_n e^{-\lambda_n \delta}$ and using what we have proved above we get

$$\left|\sum_1^N a_n e^{-\lambda_n \delta}\right| \leq CA.$$

Now make $\delta \to o$ to get the result.

It remains to prove the second part of the theorem. Now since the Laplace transform is such that

$$|F(-x)| \leq \frac{A}{x}, \; x > o$$

we have, since $f(t) \to o$ $(t \to o)$,

$$F(-x) = o(\tfrac{1}{x}), \; x \to \infty.$$

This means that $H(z) = o(1)$ for $z = -x \to -\infty$. Because H is bounded in D by Montel's theorem, it follows that $H(z) \Longrightarrow o$ as $z \to \infty$ in D which implies $F(z) = o(\frac{1}{R})$ for $|z| = R = \sqrt{q}\,\lambda_N$. Hence the residue theorem yields

$$|\sum_{1}^{N} a_n| = \frac{1}{2\pi}\, o(\tfrac{1}{R})\; 2\pi R = o(1), \text{ as } N \to \infty,$$

which proves the theorem.$\|$

§ 3. Some additional remarks to the classical high indices theorem.

A. The gap condition is best possible.

THEOREM. (Rudin, P.A.M.S. 17, 1966). Given any increasing sequence $\{\lambda_n\}$ of integers with $\liminf_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$, there exists a power series $f(x) = \Sigma a_n x^{\lambda_n}$ such that

1) $f(x)$ is monotonically increasing in $(0,1)$ and bounded,

2) Σa_n is divergent.

PROOF. Let us choose a subsequence $\{\mu_n\}$ of $\{\lambda_n\}$ such that

$$\frac{\mu_{2n+1}}{\mu_{2n}} < 1 + \frac{1}{n^2}, \quad n = 1,2,\ldots..$$

Now consider

$$f(x) = \Sigma\left(\frac{\mu_{2n+1}}{\mu_{2n}} x^{\mu_{2n}} - x^{\mu_{2n+1}}\right).$$

We claim that f is the required series. Because:

1) since its coefficients are bounded, it has radius of convergence one,

2) it has only powers $\lambda_1, \lambda_2, \ldots..$

3) the series does not converge at $x = 1$ since the coefficients do not tend to zero,

4) f is increasing in $0 \le x < 1$ for

$$f'(x) = \Sigma \frac{\mu_{2n+1}}{x}\left[x^{\mu_{2n}} - x^{\mu_{2n+1}}\right] > 0,$$

5) f is bounded for

$$0 \leq \frac{\mu_{2n+1}}{\mu_{2n}} x^{\mu_{2n}} - x^{\mu_{2n+1}} \leq \frac{\mu_{2n+1}}{\mu_{2n}} - 1 \leq \frac{1}{n^2} . \|$$

B. <u>High Indices Theorem with Reamainder Term.</u>

<u>THEOREM. Let</u> $f(x) = \Sigma a_n e^{-\lambda_n x}$ <u>converge for</u> $x > 0$ <u>where</u> $\{\lambda_n\}$ <u>is an Hadamand sequence with</u> $\lambda_1 > 0$,

$$|f(x)| \leq Ax^{\alpha} \quad (\alpha \geq 0,\ x > 0)$$

<u>then</u>

$$\left|\sum_1^N a_n\right| \leq AC_1C_2 \frac{1}{\lambda_N^{\alpha}}, \quad N = 1,2,\ldots.$$

<u>where</u> C_1 <u>depends on</u> $\{\lambda_n\}$ <u>and</u> C_2 <u>depends on</u> α.

