CONCEPTS IN MODERN MATHEMATICS - II

(TOPOLOGY)

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INTRODUCTION

This is the second in the series 'Concepts in Modern Mathematics'. This volume deals with some fundamental concepts in topology.

The first part deals with point set topology and introduces the notions of compactness, connectedness, product and quotient topologies and para-compact spaces. The second part deals with algebraic topology. Basically it is topology of polyhedra and the materials are freely drawn from the lectures given to the author at Northwestern University by Professor A.I. Weinzweig to whom the author is indebted.

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POINT SET TOPOLOGY

ALGEBRAIC TOPOLOGY

POINT SET TOPOLOGY

1. Topology and Topological Spaces

1.1. DEFINITION. Let S be a set. A family τ of subsets of S is said to be a topology for S if and only if

a) ϕ and S are in τ.

b) If $\tau_1 \subset \tau$ then $\cup \{A / A \in \tau_1\} \in \tau$, that is, the union of members belonging to any sub-family of τ is again in τ.

c) Intersection of any finite number of members of τ is again in τ.

Then, S is called the space for the topology τ and the pair (S, τ) is called a topological space. When there is no confusion, we write S for a topological space, which means that the underlying topology is understood. The sets of τ are called open relative to τ or τ - open or simply open sets when no confusion is likely to arise.

1.2. DEFINITION. If τ_1 and τ_2 are two topologies for S, we say that τ_1 is weaker than τ_2 (or τ_2 is stronger than τ_1) if $\tau_1 \subset \tau_2$.

1.3. DEFINITION. If σ is a family of subsets of S, $\tau(\sigma)$, the topology generated by σ, is the smallest topology containing σ. If $\tau = \tau(\sigma)$, then σ is a

<u>sub-basis</u> for the topology τ. If σ is a sub-basis for τ, then $A \in \tau$ if and only if $A = \phi$ or S or a union (perhaps countable) of finite intersections of elements of σ. If every element of τ is a union of elements of σ, σ is called a <u>basis</u> for τ.

1.4. <u>Examples</u>. a) S = any set. $\tau = (\phi, S)$. Then, τ is called the <u>trivial topology</u> or <u>indiscrete topology</u>.

b) Let S be any set. Let τ be the collection of all subsets of S. Then τ is called the <u>discrete topology</u>.

c) $S = \{a,b,c\}$, $\tau = \{\phi, \{a\}, \{b,c\}, S\}$.

d) S is an infinite set. Let τ be the collection of all subsets A of S such that $S \setminus A$ is finite, together with ϕ.

e) S is the set of all real numbers. The collection of all possible unions of open intervals forms a basis for the corresponding topology.

f) S is the set of all complex numbers. A basis is defined by $\{|z - z_0| < c\}$ where z, z_0 are complex numbers and c is a constant.

1.5. THEOREM. Suppose (S, τ) is a topological space. If a family B of subsets of S is a basis for τ then

a) if $x \in S$, there exists a $B \in \mathcal{B}$ such that $x \in B$.

b) if $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, then there must exist $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.

Proof. Suppose $\mathcal{B}$ is a basis for τ. Let $x \in S$. Therefore there exists $B_\alpha \in \mathcal{B}$ such that $x \in S = \bigcup_\alpha B_\alpha$. Then, $x \in B_\alpha$ for some α.

Let $B_1, B_2 \in \mathcal{B}$. Then $B_1, B_2 \in \tau$ and hence $B_1 \cap B_2 \in \tau$. There exists $B_\alpha \in \mathcal{B}$ such that $x \in B_1 \cap B_2 = \bigcup_\alpha B_\alpha$. Hence, there exists α_0 such that $x \in B_{\alpha_0} \subset B_1 \cap B_2$.

1.6. THEOREM. Let $\mathcal{B}$ be a family of subsets of a set S. If $\mathcal{B}$ satisfies (a) and (b) of Theorem 1.5, then $\mathcal{B}$ is a basis for some topology τ of S. Each element of τ is a union of elements of $\mathcal{B}$.

1.7. DEFINITION. Let S be a topological space. A subset A of S is closed if $S \setminus A$ is open.

1.8. THEOREM. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a family of closed sets in S. Then, the following sets are closed.

a) ϕ and S.

b) $\bigcap_{\lambda \in \Lambda} A_\lambda$

c) $\bigcup_{i=1}^{n} A_{\lambda_i}$.

Proof. Since ϕ and S are open, (a) is trivial from definition. (b) and (c) follow from De Morgan's laws.

1.9. DEFINITION. A subset N of a topological spaces S is called a neighborhood of a point p, if there exists an open set G such that $p \in G \subset N$.

1.10. Remark. An open set is a neighborhood of any point contained in it. A neighborhood need not be an open set.

1.11. THEOREM. A set A is open if and only if for each point $x \in A$, there exists a neighborhood N such that $x \in N \subset A$.

Proof. The necessity is trivial. To establish the sufficiency, assume that the condition holds. We assert that A is an open set. Pick any point $x \in A$. There exists a neighborhood N such that $x \in N \subset A$. By definition of neighborhood, there exists an open set $G(x)$ such that $x \in G(x) \subset N \subset A$. Then $A = \bigcup_{x \in A} G(x)$ and A is open.

1.12. DEFINITION. A neighborhood system at a point x in S is the set of all neighborhoods of x.

1.13 THEOREM. Let S be a topological space and $N(x)$ be the neighborhood system at x, for each $x \in S$. Then the following conditions are true.

a) If $N \in N(x)$, then $x \in N$,

b) If $N \in N(x)$ and $M \in N(x)$, then $N \cap M \in N(x)$

c) If $N \in N(x)$ and $N \subset M$, then $M \in N(x)$.

d) If $N \in N(x)$, then there exists an $M \in N(x)$ such that $M \in N(y)$ for each $y \in M$.

Proof. a) is obvious. To prove (b), let $x \in N \cap M$. There exist open sets G_1 and G_2 such that $x \in G_1 \subset N$, $x \in G_2 \subset M$, so that $x \in G_1 \cap G_2 \subset N \cap M$. Now, G_1 and G_2 are open sets, therefore $G_1 \cap G_2$ is an open set. To prove (c), there exists an open set G such that $x \in G \subset N$. Now $M \supset N$ and $M \supset G$ such that x is contained in G, which implies that M is a neighborhood of x. To prove (d), given $x \in N(x)$, there exists an open set G such that $x \in G \subset N$. Take $M = G$.

1.14. THEOREM. Let S be a nonempty set and for each $x \in S$, let $N(x)$ be a collection of subsets of S such that

a) If $N \in N(x)$, then $x \in N$.

b) If $N, M \in N(x)$, then $N \cap M \in N(x)$.

Let $\tau = \{\phi, \{G \subset S / x \in G, \exists\ N \in N(x) \ni x \in N \subset G\}\}$.
Then τ is a topology for S.

<u>Proof.</u> (i) $S \in \tau$ [since $x \in S \Rightarrow S \subset N(x)$.]

(ii) Let $\{G_\alpha / G_\alpha \in \tau\}$ be a collection. To prove that $\bigcup_\alpha G_\alpha \in \tau$, let $x \in \bigcup_\alpha G_\alpha$. Then, there exists α such that $x \in G_\alpha$ and $N \in N(x)$ such that $x \in N \subset \bigcup_\alpha G_\alpha$ which implies $\bigcup_\alpha G_\alpha \in \tau$.

(iii) Let $G_1, G_2, \ldots, G_n$ be a finite number of sets in τ. Let $x \in \bigcap_{i=1}^{n} G_i$. Then, there exists $N_i \in N(x)$, $i=1,\ldots,n$ such that $x \in N_i \subset G_\alpha$. By (b) of the statement of the theorem $\bigcap_{i=1}^{n} N_i \subset N(x)$ and $x \in \bigcap_{i=1}^{n} N_i \subset \bigcap_{i=1}^{n} G_i$. Hence $\bigcap_{i=1}^{n} G_i \in \tau$.

These three conditions prove that τ is a topology for S.

1.15. DEFINITION. Let A be a subset of a topological space S. Then the largest open set contained in A is called the <u>interior</u> of A and is denoted by A^o. Clearly A is open if $A = A^o$. The <u>closure</u> of A is the smallest closed set containing A and is denoted by $\bar{A}$. Notice that A is closed if and only if $A = \bar{A}$.

1.16. DEFINITION. Let $x \in S$. x is called a <u>limit point</u> of A if for each neighborhood N of x, there exists $y \neq x$ and $y \in N$. The set of all limit points of A is called the <u>derived set</u> of A and is denoted by A'.

1.17. THEOREM. Let A be a subset of S. Then A' and $A \cup A'$ are closed sets. Further $\bar{A} = A \cup A'$.

Proof. We shall first prove that A is closed if and only if $A' \subset A$. Assume A is closed. Then $S \setminus A$ is open. Let $x \in S \setminus A$. There exists N such that $x \in N \subset S \setminus A$ which implies $x \notin A'$. Hence $A' \subset A$. Assume $A' \subset A$. To prove that A is closed, we show that $S \setminus A$ is open. Let $y \in S \setminus A$. $y \notin A \supset A'$ which implies $y \notin A'$ also. Therefore, there exists a neighborhood of y which does not contain any point of A and $S \setminus A$ is open.

To prove the theorem, let $x \in A''$ (derived set of A'). Enough to prove that $A'' \subset A'$. We have to show that every neighborhood of x contains a point of A different from x. Let N be a neighborhood of x. Since x is a limit point of A', there exists a point $y \neq x$ such that $y \in N \cap A'$. Since $y \in A'$ and is a limit point of A, there exists $z \neq y$ such that $z \in N \cap A' \cap A$. Hence x is a limit point of A. Thus $x \in A'$. The proof that $A \cup A'$ is a closed set is left as an exercise.

To prove that $\bar{A} = A \cup A'$, $A \cup A'$ is a closed set containing A. By definition, $\bar{A} \subset A \cup A'$. It remains to prove that $A \cup A' \subset \bar{A}$. Since $A \subset \bar{A}$, it is enough to prove that $A' \subset \bar{A}$. If $x \in A'$, for any neighborhood $N(x)$ of x, there exists $y \in A \cap (N - \{x\}) \subset \bar{A} \cap (N - \{x\})$. Since $\bar{A}$ is closed, $A' \subset \bar{A}$

Hence $A \cup A' = \bar{A}$.

2. MAPPINGS

2.1. DEFINITION. Let X, Y be topological spaces. A function $f : X \to Y$ is said to be continuous if G open in Y implies $f^{-1}(G)$ is open in X.

2.2. THEOREM. Let X, Y be topological spaces and $f : X \to Y$. The following are equivalent.

a) f is continuous.

b) F closed in $Y \Rightarrow f^{-1}(F)$ is closed in X.

c) For each $x \in X$ and $N \in N(f(x))$, $f^{-1}(N) \in N(x)$.

d) If $x \in X$ and $N \in N(f(x))$, there exists a neighborhood $M \in N(x)$ such that $f(M) \subset N$.

e) If $A \subset X$, then $f(\bar{A}) \subset \overline{f(A)}$.

Proof. Exercise.

2.3. DEFINITION. Let X and Y be topological spaces. A one-one, onto function $f : X \to Y$ is called a homeomorphism if f and f^{-1} are both continuous.

2.4. DEFINITION. A mapping which takes open sets into open sets is called an open map.

3. CONNECTED SETS.

3.1. DEFINITION. Suppose X is a topological space and $A \subset X$. We make A into a topological space by defining open sets in A as the intersections of A with the open sets in X. The topology of A thus obtained is known as induced topology or relative topology. A is then called a topological subspace of X.

Let X be a topological space and $A \subset X$. A is said to be <u>connected</u> if there do not exist two nonempty sets A_1 and A_2 such that $A_1 \cap \bar{A}_2 = \bar{A}_1 \cap A_2 = \phi$ and $A_1 \cup A_2 = A$.

3.2. THEOREM. If A is connected, then $\bar{A}$ is connected.

<u>Proof</u>. Let $\bar{A} = B \cup C$ where B and C are both open and closed in $\bar{A}$. Then $A \cap B$ and $A \cap C$ are both open and closed in A. Since A is connected, one of $A \cap B$ and $A \cap C$ is equal to ϕ. Suppose $A \cap B = \phi$. Then $A \subset C$. Hence $\bar{A} \subset C$, since C is closed in $\bar{A}$. Thus $B = \phi$. Similarly, if $A \cap C = \phi$, then $C = \phi$. Hence, we have $\bar{A}$ is connected.

3.3. THEOREM. Let $\mathfrak{A}$ be a family of connected subsets of a topological space S. If $A_\alpha \cap A_\beta \neq \phi$ for any two sets $A_\alpha, A_\beta \in \mathfrak{A}$, then $\bigcup_\alpha A_\alpha$ is connected.

<u>Proof.</u> Let $X = \cup \{A / A \in \mathfrak{A}\}$. Suppose $X = B \cup C$ where B and C are both open and closed in X. Then $A \cap B$ and $A \cap C$ are both open and closed in A, for each A. Since A is connected, one of them should be ϕ, say $A \cap B = \phi$. Then $A \subseteq C$. Suppose $D \in \mathfrak{A}$. If $D \cap C = \phi$, then $D \subset B$ and $A \cap D \subset A \cap B = \phi$ which is a contradiction. Hence $A \cap B = \phi$ for all $A \in \mathfrak{A}$. Hence $B = \phi$. Thus $\cup \{A / A \in \mathfrak{A}\}$ is connected.

3.4. THEOREM. A continuous image of a connected set is connected.

<u>Proof.</u> Exercise.

3.5. DEFINITION. A component in a topological space is a maximal connected subset.

3.6. Remarks. a) Two components are disjoint

b) A component is a closed set.

4. COMPACTNESS.

4.1. DEFINITION. Let S be a topological space and $A \subset S$. A family of subsets $\{D_\lambda\}_{\lambda \in \Lambda}$ is called a <u>covering</u> for A if $A \subset \bigcup_{\lambda \in \Lambda} D_\lambda$. If D_λ's are open, then we call it an <u>open covering</u>.

4.2. DEFINITION. A is <u>compact</u> if every open covering of A has a finite subcovering.

4.3. DEFINITION. A family of sets is said to have <u>finite intersection property</u> if every finite subfamily has nonempty intersection.

4.4. THEOREM. A is compact if and only if every family of closed sets with finite intersection property has nonempty intersection.

<u>Proof</u>. Exercise.

4.5. DEFINITION. S is a <u>Hausdorff space</u> (or T_2-space) if for any two distinct points p, q there exist open sets U, V such that $p \in U$, $q \in V$ and $U \cap V = \phi$.

4.6. THEOREM. Let C be a compact subset of a Hausdorff space S and $x \in S \setminus C$. Then, there exist open sets U, V such that $x \in U$, $C \subset V$, $U \cap V = \phi$. In particular, C is closed.

Proof. Since S is Hausdorff, for each point $y \in C$, there exist open sets U_y, V_y such that $x \in U_y$, $y \in V_y$, $U_y \cap V_y = \phi$. The family $\{V_y\}_{y \in C}$ is an open covering of C. Since C is compact, there exist $y_1, y_2, \ldots, y_n$ such that $C \subset \bigcup_{i=1}^{n} V_{y_i}$. Let $V = \bigcup_{i=1}^{n} V_{y_i}$ and $U = \bigcap_{i=1}^{n} V_{y_i}$. Then U, V satisfy all the requirements.

4.7. THEOREM. Let A and B be disjoint compact subsets of a Hausdorff space S. Then, there exist open sets U, V such that $A \subset U$, $B \subset V$, $U \cap V = \phi$.

Proof. For each $x \in A$, there exist open sets U_x, V_x such that $x \in U_x$, $B \subset V_x$ and $U_x \cap V_x = \phi$. Now, the sequence $\{U_x\}_{x \in A}$ is an open covering of A. By the compactness of A, we obtain points $x_1, x_2, \ldots, x_n$ such that $A \subset \bigcup_{i=1}^{n} U_{x_i}$. Take $U = \bigcup_{i=1}^{n} U_{x_i}$, $V = \bigcap_{i=1}^{n} U_{x_i}$. U and V will then satisfy all the requirements.

4.8. THEOREM. A closed subset of a compact space is compact.

<u>Proof</u>. Let C be a closed subset of a compact space S. Let $\{G_\alpha\}$ be an open covering of C. Now $\{G_\alpha\}$ and $S \setminus C$ give an open covering of S. Since S is compact, there exists a finite number of these open sets $G_{\alpha_1}, \ldots, G_{\alpha_n}$ such that these sets together with $S \setminus C$ cover S. Therefore $G_{\alpha_1}, \ldots, G_{\alpha_n}$ cover C and hence C is compact.

4.9. THEOREM. A continuous image of a compact set is compact.

<u>Proof.</u> Let

$$f : X \to Y$$

be continuous and onto, where X and Y are sets, with X compact. We have to show that Y is compact. Let $\{D_\alpha\}$ be an open covering of Y. Then $f^{-1}(D_\alpha)$ is an open set in X for each α. Further $\{f^{-1}(D_\alpha)\}$ is an open covering of X. Since X is compact, there exist $f^{-1}(D_{\alpha_1}), \ldots, f^{-1}(D_{\alpha_n})$ such that $X = \bigcup_{i=1}^{n} f^{-1}(D_{\alpha_i})$. Then $D_{\alpha_1} \ldots, D_{\alpha_n}$ will oover Y and therefore Y is compact.

4.10. THEOREM. Let

$$f : X \to Y$$

be a continuous, onto function. If X is compact, Y is Hausdorff and f is one-one, then f is a homeomorphism.

<u>Proof</u>. Enough to show that f is an open map or it takes closed sets into closed sets. Let C be a closed set in X. Then C is compact. $f(C)$ is compact, since f is continuous. Since Y is Hausdorff, $f(C)$ is closed.

