MATSCIENCE REPORT 52

CONCEPTS IN MODERN MATHEMATICS-III

(Analysis)

Part I

By

K.R. UNNI*

Member, MATSCIENCE, Madras-20.

Notes by

G.N. KeshAvamurthy and N.R.Nandakumar

THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS-20.

INTRODUCTION

Being the third in the series 'Concepts in modern mathematics', this volume deals with some fundamental concepts in analysis. The first four chapters comprise the first part. Chapter 1 gives a detailed discussions of Lebesgue integrals. Basic properties of topological vector spaces are given in Chapter 2 while the results are specialized to normed linear spaces in Chapter 3 and in particular different representation theorems are given. Gelfand theory and elementary properties of Banach algebras are the contents of Chapter 4.

In the remaining Chapters, which will appear in a separate part, are discussed the existence of Haar integral on a locally compact abelian groups, duality and characters, Fourier transforms on $L_1(G)$ and $L_2(G)$ and finally Pontrjagin's duality theorem is proved.

Materials are freely drawn from the standard books included in the bibliography given at the end of part 2.

K.R.U.

C O N T E N T S

Part I

CHAPTER 1

LEBESGUE INTEGRATION

1.1. Lebesgue Measure

Let $\mathbb{R}$ denote the field of real numbers. Let $\mathbb{R}^n$ denote the n-dimensional real Euclidean space with the usual topology. $x \in \mathbb{R}^n$ means $x = (x_1, x_2, \ldots, x_n)$ where $x_i \in \mathbb{R}$. The metric in $\mathbb{R}^n$ is given by

$$d(x,y) = \left\{ \sum_{r=1}^{n} (x_r - y_r)^2 \right\}^{\frac{1}{2}} \qquad \begin{array}{l} x = (x_1, x_2, \ldots, x_n) \\ y = (y_1, y_2, \ldots, y_n) . \end{array}$$

The symbol $\prec$ will stand for $<$ or $\leq$. An <u>extended vector</u> $a = (a_1, a_2, \ldots, a_n)$ in $\mathbb{R}^n$ is one in which the components a_i could take the value $\pm \infty$.

An <u>interval</u> in $\mathbb{R}^n$ is a product of n intervals in $\mathbb{R}$. This means that if a, b are two extended vectors in $\mathbb{R}^n$, $a_r \leq b_r$, $a_r < \infty$, $b_r > -\infty$, $1 \leq r \leq n$, the interval $I = (a,b)$ $= \{ x \in \mathbb{R}^n \mid a_r \prec x_r \prec b_r;\ 1 < r \leq n \}$. An <u>open</u> interval $]a,b[$ is given by

$$]a,b[\;=\; \{ x \in \mathbb{R}^n \mid a_r < x_r < b_r;\ 1 \leq r \leq n \} .$$

and the closed interval $[a,b]$ is given by

$$[a,b] \;=\; \{ x \in \mathbb{R}^n \mid a_r \leq x_r \leq b_r;\ 1 \leq r \leq n \} .$$

I is said to be <u>degenerate</u> if $a_r = b_r$ for one or more values of r. If $a, b \in \mathbb{R}^n$, then the components of a, b are finite real numbers and the interval is said to be <u>bounded.</u>

<u>DEFINITION</u> 1. Let I be a bounded interval in $\mathbb{R}^n$. The <u>n-dimensional</u> <u>measure</u> is defined by

$$m(I) = \prod_{r=1}^{n} (b_r - a_r)$$

If I is unbounded, we define $m(I) = \infty$.

<u>Remarks</u>. 1. $m(I)=0$ if and only if I is degenerate

2. measures of an open interval, closed interval and all intermediaries are the same.

Let X be a bounded interval in $\mathbb{R}^n$. We assume that X is closed (and thus compact). We will consider only subsets of X. Let $\mathcal{J} = \mathcal{J}(X)$ denote the class of all subsets of X which are countable unions of intervals. Our object is to extend the Definition 1 to the class $\mathcal{J}$. We have

THEOREM 1. <u>Let</u> $J \in \mathcal{J}$. <u>Then</u> J <u>is a countable union of disjoint intervals. Moreover if</u> J <u>is expressed as countable disjoint union of intervals in two different ways</u>

$$J = \bigcup_{r=1}^{\infty} I_r = \bigcup_{s=1}^{\infty} I'_s \quad .$$

then

$$\sum_{r=1}^{\infty} m(I_r) = \sum_{s=1}^{\infty} m(I'_s) \qquad (*)$$

PROOF. By definition of the class $\mathcal{J}$, we have $J = \bigcup_{r=1}^{\infty} A_r$ where A_r is an interval for each r. Now set

$$I_1 = A_1$$

$$I_2 = A_2 \setminus I_1$$

Define inductively

$$I_{n+1} = A_{n+1} \setminus (I_1 \cup I_2 \cup \dots \cup I_n)$$

Thus $J = \bigcup_{r=1}^{\infty} I_r$ and the I_r's are disjoint. The first part of the theorem is immediate if we notice that each I_r is a union of finite number of intervals and hence a disjoint union of finite number of intervals.

To prove the second part of the theorem, we first observe that both series in (*) converge, since all intervals of subsets of X, it follows that

$$\sum_{r=1}^{N} m(I_r) \leq m(X) \quad \text{and} \quad \sum_{s=1}^{M} m(I'_s) \leq m(X)$$

Now suppose that the two sums are different. For convenience let us assume that $\sum_{r=1}^{\infty} m(I_r) > \sum_{s=1}^{\infty} m(I_s')$. Then there exists an integer N such that

$$\sum_{r=1}^{N} m(I_r) - \sum_{s=1}^{\infty} m(I_s') = h > 0$$

We choose open intervals $A_s \supset I_s'$ so that $m(A_s) < m(I_s') + 2^{-s-2}h$ and closed intervals $B_r \subset I_r$ so that $m(B_r) > m(I_r) - 2^{-r-2}h$. Then

$$\sum_{r=1}^{N} m(B_r) > \sum_{r=1}^{N} m(I_r) - \sum_{r=1}^{N} 2^{-r-2}\, h$$

$$> \sum_{s=1}^{\infty} m(A_s) + h + \sum_{s=1}^{\infty} 2^{-s-2}\, h - \sum_{r=1}^{N} 2^{-r-2}\, h$$

$$> \sum_{s=1}^{\infty} m(A_s).$$

Now set $K = \bigcup_{r=1}^{N} B_r$. K is then compact. Now $\{A_s\}$ is an open covering of J and hence of K. The compactness of K implies the existence of a finite number of intervals $A_1, A_2, \ldots, A_n$ such that $K \subset \bigcup_{k=1}^{n} A_k$. Then

$$\sum_{r=1}^{N} m(B_r) \le \sum_{s=1}^{n} m(A_s) \le \sum_{s=1}^{\infty} m(A_s)$$

which gives a contradiction.

In view of Theorem 1, we can make

DEFINITION 2. If $J \in \mathcal{J}$ and $J = \bigcup_{r=1}^{\infty} I_r$ is a representation as a countable disjoint union of intervals then the *measure* $m(J)$ of J is defined by

$$m(J) = \sum_{r=1}^{\infty} m(I_r).$$

Remarks 1. $\phi \in \mathcal{J}$ and $m(\phi) = 0$

2. every countable set is in $\mathcal{J}$ and its measure is zero.

3. If $J \in \mathcal{J}$, then given $\varepsilon > 0$ there exists J_o which is a finite union of intervals such that $J_o \subset J$ and $m(J_o) > m(J) - \varepsilon$.

THEOREM 2. a) $J_1 \subset J_2$ *implies* $m(J_1) \leq m(J_2)$

b) $J = \bigcup_{r=1}^{\infty} I_r$ *implies* $m(J) \leq \sum_{r=1}^{\infty} m(I_r)$

c) *if* $J_1 \subset J_2 \subset \ldots, \subset J_n \subset \ldots$ *and* $J = \bigcup_{r=1}^{\infty} J_r$ *then* $m(J) = \lim_{r \to \infty} m(J_r)$.

d) $m(J_1 \cup J_2) + m(J_1 \cap J_2) = m(J_1) + m(J_2)$

where $J, J_r \in \mathcal{J}$.

PROOF. (a) and (b) are left as exercises. We shall now prove (c). By (b), $\lim_{r \to \infty} m(J_r)$ exists and it does not exceed $m(J)$. To prove the opposite inequality, let $J_r = \bigcup_{s=1}^{\infty} I_{rs}$. Then $J = \bigcup_{r=1}^{\infty} \bigcup_{s=1}^{\infty} I_{rs}$ and this may be arranged as a single sequence $J = \bigcup_{t=1}^{\infty} I_t$. We set

$$D_1 = I_1$$

$$D_2 = I_2 \setminus D_1$$

$$\vdots$$

$$D_n = I_n \setminus (D_1 \cup D_2 \cup \dots \cup D_n)$$

Now given $\epsilon > 0$ there exists an integer N such that

$$m\left(\bigcup_{t=1}^{N} D_t\right) > m(J) - \epsilon$$

Further there exists an integer n such that

$$J_n \supset \bigcup_{t=1}^{N} I_t \supset \bigcup_{t=1}^{N} D_t$$

Hence $m(J_n) > m(J) - \epsilon$. Since ϵ is arbitrary, the result follows

We have to prove (d). If J_1, J_2 are finite unions of intervals, so is $J_2 \setminus J_1$ and $J_1 \cup J_2 = J_1 \cup (J_2 \setminus J_1)$; so

$$m(J_1 \cup J_2) = m(J_1) + m(J_2 \setminus J_1)$$

Also

$$J_2 = (J_1 \cap J_2) \cup (J_2 \setminus J_1)$$

whence

$$m(J_2) = m(J_1 \cap J_2) + m(J_2 \setminus J_1).$$

This gives the result. In the general case, $J_2 \setminus J_1$ is not necessarily in $\mathcal{J}$, nor have proved that the measure of a disjoint union is the sum of the measures. But this fact has been assured for finite number of intervals. To complete the proof, we proceed as follows.

Let

$$J_1 = \bigcup_{r=1}^{\infty} I_r, \quad J_2 = \bigcup_{s=1}^{\infty} I_s' \qquad \text{disjoint unions.}$$

Set

$$A_n = \bigcup_{r=1}^{n} I_r, \quad B_n = \bigcup_{s=1}^{n} I_s'$$

The A_n, B_n, $A_n \cup B_n$, $A_n \cap B_n$ are increasing sequence of sets whose unions are respectively J_1, J_2, $J_1 \cup J_2$ and $J_1 \cap J_2$. Further for each n,

$$m(A_n \cap B_n) + m(A_n \cup B_n) = m(A_n) + m(B_n).$$

Now let $n \to \infty$ to obtain the result.

THEOREM 3. _If_ $J_1, J_2, \ldots$ _is a countable collection of sets in_ $\mathcal{J}$, _then_

$$m\left(\bigcup_{r=1}^{\infty} J_r\right) \leq \sum_{r=1}^{\infty} m(J_r)$$

<u>and if the sets</u> J_r <u>are disjoint, then</u>

$$m\left(\bigcup_{r=1}^{\infty} J_r\right) = \sum_{r=1}^{\infty} m(J_r).$$

PROOF. If there are only two sets J_1, J_2, then

$$m(J_1 \cup J_2) \leq m(J_1) + m(J_2)$$

with equality when $J_1 \cap J_2 = \phi$.

Now let there be infinitely many sets $J_1, J_2, \ldots$. Now let $D_n = J_1 \cup J_2 \cup \ldots \cup J_n$. Then the sets D_n's form an increasing sequence whose union is J. Then by Theorem 2, we have

$$m(J) \leq \lim_{n\to\infty} m(D_n) = \lim_{n\to\infty} \sum_{r=1}^{n} m(J_r) = \sum_{r=1}^{\infty} m(J_r).$$

Again equality in the disjoint case.

<u>Exercise 1</u>. Show that every open set is in $\mathcal{J}$ (closed sets may not be there).

<u>Exercise 2.</u> If $J \in \mathcal{J}$, show that for each $\epsilon > 0$ there exists an open set $J'(\epsilon) \supset J$ such that $m(J'(\epsilon)) < m(J)+\epsilon$.

DEFINITION.3. Let X be a bounded interval and $A \subset X$. We define the <u>outer measure</u> $m^*(A)$ by

$$m^*(A) = \inf_{\substack{J \supset A \\ J \in \mathcal{J}}} m(J)$$

and the <u>inner-measure</u> $m_*(A)$ is defined by

$$m_*(A) = m(X) - m^*(X \setminus A)$$

THEOREM 4. <u>The inner and the outer measures of a set are independent of</u> X.

PROOF. We will prove only for inner measure, the case of outer measure being trivial. Let X, X' be bounded intervals containing A. Without loss of generality, we assume that $X \subset X'$. Suppose $X \supset J \supset X \setminus A$. Then $J' = J \cup (X' \setminus X) \supset X' \setminus A$ and $J = J' \cap X$ satisfies $X \supset J \supset X \setminus A$ and $J \in \mathcal{J}(X)$.

$$m(J') = m(J \cup (X' \setminus X)) = m(J) + m(X' \setminus X) = m(J) + m(X') - m(X)$$

$$\inf_{\substack{J' \in \mathcal{J}(X') \\ J' \supset X' \setminus A}} m(J') = \inf_{\substack{J \in \mathcal{J}(X) \\ J \supset X \setminus A}} m(J) + m(X') - m(X)$$

$$m^*(X' \setminus A) = m^*(X \setminus A) + m(X') - m(X)$$

$$m(X) - m^*(X \setminus A) = m(X') - m^*(X' \setminus A).$$

<u>Exercise 3</u>. Show that $m^*(A) \geq m_*(A)$ for any set A.

<u>Exercise 4.</u> If $J \in \mathcal{J}$, show that $m_*(J) = m^*(J) = m(J)$

DEFINITION 4. A set E is <u>measurable</u> if $m^*(E) = m_*(E)$ and the common value is called the measure of the set E denoted by $m(E)$.

Remark 1. $J \in \mathcal{J}$ is measurable.

Remark 2. If E is measurable, then $X \setminus E$ is measurable and $m(X \setminus E) = m(X) - m(E)$.

DEFINITION 5. If A is unbounded, we define $m(A)$ as follows. For each positive integer k, set

$$I^{(k)} = \{ x \in \mathbb{R}^n \mid |x_r| \leq k; 1 \leq r \leq n \}.$$

Let

$$A^{(k)} = A \cap I^{(k)}.$$ A is measurable if $A^{(k)}$ is measurable for each k. Then $m(A) = \lim_{k \to \infty} m(A^{(k)})$.

The remaining theorems in this section hold also for unbounded sets, but we prove them only for bounded sets.

THEOREM 5. *If A and A' are sets, then*

$$m^*(A \cup A') + m^*(A \cap A') \leq m^*(A) + m^*(A')$$

$$m_*(A \cup A') + m_*(A \cap A') \geq m_*(A) + m_*(A')$$

PROOF. Let $\epsilon > 0$ be given. Let $J, J' \in \mathcal{J}$ such that $A \subset J$, $A' \subset J'$, $M(J) < m^*(A) + \epsilon$ and $m(J') < m^*(A') + \epsilon$. Then $A \cup A' \subset J \cup J'$ and $A \cap A' \subset J \cap J'$ so that

$$m^*(A \cup A') + m^*(A \cap A') \leq m(J \cup J') + m(J \cap J') = m(J) + m(J')$$
$$\leq m^*(A) + m^*(A') + 2\epsilon.$$

ϵ being arbitrary,

$$m^*(A \cup A') + m^*(A \cap A') \leq m^*(A) + m^*(A')$$

Now

$$m_*(A \cup A') + m_*(A \cap A') = m(X) - m^*(X \setminus A \cup A') + m(X) - $$
$$- m^*(X \setminus A \cap A')$$
$$= 2m(X) - m^*((X \setminus A) \cap (X \setminus A'))$$
$$- m^*((X \setminus A) \cup (X \setminus A'))$$
$$\geq 2m(X) - m^*(X \setminus A) - m^*(X \setminus A')$$
$$= m_*(A) + m_*(A').$$

THEOREM 6. <u>If</u> $A = \bigcup_{r=1}^{\infty} A_r$, <u>then</u> $m^*(A) \leq \sum_{r=1}^{\infty} m^*(A_r)$.

PROOF. Given $\epsilon > 0$, there exists $J_r \supset A_r$ such that $m(J_r) < m^*(A_r) + \epsilon\, 2^{-r-1}$. Then $A \subset \bigcup_{r=1}^{\infty} J_r$ and

$$m^*(A) \leq m(\bigcup_{r=1}^{\infty} J_r) \leq \sum_{r=1}^{\infty} m(J_r) \leq \sum_{r=1}^{\infty} (m^*(A_r) + \frac{\epsilon}{2^{r+1}})$$
$$< \sum_{r=1}^{\infty} m^*(A_r) + \epsilon.$$

Since ϵ is arbitrary, the result follows.

THEOREM 7. <u>If</u> E <u>and</u> E' <u>are measurable sets, so are</u> $E \cup E'$ <u>and</u> $E \cap E'$. <u>Further</u> $m(E \cup E') + m(E \cap E') = m(E) + m(E')$.

PROOF. Since E is measurable, we have

$$m(E) + m(E') \leq m_*(E \cup E') + m_*(E \cap E') \leq m^*(E \cup E') + m^*(E \cap E')$$
$$\leq m(E) + m(E').$$

Thus we have equality throughout and

$$m_*(E \cup E') + m_*(E \cap E') = m^*(E \cup E') + m^*(E \cap E').$$

Since inner measure does not exceed outer measure, it follows that

$$m_*(E \cup E') = m^*(E \cup E') \text{ and } m_*(E \cap E') = m^*(E \cap E').$$

This proves our theorem.

COROLLARY. *If E and F are measurable, then $E \setminus F$ is also measurable.*

PROOF. $E \setminus F = E \cap (X \setminus F)$ which is the intersection of two measurable sets.

THEOREM 8. *Any countable union or countable intersection of measurable sets is measurable. Further $m(\bigcup_{r=1}^{\infty} E_r) \leq \sum_{r=1}^{\infty} m(E_r)$, with equality when the sets are disjoint.*

THEOREM 9. *If the sets E_r are measurable, $E_1 \subset E_2 \subset \cdots$ and $E = \bigcup_{r=1}^{\infty} E_r$, then*

$$m(E) = \lim_{r \to \infty} m(E_r)$$

and more generally if $A_1 \subset A_2 \subset \ldots.$ and $A = \bigcup_{r=1}^{\infty} A_r$, then

$$m^*(A) = \lim_{r \to \infty} m^*(A_r).$$

PROOF. Exercise.

If $E \subset \mathbb{R}^n$, $E' \subset \mathbb{R}^m$, then $E \times E' \subset \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$.

THEOREM 10. <u>If</u> E <u>and</u> E' <u>are measurable, so does</u> $E \times E'$ <u>and then</u> $m(E \times E') = m(E)m(E')$.

PROOF. Suppose $E \subset X$ and $E' \subset X'$ where X, X' are bounded intervals. For sets in $\mathcal{J}$, if $J = \bigcup I_r$ and $J' = \bigcup I_s'$ where unions are disjoint,

$$J \times J' = \bigcup I_r \times \bigcup I_s' = \bigcup \bigcup (I_r \times I_s')$$

is a countable disjoint union. Then

$$m(J \times J') = \sum \sum m(I_r).m(I_s') = \sum m(I_r). \sum m(I_s') = m(J).m(J').$$

Now let $J \supset E$, $J' \supset E'$. Then $J \times J' \supset E \times E'$ and

$$m^*(E \times E') \leq m(J \times J') = m(J).m(J').$$

The left hand side is independent of J and J'. We can choose J, J' such that $m(J)$, $m(J')$ are arbitrarily close to $m(E)$, $m(E')$ respectively. Then

$$m^*(E \times E') \leq m(E).m(E').$$

Also

$$m_*(E \times E') = m(X \times X') - m^*(X \times X' \setminus E \times E')$$

Now $X \times X' \setminus E \times E' = X \times (X' \setminus E') \cup (X \setminus E) \times E'$.

$m^*(X \times X' \setminus E \times E') \leq m(X)m(X' \setminus E') + m(X \setminus E).m(E')$.

$m(X' \setminus E') = m(X') - m(E')$; $m(X \setminus E) = m(X) - m(E)$

$$m_*(E \times E') \geq m(X)m(X') - m(X)(m(X')-m(E')) - m(E').(m(X)-m(E))$$
$$= m(E).m(E').$$

Hence

$$m^*(E\times E') \leq m(E).m(E') \text{ and } m_*(E\times E') \geq m(E).m(E').$$

$$m_*(E \times E') = m^*(E \times E') = m(E).m(E').$$

1.2 <u>Integration</u>

Let E be a measurable set in $\mathbb{R}^n$. A countable collection of disjoint measurable sets

$$\mathscr{E} = \{E_1, E_2, \dots\}$$

is called a <u>dissection</u> if their union is E.

If $\mathscr{E}' = \{E_1', E_2', \dots\}$ is another dissection of E, then the collection $\{E_i \cap E_j'\}$ is also a dissection of E <u>called the common refinement of</u> $\mathscr{E}$ <u>and</u> $\mathscr{E}'$ and is denoted by $\mathscr{E} \vee \mathscr{E}'$.

Let $\mathscr{E} = \{E_r\}_{r=1}^{\infty}$ be a dissection of the set E and let f be a real valued function on E. Set

$$B_r(f) = \sup_{x \in E_r} f(x), \quad b_r(f) = \inf_{x \in E_r} f(x)$$

and

$$h_r(f) = \sup_{x \in E_r} |f(x)| = \max\left\{ |B_r(f)|, |b_r(f)| \right\}$$

<u>Convention</u> $h_r(f)m(E_r) = 0$ if one of them zero and the other is infinite.

$\mathcal{E}$ is said to be <u>admissible (for</u> f <u>over</u> E) if

$$\sum_{r=1}^{\infty} h_r'(f)m(E_r) < \infty .$$

Let $\mathfrak{A}$ denote the set of all admissible dissections of f over E. If $\mathcal{E} \in \mathfrak{A}$, set

$$S_{\mathcal{E}}(f) = \sum_{r=1}^{\infty} B_r(f)m(E_r)$$

$$s_{\mathcal{E}}(f) = \sum_{r=1}^{\infty} b_r(f)m(E_r)$$

and call them upper and lower approximating sums for f over E.

LEMMA 1. <u>If $\mathcal{E}'$ is a refinment of an admissible dissection $\mathcal{E}$ then $\mathcal{E}'$ is also admissible and</u>

$$S_{\mathcal{E}'} \leq S_{\mathcal{E}} , \qquad s_{\mathcal{E}'} \geq s_{\mathcal{E}}$$

PROOF. Let $\left\{ E'_{p_1}, E'_{p_2}, \ldots \right\}$ be the sets of $\mathcal{E}'$ contained in E_p of $\mathcal{E}$. Then

$$B'_{p_q}(f) \leq B_p(f),\ b'_{p_q}(f) \geq b_p(f) \quad \text{for all } p \text{ and } q.$$

Also

$$\sum_q m(E'_{p_q}) = m(E_p)$$

$$(h'_{p_q}(f) \le h_p(f)$$

$$\sum_q B'_{p_q}(f)m(E'_{p_q}) \le \sum_q B_p(f)m(E'_{p_q}) = B_p(f)m(E_p)$$

$$\sum_q b'_{p_q}(f)m(E'_{p_q}) \ge \sum_q b_q(f)m(E'_{p_q}) = b_p(f)m(E_p)$$

$$\sum_q h'_{p_q}(f)m(E'_{p_q}) \le \sum_q h_q(f)m(E'_{p_q}) = h_p(f)m(E_p)$$

The results now follow by summing over p.