<u>PROOF.</u> The proof follows the same lines as that of the high indices theorem. The Laplace transform F of f is a meromorphic function in the whole plane with simple poles at λ_n and residues $-a_n$. On the negative real axis

$$|F(-x)| \leq \int_0^{\infty} |f(t)| e^{-tx} dt \leq A \int_0^{\infty} t^{\alpha} e^{-tx} dt \quad (x > 0)$$

$$= \frac{A}{x^{\alpha+1}} \int_0^{\infty} u^{\alpha} e^{-u}\, du = \frac{AC_2}{x^{\alpha+1}} .$$

Therefore, we now construct a regular function

$$H(z) = F(z)B(z)z^{1+\alpha}$$

in D. On the boundary of D

$$|H(z)| \leq C_2 A.$$

Hence by the Phragmén-Lindelöf theorem the same inequality holds in D. Rewriting it we obtain

$$|F(z)| \leq C_2 A \frac{1}{|B(z)|} \frac{1}{|z|^{1+\alpha}}$$

$$= C_1 C_2 A \frac{1}{R^{1+\alpha}} \text{ for } |z| = R = \sqrt{q}\,\lambda_N$$

where $\frac{1}{C_1}$ is the lower bound of B(z) which depends on $\{\lambda_n\}$.

Now apply the residue theorem to F with the contour $|z| = R$ to get the result.‖

C. An Identity Theorem.

THEOREM (Halasz). Let $f(x) = \Sigma a_n e^{-\lambda_n x}$ be convergent for $x > o$ and assume the sequence $\{\lambda_n\}$ with $\lambda_1 > o$ satisfies

(1) $$\lambda_{n+1} - \lambda_n \geq K > o \text{ and } \sum_1^\infty \frac{1}{\sqrt{\lambda_n}} < \infty.$$

Then

(2) $$f(x) = O(e^{-\frac{1}{x}}) \quad (x \to o+)$$

implies $f(x) \equiv o$.

PROOF. Let $\delta > o$ be fixed, so that

(*) $$\Sigma \frac{|a_n| e^{-\lambda_n \delta}}{\lambda_n} < \infty .$$

Consider the Laplace transform

$$F_\delta(z) = \int_o^\infty f(t+\delta) e^{tz} dt$$

$$= \int_o^\infty \sum_1^\infty a_n e^{-\lambda_n (t+\delta)} e^{tz} dt$$

$$= \sum_1^\infty a_n e^{-\lambda_n \delta} \int_o^\infty e^{-t(\lambda_n - z)} dt$$

$$= \sum_1^\infty \frac{a_n e^{-\lambda_n \delta}}{z-\lambda_n},$$

where inverting the order of integration and summation is justified from (*). F_δ is a meromorphic function in the plane with simple poles at $z = \lambda_n$. If z stays away from the poles, i.e., $|z-\lambda_n| \geq \frac{K}{2} > 0$ we get $F_\delta(z) = O(1)$ and on $z = -x$ $(x > 0)$ we have

$$|F_\delta(-x)| \leq \frac{A}{x} \text{ for some constant } A.$$

Now we construct the regular function

$$H_\delta(z) = zF_\delta(z)B(z)$$

in D, where B(z) is the Blaschke product defined earlier in this section and is convergent because $\Sigma \frac{1}{\sqrt{\lambda_n}} < \infty$ is assumed. Since

1) H_δ is regular in D,

2) $|H_\delta(z)| = O(|z|)$ as $|z| \to \infty$ in D,

3) $|H_\delta(z)| \leq A$ $(z = -x,\ x > 0)$,

we get by the Phragmén-Lindelöf theorem

$$(**) \qquad |H_\delta(z)| \leq A, \text{ for } z \in D \text{, and for all } \delta > 0.$$

The boundary function, for $z = -x$, is

$$H_\delta(-x) = (-x)F_\delta(-x)B(-x)$$

and hence

$$|H_{\delta_1}(-x) - H_{\delta_2}(-x)| = x|F_{\delta_1}(-x) - F_{\delta_2}(-x)|$$

$$= x\left|\int_0^\infty \left[f(t+\delta_1) - f(t+\delta_2)\right]e^{-tx}dt\right|.$$

Because $\lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to +\infty} f(x) = 0$, the term $|f(t+\delta_1)-f(t+\delta_2)|$ can be made uniformly less than ε for $\delta_1, \delta_2 < \Delta$. Therefore

$$|H_{\delta_1}(-x) - H_{\delta_2}(-x)| < \varepsilon \quad (\delta_1, \delta_2 < \Delta \ ; \ x > 0).$$

From (**) it follows that the difference $H_{\delta_1}(z) - H_{\delta_2}(z)$ is also bounded in D and in fact bounded by ε, if $\delta_1, \delta_2 < \Delta$. This means $H_\delta(z)$ converges uniformly in D as $\delta \to 0$ in fact it converges uniformly in $D \cup (-\infty < -x < 0)$.