4.11. THEOREM. (Alexander). If S is a sub-basis for the topological space X such that every covering by members of S has a finite sub-covering, then X is compact.

4.12. DEFINITION. A family of subsets of X is <u>inadequate</u> if it fails to cover X and <u>finitely inadequate</u> if no finite subfamily covers X. Then

a) Compactness means every finitely inadequate family is inadequate.

b) If $\mathcal{C}$ is the class of all finitely inadequate families of subsets of X, then $\mathcal{C}$ contains a maximal element. (this follows from Zorn's lemma).

c) If $\mathcal{B}$ is a finitely inadequate family, then by (b) there exists a maximal finitely inadequate family of open sets containing $\mathcal{B}$.

d) If $C \in \mathfrak{A}$ and $C = \bigcap_{i=1}^{n} C_i$ where C_i's are open sets, then $C_i \in \mathfrak{A}$ for some i.

e) Every covering of X by members of S has a finite subcovering. Every finitely inadequate family of S is inadequate.

<u>Proof.</u> d). First we notice that if A is an open set, $A \not\in \mathfrak{A}$. By maximality of $\mathfrak{A}$, it follows that there exists a finite number of open sets $A_1, \ldots, A_m \in \mathfrak{A}$ such that $A_1 \cup \ldots \cup A_m \cup A = X$. Therefore any set containing A does

not belong to $\sigma\tau$. Let $C = \bigcap_{i=1}^{n} C_i$ belong to $\sigma\tau$ where the C_i's are open sets. Suppose no $C_i \in \sigma\tau$, $i = 1,2,\ldots,k$. Then, by what has already been proved, we can find open sets A_{i_j} $(j = 1,2,\ldots,m_i)$ belonging to $\sigma\tau$ such that

$$C_i \cup A_{i_1} \cup \ldots \cup A_{i_{m_i}} = X, \; i = 1,2,\ldots, k.$$

Then

$$C \cup \left(\bigcup_{i,j} A_{i_j} \right) = \left(\bigcap_{i=1}^{p} C_i \right) \cup \left(\bigcup_{i=1}^{k} \bigcup_{j=1}^{m_i} A_{i_j} \right)$$

$$= \bigcap_{i=1}^{k} \left(C_i \cup \bigcup_{j=1}^{m_i} A_{i_j} \right)$$

$$= \bigcap X = X.$$

Hence $C \not\in \sigma\tau$ which yields a contradiction. Thus the result follows.

e) Let $\mathcal{B}$ be any finitely inadequate family of open sets. It is sufficient to prove that $\mathcal{B}$ is inadequate. Let $\sigma\tau$ be given by (c). We now show that $\sigma\tau$ is inadequate. The family $S \cap \sigma\tau$ of all members of $\sigma\tau$ which belong to S is finitely inadequate and hence $S \cap \sigma\tau$ is finitely inadequate. Thus, it is sufficient to prove that

$$\cup \{A / \; A \in \sigma\tau\} = \cup \; \{ A / A \in S \cap \sigma\tau \} .$$

Suppose $x \in A$. Since A is open and S is a sub-basis, A is the union of finite intersections of S. Hence x belongs to finite intersections of members of S contained in $\mathcal{O}$ and hence by(d) x belongs to a set in S which belongs to $\mathcal{O}$. This completes the proof of (e). The other parts can be easily proved.

5. PRODUCT SPACES.

5.1. DEFINITION. If σ_1, σ_2 are bases for the topologies X and Y respectively, then

$$\sigma_1 \times \sigma_2 = \left\{ A_1 \times A_2 / A_1 \in \sigma_1, A_2 \in \sigma_2 \right\}$$

is a basis for the topology of the product space $X \times Y$.

5.2. DEFINITION. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a family of sets indexed by Λ. Then, $\prod_{\lambda \in \Lambda} A_\lambda$ is the set of all functions

$$f : \Lambda \to \bigcup_{\lambda \in \Lambda} A_\lambda$$

such that

$$f(\lambda) \in A_\lambda, \quad \text{for each } \lambda \in \Lambda$$

Let $\{X_\lambda\}$ be a family of topological spaces. Then the product $\bigtimes_{\lambda \in \Lambda} X_\lambda$ is the set of all functions x on Λ such that

$$x(\lambda) = x_\lambda \in X_\lambda \text{ for each } \lambda \in \Lambda .$$

x_λ is called the λ^{th} coordinate of x. Define

$$p_\mu : \bigtimes X_\lambda \longrightarrow X_\mu$$

by

$$p_\mu(x) = x_\mu.$$

p_μ is called the projection onto the μ^{th} coordinate.

A topology is defined on $\times X_\mu$ by taking as sub-basis the sets of the form $p_\mu^{-1}(O_\mu)$, where O_μ is an open set in X_μ. The resulting topology is called the <u>product topology</u> and is the smallest in which the projections are continuous. This is called <u>Tychonoff topology</u>.

$$p_\mu^{-1}(O_\mu) = \left\{x \in \times X_\mu \,/\, x_\mu \in O_\mu\right\}.$$

5.3. <u>Exercise.</u> Show that the product of an arbittrary family of connected spaces is connected.

5.4. THEOREM. If X_λ's are Hausdorff spaces, then $\underset{\lambda\in\Lambda}{\times} X_\lambda$ is also a Hausdorff space.

<u>Proof.</u> If x and y are two distinct points of $X = \underset{\lambda\in\Lambda}{\times} X_\lambda$, then for some $\lambda=\lambda_o$ say, their components are different, that is, $x_{\lambda_o} \neq y_{\lambda_o}$. Since X_{λ_o} is Hausdorff, there exist open sets $U_{\lambda_o}, V_{\lambda_o}$ such that $x_{\lambda_o} \in U_{\lambda_o}$, $y_{\lambda_o} \in U_{\lambda_o}$, $U_{\lambda_o} \cap V_{\lambda_o} = \emptyset$. Then $p_{\lambda_o}^{-1}(U_{\lambda_o})$ and $p_{\lambda_o}^{-1}(V_{\lambda_o})$ are disjoint open sets in X and $x \in p_{\lambda_o}^{-1}(U_{\lambda_o})$, $y \in p_{\lambda_o}^{-1}(V_{\lambda_o})$. Hence X is Hausdorff.

5.5. THEOREM. The product of a family of compact sets is compact.

<u>Proof.</u> Let $X = \times X_\lambda$ be a product of topological spaces, X_λ, $\lambda \in \Lambda$. X has the product topology and X_λ's are assumed to be compact. Let $\mathcal{S}$ denote the sub-basis consisting of $p_\lambda^{-1}(U)$ where p_λ is the projection of X_λ. We shall apply Alexander's theorem to prove the result. Let $\mathfrak{A}$ be a finitely inadequate family of members of $\mathcal{S}$. Let $\mathfrak{A}_\lambda$ be the family of sets U on X_λ such that $p_\lambda^{-1}(U) \in \mathfrak{A}$. Then $\mathfrak{A}_\lambda$ is finitely inadequate. Since X_λ is compact, $\mathfrak{A}_\lambda$ is inadequate. Hence there exists $x_\lambda \in X_\lambda \setminus U$ for any $U \in \mathfrak{A}_\lambda$.

5.6. THEOREM (Wallace). If X and Y are topological spaces. A and B are compact subsets of X and Y respectively and W is a neighborhood of $X \times Y$, then there are neighborhoods U of A and V of B such that $A \times B \subset U \times V \subset W$.

Proof. For each point $(x,y) \in A \times B$, there exist open sets R, S such that $x \in R$, $y \in S$ and $R \times S \subset W$. Since for a fixed point $x \in A$, B is compact, there exist R_i, S_i, $(i = 1,2,\ldots,n)$ such that $x \in R_i$ for each i and $B \subset \bigcup_{i=1}^{n} S_i$. Let $P = \bigcap_{i=1}^{n} R_i$ and $Q = \bigcup_{i=1}^{n} S_i$. Then P and Q are open sets such that $x \in P$, $B \subset Q$. Now, use the compactness of A. Considering sets of the type $P \times Q$, there exist P_i, $Q_i (i=1,\ldots,m)$ such that $B \subset Q_i$ for each i and $\bigcup P_i \supset A$. Let $U = \bigcup_{i=1}^{m} P_i$, $V = \bigcap_{i=1}^{m} Q_i$. Then U,V will do the job.

5.7. Problem. Given a function $f: X \to Y$, where X is a topological space and f is onto, topologize Y such that f is continuous.

5.8. DEFINITION. Define U to be open in Y if and only if $f^{-1}(U)$ is open in X. The collection of all such U's is a topology for Y. This is called the quotient topology for Y. f is called the quotient map. Then f is an open map also.

5.91 THEOREM. If $f : X \to Y$ is continuous, where X and Y are topological spaces, such that f is either open or closed, then f is ~~either~~ a quotient map.

Proof. Exercise.

5.10. THEOREM. Let $f : X \to Y$, be continuous and suppose Y has the quotient topology. If $g : Y \to Z$, then gf is continuous if and only if g is continuous.

Proof. If f and g are continuous, clearly gf is continuous.

Conversely, suppose gf is continuous, we have to prove that g is continuous. Let U be an open set in Z. Now $(gf)^{-1}U$ is open in X. But $(gf)^{-1}U = f^{-1}(g^{-1}(U))$. Since $f^{-1}(g^{-1}(U))$ is open in X and Y has the quotient topology, $g^{-1}(U)$ is open in Y. This proves the theorem.

6, METRIC SPACES AND PSEUDOMETRIC SPACES.

6.1. Let X be a set. A non-negative real valued function d defined on $X \times X$ is called a metric for X if

a) $d(x,y) = d(y,x)$

b) $d(x,y) + d(y,z) \geq d(x,z)$

c) $d(x,y) = 0$ if $x = y$.

d) If $d(x,y) = 0$, then $x = y$.

If the condition (d) does not hold, then d is called a pseudo-metric.

6.2. A <u>metric</u> (pseudo-metric) <u>space</u> is a pair (X,d), where X is a set and d is a metric (pseudo-metric) for X.

6.3. <u>Notation</u>. Hereafter, X will stand for a pseudo-metric space with underlying pseudo-metric d being understood.

6.4. DEFINITION. Let $r > 0$. Then $S_r(x) = \{y/d(x,y) < r\}$ is called the <u>open sphere</u> of d-radius r, with centre x, and $\{y/d(x,y) \leq r\}$ is called the <u>closed sphere</u> of d-radius r, with centre x.

6.5. THEOREM. Let $r_1, r_2 > 0$. If $x \in S_{r_1}(x_1) \cap S_{r_2}(x_2)$, then there exists $r > 0$, such that

$$S_r(x) \subset S_{r_1}(x_1) \cap S_{r_2}(x_2).$$

<u>Proof</u>. Let $s_1 = d(x,x_1)$, $s_2 = d(x,x_2)$. Let $r = \min \{ (r_i - s_i)/i = 1,2 \}$. Then $S_r(x) \subset S_{r_1}(x_1) \cap S_{r_2}(x_2)$. Equivalently, $S_r(x) \subset S_{r_i}(x_i)$, $i = 1,2$. Let $y \in S_r(x)$. Then $d(y,x) < r$

$$d(y,x_i) \leq d(y,x) + d(x,x_i) < r + s_i < r_i - s_i + s_i = r_i.$$

For a topology on X, we take the set of all open sphere as a basis and the resulting topology is known as the <u>pseudo-metric topology</u>.

6.6. DEFINITION. Let X be a pseudo-metric space and A is a subset of X, A being nonempty. If $x \in X$, then the <u>distance</u> of A from X,

$$D(A, x) = \inf \{ d(y,x) / y \in A \} .$$

6.7. THEOREM. Let A be a fixed subset of a pseudo-metric space X. Then the distance $D(A,x)$ of x from A is a continuous function of x in the pseudo-metric topology.

Proof. Exercise.

6.8. THEOREM. The closure of a set A in a pseudo-metric space is the set of all points which are at zero distance from A.

Proof. Exercise.

6.9. DEFINITION. A space X is normal if for any two disjoint closed sets A and B, there exists open sets U, V such that $A \subset U$, $B \subset V$ such that $U \cap V = \phi$.

6.10. THEOREM. Every pseudo-metric space is normal.

Proof. Let A and B be two disjoint closed sets of a pseudo-metric space X. Let

$$U = \{ x \in X / D(A,x) - D(B,x) < 0 \}$$

$$V = \{ x \in X / D(A,x) - D(B,x) > 0 \} .$$

Then, the continuity of D gives the result.

7. LOCAL COMPACTNESS.

7.1. DEFINITION. A space X is said to be locally compact at the point x if there exists some open set U such that $x \in U$ and $\bar{U}$ is compact.

7.2. Remark. X is locally compact if it is locally compact at each point.

7.3. Examples. a) A compact space is locally compact.

b) The real line is locally compact.

7.4. THEOREM. A closed subset of a locally compact space is locally compact.

Proof. Let C be a closed subset of a locally compact space X. Pick a point $x \in C$. Since X is locally compact, there exists a U, open in X such that $x \in U$, $\bar{U}$ is compact. Then $U \cap C$ is open in C and $\bar{U} \cap C$ is closed in C and hence compact and $\bar{U}$ is compact.

7.5. THEOREM. The image of a locally compact set under an open continuous map is locally compact.

Proof. Let X be a locally compact space and $f : X \to Y$, an open, onto map. Assume f is continuous. For each point $x \in X$, there exists an open set U such that $x \in U$, $\bar{U}$ is compact. Then $f(\bar{U})$ is compact. Take the interior of $f(U)$ as the required open set containing $f(x)$.

7.6. THEOREM. The product of a finite number of locally compact spaces is locally compact.

Proof. Let X_i, $i = 1,2,\ldots,n$ be locally compact spaces. To show that $X_1, x X_2 x \ldots x X_n$ is locally compact. Let $x = (x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n$ with $x_i \in X_i$. Since X_i is locally compact, there exist U_i open in X_i such that $x_i \in U_i$, $\bar{U}_i$ is compact. Let $U = U_1 \times U_2 \times \ldots \times U_n$. Then $x \in U$ and $\bar{U} = \bar{U}_1 \times \bar{U}_2 \times \ldots \times \bar{U}_n$ is compact.

7.7. THEOREM. If the product $\times_{\lambda \in \Lambda} X_\lambda$ is locally compact, then each X_λ is locally compact and all but a finite number of X_λ's are compact.

Proof. Since the projection maps are both open and continuous, each coordinate is locally compact if $\times_{\lambda \in \Lambda} X_\lambda$ is locally compact. If $x \in \times_{\lambda \in \Lambda} X_\lambda$; there exists an open set U such that $\overline{U}$ is compact. Then, there exists an element of the basis which contains x but is contained in U. Now, $\overline{U}_\lambda$ is compact and for but a finite number of λ's , $\overline{U}_\lambda = X_\lambda$. Hence all but a finite number of X_λ's are compact.

7.8. DEFINITION. A space X is completely normal if for any two separated sets $A, B (A \neq \phi , B \neq \phi , \overline{A} \cap B = A \cap \overline{B} = \phi)$ there exist open sets U,V such that $A \subset U$, $B \subset V$, $U \cap V = \phi$.

7.9. Exercise. Show that a metric space is completely normal.

8. PARACOMPACT SPACES

8.1. DEFINITION. Let X be a topological space. A covering $\{V_\beta\}$ of X is called a refinement of the covering $\{U_\alpha\}$, if for each V_β of $\{V_\beta\}$, there is a U_α of $\{U_\alpha\}$ such that $V_\beta \subset U_\alpha$.

8.2. DEFINITION. A covering $\{U_\alpha\}$ of the space X is <u>locally finite</u> if for each $x \in X$, there is an open set containing x and intersecting only a finite number of the sets U_α.

8.3. DEFINITION. X is <u>paracompact</u> if X is Hausdorff and if every open covering of X has a locally finite refinement.

8.4. THEOREM. The homeomorphic image of a paracompact space is paracompact.

<u>Proof.</u> Let X be a paracompact space and $f : X \longrightarrow Y$ be a homeomorphism. To show that Y is paracompact, let $\{U_\alpha\}$ be an open covering of Y. Since f is continuous, $\{f^{-1}(U_\alpha)\}$ is an open covering of X. Since X is paracompact there exists a locally finite refinement of $\{f^{-1}(U_\alpha)\}$, say $\{V'_\beta\}$. Let $V_\beta = f(V'_\beta)$. Since f is a homeomorphism, $\{V_\beta\}$ is a refinement by open sets of $\{U_\alpha\}$. To complete the proof, we have to show that $\{V_\beta\}$ is locally finite. Let $y \in Y$. There exists $x \in X$ such that $f(x) = y$. Then, there exists an open set W such that $x \in W$, W intersects only a finite number of V'_β. Then $f(W)$ is the required open set containing y.

8.5. <u>Remarks</u>. a) in terms of refinement, we state that the space is compact if and only if every open covering has a finite refinement.

b) A compact space is paracompact.

8.6. THEOREM. Every paracompact space is regular.

Proof. Let X be a paracompact space. Let C be closed in X and $p \in X \setminus C$. For each point $x \in C$, there exist open sets U_x and V_x such that $x \in V_x$, $p \in U_x$ such that $V_x \cap U_x = \phi$. The sets V_x together with $X \setminus C$ give an open covering of X. Since X is paracompact, we have an open locally finite refinement $\{V_\beta\}$. Now let V be the union of the sets V_β which intersect C. Then V is an open set containing C. There exists an open set W containing p, intersecting only a finite number of V_β's, say $V_{\beta_1}, \ldots, V_{\beta_n}$ and each such V_{β_i} which meets C is contained in some V_{x_i}. Let $U = W \cap (\cap U_{x_i})$. Then U and V are the required open sets.