DEFINITION 6. The <u>upper</u> and <u>lower</u> <u>integrals</u> are defined by

$$\int_E^* f = \inf_{\mathscr{E} \in \mathfrak{A}} S_{\mathscr{E}}(f)$$

$$\int_{*E} f = \sup_{\mathscr{E} \in \mathfrak{A}} s_{\mathscr{E}}(f).$$

THEOREM 11. $\int_E^* f \ge \int_{*E} f$

PROOF. Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two admissible dissections. Let $\mathscr{E} = \mathscr{E}_1 \vee \mathscr{E}_2$ the common refinment, then

$$s_{\mathscr{E}_2}(f) \leq s_{\mathscr{E}}(f) \leq S_{\mathscr{E}}(f) \leq S_{\mathscr{E}_1}(f),$$

$$\sup_{\mathscr{E} \in \mathfrak{A}} s_{\mathscr{E}}(f) \leq \inf_{\mathscr{E} \in \mathfrak{A}} S_{\mathscr{E}}(f),$$

$$\int_{*E} f \leq \int_{E}^{*} f.$$

DEFINITION 7. f is said to be <u>integrable over</u> E if $\int_{E}^{*} f = \int_{*E} f$ and the integral of f is the common value. We write

$$I(f) = \int_E f dx \quad \text{or} \quad \int_E f.$$

THEOREM 12. <u>A necessary and sufficient condition that f should be integrable over E is that given</u> $\varepsilon > 0$ <u>there exists an admissible dissection</u> $\mathscr{E} \in \mathfrak{A}$ <u>such that</u>

$$S_{\mathscr{E}} - s_{\mathscr{E}} < \varepsilon.$$

PROOF. If there is such a dissection, then f is integrable is trivial. Suppose f is integrable, there exists dissection $\mathscr{E}_1$ such that

$$S_{\mathscr{E}_1}(f) - I(f) < \frac{\varepsilon}{2}$$

and a dissection $\mathscr{E}_2$ such that

$$I(f) - s_{\mathscr{E}_2}(f) < \frac{\varepsilon}{2}$$

Let $\mathscr{E} = \mathscr{E}_1 \vee \mathscr{E}_2$. Then

$$S_{\mathscr{E}} - s_{\mathscr{E}} \leq S_{\mathscr{E}_1}(f) - s_{\mathscr{E}_2}(f) < \varepsilon .$$

REMARK. If there is a dissection such that $S_{\mathscr{E}} = s_{\mathscr{E}}$ then f is integrable over E and

$$\int_E f = S_{\mathscr{E}} = s_{\mathscr{E}}$$

<u>Examples</u> 1. Let f be a generalized step function, i.e., it takes a countable set of non-zero values on E say $f_1, f_2, \ldots, f_n, \ldots$. Let $E_r = \{ x \mid f(x) = f_r \}$. If

$$\sum |f_r| \cdot m(E_r) < \infty$$

then $\mathscr{E} = \{ E_1, E_2, \ldots \}$ is admissible and $E = \bigcup_{r=1}^{\infty} E_r$

$$\int_E f = \sum f_r m(E_r)$$

2. Let ϕ_E be the characteristic function on E. Then ϕ_E is integrable and

$$\int_E \phi_E = \sum m(E_r) = m(E)$$

3. If f is a constant function say $f = c$ then

$$\int f = cm(E).$$

THEOREM 13. <u>Let E, E' be measurable sets such that $E \subseteq E'$. Let f and g be defined on E and E' respectively. Suppose</u>

$$g(x) = \begin{cases} f(x) & x \in E \\ 0 & x \in E' \setminus E. \end{cases}$$

<u>Then f is integrable over E iff g is integrable over E'</u> and

$$\int_E f = \int_{E'} g .$$

PROOF. If $\mathcal{E} = \{E_1, E_2, \ldots\}$ is an admissible dissection on E, then $\{E' - E_1, E' - E_2, \ldots\}$ is an admissible dissection of E'. If $\{E_1', E_2', \ldots\}$ is an admissible dissection, then $\{E \cap E_1', E \cap E_2', \ldots\}$ is an admissible dissection of E.

COROLLARY. $A \subset E$, <u>then</u>

$$\int_A f = \int_A f \phi_A$$

THEOREM 14. <u>If</u> $f = g$ a.e., <u>then</u> $\int^* f = \int^* g, \int_* f = \int_* g$ <u>and</u> $\int f = \int g$.

PROOF. Let E_o be the set in which $f \neq g$. Let $\mathscr{E} = \{E_1, E_2, \ldots\}$. Then

$$\{E_o, E_1 \cap (E-E_o), E_2 \cap (E-E_o), \ldots\} = \mathscr{E}'$$

is an admissible dissection of E. Then

$$\begin{aligned} S_{\mathscr{E}'}(f) &\leq B_o'(f)m(E_o) + B_1'(f)m(E_1 \cap (E-E_o)) \\ &\quad + \ldots + B_r'(f)m(E_r \cap (E-E_o)) + \ldots \\ &= B_1'(g)m(E_1) + \ldots + B_r(g)m(E_r) + \ldots \\ &< S_{\mathscr{E}}(g) \end{aligned}$$

from which it follows that

$$S_{\mathscr{E}'}(f) \leq S_{\mathscr{E}}(g).$$

Similarly,

$$S_{\mathscr{E}'}(g) \leq S_{\mathscr{E}}(f).$$

Then

$$\int_E^* f = \int_E^* g$$

Similarly

$$\int_{*E} f = \int_{*E} g.$$

THEOREM 15. (a) <u>If</u> $f \geq g$, <u>then</u>

$$\int_E^* f \geq \int_E^* g, \quad \int_{*E} f \geq \int_{*E} g \quad \text{and} \quad \int_{*E} f \geq \int_E g$$

(b) <u>If</u> $f_1, f_2, \ldots, f_k$ <u>are integrable and</u> $c_1, c_2, \ldots, c_k$ <u>are constants then</u> $c_1 f_1 + c_2 f_2 + \ldots + c_k f_k$, <u>are integrable and</u>

$$\int_E (c_1 f_1 + \ldots + c_k f_k) = \sum_{i=1}^{k} c_i \int_E f_i \, .$$

(c) <u>If</u> $E = \bigcup_{i=1}^{n} E_i$ <u>disjoint union, then</u>

$$\int_E f = \sum_{i=1}^{n} \int_{E_i} f$$

PROOF. a) is trivial since for any admissible dissection $\mathscr{E}$ of E. $S_{\mathscr{E}}(f) \geq S_{\mathscr{E}}(g)$ and $s_{\mathscr{E}}(f) \geq s_{\mathscr{E}}(g)$

b) Let f and g be integrable over E and c a constant. Let $\mathscr{E}$ be an admissible dissection of f over E. Then

$$B_r(cf) = cB_r(f), \; b_r(cf) = cb_r(f), \; h_r(cf) = ch_r(f),$$

hence

$$\sum h_r(cf) m(E_r) = c \sum h_r(f) m(E_r) < \infty \Rightarrow \mathscr{E} \text{ is}$$

admissible for cf also.

If $c > 0$, $S_{\mathscr{E}}(cf) = cS_{\mathscr{E}}(f)$, $s_{\mathscr{E}}(cf) = cs_{\mathscr{E}}(f)$

$c < 0$, $S_{\mathscr{E}}(cf) = cs_{\mathscr{E}}(f)$, $s_{\mathscr{E}}(cf) = cS_{\mathscr{E}}(f)$.

and the integrability of f follows and $\int cf = c \int f$.

Let f and g be integrable over E. Let $\mathscr{E}'$ and $\mathscr{E}''$ be admissible dissections for f, g respectively over E. Let $\mathscr{E} = \mathscr{E}' \vee \mathscr{E}''$. Then $\mathscr{E}$ is admissible for f and g. Consider the dissection $\mathscr{E}$

$$B_r(f+g) \leq B_r(f)+B_r(g), \quad b_r(f+g) \geq b_r(f)+b_r(g),$$
$$h_r(f+g) \leq h_r(f)+h_r(g).$$

Hence, $\mathscr{E}$ is admissible for $f+g$ over E also

$$S_{\mathscr{E}}(f+g) \leq S_{\mathscr{E}}(f) + S_{\mathscr{E}}(g) \leq S_{\mathscr{E}'}(f) + S_{\mathscr{E}'}(g)$$

and

$$s_{\mathscr{E}}(f+g) \geq s_{\mathscr{E}}(f) + s_{\mathscr{E}}(g) \geq s_{\mathscr{E}'}(f) + s_{\mathscr{E}'}(g).$$

Given $\varepsilon > 0$, choose $\mathscr{E}'$, $\mathscr{E}''$ such that $S_{\mathscr{E}'}(f) - s_{\mathscr{E}'}(f) < \varepsilon$ and $S_{\mathscr{E}''}(g) - s_{\mathscr{E}''}(g) < \varepsilon$, $S_{\mathscr{E}}(f) - s_{\mathscr{E}}(f) < \frac{\varepsilon}{2}$, $S_{\mathscr{E}}(g) - s_{\mathscr{E}}(g) < \frac{\varepsilon}{2}$. Hence

$$S_{\mathscr{E}}(f+g) - s_{\mathscr{E}}(f+g) \leq S_{\mathscr{E}}(f) - s_{\mathscr{E}}(f) + S_{\mathscr{E}}(g) - s_{\mathscr{E}}(g) < \varepsilon.$$

Therefore $f+g$ is integrable, further $\int(f+g) = \int f + \int g$.

(c) Let ϕ_{E_i} be the characteristic function of E_i, $i = 1,2,\ldots, n$. Then $f = \sum_{i=1}^{n} f\,\phi_{E_i}$ and $\int_{E_i} f = \int_E f\phi_{E_i}$.

Hence

$$\int_E f = \int_E \sum_{i=1}^{n} f\phi_{E_i} = \sum_{i=1}^{n} \int_E f\phi_{E_i} = \sum_{i=1}^{n} \int_{E_i} f.$$

DEFINITION 8. For any real valued function f we define the <u>positive part</u> f^+ by

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$$

The negative part $f = f^+ - f^-$ equivalently

$$f^-(x) = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

Now $|f| = f^+ + f^-$.

THEOREM 16. *If* f *is integrable, so are* f^+, f^-, $|f|$ and

$$\left| \int_E f \right| \leq \int_E |f| \quad .$$

PROOF. Let $\mathscr{E}$ be an admissible dissection for f over E. Now for each r, $h_r(f^+) \leq h_r(f)$ and $\mathscr{E}$ is admissible for f^+. Also

$$B_r(f^+) - b_r(f^+) \leq B_r(f) - b_r(f)$$

then

$$S_{\mathscr{E}}(f^+) - s_{\mathscr{E}}(f^+) \leq S_{\mathscr{E}}(f) - s_{\mathscr{E}}(f).$$

Hence f^+ is integrable if f is.

Now if f is integrable, $(-1)f$ is integrable and $((-1)f)^+$ is integrable. But $((-1)f)^+ = f^-$ and hence f^- also integrable.

$$|f| = f^+ + f^- \text{ is also integrable.}$$

$$\left|\int_E f\right| = \left|\int_E (f^+ - f^-)\right| = \left|\int_E f^+ - \int_E f^-\right| \leq \int_E f^+ + \int_E f^-$$

$$= \int_E (f^+ + f^-).$$

Let f, g be two functions which are integrable. Define

$$f \vee g = \frac{f+g+|f-g|}{2}, \quad f \wedge g = \frac{f+g-|f-g|}{2}$$

then $f \vee g$ and $f \wedge g$ are integrable.

Notice

$$f \vee g = \max(f,g),$$

$$f \wedge g = \min(f,g).$$

Exercise 1. If E is measurable and A is a subset of E, such that $m^*(A) < \infty$, then

$$\int_E^* \phi_A = m^*(A), \quad \int_{*E} \phi_A = m_*(E)$$

Exercise 2. If f is integrable over E, and E' is a measurable subset of E, then f is integrable over E'.

<u>Exercise 3</u>. If $f \geq 0$, then $\int f = 0$, iff $f = 0$, a.e.

1.3. <u>Measurable Functions</u>

Let E be a measurable set and f an extended real valued function. We say that f is measurable if for each real number a, the set

$$\{ x \mid f(x) > a \}$$

is measurable.

The following conditions are equivalent

(1) Sets $\{ x \mid f(x) < a \}$ are measurable

(2) Sets $\{ x \mid f(x) \geq a \}$ are measurable

(3) Sets $\{ x \mid f(x) > a \}$ are measurable

(4) Sets $\{ x \mid f(x) \leq a \}$ are measurable

(1) and (2) are equivalent by complementation. So are (3) and (4). Assume (4). Let a be a real number. Then the sets

$$\{ x \mid f(x) \leq a - \tfrac{1}{r} \}$$

are measurable. Now

$$\{ x \mid f(x) < a \} = \bigcup_{r=1}^{\infty} \{ x \mid f(x) \leq a - \tfrac{1}{r} \} .$$

Hence (4) implies (1). Assume (2), then

$$\{x \mid f(x) > a\} = \bigcup_{r=1}^{\infty} \{x \mid f(x) \geq a + \frac{1}{r}\}$$

and hence measurable i.e., (2) $\Longrightarrow$ (3).

Remark 1. $\{x \mid a \leq f(x) \leq b\} = \{x \mid f(x) \geq a\} \cap \{x \mid f(x) \leq b\}$ is measurable.

Remark 2. If $f = g$ a.e., f is measurable iff g is measurable

$$\{x \mid f(x) > a\} = [\{x \mid g(x) > a\} \cup E_1] \setminus E_2$$

where E_1, E_2 are sets in which $f \neq g$ and of measure zero.

Exercise 17. If f and g are measurable functions and c is a constant, then the functions $f+g$, $f+c$, cf, f^2, $f^{1/2}$, fg, $f \vee g$, $f \wedge g$, f^+, f^-, $|f|$ are all measurable.

Exercise 18. Continuous functions are measurable.

DEFINITION 8'. f is said to be dominated in E, if there is an integrable function g such that $|f| \leq g$ on E.

THEOREM 17. $E \subset \mathbb{R}^n$. A necessary and sufficient condition that f is integrable over E is that f is measurable and dominated in E.

PROOF. Let f be measurable and dominated by g. Take any admissible dissection $\mathscr{E}$ for g. Given $\varepsilon > 0$ choose p such that

$$\sum_{p+1}^{\infty} h_r(g)\, m(E_r) < \frac{\varepsilon}{3} \quad .$$

Let $E_o = E_1 \cup E_2 \cup \ldots . \cup E_p$. Choose N integer such that $\frac{m(E_o)}{N} < \frac{\varepsilon}{3}$. Consider the dissection $\mathcal{E}'$ obtained by the intersection of the sets in $\mathcal{E}$ with the sets

$$\left\{ x \,\middle|\, \frac{p}{N} \leq f(x) \leq \frac{p+1}{N} \right\}$$

where p is an integer and the sets on which f is $+\infty$ or $-\infty$. This dissection is admissible for f. Then

$$S_{\mathcal{E}'} - s_{\mathcal{E}'} = \left(\sum_1 + \sum_2\right)(B_r(f) - b_r(f))m(E_r')$$

where

$$\sum_1 = \sum_{E_r' \subset E_o}, \quad \sum_2 = \sum_{E_r' \subset E \setminus E_o}.$$

Then

$$\sum_1 (B_r(f) - b_r(f))m(E_r') \leq \sum \frac{1}{N} m(E_r') = \frac{1}{N} m(E_o) < \frac{\varepsilon}{3}$$

$$\sum_2 (|B_r(f)| + |b_r(f)|)m(E_r') \leq \sum 2\, h_r(g)m(E_r')$$

$$= 2 \sum_{p+1}^{\infty} h_r(g)m(E_r') < \frac{2\varepsilon}{3}.$$

Hence $S_{\mathcal{E}} - s_{\mathcal{E}'} < \varepsilon$, implies f is integrable.

THEOREM (Egorov) 18. <u>Let E be of finite measure, $\{f_n\}$ a sequence of measurable functions such that</u> $f_n \rightarrow f$ a.e. in E. <u>Then given</u> $\varepsilon > 0$ <u>there exists</u>

<u>a subset</u> $F \subset E$ <u>such that</u> $m(E \setminus F) < \epsilon$ <u>and</u> $f_n \to f$ <u>uniformly</u>.

PROOF. We assume without loss of generality that $f_n \longrightarrow f$ everywhere on E (otherwise, we have to start with E_o where $f_n \to f$. Let

$$g_r = |f_r - f| .$$

Define

$$E_{pq} = \left\{ x \in E \,\middle|\, g_r(x) < q^{-1}, \text{ for } r \geq p \right\}$$

for a fixed q. The sets $E_{1q}, E_{2q}, \ldots,$ is an increasing sequence whose union is E. Hence $\lim_{p \to \infty} m(E_{pq}) = m(E)$. There exists an integer $p(q)$ such that $m(E \setminus E_{p(q)q}) < \frac{\varepsilon}{2^q}$. Now let

$$F = \bigcap_{q=1}^{\infty} E_{p(q)q}.$$

In $F, g_r < q^{-1}$ if $r \geq p(q)$. Therefore $f_r \to f$ uniformly in F. Since $E \setminus F \subset \bigcup (E \setminus E_{p(q)q})$

$$m(E \setminus F) \leq \sum_{q=1}^{\infty} m(E \setminus E_{p(q)q}) < \sum_{q=1}^{\infty} \frac{\varepsilon}{2^q} = \varepsilon .$$

REMARK. Theorem does not hold if E is not of finite measure. (Find an example).

DEFINITION 9. A family of functions $\{f_\alpha\}_{\alpha \in \Delta}$ is said to be dominated by g in E if g is integrable over E and $|f_\alpha| \leq g$ for all α.

THEOREM 19. (Lebesgue dominated Convergence Theorem.) If $\{f_r\}$ is a dominated sequence of integrable functions such that $f_r \to f$ as $r \to \infty$ then f is integrable and

$$\lim_{r \to \infty} \int |f_r - f| = 0$$

i.e.

$$\int f = \lim_{r \to \infty} \int f_r .$$

PROOF. f is clearly measurable. If f_r is dominated by ϕ then f is also dominated by ϕ and hence integrable. Let

$$g_r = |f - f_r|$$

clearly $g_r \to 0$ as $r \to \infty$ and $g_r < 2\phi$. Choose an admissible dissection $\{E_1, E_2, \ldots\}$ for ϕ over E consisting of sets of finite measure. Given $\varepsilon \gtrsim 0$ there exists an integer such that

$$\sum_{r=N+1}^{\infty} h_r(\phi) m(E_r) < \frac{\varepsilon}{4}$$

Let $E' = \bigcup_{r=1}^{N} E_r$ and $h = \sup h_r(\phi)$ where supremum is taken for varying r, 1 to ∞ where $h_r(\phi) < \infty$. By Egorov's theorem, $g_r \longrightarrow 0$ uniformly on a subset E'' of E' such that

$$m(E' - E'') < \frac{\varepsilon}{8h} .$$

There exists r' such that $g_r < \dfrac{\varepsilon}{4m(E')}$ if $r \geq r'$. Then

$$\int_E g_r = \int_{E''} g_r + \int_{E' \setminus E''} g_r + \int_{E \setminus E'} g_r .$$

Now

$$\left| \int_{E''} g_r \right| \leq \frac{\varepsilon}{4m(E')} \int_{E''} 1 = \frac{\varepsilon\, m(E'')}{4m(E'')} = \frac{\varepsilon}{4} .$$

Similarly

$$\left| \int_{E' \setminus E''} gr \right| < \frac{\varepsilon}{4}$$

and

$$\left| \int_{E \setminus E''} g_r \right| \leq 2 \int_{E \setminus E'} \phi < 2 \sum_{r=1}^{\infty} h_r(\phi)\, m(E_r) < \frac{\varepsilon}{2} .$$

Hence

$$\int_E g_r < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon .$$

THEOREM 20. (Lebesgue Theorem on Bounded Convergence). If E is of finite measure and $\{f_r\}$ is a sequence of integrable functions such that $|f_r| \leq a < \infty$ and $f_r \to f$ as $r \to \infty$ then

$$\lim_{r \to \infty} \int f_r = \int f$$

i.e.,

$$\lim_{r \to \infty} \int f_r = \int \left(\lim_{r \to \infty} f_r \right)$$

PROOF. This is immediate from the Lebesgue dominated convergence theorem.

Example. Consider the functions f_r defined by

$$f_r(x) = \begin{cases} r^2 x, & 0 \leq x \leq \frac{1}{r} \\ 2r - r^2 x, & \frac{1}{r} < x \leq \frac{2}{r} \\ 0, & \frac{2}{r} < x \leq 1. \end{cases}$$

Then

$$\int_0^1 f_r = \int_0^{\frac{1}{r}} r^2 x\,dx + \int_{\frac{1}{r}}^{\frac{2}{r}} (2r - r^2 x)\,dx = \frac{1}{2} + \frac{1}{2} = 1$$

$$\int f_r \longrightarrow 1 \quad \text{but} \quad f_r \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty.$$

Hence $\int \lim f_r \neq \lim \int f_r$.

Define

$$I^{(k)} = \left\{ x \in R^n \,\middle|\, |x_r| \leq k,\ r = 1, 2, \ldots, n \right\}$$

$$E^{(k)} = E \cap I^{(k)}$$

and

$$f^{|k|}(x) = \begin{cases} f(x) & \text{if } f(x) \leq k \\ k & \text{if } f(x) > k. \end{cases}$$

Remark. If $f \geq 0$ then $\lim \int_{E^{(p)}} f^{[q]}$ may be finite or infinite and independent of the way $p, q \to \infty$.

THEOREM 21. *Let f be nonnegative. A necessary and sufficient condition that f should be integrable over E is that $f^{[q]}$ is integrable over $E^{(p)}$ for all p and q and that $\lim \int_{E^{(p)}} f^{[q]}$ should be finite. If f is integrable then $\int f = \lim \int_{E^{(p)}} f^{[q]}$*

PROOF. Suppose f is integrable, then $f^{[q]}$ is obviously measurable and is dominated by f and hence integrable over any measurable subset of E.

Suppose $\int_{E^{(p)}} f^{[q]}$ exists (s.y a_{pq}) for all p and q and $a = \lim a_{pq}$ (finite). Let

$$E_r = \{x \mid 2^r \leq f(x) < 2^{r+q}\} \quad (-\infty < r < \infty)$$

$$E_{-\infty} = \{x \mid f(x) = 0\}$$

$$E_\infty = \{x \mid f(x) = \infty\}.$$

Obviously $m(E_\infty) = 0$. Otherwise $a = \infty$. Also

$$\sum_{r=q'}^{q} 2^{r+1}\, m(E_r^{(q)}) \leq 2a_{pq} < 2a .$$

Making $q' \to -\infty$, $q \to \infty$ and $p \to \infty$, we have

$$\sum_{r=-\infty}^{\infty} 2^{r+1}\, m(E_r) \leq 2a.$$

Now the function F defined by

$$F = \begin{cases} 0 & \text{on } E_{-\infty} \\ 2^{r+1} & \text{on } E_r \\ \infty & \text{on } E_{\infty} \end{cases}$$

is integrable. Since f is dominated by F and is measurable being the limit of sequence of measurable functions, f is integrable.

THEOREM 22. (Fatou) <u>If</u> $f_r \to 0$ <u>for all</u> r <u>and</u> $f = \liminf f_r$, <u>then</u>

$$\int f \leq \liminf \int f_r.$$

PROOF. Set $g_r = \inf f_s$, $s \geq r$ then $f = \lim g_r$

$$\int_{E^{(p)}} f^{[q]} = \lim \int_{E^{(p)}} g_r^{[q]}$$

$$= \liminf \int_{E^{(p)}} g_r^{[q]} \leq \liminf \int_{E^{(p)}} f_r^{[q]}$$

$$\leq \liminf \int_E f_r$$

i.e.,

$$\int_E f \leq \liminf \int_E f_r.$$

1.4 Fubini's theorem.

DEFINITION 10. Let $C \subset A \times B$. By a section of C by $a \in A$, we mean the set

$$\{ b \mid (a,b) \in C \}.$$

The projection C on B is the set $\{ b \mid (a,b) \in C$ for some $a \in A \}$.

Let x be any point in $\mathbb{R}^n$ and y any point in $\mathbb{R}^p$. Let E be a measurable subet of $\mathbb{R}^{n+p}$. The section of E by x is denoted by $E(x)$.

Let $f(x,y)$ be defined in $\mathbb{R}^{n+p}$. If the integral $\int f(x,y)\,dy$ exists, we write $g(x) = \int f(x,y)\,dy$ (This integral is taken over $E(x)$. $\int g$ is taken over some set in $\mathbb{R}^n$ containing the projection of E on $\mathbb{R}^n$. m will stand for Lebesgue measure.

THEOREM 23. *If* E *is of finite measure in* $\mathbb{R}^{n+p}$, *then for almost all* x, $E(x)$ *is measurable and of finite measure in* $\mathbb{R}^p$, $m(E(x))$ *is an integrable function of* x *and*

$$\int m(E(x))\,dx = m(E)$$

PROOF. (i) If E is a bounded interval, the result is trivial. $E(x)$ is measurable for all x, and $m(E(x))$ is a constant multiple of characteristic function of the projection of E on $\mathbb{R}^n$.

(ii) If E is a finite union of bounded intervals then the result follows from (i) by addition.

(iii) Let E be a (bounded) set in $\mathcal{J}(\mathbb{R}^{n+p})$ then for each x, $E(x)$ is a bounded set in $\mathcal{J}(\mathbb{R}^{p})$ and hence is measurable and of finite measure. If $E = \bigcup I_r$ then $E(x) = \bigcup I_r(x)$. $E_N = \bigcup_{r=1}^{N} I_r$, $E_N(x) = \bigcup_{r=1}^{N} I_r(x)$. Then $m(E_N) \longrightarrow m(E)$ and $m(E_N(x)) \longrightarrow m(E(x))$ as $N \to \infty$ and $E(x)$ is measurable. By Lebesgue theorem on bounded convergence.

$$\lim m(E_N(x))dx = \lim \int m(E_N(x))dx$$

$$\int m(E(x))dx = m(E)$$

(iv) If E is the complement with respect to a bounded interval of a set in $\mathcal{J}$, (iii) gives the result.

(v) Let E be a bounded measurable set. Given $\varepsilon > 0$ there exists sets $J \supset E$ with $J \in \mathcal{J}$, $K^c \in \mathcal{J}$ with $m(J)-m(K) < \varepsilon$. Take a sequence J_r, K_r of sets such that $J_r \subset J_{r+1}$, $K_r \subset K_{r+1}$ for all r and $m(J_r)-m(K_r) \longrightarrow 0$ as $r \to \infty$. Then

$$\int (m(J_r(x))-m(K_r(x))dx \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty.$$

Since $m(J_r(x))-m(K_r(x))$ is a decreasing function of x for all r and the limit exists and the limit is zero a.e. Hence $E(x)$ is measurable and

$$m(E(x)) = \lim_{r \to \infty} m(E_r(x)) \text{ a.e.}$$

By Lebesgue bounded convergence theorem

$$\int m(E(x))\,dx = \int \lim m(J_r(x))\,dx = \lim \int m(J_r(x))\,dx$$
$$= m(E).$$

(vi) If E is unbounded, we first consider E_k and then take the limit as $k \to \infty$.