The limit function $H(z)$ (say) is analytic and bounded by A in D. Moreover on the boundary of D

$$H(-x) = (-x)\int_0^\infty f(t)e^{-tx}dt \; B(-x),$$

so

$$|H(-x)| \leq x \int_{o}^{\infty} e^{-\frac{1}{t}} e^{-tx} dt = O(e^{-C\sqrt{x}})^{*} \quad (C > o).$$

Therefore $H(z^2)$ is regular and bounded in Re $z > o$, and continuous as z tends to the boundary, also

$$|H(z^2)| \leq \text{Const.} \cdot e^{-C|z|}$$

on the imaginary axis. By a theorem (Titchmarsh, p.185) $H(z^2) \equiv o$, so

$$H_\delta(\lambda_n) \to o \quad (\delta \to o, \ n \text{ fixed})$$

i.e.,

$$\lambda_n a_n e^{-\lambda_n \delta} \cdot B'(\lambda_n) \to o \quad (\delta \to o, \ n \text{ fixed})$$

which is possible only when $a_n = o$ $(n = 1,2,\ldots)$. Hence the theorem. ||

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* This can be seen by splitting the integral from o to 1 and 1 to ∞, and taking the maximum of the integrand in the first integral.

REMARK. Similarly it is proved that $f \equiv 0$, if in the theorem the second condition in (1) is replaced by $\Sigma\lambda_n^{-\rho/\rho+1} < \infty$ and (2) is replaced by

$$f(x) = \mathcal{O}(e^{-\frac{1}{x^{\rho}}}) \quad (\rho > 0,\ x \to 0+).$$

§ 4. Borel Summable Series with Gaps.

Hitherto, in our theorems we considered Dirichlet series on their boundary of convergence. Now we begin by studying entire functions with gaps in their power series expansion.

Assume $f(z) = \Sigma a_n z^n$ is an entire function with gaps, i.e.

$$a_n = 0 \text{ for } n \neq \lambda_k, \text{ where } \Sigma\frac{1}{\lambda_k} < \infty,$$

and also $f(x) = \mathcal{O}(e^x)$ $(x \to \infty)$. Consider the auxiliary function

$$F(z;T) = \int_0^T f(t)t^{-z-1}dt$$

where $0 < T < \infty$ is fixed. Notice that $a_0 = 0$ so that the integral converges if Re $z < 1$ and defines an analytic function in Re $z < 1$.

We first construct an analytic extension to Re $z \geq 1$. Since $f(t) = \sum_{1}^{\infty} a_n t^n$ converges uniformly in $[o,T]$, we have for Re $z < 1$

$$
\begin{aligned}
F(z;T) &= \int_{o}^{T} \sum_{1}^{\infty} a_n t^n t^{-z-1} dt \\
&= \sum_{1}^{\infty} a_n \int_{o}^{T} t^{n-z-1} dt \\
&= -T^{-z} \sum_{1}^{\infty} \frac{a_n T^n}{z-n} .
\end{aligned}
$$

The series on the right defines a meromorphic function in the plane, so we have continued F into the plane. F has simple poles at $z = n$ with residues $-a_n$.

If z stays away from the poles, i.e. for $|z-n| \geq \eta > o$, we have

$$|F(z;T)| \leq T^{-x} \frac{D(T)}{\eta} \text{ with } D(T) = \Sigma |a_n| T^n.$$

We now form the Blaschke product

$$B(z) = \prod_{k=1}^{\infty} \frac{\lambda_k - z}{\lambda_k + z} = \prod_{k=1}^{\infty} (1 - \frac{2z}{\lambda_k + z})$$

convergent for $z \neq -\lambda_k$ $(k = 1,2, \ldots .)$ and defining a regular function Re $z \geq o$, which is bounded by 1 on Re $z = o$.