8.7. THEOREM. A closed subset of a paracompact space is paracompact.

Proof. Let X be paracompact. Let A be a closed subset of X. Let $\{A_\alpha\}$ be an open covering of A. Now, the family $\{A_\alpha\}$ together with $X \setminus A$ gives an open covering of the paracompact space X. We can find therefore a locally finite refinement, say $\{U_\beta\}$. Then, $\{A \cap U_\beta\}$ is a locally finite refinement and A is paracompact.

8.8. THEOREM. Any locally compact Hausdorff space that is a union of a countable number of compact sets is paracompact.

Proof. Let $X = \bigcup_{i=1}^{\infty} C_i$, where C_i's are compact and X is locally compact and Hausdorff. Without loss of generality, we assume $C_n \subset C_{n+1}$ for $n = 1,2,\ldots$ (otherwise , let $C_n' = \bigcup_{i=1}^{n} C_i$). We shall first show that X is the union of the sets W_n such that $\overline{W}_n$ is compact, $\overline{W}_n \subset W_{n+1}$. For each point $x \in C_1$, since X is locally compact, there exists an open set V_x such that $x \in V_x$, $\overline{V}_x$ is compact. Now V_x is an open covering of C_1. Use the compactness of C_1 to obtain a finite number of sets $V_{x_1},\ldots,V_{x_n}$ such that $C_1 \subset \bigcup_{i=1}^{n} V_{x_i}$. Let $W_1 = \bigcup_{i=1}^{n} V_{x_i}$. Then, $\overline{W}_1$ is compact.
is defined for all m n. With the property that W_m
Now assume that W_m is open, $\overline{W}_m$ is compact and $\overline{W}_m \subset W_{m+1}$. Now consider the compact set $\overline{W}_{n-1} \cup C_n$ and obtain W_n is exactly the same way as W_1, from C_1. Set $K_n = \overline{W}_n - W_{n-1}$. K_n is compact. Let $\{U_\alpha\}$ be an open covering of X. Let x be an arbitrary point of K_n. We can find an open set V_x containing x and contained in one of the U_α's containing x. Now, $\{V_x\}$ covers K_n. The V_x's can be chosen to lie in $W_{n+1} - \overline{W}_{n-2}$. Then, the V_x's give a covering of K_n. Since K_n is compact, we can find a finite number, $V_{x_1},\ldots,V_{x_n}$ to cover K_n. Hence the result follows.

ALGEBRAIC TOPOLOGY

1. CONVEX SETS

1.1. Preliminary notation. V is an arbitrary vector space over the real numbers R. If p and q are any two vectors in V, the line segment joining p and q is the set of vectors

$$pq = \{\alpha p + (1-\alpha) q / 0 \leq \alpha \leq 1\} .$$

Clearly

$$pq = qp$$

and if $p = q$,

$$pq = \{p\}$$

If A and B are two non-empty subsets of V, their join is defined as the set

$$AB \equiv \bigcup_{\substack{a \in A \\ b \in B}} ab$$

Then

$$AB = BA$$
$$AB \supseteq A$$
$$AB \supset B$$

and if $C \subset A$, $D \subset B$,

$$CD \subset AB.$$

Whenever the set A (or B) consists of a single point $A = \{a\}$, we write AB as aB.

For convenience, we take $\phi \cdot A = A$, where ϕ is the empty set.

1.2. DEFINITION. The subset $A \subset V$ is <u>convex</u> if it contains the join of any two of its subsets (i.e.) if $C \subset A$, $D \subset A$, then $CD \subset A$.

1.3. <u>Remark</u>. A set is convex if and only if it contains the join of any two of its points.

1.4. <u>Examples</u>. a) Any subspace of V is convex. In fact, a coset of a subspace of V is convex.

b) The line-segment pq joining any two points p, q of V is convex. $\{p\}$ is convex for all $p \in V$.

c) The empty set is convex.

1.5. <u>Exercises</u>. a) The ε-neighborhood, $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ of a convex subset A of $\mathbb{R}^n$,

$$N(A, \varepsilon) = \{ p \in \mathbb{R}^n / \| p-A \| < \varepsilon \}$$

is an open convex set contining A.

b) The closure $\bar{A}$ of any convex set A in $\mathbb{R}^n$ is convex. Use a) and b) to show that the unit ball B^n in $\mathbb{R}^n$ is convex.

c) For any open set $U \subset \mathbb{R}^n$, the set $p\,U \setminus \{p\}$ is open. Moreover, when $\bar{U}$ is compact, then so is $p\bar{U}$ and $\overline{p\,U \setminus \{p\}} = p\bar{U}$. Further, if U is convex, so is $p\,U \setminus \{p\}$ provided $p \notin U$.

1.6. PROPOSITION. The intersection of any family of convex sets is convex.

1.7. COROLLARY. Every subset $A \subset V$ is contained in a smallest convex set.

1.8. DEFINITION. The smallest convex set $[A]$ containing a given set $A \subset V$ is called the <u>convex hull</u> of A or $[A]$ is spanned by A.

1.9. THEOREM. a) A is convex if $A = [A]$

b) $AB \subset [A \cup B]$

c) If A and B are convex sets, then $AB = [A \cup B]$.

<u>Proof.</u> a) and b) follow from definition. We shall prove c). It is enough to show that AB is convex if A and B are convex.

Let p_1, p_2 be any two points of AB. Then, $p_i = \alpha_i a_i + (1-\alpha_i)b_i$, where $a_i \in A$, $b_i \in B$, $0 \leq \alpha_i \leq 1$, $i=1,2$. Any point of $p_1 p_2$ is of the form

$$\begin{aligned}
\alpha p_1 + (1-\alpha)p_2 &= \alpha(\alpha_1 a_1 + (1-\alpha_1)b_1) + (1-\alpha)(\alpha_2 a_2 + (1-\alpha_2)b_2) \\
&= \alpha\alpha_1 a_1 + \alpha(1-\alpha_1)b_1 + (1-\alpha)\alpha_2 a_2 + (1-\alpha)(1-\alpha_2)b_2 \\
&= \alpha\alpha_1 a_1 + (1-\alpha)\alpha_2 a_2 + \alpha(1-\alpha_1)b_1 + (1-\alpha-\alpha_2+\alpha\alpha_2)b_2 \\
&= \beta\left[\frac{\alpha\alpha_1}{\beta} a_1 + (1-\frac{\alpha\alpha_1}{\beta})a_2\right] + \left[\frac{\alpha(1-\alpha_1)}{(1-\beta)} b_1 + \right. \\
&\quad \left. + (1 - \frac{\alpha(1-\alpha_1)}{1-\beta} b_2)\right](1-\beta) \\
&\in AB .
\end{aligned}$$

where $\beta = \alpha\alpha_1 - \alpha\alpha_2 + \alpha_2$.

1.10. COROLLARY. If C is convex and $A \subset C$, then $[A] \subset C$. If further, $B \subset C$, then $AB \subset C$.

1.11. COROLLARY. If A,B,C are convex sets, then

$$A(BC) = (AB)C.$$

Proof. Each is equal to $[A \cup B \cup C]$.

1.12. Remark. The join of any finite number of convex sets is independent of the order in which the joins are taken. We thus write $A_1 \cdot \ldots \cdot A_k$ for $[A_1 \cup \ldots \cup A_k]$ where $A_1, \ldots, A_k$ are convex. In particular, if $A_i = \{a_i\}$, then $a_1 \cdot \ldots \cdot a_k$ is the convex set spanned by $\{a_1, \ldots, a_k\}$. The subscripts $i = 1,2,\ldots,k$ are used to designate the points, not order them.

1.13. DEFINITION. A barycentric function on a set A is a function

$$\beta : A \to I$$

such that $\beta(a) = 0$ for all but a finite number of the points in A and

$$\sum_{a \in A} \beta(a) = 1.$$

The finite set of points of A, $\{a_0, a_1, \ldots, a_k\}$ on which the barycentric function p is different from zero is called the support of p, $s(p)$. The integer k is called the degree of p, $d(p)$. The set of barycentric functions on a set A will be denoted by $\mathcal{B}(A)$.

1.14. PROPOSITION. For any set $A \subset V$, there is a natural projection

$$\eta : \mathcal{B}(A) \to V$$

given by

$$p \to \sum_{a \in A} p(a)(a)$$

whose image is precisely $[A]$, the convex hull of A.

Proof. The natural projection is certainly welldefined. We shall now show that the image of the natural projection $\subset [A]$ and is convex. Since the image contains A, it must then be $[A]$. If $d(p) = 0$, then $\eta(p) = a \in A \subset [A]$, where $\{a\} = s(p)$. Suppose that $\eta(p) \in [A]$, whenever $d(p) \leq k-1$ and consider any $p \in \mathcal{B}(A)$ of degree k. Let $s(p) = \{a_1, \ldots, a_k\}$. If $\underline{a}_o$ is the unique barycentric function with $\underline{a}_o(a_o) = 1$, then

$$\frac{1}{(1 - \underline{p}(a_o))} (\underline{p} - \underline{p}(a_o)\, \underline{a}_o)$$

is a barycentric function of degree $k-1$ with support $\{a_1, \ldots, a_k\}$. Hence its image under η is a point $q \in [A]$, and

$$\eta(\underline{p}) = \sum_{a \in A} \underline{p}(a)\, a = \sum_{i=0}^{k} \underline{p}(a_i)\, a_i$$

$$= \underline{p}(a_o)\, a_o + (1 - \underline{p}(a_o))\, q \in [A] .$$

That the image under η is convex follows from the facts that for any two $\underline{p}, \underline{q} \in \mathcal{B}(A)$,

$$(1 - \alpha)\, \underline{p} + \alpha\, \underline{q} \in \mathcal{B}(A)$$

for any $\alpha \in I$ and moreover

$$\eta\left[(1-\alpha)\underline{p} + \alpha\, \underline{q}\right] = (1-\alpha)\, \eta(\underline{p}) + \alpha\eta(\underline{q}).$$

1.15. Remarks. a) If $R(A)$ denotes all the real valued functions on A with finite support, that is, which assume non-zero values in only finitely many elements of A, then the natural projection defined above extends to a projection

$$\eta : \mathcal{R}(A) \longrightarrow V$$

whose image is the subspace generated or spanned by A.

b) Denoting by $\eta(A)$ the subset of $\mathcal{R}(A)$ consisting of these functions $\underline{p}$ such that $\sum_{a \in A} \underline{p}(a) = 1$, the image of $\eta(A)$ under η is called the smallest flat set containing A, where by flat set we mean a coset of a subspace of V. The dimension of a flat set is the dimension of the subspace of which it is a coset.

c) $\mathcal{R}(A)$ is in fact the vector space generated by A. $\mathcal{B}(A)$ is a convex subset of $\mathcal{R}(A)$, the convex hull of $\underline{A} = \{\underline{a} \in \mathcal{B}(A) \,/\, \underline{a}(a) = 1,\ a \in A\}$; $\eta(A) \supset \mathcal{B}(A)$ and $\eta(A)$ is the coset $\underline{a} + \eta'(A)$ where $\underline{a} \in \underline{A}$ and $\eta'(A)$ is the kernel of the projection,

$$\mathcal{R}(A) \longrightarrow \mathbb{R}$$
$$\underline{p} \rightarrow \sum_{a\in A} \underline{p}(a).$$

d) Recall that the set A is linearly independent, if the natural projection $\eta : \mathcal{R}(A) \rightarrow V$ is 1:1 .

1.16. DEFINITION. The set $A \subset V$ is <u>affinely linearly independent</u> if the natural projection

$$\eta : \eta(A) \longrightarrow V$$

is 1 : 1 . We shall use a.l.i. for brevity.

1.17. PROPOSITION. The set A is affinely linearly independent (a.l.i.) if the only function $r \in \mathcal{R}(A)$ with the property that

$$\eta(\underline{r}) = 0 \text{ and } \sum_{a\in A} \underline{r}(a) = 0$$

is the zero function $\underline{p} = 0$.

<u>Proof</u>. If there is such a non-zero function $\underline{r}$, then for any $\underline{p} \in \eta(A)$,

$$\underline{p} + \underline{r} \in \eta(A)$$

and

$$\eta(\underline{p}) = \eta(\underline{p} + \underline{r})$$

so that η is not 1 : 1 .

Conversely, if

$$\eta(\underline{p}) = \eta(\underline{q})$$

Then

$$\eta(\underline{p} - \underline{q}) = 0$$

and

$$\sum_{a \in A} (\underline{p} - \underline{q})\ (a) = 0$$

where $\underline{p} - \underline{q} \in \mathcal{R}(A)$.

1.18. COROLLARY. A is a.l.i. if every finite subset of A is a.l.i.

Proof. This follows from the fact that all functions have finite support.

Exercises. a) The set A is a.l.i. if for any $a \in A$, the set

$$A - a = \{ b - a \ / \ b \in A \}$$

is linearly independent.

b) The set $A = \{ a_0, a_1, \ldots, a_k \}$ is a.l.i. if the smallest flat set contining A has dimension k.

1.19. Remarks. a) A linearly independent set is a.l.i. but not conversely. If v is a non-zero vector, then $\{v, -v\}$ is a a.l.i. but not linearly independent.

b) The points in the convex hull $[A]$ of an a.l.i. set A are in 1.1 correspondence with the set $\mathcal{B}(A)$ of barycentric functions on A.

2. SIMPLEXES.

2.1. DEFINITION. The convex hull of an a.l.i. set of (k+1) points $A \equiv \{a_0, a_1, \ldots, a_k\}$ is called the <u>k-simplex</u> spanned by A.

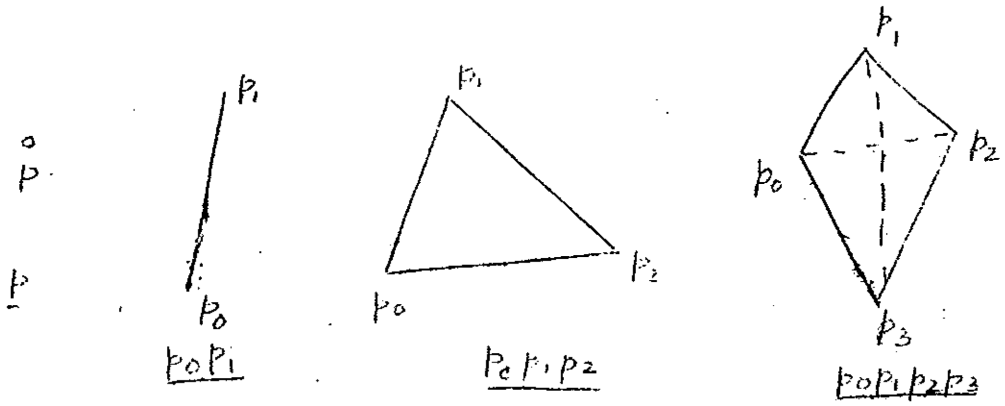
The elements of A are called the vertices of the simplex. We usually denote a simplex by Greek letters $\bar{\sigma}$, $\bar{\tau}$, etc. If we wish to emphasize that $\bar{\sigma}$ is a k-simplex, we write $\bar{\sigma}^k$. If we identity 3-space with $\mathbb{R}^3$,

a 0-simplex is just a point,

a 1-simplex is a line segment,

a 2-simplex is a triangle,

a 3-simplex is a tetrahedron and so on



2.2. PROPOSITION. The points of a simplex are in 1 : 1 correspondence with the barycentric functions in the vertices. This correspondence is effected by natural projection.

2.3. PROPOSITION. A simplex completely determines its vertices.

Proof. Let $\bar{\sigma} = a_0\, a_1, \ldots, a_k$ be any simplex. We will show that any point of $\bar{\sigma}$ other than the vertices is the midpoint of two distinct points of $\bar{\sigma}$. This will completely characterise vertices.

For any point $p \in \bar{\sigma}$, let $\underline{p}$ denote the unique barycentric function on the vertices such that $\eta(\underline{p}) = p$. $\underline{p}$ never assumes the value 1 if p is not a vertex. If

$$p = \frac{1}{2} q_1 + \frac{1}{2} q_2$$

where $q_1 \neq q_2$, $q_1, q_2 \in \bar{\sigma}$, then for some vertex a_i

$$\underline{q}_1(a_i) \neq \underline{q}_2\,(a_i).$$

Hence,

$$0 < \underline{q}_1(a_i) + \underline{q}_2\,(a_i) < 2$$

from which it follows that

$$0 < \underline{p}\,(a_i) < 1$$

and p is not a vertex.

Conversely, if p is not a vertex, then for any $a_i \in S(\underline{p})$, $0 < \underline{p}\,(a_i) < 1$. Then,

$$p = \frac{1}{2} q_1 + \frac{1}{2} q_2$$

where $q_i = p + (-1)^i \alpha (p_i - p)$, $i = 1,2$, are distinct elements of $\bar{\sigma}$ and $\alpha = \min(\underline{p}(a_i), 1-\underline{p}(a_i))$

Any nonempty subset of the vertices of a simplex $\bar{\sigma}^k = a_o a_1, \ldots, a_k$, say, $\{a_{i_o}, a_{i_1}, \ldots, a_{i_m}\}$ is certainly a.l.i. and hence spans a simplex $\bar{\tau}^m$ called an m-face of $\bar{\sigma}^k$. Clearly $\bar{\tau}^m \subset \bar{\sigma}^k$. We denote this face relation by $\prec$, i.e., $\bar{\tau}^m \prec \bar{\sigma}^k$. Since any face of $\bar{\tau}^m$ is also a face of $\bar{\sigma}^k$, The face relation defines a partial ordering among the faces of $\bar{\sigma}^k$. If $m = k$, $\bar{\tau}^m = \bar{\sigma}^k$. Otherwise, $\bar{\tau}^m$ is called a proper face of $\bar{\sigma}^k$. In general, there are $k + 1\, C_{m+1}$ m-faces of $\bar{\sigma}^k$.