THEOREM 24. If $f(x,y)$ is integrable over E, then for almost all x, $f(x,y)$ is integrable over $E(x)$.

$$g(x) = \int_{E(x)} f(x,y)\,dy$$

then g is integrable and $\int g = \int f$.

$$\left(\text{or} \int \left\{ \int_{E(x)} f(x,y)\,dy \right\} dx = \iint f(x,y)\,dx\,dy \right)$$

THEOREM 25. If $f(x,y)$ is measurable (in $\mathbb{R}^{n+p}$) and if $\int \left\{ \int |f(x,y)|\,dx \right\} dy$ exists then

$$\int \left\{ \int f(x,y)\,dx \right\} dy = \int \left\{ \int f(x,y)\,dy \right\} dx.$$

1.5 <u>The Class L_p</u>.

Let $p > 0$ $L_p(E) =$ class of all measurable functions f such that

$$\int_E |f|^p < \infty$$

We will identify two functions which differ only on a set of measure zero.

If $p > 1$ we define q by $\frac{1}{p} + \frac{1}{q} = 1$

If $f \in L_p$ and c is a constant, then clearly $cf \in L_p$ and if $f, g \in L_p$ then

$$|f + g|^p \leq 2^p(|f|^p + |g|^p)$$

which gives $f+g \in L_p(E)$ as well and hence L_p is a vector space.

We set

$$\|f\|_p = \left\{ \int_E |f|^p \right\}^{1/p}$$

If c is a constant

$$\|cf\|_p = |c| \; \|f\|_p .$$

$\|f\|_p = 0 \Longrightarrow \int_E |f|^p = 0 \Longrightarrow f = 0$ a.e. Since we are not distinguishing functions which differ on a set of measure zero, we have $\|f\|_p = 0 \Longrightarrow f = 0$.

LEMMA. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ *and* $x > 0, y > 0$ *then*

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q} .$$

PROOF. Consider the function

$$f(t) = t^p - pt - 1 + p$$

$$f'(t) = pt^{p-1} - p = p(t^{p-1} - 1).$$

If

$$t > 1, f'(t) > 0$$

$$t < 1, f'(t) < 0$$

$$\text{and} \quad f'(1) = 0$$

$$f(1) = 0, f'(t) > 0 \text{ for } t > 1 \Rightarrow f(t) \geq 0,$$

$t \geq 1$ when $t < 1$, $f'(t) < 0 \Rightarrow f(t)$ is decreasing i.e.,

$$f(t) \geq f(1) = 0 \text{ for all } t \geq 0$$

or

$$t^p - pt - 1 + p \geq 0$$

$$\frac{t^p}{p} \geq t + \frac{1}{p} - 1$$

or

$$t \leq \frac{1}{q} + \frac{t^p}{p}$$

Let $t = \dfrac{x}{y^{q/p}}$ and obtain $xy \leq \dfrac{x^p}{p} + \dfrac{y^p}{q}$.

To prove (1)

$$\int_E |fg| \le \left(\int_E |f|^p\right)^{\frac{1}{p}} \left(\int_E |g|^q\right)^{\frac{1}{q}} \quad f \in L_p, g \in L_q$$

(2)

$$\left(\int_E |f+g|^p\right)^{\frac{1}{p}} \le \left(\int_E |f|^p\right)^{\frac{1}{p}} + \left(\int_E |g|^p\right)^{\frac{1}{p}}, \quad f, g \in L_p.$$

(1) Put

$$x \equiv \frac{|f|}{\left(\int |f|^p\right)^{\frac{1}{p}}}, \quad y = \frac{|g|}{\left(\int |g|^q\right)^{\frac{1}{q}}}$$

$$\frac{|fg|}{\left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}} \le \frac{|f|^p}{p\left(\int |f|^p\right)} + \frac{|g|^q}{q\left(\int |g|^q\right)}$$

Integrating, we have

$$\frac{\int |fg|}{\left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}} \le \frac{\int |f|^p}{p\left(\int |f|^p\right)} + \frac{\int |g|^q}{q\left(\int |g|^q\right)}$$

i.e.

$$\int |fg| \le \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}$$

2)

$$
\begin{aligned}
\int |f+g|^p &= \int |f+g|\,|f+g|^{p-1} \\
&\leq \int |f|\,|f+g|^{p-1} + \int |g|\,|f+g|^{p-1} \\
&\leq \left(\int |f|^p\right)^{1/p} \left(\int |f+g|^{(p-1)q}\right)^{1/q} \\
&\qquad + \left(\int |g|^p\right)^{1/p} \left(\int |f+g|^{(p-1)q}\right)^{1/q} \\
&= \left(\int |f|^p\right)^{1/p} \left(\int |f-g|^p\right)^{1/q} \\
&\qquad + \left(\int |g|^p\right)^{1/p} \left(\int |f+g|^p\right)^{1/q} .
\end{aligned}
$$

Dividing by $\left(\int |f+g|^p\right)^{1/q}$ we obtain the result.

DEFINITION 11. Let $\{f_n\}$ be a sequence of functions belonging to the class L_p. If $\|f_n - f\|_p$ as $n \to \infty$ we say $f_n \to f$ in the mean (with index p).

DEFINITION 12. $\{f_n\}$ is a Cauchy sequence or fundamental sequence if $\|f_m - f_n\| \to 0$ as $m,n \to \infty$ or given $\varepsilon > 0$ there exists $n_o(\varepsilon)$ such that

$$\|f_m - f_n\|_p < \varepsilon \text{ if } m,n \geq n_o(\varepsilon) .$$

THEOREM 26. If $\{f_n\}$ is a Cauchy sequence in L_p, then there exists $f \in L_p$ such that $f_n \to f$ in mean.

DEFINITION 13. A family of functions $\{f_\alpha\}$ is said to be p-dominated if there is a function $g \in L_p$ such that $|f_\alpha| \leq g$ for all α.

THEOREM 27. <u>If</u> $\{f_n\}$ <u>is a p-dominated sequence of functions in</u> L_p <u>and if</u> $f_n \to f$ <u>a.e. on</u> E, <u>then</u> $f \in L_p$ <u>and</u>

$$\|f_n - f\|_p \to 0 \quad \text{as} \quad n \to \infty .$$

1.5 <u>General Measures.</u>

In this section we will define measures more general than Lebesgue measures.

DEFINITION 14. Let X be an arbitrary set and R any class of subsets of X. R is said to be a <u>ring</u> if $E, F \in R$, then $E \cup F \in R$ and $E \setminus F \in R$.

<u>Remark</u>. (i) $\phi \in R$. Since $\phi = E \setminus E \in R$ if $E \in R$.

(ii) $E \Delta F \in R$ whenever $E, F \in R$. Follows from $E \Delta F = (E \setminus F) \cup (F \setminus E)$.

(iii) $E \cap F \in R$, whenever $E, F \in R$. This follows from $E \cap F = (E \cup F) \setminus (E \Delta F)$.

(iv) $\bigcup_{i=1}^{n} E_i \in R$ and $\bigcap_{i=1}^{n} E_i \in R$ if $E_i \in R$ $i = 1, 2, \ldots n$.

DEFINITION 15. Let $\mathcal{S}$ be any nonempty collection of subsets of X. Then $\mathcal{S}$ is said to be a <u>σ-ring</u> if $\mathcal{S}$ is a ring and if $E_i \in \mathcal{S}$ for $i = 1, 2, \ldots,$ implies $\bigcup_{i=1}^{\infty} E_i \in \mathcal{S}$.

<u>Notation</u>. Let $\mathscr{E}$ be any class of subsets of X. We denote by $R(\mathscr{E})$ $(\mathscr{S}(\mathscr{E}))$ the ring (σ-ring) generated by $\mathscr{E}$ i.e. the smallest ring (σ-ring) that contains $\mathscr{E}$.

DEFINITION 16. A <u>measure</u> is an extended real valued, non-negative and countably additive set function μ defined on a σ-ring $\mathscr{S}$ of subsets of X such that $\mu(\phi) = 0$. That is if $E \in \mathscr{S}$ then $\mu(E) \geq 0$ and if $E_i \in \mathscr{S}$ for $i = 1,2,\ldots,$ and $E_i \cap E_j = \phi$ for $i \neq j$ then $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.

<u>Remark</u>. We could have defined a measure on a ring of subsets of X as well, and then extended the measure to the generated σ-ring. For our purposes it is enough if it is defined on a σ-ring. The interested reader is referred to P.R.Halmas.

It is immediate that we can define measurability, integrability, integrals of functions with respect to μ, in just the same way as we did for Lebesgue measure. The integral of a function f over $E \in \mathscr{S}$ is denoted by $\int_E f\, d\mu$ or $\int_E f(x)\, d\mu(x)$. The measure μ is finite if $\mu(X) < \infty$ and is σ-finite if X is a countable union of sets of finite μ-measures.

DEFINITION 17. A function μ, defined on a σ-ring $\mathscr{S}$ of subsets of X, is a <u>signed measure</u> if it is of the form

$$\mu(E) = \mu_1(E) - \mu_2(E) \qquad E \in \mathcal{S} ,$$

where μ_1 and μ_2 are measures and at least one of μ_1 and μ_2 is finite.

DEFINITION 18. A function μ, defined on a σ-ring $\mathcal{S}$ of subset of X, is a <u>complex measure</u> if it is of the form

$$\mu(E) = \mu_1(E) + i\mu_2(E), \quad E \in \mathcal{S}$$

where μ_1 and μ_2 are signed measures.

<u>Note</u>. To avoid confusion we use positive measures for ordinary measures and real measures for signed measures.

A complex measure is a linear combination of at most four positive measures. Integrals with respect to real measures and complex measures are defined in the obvious way in terms of integrals with respect to the appropriate positive measures. For example, if $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ where μ_i are positive measures then f is said to be integrable w.r.t μ if it is so w.r.t. μ_i, $i = 1,2,3,4$ and we have

$$\int f\, d\mu = \int f\, d\mu_1 - \int f\, d\mu_2 + i\int f\, d\mu_3 - i\int f\, d\mu_4.$$

If μ is a real or complex measure, then $|\mu|$, defined by

$$|\mu|(E)(\ ) = \sup \sum |\mu(E_i)| \quad \text{where} \quad E = \bigcup_{i=1}^{n} E_i$$

$E_i \cap E_j = \phi$, $i \neq j$, and the sup is taken over all such finite unions, is a positive measure. The measure $|\mu|$ is the <u>total variation</u> of μ. If μ is real then

$$\mu^+ = \frac{|\mu| + \mu}{2}, \quad \mu^- = \frac{|\mu| - \mu}{2}$$

are obviously positive measures. They are respectively called the <u>positive</u> and the <u>negative</u> variation of μ and also

$$\mu = \mu^+ - \mu^- \quad \text{and} \quad |\mu| = \mu^+ + \mu^- .$$

The representation of μ in terms of μ^+ and μ^- is the Jordan decomposition of μ.

The relation between measurability and continuity are most interesting and have been studied in locally compact spaces. Here we shall introduce some basic results of measurability theory in a locally compact space.

<u>Notation.</u> We denote by

X - a locally compact Hausdorff space.

$\mathcal{C}$ - the class of all compact subsets of X.

$\mathcal{C}_0$ - the class of all compact subsets of X which are also G_δ

$\mathcal{S}$ - the σ-ring generated by $\mathcal{C}$.

$\mathcal{S}_0$ - the σ-ring generated by $\mathcal{C}_0$

$\mathcal{U}$ - the class of all open sets contained in $\mathcal{S}$.

$\mathcal{U}_0$ - the class of all open sets contained in $\mathcal{S}_0$.

We shall call the elements of $\mathcal{S}$ the <u>Borel-sets</u> of X and the elements of $\mathcal{S}_o$, the <u>Baire-sets</u> of X. A real valued function on X is said to be Borel (Baire) measurable if it is measurable with respect to the σ-ring $\mathcal{S}$ ($\mathcal{S}_o$). It is immediate that every Baire-set is a Borel-set.

DEFINITION 19. A <u>Borel measure (Baire measure)</u> is a non-negative measure μ defined on the class $\mathcal{S}$ ($\mathcal{S}_o$) of Borel sets (Baire sets) such that $\mu(c) < \infty$ ($\mu(C_o) < \infty$) for every $C \in \mathcal{C}$ (for every $C_o \in \mathcal{C}_o$).

DEFINITION 20. A set $E \in \mathcal{S}$ is said to be <u>outer regular</u> with respect to the Borel measure μ, if

$$\mu(E) = \inf \{ \mu(U), E \subset U \text{ and } U \in \mathcal{U} \}$$

E is said to be <u>inner regular</u> if

$$\mu(E) = \sup \{ \mu(C), C \subset E \text{ and } C \in \mathcal{C} \}$$

and E is <u>regular</u> if it is both inner and outer regular. A measure μ is saidto be <u>regular</u> if each $E \in \mathcal{S}$ is regular. We can similarly defines regularity for Baire measures.

Every Borel measure μ defines a Baire measure ν in an obvious manner. ν is defined by

$$\nu(B_o) = \mu(B_o) \quad \text{for every Baire set } B_o.$$

It is not difficult to prove that every Baire measure is regular. Also every Baire measure can be extended to a unique regular Borel measure.

1.6. Measurable Transformations.

A measurable space is a set X together with a σ-ring $\mathcal{S}$ of subsets of X such that $\bigcup_{S\in\mathcal{S}} S = X$. We denote a measurable space by $(X, \mathcal{S})$. A measure space is a measurable space $(X, \mathcal{S})$ together with a measure μ defined on $\mathcal{S}$. A measure space is denoted by $(X, \mathcal{S}, \mu)$.

Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be measurable spaces. Let T be a transformation of X into Y i.e. T is a function which assigns a unique point of Y to every point of X. Then T assigns, in an obvious way, a real valued function f on X to every real valued function g on Y, f is defined by $f(x) = g(T(x))$, $x \in X$. We write $f = gT$.

DEFINITION 21. Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be measurable spaces and T a transformation from X into Y. Then T is said to be a <u>measurable transformation</u> if for every $F \in \mathcal{T}$, $T^{-1}(F)$ is in $\mathcal{S}$. That is, the inverse image of every measurable set of $(Y, \mathcal{T})$ is a measurable set of $(X, \mathcal{S})$.

We denote by $T^{-1}(\mathcal{T})$ the class of all subsets of X which have the form $T^{-1}(F)$ for some $F \in \mathcal{T}$. Then $T^{-1}(\mathcal{T})$ is a σ-ring contained in $\mathcal{S}$.

A measurable transformation T from $(X, \mathcal{S})$ into $(Y, \mathcal{T})$ assigns in an obvious way a measure ν on $\mathcal{T}$ to every measure μ on $\mathcal{S}$, ν is defined for every $F \in \mathcal{T}$ by $\nu(F) = \mu(T^{-1}(F))$. We write $\nu = \mu T^{-1}$.

We conclude this section by stating a theroem without proof. The reader is again referred to P.R.Halmos' book for proof.

THEOREM 28. If T is a measurable transformation from a measure space $(X, \mathcal{S}, \mu)$ into a measurable space $(Y, \mathcal{T})$ and if g is an extended real valued measurable function on Y, then

$$\int g \, d(\mu T^{-1}) = \int (gT) \, d\mu,$$

in the sense that if either integral exists so does the other and the two are equal.

COROLLARY. If F is a measurable subset of Y then an application of the above theorem to the function $\chi_F g$ yeilds the relation

$$\int_F g(y) \, d\mu T^{-1}(y) = \int_{T^{-1}(F)} g(T(x)) \, d\mu(x).$$

We observe that either side of the above relation is obtained from the other by the formal substitution $y = T(x)$.

TOPOLOGICAL VECTOR SPACES

2.1. Topological Vector Space.

Let $\mathbb{K}$ denote the real or complex number field.

DEFINITION 1. Let E be a vector space over $\mathbb{K}$. A topology τ on E is said to be compatible with the algebraic structure of E if the vector space operations $(x,y) \in E \times E \longrightarrow x+y \in E$ and $(\lambda,x) \in K \times E \longrightarrow \lambda x \in E$ are continuous. A topological vector space is a vector space ~~space~~ together with a compatible topology.

PROPOSITION 1. If E is a topological vector space, the mapping $x \in E \longrightarrow \alpha x+a \in E$ is a homeomorphism of E for each fixed $\alpha \neq 0$, $\alpha \in \mathbb{K}$ and $a \in E$.

PROOF. Since a continuous function of two variables is continuous in each of the variables, it follows from Definition 1 that the mappings $x \longrightarrow \alpha x$ for fixed $\alpha \in \mathbb{K}$ and $x \longrightarrow x+a$ for fixed $a \in E$ are continuous. Therefore the composite map $x \longrightarrow \alpha x \longrightarrow \alpha x+a$ is also continuous. If further $\alpha \neq 0$ and $y = \alpha x+a$, then $x = \frac{1}{\alpha}(y-a)$. Hence the mapping $x \rightarrow \alpha x+a$ is invertible and its inverse $y \longrightarrow \frac{1}{\alpha}\, y - \frac{1}{\alpha}\, a$ is continuous. Hence $x \longrightarrow \alpha x+a$ is a homeomorphism.

Remark 1. The mapping $x \in E \longrightarrow -x \in E$ is continuous and so is $(x,y) \in E \times E \longrightarrow x-y \in E$.

Remark 2. If U is a neighborhood of 0 then $U+a$ is a neighborhood of a and αU is a neighborhood of 0 for each $\alpha \neq 0$.

Remark 3. In a topological vector space, it is enough to deal with neighborhoods of 0.

DEFINITION 2. A subset A of a vector space E is <u>convex</u> if $x, y \in A \Longrightarrow \lambda x+\mu y \in A$ whenever $\lambda \geq 0$, $\mu \geq 0$ and $\lambda+\mu = 1$. It is <u>balanced</u> if $x \in A \Longrightarrow \lambda x \in A$ for $|\lambda| \leq 1$. It is <u>symmetric</u> if $A = -A$. It is <u>absolutely convex</u> if it is balanced and convex.

<u>Remark 1</u>. If A is convex, so is $x+\lambda A$ for each $x \in E$, $\lambda \in \mathbb{K}$.

<u>Remark 2</u>. Any intersection of convex sets is convex.

<u>Remark 3</u>. A is absolutely convex if and only if $x, y \in A \Rightarrow \lambda x+\mu y \in A$ when $|\lambda| + |\mu| \leq 1$.

<u>Exercise 1</u>. Let A be a nonempty absolutely convex set. Then

(i) $0 \in A$

(ii) $\lambda A \subset \mu A$ if $|\lambda| \leq |\mu|$

(iii) $\sum_{i=1}^{n} \lambda_i A = \left(\sum_{i=1}^{n} |\lambda_i|\right) A$ for all $\lambda_i \in \mathbb{K}$.

DEFINITION 3. A subset A of a vector space E is <u>absorbent</u> if for each $x \in E$ there is some $\lambda > 0$ such $x \in \mu A$ for all μ with $|\mu| \geq \lambda$.

<u>Remark 1</u>. A finite intersection of absorbent sets is absorbent.

<u>Remark 2</u>. An absolutely convex set is absorbent if and only if it spans E. This is equivalent to

$$E = \bigcup_{\lambda > 0} \lambda A \qquad \text{or} \qquad E = \bigcup_{n=1}^{\infty} nA$$

PROPOSITION 2. If V is a neighborhood of 0, then

(i) V is absorbent

(ii) there is a neighborhood W of 0 such that $W+W \subset V$.

(iii) there is a balanced neighborhood $W \subset V$.

PROOF. (i) Let $x \in E$ be given. If $f(\lambda) = \lambda x$ then f is continuous at $\lambda = 0$ and so there is $\epsilon > 0$ such that $\lambda \in \mathbb{K}$, $|\lambda| < \epsilon \Longrightarrow \lambda x \in V$. Then $x \in \mu V$ for $|\mu| \geq \epsilon^{-1}$.

(ii) The mapping $(x,y) \in E \times E \longrightarrow x+y \in E$ is continuous at $(0,0)$. There exist neighborhoods W_1, W_2 of 0 in E such that $x \in W_1$, $y \in W_2$ and $W_1 + W_2 \subset V$. Set $W = W_1 \cap W_2$.

(iii) The mapping $(\lambda, y) \in \mathbb{K} \times E \longrightarrow \lambda x \in E$ is continuous at $(0,0)$. So there exists $\delta > 0$ and a neighborhood U of 0 in E such that $|\lambda| \leq \delta$, $x \in U \Rightarrow \lambda x \in V$. Set $W = \{\lambda x \mid \lambda \in K, |\lambda| \leq \delta, x \in U\}$. Then W is a balanced neighborhood contained in V. (Verify)

Remark. Symmetric neighborhoods form a basis of neighborhoods at 0.

2.2.2 Seminorms.

DEFINITION 4. A seminorm p on a vector space E is a non-negative real valued function on E such that

(i) $p(x) \geq 0$

(ii) $p(\lambda x) = |\lambda| p(x)$

(iii) $p(x+y) \leq p(x) + p(y)$

for all $x, y \in E$, $\lambda \in K$. Clearly $p(0) = 0$ and $|p(y) - p(x)| \leq p(y-x)$.

A norm p on E is a seminorm such that $p(x) = 0 \Longrightarrow x = 0$.

DEFINITION 5. Let p be a seminorm on a vector space E. If $a \in E$, $r \geq 0$, the <u>open</u> ball $B_{r,p}(0)$ with centre a and radius r is defined by

$$B_{r,p}(a) = \left\{ x \in E \,\middle|\, p(x-a) < r \right\}$$

and the <u>closed</u> ball $\bar{B}_{r,p}(a)$ is defined by

$$\bar{B}_{r,p}(a) = \left\{ x \in E \,\middle|\, p(x-a) \leq r \right\}.$$

In a seminormed space E, we usually write $B_r(a)$ and $\bar{B}_r(a)$ for $B_{r,p}(a)$ and $\bar{B}_{r,p}(a)$ respectively.

DEFINITION 6. Let $\Gamma = \{ p_i \}_{i \in I}$ be a family of seminorms on a vector space E. The collection of all sets of the form $B_{r,p_i}(a)$ where $a \in E$, $r > 0$, $i \in I$ may be taken as a sub-basis for a topology τ_Γ on E and it is called the <u>natural topology</u> defined on E by Γ.

<u>Remark</u>. Each p_i is continuous in τ_Γ .

PROPOSITION 3. <u>Let</u> $\varepsilon > 0$, $i_1, i_2, \ldots, i_n \in I$ <u>and</u> $a \in E$. <u>Set</u>

$$V_{\varepsilon, i_1, \ldots\ldots, i_n}(a) = \bigcap_{j=1}^{n} B_{\varepsilon, p_{i_j}}(a)$$

<u>and</u>

$$\bar{V}_{\varepsilon, i_1, \ldots, i_n}(a) = \bigcap_{j=1}^{n} \bar{B}_{\varepsilon, p_{i_j}}(a) .$$

<u>Then the collection of all</u> $V_{\varepsilon, i_1, \ldots, i_n}(a)$ $(\bar{V}_{\varepsilon, i_1, \ldots, i_n}(a))$ <u>is a basis of open (closed) neighborhoods at a for</u> τ_Γ. <u>Also E is a topological vector space with respect to</u> τ_Γ.

Proof. Let $a \in E$, $b_1, b_2, \ldots . b_n \in E$, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n > 0$ $i_1, i_2, \ldots, i_n \in I$ be such that

$$a \in \bigcap_{j=1}^{n} B_{\varepsilon_j, p_{i_j}}(b_j)$$

Let $\delta = \min_j \{ \varepsilon_j - p_{i_j}(a-b_j) \}$. If $x \in V_{\delta, i_1, \ldots, i_n}(a)$ then $p_{i_j}(x-a) < \delta$ so that $p_{i_j}(x-b_j) \leq p_{i_j}(x-a) + p_{i_j}(a-b_j) < \delta + p_{i_j}(a-b_j) \leq \varepsilon_j$ so that

$$V_{\delta, i_1, \ldots, i_n}(a) \subseteq B_{\varepsilon_j, p_{i_j}}(b_j) \text{ for each } j.$$

Hence

$$a \in V_{\delta, i_1, \ldots, i_n}(a) \subset \bigcap_{j=1}^{n} B_{\varepsilon_j, p_{i_j}}(a).$$

Further we have

$$a \in V_{\frac{\delta}{2}, i_1, \ldots, i_n}(a) \subset \overline{V}_{\frac{\delta}{2}, i_1, \ldots, i_n}(a) \subseteq V_{\delta, i_1, \ldots, i_n}(a)$$

This will prove the first part of the proposition.

It remains to show that τ_Γ is compatible with the algebraic structure of E. Let $a, b \in E$. Set $c = a+b$. Consider a sub-basis neighborhood $B_{\varepsilon, p_i}(c)$ of c. If $x \in B_{\varepsilon/2, p_i}(a)$, $y \in B_{\varepsilon/2, p_i}(b)$ then

$$p_i(x+y-c) \leq p_i(x-a) + p_i(y-b) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

so that $x+y \in B_{\varepsilon,p_i}(c)$. This proves the continuity of addition. Let $\alpha \in \mathbb{K}$, $a \in E$ set $b = \alpha a$. Pick a sub-basis neighborhood $B_{\varepsilon,p_i}(b)$ of b. If $\varepsilon > 0$, $\gamma > 0$, $\delta > 0$, $|\lambda - \alpha| < \gamma$, $x \in B_{\delta,p_i}(a)$, then

$$p(\lambda x - \alpha a) = p_i\left[\lambda(x-a) + (\lambda-\alpha)\ a\right] \leq |\lambda|\ p_i\ (x-a) + |\lambda-\alpha|\ p_i(a)$$

Now $|\lambda| \leq |\lambda-\alpha| + |\alpha| < \gamma + |\alpha|$ so that $p_i(\lambda x - \alpha a) < (\gamma + |\alpha|)\,\delta + \gamma p_i(a)$ $\leq \varepsilon$, if we choose γ, δ small enough. i.e., $\lambda x \in B_{\varepsilon,p_i}(a)$.