Therefore,

$$H(z;T) = F(z;T)\ B(z)$$

is regular in Re $z \geq o$, and its bound on Re $z = o$ is

$$|H(z;T)| \leq A\int_0^T \frac{|f(t)|}{t}\, dt \leq \frac{Be^T}{T}$$

where B does not depend on T. Its growth in Re $z \geq o$ is determined by

$$|H(z;T)| \leq T^{-x}D'(T)$$

where $D'(T)$ does not depend on z. In particular, H is of exponential type in Re $z \geq o$. Now by applying our theorem in Ch. II, § 1, B, we get

$$\begin{aligned} |H(z;T)| &\leq B\,\frac{e^T}{T}\,\frac{1}{T^x} \quad \text{Re } z \geq o \\ &= B\,\frac{e^T}{T^{x+1}} \end{aligned}$$

If we put $z = n = \lambda_m$, we get

$$|a_n|\ |B'(\lambda_m)| \leq B\ \frac{e^T}{T^{n+1}}$$

or

$$|a_n| \leq B.2\lambda_m p_m\ \frac{e^T}{T^{n+1}}$$

(where $\{p_m\}$ is the sequence studied in Ch. II, § 3, C) valid for every $T > o$. Choose the optimal $T = 1+n$, so we get by using Stirling's formula that

$$|a_n| \leq 2B_1 \lambda_m p_m\ \frac{\sqrt{n}}{(n+1)!}$$

or

$$|a_n| \leq Cp_m \frac{\sqrt{n}}{n!} .$$

Thus we have proved the following.

THEOREM. If $f(z) = \Sigma a_n z^n$ is entire, $a_n = o$ for $n \neq \lambda_k$ with $\Sigma \frac{1}{\lambda_k} < \infty$, then $f(x) = \mathcal{O}(e^x)$ implies

$$|a_n| \leq C\ p_m \frac{\sqrt{n}}{n!}$$

where

$$p_m = \prod_{k \neq m} \left| \frac{\lambda_k + \lambda_m}{\lambda_k - \lambda_m} \right| \quad , \; n = \lambda_m .$$

COROLLARY. If $\lambda_{k+1} - \lambda_k \geq \theta \sqrt{\lambda_k}$ ($\theta > 0$, $k = 1,2,\ldots$) then we have under the above conditions

$$a_n = O(1) \frac{e^{C\sqrt{n}}}{n!} \quad \text{for some } C > 0.$$

As am application to Borel summability we prove the following theorem.

THEOREM. If Σa_n is Borel summable where

(*) $$a_n = 0 \text{ for } n \neq \lambda_k \text{ with } \lambda_{k+1} - \lambda_k \geq \theta\sqrt{\lambda_k} \; (\theta > 0)$$

then

$$a_n = O(1) e^{C\sqrt{n}} .$$

PROOF. By the hypotheses we have

(1) $$e^{-x} \Sigma \frac{s_n x^n}{n!} \to s \quad (x \to \infty).$$

But from Ch. I, § 1, D, we have $B \Longrightarrow B'$. Hence

(2) $$e^{-x}\Sigma\frac{S_n x^n}{n!} \to s \quad (x \to \infty)$$

where $S_n = s_{n-1}$. Subtracting (2) from (1) we obtain

$$e^{-x}\Sigma \frac{x^n}{n!}a_n \to 0 \text{ as } x \to \infty$$

i.e.,

$$\Sigma \frac{x^n}{n!} a_n = \mathcal{O}(e^x).$$

From the corollary, it follows that

$$a_n = \mathcal{O}(1)e^{C\sqrt{n}}\,.\,\|$$

A result of Pitt states that the series Σa_n is then actually convergent. This is the unrestricted high indices theorem for Borel summability.

<u>THEOREM</u>. <u>If</u> Σa_n <u>is Borel summable and</u> (*) <u>is fulfilled then</u> Σa_n <u>is convergent</u>.

§ 5. The Θ - B $\rightarrow$ K Theorem

For a change we shall leave our subject of gap series, and prove a Tauberian theorem of the other type i.e., when the Tauberian condition is a growth condition.

THEOREM. (Hardy and Littlewood) If Σa_n is Borel summable and $a_n = \Theta(\frac{1}{\sqrt{n}})$, then Σa_n is convergent.