Sometimes, we consider the empty set as the -1 face of a simplex and then the simplex is said to be augmented. If $a_i \notin s(\underline{p})$ for some $p \in \bar{\sigma}^k$, then $\underline{p}(a_i) = 0$ and the restriction of $\underline{p}$ to the remaining vertices again defines a barycentric function which corresponds to a (k-1)-face $\bar{\sigma}_i = a_o \ldots \hat{a}_i \ldots a_k$ of $\bar{\sigma}^k$. Thus $p \in a_o \ldots \hat{a}_i \ldots a_k$ and conversely if $p \in a_o \ldots \hat{a}_i \ldots a_k$, then the barycentric function of p on the vertices of $\bar{\sigma}^k$ takes on the value 0 at a_i. More generally if $\bar{\tau}^m \prec \bar{\sigma}^k$, then the barycentric function of p in $\bar{\tau}^m$, on the vertices of $\bar{\sigma}^k$ is different from zero only on the vertices of $\bar{\tau}^m$. Hence we can identify the barycentric function of a point $p \in \bar{\tau}^m \prec \bar{\sigma}^k$ on the vertices of $\bar{\tau}^m$ with that on the vertices of $\bar{\sigma}^k$.

2.4. Remark. We hereafter identify each point of a simplex with the corresponding barycentric function.

2.5. DEFINITION. The closure of a point p of a simplex $\bar{\sigma} \equiv a_o \dots a_i \dots a_k$ is the face $\phi(p) = \lfloor s(p) \rfloor$ of σ.

2.6. Remark. ϕ may be regarded as an operator function from the points of the simplex to its faces.

2.7. Properties of the closure function

a) Any face of $\bar{\sigma}$ which contains p contains $\phi(p)$ so that $\phi(p)$ is the smallest face of $\bar{\sigma}$ containing p. In particular, if $q \in \phi(p)$, then $\phi(q) \prec \phi(p)$.

b) The closure operator is onto; that is, if $\bar{\sigma} = a_o \dots a_k$ and $\bar{\tau}^m = a_{i_o} \dots a_{i_m}$ is any face of $\bar{\sigma}$, then $\phi(e) = \bar{\tau}^m$ where $e = \frac{1}{m+1}(a_{i_o} + \dots + a_{i_m})$.

c) Let $\bar{\tau}_1$ and $\bar{\tau}_2$ be any two faces of $\bar{\sigma}$. If $p \in \bar{\tau}_1 \cap \bar{\tau}_2$, then $\phi(p) \prec \bar{\tau}_1$ and $\phi(p) \prec \bar{\tau}_2$ so that $\phi(p) \subset \bar{\tau}_1 \cap \bar{\tau}_2$. In this case, $\bar{\tau}_1 \cap \bar{\tau}_2 \neq \phi$. Let $\bar{\tau}$ be the largest face of $\bar{\sigma}$ common to $\bar{\tau}_1$ and $\bar{\tau}_2$. For any point $p \in \bar{\tau}_1 \cap \bar{\tau}_2$, $\phi(p) \prec \bar{\tau}$. By (b), $\bar{\bar{\tau}} = \phi(q)$ for some $q \in \bar{\tau}_1 \cap \bar{\tau}_2$. Then $\bar{\tau} = \bar{\tau}_1 \cap \bar{\tau}_2$.

2.8. DEFINITION. A point p of the simplex $\bar{\sigma}^k$ is an <u>interior point</u> of $\bar{\sigma}^k$ if $\phi(p) = \bar{\sigma}^k$. Otherwise $\phi(p)$ is a proper face of $\bar{\sigma}^k$ and p is a <u>frontier point</u> of $\bar{\sigma}^k$. The set of all interior points of $\bar{\sigma}^k$, denoted by σ^k is called the <u>interior</u> of $\bar{\sigma}^k$. The <u>frontier</u> $\dot{\sigma}^k$ of $\bar{\sigma}^k$ is the set of frontier points of $\bar{\sigma}^k$. Then, $\bar{\sigma}^k$ is the disjoint union of the interiors of all its faces and $\dot{\sigma}^k$ is the disjoint union of all the interiors of all proper faces of $\bar{\sigma}^k$.

(i.e.) $\bar{\sigma}^k = \sigma^k \cup \dot{\sigma}^k$

and

$$\sigma^k \cap \dot{\sigma}^k = \phi .$$

2.9. <u>Remarks</u>. Let $\bar{\sigma}$ be a simplex. Then

(a) $\bar{\sigma} = \sigma \cup \dot{\sigma}$ and $\sigma \cap \dot{\sigma} = \phi$.

(b) ~~If~~ p is an interior point of $\phi(p)$. Every point of $\bar{\sigma}^k$ is the interior of a unique face of $\bar{\sigma}^k$ and every frontier point is the interior of a unique proper faces of $\bar{\sigma}^k$.

(c) $\bar{\sigma}^k$ is the disjoint union of all its faces. $\dot{\sigma}^k$ is the disjoint union of interiors of all proper faces of $\bar{\sigma}^k$.

2.10. DEFINITION. The interior of the simplex $\bar{\sigma}^k = a_0 \ldots a_k$ is called the open <u>simplex</u> spanned by the vertices $\{a_0, \ldots, a_k\}$. We denote this by $\sigma^k \equiv (a_0, \ldots, a_k)$. An <u>open face</u> of a simplex is the interior of a face of the simplex. Thus, we have the following:

2.11. PROPOSITION. Any simplex is the union of all its faces and the disjoint union of all open faces. The frontier of a simplex is the union of all the proper faces of the simplex and the disjoint union of all its open proper faces. Any two faces of (the frontier of) a simplex are disjoint or intersect in a common face. If the simplex is argmented, we say that any two faces of (the frontier of) a simplex intersect in a common face.

3. THE NATURAL TOPOLOGY OF A SIMPLEX

3.1. DEFINITION. Let $e_1, \ldots, e_{k+1}$ be the unit basis vector in $\mathbb{R}^{k+1}$, where $e_1 = \langle 1,0,\ldots,0 \rangle$, $e_2 = \langle 0,1,\ldots,0 \rangle$, $e_3 = \langle 0,0,1,\ldots,0 \rangle$ and so on. The simplex $e_1 \ldots e_{k+1} = \sum_{k+1}^{+k}$ is called the <u>representative k-simplex</u> in $\mathbb{R}^{k+1}$

Any function $\phi : \{e_1, \ldots, e_{k+1}\} \longrightarrow \mathbb{R}^n$ has a unique linear extension

$$\mathbb{R}^{k+1} \longrightarrow \mathbb{R}^n$$

defined by

$$\phi \langle \beta_1 \ldots \beta_{k+1} \rangle = \sum_{i=1}^{k+1} \beta_i \, \phi \, e_i.$$

Thus, the image of $\bar{\Sigma}^k$ under ϕ is just $\phi(e_1)\dots\phi(e_{k+1})$. If $\bar{\sigma}$ is any simplex in $\mathbb{R}^n$, with $\phi(e_i) \in \bar{\sigma}$, then $\phi(\bar{\Sigma}^k) \subset \bar{\sigma}$ so that we have a function

$$\phi' : \bar{\Sigma}^k \to \bar{\sigma}$$

called the <u>linear extension of the vertex map</u>.

$$\phi : \{e_1, \dots, e_{k+1}\} \to \bar{\sigma}.$$

If $\phi\, e_i$ is vertex of $\bar{\sigma}$ for $i = 1, \dots, k+1$, then ϕ' is called a <u>simplicial map</u>. The barycentric function of any point $p = \langle \alpha_1, \dots, \alpha_{k+1} \rangle$ of $\bar{\Sigma}^k$ is given by

$$p(e_i) = \alpha_i, \; i = 1, 2, \dots, k+1.$$

Then

$$\phi'(p) = p\,(e_1)\,\phi\,(e_1) + \dots + p(e_{k+1})\,\phi\,(e_{k+1}).$$

3.2. DEFINITION. A function $\phi : \bar{\sigma} \to \bar{\tau}$, where $\bar{\sigma}$ and $\bar{\tau}$ are simplexes, (not necessarily in the same vector space) is linear, if it is the linear extension of its restriction to the vertices of $\bar{\sigma}$, (i.e.), if $\bar{\sigma} = a_o \dots a_k$ and $p \in \bar{\sigma}$ is any point, then

$$\phi(p) = p(a_o)\;\phi\,(a_o) + \dots + p(a_k)\;\phi\,(a_k).$$

If moreover ϕ takes vertices of $\bar{\sigma}$ into vertices of $\bar{\tau}$, ϕ is called simplicial. A simplicial map which is 1:1 and onto is called an isomorphism.

3.2. PROPOSITION. Let $\phi: \bar{\sigma} \to \bar{\tau}$ be a simplicial map.

a) If $\bar{\sigma}_1 \prec \bar{\sigma}$, then $\phi(\bar{\sigma}_1) \prec \phi(\bar{\sigma})$.

b) If $\bar{\sigma}_1 \prec \bar{\sigma}_2$, then $\phi(\bar{\sigma}_1) \prec \phi(\bar{\sigma}_2)$.

c) If $\bar{\tau}_1 \prec \bar{\tau}_2$, then $\phi^{-1}(\bar{\tau}_1) \prec \bar{\sigma}$.

d) If ϕ is 1:1 on the vertices of $\bar{\sigma}$, then ϕ is 1:1 and takes frontier of $\bar{\sigma}$ onto frontier of $\bar{\tau}$.

e) If ϕ takes the vertices of $\bar{\sigma}$ onto the vertices of $\bar{\tau}$, then ϕ is onto and taken interior points.

f) If ϕ takes vertices of $\bar{\sigma}$ onto the vertices of $\bar{\tau}$ 1:1, then ϕ is 1:1 and onto and the inverse map is also simplicial.

g) The composition of two simplcial map is again a simplicial map.

3.4. DEFINITION. Two simplexes are isomorphic if there exhists an isomorphism between them.

3.5. PROPOSITION. Two simplexes are isomorphic if and only if they have the same number of vertices. In particular, every k-simplex is isomorphic to the representative k-simplex.

3.6. PROPOSITION. An isomprphism carries interior points into interior points and frontier points into frontier points.

The representative k-simplex $\bar{\Sigma}^k$ inherits a topology from $\mathbb{R}^{k+1}$ and in this topology $\bar{\Sigma}^k$ is a compact metric spaces. Any simplex $\bar{\sigma}^k = a_o \ldots a_k$ is isomorphic to $\bar{\Sigma}^k$ so that there exists an isomorphism $\phi: \bar{\Sigma}^k \to \bar{\sigma}^k$. We assign to $\bar{\sigma}^k$ the topology which makes ϕ a homeomorphism and show that it is independent of ϕ. The topology of $\bar{\Sigma}^k$ is given by the metric of $\mathbb{R}^{k+1}$ restricted to $\bar{\Sigma}^k$. We set

$$\| \phi p_1 - \phi p_2 \| = \| p_1 - p_2 \|$$

where p_1, p_2 are any two points of $\bar{\Sigma}^k$. The coordinates of $p_i \in \bar{\Sigma}^k \subset \mathbb{R}^{k+1}$ are given by the barycentric functions,

$$p_i = \langle p_i(e_1), \ldots, p_i(e_{k+1}) \rangle \quad , \; i = 1,2$$

so that

$$\|p_1 - p_2\| = \left\{ \sum_{i=1}^{k+1} (p_1(e_i) - p_2(e_i))^2 \right\}^{1/2} .$$

By the definition of ϕ ,

$$\phi p_i(\phi(e_j)) = p_i(e_j), \qquad \begin{array}{l} j = 1,2,\ldots,k+1 \\ i = 1,2 \end{array}$$

so that

$$\left\{\sum_{i=1}^{k+1} (p_1(e_i)-p_2(e_i))^2\right\}^{1/2} = \left\{\sum_{i=1}^{k+1} (\phi p_1(\phi e_i)-\phi p_2(\phi e_i))^2\right\}^{1/2}$$

$$= \left\{\sum_{i=0}^{k} (\phi p_1(a_i)- \phi p_2(a_i))^2\right\}^{1/2}$$

$$= \left\{\sum_{i=0}^{k} (v_1(a_i) - v_2(a_i))^2\right\}^{1/2}$$

where $v_i = \phi p_i$ are any two points of $\bar{\sigma}$. Thus

(*) $$\|v_1 - v_2\| = \left\{\sum_{i=0}^{k} (v_1(a_i) - v_2(a_i))^2\right\}^{1/2}$$

is independent of ϕ.

3.7. DEFINITION. The metric defined by (*) for the simplex $\bar{\sigma}^k$ is called the natural metric in $\bar{\sigma}^k$.

3.8. Remark. If $\bar{\tau}$ is a face of the simplex $\bar{\sigma}$, then the natural metric of $\bar{\tau}$ is just the restriction to $\bar{\tau}$ of the natural metric of $\bar{\sigma}$. Hence the natural topology of $\bar{\tau}$ agrees with the topology inherited from $\bar{\sigma}$.

3.9. Exercise. If $\bar{\sigma}^k$ is a simplex in $\mathbb{R}^n$ the topology inherited from $\mathbb{R}^n$ is the same as the natural topology of $\bar{\sigma}^k$.

3.10. PROPOSITION. A linear map from one simplex to another is continuous in the natural topologies.

<u>Proof</u>. Let $\bar{\Sigma}^k$ and $\bar{\Sigma}^n$ be representative simplexes in $\mathbb{R}^{k+1}$ and $\mathbb{R}^{n+1}$ respectively and $\phi : \bar{\Sigma}^k \to \bar{\Sigma}^n$, a linear map. Since ϕ is defined by the linear extension

$$\phi : \mathbb{R}^{k+1} \longrightarrow \mathbb{R}^{n+1}$$

which is continuous, ϕ is also continuous in the inherited topologies. More generally, let

$$\phi : \bar{\sigma}^k \longrightarrow \bar{\tau}^n$$

be a linear map, and

$$\psi_1 : \bar{\Sigma}^k \longrightarrow \bar{\sigma}^k$$

$$\psi_2 : \bar{\Sigma}^n \longrightarrow \bar{\tau}^n$$

isomorphisms. There is a unique linear map

$$\phi' : \bar{\Sigma}^{-k} \longrightarrow \bar{\Sigma}^n$$

such that

$$\phi = \psi_2 \phi' \psi_1^{-1}.$$

Since all the maps on the right are continuous, (in fact ψ_1 and ψ_2 are homeomorphiams) ϕ is also continuous.

3.11. <u>Remarks</u>. a) The frontier of a simplex is a union of finitely many closed sets (viz) faces and hence closed and so is a compact subset of the simplex. The interior of the simplex, as the complement of a closed set, is open.

b) A k-simplex in $\mathbb{R}^n$ is a compact subset and hence its frontier which is also compact, is closed in $\mathbb{R}^n$. The interior of the simplex is not in general open in $\mathbb{R}^n$.

3.12. DEFINITION. A point $v \in V$ is <u>independent</u> of the set $A \subset V$ if $v \notin A$ and $va \cap A = \{a\}$ for all $a \in A$.

3.13. LEMMA. If the point $v \in \mathbb{R}^n$ is independent of $A \subset \mathbb{R}^n$ and if A is compact, then in vA (as a subspace of $\mathbb{R}^n$) sets of the form

(i) $\{\alpha v + (1-\alpha)\, a / \alpha \in U,\ a \in A\}$

(ii) $vV \setminus \{v\}$

are open, whenever U is an open set in I and V is an open set in A. Moreover, these two types of sets together generate the open sets in vA.

<u>Proof</u>. Exercise.

3.14. COROLLARY. If $u, v \in \mathbb{R}^n$ are independent of the subsets A, B respectively of $\mathbb{R}^n$, and if $\phi : A \to B$ is a continuous map and if moreover A is compact, then the map

$$\phi' : uA \longrightarrow vB$$

defined by

$$\alpha u + (1-\alpha)\, a \longrightarrow \alpha v + (1-\alpha)\, \phi a$$

is continuous. If ϕ is open, or 1:1, or onto, then so is ϕ'.

3.15. Exercise. Given an example to show that the hypothesis that A be compact is essential.

Let A be convex subset of V such that the flat set spanned by A has dimension n. Then A contains an a.l.i. set $\{a_o, \ldots, a_n\}$ which spans the flat set spanned by A. Choose any isomorphism

$$\phi : o\, e_1 \ldots e_n \rightarrow a_o \ldots a_n$$

where $e_1, \ldots, e_n$ are the unit basis vectors in $\mathbb{R}^n$. This extends linearly to a function

$$\phi' : \mathbb{R}^n \longrightarrow V$$

where

$$\phi' \langle \alpha_1, \ldots, \alpha_n \rangle = \sum_{i=1}^{n} \alpha_i \left(\phi(e_i) - \phi(o) \right)$$

and ϕ' maps $\mathbb{R}^n$, 1:1, onto the n-flat set spanned by A. Let $A' = \phi'^{-1}(A)$. Then, $A' \supset o\, e_1 \ldots e_n$ and hence has a non-empty interior (as a subset of $\mathbb{R}^n$). Moreover, A' is convex.

3.16. LEMMA. Let B be a compact convex subset of $\mathbb{R}^n$ with a non-empty interior and $p_o \in \text{Int } B = B^o$. Then p_o is independent of $\dot{B} = B \setminus B^o$ and $B = p_o \dot{B}$.

Proof. For any point $p \in B$, $p B^o \setminus \{p\}$ is an open subset of B and

$$B^o - \{p\}\ \cancel{B^o} \subset pB^o \setminus \{p\} .$$

If $p \notin B^o$, then $p B^o \setminus \{p\} = B^o$. If $b \in \dot{B}$, $p_o \in B^o$, then $p_o b \cap \dot{B} = \{b\}$ which implies that p_o is independent of $\dot{B}$. For any $p \in B$, consider the real-valued function.

$$(1-\alpha)p_o + \alpha p \longrightarrow \alpha.$$

Let α_o be the maximum value of this function at b_o say. Then, clearly b_o is a frontier point of B; $\alpha_o \geq 1$ and $p = \left(\frac{\alpha_o - 1}{\alpha_o}\right) p_o + \frac{1}{\alpha_o} b_o$ is a frontier point of $p_o \dot{B}$.