<u>Exercise 2</u>. The family $\Gamma = \{p_i\}_{i \in I}$ of seminorms is directed if for any $i_1, i_2 \in I$ there exists $i \in I$, $\lambda \in \mathbb{R}$, $\lambda > 0$ such that $p_{i_j} \leq \lambda p_i$, $j = 1,2$. If Γ is a directed family of seminorms on E, show that for each fixed $a \in E$, the collection of all $B_{\varepsilon,p_i}(a)$ ($\bar{B}_{\varepsilon,p_i}(a)$) is a basis of open (closed) neighborhoods at a, with respect to τ_Γ, where $\varepsilon > 0$ and $i \in I$.

2.3. <u>Locally convex spaces</u>.

DEFINITION 7. Let V be an absorbent convex set in a vector space E. The <u>Minkowsky function</u> p on V is defined by

$$p(x) = \inf\left\{\lambda \in \mathbb{R} \,|\, \lambda > 0,\ x \in \lambda V\right\}.$$

PROPOSITION 4. <u>If</u> p <u>is the Minkowsky function of</u> <u>V</u>, then

(i) $0 \leq p(x) < \infty$ for $x \in E$.

(ii) $p(x+y) \leq p(x)+p(y)$.

(iii) $p(\lambda x) = \lambda p(x)$, $x \in E$, $\lambda \in \mathbb{R}$, $\lambda \geq 0$.

(iv) $p(\lambda x) = |\lambda| p(x)$, $x \in E$, $\lambda \in \mathbb{K}$, if V is balanced.

(v) $\{x \in E \mid p(x) < 1\} \subset V \subset \{x \in E \mid p(x) \leq 1\}$

(vi) If E is a topological vector space, then

$$\{x \in E \mid p(x) < 1\} = V \text{ if } V \text{ is open, and}$$
$$\{x \in E \mid p(x) \leq 1\} = V \text{ if } V \text{ is closed.}$$

PROOF. i) is immediate from the absorbing property of V. ii) Let $x,y \in E$. Choose $\lambda,\mu \in \mathbb{R}$, $\lambda,\mu > 0$ such that $\lambda < p(x)+\varepsilon$, $\mu < p(y)+\varepsilon$, $x \in \lambda V$, $y \in \mu V$. Then $x+y \in (\lambda+\mu)V$ so that $p(x+y) \leq \lambda+\mu < p(x)+p(y)+2\varepsilon$. Since ε is arbitrary $p(x+y) \leq p(x)+p(y)$. iii) As in ii), it is easy to verify that $p(\lambda x) \leq \lambda p(x)$ if $\lambda > 0$. Then $\frac{1}{\lambda} > 0$ and then $p(x) = p(\frac{1}{\lambda}\cdot\lambda x) \leq \frac{1}{\lambda}p(\lambda x)$ so that $\lambda p(x) \leq p(\lambda x)$. Hence $p(\lambda x) = \lambda p(x)$. If $\lambda = 0$, $p(\lambda x) = \lambda p(x) = 0$ is trivial.

iv) Suppose V is balanced. Let $\lambda \in \mathbb{K}$ with $|\lambda| = 1$. Then $\lambda V = V$. Let $x \in E$. Pick $\mu > 0$. Then $x \in \mu V$ if and only if $\lambda x \in \mu(\lambda V) = \mu V$ and hence $p(\lambda x) = p(x)$. If $\mu = \lambda\nu$ where $\nu = |\mu|$, $|\lambda| = 1$ then $p(\mu x) = p(\lambda\nu x) = p(\nu\lambda x) = \nu p(\lambda x) = |\mu|\, p(x)$.

v) If $x \in E$, $p(x) < 1$, there is $\lambda \in \mathbb{R}$, $0 < \lambda < 1$, such that $x \in \lambda V$. Since $0, \frac{x}{\lambda} \in V$ then $x = \lambda\frac{x}{\lambda} + (1-\lambda)0 \in V$ by convexity of V. Moreover, if $x \in V = 1V$ then $p(x) \leq 1$.

vi) Let V be open and $x \in V$. Now because $\lambda \in \mathbb{K} \longrightarrow \lambda x \in E$ is continuous at $\lambda = 1$, we can choose $\delta > 0$ such that $|\lambda - 1| \leq \delta \Rightarrow \lambda x \in V$. Then $(1+\delta)x \in V \Rightarrow p(x) \leq \frac{1}{1+\delta} < 1$. Now let V be closed and $x \in E$, $p(x) \leq 1$. If $\Theta \in K$, $|\Theta| < 1$ then $p(\Theta x) < 1$ hence $\Theta x \in V$. Since $\Theta x \longrightarrow x$ as $\Theta \longrightarrow 1$ and V is closed, we get $x \in V$.

DEFINITION 8. A topological vector space E is locally convex if there exists a basis of convex neighborhoods of 0.

PROPOSITION 5. A seminorm on a topological vector space E is continuous on E if and only if p is continuous at 0. (Exercise)

PROPOSITION 6. On a locally convex space, the absolutely convex neighborhodds form a basis of neighborhoods of 0. (Exercise)

PROPOSITION 7. If E is a topological vector space whose topology is defined by a family $\Gamma = \{p_i\}_{i \in I}$ of seminorms, then E is locally convex space and p_i is continuous. Conversely, if E is a locally convex space, its topology is defined by the collection of all continuous seminorms.

PROOF. Suppose the topology of E is defined by Γ. If p is a seminorm on E, $a \in E$, $\varepsilon > 0$ then the set $\{x \mid p(x-a) < \varepsilon\}$ is convex. By the definition of τ_Γ, the topology of E has an open sub-basis, hence an open basis, formed by convex sets, so E is locally convex. Moreover, each p_i is continuous.

Conversely let E be locally convex with topology τ. Let Γ be the collection of all τ-continuous seminorms on E. If $p \in \Gamma$, $a \in E$, $\varepsilon > 0$ then the set $\{x \in E \mid p(x-a) < \varepsilon\}$ is τ-open since $x \to p(x-a)$ is τ-continuous. Then it follows that every τ_Γ-open set is τ-open. Hence $\tau_\Gamma \leq \tau$. Let V be a neighborhood of 0. Choose an absolutely convex τ-neighborhood U of 0 such that $U \subset V$ and p the Minkowsky function of U. Then p is a seminorm. Moreover, $x \in U \Rightarrow p(x) \leq 1$, hence $x \in rU \Rightarrow p(x) \leq r$ for $r > 0$. Hence p is continuous at 0 and hence it is continuous on E. So $p \in \Gamma$. On the other hand, $x \in E$, $p(x) \leq 1 \Rightarrow x \in U \subset V$ i.e. $\{x \in E \mid p(x) < 1\} \subset V$. Therefore V contains a τ_Γ-open subset containing 0 and is a τ_Γ neighborhood of 0. Thus every τ-neighborhood of 0 is a τ_Γ-neighborhood of 0. Hence $\tau \leq \tau_\Gamma$. Therefore $\tau = \tau_\Gamma$.

PROPOSITION 8. *If $\mathcal{U}$ is a base of neighborhoods of 0 in a topological vector space E, then E is separated (i.e. Hausdorff) if and only if*

$$\bigcap_{U \in \mathcal{U}} U = \{0\}$$

In particular, if the topology of E is τ_Γ, then E is separated if and only if for each nonzero $x \in E$ there is some $p \in \Gamma$ such that $p(x) > 0$.

PROOF. If E is separated and $x \neq 0$, there is some $U \in \mathcal{U}$ with $x \notin U$ so that

$$\bigcap_{U \in \mathcal{U}} U = \{0\}$$

Conversely, if $\bigcap U = \{0\}$ and $x \neq y$, then there is some U such that $x-y \notin U$. By Proposition 2 there is a balanced neighborhood W such that $W+W \subset U$. Then $x+W$ and $y+W$ are disjoint neighborhoods of x and y respectively. Hence E is separated

PROPOSITION 9. <u>A locally convex space E is metrizable if and only if it is separated and there is a countable base of neighborhoods of 0. The topology of metrizable space can always be defined by a metric that is invariant under translation.</u>

PROOF. If E is metrizable, it is separated and has a countable base of neighborhoods of 0.

If E has a countable base of neighborhoods of 0, each neighborhood contains an absolutely convex neighborhood and so there is a base $\{U_n\}$ of absolutely convex neighborhoods. Let p_n be the Minkowsky function of U_n. Put

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} \inf(p_n(x), 1)$$

Then $f(x+y) \leq f(x)+f(y)$, $f(-x) = f(x)$ and if $f(x) = 0$, then $p_n(x) = 0$ for all n and so $x = 0$, since E is separated. Define d by

$$d(x,y) = f(x-y)$$

then d is a metric and $d(x+z,y+z) = d(x,y)$ so that d is invariant under translation. In this metric topology, the sets

$$V_n = \left\{ x \mid f(x) < 2^{-n} \right\}$$

form a base of neighborhoods. But V_n is open in the original topology since each p_n and so f is continuous. Also $V_n \subset U_n$. Hence d defines the original topology on E.

COROLLARY. If the topology on the separated space E is the coarsest convex topology making a sequence of absolutely convex sets neighborhoods, then E is metrizable.

2.4. Linear Mappings.

Let E and F be vector spaces over $\mathbb{K}$. $f:E \to F$ is *linear* if

$$f(\alpha x+\beta y) = \alpha f(x)+\beta f(y)$$

f is 1:1 if $f^{-1}(0) = \{0\}$. In general $f^{-1}(0)$ is a sub-space of E.

Let L denote the set of linear mappings of E into F. L is a vector space over $\mathbb{K}$ if

$$(f+g)(x) = f(x)+g(x)$$
$$(\lambda f)(x) = \lambda f(x)$$

A linear mapping of E into $\mathbb{K}$ is called a <u>linear form</u> on E. E* will denote the set of all linear forms on E.

PROPOSITION 10. <u>A linear mapping</u> $f: E \to F$ <u>is continuous if and only if it is continuous at</u> 0.

PROOF. For each $a \in E$, a neighborhood of $f(a)$ is given by $f(a)+V$ where V is a neighborhood of 0 in F. If f is continuous at 0, there exists a neighborhood U of 0 in E such that $f(U) \subset V$. Then $f(a+V) \subset f(a)+V$ and f is continuous at a. The converse is trivial.

COROLLARY. <u>If</u> E <u>and</u> F <u>are normed linear spaces and</u> $f: E \to F$ <u>is linear, then</u> f <u>is continuous if and only if there is</u> $\alpha > 0$ <u>such that</u> $\|f(x)\| \leq \alpha \|x\|$ <u>for all</u> $x \in E$.

PROOF. Exercise.

<u>Exercise 3</u>. If f is a non-zero linear form on E, then $f^{-1}(0)$ is a maximal subspace of E. Conversely to each maximal subspace H of E there exists a linear form f on E such that $f^{-1}(0) = H$.

PROPOSITION 11. *A linear form* f *on a topological vector space* E *is continuous if and only if* $f^{-1}(0)$ *is closed in* E.

PROOF. Suppose $f^{-1}(0) = H$ is closed. Set $V = \{x \mid |f(x)| < 1\}$. If $f \neq 0$, choose $a \in E$ such that $f(a) = 1$. Then there is a balanced neighborhood U such that $(a+U) \cap H = \emptyset$ We assert that $U \subset V$. Suppose that $x \in U$ and $|f(x)| \geq 1$. Then $y = \frac{-x}{f(x)} \in U$ and $f(a+y) = 0$ so that $(a+U) \cap H \neq \emptyset$. Contradictions. Then $x \in \delta U \Rightarrow |f(x)| < \delta$ for $\delta > 0$. Hence f is continuous at 0 and hence continuous on E.

DEFINITION 9. The *dual space* E' of a topological vector space E is the subspace of E^* consisting of all continuous linear forms on E.

PROPOSITION 12. *Let* A *be an open convex subset of a locally convex space* E *and let* M *be a vector sub-space of* E *such that* $A \cap M = \emptyset$. *Then there exists a closed hyperplane containing* M *and not meeting* A.

LEMMA 1. *If* M *is a vector subspace of* E, *so is* $\overline{M}$.

LEMMA 2. *In a topological vector space, a hyperplane is either closed or dense.*

PROOF. Exercise.

LEMMA 3. Suppose that E is a real locally convex space, A an open convex subset of E and H a vector subspace not meeting A. Then H is either a hyperplane or there exists $x \notin H$ such that the vector subspace spanned by x and H does not meet A.

PROOF. Let $C = H + \bigcup_{\lambda > 0} \lambda A$. Then C is open $-C = H + \bigcup_{\lambda < 0} \lambda A$ and $C \cap -C = \phi$. For if $x \in C \cap -C$, then $x = h + \lambda a = h' - \lambda' a'$ for some $h, h' \in H$, $a, a' \in A$ and $\lambda, \lambda' > 0$ and so $\lambda a + \lambda' a' \in H$. But since A is convex, $\lambda a + \lambda' a' \in (\lambda + \lambda') A$ which does not meet H.

i) Suppose that $H \cup C \cup -C \neq E$. There is some $x \notin H$ with $x \notin C \cup -C$. If the vector subspace spanned by x and H meets A say in y, then for some $\lambda \neq 0$, $x \in \lambda y + H \subset C \cup -C$. Hence the vector subspace does not meet A.

ii) Suppose $H \cup C \cup -C = E$. If H is not a hyperplane there is some point $a \in C$ so that H and a together does not span E, hence there is some point $b \in -C$ not in the span of H and a. Let $f(\lambda) = (1-\lambda)a + \lambda b (0 \leq \lambda \leq 1)$. Now f is continuous and C is open and so $I = f^{-1}(C)$ and $J = f^{-1}(-C)$ are open in $[0,1]$. Also $0 \in I$, $1 \in J$ and $I \cap J = \emptyset$. Let

$$\alpha = \operatorname{Sup} \{ \lambda \mid \lambda \in I \}$$

Then $\alpha \in \overline{I} \cap (\overline{\sim I}) \subset (\overline{\sim J}) \cap (\overline{\sim I}) = \sim J \cap \sim I$. Hence $f(\alpha) \notin C \cup -C$. Thus $f(\alpha) \in H$ i.e. $(1-\alpha)a + \alpha b \in H$. But b is not in the span of a and H. Therefore H is a hyperplane.

LEMMA 4. <u>Suppose that</u> E <u>is a complex vector space and</u> H <u>is a real hyperplane in</u> E. <u>Then</u> $H \cap iH$ <u>is a complex hyperplane in</u> E.

PROOF. Suppose $a \notin H \cap iH$ and suppose for example $a \notin H$. Then $ia \notin iH$, which is a real hyperplane and so $a = \alpha ia+b$ with α real and $b \in iH$, then $(1+\alpha_i)b = (1+\alpha^2)a \notin H$, and so $b \notin H$. Now if $x \in E$, $x = \beta b+y$ with β real and $y \in H$, and then $y = \gamma ib+z$ with γ real and $z \in iH$. Hence $z \in H$, thus $x = (\beta+\gamma i)b+z$ say with $z \in H \cap iH$. Therefore $H \cap iH$ is a complex hyperplane in E.

PROOF OF PROPOSITION 12. First consider the case when E is a real vector space. Let $\mathcal{E}$ be the set of all vector subspaces of E containing M and not meeting A. Apply the maximal axiom to the chain $\mathcal{C} = \{M\}$, there is a maximal chain $\mathcal{M}$ in E with $\mathcal{C} \subset \mathcal{M} \subset \mathcal{E}$. Let H be the union of all sets of $\mathcal{M}$. Then clearly H is a vector subspace of E not meeting A. By Lemma 3, it is a hyperplane, because the other possibility would contradict the maximality of $\mathcal{M}$. Also H is closed because otherwise it is dense in E and meets every open set including A.

If E is a complex vector space, it is also a real vector space and so there is a real closed hyperplane K containing M and not meeting A. Then $H = K \cap iK$ is a complex closed hyperplane containing $M \cap iM = M$ and not meeting A.

COROLLARY. *Every closed vector subspace of a convex space is the intersection of closed hyperplanes containing it.*

2.5. Extension of a linear form.

PROPOSITION 13. *Let* p *be a real valued function on the real vector space* E *such that*

$p(x+y) \leq p(x)+p(y)$ *and* $p(\lambda x) = \lambda p(x)$, $\lambda \geq 0$, $\lambda \in \mathbb{R}$.

Let $F \subset E$ *be a vector subspace and* g *a linear form on* F *satisfying* $g(x) \leq p(x)$ *for any* $x \in F$. *Then there is a linear form* f *on* E *extending* g *and satisfying* $f(x) \leq p(x)$ *for all* $x \in E$.

PROOF. Let $a \in E \setminus F$ and $F_1 = F \oplus \mathbb{R}a$, the vector subspace generated by a and F. Each element $y \in F_1$ has the unique representation $y = x+\lambda a$ where $x \in F$, $\lambda \in \mathbb{R}$. Define g_1 by $g_1(y) = g(x)+\lambda\alpha$ where α is to be determined. Then g_1 is a linear form on F_1 extending g. We now choose α so that $g_1(y) \leq p(y)$ for all $y \in F_1$, or $g(x)+\lambda\alpha \leq p(x+\lambda\alpha)$. If $x_1, x_2 \in F$, we have

$$g(x_1)+g(x_2) = g(x_1+x_2) \leq p(x_1+x_2) \leq p(x_1-a)+p(x_2+a)$$

so that $g(x_1) - p(x_1-a) \leq p(x_2+a) - g(x_2)$.

Let $\quad \mu = \operatorname{Sup} \{ g(x_1) - p(x_1 - a) \mid x_1 \in F \}$

$$\nu = \inf \{ f(x_2 + a) - p(x_2) \mid x_2 \in F \}$$

Now choose α such that $\mu \leq \alpha \leq \nu$. We assert that this α will work.

Now let $\emptyset$ be such that (1) $\emptyset$ is a linear form defined on a vector subspace $D_\emptyset$ of E containing F (2) $\emptyset$ is an extension of g,(3) $\emptyset$ satisfies $\emptyset(x) \leq p(x)$ for all $x \in D_\emptyset$. The set Φ of all such $\emptyset$ is partially ordered as follows: If $\emptyset_1, \emptyset_2 \in \Phi$, then $\emptyset_1 \leq \emptyset_2$ if $D_{\emptyset_1} \subset D_{\emptyset_2}$ and $\emptyset_2$ extends $\emptyset_1$. The maximal principle applied to Φ gives a linear form g_1 which extends g and $g_1(x) \leq p(x)$. If $D_{g_1} \neq E$, we can extend g_1 as in the first paragraph which will contradict the maximality of g_1. This completes the proof.

LEMMA. <u>Let</u> q <u>and</u> p <u>be seminorms on the vector space</u> E. <u>Then</u> $q \leq p$ <u>if and only if</u> $x \in E$, $p(x) \leq 1 \Rightarrow q(x) \leq 1$.

PROOF. If $q \leq p$, then clearly $p(x) \leq 1 \Rightarrow q(x) \leq 1$. Conversely assume $p(x) \leq 1 \Rightarrow q(x) \leq 1$. Let $x \in E$ such that $p(x) > 0$. Then $p\left(\frac{x}{p(x)}\right) = 1 \Rightarrow q\left(\frac{x}{p(x)}\right) \leq 1$, hence $q(x) \leq p(x)$. If $p(x) = 0$, then $p(\lambda x) = 0 \leq 1 \Rightarrow |\lambda| \cdot q(x) = q(\lambda x) \leq 1 \Rightarrow q(x) \leq \frac{1}{|\lambda|}$ for every $\lambda \in \mathbb{K}$, $\lambda \neq 0$. Letting $\lambda \rightarrow \infty$, we get $q(x) = 0$ also. Hence $q(x) = p(x) = 0$.

PROPOSITION 14. (Hahn-Banach) Suppose that p is a seminorm on a vector space E and that f is a linear form on a vector subspace M of E such that $|f(x)| \leq p(x)$ for all $x \in M$. Then there is a linear form f_1 on E extending f with $|f_1(x)| \leq p(x)$ for all $x \in E$.

PROOF. Let E have the topology determined by the seminorm p. Let $U = \{x \mid p(x) < 1\}$. Suppose $f \neq 0$. Let $a \in E$ suchthat $f(a) = 1$ and $A = a+U$. Then A is open and convex. Put $N = f^{-1}(0)$. If $x \in U$, then $|f(x)| < 1$. Hence $A \cap N = \emptyset$. There is then a closed hyperplane H containing N but not meeting A. Let f_1 be the linear form on E with $H = f_1^{-1}(0)$ and $f_1(a) = 1$. f_1 then extends f.

COROLLARY 1. Any continuous linear form defined on a vector subspace of a locally convex space has a continuous extension.

COROLLARY 2. If $a \in E$ and p is a seminorm on E, there is a linear form f on E with $|f(x)| \leq p(x)$ for all $x \in E$ and $f(a) = p(a)$.

COROLLARY 3. If E is separated with dual E', then $f(a) = 0$ for all $f \in E' \Longrightarrow a = 0$.

PROPOSITION 15. *Let* E *be a locally convex space. Suppose that* A *and* B *are disjoint convex sets and* A *is open. Then there is a continuous linear form* f *such that* f(A) *and* f(B) *are disjoint*.

PROOF. The set A-B is open and convex and does not contain the origin. There is a closed hyperplane $H = f^{-1}(0)$ containing the vector subspace $\{0\}$ and not meeting A-B. The linear form f is continuous since H is closed and f(A) and f(B) do not meet.

LEMMA. *Any non-zero linear form on* E *is an open map*.

PROOF. Let A be an open set in E and $x \in A$. Then $A-x$ contains a neighborhood of 0 and so is absorbent. If f is a nonzero linear form on E there is some $a \in E$ with $f(a) = 1$ and then there is some $\alpha > 0$ with $\lambda a \in A-x$ for $|\lambda| \leq \alpha$. Then $f(x)+\lambda \in f(A)$ for $|\lambda| \leq \alpha$. Hence f(A) is open.

COROLLARY 1. *If* B *is a convex subset of a locally convex space and* $a \notin \overline{B}$ *then there is continuous linear form* f *with* $f(a) \notin \overline{f(B)}$.

PROOF. Since $a \notin \overline{B}$ there is an absolutely convex neighborhood U of 0 such that $(a+U) \cap B = \emptyset$. By Proposition 15, there exists a continuous linear form f such that $f(a+U) \cap f(B) = \emptyset$. But $f(a+U)$ is open. Hence $f(a) \notin \overline{f(B)}$

COROLLARY 2. *If* B *is an absolutely convex subset of a locally convex space and* $a \notin \overline{B}$ *, then there is a continuous linear form* f *such that* $|f(x)| \leq 1$ *for all* $x \in B$ *and* $f(a) > 1$.

PROOF. By Corollary 1, there is a continuous linear form g such that $\overline{g(a)} \not\subset \overline{g(B)}$. Then $\overline{g(B)}$ is an absolutely convex set so that $\sup \{ |g(x)| \mid x \in B \} < |g(a)|$. Let $\alpha = \sup \{ |g(x)| \mid x \in B \}$. Set $f = \frac{|g(a)|}{\alpha g(a)} g$ if $\alpha \neq 0$ and $f = \frac{2}{g(a)} g$ if $\alpha = 0$.

COROLLARY 3. *Let* E *be a real locally convex space. If* A *and* B *are disjoint convex subsets of* E *and* A *is open then there is a continuous linear form* f *and a constant* α *with* $f(x) > \alpha$ *for all* $x \in A$ *and* $f(x) \leq \alpha$ *for all* $x \in B$.

PROOF. By Proposition 15, there exists continuous linear form f with $f(A) \cap f(B) = \emptyset$. $f(A)$ and $f(B)$ are convex sets and $f(A)$ is open in $\mathbb{R}$. We may suppose that

$$\sup \{ f(x) \mid x \in B \} \leq \inf \{ f(x) \mid x \in A \}$$

(if necessary multiply by -1). Put $\alpha = \sup \{ f(x) \mid x \in B \}$, $f(x) \leq \alpha$ for all $x \in B$. Since A is open $f(x) > \alpha$ for all $x \in A$.

PROPOSITION 16. *Let* E *be a real locally convex space. Suppose that f is a linear form on a vector subspace M of E and that f(x) > 0 on the (non-empty) intersection of M with an open convex set A. Then there is a linear form* f_1 *extending f with* $f_1(x) > 0$ *on* A.