Preliminary Remark. If Θ-condition is replaced by o-condition then the above theorem follows from gap theorem.

PROOF. Choose a sequence $\{n_k\}$ such that $n_{k+1} - n_k = [\sqrt{n_k}]$. Let the partial sums of Σa_n be s_n. Now put

$$S_n = s_{n_k} \quad \text{if } n_k \leq n < n_{k+1}$$

and

$$S_n = \sum_{0}^{n} A_k$$

where $\sum_{0}^{\infty} A_k$ is a new series with partial sums S_n. This series satisfies the gap condition for Borel summability. Moreover

$$|s_n - \mathbf{s}_n| = |s_n - s_{n_k}|$$

$$\leq \sum_{j=n_{k+1}}^{n} |a_j|$$

$$= o(1) \frac{1}{\sqrt{n_k}} \sqrt{n_k}$$

$$= o(1).$$

Hence $\{S_n\}$ is Borel summable. Also $\{S_n\}$ is Borel summable to s because

$$e^{-x} \Sigma \frac{S_n x^n}{n!} = e^{-x} \Sigma \frac{S_n - s_n}{n!} x^n + e^{-x} \Sigma \frac{s_n x^n}{n!}$$

where the first summation on the right hand side tends to zero as $x \to +\infty$ and the second summation tends to s as $x \to +\infty$ by our assumption. Therefore it follows by gap theorem that $\{S_n\}$ is convergent which in turn implies $\{s_n\}$ is convergent.||

We give here a proof of the Hardy-Littlewood theorem by Jurkat (1956) which employs complex variable techniques. We use the following notation

$$\sigma_n^{(q)} = \frac{1}{q^n} \sum_{k=0}^{n} \binom{n}{k} (q-1)^{n-k} s_k$$

where $q \geq 1$ is a fixed number.

DEFINITION. We say Σa_n is E_q-summable to s if $\sigma_n^{(q)} \to s$ as $n \to \infty$.

Since the proof of the theorem is fairly long, we present the proof in three steps in the following way.

a) B-summability + $a_n = O(\frac{1}{\sqrt{n}}) \Longrightarrow s_n = O(1)$

b) B-summability + $s_n = O(1) \Longrightarrow E_q$-summability for every $q > 1$.

c) E_q-summability for all $q > 1$ + $a_n = O(\frac{1}{\sqrt{n}}) \Longrightarrow \Sigma a_n$ is convergent

BROOF of a). This is evident if in the preliminary remark o is read as O . But we give a straightforward proof.

First we claim that for $n, \nu \geq o$,

(*)) $$|s_n - s_\nu| \leq L \frac{|n-\nu|}{\sqrt{n+1}} .$$

since $a_n \leq \frac{k}{\sqrt{n+1}}$, $s_n \leq k\sqrt{n+1}$. In fact

1) if $\nu \leq \frac{n}{2}$, i.e. $n-\nu \geq \frac{n}{2}$, we have

$$|s_n - s_\nu| = 2k\sqrt{n+1} \leq k'' \frac{n-\nu}{\sqrt{n+1}}$$

ii) if $n \geq \nu \geq \frac{n}{2}$

$$|s_n - s_\nu| = \left| \sum_{k=\nu+1}^{n} a_k \right|$$
$$\leq K \frac{(n-\nu)}{\sqrt{\nu+1}} \leq C \frac{n-\nu}{\sqrt{n+1}} .$$

iii) if $\nu > n$,

$$|s_n - s_\nu| \leq K \frac{(\nu - n)}{\sqrt{n+1}} .$$

Thus

$$|s_n - s_\nu| \leq C \frac{|n-\nu|}{\sqrt{n+1}} . \quad \|$$

PROOF of b). This is the deepest part of the proof and we shall employ our complex variable techniques to prove it. We start with two auxiliary lemmas.

LEMMA 1. *Let* g *be regular and bounded in the parabolic region* $P = \{z \mid |z| - \operatorname{Re} z < M\}$ *with* $\lim_{x \to \infty} g(x) = 0$. *Then* $g(z) \longrightarrow 0$ *uniformly within every parabolic region* $P' = \{z \mid |z| - \operatorname{Re} z < M'\}$ *for every* $M' < M$.