3.17. COROLLARY. If the set A' is compact, then, for any interior point p' of A' (such a point certainly exists) the corresponding point $\phi' p' = p \in A$ is independent of A^o and $A = pA^o$ where $A^o = \phi' A^{o\prime}$.

3.18. THEOREM. If B is a compact convex subset of $\mathbb{R}^n$ with nonempty interior, then the boundary is homeomorphic to S^{n-1}, the unit ball in $\mathbb{R}^n$.

Proof. Choose an interior point p of B. Then, $b-p$ is never zero for $b \in B^o$ so that the function

$$\psi : B^o \longrightarrow S^{n-1}$$

$$b \longrightarrow \frac{b-p}{\|b-p\|}$$

is well defined and continuous.

For some $\varepsilon > 0$, $N(p, \varepsilon) \subset B$, so that for any $s \in S^{n-1}$, $(p + \varepsilon s) \in B$ and by the previous lemma, has a unique representation

$$p + \varepsilon s = (1 - \alpha) p + \alpha b$$

for some $\alpha \in I$, $\alpha \neq 0$, $b \in B^o$. This gives rise to a function

$$\phi : S^{n-1} \to B^o$$

$$s \to b$$

such that

$$b - p = \frac{\varepsilon}{\alpha} s.$$

Hence $\psi = I_{S^{n-1}}$ and $\phi \psi = I_{B^o}$. Thus, ψ is 1:1 and onto and since B^o is compact, ψ is a homeomorphism.

3.19. COROLLARY. The homeomorphism extends to a homeomorphism of B onto B^n, the unit ball in $\mathbb{R}^n$, which takes interior of B onto the interior of B^n.

Proof. Since S^{n-1} is the boundary of the compact convex subset $B^n \subset \mathbb{R}^n$ and 0 is an interior point of B^n, the result follows.

3.20. COROLLARY. If A is a compact convex subset in $\mathbb{R}^n$ then A is homeomorphic (as a subspace) to the unit ball $B^k \subset \mathbb{R}^k$ for some k.

Proof. Since the function ϕ' is continuous, the result follows.

Applying the above to a k-simplex $\bar{\sigma}^k$, we may choose as the maximal a.l.i. set, the vertices of $\bar{\sigma}^k$. It follows therefore that any interior point p of $\bar{\sigma}^k$ is independent of $\overset{\circ}{\sigma}{}^k$ and $\bar{\sigma}^k = p\,\overset{\circ}{\sigma}{}^k$. Moreover, $\bar{\sigma}^k$ is homeomorphic to the unit ball B^k which takes, the interior of $\bar{\sigma}^k$ homeomorphically onto the interior of B^k and $\overset{\circ}{\sigma}{}^k$ homeomorphically onto S^{n-1}.

> 3.21. DEFINITION. A topological space Y homeomorphic to the unit n-ball B^n in $\mathbb{R}^n$ is called an n-cell. If Y is a subspace of X, then Y is an n-cell in X. More particularly, Y is a closed n-cell. If Y is homeomorphic to the interior of B^n, then Y is an open n-cell. Thus a compact, convex subset of R^n is a k-cell for some k. A k-simplex is a k-cell and an open k-simplex is an open k-cell. The interior of a k-simplex is an open k-cell in the simplex.

4. THE BARYCENTRIC SUBDIVISION OF A SIMPLEX

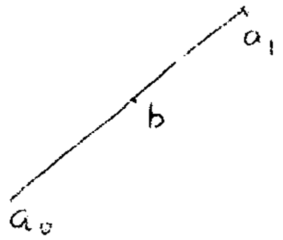
The k-simplex provides a useful representation of the k-cell. One advantage of this representation is that it can be easily decomposed into smaller k-simplexes. We assume throughout that the simplexes are augmented.

4.1. DEFINITION. The barycenter of a simplex is the point whose barycentric function takes on the same value at each vertex. Thus, if $\bar{\sigma}^k \equiv a_o \ldots a_k$, the barycenter $b(\bar{\sigma}^k)$ is the point $b(\bar{\sigma}^k)(a_i) = \frac{1}{k+1}$, $i = 0,\ldots,k$ or $b(\bar{\sigma}^k) = \frac{1}{k+1} \sum_{i=0}^{k} a_i$. The barycenter is sometimes called the <u>centroid</u>.

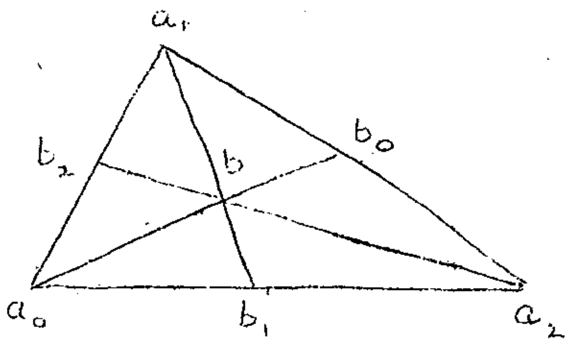
4.2. DEFINITION. <u>The barycentric subdivision</u> of a simplex $\bar{\sigma}^k \equiv a_o \ldots a_k$ is defined inductively as the collection of k-simplexes of the form $b\,\bar{\tau}_i^{k-1}$, where b is the barycenter of $\bar{\sigma}^k$ and $\bar{\tau}_i^{k-1}$ is any (k-1) simplex in the barycentric subdivision of the (k-1) face $\bar{\sigma}_i = a_o \ldots \hat{a}_i \ldots a_k$, $i = 0,1,\ldots,k$, of $\bar{\sigma}^k$. To complete the definition (start the induction) we define the barycentric subdivision of a (-1)-simplex to consist of the (-1) simplex itself.

The barycentric subdivision of the simplex $\bar{\sigma}^k$ will be denoted by $Sd\,\bar{\sigma}^k$.

It follows from the definition that the barycentric subdivision of a 0-simplex consists of a 0-simplex itself. For a 1-simplex a_oa_1, $Sd\, a_oa_1 = \{a_o b, a_1 b\}$. For a 2-simplex $a_oa_1a_2$, $Sd\, a_oa_1a_2 = \{a_ob_2b, a_1b_2b, a_ob_1b, a_2b_1b, a_2b_ob, a_1b_ob\}$.



Sd $a_0 a_1$



Sd $a_0 a_1 a_2$

Inductively, the number of k-simplexes in Sd $\bar{\sigma}^k$ is $(k+1)!$.

4.3. Some Properties of the Barycentric subdivision.

Let $\bar{\sigma}^k = a_0 a_1 \ldots a_k$ be any simplex. The properties we wish to consider are trivial for $k = -1$ or 0. Assume true ~~for~~ for $m < k$. We shall prove for $m = k$, to complete the induction argument. A face of any simplex in Sd $\bar{\sigma}^k$ is referred to as a face Sd $\bar{\sigma}^k$. We denote by $\bar{\sigma}_i$, the simplex $a_0 \ldots \hat{a}_i \ldots, a_k$, $i = 0, \ldots, k$ and by b, the barycenter of $\bar{\sigma}^k$.

a) the union of all the simplexes of Sd $\bar{\sigma}^k$ is $\bar{\sigma}^k$.

Since b is independent of $\bar{\sigma}^k$, we have $\bar{\sigma}^k = b\dot{\sigma}^k = \bigcup_i b\bar{\sigma}_i^{\,i} =$

$$\bigcup_i \quad \bigcup_{\tau_i \in \text{Sd } \bar{\sigma}_i} b\,\bar{\tau}_i \,.$$

b) The simplexes of Sd $\bar{\sigma}^k$ are of the form $b_0 b_1 \ldots b_k$ where $b_i = b_i(\bar{\sigma}^i)$, the barycenter of some i-face of $\bar{\sigma}^k$ and $\bar{\sigma}^0 < \bar{\sigma}^1 < \bar{\sigma}^2 < \ldots < \bar{\sigma}^k$ and conversely, (i.e.), the faces of $\bar{\sigma}^k$ are partially ordered by the facing relation. The simplexes of $Sd\ \bar{\sigma}^k$ are then in 1:1 correspondence with the maximal linearly ordered sets of nonempty faces of $\bar{\sigma}^k$. Since every simplex of $Sd\ \bar{\sigma}^k$ is of the form $b\ \bar{\tau}_j$, where $\bar{\tau}_j$ is a simplex of $Sd\ \bar{\sigma}_j$, the assertion follows from induction.

c) The faces of $Sd\ \bar{\sigma}^k$ are of the form $b_{i_o} \cdots b_{i_r}$ where b_{i_j} is the barycenter of an i_j-face $\bar{\sigma}^{i_j}$ of $\bar{\sigma}^k$ and $\bar{\sigma}^{i_o} \underset{\neq}{\prec} \bar{\sigma}^{i_1} \underset{\neq}{\prec} \cdots \underset{\neq}{\prec} \bar{\sigma}^{i_r}$.

d) The intersection of a face $\bar{\rho}$ of $Sd\ \bar{\sigma}^k_o$ and a face $\bar{\omega}$ of $\bar{\sigma}^k$ is a face of $\bar{\rho}$ and a face of $Sd\ \bar{\omega}$. For $\bar{\rho}$ is spanned by the barycenters of a linearly ordered sequence of faces of $\bar{\sigma}^k$, $\bar{\sigma}^{i_o} \underset{\neq}{\prec} \bar{\sigma}^{i_1} \underset{\neq}{\prec} \cdots \underset{\neq}{\prec} \bar{\sigma}^{i_r}$ and $\bar{\rho} \cap \bar{\omega}$ is the spanned by the barycenters of these faces in the sequence, which are faces of $\bar{\sigma}^{i_r} \cap \bar{\omega}$ (i.e.)

$$\bar{\rho} \cap \bar{\omega} = b_{i_o} \cdots b_{i_r},$$

where $b_{i_j} = b_{i_j}(\bar{\sigma}^{i_j})$ and $\bar{\sigma}^{i_o} \prec \bar{\sigma}^{i_1} \prec \cdots \prec \bar{\sigma}^{i_s} \prec \bar{\sigma}^{i_r} \cap \omega$

$\underset{\neq}{\prec} \bar{\sigma}^{i_{s+1}} \cdots \prec \bar{\sigma}^{i_r}$.

e) A (k-1)-face $\bar{\tau}$ of $Sd\ \bar{\sigma}^k$ is a face of exactly two k-simplexes of $Sd\ \bar{\sigma}^k$ if $\bar{\tau}$ contains interior points of $\bar{\sigma}^k$. Otherwise, $\bar{\tau}$ is a face of exactly one k-simplex of $Sd\ \bar{\sigma}^k$.

A (k-1)-simplex is spanned by the barycenters of a linearly ordered sequence of k nonempty faces of $\bar{\sigma}^k$, say $\bar{\tau} = b_o b_1 \cdots b_{i-1}\ b_{i+1} \cdots b_k$, $i \leq k$. If $i < k$, then $\bar{\sigma}^{i-1}$

is a face of exactly two i-faces of $\bar{\sigma}^{i+1}$, $\bar{\sigma}_1^i$, $\bar{\sigma}_2^i$ where $b_j = b_j(\bar{\sigma}^j)$. Then $\bar{\tau}$ is a face of

$$b_o \ldots b_{i-1} b_i^1 \ldots b_k$$

and

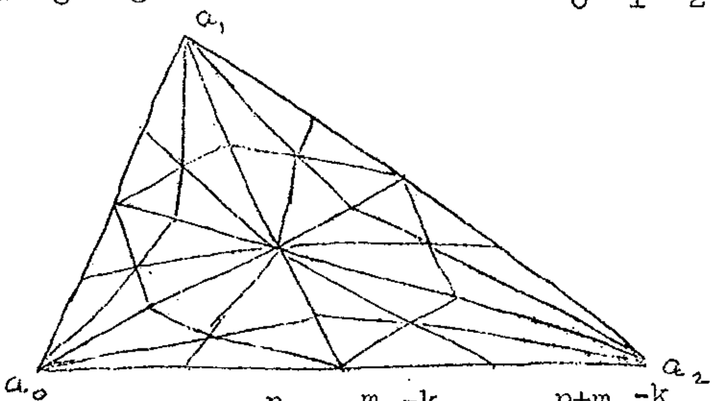
$$b_o \ldots b_{i-1} b_i^2 \ldots b_k.$$

Moreover, $\bar{\tau}$ contains interior points of $\bar{\sigma}^k$ if $b_k = b$ is a vertex. On the other hand, if $i = k$, then $\bar{\tau}$ contains no interior points and $\bar{\tau}$ is a face of the simplex $b_o \ldots b_k$ and no other.

f) Any two faces of $Sd\ \bar{\sigma}^k$ intersect in a common face. In fact, if ρ_1, ρ_2 are spanned by the barycenters of the sequences $\bar{\sigma}^{i_o} < \bar{\sigma}^{i_1} < \ldots . < \bar{\sigma}^{i_r}$, $\bar{\tau}^{j_o} < \bar{\tau}^{j_1} \ldots < \bar{\tau}^{j_s}$, then $\rho_1 \cap \rho_2$ is spanned by the barycenters of those faces occurring in both sequences.

4.4. DEFINITION. The n^{th} barycentric subdivision of of a k-simplex is the collection of all simplexes arising from the barycentric subdivision of every k-simplex in the $(n-1)^{th}$ barycentric subdivision of the simplex and is denoted by $Sd^n\ \bar{\sigma}^k$, where $\bar{\sigma}^k$ is the k-simplex. We set $Sd^o\ \bar{\sigma}^k = \{ \bar{\sigma}^k \}$ and any face of any simplex in $Sd^n\ \bar{\sigma}^k$ is called a face of $Sd^n\ \bar{\sigma}^k$.

The following figure illustrates $Sd^2\, a_0\, a_1\, a_2$.



4.5. <u>Remark</u>. $Sd^n\, (Sd^m\, \bar{\sigma}^k) = Sd^{n+m}\, \bar{\sigma}^k$.

4.6. <u>Exercises</u>. a) The number of k-simplexes in $Sd^n\, \bar{\sigma}^k$ is $(k+1\,!\,)^n$

b) The union of all the simplexes of $Sd^n\, \bar{\sigma}^k$ is $\bar{\sigma}^k$.

c) The faces of $Sd^n\, \bar{\sigma}^k$ are partially ordered by the facing relation $\prec$. Show that the faces of $Sd^n\, \bar{\sigma}^k$ are spanned by the barycenters of a linearly ordered sequences of faces of $Sd^{n-1}\, \bar{\sigma}^k$.

d) If $\bar{\sigma}^i$ is any face of $\bar{\sigma}^k$, any face of $Sd^n\, \bar{\sigma}^k$ contained in $\bar{\sigma}^i$ is a face of $Sd^n\, \bar{\sigma}^i$ and conversely.

4.7. DEFINITION. The <u>closure</u> of a point $p \in \bar{\sigma}^k$ in $Sd^m\, \bar{\sigma}^k$ is the smallest face of $Sd^m\, \bar{\sigma}^k$ containing p. This is well-defined, since the intersection of any two faces of $Sd^m\, \bar{\sigma}^k$ is again a face of $Sd^m\, \bar{\sigma}^k$. We denote the closure of p in $Sd^m\, \bar{\sigma}^k$ by $Cl\,(p, Sd^m\, \bar{\sigma}^k)$ or simply $Cl_m(p)$.

4.8. Exercises. a) Let $p \in \bar{\sigma}^k$ and $\bar{\tau}$ any face of $Sd^m \bar{\sigma}^k$ with $p \in \bar{\tau}$. Then, with respect to $\bar{\tau}$, we have defined $C\ell\,(p) \prec \bar{\tau}$. Show that $C\ell\,(p) = C\ell_m(p)$.

b) For any point $p \in \bar{\sigma}^k$, $C\ell_m(p)$ is a face of any face of $Sd^m \bar{\sigma}^k$ which contains p.

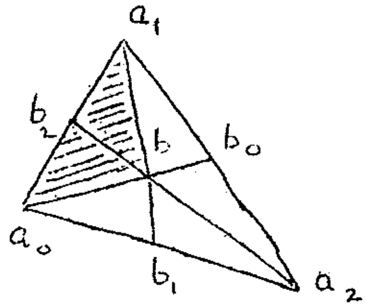
4.9. DEFINITION. Let $V_m(\bar{\sigma}^k)$ denote the set of all vertices of $Sd^m(\bar{\sigma}^k)$, $m \geqslant 0$. An m-standard map on $\bar{\sigma}^k$ is a function $\phi : V_m(\bar{\sigma}^k) \longrightarrow V_o(\bar{\sigma}^k)$ such that $\phi(v)$ is a vertex of $C\ell_o(v)$ for all $v \in V_m(\bar{\sigma}^k)$. An m-standard map extends to a simplicial map of $Sd^m \bar{\sigma}^k$.

We de not distinguish between the standard map and its extension on any $\bar{\sigma}^k$ of $Sd^m \bar{\sigma}^k$. In general, we drop the reference to the order of subdivision and speak of a standard map.

4.10. DEFINITION. Let v be a vertex of $Sd^m \bar{\sigma}^k$, $v \in V_m(\bar{\sigma}^k)$. The union of those simplexes of $Sd^m \bar{\sigma}^k$ with v as a vertex, is a closed subset of $\bar{\sigma}^k$ (the finite union of compact sets is compact) called the closed star of v in $Sd^m \bar{\sigma}^k$. We denote this set by $\overline{St}\,(v, Sd^m \bar{\sigma}^k)$ or simply $\overline{St}_m(v)$. The union of those simplexes of $Sd^m \bar{\sigma}^k$, for which v is not a vertex is laso a closed subset of $\bar{\sigma}^k$. The complement of this set is an open set of $\bar{\sigma}^k$ containing v, called the open star of v in $Sd^m \bar{\sigma}^k$ or simply star of v in $Sd^m \bar{\sigma}^k$. This is denoted by $St\,(v, Sd^m \bar{\sigma}^k)$ or $St_m(v)$.