PROOF. Let $N = f^{-1}(0)$. Then $A \cap N = \emptyset$ (since $f > 0$ in $A \cap M$). By Proposition, there is ahyperplane H containing N not meeting A. Let $a \in A \cap M$. Define f_1 by $f_1^{-1}(0) = H$ and $f_1(a) = f(a)$. Then f_1 extends f. We shall now show that $f_1(x) > 0$ for all $x \in A$. Suppose not.
Let $f_1(a) = \lambda > 0$ and $f_1(b) = -\mu \leq 0$, $b \in A$. Since A is convex.

$$\frac{\mu a+\lambda b}{\lambda+\mu} \in A \text{ and } f_1\left(\frac{\mu a}{\lambda+\mu} + \frac{\lambda b}{\lambda+\mu}\right) = \frac{\mu}{\lambda+\mu} f_1(a) + \frac{\lambda}{\lambda+\mu} f_1(b)$$

Therefore $\frac{\mu a}{\lambda+\mu} + \frac{\lambda b}{\lambda+\mu} \in H \therefore H \cap A \neq \emptyset$ contradiction.

2.6. Duality and weak topology.

Let E be a locally convex space and E^* the algebraic dual of E, E' is the continuous dual of E. $\because$ E' is a vector subspace of E^*.

To each $x \in E$, define $\tilde{x} : E' \longrightarrow \mathbb{K}$ by $\tilde{x}(f) = f(x)$. Then $\tilde{x}$ is a linear form on E' i.e. $\tilde{x} \in E'^*$. Thus we have a map $x \in E \longrightarrow \tilde{x} \in E'^*$. If E is separated, $\tilde{x}(f) = \tilde{y}(f)$ for all $f \in E'$ if and only if $f(x) = f(y)$ for all $f \in E'$, if and only if $x = y$. Then E is isomorphic to a subspace of E'^*.

PROBLEM. Topologize E' such that E is the continuous dual of E'.

NOTATION. $x, y, z, \ldots$ will denote the elements of E, whereas $x', y', z', \ldots$ will denote elements of E'. Write $\langle x, x' \rangle$ for $x'(x)$. Then $\langle x, x' \rangle$ is a bilinear form on $E \times E'$. Also we have

1. If $x \neq 0$, there is an $x' \in E'$ such that $\langle x, x' \rangle \neq 0$
2. If $x' \neq 0$, there is an $x \in E$ such that $\langle x, x' \rangle \neq 0$.

This is the same as

(1') $\langle x, x' \rangle = 0$ for all $x' \in E' \Longrightarrow x = 0$

(2') $\langle x, x' \rangle = 0$ for all $x \in E' \Longrightarrow x' = 0$.

Let E, E' be vector spaces over the same field $\mathbb{K}$. Let $\langle x, x' \rangle$ be a nondegenerate bilinear form on $E \times E'$. $x' \in E'$ gives rise to a linear form on E given by $f(x) = \langle x, x' \rangle$. Then f is 1:1 and E' is also isomorphic to a vector subspace of E^*. Similarly E is also isomorphic to a vector subspace of E'^*.

DEFINITION. (E, E') is called a <u>dual pair</u>. If (E, E') is a dual pair, so is (E', E).

Examples 1. If E is a separated locally convex space with dual E', then (E,E') is a dual pair.

Examples 2. For a vector space E with algebraic dual E^*, (E,E^*) is a dual pair.

Let (E,E') be a dual pair. To each $x' \in E'$, set $p(x) = |\langle x,x'\rangle|$ for all $x \in E$. Then p is a seminorm on E. The coarsest topology on E making all these seminorms continuous is the weak topology on E determined by E' and is denoted by $\sigma(E,E')$. It is the coarsest topology on E for which all the linear forms in E' are continuous. In $\sigma(E,E')$ the sets $\{x \mid \sup_{1\le i\le n} |\langle x,x_i'\rangle| < \varepsilon\}$ $x_i' \in E'$, form a base of closed neighborhoods of 0. Now $\sigma(E,E')$ is convex and separated.

The dual of E under $\sigma(E,E')$ contains E'. We shall show that it is precisely E'.

LEMMA. *If f_o, f_1, ... f_n are linear forms defined on a vector space E, then either f_o is a linear combination of f_1, ... f_n or there is $a \in E$ such that $f_o(a) = 1$ and $f_i(a) = 0$ for $i = 1,2,..n$.*

PROOF. For n = 0, the result is trivial. Assume it is true for n-1. Then for each i, $1 \le i \le n$, f_i is not a linear combination of f_1, ... f_{i-1}, f_{i+1},...f_n. Then by induction hypothesis, there exists $a_j \in E$, $j = 1,2,..n$ such that

$f_i(a_j) = 0$ for $i \neq j$ and $f_i(a_i) = 1$. For each $x \in E$,

$$x - \sum_{1 \leq i \leq n} f_i(x) a_i \quad \in \bigcap_{1 \leq i \leq n} f_i^{-1}(0) = N$$

Then there is an element $a \in N$ such that $f_o(a) = 1$ or $f_o(y) = 0$ for all $y \in N$. In the latter case we have

$$f_o(x) = \sum_{1 \leq i \leq n} f_o(a_i) f_i(x) \quad \text{for all } x \in E$$

which implies that

$$f_o = \sum_{1 \leq i \leq n} \lambda_i f_i \quad \text{where } \lambda_i = f_o(a_i)$$

COROLLARY. *If* $f_1, \ldots f_n$ *are linearly independent linear forms on a vector space* E, *then there are elements* $a_1, \ldots a_n \in E$ *such that* $f_i(a_j) = \delta_{ij}$.

PROPOSITION 17. *For a dual pair* (E,E'), *the dual of* E *under* $\sigma(E,E')$ *is* E'.

PROOF. Let f be a linear form on E continuous under $\sigma(E,E')$. Then $|f(x)| \leq \alpha < 1$ on some neighborhood $U = \{x \mid \sup_{1 \leq i \leq n} |\langle x, x_i' \rangle| \leq 1\}$ where $x_i' \in E'$. Then by lemma, f is either a linear combination of $x_1', x_2' \ldots x_n'$ or there is some $a \in E$ such that $f(a) = 1$ and $x_i'(a) = 0$ for $i = 1,2,\ldots n$. Then $a \in U$ and $f(a) > \alpha$, contradiction. Hence

$$f = \sum_{1 \leq i \leq n} \lambda_i x_i'$$

PROPOSITION 18. *If (E,E') is a dual pair, and A is a convex subset of E, then $\bar{A}$ is the same for every topology of the dual pair (E,E').*

PROOF. We shall show that if the closure $\bar{A}$ under any topology ξ is the same as the closure $\bar{A}(\sigma)$ under $\sigma(E,E')$. Since ξ is finer than σ, $\bar{A} \subset \bar{A}(\sigma)$. Let $a \notin \bar{A}$. Then there is a continuous linear form f such that $f(a) \notin \overline{f(A)}$ i.e. there exists $x' \in E'$ such that $\langle a, x' \rangle \notin \overline{\langle A, x' \rangle}$. There is a $\delta \geq 0$ such that $|\langle a-x, x' \rangle| \geq \delta$ for all $x \in A$. Let $U = \{x \mid |\langle x, x' \rangle| < \delta\}$. Then U is a neighborhood in σ and $a+U$ does not meet A. This means $a \notin \bar{A}(\sigma)$ i.e. $\bar{A}(\sigma) \subset \bar{A}$.

2.7. Polar Sets.

DEFINITION. Let (E,E') be a dual pair. If A is a subset of E, the subset of E' consisting of those x' such that

$$\sup_{x\in A} |\langle x,x'\rangle| \leq 1$$

is called the *polar* of A and is denoted by A°

PROPOSITION 19. *Let (E,E') be a dual pair. Then polars in E' of subsets of E have the following properties*

(i) A° is absolutely convex and $\sigma(E',E)$-closed.

(ii) If $A\subset B$, then $B^{\circ}\subset A^{\circ}$.

(iii) If $\lambda \neq 0$, then $(\lambda A)^{\circ} = \frac{1}{|\lambda|}A^{\circ}$

(iv) $\left(\bigcup_{\alpha} A_{\alpha}\right)^{\circ} = \bigcap_{\alpha} A_{\alpha}^{\circ}$

PROOF. Exercise.

Notice that $A^{\circ} = \bigcap_{x\in A} \{x' \mid |\langle x,x'\rangle| \leq 1\}$

There are important special cases of polar sets. If M is a vector subspace of E, then $\sup_{x \in M} |\langle x,x' \rangle| \leq 1$ implies $\langle x,x' \rangle = 0$ for all $x \in M$. Hence M^o consists of these elements of E' that vanish on M and so is a vector subspace of E' orthogonal to M. If E is a separated locally convex space, a subset A' of its dual E' is equicontinuous if and only if there is a neighborhood U of 0 with $\sup |\langle x,x' \rangle| \leq 1$ for all $x \in U$ and $x' \in A'$. Thus A' is equicontinuous if and only if it is contained in the polar of some neighborhood.

PROPOSITION 20. <u>If</u> E <u>is a separated locally convex space and</u> <u>is a base of neighborhoods, then the dual of</u> E <u>is</u> $\bigcup_{U \in \mathcal{U}} U^o$ <u>(the polars being taken in</u> E*).

continuou

PROOF. The linear form $x^* \in E^*$ is continuous if and only if there is some neighborhood $U \in \mathcal{U}$ with $|\langle x,x' \rangle| \leq 1$ on U.

DEFINITION. If (E,E') and (E',F) are dual pairs and A is a subset of E, the polar A^{oo} of A^o in F is called the <u>bipolar</u> of A.

<u>Remark</u>. If $E \subset F \subset E'^*$, then $A \subset A^{oo}$. For, $z \in A^{oo}$, if and only if $|\langle z,x' \rangle| \leq 1$ whenever $x' \in A^o$, i.e. whenever $\sup_{x \in A} |\langle x,x' \rangle| \leq 1$. Thus $z \in A^{oo}$ if and only if

$|\langle z,x' \rangle| \leq \sup_{x \in A} \{|\langle x,x' \rangle|\}$. Since $A \subset E \subset F$, this implies $A \subset A^{oo}$

oo.

PROPOSITION 21. *Let* (E,E') *be a dual pair and* F *a vector subspace of* E'^* *containing* E. *Then the bipolar* A^{oo} *in* F *of a subset* A *of* E *is the* $\sigma(F,E')$-*closed absolutely convex envelope of* A.

PROOF. Let B be the $\sigma(F,E')$-closed absolutely convex envelope of A. Then A^{oo} is $\sigma(F,E')$-closed absolutely convex subset containing A and therefore $B \subset A^{oo}$. If $a \notin B$, then there is a continuous linear form $x' \in E'$ with $|\langle x,x'\rangle| \leq 1$ for all $x \in B$ and $|\langle a,x'\rangle| > 1$. Now $A \subset B$ and so $x' \in A^o$ thus $a \notin A^{oo}$. Hence $A^{oo} \subset B$ and so $A^{oo} = B$.

COROLLARY 1. *If* E *is a separated locally convex space with dual* E' *and* A *is a subset of* E, *then the bipolar* A^{oo} *in* E *is the closed absolutely convex envelope of* A.

COROLLARY 2. *Under the conditions of the proposition ,the polar of* A^{oo} *in* E' *is* A^o.

PROOF. By proposition, the polar of A^{oo} in E' is $\sigma(E',F)$-closed absolutely convex envelope of A^o. Now A^o is absolutely convex and $\sigma(E',F)$-closed; also $\sigma(E',F)$ is finer than $\sigma(E',E)$. Hence A^o is also $\sigma(E',F)$-closed. Thus A^o is the polar of A^{oo}.

COROLLARY 3. <u>If</u> (E,E') <u>is a dual pair and if, for each</u> α, A_α <u>is</u> $\sigma(E,E')$-<u>closed absolutely convex subset of</u> E <u>then</u> $(\bigcap_\alpha A_\alpha)^\circ$ <u>is the</u> $\sigma(E',E)$-<u>closed absolutely convex envelope of</u> $\bigcup_\alpha A_\alpha^\circ$.

PROOF. Taking polars in E of subsets of E'.

$$\left(\bigcup_\alpha A_\alpha^\circ\right)^\circ = \bigcap_\alpha A_\alpha^{\circ\circ} = \bigcap_\alpha A_\alpha$$

Hence $\left(\bigcup_\alpha A_\alpha^\circ\right)^{\circ\circ} = \left(\bigcap_\alpha A_\alpha\right)^\circ$ and the result follows from the proposition.

2.8. <u>Finite dimensional subspaces</u>.

Let E be an n-dimensional vector space with a basis $e_1, e_2 \ldots e_n$. There is a dual base $e_1^*, \ldots e_n^*$ in the algebraic dual E^* of E, with the property $\langle e_i, e_j^* \rangle = \delta_{ij}$. For any element $x \in E$ can be uniquely written in the form $\sum_{1\le i\le n} \lambda_i e_i$ and put $\langle x, e_i^* \rangle = \lambda_i$. Clearly e_i^* are linearly independent. They also span E^*.

If E is finite dimensional and (E,E') is a dual pair, then $E' = E^*$. For E and E^* have the same dimension and so have E' and E'^*. Since $E' \subset E^*$ and $E \subset E'^*$, all must have the same dimension.

PROPOSITION.22. <u>A finite dimensional vector space has only one topology under which it is a separated locally convex space</u>.

PROOF. We show that for a finite dimensional separated locally convex space E, its topology is identical with $\sigma(E,E^*)$. Since dual of E is E^*, the topology is certainly finer than $\sigma(E,E^*)$. Now let $e_1,\dots,e_n$ be any base of E and $e_1^*,\dots e_n^*$ the corresponding dual base of E^*. Let U be an absolutely convex neighborhood in E. There is some $\mu>0$ with $e_i \in \mu U$ for $1 \leq i \leq n$. Then

$$V = \left\{ x \,\middle|\, \sup_{1\leq i\leq n} |\langle x, e_i^*\rangle| \leq (\mu n)^{-1} \right\}$$

is a $\sigma(E,E^*)$-neighborhood if $x = \Sigma\lambda_i e_i \in V$,

$$x \in \sum_{1\leq i\leq n} |\lambda_i|\,\mu U = \sum |\langle x, e_i^*\rangle|\,\mu U \subset n(\mu n)^{-1}\,\mu U = U.$$

Thus the given topology is coarser than $\sigma(E,E^*)$ and so identical.

PROPOSITION 23. <u>Let</u> M <u>be a finite dimensional vector subspace of a locally convex separated space</u>. <u>Then</u> M <u>is closed in</u> E <u>and the topology induced on</u> M <u>is the euclidean topology</u>.

PROOF. The second part follows from Proposition 22. To show M is closed: If $e_1, \ldots, e_n$ is a base of M and if $a \notin M$, then regarding $a, e_1, \ldots, e_n$ as linear forms on the dual E' of E, by lemma there is some $x' \in E'$ with $\langle a, x' \rangle = 1$ and $\langle e_i, x' \rangle = 0$ for $i = 1, 2, \ldots, n$. Let $U = \{ x \,|\, |\langle x, x' \rangle| < 1 \}$. Then U is a neighborhood of 0 and a+U does not meet M, for $\langle a+u, x' \rangle = \langle a, x' \rangle + \langle u, x' \rangle = 1 + \langle u, x' \rangle \neq 0$. Hence M is closed.

2.9. Transpose of a linear map.

Let (E,E') and (F,F') be dual pairs. Let $t : E \longrightarrow F$ be a linear transformation. Then $\langle tx, y' \rangle$ is a bilinear form of the two variables x,y'. For each fixed $y' \in F'$ let $t'(y')$ be the linear form on E defined by

$$\langle x, t'(y') \rangle = \langle tx, y' \rangle \quad \text{for all } x \in E.$$

Then $t'(y') \in E^*$. Then t' is a linear transformation of F' into E*. t' is called the adjoint, conjugate, dual or transpose.

PROPOSITION 24. *Let* (E,E') *and* (F,F') *be dual pairs. Let* t *be a linear transformation of* E *into* F *with transpose* t'. *Then* $t'(F') \subset E'$ *if and only if* t *is continuous in the weak topologies* $\sigma(E,E')$ *and* $\sigma(F,F')$.

PROOF. Assume t is continuous. Let $y' \in F'$ be fixed. Then $\langle tx, y' \rangle$ is a continuous linear form on E. Hence $t'(y') \in E'$ or $t'(F') \subset E'$.

Conversely suppose $t'(F') \subset E'$. Let

$V = \left\{ y \in F \,\middle|\, \sup_{1 \leq i \leq n} |\langle y, y_i' \rangle| \leq 1 \right\}$ any $\sigma(F,F')$-neighborhood. Take

$$U = \left\{ x \in E \,\middle|\, \sup_{1 \leq i \leq n} |\langle x, t'(y_i') \rangle| \leq 1 \right\}$$

Then U is a $\sigma(E,E')$-neighborhood and $t(U) \subset V$. Hence t is continuous.

DEFINITION. t *is said to be weakly continuous if it is continuous in the topologies* $\sigma(E,E')$ *and* $\sigma(F,F')$.

COROLLARY. *If* t *is weakly continuous so is* t'

PROPOSITION 25. *If* t *is a continuous linear mapping of the separated locally convex space* E *(with dual* E'*) into the separated locally convex space* F *(with dual* F'*) then it is also continuous when* E *and* F *have the associated weak topologies* $\sigma(E,E')$ *and* $\sigma(F,F')$.

PROOF. Let $y' \in F'$ be fixed (but arbitrary). $\langle tx, y' \rangle$ is continuous on E. $t'(y') \in E'$ i.e. $t'(F') \subset E'$. Hence t is continuous.

LEMMA. *Let* (E,E'), (F,F') *be dual pairs. Let t be a weakly continuous linear mapping of E into F. Let t' be its transpose. Then* $A \subset E$, *then*

$$\left(t(A)\right)^{\circ} = t'^{-1}(A^{\circ})$$

PROOF. Each of these is the set of all $y' \in F'$ such that $|\langle tx, y' \rangle| = |\langle x, t'y' \rangle| \leq 1$ for all $x \in A$.

CHAPTER 3

MORE ON NORMED LINEAR SPACES

3.1 Now we shall specialize on normed linear spaces.

DEFINITION 1. A normed linear space X is a vector space over $\mathbb{R}$ or $\mathbb{K}$ on which is defined a non-negative function called the <u>norm</u> (norm of x being denoted by $\|x\|$ such that

$$\|x\| = 0 \text{ iff } x = 0$$

$$\|x + y\| \leq \|x\| + \|y\|$$

$$\|\alpha x\| \leq |\alpha| \, \|x\|$$

for all vectors x, y and scalars α.

X becomes a metric space if we define $\rho(x,y) = \|x - y\|$ and is called a <u>Banach Space</u> if it is complete in this metric.

<u>Example</u>. $C[a,b]$ = Set of all continuous real valued functions on $[a,b]$. If $f, f_1, f_2 \in C[a,b]$ define

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(\alpha f)(x) = \alpha f(x).$$

$C[a,b]$ then becomes a vector space. A norm is defined by

$$\|f\| = \max_{[a,b]} |f(x)|$$

and obtain a Banach space.

DEFINITION 2. Let M be subset of a normed linear space X. M is called a <u>linear manifold</u> if $x,y \in M$, α,β scalars, then $\alpha x + \beta y \in M$.

M is a subspace of X if M is a closed linear manifold.

Let E be a vector space and M a linear subspace of E. Two elements x,y are said to be equivalent, $x \sim y$, if $x - y \in M$. If $x + M$, $y + M$ are two cosets, then the above equivalence relation tells us that either two cosets are identical or disjoint. The set of all cosets is denoted by E/M. It is made a vector space by defining addition and scalar multiplication by

$$(x + M) + (y + M) = x + y + M$$
$$\alpha(x + M) = \alpha x + M.$$

PROPOSITION 1. *Let M be a subspace of a normed linear space X. The norm in X/M is defined by*

$$\|y\| = \text{g.l.b.} \left\{ \|x\| \;\middle|\; x \in y \right\} \quad \textit{for} \quad y \in X/M.$$

If X is complete, then X/M is also complete.

<u>PROOF</u>. 1) By the definition of norm, if $y \in X/M$, then $\|y\| = 0$ iff there exists $\{x_n\} \in y$ such that $\|x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Since y is closed, $\|y\| = 0$ iff $0 \in y$ so that $\|y\| = 0$ iff $y = M$. The other axioms of the norm can be easily checked.

2) Suppose X is complete. If $\{y_n\}$ is a Cauchy sequence in X/M, we can suppose, by passing onto a subsequence if necessary, that

$$\|y_{n+1} - y_n\| < \frac{1}{2^n}$$

We can then choose inductively a sequence $x_n \in y_n$ such that $\|x_{n+1} - x_n\| < \frac{1}{2^n}$ for $\rho(x_n, x_{n+1}) = \rho(y_n, y_{n+1}) < \frac{1}{2^n}$. Then $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x_0 \in X$ such that $x_n \rightarrow x_0$. Let y_0 be the coset containing x_0. Then $y_n \rightarrow y_0$ (check). By the property of the Cauchy sequence $\{y_n\}$ converges to y_0 and X/M is complete.

DEFINITION.3. Let X, Y be normed linear spaces. A function $T : X \rightarrow Y$ is called a transformation. T is said to be <u>linear</u> if $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$ for $x_1, x_2 \in X$ and α_1, α_2 scalars. T is said to be <u>bounded</u> if there exists $M > 0$ such that

$$\|T(x)\| \leq M \|x\| \quad \text{for all } x \in X.$$

PROPOSITION 2. <u>Let</u> X, Y <u>be normed linear spaces and let</u> $T: X \to Y$ <u>be a linear transformation. Then</u>

a) if T is continuous at x_o, then T is continuous on X

b) T is continuous iff it is bounded.

PROOF. a) $x_n \to x_o$ <u>implies</u> $T(x_n) \to T(x_o)$. Now suppose $y_o \in X$ and $y_n \to y_o$. Then, by the linearity of T

$$T(y_n) = T(y_n - y_o + x_o + y_o - x_o)$$
$$= T(y_n - y_o + x_o) + T(y_o) - T(x_o).$$

Since $y_n - y_o + x_o \to x_o$, $T(y_n - y_o + x_o) \to T(x_o)$ so that

$$T(y_n) \to T(x_o) + T(y_o) - T(x_o) = T(y_o)$$

b) i) if T is bounded there exists $M > 0$ such that

$$\|T(x)\| \leq M\|x\| .$$

Hence $\|T(x) - T(x_o)\| \leq \|T(x - x_o)\| \leq M\|x - x_o\|$, from which follows the continuity.

ii) If T is not bounded, then for each n there exists x_n such that $\|T(x_n)\| > \|x_n\| n$. Let $y_n = \dfrac{x_n}{n\|x_n\|}$.

Then $\|y_n\| = \frac{1}{n}$ and $\|T(y_n)\| > 1$. Hence $\|y_n\| \to 0$ but $T(y_n) \not\to T(0) = 0$. Hence T is not continuous at 0.

<u>Notation</u>. $B(X,Y)$ will denote the set of all bounded linear transformations of X into Y.

If $T, T_1, T_2 \in B(X,Y)$ and $\alpha \in \mathbb{K}$, we define $T_1 + T_2, \alpha T$ by

$$(T_1 + T_2)(x) = T_1(x) + T_2(x)$$

$$(\alpha T)(x) = \alpha T(x), \quad x \in X.$$

Then $B(X,Y)$ becomes a vector space. If $T \in B(X,Y)$ notice that there exists $M > 0$ such that

$$\|T(x)\| \leq M\|x\| \quad \text{for all } x \in X.$$

We define a norm by any one of the following :

$$\text{(i)} \quad \|T\| = \text{g.l.b.} \left\{ M \;\middle|\; \|T(x)\| \leq M\|x\| \right\}$$

$$\text{(ii)} \quad \|T\| = \underset{x \neq 0}{\text{l.u.b.}} \frac{\|T(x)\|}{\|x\|}$$

$$\text{(iii)} \quad \|T\| = \underset{\|x\| = 1}{\text{l.u.b.}} \|T(x)\|$$

It is easy to verify that (i), (ii) and (iii) are equivalent. As a consequence of this definition it follows that

$$\|T(x)\| \leq \|T\| \; \|x\| .$$

PROPOSITION 3. $B(X,Y)$ <u>is complete, if</u> Y <u>is complete.</u>

PROOF. Let $\{T_n\}$ be a Cauchy sequence in $B(X,Y)$. Then given $\varepsilon > 0$, there exists $n_o(\varepsilon)$ such that $\|T_m - T_n\| < \varepsilon$ for $m,n \geq n_o(\varepsilon)$. Then for each $x \in X$, $\|T_m(x) - T_n(x)\| < \varepsilon \|x\|$ so that $\{T_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete, there exists $T(x) \in Y$ such that $T_n(x) \longrightarrow T(x)$. Thus we define a function

$$T : X \longrightarrow Y$$

by

$$T(x) = \lim_{n \to \infty} T_n(x).$$

Now

1)
$$\begin{aligned} T(\alpha_1 x_1 + \alpha_2 x_2) &= \lim_{n\to\infty} T_n(\alpha_1 x_1 + \alpha_2 x_2) \\ &= \lim_{n\to\infty} \left[\alpha_1 T_n(x_1) + \alpha_2 T_n(x_2)\right] \\ &= \alpha_1 \lim_{n\to\infty} T_n(x_1) + \alpha_2 \lim_{n\to\infty} T_n(x_2) \\ &= \alpha_1 T(x_1) + \alpha_2 T(x_2) \end{aligned}$$

2) Since $\|T_{n+p} - T_n\| < 1$ for all $n > N$ and all p,

$$\|T_{N+p}\| \leq \|T_N\| + 1 \quad \text{for} \quad p \geq 1$$

and

$$\|T\| = \lim_{n \to \infty} \|T_{N+p}\| < \|T_N\| + 1$$

and hence T is bounded.