PROOF. It is easy to check that if $z \in P$ then $\sqrt{z}$ is in a half strip. Now $g(z) = g(w^2)$ is regular and bounded in the half strip, and tends to zero for $u \to +\infty$ $(w = u+iv)$ by Montel's theorem for a half strip $g(w^2) \Longrightarrow 0$ for $z \to \infty$ in a substrip i.e. $g(z) \Longrightarrow 0$ as $z \to \infty$ in a subparabola. ||

LEMMA 2. We have

$$e^{(q-1)z} f(z) = \sum_{n=0}^{\infty} \frac{\sigma_n^{(q)} q^n}{n!} z^n$$

where $\quad f(z) = \sum \frac{s_n z^n}{n!}$

PROOF. L.H.S. is

$$\sum_{n=0}^{\infty} \frac{(q-1)^n}{n!} z^n \cdot \sum_{n=0}^{\infty} \frac{s_n z^n}{n!} = \sum A_n z^n$$

with

$$A_n = \sum_{k=0}^{n} \frac{s_k}{k!} \frac{(q-1)^{n-k}}{(n-k)!}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{s_k}{n!} (q-1)^{n-k},$$

which proves the Lemma. ||

The proof of b) will be completed if we prove the following theorem.

<u>THEOREM</u>. <u>If Σa_n is B-summable to zero and</u> $s_n = O(1)$, <u>then</u> Σa_n is <u>E_q-summable for every fixed</u> $q > 1$.

<u>PROOF</u>. Consider the auxiliary function

$$g(x) = e^{-z} f(z).$$

By lemma 1, $g(z) \to 0$ as $z \to \infty$ in every fixed parabolic region of the plane.

Next we apply Cauchy's formula to the identity of Lemma 2.

$$\sigma_n^{(q)} = \frac{n!}{q^n} \frac{1}{2\pi i} \int_{|z| = \frac{n}{q}} \frac{e^{(q-1)z} f(z)}{z^{n+1}} \, dz .$$

We split the above integral as a sum of two integrals, so that

$$(**) \qquad \sigma_n^{(q)} = \frac{n!}{2\pi} \left[\int_{|\varphi| \leq \frac{m}{\sqrt{n}}} + \int_{|\varphi| \geq \frac{m}{\sqrt{n}}} \right] \frac{e^{(q-1)z} f(z)}{(zq)^n} \, d\varphi$$

$$= A_{n,m} + B_{n,m} \text{ (say)}$$

where $z = re^{i\varphi}$ and m, n are integers such that $\frac{m}{\sqrt{n}} \leq \pi$. We estimate A and B separately.

Estimation of $B_{n,m}$. In the second integral of (**) we merely use the fact that $|f(z)| = O(e^{|z|})$. So

$$|B_{n,m}| = O(1)\frac{n!}{n^n} e^{\frac{n}{q}} \int_{|\varphi| > \frac{m}{\sqrt{n}}} e^{(q-1)\frac{n}{q}\cos\varphi} \, d\varphi$$

$$= O(1) \frac{n!}{n^n} e^{\frac{n}{q}+(q-1)\frac{n}{q}} \int_{|\varphi| > \frac{m}{\sqrt{n}}} e^{(q-1)\frac{n}{q}(-1+\cos\varphi)} \, d\varphi$$

$$\leq O(1) \sqrt{n} \int_{|\varphi| > \frac{m}{\sqrt{n}}} e^{-K\varphi^2 n} d\varphi$$

where $\frac{q-1}{q}(1-\cos\varphi) \geq K\varphi^2 \, (\theta \leq \varphi \leq \pi)$ for some K and the value of the integral is of $O(1)\frac{1}{m\sqrt{n}}$. Hence

$$|B_{n,m}| = O(1)\,\frac{1}{m}$$

where $O(1)$ holds uniformly with respect to m and n. This implies for a given ε, we can choose $m = M$ fixed and for all sufficiently large n such that

$$|B_{n,M}| < \frac{\varepsilon}{2}$$

<u>Estimation of</u> $A_{n,M}$. For a fixed M, we claim that all the values of z needed in the first integral of (**) lie within a parabolic region, because

$$|z| - \operatorname{Re} z = \frac{n}{q} - \frac{n}{q}\cos\varphi = \frac{n}{q}(1-\cos\varphi)$$

$$\leq \frac{n}{q} C\varphi^2 \quad \leq \quad \frac{n}{q} C \frac{M^2}{n} = \frac{CM^2}{q}$$

where $\frac{CM^2}{q}$ is a fixed number.