4.11. <u>Exercise</u>. Prove that for any vertex $v \in V_m(\bar{\sigma}^k)$ the topological closure of $St_m(v)$ is just $\overline{St}_m(v)$ and the topological interior of $\overline{St}_m(v)$ is just $St_m(v)$.

4.12. <u>Example</u>. Consider the 2-simplex Sd $a_0a_1a_2$. $St_1(b)$ = interior of $a_0a_1a_2$, while $\overline{St}_1(b)$ is all of $a_0a_1a_2$. The shaded portion in the adjacent figure represents $St_1(b_2)$. $\overline{St}_1(b_2)$ is obtained by adjoining $a_0\ b \cup ba_1$.



4.13. <u>Exercise.</u> Let $v \in V_m(\bar{\sigma}^k)$. Show that $St_m(v)$ is the union of all <u>open</u> faces of $Sd^m\ \bar{\sigma}^k$ with v as a vertex.

4.14. PROPOSITION. Let $v_0, \ldots, v_r \in V_m(\bar{\sigma}^k)$ where $\bar{\sigma}^k$ is any k-simplex. Then the sets $St_m(v_0), \ldots, St_m(v_r)$ have a non-empty intersection if the vertices $v_0, \ldots, v_r$ span a face of $Sd^m\ \bar{\sigma}^k$. In this case, $St_m(v_0) \cap \ldots \cap St_m(v_r) \supset \bar{\tau}^r$, where $\bar{\tau}^r = v_0 \ldots v_r$.

<u>Proof.</u> The point $p \in St_m(v)$ for any vertex $v \in V_m(\bar{\sigma}^k)$ if v is a vertex of $cL_m(p)$.

4.15. COROLLARY. $St_m(v_0) \cap \ldots \cap St_m(v_r)$ is the union of all open simplexes of which $v_0 \ldots v_r$ is a face.

4.16. LEMMA. The diameter of a simplex in $\mathbb{R}^n$ is the length of the longest edge or 1-face.

Proof. Let $\bar{\sigma}^k = a_o \ldots a_k$ be a k-simplex in $\mathbb{R}^n$ $(n \geqslant k)$. Since $\bar{\sigma}^k$ is compact, the diameter is assumed by some pair of points of $\bar{\sigma}^k$; that is, a line segment in $\bar{\sigma}^k$. However, for any point $p \in \bar{\sigma}^k$, the closed ball about p, with radius = $\max\limits_i \| p - a_i \|$ is convex and contains all the vertices of $\bar{\sigma}^k$ and contains therefore . The maximum distance from p is thus achieved by a vertex. The longest line segment in $\bar{\sigma}^k$ must therefore have a vertex as an end point. By the same argument, it must have a vertex at each end and hence is an edge.

4.17. LEMMA. If the diameter of the simplex $\bar{\sigma}^k = a_o \ldots a_k$ in $\mathbb{R}^n$ is d, then the diameter of each simplex of Sd $\bar{\sigma}^k$ is $\leqslant \frac{kd}{k+1}$.

Proof. Let $b = b(\bar{\sigma}^k)$. Then,

$$b - a_i = \frac{1}{k+1} (a_o + \ldots + a_k) - a_i$$

$$= \frac{1}{k+1} \left[(a_o - a_i) + \ldots + (a_k - a_i) \right]$$

from which

$$\| b - a_i \| \leqslant \frac{1}{k+1} (\| a_o - a_i \| + \ldots + \| a_k - a_i \|) \leqslant \frac{kd}{k+1} .$$

The ball of radius $\frac{kd}{k+1}$, about b contains every vertex of $\bar{\sigma}^k$ and hence contains $\bar{\sigma}^k$. Thus, every edge of Sd $\bar{\sigma}^k$ with b a vertex, must have length $\leqslant \frac{kd}{k+1}$. Any of the remaining edges of Sd $\bar{\sigma}^k$ are edges of the subdivision of one of the (k-1)-faces of of $\bar{\sigma}^k$ and by induction hypothesis has diameter

$\leq \frac{k-1}{k}\, d \leq \frac{k}{k+1}\, d$ and this completes the proof.

4.18. THEOREM. For any k-simplex $\bar{\sigma}^k$ in $\mathbb{R}^n$ and any $\varepsilon > 0$, m can be chosen so that the diameter of every simplex in $Sd^m\, \bar{\sigma}^k$ is $\leq \left(\frac{k}{k+1}\right)^m d$, where d is the diameter of $\bar{\sigma}^k$. Choose m sufficiently large such that $\left(\frac{k}{k+1}\right)^m d < \varepsilon$.

5. AN INTRODUCTION TO DIMENSION

5.1. DEFINITION. Let $\mathcal{U} = \{ U_\lambda \,/\, \lambda \in \Lambda \}$ be a covering of the topological space X. The largest integer n for which there are n distinct indices $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Lambda$ with $U_{\lambda_1} \cap \ldots \cap U_{\lambda_n} \neq \phi$ is called the <u>order of the covering</u>. If there is no such largest integer, then the order of the covering is infinite.

5.2. <u>Remark</u>. We do not assume $U_{\lambda_1} \neq U_{\lambda_2}$ for $\lambda_1 \neq \lambda_2$. For any space X, the covering $\{X_\lambda \,/\, \lambda \in \Lambda\}$ where $X_\lambda = X$ for all $\lambda \in \Lambda$ has order equal to the cardinality of Λ if Λ is finite, otherwise infinite. However, X is connected if and only if there is only one covering by nonempty open sets, of order 1, (viz), $\{ X \}$. The order of any covering of the null space ϕ is 0.

5.3. DEFINITION. A covering $\mathcal{U} = \{U_\lambda / \lambda \in \Lambda\}$ for the topological space X is said to <u>refine</u> the covering $\mathcal{V} = \{V_\mu / \mu \in M\}$ of X if there is a function $f : \Lambda \to M$ such that $U_\lambda \subset V_{f(\lambda)}$ for all $\lambda \in \Lambda$. We say that f effects the <u>refinement</u>. If $\mathcal{U}$ refines $\mathcal{V}$, then $\mathcal{U}$ is called a <u>refinement</u> of $\mathcal{V}$. If further $\Lambda = M$, any refinement effected by the identity of Λ is called a <u>direct refinement</u>. If every (open) covering of X has a (open) refinement with a given property p, then X is said to have <u>arbitrarily fine</u> (open) coverings with the property p.

5.4. DEFINITION. The dimension of a topological space X, is the least integer n for which the space has arbitrarily fine coverings of order n+1 . This definition is clearly topologically invariant.

5.5. DEFINITION. For a metric space (X,d), the supremum of the diamters of the sets of a covering $\mathcal{U}$ of X is called the <u>mesh of the covering</u> $\mathcal{U}$. For any $\varepsilon > 0$, the the covering $\eta(\varepsilon) = \{N(p, \varepsilon) / p \in X\}$ is an open covering of mesh $\leq \varepsilon$.

5.6. THEOREM. Let $\langle X,d \rangle$ be a compact metric space. For any covering $\mathcal{U} = \{U_\lambda / \lambda \in \Lambda\}$ of X by open sets, there exists a positive real number ε such that any covering of X of mesh less than ε refines $\mathcal{U}$.

Proof. Since X is compact, we may choose a finite sub-covering $\{U_{\lambda_1}, \ldots, U_{\lambda_n}\}$ of $\mathfrak{U}$. For each $i = 1,2,\ldots,n$, the function

$$L_i : X \longrightarrow R^+$$

where

$$L_i(p) = d(p, X \setminus U_{\lambda_i})$$

is continuous, hence the function

$$L : X \longrightarrow R^+$$

where

$$L(p) = \max_i L_i(p)$$

is also continuous. Set

$$\varepsilon = \inf_{p \in X} L(p).$$

Every point $p \in U_{\lambda_i}$ for some i, so that $L(p_i) > 0$. But X is compact, hence $\varepsilon > 0$ and for any $p \in X$,

$$d(p, X \setminus U_{\lambda_i}) \geqslant \varepsilon$$

for some $i = 1,2,\ldots,n$. But this means that if $p \in V$ and V has diameter less than ε, then $V \subset U_{\lambda_i}$. The assertion is proved.

5.7. SPERNER'S LEMMA. A standard map on $\bar{\sigma}^k$ is an isomorphism on an odd number of simplexes of $Sd^m\, \bar{\sigma}^k$.

Proof (By induction). Let

$$\phi : V_m(\bar{\sigma}^k) \longrightarrow V_o(\bar{\sigma}^k)$$

be a standard map. A k-simplex of $Sd^m \bar{\sigma}^k$ on which ϕ is an isomorphism is called regular. A (k-1)-face of $Sd^m \bar{\sigma}^k$ is regular if ϕ takes it isomorphically onto the (k-1)-face $a_1 a_2 \cdots a_k$ of $\bar{\sigma}^k$. We must show that the number of regular k-simplexes is odd.

If a simplex $v_o \cdots v_k$ of $Sd^m \bar{\sigma}^k$ has a regular (k-1)-face, $v_1, \ldots, v_k$, where we assume that

$$\phi(v_i) = a_i, \; i = 1,2,\ldots,k \tag{1}$$

then either

$$\phi(v_o) = a_o$$

in which case $v_o \cdots v_k$ is regular or

$$\phi(v_o) = a_i$$

for some $i = 1,2,\ldots,k$ and $v_o \cdots \hat{v}_i \cdots v_k$ is regular. (We use i only to label the vertex and not order the vertices). Thus, a regular simplex has exactly one regular face while all other simplexes have two regular faces or none at all. Hence, if $n(\bar{\tau}^k)$ is the number of regular (k-1)-faces of the simplex $\bar{\tau}^k$ of $Sd^m \bar{\sigma}^k$, the parity of the number

$$n = \sum_{\bar{\tau}^k \in Sd^m \bar{\sigma}^k} n(\bar{\tau}^k) \qquad (2)$$

is the same as the number of regular k-simplexes. In the summation (2), any regular (k-1) face containing interior points of $\bar{\sigma}^k$ is the face of exactly two k-simplexes of $Sd^m \bar{\sigma}^k$ and hence is counted twice. On the other hand, a (k-1) face of $Sd^m \bar{\sigma}^k$ on the frontier of $\bar{\sigma}^k$ is counted only once. The parity of n therefore is equal to the parity of the number of regular (k-1) faces on the frontier of $\bar{\sigma}^k$.

Any (k-1) face lying in $\dot{\bar{\sigma}}^k$ must die in some face of $\bar{\sigma}^k$, say $a_0 \ldots \hat{a}_i \ldots a_k$. No vertex of such a face is mapped onto a_i, hence cannot be regular unless $i = 0$, that is, all the regular (k-1) faces contained in $\bar{\sigma}_0$. The standard map ϕ gives rise to a standard map

$$\phi_0 : V_m(\bar{\sigma}_0) \longrightarrow V_0(\bar{\sigma}_0)$$

and the regular simplexes of this are just the regular (k-1) faces of $Sd^m \bar{\sigma}^k$ contained in $\bar{\sigma}_0$. The result follows by induction.

5.8. LEMMA. A k-cell has dimension $\leq$ k.

Proof. Consider a representation of the k-cell as the k-simplex $\bar{\sigma}^k$ in $\mathbb{R}^n$ $(n \geqslant k)$. Choose m sufficiently large such that the mesh of the open star covering

$$\mathcal{S}_m = \{ St(v)/v \in V_m(\bar{\sigma}^k) \}$$

can be arbitrarily small. Moreover, for any k-simplex $v_0 \ldots v_k$

of $Sd^m \bar{\sigma}^k$,

$$St(v_o) \cap \ \ldots \ \cap St(v_k)$$

it not empty so that the order of this covering is at least k+1. The order cannot exceed k+1, since k is the largest integer for which there are k-faces of $Sd^m \bar{\sigma}^k$.

5.9. LEMMA. Every finite open covering of a normal space has a closed direct refinement.

Proof. Let $\mathfrak{U} = \{ \mathfrak{U}_i / i = 1,2,\ldots,n \}$ be any finite open covering of the normal space. X. The assertion is trivial for m = 1. Assume by induction that the assertion is true for m-1 and set

$$\mathfrak{U}' = \bigcup_{k=2}^{m} \mathfrak{U}_k.$$

Then $X = \mathfrak{U}' \cup \mathfrak{U}_1$ so that $X \setminus \mathfrak{U}_1$ and $X \setminus \mathfrak{U}'$ are disjoint closed sets. By normality, there are disjoint open sets

$$V' \supset X \setminus \mathfrak{U}' \ , \ V_1 \supset X \setminus \mathfrak{U}_1$$

The complements $X \setminus V' = A'$, $X \setminus V_1 = A_1$ of these sets are closed and satisfy $A' \subset \mathfrak{U}'$, $A_1 \subset U_1$ and $A' \cup A_1 = X$. Applying induction hypothesis to $\{ A' \cap \mathfrak{U}_2, \ldots, A' \cap \mathfrak{U}_m \}$ of A', (which is closed in X and hence normal in the relative topology) there are closed sets $A_2, \ldots, A_m$ such that $A_i \subset \mathfrak{U}_i$, $i = 2,\ldots,m$ and $A' = \bigcup_{k=2}^{m} A_k$. Then the covering $\{ A_1, A_2, \ldots, A_k \}$ is a closed direct refinement of the covering $\mathfrak{U}$.

5.10. LEBESGUE SPERNER LEMMA. Any direct refinement of the covering by open stars $\mathcal{S}_o$ of the simplex $\bar{\sigma}^k \equiv a_o \dots a_k$ has order $\geqslant k+1$.

Proof. Let $\mathfrak{U} = \{ \mathfrak{U}_o, \dots, \mathfrak{U}_k \}$ be any open direct refinement of $\mathcal{S}_o$, where $\mathfrak{U}_i \subset St(a_i)$, $i = 0,1,\dots,k$. By above lemma, we obtain a direct closed refinement

$$r = \{ A_o, A_1, \dots, A_k \}$$

with

$$A_i \subset \mathfrak{U}_i , \quad i = 0,1, \dots, k.$$

Let

$$V = V_m(\bar{\sigma}^k), \; V_i' = V \cap A_i, \; V_i = V_i' \setminus \bigcup_{j=0}^{i-1} V_j' ,$$

$$i = 0,1,\dots,k.$$

The sets V_i are mutually disjoint and $V = \bigcup_{i=0}^{k} V_i$. Define a standard map by

$$\phi(V_i) = a_i.$$

Then, by Sperner's lemma, there is at least one regular k-simplex $\bar{\tau}^k = v_o \dots v_k$ in $Sd^m \bar{\sigma}^k$ where

$$\phi(v_i) = a_i .$$

But then $v_i \in A_i$, $i = 0, 1, \dots k$. If m is sufficiently large so that the diameters of the simplexes of $Sd^m \bar{\sigma}^k$ are less than ε where $\varepsilon = \min_{i=0,\dots,k} \| A_i, \bar{\sigma}^k \setminus \mathfrak{U}_i \|$, then any point within ε of A_i must lie in $\mathcal{U}_i$ and hence any simplex of $Sd^m \bar{\sigma}^k$ with a point in A_i must lie entirely in $\mathcal{U}_i$. In particular, the regular simplex $\bar{\tau}^k$ lies in each of the sets U_i and hence the order of covering $\mathcal{U}$ must be $k+1$.

5.11. THEOREM. A k-cell has dimension k.

Proof. By Lemma 5.8, the dimension is $\leq k$. To complete the proof, it is enough to produce a covering of $\bar{\sigma}^k$, with the property that every refinement has order $\geqslant k+1$. This will prove that the dimension is at least k. The covering $\mathcal{S}_0$ by open stars of $\bar{\sigma}^k = Sd^0 \bar{\sigma}^k$ has this property. For, let $\mathcal{U} = \{U_\lambda / \lambda \in \Lambda\}$ be any refinement of $\mathcal{S}_0$. We construct a direct refinement w of $\mathcal{S}_0$ as follows:

$$w_i = \bigcup \{ U_\lambda \in U / U_\lambda \subset St(a_i),\ U_\lambda \not\subset St(a_j) \text{ for } j < i \}$$

Since U is a refinement of $\mathcal{S}_0$, this process exhausts all the sets of U so that w is indeed a covering of $\bar{\sigma}^k$. Moreover each of the sets w_i is open and

$$w_i \subset St(a_i),\ i = 0,1,\ldots,k.$$

Clearly, the order of w is not greater than that of U and hence the result follows.

5.12. LEMMA. A closed subset of a topological space of dimension n has dimension $\leq$ n.

Proof. Let A be a closed subspace of the n-dimensional space X and let U be an open covering of A. Let U be any covering of X which induces $\mathfrak{U}$, that is,

$$\mathfrak{U} \equiv \{ u' \cap A \;/\; u' \in \mathfrak{U}' \} .$$

Such a covering exists, for every set of $\mathfrak{U}$ is the intersection of some open sets of A, with A and the collection of all these open sets together with $X \setminus A$ constitutes such a covering. Since X has dimension n, there is an open refinmment $\mathcal{V}'$ of $\mathfrak{u}'$ of order $\leq n+1$ and the restriction V of $\mathcal{V}'$ to A yields a refinement of $\mathfrak{U}$ of order $\leq n+1$. Hence dimension of $A \leq n$.