3) We show that $\|T_n - T\| \to 0$ as $n \to \infty$. Now for $n > n_o(\varepsilon)$

$$\|T_{n+p} - T_n\| < \varepsilon .$$

For $\|x\| = 1$

$$\|T(x) - T_n(x)\| = \lim_{p \to \infty} \|T_{n+p}(x) - T_n(x)\|$$

$$\leq \lim_{p \to \infty} \|T_{n+p} - T_n\| < \varepsilon .$$

Hence $\|T_n - T\| < \varepsilon$ if $n \geq n_o(\varepsilon)$. Thus $B(X,Y)$ is complete.

If Y = field of complex numbers then Y is complete ($\|Y\| = |Y|$), we write $B(X,Y) = X^*$ called the conjugate space or dual space of X. An element of X^* is called a <u>bounded linear functional</u>.

COROLLARY. X^* <u>is always complete</u>.

<u>Hahb Banach Extension Theorem</u>:

PROPOSITION 4. <u>Let</u> M <u>be a subspace of normed linear space</u> X. <u>Then every bounded linear functional on</u> M <u>can be extended to the whole of</u> X <u>with preservation of the norm i.e., if</u> $T \in M^*$ <u>there exists</u> $S \in X^*$ <u>such that</u>

$$S(x) = T(x) \quad \text{for all} \quad x \in M$$

<u>and</u>

$$\|S\|_X = \|T\|_M$$

COROLLARY. <u>Given</u> $x_o \in X$, $x_o \neq 0$, <u>there exists</u> $T \in X^*$ <u>such that</u>

$$T(x_o) = \|x_o\|, \quad \|T\| = 1.$$

3.2 DEFINITION 4. A Euclidean space E is a vector space in which a function of two variables x,y denoted by (x,y) called <u>inner product</u>, is defined, satisfying

a) $(x,x) > 0$ if $x \neq 0$, $(x,x) = 0$ if $x = 0$

b) $(x,y) = \overline{(y,x)}$

c) $(\lambda x,y) = \lambda(x,y)$

d) $(x_1 + x_2,y) = (x_1,y) + (x_2,y)$.

REMARK. Define $\|x\| = (x,x)^{1/2}$. If E is complete in this norm then E is called a <u>Hilbert space.</u>

DEFINITION 5. A Hilbert space H is a Banach space in which the norm satisfies an additional requirement viz.

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

for all $x, y \in H$. An inner product (x,y) is then defined by

$$4(x,y) = \|x + y\|^2 - \|x - y\|^2 + \|x + iy\|^2 - \|x - iy\|^2.$$

PROPOSITION 5. <u>The above two definitions for a Hilbert space are equivalent</u>.

<u>Proof</u>. Exercise.

We only give some properties necessary to prove the equivalence which will also be used elsewhere.

PROPOSITION 6. <u>Let</u> H <u>be Hilbert space. If</u> $x, y \in H$ <u>and</u> λ <u>is a scalar, then</u>

a) $(x, \lambda y) = \bar{\lambda}(x,y)$

b) $|(x,y)| \leq \|x\| \, \|y\|$.

PROOF. a) $(x,\lambda y) = \overline{(\lambda y,x)} = \bar{\lambda}\overline{(y,x)} = \bar{\lambda}(x,y)$

b) Let λ be any complex number. Then

$$\begin{aligned} 0 &\leq (x - \lambda y, x - \lambda y) \\ &= (x,x) + (x, -\lambda y) + (-\lambda y,x) + (-\lambda y,-\lambda y) \\ &= (x,x) - \bar{\lambda}(x,y) - \lambda\overline{(x,y)} + \lambda\bar{\lambda}(y,y). \end{aligned}$$

Assuming $y \neq 0$, set $\lambda = \dfrac{(x,y)}{(y,y)}$. Then

$$0 \leq (x,x) - \frac{\overline{(x,y)}(x,y)}{(y,y)}$$

which gives

$$(x,x)\cdot(y,y) - (x,y)\cdot\overline{(x,y)} \geq 0$$

or

$$|(x,y)|^2 \leq (x,y)(y,y) = \|x\|^2\|y\|^2 .$$

DEFINITION 6. Two elements x,y in a Hilbert space H are said to be <u>orthogonal</u> if the inner product $(x,y) = 0$. Then we write $x \perp y$. Two subsets S_1,S_2 in H are said to be <u>orthogonal</u> if $x_1 \perp x_2$ for all $x_1 \in S_1$ and $x_2 \in S_2$. If M is a subspace of H the set of all elements of H that are orthgonal to M is denoted by $M^{\perp}$ and is called the orthogonal complement of M.

PROPOSITION 7. $M^{\perp}$ *is a subspace of* H.

PROPOSITION 8. *Let* M *be a subspace of a Hilbert space* H *and* $x \in H \setminus M$. *Let*

$$d = \text{g.l.b.} \left\{ \|x - y\| \;\middle|\; y \in M \right\}.$$

Then there exists a unique y_0 *in* M *such that*

$$d = \|x - y_0\| .$$

Further $x - y_0 \in M^{\perp}$.

The proof is straight forward. We have the following corollary.

COROLLARY. *If* M *is a subspace of a Hilbert space* H, *then every element* $x \in H$ *can be uniquely represented as*

$$x = x_1 + x_2$$

where $x_1 \in M$, $x_2 \in M^{\perp}$.

3.3 *Riesz Representation Theorem.*

PROPOSITION 9. *If* y *is a fixed element in a Hilbert space* H *and* T_y *is defined by*

$$T_y(x) = (x,y) \quad \text{for all } x \in H,$$

then $T_y \in H^*$.

PROOF. Let α_1, α_2 be scalars and $x_1, x_2 \in H$. Then

$$\begin{aligned} T_y(\alpha_1 x_1 + \alpha_2 x_2) &= (\alpha_1 x_1 + \alpha_2 x_2, y) \\ &= \alpha_1 (x_1, y) + \alpha_2 (x_2, y) \\ &= \alpha_1 T_y(x_1) + \alpha_2 T_y(x_2). \end{aligned}$$

Hence T_y is linear. Further

$$|T_y(x)| = |(x,y)| \leq \|x\| \, \|y\|$$

so that

$$\|T_y\| \leq \|y\| \quad .$$

Hence T_y is continuous which proves the theorem.

PROPOSITION 10. <u>Every bounded linear functional</u> T <u>on a Hilbert space</u> H <u>can be expressed uniquely in the form</u>

$$T(x) = (x,y) \quad \text{for all} \quad x \in H$$

<u>where</u> y <u>is a fixed point of</u> H <u>and</u>

$$\|T\| = \|y\|$$

PROOF. Given a bounded linear functional T, let

$$M = \left\{ x \in H \,\middle|\, T(x) = 0 \right\}.$$

If $M = H$, take $y = 0$. Suppose $M \neq H$, then $M^{\perp} \neq 0$. There exists an element $z \neq 0$, $z \in H$, $z \notin M$. We notice that $T(z) \neq 0$. If x is any element in H, let

$$u = x - \frac{T(x)}{T(z)}\, z$$

then $u \in M$ and $z \in M$. That is $(u,z) = 0$, which gives

$$\left(x - \frac{T(x)}{T(z)}\, z,\ z\right) = 0$$

or

$$(x,z) - \frac{T(x)}{T(z)}\,(z,z) = 0$$

or

$$T(x) = \frac{T(z)}{(z,z)}\,(x,z) = \left(x,\ \overline{\frac{T(z)}{(z,z)}}\ z\right) .$$

Now take $y = \overline{\dfrac{T(z)}{(z,z)}}\, z$. This y is the required one. By the previous proposition

$$\| T \| \leq \| y \| ,$$

also from

$$T(y) = (y,y) = \| y \|^2$$

it follows that

$$\|T\| \geq \|y\| .$$

Hence

$$\|T\| = \|y\| .$$

This completes the proof.

3.4 Consider the set of all measurable functions which are complex valued, defined on a measurable set E of finite or infinite measure. Let L^2 denote the set of all these functions which are square integrable i.e. $\int_E |f|^2 dx < \infty$. Then by Schwarz's inequality f is integrable on the subsets of E of finite measure.

Define for $f,g \in L^2$

$$(f,g) = \int_E f(x)\overline{g(x)}dx,$$

$$\|f\| = (f,f)^{1/2} .$$

Then (f,g) is an inner product in L^2 and L^2 is complete under the above norm. So L^2 is a Hilbert space. So from what we have proved in Proposition 10 - it follows that if T is a bounded linear functional on L^2 then there exists a unique function $g \in L^2$ such that

$$T(f) = \int_E f(x)\,\overline{g(x)}\,dx \quad \text{for all} \quad f \in L^2$$

and $\|T\| = \|g\|$.

We denote by $L^p (p \geq 1)$ the class of all measurable complex valued functions which are such that $\int_E |f(x)|^p dx < \infty$.

Define

$$\|f\| = \left(\int_E |f(x)|^p dx\right)^{1/p} \qquad (1 \leq p < \infty).$$

We understand by L^∞ the space of all measurable functions which are bounded or are equal a.e. to bounded functions. We define the norm

$$\|f\| = \text{true max}\, |f(x)|, \quad f \in L^\infty$$

i.e. $\|f\|$ is the smallest value of M for which $|f(x)| \leq M$ a.e. If $p > 1$, let q be define by $p^{-1} + q^{-1} = 1$.

3.5 Let $b > a$ be real numbers and $P([a,b])$ be the set of all partitions of $[a,b]$.

DEFINITION 7. A function f(real or complex valued) defined on $[a,b]$ is said to be of <u>bounded variation</u> if

$$V(f;a,b) = \underset{\pi \in P([a,b])}{\text{l.u.b.}} \left\{ \sum_{t_j \in \pi} |f(t_j) - f(t_{j-1})| \right\}$$

is finite and $V(f;a,b)$ is called the <u>total variation</u> of f. The class of all functions of bounded variation on $[a,b]$ is denoted by $BV[a,b]$.

REMARK 1. If $f \in BV[a,b]$ then there exist two functions f_1, f_2 which are non-negative non-decreasing such that $f = f_1 - f_2$.

REMARK 2. If $f \in BV[a,b]$ then f is bounded on $[a,b]$.

DEFINITION.8. Let f and σ (real or complex valued) be two functions defined on $[a,b]$ where $a < b$. Let $\pi = \{a = t_0, \ldots, t_n = b\} \in P([a,b])$ and let $\alpha = \{s_1, s_2, \ldots, s_n\}$ where $s_k \in [t_{k-1}, t_k]$, $k = 1,2,\ldots,n$. We define the Stieltjes integral of f with respect to σ to be the limit of the sums

$$S_{\pi,\alpha} = \sum_{k=1}^{n} f(s_k)\,[\sigma(t_k) - \sigma(t_{k-1})]$$

when $\|\pi\| = \max_{1 \leq k \leq n} (t_k - t_{k-1}) \longrightarrow 0$, and is denoted by

$$\int_a^b f(x)\,d\sigma(x).$$

DEFINITION 9. Let $C\,[a,b]$ be the set of all continuous real valued functions on $[a,b]$. Then the mapping $L: C\,[a,b] \longrightarrow \mathbb{R}$ is said to be linear functional, if

1) $L(f_1 + f_2) = L(f_1) + L(f_2)$

2) $L(cf) = cL(f), \quad c \in \mathbb{R}$.

and it is bounded (hence continuous) if

3) There exists a constant M such that

$$|L(f)| \leq M \|f\|$$

where $\|f\| = \max\limits_{a \leq x \leq b} |f(x)|$.

The smallest of all such bounds M is denoted by $\|L\|$ and is called the norm of the linear functional.

Now we prove the following lemma which we utilise in the proof of the next proposition.

LEMMA 11. *Let L be a bounded linear functional on $C\,[a,b]$. If $\{f_n\}$ and $\{g_n\}$ are two increasing sequences belonging to $C[a,b]$, which tend to the same limit, then the sequences $\{L(f_n)\}$ and $\{L(g_n)\}$ also tend to the same limit.*

PROOF. Without loss of generality we assume that the sequences are strictly increasing (otherwise we consider the sequences $\{f_n - \frac{1}{n}\}$, $\{g_n - \frac{1}{n}\}$). For each fixed m there should exist an n such that $f_m < g_n$ for all $n' \geq n$. Suppose not, then the sets $K_n = \{x \mid f_m(x) \geq g_n(x)\}$ form a nested sequence of closed, nonempty sets. Hence there exists a point $x_o \in K_n$ for all n. In turn, we have $f_n(x_o) \geq \lim g_n(x_o) = f(x_o)$ contrary to the hypothesis. By the same argument there exists for each m an n such that $g_m < f_n$ for all $n' \geq n$. Now we can form an increasing sequence $f_{m_1} < g_{m_2} < f_{m_3} < g_{m_4} < \ldots$ tending to f. Hence, the sequences $\{L(f_n)\}$ and $\{L(g_n)\}$ tend to the same limit.

PROPOSITION 12. A continuous linear functional L defined on $C[a,b]$ can be extended to a wider class of functions.

PROOF. Let $\{f_n\}$ be an increasing and bounded sequence of continuous functions on $[a,b]$ which tends to a bounded function f. Now we extend the functional L to this f which may not be continuous. Consequently, the sequence of values $\{Lf_n\}$ tend to a finite limit. The values of the series

$$\sum_{n=1}^{\infty} |L(f_{n+1}) - L(f_n)|$$

correspond to the partial sums of the series

$$\sum_{n=1}^{\infty} \pm \left[f_{n+1}(x) - f_n(x)\right]$$

by means of L (where the signs are suitably chosen). But

$$\left|\sum_{n=1}^{k} \pm \left[f_{n+1}(x) - f_n(x)\right]\right| \leq f(x) - f_1(x) = B(\text{say}),$$
$$k = 1,2,\ldots .$$

Hence

$$\sum_{n=1}^{k} |L(f_{n+1}) - L(f_n)| \leq B \| L \|, \; k = 1,2,\ldots$$

and so

$$L(f_1) + \sum_{n=1}^{\infty} (L(f_{n+1}) - L(f_n))$$

converges absolutely and partial sums being Lf_n tend to a finite limit which we denote by Lf. This is justified from the previous lemma. Hence L is defined uniquely to every bounded function which is the limit of an increasing sequence of continuous functions. If f and g are of the above type then so is $f + g$ and $L(f+g) = L(f) + L(g)$. But the difference $f - g$ neither is of the type f nor $-f$. To justify to write

$$L(f - g) = L(f) - L(g)$$

consider the relation $f - g = f_1 - g_1$ which is equivalent to writing $f + g_1 = f_1 + g$; so

$$L(f) + L(g_1) = L(f + g_1) = L(f_1 + g) = L(f_1) + L(g)$$

which yields in turn **the desired.**

Since (1) and (2) properties of the bounded linear functional are evident, it remains to prove that it is bounded, i.e., we show

$$|L(f - g)| \leq \|L\| \mu$$

where $\mu = \sup_{a \leq x \leq b} |f(x)-g(x)|$ and f,g are the limits of the increasing sequences $\{f_n\}$ and $\{g_n\}$ respectively.

Now set

$$F_n(x) = \begin{cases} f_n(x) & \text{when } |f_n(x) - g_n(x)| \leq \mu \\ g_n(x) + \mu & \text{when } f_n(x) - g_n(x) > \mu \\ g_n(x) - \mu & \text{when } f_n(x) - g_n(x) < -\mu. \end{cases}$$

It is easy to verify that $\{F_n\}$ is an increasing sequence of continuous functions and tends to f. Further

$$\sup_{a \leq x \leq b} |F_n(x) - g_n(x)| \leq \mu .$$

Hence

$$\begin{aligned} |L(f-g)| &= |\lim(L(F_n) - L(g_n))| \\ &= \lim |L(F_n) - L(g_n)| \\ &\leq \|L\| \end{aligned}$$

which proves the theorem.

For our further purposes we remark that the class under consideration contains apart from continuous functions, simple discontinuous functions and their finite linear combinations. In particular the characteristic function $f_{c,d}$ of the closed interval $[c,d] \subset [a,b]$ also belongs to the class being the limit of decreasing sequence of continuous functions $\{ f_n \}$ where f_n is zero outside $(c - \frac{1}{n}, d + \frac{1}{n})$ and equal to 1 on $[c,d]$.

PROPOSITION 13. (Riesz representation theorem).

<u>For every bounded linear functional</u> L <u>on</u> $C[a,b]$ <u>there exists a function</u> $\sigma \in BV [a,b]$ <u>such that</u>

$$L(f) = \int_a^b f(x)\,d\sigma(x)$$

<u>and</u> $V[\sigma;a,b] = \|L\|$. <u>Conversely the integral of the type defines a linear functional.</u>

PROOF. The converse is evident. For any $f \in C[a,b]$ and $\sigma \in BV[a,b]$ we have

$$\left|\int_a^b f(x)\,d\sigma(x)\right| \leq \|f\|\cdot V(\sigma;a,b).$$

This satisfies all the three properties of the bounded linear functional.

Reciprocally, the bounded linear functiomal L can be extended to a characteristic function $f_{c,d}$ of the internal $[c,d] \subset [a,b]$. Now we define

$$\sigma(x) = \begin{cases} 0, & x = a \\ L(f_{a,x}), & a < x \leq b \end{cases}$$

where $f_{a,x}$ is the characteristic function of the interval $[a,x]$ We claim that the function $\sigma \in BV[a,b]$ and $V(\sigma;a,b) \leq \|L\|$. To prove this consider a partition $\pi = \{a = x_0,\ldots,x_n = b\} \in P([a,b])'$, and the expression

$$\sum_{k=1}^{n} |\sigma(x_k) - \sigma(x_{k-1})|, \quad x_k \in \pi.$$

This is the value of L at

$$f(x) = \varepsilon_1 f_{a,x_1}(x) + \sum_{k=2}^{n} \varepsilon_k \left[f_{a,k}(x) - f_{a,k-1}(x) \right]$$

where ε_k equals to 1,0 or -1 according as the sign of $\sigma(x_k) - \sigma(x_{k-1})$. The function f being the finite linear combination of the functions $f_{a,k}$ belongs to the class under consideration and further $|f| \leq 1$. Hence

$$\sum_{k=1}^{n} |\sigma(x_k) - \sigma(x_{k-1})| = L(f) \leq \|L\|.$$

Since r.h.s. is independent of π we have

$$(*) \qquad V(\sigma;a,b) \leq \|L\| .$$

Let $f \in C[a,b]$ and let π be as above. Let $s_k \in [x_{k-1},x_k]$ $k = 1,2,\ldots,n$. Now define the step function

$$\varphi(x) \doteq \begin{cases} f(s_k), & x_{k-1} < x \leq x_k \\ f(s_1), & x = 1 \end{cases}$$

which could be written as

$$\varphi(x) = f(s_1)f_{a,x_1}(x) + \sum_{k=2}^{n} f(s_k)\left[f_{a,x_k}(x) - f_{a,x_{k-1}}(x)\right] .$$

So φ belongs to the class under consideration. Thus

$$L(\varphi) = \sum_{k=1}^{n} f(s_k)\left[\sigma(x_k) - \sigma(x_{k-1})\right]$$

where observing that $\sigma(x_0) = \sigma(a) = 0$. The r.h.s. of the expression is exactly the definition of Stieltjes integral when $\|\pi\| \longrightarrow 0$. Let ω be the maximum oscillation of f on the subdivision intervals. Then $|f(x) - \varphi(x)| \quad \omega$ and hence

$$|L(f) - L(\varphi)| \leq |L(f - \varphi)| \leq \omega \|L\| .$$

For $\|\pi\| \longrightarrow 0$, $\omega \longrightarrow 0$; so $L(\varphi) \longrightarrow L(f)$ which means

$$L(\varphi) \longrightarrow L(f) = \int_a^b f(x)\,d\sigma(x) .$$

Also $V(\sigma;a,b)$ is a bound for L, hence

$$V(\sigma;a,b) \geq \|L\| .$$

Combining (*) with the above yields the result.

At this stage we state the above theorem more generally in terms of Radon measures* more general than Borel measures. The interested readers may refer to R.E.Edwards, Functional Analysis, Page 203.

PROPOSITION 14. _Let_ L _be a continuous linear functional on_ $C(T)$ - _the vector space of continuous real valued functions on a locally compact space_ T. _There exists a Radon measure_ μ _on_ T _having a compact support and such that_

$$L(f) = \mu(f) = \int_T f(t)\,d\mu(t).$$

Conversely, the integral of the above form where $f \in C(T)$ _and_ μ _a Radon measure represents a linear functional_.

---

*A Radon measure μ on T is a linear functional on the class of continuous functions on T with compact support which is continuous in the following sense: Given $\varepsilon > 0$ and a compact set $K \subset T$ there exists a $\delta > 0$ such that $|\mu(f)| < \varepsilon$ whenever f has its support in K and $|f(x)| < \delta$ for $x \in K$.

3.6 $\underline{L^2(a,b)}$:

Let

$$(f,g) = \int_a^b f(t)\overline{g(t)}\,dt, \quad f,g \in L^2.$$

$$\|f\|_2 = (f,f)^{\frac{1}{2}} = \left(\int_a^b |f(x)|^2\,dx\right)^{\frac{1}{2}}$$

$$|(f,g)| \leq \int_a^b |f(x)\overline{g(x)}|\,dx \leq \left(\int_a^b |f(x)|^2\,dx\right)^{\frac{1}{2}} \left(\int_a^b |g(x)|^2\,dx\right)^{\frac{1}{2}}$$

$$|(f,g)| \leq \|f\|_2 \; \|g\|_2$$

Then

$$(f_1 + f_2, g) = (f_1,g) + (f_2,g)$$

$$(f, g_1 + g_2) = (f,g_1) + (f,g_2)$$

$$(\lambda f, g) = \lambda(f,g), \quad (g,f) = \overline{(f,g)}$$

$$(f,\lambda g) = \overline{\lambda}(f,g)$$

$$\begin{aligned}
\|f + g\|^2 = (f+g,f+g) &= (f,f) + (g,g) + (g,f) + (f,g) \\
&= (f,f) + (g,f) + 2\,\mathrm{Re}\,(f,g) \\
&\leq (f,f) + (g,g) + 2|f,g)| \\
&\leq \|f\|^2 + \|g\|^2 + 2\|f\|\,\|g\| .
\end{aligned}$$

Hence

$$\|f + g\| \leq \|f\| + \|g\|.$$

f_n is said to converge strongly (or in the norm) if $\|f_n - f\| \to 0$ $n \to \infty$.

RIESZ-FISCHER THEOREM.15. <u>If a sequence</u> $\{f_n\}$ <u>is given, then in order that there exists a function</u> f <u>such that</u> $f_n \to f$, <u>it is necessary and sufficient that</u> $\|f_n - f\| \to 0$ <u>as</u> $m,n \to \infty$, i.e. f_n <u>is a Cauchy sequence.</u>

PROOF. If $f_n \to f$ strongly, given $\varepsilon > 0$ there exists n_0 such that

$$\|f_n - f\| < \varepsilon/2 \text{ if } n \geq n_0 .$$

THEOREM. L^2 <u>is complete</u>.

PROOF. Let $\{f_n\}$ be a Cauchy sequence in L^2. i.e., $\|f_m - f_n\| \to 0$ as $m,n \to \infty$. Choose integers $m_1 < m_2 < \dots$ such that $n > m_k$ implies

$$\|f_n - f_{m_k}\| < 2^{-k}.$$

In particular

$$\|f_{m_{k+1}} - f_{m_k}\| < 2^{-k}$$

$$\int_E |f_{m_{k+1}}(x) - f_{m_k}(x)|dx \le \left(\int_E |f_{m_{k+1}}(x) - f_{m_k}(x)|^2\, dx\right)^{\frac{1}{2}} \left(\int_E dx\right)^{\frac{1}{2}}$$

$$\le \sqrt{m(E)}\ \|f_{m_{k+1}} - f_{m_k}\|$$

$$< \frac{\sqrt{m(E)}}{2^k}\ .$$

For every measurable subset E such that $m(E)$ is finite

$$\sum_{k=1}^{\infty} \int_E |f_{m_{k+1}}(x) - f_{m_k}(x)|dx$$

converges.

Now apply B.Levi's theorem to obtain the absolute convergence almost everywhere of

$$\sum (f_{m_{k+1}}(x) - f_{m_k}(x)).$$

Therefore $\{f_{m_k}\}$ converges a.e. (say to f)

$$\|f_{m_k}\| \le \|f_{m_k} - f_{m_1}\| + \|f_{m_1}\| \le \|f_{m_1}\| + \frac{1}{2}$$

By Fatou's lemma $f \in L^2$. Similarly $\|f - f_{m_k}\| \to 0$ as $k \to \infty$.

For fixed $n > m_r$, pick $k > r$, then

$$\|f_n - f_{m_k}\| \leq \|f_n - f_{m_r}\| + \|f_{m_r} - f_{m_k}\| \leq 2^{-r+1}.$$

Let $k \to \infty$.

$$\|f - f_n\| \leq 2^{-r+1}.$$

Let $r \to \infty$ i.e. $n \to \infty$

$$\|f - f_n\| \to 0.$$

If f^* is also a limit of f_n, then

$$\|f - f^*\| \leq \|f - f_n\| + \|f_n - f^*\| \longrightarrow 0$$

$$\|f - f^*\| = 0 \Longrightarrow f = f^* \text{ a.e.}$$

DEFINITION 10. A sequence of functions $\{f_n\}$ in L^2 converges weakly to $f \in L^2$ if

$$(f_n, g) \to (f, g)$$

for all $g \in L^2$.