Therefore, by Lemma (1), $g(z) \Longrightarrow o$ in this parabolic region as $z \to \infty$. Hence

$$|A_{n,M}| = O(1)\, n! \int_{|\varphi| \leq \frac{m}{\sqrt{n}}} \frac{e^{q|z|}}{n^n}\, d\varphi$$

$$= o(1)\; n! \frac{e^n}{n^n} \frac{1}{\sqrt{n}}$$

$$= o(1)$$

$$= \frac{\varepsilon}{2} \text{ for } n > N.$$

Thus it follows

$$|\sigma_n^{(a)}| \leq \varepsilon \quad (n > N). \;\|$$

PROOF of c). Without loss of generality we could assume

(***) $$\sigma_n^{(q)} \to o \text{ as } n \to \infty$$

for q > 1 fixed. From the result a)

$$|\sigma_m^{(q)} - s_n| \leq \frac{1}{q^m} \sum_{k=0}^{m} \binom{m}{k} (q-1)^{m-k} |s_k - s_n|$$

$$\leq \frac{L}{\sqrt{n+1}} \frac{1}{q^m} \sum_{k=0}^{m} \binom{m}{k} (q-1)^{m-k} |k-n| .$$

It can be shown by arithmetical computations (see Jurkat, p.281, Arch. Math. 1956) for $m = [qn]$, that

$$\frac{1}{q^m} \sum_{k=0}^{m} \binom{m}{k} (q-1)^{m-k} |k-n| \leq N \sqrt{\frac{q-1}{q}} \sqrt{n} .$$

Hence we obtain

$$|\sigma_m^{(q)}-s_n| \leq L\sqrt{\frac{q-1}{q}}\ , \text{ (n big enough).}$$

Using (***) in the above, it follows that

$$\overline{\lim_n}|s_n| \leq L\ \frac{q-1}{q} \text{ for every } q > 1,$$

which implies $s_n \to o$ $(n \to \infty)$.

§ 6. The High Indices Theorem for absolute convergence.

This is an analogue of the classical high indices theorem.

THEOREM (Zygmund, 1944). Let $f(x) = \Sigma a_n e^{-\lambda_n x}$ be convergent for $x > o$ and $\frac{\lambda_{n+1}}{\lambda_n} \geq q > 1$ with $\lambda_1 > o$. Then

$$(1) \qquad \sum_1^\infty |a_n| \leq C \int_0^\infty |f'(x)|dx$$

where C depends only on $\{\lambda_n\}$.

PROOF. (Halasz) First of all it is clear that $\int_0^\infty |f'(x)|dx < \infty$ implies $\lim_{x \to o} f(x) = s$ exists. It is sufficient to prove the theorem for the special case $s = o$. In the general case consider the auxiliary function $g(x) = f(x) - se^{-\lambda_1 x}$ with $\lim_{x \to \infty} g(x) = o$. If (1) holds for the functions with limit zero we have

$$|a_1-s| + \sum_2^\infty |a_n| \leq C \int_0^\infty |g'(x)|dx$$
$$\leq C \int_0^\infty |f'(x)|dx + C|s|.$$