5.13. THEOREM. $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $n = m$.

Proof. We first remark that any compact subset of $\mathbb{R}^n$, being bounded, is contained in some n-cell of $\mathbb{R}^m$ and hence has dimension $\leq n$. Any homeomorphism of $\mathbb{R}^n$ to $\mathbb{R}^m$ with $m < n$ would map the unit n-ball B^n homeomorphically into a compact subset B of $\mathbb{R}^m$. Since the dimension is topologically invariant the dimension of B would be n. But this is impossible since as a subset of $\mathbb{R}^m$, the dimension of B is $\leq m < n$. Hence $\mathbb{R}^n$ is never homeomorphic to $\mathbb{R}^m$ when $m < n$. Interchanging m and n, we have the result.

5.14. DEFINITION. For a compact metric space $\langle X, d \rangle$, the number ε defined as in the Lebesgue Sperner Lemma, for the finite covering $\{ \mathfrak{u}_1, \ldots, \mathfrak{u}_n \}$ is called the <u>Lebesgue number</u> of the covering.

Map means a continuous mapping

5.15. Remark. For a compact metric space, to show the existence of arbitrary fine coverings with a given property p, it suffices to show the existence of coverings with property p of arbitrarily small mesh.

5.16. DEFINITION. ~~A subset A of X is called a retraction if there exists a map~~

$$\cancel{r : X \longrightarrow A .}$$

A subspace Y of a topological space X is called a retract if there exists a mapping or function

$$r : X \longrightarrow Y$$

such that r/Y is the identity on Y, (i.e.),

$$r(y) = y$$

for all $y \in Y$. The function r is called retraction.

5.17. DEFINITION. Let X and Y be topological spaces. Two functions $f, g : X \longrightarrow Y$ are said to be homotopic if there exists a function $h : X \times I \longrightarrow Y$ such that

$$h(x,0) = f(x) \qquad x \in X.$$
$$h(x,1) = g(x)$$

We write $f \simeq g$. If instead of X in previous definition, there exists a neighborhood $\mathfrak{U}$ such that Y is a retract of $\mathfrak{U}$, then Y is called a neighborhood retract.

5.18. DEFINITION. Combine the retractions

$$r : X \longrightarrow Y$$
$$r : \mathfrak{U} \longrightarrow Y$$

with the inclusions

$$i : Y \longrightarrow X$$
$$i : Y \longrightarrow \mathcal{U}$$

and we get the compositions

$$ir : X \longrightarrow X$$

$$ir : \mathcal{U} \longrightarrow \mathcal{U}$$

If this map is homotopic to the identity, then Y is called a (neighborhood) <u>deformation retract of</u> X. Moreover, if the points of Y are left fixed, that is $h(y,t) = y$ for all $t \in [0,1]$ and $y \in Y$, then Y is a strong (neighborhood) <u>deformation retract of</u> X.

5.19. <u>Examples</u>. a) Any point of a topological space X is a retract of X.

b) The equatorial plane is a retract of the n-ball under affine projection.

c) S^{n-1} is a neighborhood deformation retract of B^n.

5.20. BROUWER FIXED POINT THEOREM. Every map of an n-cell into itself has a fixed point.

<u>Proof</u>. Consider the representation of an n-cell as an n-simplex, i.e., $\bar{\sigma}^n \equiv a_o \dots a_n$, and let

$$f : \bar{\sigma}^n \longrightarrow \bar{\sigma}^n$$

be any map.

The sets

$$C_i = \{ p \in \bar{\sigma}^n \,/\, f(p)\,(a_i) < p(a_i) \}, \; i = 0,1,\dots,n.$$

are open subsdts of $\bar{\sigma}^n$.

$$C_i \subset St(a_i), \; i = 0,1,\ldots,n;$$

since $p \in C_i$,

$$p(a_i) > f(p)(a_i) \geqslant 0$$

and hence

$$p \subset St(a_i) .$$

If $C = \{ C_0, C_1, \ldots, C_n \}$ is a covering of $\bar{\sigma}^n$, then by Lebesgue Sperner Lemma, there is

$$p \in \bigcap_{i=0}^{n} C_i$$

such that

$$f(p)(a_i) < p(a_i), \quad i = 0,1,\ldots,n.$$

By adding

$$\sum_{i=0}^{n} f(p)(a_i) < \sum_{i=0}^{n} p(a_i).$$

Since both the sums are equal to unity, we have a contradiction. Hence the C_i's do not cover the whole space. Therefore, there exists p such that

$$p \in \bar{\sigma}^n \setminus \bigcup_{i=0}^{n} C_i$$

for which

$$f(p)(a_i) \geqslant p(a_i), \quad i = 0,1,\ldots,n.$$

Summing both sides over $i = 0,1,\ldots,n,$ both sides would be equal to unity, hence all the inequalities should be equalities. Hence

$$p(a_i) = f(p)\ (a_i),\ i = 0,1,\ldots,n.$$

Hence

$$f(p) = p$$

which implies that p is a fixed point.

5.21. THEOREM. S^{n-1} is not a retract of B^n.

Proof. If

$$r : B^n \longrightarrow S^{n-1}$$

is a retraction, then the map

$$B^n \longrightarrow B^n$$

given by

$$p \longrightarrow -r(p)$$

has no fixed point, a contradiction to the Brouwer fixed point theorem. Hence the result.

5.22. DEFINITION. If the topological vector space has a base of convex neighborhood of the origin, the space is called a locally convex topological vector space, or in short, convex, space.

5.23. THEOREM. S^{n-1} is a neighborhood deformation retract of B^n.

Proof. Let p_o be an interior point of B^n. Then p_o is independent of S^{n-1} and $B^n = p_o\ S^{n-1}$. If $p \in B^n$, then

$p = \alpha p_o + (1-\alpha)q$, $q \in S^{n-1}$. Then

$$\langle p, t\rangle \longrightarrow \alpha t\, p_o + (1-\alpha t)q$$

is a deformation retraction of $B^n \setminus \{p_o\}$ onto S^{n-1}.

5.24. DEFINITION. A space X is said to be <u>contractible</u> if any point of X is a deformation retract of X. The deformation retraction is called <u>contraction</u>.

5.25. <u>Example</u>. The map $\langle p, t\rangle \longrightarrow tp$ defines a contraction of $\mathbb{R}^n$ onto the origin. Its restriction to B^n gives a contraction of B^n onto the origin.

5.26. <u>Exercise</u>. If a point is a deformation retract of X, so is every point.

5.27. THEOREM. S^n is not contractible.

<u>Proof</u>. If possible, let

$$F : S^n \times I \longrightarrow S^n$$

be a contraction and let p_o be an interior point of B^{n+1}. Then, p_o is independent of S^n and $B^{n+1} = p_o S^n$. Then, the function

$$\alpha p_o + (1-\alpha)q \longrightarrow F(q,\alpha)$$

is a retraction of B^{n+1} onto S^n which gives a contradiction.

5.28. <u>Exercises</u>. a) A point of B^n has arbitrary small neighborhoods $\mathcal{U}$ such that $B^n \setminus \mathcal{U}$ is a retract of B^n, if p is a frontier point.

b) S^{n-1} has dimension $n-1$.

6. LINEAR SIMPLICIAL POLYHEDRON

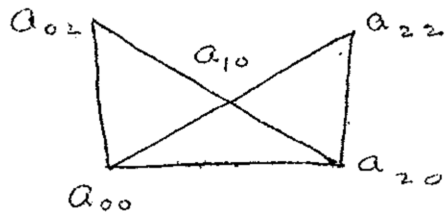
6.1. <u>A linear simplicial polyhedron</u> P in a vector space represented as the union of a collection of simplexes in V such that

a) any two simplexes forming the polyhedron are disjoint or intersect in a common face.

b) A subset of P is closed if it intersects each simplex making up P in a set closed in the natural topology.

<u>Example</u>. Consider $a_{ij} = \langle i,j \rangle \in \mathbb{R}^2$; then

$$a_{00}\ a_{20}\ a_{22} \cup a_{00}\ a_{20}\ a_{02}$$



is not a polyhedron, because their intersection is not a common face. But

$$a_{00}\ a_{11}\ a_{02} \cup a_{00}\ a_{20}\ a_{11} \cup a_{20}\ a_{22}\ a_{11}$$

is a polyhedron.

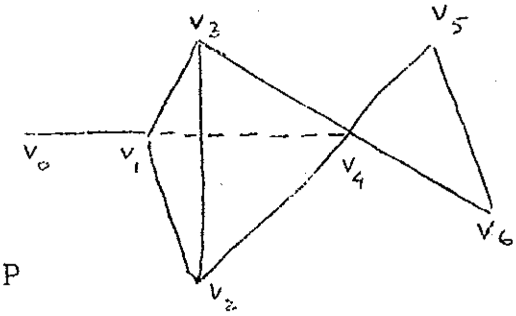
Condition (b) asserts that the topology of the polyhedron is the finest topology which induces the natural topology in each of the simplexes. It is called the natural topology of the polyhedron or the <u>Whitehead topology</u>, after J.H.C.Whitehead.

6.2. DEFINITION. If a polyhedron is made up of a finite number of simplexes, it is called a <u>finite</u> polyhedron. Otherwise it is an <u>infinite polyhedron</u>. If one of the simplexes making up the polyhedron has dimension n, but none of the simplexes has dimension $>$ n, then n is called the dimension of the polyhedron and the polyhedron is called an n-polyhedron. Otherwise it is an ∞-polyhedron. Thus, a k-simplex and the nth barycentric subdivision of a k-simplex are finite k-polyhedra. On the other hand, the real numbers $\mathbb{R}$ can be represented as the union of 1-simplexes spanned by consecutive integers. This is an infinite 1-polyhedron. Observe that an ∞-polyhedron is necessarily infinite, since a simplex is always finite dimensional.

6.3. Notation. Any face of any simplex of a polyhedron will be referred to as a simplex of the polyhedron. If every simplex of a polyhedron is augmented, we say that the polyhedron is augmented. Then the intersection of any two simplexes of a polyhedron is again a simplex (common face) of the polyhedron. Henceforth, we assume that all polyhedra are augmented.

6.4. DEFINITION. A subset Q of P which is a union of simplexes of P is called a sub-polyhedron of P. Q itself is a polyhedron. In particular, any k-simplex of a polyhedron P is a finite k-subpolyhedron of P. The union of all k-simplexes $(k \leq n)$ of P is an n-subpolyhedron called the n-skeleton or n-section of P and is denoted by ${}_nP$. Thus, if $P = \bar{\sigma}^m$, then ${}_{n-1}P = \dot{\sigma}^n$.

Again, if P is the polyhedron represented in the adjacent figure, then P is a finite 3-polyhedron which is the union of a 1-simplex $v_0 v_1$, a 2-simplex $v_4 v_5 v_6$ and a 3-simplex $v_1 v_2 v_3 v_4$. Then ${}_?P$ consists of all points of P



not in the interior of $v_1 v_2 v_3 v_4$. For any polyhedron P, ${}_oP$ denotes the vertices of P.

6.5. DEFINITION. Let X be a subset of P. The union of all simplexes of P which are disjoint from X is a subpolyhedron and hence a closed subset of P. Its complement is an open set in P containing X and is called star of X, written St. X. On the other hand, the union of all the simplexes of P which have nonempty intersection with X is a subpolyhedron and a closed subset containing X is called the closed star of X, written $\overline{\text{St}}\, X$. If $X = \{x\}$ consisting of a single point, we may write $\overline{\text{St}}\,\{x\}$, $\overline{\text{St}}\, x$ and so on. We say that P is star-finite if $\overline{\text{St}}\ p$ is a finite subpolyhedron, for each point $p \in P$.

6.6. Examples. a) Consider $\mathbb{R}^2$. Let a_o be the origin and $a_j = \langle \cos \frac{\pi}{j}, \sin \frac{\pi}{j} \rangle$, $j = 1,2,\ldots$. Then the polyhedron P is the union of all the simplexes $\{ a_o a_n / n \geqslant 1 \}$. P is an infinite 1-polyhedron. But P is not star-finite, since $\overline{\text{St}}\ a_o$ is not finite.

b) Let $\mathbb{R}^\infty$ denote the real vector space generated by the non-negative integers and let e_i be the basis element corresponding to the integer i. Then $\{ e_i / i = 0,1,\ldots \}$ is a basis for $\mathbb{R}^\infty$. The polyhedron P' is the set of all simplexes

$$\left\{ \bar{\sigma}_n^{2n-2} = e_j \, e_{j+1} \cdots e_{j+2n-2} \;/\; j = \frac{n^2 - 3n + 4}{2},\ n = 1,2,\ldots \right\}$$

with the Whitehead topology. That is,

$$P' = e_1 \cup e_1 e_2 e_3 \cup e_2 e_3 e_4 e_5 e_6 \cup e_4 e_5 e_6 e_7 e_8 e_9 e_{10} \cup \cdots .$$

Clearly, this star-finite.

6.7. PROPERTIES. We may think of St as an operator or function defined on subsets of P into some special open sets of P. It has the following properties:

a) If $X_1 \subset X_2 \subset P$, then $St\, X_1 \subset St\, X_2$

b) $St \bigcup_{\lambda \in \Lambda} \{ X_\lambda \} = \bigcup_{\lambda \in \Lambda} St\ X_\lambda$.

c) $St\,(St\, X) = St\, X$.

6.8. DEFINITION. Let p be any point of the polyhedron P and $\bar{\sigma}_1$ any simplex of P containing p. Regarding p as a point of $\bar{\sigma}_1$, the closure of p is a face of $\bar{\sigma}_1$, which is denoted by $Cl(p,\bar{\sigma}_1)$. If p lies also in $\bar{\sigma}_2$, then $\bar{\sigma}_1 \cap \bar{\sigma}_2$ is a face of $\bar{\sigma}_1$ containing p, so that $Cl(p,\bar{\sigma}_1)$ is a face of $\bar{\sigma}_1 \cap \bar{\sigma}_2$. Hence $Cl(p,\bar{\sigma}_1) = Cl(p,\bar{\sigma}_2)$ is independent of $\bar{\sigma}_1$ and define it to be the closure of p, denoted by $Cl(p)$. It is the smallest simplex of P containing p and p is an interior point of the simplex. The closure Cl is thus a function defined from the points of P to the simplexes of P. If $\bar{\sigma}$ is a simplex, then $Cl^{-1}\bar{\sigma} = \sigma$ (= interior of $\bar{\sigma}$). Therefore, P is a disjoint union of the interiors of simplexes of P.

6.9. Exercises. a) Let p be any point in a polyhedron P and let $\bar{\sigma} = Cl\ p$. Prove that $St\ p = St\ \sigma$.

b) Let $\{ v_o, v_1, \ldots, v_k \}$ be a set of (distinct) vertices of P. Then the stars of the vertices, $St\ v_o, \ldots, St\ v_k$, have a nonempty intersection if $v_o, \ldots, v_k$ is a simplex $\bar{\tau}$ of P and then

$$St\ v_o \cap \ldots \cap St\ v_k = St\ \tau .$$

6.10. PROPERTIES OF THE CLOSURE FUNCTION. If $X \subset P$,

$$Cl\ X = \bigcup_{x \in X} Cl\ x,$$

a) $Cl\ X$ is the union of all simplexes $\bar{\sigma}$ of P whose interiors meet X, that is, $X \cap \sigma \neq \phi$.

b) If $X_1 \subset X_2 \subset P$, then $Cl\ X_1 \subset Cl\ X_2$.

c) $Q \subset P$ is a subpolyhedron if $\mathrm{Cl}\ Q = Q$.

d) $\mathrm{Cl}(\mathrm{Cl}\ X) = \mathrm{Cl}\ X$

e) $\mathrm{Cl} \bigcup_{\lambda \in \Lambda} X_\lambda = \bigcup_{\lambda \in \Lambda} \mathrm{Cl}\ X_\lambda$

f) $\mathrm{Cl} \bigcap_{\lambda \in \Lambda} X_\lambda \subset \bigcap_{\lambda \in \Lambda} \mathrm{Cl}\ X_\lambda$

g) $\mathrm{Cl}\ \mathrm{St}\ X = \overline{\mathrm{St}}\ X$

h) $x \in \mathrm{St}\ X$ if and only if $(\mathrm{Cl}\ x) \cap X \neq \phi$.

6.11. PROPOSITION. The union of a family of <u>subpolyhedra</u> of ~~P~~ a polyhedron P is again a subpolyhedron of P. The <u>intersection</u> of any family of (finite) subpolyhedra is a (finite) subpolyhedron. The first statement follows from the definition. The second follows from the above properties.

6.12. COROLLARY. For any X, $\mathrm{Cl}\ X$ is the intersection ~~of all~~ of all subpolyhedra of P containing X and hence is the smallest subpolyhedron of P containing X.

6.13. <u>Remarks</u>. a) $\mathrm{St}\ X$ is the complement of the union of all polyhedra of P disjoint from X and hence is the complement of the largest subpolyhedron of P disjoint from X.

b) If P and Q are polyhedra in the same vector space V, then $P \cap Q$ and $P \cup Q$ are defined but <u>need not be</u>

polyhedra. However, if either one is a polyhedron, then both are.

6.14. PROPOSITION. Let $\{ P_\lambda / \lambda \in \Lambda \}$ be a family of polyhedra such that $P_{\lambda 1} \cap P_{\lambda_2}$ is a polyhedron for any $\lambda_1, \lambda_2 \in \Lambda$. Then $\bigcup_{\lambda \in \Lambda} P_\lambda$ defines a polyhedron P and P_λ is a subpolyhedron of P for all $\lambda \in \Lambda$.

6.15. Notation. If P is a polyhedron, the topological space represented by P is denoted by $|P|$. If Q is a subpolyhedron of P, then $|Q|$ is a subset of $|P|$. Two different polyhedra P_1 and P_2 may have the same topological space $|P|$. We then write for any $X \subset P$,

$$\begin{aligned} St(X,P) &= St_p(X) \\ \overline{St}(X,P) &= \overline{St}_p(X) \\ Cl(X,P) &= Cl_p(X). \end{aligned}$$

If Q is a subpolyhedron of P and $X \subset |P|$,

$$St_Q(X) = St_Q(X \cap |Q|).$$

6.16. DEFINITION. The polyhedron P' is said to be a simplicial subdivision of the polyhedron P if $|P| = |P'|$, and for every simplex $\bar{\sigma}$ of P', $Cl_P(|\bar{\sigma}|)$ is a simplex of P.