PROPOSITION 16. <u>If $f_n \to f$ strongly then $f_n \to f$ weakly but not conversely.</u>

A linear functional T on L^2 is said to be bounded if there exists $M > 0$ such that

$$|T(f)| \leq M \|f\| \text{ for all } f \in L^2.$$

Then the following are equivalent.

1) T is continuous at 0

2) T is continuous

3) T is bounded.

We define g.l.b. of all such M to be the norm of T.

$$\|T\| = \sup_{\|f\| \neq 0} \frac{|T(f)|}{\|f\|} = \sup_{\|f\| = 1} |T(f)| .$$

For a fixed $g \in L^2$, define $T_g(f) = (f,g)$ for all $f \in L^2$.

<u>Claim</u>: T_g is a bounded linear functional and

$$\|T_g\| = \|g\|$$

$$|T_g(f)| \leq |(f,g)| \leq \|f\| \, \|g\|$$

$$\|T_g\| \leq \|g\|.$$

Then $T_g(g) = (g,g) = \|g\|^2 \Rightarrow \|T_g\| \geq \|g\|$.

PROPOSITION 17. <u>If</u> T <u>is a bounded linear functional on</u> L^2, <u>then there exists a</u> $g \in L^2$ <u>such that</u> $T = T_g$.

PROOF. Choose $\{g_n\}$ such that $\|g_n\| = 1$. $\|T(g_n)\| \to \|T\|$. Without loss of generality we may assume that $T(g_n) = 0$ (otherwise multiply by a factor $e^{i\theta}$)

$$|T_{g_n} + T_{g_m}| = |T(g_n + g_m)| \leq \|T\| . \|g_n + g_m\|$$

$$\|g_n - g_m\|^2 = 2\|g_n\|^2 + 2\|g_m\|^2 - \|g_n + g_m\|^2$$

$$\leq 4 - \frac{1}{\|T\|^2} |Tg_n + Tg_m|^2$$

$$\to 4 - \frac{1}{\|T\|^2} 4\|T\|^2 = 0.$$

Since L^2 is complete, there exists $g^* \in L^2$ such that $g_n \to g^*$, $\|g^*\| = 1$, $Tg^* = \|T\|$. Put $g = \|T\| g^*$. This is the g we want. To show $T(f) = T_g(f)$ for all $f \in L^2$.

<u>Case (i)</u>. $T_g(g^*) = (g^*, g) = (g^*, \|T\| g^*) = \|T\| = T(g^*)$

<u>Case (ii)</u>. Suppose $T(f) = 0$, $f \in L^2$.

$T(g^*) = T(g^* - \lambda f)$ where λ is a scalar.

$$\|T\|^2 = |T(g^*)|^2 = |T(g^* - \lambda f)|^2$$

$$\leq \|T\|^2 \|g^* - \lambda f\|^2$$

$$= \|T\|^2 (g^* - \lambda f, g^* - \lambda f)$$

$$= \|T\|^2 [(g^*, g^*) - \lambda(f, g^*) - \bar{\lambda}(g^*, f) + \lambda\bar{\lambda}(f, f)]$$

$$- \lambda\overline{(g^*, f)} - \bar{\lambda}(g^*, f) + \lambda\bar{\lambda}(f, f) \geq 0.$$

Put $\lambda = \frac{(g^*, f)}{(f, f)}$.

Then we get $\quad - \dfrac{(g^*,f)\overline{(g^*,f)}}{(f,f)} \geq 0$

$$|(g^*,f)| = 0 \quad \text{so that} \quad (f,g^*) = 0$$

$$(f,\|T\|g^*) = 0 \text{ i.e., } (f,g) = 0$$

Thus $\quad T(f) = 0 \Rightarrow T_g(f) = 0$, hence $T = T_g$.

Case (iii). Let $f \in L^2$; let $\lambda = \dfrac{T(f)}{T(g^*)}$ and set

$$f_o = f - \frac{T(f)}{T(g^*)} g^*$$

Then $T(f_o) = 0$ and $f = f_o + \lambda g^*$

$$T_g(f) = T_g(f_o) + \lambda T_g(g^*) = T(f_o) + \lambda T(g^*)$$

$$= \frac{T(f)}{T(g^*)} T(g^*) = T(f) .$$

Hence $T_g = T$.

Functions of bounded variation (Read from my notes).

3.7 Absolute continuity

DEFINITION 11. A function f defined on (a,b) is said to be absolutely continuous (A.C.) if

$$\left| f(\beta_k) - f(\alpha_k) \right| \longrightarrow 0 \quad \text{as} \quad \sum (\beta_k - \alpha_k) \longrightarrow 0$$

where (α_k, β_k) denotes a system (finite or infinite) of non-overlapping intervals.

DEFINITION 12. F is said to satisfy <u>Lipschitz condition</u> if there exists $c > 0$ such that

$$|F(x') - F(x'')| < c|x' - x''|$$

<u>Remark.</u> A function which satisfies Lipschitz condition is A.C.

PROOF. $\sum\{f(\beta_k) - f(\alpha_k)\} \leq \sum|f(\beta_k) - f(\alpha_k)|$

$$\leq c\sum(\beta_k - \alpha_k).$$

PROPOSITION 18. <u>A function F is A.C. if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that</u>

$$\sum|F(\beta_k) - F(\alpha_k)| < \varepsilon .$$

<u>whenever</u> $\sum(\beta_k - \alpha_k) < \delta$ <u>where the intervals</u> (α_k, β_k) <u>do not overlap</u>.

PROOF. Suppose F is absolutely continuous. We shall show that given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum|F(\beta_k) - F(\alpha_k)| < \varepsilon .$$

whenever $\sum(\beta_k - \alpha_k) < \delta$.

Assume false: Then there exists some $\varepsilon > 0$ and a system of intervals (non-overlapping) (α_k, β_k) such that

$$\sum (\beta_k = \alpha_k) \longrightarrow 0 \text{ and}$$

$$\sum |F(\beta_k) - F(\alpha_k)| \geq \varepsilon .$$

Now decompose the intervals (α_k, β_k) into two parts according to the sign of $F(\beta_k) - F(\alpha_k)$. Then the intervals on one of these parts will satisfy

$$\sum (\beta_k - \alpha_k) \longrightarrow 0$$

but

$$\left| \sum (F(\beta_k) - F(\alpha_k)) \right| \geq \varepsilon/2$$

contradiction to the AC of F.

Converse is trivial.

PROPOSITION 19. <u>A necessary and sufficient condition that</u> F <u>be the indefinite integral is that it is absolutely continuous</u>.

PROOF. Suppose f is integrable in $[a, b]$ and

$$F(x) = \int_a^x f(t)\,dt.$$

Case (i). If F is bounded then there exists $c > 0$ such that $|f(x)| \leq c$ for all $a \leq x \leq b$. Then

$$|F(\beta) - F(\alpha)| = \left| \int_{\alpha}^{\beta} f(t)\,dt \right| \leq c(\beta - \alpha)$$

and the absolute continuity of F is immediate.

Case (ii). If f is not bounded, let $\varepsilon > 0$ be given. We can decompose f into a sum of two functions g and h where g is bounded and integrable and $\int h\,dx < \varepsilon/2$. i.e., $f(x) = g(x) + h(x)$ (to see that this is possible :

$$f_n(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq n \\ n & \text{if } |f(x)| \not\leq n. \end{cases}$$

Then

$$f(x) = f_n(x) + \underbrace{f(x) - f_n(x)}_{h_n(x)} .$$

Let c be the bound of g. Then for any system of non-overlapping intervals (α_k, β_k) such that

$$\left| \sum [F(\beta_k) - F(\alpha_k)] \right| = \left| \sum \int_{\alpha_k}^{\beta_k} f(t)\,dt \right|$$

$$\leq \sum \int_{\alpha_k}^{\beta_k} |g(t)|dt + \sum \int_{\alpha_k}^{\beta_k} hdt$$

$$\leq c \sum (\beta_k - \alpha_k) + \mathcal{E}/2 < c\frac{\mathcal{E}}{2c} + \frac{\mathcal{E}}{2} = \mathcal{E} .$$

Thus F is absolutely continuous.

Suppose F is absolutely continuous. Given $\mathcal{E} > 0$, choose a $\delta > 0$ such that for any system of non-overlapping intervals (α_k, β_k)

$$\sum | F(\beta_k) - F(\alpha_k)| < \mathcal{E}$$

whenever $\sum (\beta_k - \alpha_k) < \delta$. We claim F is of bounded variation.

3.8 LEMMA 20. <u>A necessary and sufficient condition that a function</u> $F(x)$ <u>be an integral of an element</u> $f(x) \in L^p$ $(1 < p < \infty)$ <u>is that the sum</u>

$$(*) \qquad \sum_{k=1}^{m} \frac{|F(x_k) - F(x_{k-1})|^p}{(x_k - x_{k-1})^{p-1}}$$

<u>formed for every system of points</u> $x_0 < x_1 < \cdots < x_m$ <u>lying in</u> $[a,b]$ <u>have a finite least upper bound</u>.

PROOF. Necessity: Assume F is the integral of $f(x)$

$$|F(x_k) - F(x_{k-1})| \leq \int_{x_{k-1}}^{x_k} |f(x)|dx$$

$$\leq (x_k - x_{k-1})^{1-\frac{1}{p}} \left(\int_{x_{k-1}}^{x_k} |f(x)|^p dx\right)^{\frac{1}{p}}$$

$$\frac{|F(x_k) - F(x_{k-1})|^p}{(x_k - x_{k-1})^{p-1}} \leq \int_{x_{k-1}}^{x_k} |f(x)|^p dx.$$

$$(**) \quad \sum_{k=1}^{m} \frac{|F(x_k) - F(x_{k-1})|^p}{(x_k - x_{k-1})^{p-1}} \leq \int_a^b |f(x)|^p dx$$

<u>Sufficiency</u>: Assume the sums (*) are bounded. Let B^p be l.u.b. Let (α_k, β_k) be a system of non-overlapping intervals. By Hölders inequality for the (sum)

$$\sum |F(\beta_k) - F(\alpha_k)| \leq \left[\sum \frac{|F(\beta_k) - F(\alpha_k)|^p}{(\beta_k - \alpha_k)^{p-1}}\right]^{\frac{1}{p}} \left[\sum(\beta_k - \alpha_k)\right]^{\frac{p-1}{p}}$$

$$\leq B \left[\sum (\beta_k - \alpha_k)\right]^{\frac{p-1}{p}}.$$

This implies $F(x)$ is absolutely continuous. Therefore $F'(x)$ exists a.e. and $F(x)$ is its indefinite integral.

Now this derivative is the limit of a sequence of functions $f_n(x)$. For example divide $[a,b]$ into 2^n equal segments and in each of these define $f_n(x)$ by

$$\frac{F(\beta) - F(\alpha)}{\beta - \alpha}.$$

Now the sum (*) which corresponds to the decomposition considered is precisely the integral of $|f_n|^p$ over $[a,b]$. Then by Fatou's lemma, $|F'(x)|^p$ is integrable and this integral does not exceed B^p.

<u>Remark</u>.

$$B^p = \int_a^b |f(x)|^p dx .$$

PROOF. Already proved

$$\int_a^b |f(x)|^p dx \leq B^p.$$

Also from (**) we have

$$B^p \leq \int_a^b |f(x)|^p dx.$$

PROPOSITION 21. <u>Let</u> T <u>be a bounded linear functional on</u> $L^p(a,b)$ $(1 \leq p < \infty)$. <u>Then there exists</u> $f \in L^q(a,b)$ <u>such that</u>

$$T(g) = \int_a^b f(t)g(t)dt, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

PROOF. Case 1: When $1 < q < \infty$. Let $F(x)$ equal T_{g_x} where g_x is the characteristic function of (a,x). We shall show that $F(x)$ satisfies $(*)$ and is the indefinite integral of a function $f(x)$ which belongs to L^p.

Consider the step function $\varphi(\xi)$ which assumes the values

$$\frac{|F(x_k) - F(x_{k-1})|^{p-1}}{(x_k - x_{k-1})^{p-1}} \operatorname{sgn}\left[F(x_k) - F(x_{k-1})\right] \quad \text{on } (x_{k-1}, x_k)$$

$$\begin{aligned} T\varphi &= T\left(\sum \frac{|F(x_k) - F(x_{k-1})|^{p-1}}{(x_k - x_{k-1})^{p-1}} \operatorname{sgn}(F(x_k) - F(x_{k-1})) \cdot (g_{x_k} - g_{x_{k-1}})\right) \\ &= \sum \frac{|F(x_k) - F(x_{k-1})|^{p-1}}{(x_k - x_{k-1})^{p-1}} \operatorname{sgn}(F(x_k) - F(x_{k-1}))(Tg_{x_k} - Tg_{x_{k-1}}) \\ &= \sum \frac{|F(x_k) - F(x_{k-1})|^{p}}{(x_k - x_{k-1})^{p-1}} \end{aligned}$$

$$T\varphi \leq \|T\| \|\varphi\|_q = \|T\| \cdot \left(\int_a^b |\varphi(\xi)|^q d\xi\right)^{\frac{1}{q}}$$

$$= \|T\| \left[\sum \frac{|F(x_k) - F(x_{k-1})|^{q(p-1)}}{(x_k - x_{k-1})^{q(p-1)}} (x_k - x_{k-1}) \right]^{\frac{1}{q}}$$

$$= \|T\| \left[\sum \frac{|f(x_k) - F(x_{k-1})|^{p}}{(x_k - x_{k-1})^{p-1}} \right]^{\frac{1}{q}}$$

$$\sum \frac{|F(x_k) - F(x_{k-1})|^{p}}{(x_k - x_{k-1})^{p-1}} \leq \|T\| \left[\sum \frac{|F(x_k) - F(x_{k-1})|^{p}}{(x_k - x_{k-1})^{p-1}} \right]^{\frac{1}{q}}$$

or

$$\sum \frac{|F(x_k) - F(x_{k-1})|^{p}}{(x_k - x_{k-1})^{p-1}} \leq \|T\|^p .$$

Therefore $F(x)$ is the indefinite integral of a function $f(x) \in L^p$ and such that

$$(1) \qquad \int_a^b |f(x)|^p dx \leq \|T\|^p .$$

We claim if g is a step function $\in L^q$

$$(*) \qquad Tg = \int_a^b g(t)F'(t)dt \qquad \text{(check)}$$

Since step functions are dense in L^q, (*) holds for all $g \in L^q$. Now

$$Tg = \int_a^b f(t)g(t)dt \qquad g \in L^q$$

$$|Tg| \leq \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q dt\right)^{\frac{1}{q}}$$

$$\|T\| \leq \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}}. \tag{2}$$

Hence

$$\|T\|^p = \int_a^b |f(t)|^p dt .$$

Case (ii): When $p = 1$. For every pair of points $x_1, x_2 \in (a,b)$ we have

$$|F(x_2) - F(x_1)| = |T(g_{x_2} - g_{x_1})| \leq \|T\| \int_a^b |g_{x_2}(\xi) - g_{x_1}(\xi)| d\xi$$

$$= \|T\| \, |x_2 - x_1|$$

i.e. $F(x)$ satisfies the Lipschitz condition. Therefore $F(x)$ is the indefinite integral of a function $f(x)$ such that $|f(x)| \leq M$. We can show as in the preceding case that

$$|Tg| = \int_a^b g(x)f(x)dx = (g,f).$$

It follows that

$$|Tg| \leq (\text{true max}\,|f(x)|\,)\int_a^b |g(x)|dx$$

hence

$$\|T\| \leq \text{true max}\ |f(x)|$$

Also we have $|f(x)| \leq \|T\|$. Therefore $\|T\| = \|f\|$.

This comples the proof.

CHAPTER 4

NORMED AND BANACH ALGEBRAS

4.1 DEFINITION 1. A set X is called a <u>normed algebra</u> over the complex field $\mathbb{K}$ if

(i) X is a normed linear space

(ii) X is a ring with respect to two internal operations, the addition being the vector space addition in (i)

(iii) $\lambda(xy) = (\lambda x)y = x(\lambda y)$

(iv) $\|xy\| \leq \|x\| \, \|y\|$, $x, y \in X, \lambda \in \mathbb{K}$.

If, in addition, X is Banach space, then X is called <u>a Banach algebra</u>.

<u>Example 1.</u> X = Banach space L(X,X) = set of all bounded linear transformations on X. Multiplication is defined by $AB(x) = A(B(x))$ for $A, B \in L(X,X)$ and the norm is defined by

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} \qquad A \in L(X,X)$$

<u>Example 2.</u> $X = C[a,b]$, complex valued continuous functions on $[a,b]$. Multiplication is defined in the pointwise fashion and $\|f\| = \max_{a \leq x \leq b} |f(x)|$, $f \in X$.

Example 3. Let K be a compact Hausdorff space and let $\mathscr{C}(K)$ be the set of all complex valued continuous functions on K. Then under pointwise operations and sup-norm

$$\|f\| = \sup_{x\in K} |f(x)|, \quad f \in \mathscr{C}(K).$$

$\mathscr{C}(K)$ is a Banach algebra.

Example 4. W = set of all absolutely convergent trigonometrical series, $x(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$ and the norm of of any $x(t)$ in W is defined by

$$\|x(t)\| = \|\sum_{n=-\infty}^{\infty} c_n e^{int}\| = \sum_{n=-\infty}^{\infty} |c_n|$$

with multiplication taken as Cauchy product.

Example 5. A = set of all functions analytic in the open unit disk in the complex plane and continuous in the closed unit disk. Multiplication is defined pointwise and

$$\|f\| = \max_{|z|\leq 1} |f(z)| = \max_{|z|=1} |f(z)|, \quad f\in A$$

Example 6. P_n = space of all polynomials of degree less than or equal to n. If $f,g \in P_n$, $f(t) = \sum_0^n b_\ell t^\ell$
$g(t) = \sum_0^n a_j t^j$, we take

$$g(t)f(t) = \sum_0^n C_k t^k$$

where $C_k = \sum_{j+\ell=k} a_j b_\ell$. Define $\|\sum_{j=o}^n a_j t^j\| = \sum_{j=o}^n |a_j|$

Example 7. $G = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ any finite group $L_1(G)$ denotes the class of all complex valued functions on G. Multiplication is defined by $*$ and the product of two functions $f,g \in L_1(G)$ is defined by

$$(f*G)(\sigma_k) = \sum_{\sigma_i \sigma_j = \sigma_k} f(\sigma_i) g(\sigma_j)$$

Norm of f is

$$\|f\| = \sum_{i=1}^n |f(\sigma_i)|$$

<u>Example 8</u>. Z = integers and $L_1(Z)$, the set of all complex valued functions f on Z such that $\sum_{n=-\infty}^{\infty} |f(n)| < \infty$. If $f, g \in L_1(Z)$ multiplication $f*g$ is defined by

$$(f*g)(n) = \sum_{m=-\infty}^{\infty} f(n-m)g(m)$$

and

$$\|f\| = \sum_{n=-\infty}^{\infty} |f(n)|.$$

DEFINITION 2. Let X. be a Banach space and D an open set in $\mathbb{K}$. A function $x: D \to X$ is said to be analytic in D if

$$x'(\lambda_0) = \lim_{\lambda \to \lambda_0} \frac{x(\lambda) - x(\lambda_0)}{\lambda - \lambda_0}$$

exists for all λ_0 in D, where the limit is taken in the norm topology of X.

LIOUVILLE'S THEOREM: <u>Let</u> $x: K \to X$ <u>where</u> X <u>is a Banach space. If</u> x <u>is analytic in the entire complex plane and bounded</u> i.e. $\|x(\lambda)\| \leq M$ <u>for all</u> $\lambda \in \mathbb{K}$, <u>then</u> x <u>is a constant</u>.

PROOF. Let f be an arbitrary bounded linear functional on X. Then there exists $k \geq o$ such that $|f(y)| \leq k\|y\|$ for all $y \in X$. Now f_x is analytic and $|f(x(\lambda))| \leq k\|x(\lambda)\| \leq kM$. By Liouville's theorem for a single complex variable, fx is a constant. If $\alpha, \beta \in K$, then $f(x(\alpha)) = f(x(\beta))$ which by linearity of f gives $f(x(\alpha)-x(\beta)) = o$. Since f is any bounded linear functional, it is a consequence of Hahn Banach theorem that $x(\alpha)-x(\beta) = 0$ or $x(\alpha) = x(\beta)$ for $\alpha, \beta \in K$.

Exercise 1. Show that in a normed algebra the ring multiplication is continuous.

PROPOSITION 1. Let X be a nonzero Banach algebra with identity e. If $x \in X$ and $\|e-x\| < 1$, then

1. x is a unit (i.e. x has an inverse) and
2. $x^{-1} = e + \sum_{1}^{\infty} (e-x)^n$.

PROOF. The series in the right hand side of (2) converges normally since $\|e-x\| < 1$. Now $x = e-(e-x)$ so that

$$[e-(e-x)]\cdot\left[e+\sum_{1}^{\infty}(e-x)^n\right] = e+(e-x)+(e-x)^2+\dots$$
$$-(e-x)-(e-x)^2-\dots$$
$$= e.$$

PROPOSITION 2. <u>Let</u> X <u>be a Banach algebra with identity. Let</u> λ <u>be a complex number such that</u> $\|x\| < |\lambda|$. <u>Then</u> $x-\lambda e$ <u>is a unit</u>

PROOF. Now $\|e-(e-\lambda^{-1}x)\| = \|\lambda^{-1}x\| = \frac{\|x\|}{|\lambda|} < 1$ which implies that $e-\lambda^{-1}x$ is a unit. Then $x-\lambda e = -\lambda(e-\lambda^{-1}x)$ is also a unit.

<u>Remark</u>. $(\lambda e-x)^{-1} = \sum_{n=1}^{\infty} \lambda^{-n} x^{n-1}$

PROPOSITION 3. <u>In a Banach algebra</u> X <u>with identity</u> e, <u>the set</u> U <u>of all units is open and the operation of inversion is continuous on</u> U .

PROOF. First we notice $e \in U$ and the unit sphere $S_1(e) = \{ x \in X \mid \|e-x\| < 1\} \subset U$. Let $x \in U$. Then $xx^{-1} = e$. Since the ring multiplication is continuous, there exists a neighbourhood $N(x)$ of x such that $N(x)x^{-1} \subset S_1(e)$. Let $y \in N(x)$. Then $yx^{-1} \in S_1(e)$ and hence is a unit. Therefore there exists $z \in X$ such that $(yx^{-1})z = y(x^{-1}z) = e$, $zyx^{-1} = e$ or $(x^{-1}z)y = e$. Hence $y \in U$. Thus $N(x) \subset U$ and U is open.

It remains to show that the operation of taking inverses is continuous in U. Let $x_n \in U$ such that $x_n \to x$. This implies that $x^{-1}x_n \to e$ or for any $\varepsilon > o$ there exists an integer $N(\varepsilon)$ such that

$$\|x^{-1}x_n - e\| < \varepsilon \text{ for } n > N.$$

Choose N_1 such that $\|x^{-1}x_n - e\| < 1$ for $n > N_1$ and consider the series $e + \sum_{k=1}^{\infty}(e - x^{-1}x_n)^k$ for $n > N_1$. This series converges to $(x^{-1}x_n)^{-1} = x_n^{-1}x$. From the absolute convergence, then

$$\|e - x_n^{-1}x\| \leq \sum_{k=1}^{\infty} \|e - x^{-1}x_n\|^k = \sum_{k=1}^{\infty} \|x^{-1}(x - x_n)\|^k$$

(*)

$$\|x^{-1}\|^k$$

$$\leq \sum_{k=1}^{\infty} \|x-1\|^k \|x - x_n\|^k$$

Since $\|x - x_n\| \to o$ we can choose N_2 such that $n > N_2$ will make (*) sufficiently small or $\|e - x_n^{-1}x\| \to o$ which means $x_n^{-1} \to x^{-1}$. This establishes the continuity.

DEFINITION.3. Let $\lambda \in K$. If $x - \lambda e$ is a unit, then λ is called a <u>regular point</u> of x. The set of all non-regular points of x is called the <u>spectrum</u> of x and is denoted by $\sigma(x)$.

PROPOSITION 4. Let X be a Banach algebra with identity. If $x \in X$ then $\sigma(x)$ is a nonempty compact subset of $\mathbb{K}$.

PROOF. First we prove that $\sigma(x)$ is closed. Enough to show the set of regular points is open. Let λ_0 be a regular point of x. Then $x_{\lambda_0} = x-\lambda_0 e \in U$. Since U is open there is a neighbourhood $N(x_{\lambda_0})$ of x_{λ_0} such that $N(x_{\lambda_0}) \subset U$.

Since the mapping $\lambda \in \mathbb{K} \to x-\lambda e \in X$ is continuous there exists a neighbourhood $N(\lambda_0)$ of λ_0 such that $\lambda \in N(\lambda_0)$ implies $x-\lambda e \in N(x_{\lambda_0}) \subset U$. Hence, each point of $N(\lambda_0)$ is a regular point and thus the set of all regular points is open which implies $\sigma(x)$ is closed.

Since $\|x\| < |\lambda|$ implies that $x-\lambda e \in U$, it is clear that $\sigma(x)$ is contained in the closed disk of radius $\|x\|$. $\sigma(x)$ is thus closed and bounded and hence compact.