Hence,

$$\sum_1^\infty |a_n| \leq C \int_0^\infty |f'(x)|dx + |s|(C+1).$$

However,

$$s = - \int_0^\infty f'(x)dx \text{ and so } |s| \leq \int_0^\infty |f'(x)|dx$$

which implies

$$\sum_1^\infty |a_n| \leq (2C+1) \int_0^\infty |f'(x)|dx.$$

It suffices to prove the theorem for the real coefficients. Now consider such n for which $a_n a_{n+1} < o$ with corresponding $\lambda_n \lambda_{n+1}$ and form their geometric mean $\sqrt{\lambda_n \lambda_{n+1}}$.. It is easy to see this new sequence $\sqrt{\lambda_k \lambda_{k+1}} = \mu_k$ (say) is an Hadamard sequence, and in fact along with $\{\lambda_n\}$ also forms an Hadamard sequence. Form the Blaschke product

$$P(z) = \prod_{1}^{\infty} \frac{\mu_k - z}{\mu_k + z} .$$

P(z) has change of sign exactly at $z = \mu_k$. Therefore the sequence $a_n \cdot P(\lambda_{n+1})$ has a constant sign throughout. Hence,

$$\sum_1^N |a_n| = \sum_1^N |a_n P(\lambda_n)| \cdot \frac{1}{|P(\lambda_n)|}$$

$$\leq C_1 |\sum_1^N a_n P(\lambda_n)|$$

where $C_1 = C_1(\{\lambda_n\})$ is the bound of the Blaschke product. Hence it remains to estimate $\sum_1^N a_n P(\lambda_n)$.

Since $\lim_{x\to o} f(x) = o$, it follows from the ordinary high indices theorem that the coefficients are bounded i.e. $a_n = O(1)$. Now consider $F(z)P(z)$ where

$$F(z) = \int_o^\infty f(t)e^{tz}dz = -\sum_1^\infty \frac{a_n}{z-\lambda_n}$$

is meromorphic in the plane with simple poles at $z = \lambda_n$. So $F(z)P(z)$ is also meromorphic in the plane with simple poles at $z = \lambda_n$ and $z = -\mu_k$ and has residues $-a_nP(\lambda_n)$ at $z = \lambda_n$ and $F(-\mu_k)p_k$ (where p_k is the residue of P at $z = -\mu_k$) at $z = -\mu_k$. By the residue theorem we have therefore

$$(2) \quad \frac{1}{2\pi i} \int_{|z|=R=4\sqrt{q}\,\lambda_N} F(z)P(z)dz = -\sum_1^N a_nP(\lambda_n) + \sum_1^K F(-\mu_k)p_k.$$

Estimation of l.h.s. of (2). As in the classical high indices theorem

$$|F(z)| \leq C_2 \frac{1}{R} \sup f(x)$$

$$\leq C_2 \frac{1}{R} \int_o^\infty |f'(x)|dx$$

where C_2 depends only on $\{\lambda_n\}$. Hence

(3) $$|\text{l.h.s.}| \leq C_1C_2 \int_0^\infty |f'(x)|\,dx.$$

<u>Estimation of second summation on the right of (2).</u>

We have

$$F(-\mu_k) = \int_0^\infty f(t)e^{-\mu_k t}\,dt.$$

Hence

$$\sum_{k=1}^{K} F(-\mu_k)p_k = \int_0^\infty f(t) \sum_{k=1}^{K} \left(p_k e^{-\mu_k t} \right) dt.$$

Now integrating by parts we obtain

$$\left| \sum_{k=1}^{K} F(-\mu_k)p_k \right| = \left| \int_0^\infty f'(t) \cdot \sum_{k=1}^{K} \frac{p_k}{\mu_k} e^{-\mu_k t}\,dt \right|$$

(4) $$\leq \sup_{t>0} \left| \sum_{k=1}^{K} \frac{p_k}{\mu_k} e^{-\mu_k t} \right| \int_0^\infty |f'(t)|\,dt$$

$$\leq \sup_{K} \left| \sum_{k=1}^{K} \frac{p_k}{\mu_k} \right| \int_0^\infty |f'(t)|\,dt.$$

But

$$1 - \sum_{1}^{K} \frac{p_k}{\mu_k} = \frac{1}{2\pi i} \int_{|z|=R} \frac{P(z)}{z}\, dz$$

which implies

(5) $$\left|\sum_{1}^{K} \frac{p_k}{\mu_k}\right| \leq C_3 1+1$$

where C_3 depends only on $\{\lambda_n\}$. From (2) - (5) the theorem follows.‖

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