It remains to show that $\sigma(x)$ is not empty. If C_R denotes the set of all regular points of x then x may be thought of as mapping $C_R \to X$ given by $\lambda \to (x-\lambda e)^{-1}$. We assert that $x(\lambda) = (x-\lambda e)^{-1}$ is analytic in the set of all regular points. If λ_1, λ_2 are any two regular points, then

$$x(\lambda_1)^{-1}x(\lambda_2) = (x-\lambda_1 e)\, x(\lambda_2) = [(x-\lambda_2 e)+(\lambda_2-\lambda_1)e]\, x(\lambda_2)$$

$$= e+(\lambda_2-\lambda_1)x(\lambda_2)$$

so that

$$x(\lambda_2) - x(\lambda_1) = (\lambda_2-\lambda_1)x(\lambda_1)x(\lambda_2)$$

from which we obtain

$$\lim_{\lambda\to\lambda_0} \frac{x(\lambda)-x(\lambda_0)}{\lambda - \lambda_0} = x(\lambda_0)^2 = (x-\lambda_0 e)^{-2}.$$

Now suppose that $\sigma(x) = \emptyset$. Then $x(\lambda) = (x-\lambda e)^{-1}$ is analytic throughout K and

$$\lim_{\lambda\to\infty} (x-\lambda e)^{-1} = \lim_{\lambda\to\infty} \frac{1}{\lambda} (\frac{1}{\lambda} x-e)^{-1} = 0.$$

By Liouville's theorem $(x-\lambda e)^{-1}$ is identically 0. This is impossible since 0 has no inverse. Thus $\sigma(x) \neq \emptyset$.

PROPOSITION 5. <u>Let X be a complex Banach algebra with identity. Then X is isomorphic to K if one of the following conditions is satisfied.</u>

(1) <u>X is a division algebra</u>

(2) $\|xy\| = \|x\|\cdot\|y\|$ <u>for all</u> $x,y\in X$

(3) $\|x^{-1}\| \leq \|x\|^{-1}$, x <u>invertible</u>.

PROOF. (1) Let $x \in X$. Then $\sigma(x) \neq \emptyset$. Let $\lambda \in \sigma(x)$ and consider $x-\lambda e$. Now $(x-\lambda e)^{-1}$ does not exist. Since X is a division algebra, the only element which has no inverse is 0. Hence $x-\lambda e = 0$ or $x = \lambda e$. Thus every element of X is of the form λe and we have the map $X \to \mathbb{K}$ defined by $\lambda e \to \lambda$ which is clearly an isomorphism.

(3) We shall prove now (3). If we **can** show that every non-zero element is invertible, then (1) applies and the result follows. Let

$$A_\rho = \{x \mid \|x\| > \rho\}$$

where $\rho > 0$. Then the set A_ρ is connected. Let U_ρ denote the set of all invertible elements in A_ρ i.e. $U_\rho = U \cap A_\rho$. Then U_ρ is open in A_ρ. Let $x_n \in U_\rho$ and $x_n \to x$ in X. Since

$$\|x_m^{-1}-x_n^{-1}\| = \|x_m^{-1}(x_n-x_m)x_n^{-1}\| \leq \frac{1}{\rho^2}\|x_n - x_m\|$$

the sequence $\{x_n^{-1}\}$ converges. Clearly $x_n^{-1} \to x^{-1}$. Thus $x \in U_\rho$ and U_ρ is closed. Since $\rho e \in U_\rho$, $U_\rho \neq \emptyset$. By connectedness $U_\rho = A_\rho$ and thus A_ρ contains invertible elements only.

(2) To prove (2) we assume that $\|e\| = 1$. Then if x is invertible, then $1 = \|e\| = \|x^{-1}x\| = \|x^{-1}\| \|x\|$ so that $\|x^{-1}\| = \|x\|^{-1}$ and now (3) applies.

This completes the proof.

4.2 Gelfand Representation

Throughout X will denote a commutative Banach algebra with identity e and $\|e\| = 1$.

We remark that the closure of a proper ideal in X is again a proper ideal.

Let I be a closed proper ideal in X. Then $\frac{X}{I}$ is again an algebra with identity. If we introduce the norm

$$\|x+I\| = \inf_{y \in I} \|x+y\|$$

then $\frac{X}{I}$ becomes a Banach space. It is easy to check

$$\|(x+I)(y+I)\| \leq \|x+I\| \; \|y+I\|$$

Hence $\frac{X}{I}$ is a Banach algebra under this norm. Then clearly the natural homomorphism is norm-diminishing

$$\|x+I\| \leq \|x\|$$

PROPOSITION 6. <u>Every maximal ideal in</u> X <u>is closed</u>.

PROOF. The closure $\overline{M}$ of a maximal ideal M in X is also a proper ideal of X. Now by maximality of M, $\overline{M} = M$ which implies that M is closed.

PROPOSITION 7. <u>Let</u> M <u>be a maximal ideal in</u> X. <u>Then</u> $\frac{X}{M} \cong \mathbb{K}$ <u>and if</u> $\emptyset$ <u>is the homomorphism of</u> X <u>onto</u> $\mathbb{K}$, <u>then</u> $|\emptyset(x)| \leq \|x\|$ <u>for all</u> $x \in X$.

PROOF. Since X is a commutative ring with identity and M is a maximal ideal, $\frac{X}{M}$ is a field. Now $\frac{X}{M}$ is a Banach algebra and it is also a field. Hence $\frac{X}{M} \cong \mathbb{K}$.

If $\emptyset$ is a homomorphism of X onto $\mathbb{K}$, let M be its kernel. Now since $\frac{X}{M} \cong \mathbb{K}$, there exists a scalar $\hat{x}(M)$ such that $x+M = \hat{x}(M)(e+M)$, and the isomorphism is explicitly given by

$$X \longrightarrow \frac{X}{M} \xrightarrow{\text{iso}} K$$

$$x \longrightarrow x+M = \hat{x}(M)(e+M) \longrightarrow \hat{x}(M)$$

so that $\emptyset(x) = \hat{x}(M)$. Now from the definitions of $\|x+M\|$

it follows that

$$\|x\| \geq \|x+M\| = \|\hat{x}(M)(e+M)\| = |\hat{x}(M)|\cdot\|e+M\| \qquad (*)$$

Now $\|e+M\| \leq \|e\| = 1$. Suppose $\|e+M\| < 1$. Then there exists $y \in e+M$ with $\|y\| < 1$. Then $y = e+x$ for some $x \in M$ and then $-x$ will be a unit. This is impossible since M is proper. Thus $\|e+M\| = 1$. Then (*) gives

$$\|x\| \geq \|\hat{x}(M)\|\cdot\|e+M\| = \|\hat{x}(M) = |\phi(x)|$$

This completes the proof.

Remark 1. ϕ satisfies the following properties

$$\phi(e) = 1,\ \phi(x+y) = \phi(x)+\phi(y),\ \phi(\lambda x) = \lambda\phi(x)$$

$$\phi(xy) = \phi(x)\ \phi(y) \text{ and } |\phi(x)| \leq \|x\|$$

ϕ is called a multiplicative linear functional on X.

Remark 2. There exists a one to one correspondence between the set of all maximal ideals of X and the set of all multiplicative linear functionals on X.

Let $\mathfrak{m}$ denote the set of all maximal ideals of X. Let ϕ be the multiplicative linear functional on X with kernel M. Then to each $x \in X$ we define a map

$$\hat{x}: \mathfrak{m} \to \mathbb{K} \text{ by } \hat{x}(M) = \phi(x), \quad M \in \mathfrak{m}$$

Then we have $\hat{e} = 1$, $(x+y)^\wedge = \hat{x}+\hat{y}$, $(\lambda x)^\wedge = \lambda\hat{x}$, $(xy)^\wedge = \hat{x}\hat{y}$. Further if $M, N \in \mathfrak{m}$ and $M \neq N$, then there exists $x \in X$ such that $\hat{x}(M) \neq \hat{x}(N)$. This is trivial if we notice that $\hat{x}(M) = 0$ if and only if $x \in M$.

We shall now topologize $\mathfrak{m}$ so that a Hausdorff topology results. Let $\varepsilon > 0$ be arbitrary and let $x_1, \ldots, x_n$ be a finite number of elements of X. If $M_0 \in \mathfrak{m}$ then the class of sets

$$V(M_0, x_1, \ldots, x_n, \varepsilon) = \{M \in \mathfrak{m} \mid |\hat{x}_k(M) - \hat{x}_k(M_0)| < \varepsilon, \ 1 \leq k \leq n\}$$

form a fundamental system of neighborhoods of M_0. It is easy to check that these sets do form a fundamental system of neighbourhoods. If $M_1, M_2 \in \mathfrak{m}$ such that $M_1 \neq M_2$, then there exists $x \in X$ such that $\hat{x}(M_1) \neq \hat{x}(M_2)$. Let $\varepsilon = \frac{1}{2}|\hat{x}(M_1) - \hat{x}(M_2)|$. If $V(M_1, x, \frac{\varepsilon}{2}) \cap V(M_2, x, \frac{\varepsilon}{2}) \neq \emptyset$, we can find $M \in \mathfrak{m}$ such that $|\hat{x}(M_1) - \hat{x}(M)| < \frac{\varepsilon}{2}$ and $|\hat{x}(M_1) - \hat{x}(M_2)| < \frac{\varepsilon}{2}$ and then $|\hat{x}(M_1) - \hat{x}(M_2)| \leq |\hat{x}(M_1) - \hat{x}(M)| + |\hat{x}(M) - \hat{x}(M_2)| < \varepsilon$ which is a contradiction. The topology thus introduced is called the <u>Gelfand topology</u>.

PROPOSITION 8. <u>The maximal ideal space $\mathfrak{M}$ is compact with respect to Gelfand topology.</u>

PROOF. For each $x \in X$, let $S_x = \{ z \mid |z| \leq \|x\| \}$. Then S_x is compact for each x. By Tychonoff theorem $S = \prod_{x \in X} S_x$ is compact with Tychonoff topology. Notice that an element α in S is a function on X such that $\alpha_x \in S_x$. Then an element of the basis for the topology of S is given by

$$W(\alpha^0, x_1, \ldots, x_n, \varepsilon) = \{ \alpha \in S \mid |\alpha_{x_i} - \alpha^0_{x_i}| < \varepsilon, \; i=1,2,\ldots,n \}$$

consider the mapping $g: \mathfrak{M} \to S$ given by $g(M) = \alpha$ where $\alpha_x = \hat{x}(M)$. Since $|\hat{x}(M)| \leq \|x\|$, we have $\alpha_x \in S_x$. We now claim that g is a homeomorphism of $\mathfrak{M}$ onto a closed subset of S. Since S is compact, this will imply the $g(\mathfrak{M})$ is also compact. Let $S_1 = g(\mathfrak{M})$. Now g is 1:1. If $M_1 \neq M_2$, there exists $x_0 \in X$ such that $\hat{x}_0(M_1) \neq \hat{x}_0(M_2)$ so that $g(M_1) \neq g(M_2)$.

To show g is continuous. Let $W(\alpha^0, x_1, \ldots x_n, \varepsilon)$ be a basis element in S_1. Then $g^{-1}(W) = \{ M \in \mathfrak{M} \mid |\hat{x}_n(M) - \alpha^0_k| < \varepsilon, 1 \leq k \leq n \}$. Since g is onto S_1, there must exist $M \in \mathfrak{M}$ such that $\hat{x}_k(M_0) = \alpha^0_k$, $g^{-1}(W) = V(M_0, x_1, \ldots, x_n, \varepsilon)$. This will imply g is continuous.

To show that S_1 is closed. Let $\alpha^{\circ} \in \overline{S_1}$. To this end consider the mapping $\phi: X \to \mathbb{K}$ given by $\phi(x) = \alpha^{\circ}_x$. First we notice that ϕ is a nontrivial functional and hence onto. Suppose $\alpha^{\circ}_x \equiv 0$, then for any $x_1 \ldots, x_n \in X$ and for any $\varepsilon > 0$ there exists $M \in \mathfrak{M}$ such that $|\hat{x}_k(M) - \alpha^{\circ}_{x_k}| = |\hat{x}_k(M)| < \varepsilon$. In particular, $|e(M)| < \varepsilon$ which is a contradiction. Thus $\phi \not\equiv 0$. Clearly ϕ is a multiplicative linear functional on X and thus it is onto C. If M_0 is the kernel of the homomorphism ϕ, then $\frac{X}{M_0} \cong \mathbb{K}$. Since $\mathbb{K}$ is a field, M_0 must be a maximal ideal. Then $\phi(x) = \hat{x}(M_0) = \alpha^{\circ}_x$. Hence $g(M_0) = \alpha^{\circ} \in S_1$. S_1 is thus closed. This completes the proof.

<u>Remark</u>. $x: \mathfrak{M} \to \mathbb{K}$ is continuous and $\|\hat{x}\|_\infty \le \|x\|$

where $$\|\hat{x}\|_\infty = \sup_{M \in \mathfrak{M}} |\hat{x}(M)|.$$

PROPOSITION 9. <u>The mapping</u> $x \to \hat{x}$ <u>is a norm-diminishing homomorphism of</u> X <u>onto a separating subalgebra</u> $\hat{X}$ <u>of</u> $\mathscr{C}(\mathfrak{M})$.

DEFINITION 4. $\hat{x}$ is called the <u>Gelfand transform</u> of x, $\hat{X}$ is the Geldfand transform of X and the mapping $x \to \hat{x}$ the <u>Gelfand representation</u> of X.

PROPOSITION 10. <u>Let</u> $x \in X$. <u>Then</u> $\sigma(x) = \{\hat{x}(M) \mid M \in \mathfrak{M}\}$

PROOF. Let $\hat{x}(M) = \lambda$ where $M \in \mathfrak{M}$. Then $(x-\lambda e)^{\wedge}(M) = \hat{x}(M) - \lambda\hat{e}(M) = 0 \Rightarrow x-\lambda e \in M \Rightarrow x-\lambda e$ is not a unit. Therefore $\lambda \in \sigma(x)$ or $\{\hat{x}(M) \mid M \in \mathfrak{M}\} \subset \sigma(x)$. Conversely suppose $\lambda \in \sigma(x)$. Then $x-\lambda e$ is not a unit. Then there exists some maximal ideal M such that $x-\lambda e \in M$ so that $(x-\lambda e)^{\wedge}(M) = 0$ or $\hat{x}(M) = \lambda$.

DEFINITION 5. The real number $r_\sigma(x) = \sup\limits_{\lambda \in \sigma(x)} |\lambda|$ is said to be the <u>spectral radius</u> of **x**.

PROPOSITION 11. $r_\sigma(x) = \lim\limits_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \|\hat{x}\|_\infty$

PROOF. $r_\sigma(x) = \sup\limits_{\lambda \in \sigma(x)} |\lambda| = \sup\limits_{M \in \mathfrak{M}} |\hat{x}(M)| = \|\hat{x}\|_\infty$

If $\lambda \in \sigma(x)$, then $\lambda^n \in \sigma(x^n)$; for otherwise $(x^{n-1}+\lambda x^{n-2}+\ldots+\lambda^{n-1}e)\times(x^n-\lambda^n e)^{-1}$ will be an inverse of $x-\lambda e$ which gives a contradiction. Hence $|\lambda|^n \leq \|x^n\|$ which implies

$$r_\sigma(x) = \lim\inf \|x^n\|^{\frac{1}{n}}.$$

On the other hand, if $|\lambda| > r_\sigma(x)$, $(\lambda e-x)^{-1}$ exists from the definition of $r_\sigma(x)$. Since $(\lambda e-x)^{-1}$ is a function analytic on the set of regular points, we can expand into a Laurent series

$$(\lambda e-x)^{-1} = \sum_{n=1}^{\infty} \lambda^{-n} x^{n-1} , \qquad |\lambda| > r_\sigma(x).$$

As this series converges for each λ , its general term must be bounded so that $\|x^n\|^{\frac{1}{n}} \leq \mu^{\frac{1}{n}}|\lambda|^{\frac{n}{n+1}}$ for some $\mu > 0$.

Thus we get

$$\lim\sup \|x^n\|^{\frac{1}{n}} \leq r_\sigma(x)$$

Hence $r_\sigma(x) \leq \lim\inf \|x^n\|^{\frac{1}{n}} \leq \lim\sup \|x^n\|^{\frac{1}{n}} \leq r_\sigma(x)$. This complets the proof.

DEFINITION 6. The <u>radical of</u> X is the set $\bigcap_{M\in\mathfrak{M}} M$. X is said to be <u>semisimple</u> if the radical of X is $\{0\}$

Notice that $x \in \bigcap_{M\in\mathfrak{M}} M \Leftrightarrow \hat{x}(M) = 0$ for all $M\in\mathfrak{M}$

$$\Leftrightarrow \sup_{M\in\mathfrak{M}} |\hat{x}(M)| = 0$$

$$\Leftrightarrow \lim \|x^n\|^{\frac{1}{n}} = 0$$

i.e. x is in the radical of X if and only if its spectral radius is 0.

DEFINITION 7. x is called <u>nilpotent</u> if there exists n such that $x^n = 0$ and it is <u>topologically nilpotent</u> (or generalized nilpotent) if $\lim\|x^n\|^{\frac{1}{n}} = 0$

<u>Remark</u> 1. The radical of X is the set of all topologically nilpotent elements of X.

Remark.2. Gelfand representation is an isomorphism if and only if X is semi simple.

An example. Let $W = \left\{ x(t) = \sum_{-\infty}^{\infty} c_n e^{int} \,\middle|\, \sum_{-\infty}^{\infty} |c_n| < \infty \right.$ with $\|x\| = \sum_{-\infty}^{\infty} |c_n|$. Then W is a commutative Banach algebra with identity. We shall first determine $\mathfrak{M}$. Let $x_0 = e^{it}$ and let $\hat{x}_0(M) = \alpha$. Then $x_0^{-1} = e^{-it}$ and $\hat{x}_0^{-1}(M) = \alpha^{-1}$. Then we have

$$|\alpha| = |\hat{x}_0(M)| \leq \|x_0\| = 1$$

$$|\alpha^{-1}| = |\hat{x}_0^{-1}(M)| \leq \|x_0^{-1}\| = 1$$

which show that $|\alpha| = 1$ so that there exists $t_0 \in [0, 2\pi]$ such that $\alpha = e^{it_0}$. Then the mapping ϕ given by $\phi: W \to \frac{W}{M} \xrightarrow{\text{iso}} \mathbb{K}$ which sends $x \to \hat{x}(M)$ gives $x(t) = \sum_{-\infty}^{\infty} c_n e^{int} \to \sum c_n e^{int_0} = \hat{x}(M)$ for ϕ is homomorphism and for any finite sum $S_N \in W, S_N = \sum_{n=N}^{N} c_n e^{int}$ we have $\hat{S}_N(M) = \sum_{n=-N}^{N} c_n e^{int_0}$ so that $\|\hat{S}_N(M) - \hat{x}(M)\| \leq \|S_N - x\|$ which also establishes the continuity of ϕ. Thus M consists of those $x \in W$ such that $x(t_0) = o$.

Conversely let $t_o \in [0, 2\pi)$ and consider the set M $M = \{x \in W \mid x(t_o) = 0\}$. M is clearly an ideal of W. Let I be an ideal in W containing M properly. Then there exists $y \in I$, $y \notin M$. By definition of M, we have $y(t_o) \neq 0$. If $z \in W$ we can write

$$z(t) = \frac{z(t_o)}{y(t_o)} y(t) + \left\{ z(t) - \frac{z(t_o)}{y(t_o)} y(t) \right\}.$$

Since $y \in I$ and $z(t) - \frac{z(t_o)}{y(t_o)} y(t) \in M \subset I$, we must have $I = W$. Hence M is maximal.

WIENER'S THEOREM. <u>If</u> $\sum_{-\infty}^{\infty} c_n e^{int}$ <u>is absolutely convergent and vanishes nowhere, then</u> $\frac{1}{\sum_{-\infty}^{\infty} c_n e^{int}}$ <u>can be expanded in an absolutely convergent trigonometric series</u>.

PROOF. If $x(t) \neq 0$ for every t, then $x \notin M$ for any M so that x is a unit or $\frac{1}{x(t)} \in W$.

4.3 Adjunction of an Identity.

PROPOSITION. <u>If</u> X <u>is a complex algebra without identity, then</u> X <u>can be extended to an algebra</u> $\hat{X}$ <u>with identity. If</u> X <u>is a normed algebra (Banach algebra) so is</u> $\hat{X}$.

PROOF. Let $\hat{X}$ denote the set of all pairs (α, x) where $\alpha \in \mathbb{K}, x \in X$. Define the algebraic operations in $\hat{X}$ by

$$\beta(\alpha, x) = (\beta\alpha, \beta x), \qquad \beta \in \mathbb{K}$$

$$(\alpha_1, x_1) + (\alpha_2, x_2) = (\alpha_1 + \alpha_2,\ x_1 + x_2)$$

$$(\alpha_1, x_1)(\alpha_2, x_2) = (\alpha_1\alpha_2, \alpha_1 x_2 + \alpha_2 x_1 + x_1 x_2).$$

Then $\hat{X}$ is an algebra with identity $(1,0)$ and the map $x \to (0,x)$ is an isomorphism. Identifying x with $(0,x)$ and setting $e = (1,0)$ we have

$$(\alpha, x) = \alpha e + x$$

If X is a normed algebra, let $\|\alpha e + x\| = |\alpha| + \|x\|$ then $\hat{X}$ is clearly a normed algebra. Now suppose X is a Banach algebra. Let $\{\alpha_n e + x_n\}$ be a Cauchy sequence in $\hat{X}$. Then given $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that $\|(\alpha_n e + x_n) - (\alpha_m e + x_m)\| < \varepsilon$ if $m, n \geq N$. This is the same as $|\alpha_n - \alpha_m| + \|x_n - x_m\| < \varepsilon$ which implies that $\{\alpha_n\}$ and $\{x_n\}$ are Cauchy sequence in C and X respectively. Since C and X are complete, there exist $\alpha \in C$, $x \in X$ such that $|\alpha_n - \alpha| \to 0$ and $\|x_n - x\| \to 0$ as $n \to \infty$. Hence $\|\alpha_n e + x_n) - (\alpha e + x)\| \to 0$ as $n \to \infty$ Hence $\hat{X}$ is also a Banach algebra.

4.4 Topological divisors of zero.

DEFINITION 8. An element x in a Banach algebra X is called a topological zero divisor of X if there is a sequence $\{y_n\}$ in X such that $\inf\|y_n\|=o$ and $\lim xy_n = o$. The modulus of integrity $\omega(x)$ of $x \in X$ is defined by

$$\omega(x) = \inf_{y \neq o} \frac{\|xy\|}{\|y\|} .$$

Notice x is a topological zero divisior iff $\omega(x) = 0$.

PROPOSITION. Any topologically nilpotent element in X is also a topological zero divisor.

PROOF. Suppose x is topologically nilpotent. Then its spectral radius is zero. Hence there exists complex members $\lambda_n \to o$ such that $x-\lambda_n e$ are invertible. Let $y_n=(x-\lambda_n e)^{-1}$. Then

$$xy_n = (x-\lambda_n e)y_n+\lambda_n y_n = e+\lambda_n y_n .$$

Since x is not invertible, so is xy_n. Hence $|\lambda_n|\cdot\|y_n\| \geq 1$. Hence $\|y_n\| \to \infty$. But then

$$\frac{\|xy_n\|}{\|y_n\|} \leq \frac{1}{\|y_n\|} + \frac{|\lambda_n|\,\|y_n\|}{\|y_n\|} \to 0 .$$

Hence $\omega(x) = 0$ and x is a topological zero divisor.

PROPOSITION (Arens). z <u>is a topological zero divisor of</u> X <u>if and only if</u> z <u>is not invertible in any Banach algebra extension</u> Y <u>of</u> X.

PROOF. The necessity is trivial. We shall prove the sufficiency. Assume that z is not a topological zero divisor. Then $\omega(z) \neq 0$. Choose $\rho > \frac{1}{\omega(x)}$. Let y be the algebra of all formal power series.

$$x(t) = x_0 + x_1 t + x_2 t^2 + \ldots .$$

where $x_0, x_1, x_2, \ldots \in X$ such that $\Sigma \|x_n\| \rho^n < \infty$. Define $\|x\| = \Sigma \|x_n\| \rho^n$. Then Y is a normed algebra. Let $\bar{Y}$ be the completion of Y and let I be a closed ideal in $\bar{Y}$ generated by e-zt. Set $X_1 = \frac{\bar{Y}}{I}$. Then t is an inverse of z in X_1 and X_1 is an extension algebra of X. Let $y \in X$ and $x(t) \in Y$. Then

$$\|y+(e-zt)x(t)\| = \|y+x_0+(x_1-x_0 z)t + (x_2-x_1 z)t^2+\ldots\|$$

$$= \|y+x_0\| + \|x_1 - x_0 z\|\rho + \|x_2 - x_1 z\|\rho^2 + \ldots$$

$$\geq (\|y\|-\|x_0\|) + (\|x_0 z\|-\|x_1\|)\rho + (\|x_1 z\|-\|x_2\|)\rho^2 + \ldots$$

$$= \|y\| - \|x(t)\| + (\|x_0 z\|+\|x_1 z\|\rho + \ldots)\rho$$

$$\geq \|y\| + [\omega(z)\rho - 1]\cdot\|x(t)\| \geq \|y\|$$

Since the elements of the form (e-zt) x(t) are dense in I so the infimum of the left hand side is the Y_1-norm $|||y|||$ of y. Thus $|||y||| \geq \|y\|$ The opposite inequality is evident and thus $|||y||| = \|y\|$. Thus X is embedded in X_1 isometrically.