This means that the simplexes of P' are 'smaller' than those of P.

6.17. Remark. For any subset $X \subset |P| = |P'|$, where P' is a simplicial subdivision of P, it follows that $Cl_{P'}(X)$ $Cl_P(X)$.

6.18. Notation. When two polyhedra P and P' are given with same space $|P| = |P'|$, then for every subpolyhedron $Q \subset P$, $Cl_{P'}(|Q|)$ is a subpolyhedron of P'. Then, $Cl_{P'}$ gives rise to a function from the subpolyhedra of P into the subpolyhedra of P'. We denote this by $Cl(\ ;P,P')$, that is $Cl(Q, P, P') = Cl_{P'}(|Q|)$ it being understand that $|P| = |P'|$ and $Q \subset P$. We will write $Cl_{P'}\,Q$ when there is no confusion.

6.19. PROPERTIES. Let Q_α, Q_β be subpolyhedra of P and Q'_α, Q'_β those of P'.

a) If $Q'_\alpha \subset Q'_\beta$, then $Cl_P(Q'_\alpha) \subset Cl_P(Q'_\beta)$.

b) $Cl_P(Q'_\alpha \cup Q'_\beta) = Cl_P(Q'_\alpha) \cup Cl_P(Q'_\beta)$.

c) $Cl_P(Q'_\alpha \cap Q'_\beta) \subset Cl_P(Q'_\alpha) \cap Cl_P(Q'_\beta)$

If moreover, P' is a simplicial subdivision of P,

d) if $|Q_\alpha| = |Q'_\alpha|$, then $Cl_P(Q'_\alpha) = Q_\alpha$

e) $|Cl_{P'}Q_\alpha| = |Q_\alpha|$ so that $Cl_{P'}\,Q_\alpha$ is a simplicial subdivision of Q_α. We say that $Cl_{P'}\,Q_\alpha$ is a simplicial subdivision of Q_α induced by P'

f) $Cl_{P'}(Q_\alpha \cap Q_\beta) = Cl_{P'}(Q_\alpha) \cap Cl_{P'}(Q_\beta)$.

g) $Cl\ (Cl_{P'} Q_\alpha) = Q_\alpha$.

h) Q'_α is a subpolyhedron of $Cl_{P'}(Cl_P\ Q'_\alpha)$ and if $Q'_\beta \subset Q'_\alpha$, then $Cl_{P'}(Cl_P\ Q'_\beta) \subset Cl_{P'}(Cl_P Q'_\alpha)$.

Suppose P'' is a simplicial subdivision of P', which is again a simplicial subdivision of P, then for any subpolyhedron Q'' of P'', it is clear that

i) $Cl\left(Cl\ (Q''; P'', P'); P', P \right) = Cl\ (Q; P'', P)$.

This means that P'' is a simpoicial subdivision of P. If Q is a subpolyhedron of P, it is also true that

j) $Cl\left(Cl(Q; P, P')\ ; P', P''\right) = Cl\ (Q; P, P'')$

6.20. Remark. If P' is a simplicial subdivision of P, for any $X \subset |P|$ and Q, a subpolyhedron of P not meeting X, $Cl_{P'} Q$ is a subpolyhedron of P' not meeting X and so

$$|St_{P'}\ X| \subset |St_P X|.$$

6.21. DEFINITION. Let Q be a subpolyhedron of a polyhedron P and let P' be a simplicial subdivision of P. If every simplex of Q is also a simplex of P', then Q is a subpolyhedron of P' and

$$Cl_{P'}\ \bar{\sigma} = \bar{\sigma} = Cl_P \bar{\sigma}$$

for any simplex $\bar{\sigma} \in Q$. We then say that P' is a <u>simplicial subdivision of P relative to</u> Q and is denoted by

$(P,Q)'$.

6.22. DEFINITION. The barycentric subdivision of a polyhedron P is the simplicial subdivision $\mathrm{Sd}\, P$ resulting from the barycentric subdivision of each of the simplexes. The n-th barycentric subdivision of P, denoted by $\mathrm{Sd}^n\, P$, is defined inductively as $\mathrm{Sd}(\mathrm{Sd}^{n-1}\, P)$ and $\mathrm{Sd}^o\, P = P$.

A common procedure for defining subdivision of a polyhedron P is inductively on the n-skeleton $_nP$ of P. Any subdivision of P leaves $_oP$ unaltered. We may assume $_nP$ has already been subdivided and describe the subdivision of any $(n+1)$-simplex $\bar{\sigma}^{n+1}$ taking care to see that the subdivision of σ^{n+1} induced by $_nP$ is preserved. We shall not describe barycentric subdivision of a polyhedron P with respect to a subpolyhedron Q. Let $\bar{\sigma}$ be any simplex and σ' any simplicial subdivision of σ. Then, the simplicial subdivision of $\bar{\sigma}$ relative to σ' is the collection of all simplexes $b\,\bar{\tau}$ where b is the barycenter of $\bar{\sigma}$ and $\bar{\tau}$ is a simplex in σ'. We denote this by $b\,\sigma'$. If $\sigma' = \mathrm{Sd}\,\sigma$ (barycentric subdivision of σ), then

$$b\,\sigma' = b\ \mathrm{Sd}\,\sigma = \mathrm{Sd}\,\sigma .$$

Suppose $\mathrm{Sd}(P,Q)$ has already been defined on $_nP$ and let $\bar{\sigma}^{n+1}$ be any $(n+1)$-simplex in P. If $\bar{\sigma}^{n+1}$ is a simplex of Q, $\bar{\sigma}^{n+1}$ is also in $\mathrm{Sd}(P,Q)$. If $\bar{\sigma}^{n+1}$ does not belong to Q, its frontier σ^{n+1} is a subpolyhedron of $_nP$. Let σ' be the subdivision of σ induced by $\mathrm{Sd}(P,Q)$. Then the subdivision of $\bar{\sigma}^{n+1}$ is $b\,\sigma'$, where b is the barycenter of $\bar{\sigma}^{n+1}$.

Inductively define $Sd^n(P,Q)$.

6.23. Notation. The relation " $\bar{\tau}$ is a face $\bar{\sigma}$ " $(\bar{\tau} \prec \bar{\sigma})$ defines a partial ordering among the simplexes of the polyhedron P.

6.24. DEFINITION. Let Q be a subpolyhedron of P. Then, a proper sequence in (P,Q) is a finite linearly ordered sequence.

$$\bar{\sigma}_o \prec \bar{\sigma}_1 \prec \cdots \prec \bar{\sigma}_k.$$

of simplexes of P, with $\bar{\sigma}_o$, and only $\bar{\sigma}_o$, a simplex of Q. ($\bar{\sigma}_o$ may be the empty set).

6.25. Remark. If $b_i = b(\bar{\sigma}_i)$ is the barycenter of $\bar{\sigma}_i$, $i = 1,\ldots,k$ then $\bar{\sigma}_o\, b_1\, b_2 \ldots b_k$ is a simplex of $Sd(P,Q)$ of dimension $= k + \dim \bar{\sigma}_o$ and every simplex in $Sd(P,Q)$ is obtained in this manner. Thus, there exists a $1:1$ correspondence between the simplexes of $Sd(P,Q)$ and the proper sequences in (P,Q).

6.26. PROPOSITION. Let Q be a subpolyhedron of P. Then a) If R is a subpolyhedron of P and $R \cap Q$ is atmost a 0-polyhedron, then the subdivision of R induced by $Sd^n(P,Q)$ is $Sd^n R$.

b) $R = P \setminus St_P Q$ is disjoint from Q.

c) $\overline{St}(Q, Sd(P,Q))^- \subset St(Q,P)$

6.27. LEMMA. Let $\bar{\sigma}^n$ be a simplex in $\mathbb{R}^m$ of diameter d and $\bar{\sigma}_o$ an $(n-1)$-face of $\bar{\sigma}^n$. If $\bar{\tau}$ is any simplex of the k-th barycentric subdivision $Sd^k(\bar{\sigma}^n, \bar{\sigma}_o)$, which meets $\bar{\sigma}_o$, that is, $\bar{\sigma}_o \cap \bar{\tau} \neq \phi$, then every point of $\bar{\tau}$ is with $(\frac{n}{n+1})^k d$ of $\bar{\sigma}_o$.

Proof. (By induction). Trivial for $k = 0$. Assume it is true for $k-1$ $(k > 0)$. Since, $\bar{\tau}$ and $\bar{\sigma}_o$ are both convex sets, it is enough to show that each vertex of $\bar{\tau}$ is within $(\frac{n}{n+1})^k d$ of $\bar{\sigma}_o$, that is, d_k of $\bar{\sigma}_o$. Then $\bar{\tau} = \bar{\nu}_o b_1 b_2 \dots b_t$ where $\bar{\nu}_o$ is a nonempty face of $\bar{\sigma}_o$, b_i $(i = 1,2,\dots,t)$ is the barycenter of a simplex $\bar{\nu}_i$ of $Sd^{k-1}(\bar{\sigma}^n, \bar{\sigma}_o)$ such that $\bar{\nu}_o < \bar{\nu}_1 < \dots < \bar{\nu}_t$ (a proper sequence). Consider $\bar{\nu}_i$ for $i = 1,2,\dots,t$ and let

$$\bar{\nu}_i = v_o \dots v_s .$$

where at least v_o is a vertex of $\bar{\sigma}_o$. Then, by the induction hypothesis, there exist $w_j \in \bar{\sigma}_o$, $j = 1,2,\dots, s$, such that

$$\| v_j - w_j \| \leq d_{k-1}, \; j = 1,2,\dots s .$$

Let $e_i = \frac{1}{s+1}\left[v_o + w_1 + \dots + w_s\right]$. Then e_i is a point of $\bar{\sigma}_o$ and

$$\| b_i - e_i \| = \| \frac{1}{s+1} (V_0+V_1+\ldots+V_s) - \frac{1}{s+1} (V_0+W_1+\ldots+W_s) \|$$

$$= \frac{1}{s+1} \| (V_0-W_0) + (V_1-W_1) + \ldots + (V_s-W_s) \|$$

$$\leq \frac{1}{s+1} s.d_{k-1} = \frac{s}{s+1} d_{k-1} = \frac{s}{s+1} \left(\frac{n}{n+1}\right)^{k-1} d$$

$$< \left(\frac{n}{n+1}\right)^k d = d_k.$$

6.28. LEMMA. Let $\mathcal{U}$ be an open covering of the simplex $\bar{\sigma}^m = a_0 a_1 \ldots a_n$ $(n > 0)$ such that, for every vertex a_i of $\bar{\sigma}_0 = a_1 a_2, \ldots, a_n$, there is prescribed a set $\mathcal{U}_i \supset \bar{\sigma}_0$. Then for some m, the covering by closed stars of $Sd^m(\bar{\sigma}^n, \bar{\sigma}_0)$ refines $\mathcal{U}$ and

$$\overline{St}\,(a_i, Sd^m(\bar{\sigma}^n, \bar{\sigma}_0)) \subset \mathcal{U}_i \quad i = 1, 2, \ldots, n.$$

Proof. $\bar{\sigma}^n$ is compact. Thus, we may assume $\mathcal{U}$ is a finite open covering with Lebesgue number $\varepsilon > 0$. Let $\mathcal{U} = \bigcap_{i=1}^{n} \mathcal{U}_i$ and choose k sufficiently large such that $\left(\frac{n}{n+1}\right)^k d < \rho(\bar{\sigma}_0, \bar{\sigma}^n \setminus \mathcal{U})$ where ρ is the natural metric in $\bar{\sigma}^n$ and d is the diameter of $\bar{\sigma}^n$. Set

$$St\,(\quad) = St\quad, Sd^k(\bar{\sigma}^n, \bar{\sigma}_0)$$

Then $\overline{St}_k(\bar{\sigma}_0) \subset \mathcal{U}$.

For any vertex v of $Sd^{k+1}(\bar{\sigma}^n, \bar{\sigma}_0)$ contained in $|St_k(\bar{\sigma}_0)|$, every open simplex of $Sd^{k+1}(\bar{\sigma}^n, \bar{\sigma}_0)$ with vertex v is contained in a simplex of $St_k(\bar{\sigma}_o)$. Then

$$|St_{k+1}(v)| \subset |St_k(\bar{\sigma}_o)|$$

and therefore

$$|\overline{St}_{k+1}(v)| \subset |St_k(\bar{\sigma}_o)| \subset \mathcal{U}$$

Let $B = \bar{\sigma}^n \setminus St_{k+1}(\bar{\sigma}_o)$. Let d_o be the maximum diameter of any simplex in B. Given $\varepsilon' > o$, we can choose k_1 sufficiently large so that $\left(\frac{n}{n+1}\right)^{k_1} d_o < \varepsilon'$. Then, if $\varepsilon' < \varepsilon/2$, $m = k + k_1 + 1$ is the required number. Let v be any vertex of $Sd^m(\bar{\sigma}^n, \bar{\sigma}_o)$. If $v \in |St_{k+1}(\bar{\sigma}_o)| \subset |St_k(\bar{\sigma}_o)| \subset \mathcal{U}$, then

$$|\overline{St}_m(v)| \subset |\overline{St}_{k+1}(v)| \subset |St_k(\bar{\sigma}_o)| \subset \mathcal{U} .$$

If $v \notin |\overline{St}_{k+1}(\bar{\sigma}_o)|$, then

$$|\overline{St}_m(v)| \subset |\overline{St}_{k+1}(v)| \subset B.$$

Since the diameter of $|St_m(v)| < \varepsilon$, there is a set $\mathcal{U}_v \in \mathcal{U}$ such that

$$|\overline{St}_m(v)| \subset \mathcal{U}_v.$$

This completes the proof.

6.29. THEOREM. (Refinement theorem). Let $\mathcal{U}$ be an open covering of the polyhedron P. There exists a simplicial subdivision P' of P such that the covering by closed stars of vertices of P' refines $\mathcal{U}$.

<u>Proof.</u> Every vertex v of P belongs to one of the sets U_v in $\mathcal{U}$. We proceed inductively. Suppose now that ${}_nP'$ has been defined (where ${}_nP'$ is the simplicial subdivision of ${}_nP$) and for each vertex a of ${}_nP'$, there is an open set $U_a \in \mathcal{U}$ such that

$$\overline{St}\,(a, {}_nP') \subset U_a .$$

To simplify the proof, we use the following notation. The letter a, with or without subscripts will represent a vertex in ${}_nP'$. $\bar{\sigma}$ will be an $(n+1)$-simplex in P, $\dot{\sigma}$ its frontier, b is the barycenter of $\bar{\sigma}$, $\dot{\sigma}'$ is the simplicial subdivision of $\dot{\sigma}$ induced by ${}_nP'$ and $\bar{\tau}$ is an $(n+1)$ simplex in the centric subdivision $b\dot{\sigma}'$, that is, $\bar{\tau} = b\,\bar{\tau}_0$, where $\bar{\tau}_0 = a_1 \ldots, a_n$ is a simplex in ${}_nP'$. Finally, $\bar{\sigma}'$ and $\bar{\tau}'$ will be the subdivisions $\bar{\sigma}$ and $\bar{\tau}$ respectively induced by ${}_{n+1}P$, so that our task is to describe $\bar{\sigma}'$ and $\bar{\tau}'$.

The covering $\mathcal{U}$ contains a finite subcovering $\mathcal{U}'$ of $\bar{\sigma}$ with Lebesgue number ε, which we may assume contains all sets of the form U_a for every vertex a of $\dot{\sigma}'$. $\mathcal{U}'$ is also a covering of $\bar{\tau}$, containing the sets $U_{a_1}, \ldots, U_{a_n}$ and satisfies the conditions of the previous lemma. The Lebesgue

number of $\mathcal{U}'$, as a covering for $\bar{\tau}$, $\varepsilon'_{\bar{\tau}} \geq \varepsilon$.
The lemma is applied to $\mathcal{U}'$ (as a covering of $\bar{\tau}$) using $\varepsilon/2$ instead of ε' (in the proof of that lemma) there is an integer $m' = m_\tau$ such that the covering by closed stars of $Sd^{m'}(\bar{\tau}, \bar{\tau}_o)$ refines $\mathcal{U}'$ and

$$\left| \bar{St}\,(a_i, Sd^{m'}\,(\bar{\tau}, \bar{\tau}_o) \right| \subset \mathcal{U}_{a_i}, \quad i = 1, 2, \ldots, n$$

and the diameter of every simplex in $Sd^{m'}(\bar{\tau}, \bar{\tau}_o)$ is less than $\varepsilon/2$. Let $m = \max \left\{ m' = m_\tau \,/\, \bar{\tau} \in b\,\dot{\sigma}' \right\}$. Then, $\bar{\bar{\sigma}}' = Sd^m\,(b\,\dot{\sigma}', \dot{\sigma}')$ is the required subdivision.

For each vertex a of ${}_nP'$,

$$\left| \bar{St}\,(a, {}_{n+1}P') \right| = \bigcup \left| \bar{St}(a, \bar{\sigma}') \right| \qquad (1)$$

where $\bar{\sigma}$ ranges over all (n+1)-simplexes containing a. Also

$$\left| \bar{St}\,(a, \bar{\bar{\sigma}}') \right| = \bigcup \left| \bar{St}\,(a, \bar{\tau}') \right| \qquad (2)$$

where $\bar{\tau}$ ranges over all simplexes of $b\,\dot{\sigma}'$ which have a as a vertex. But

$$\bar{\bar{\tau}}' = Sd^m\,(\bar{\tau}, \bar{\tau}_o),$$

so that

$$\left| \bar{St}\,(a, \bar{\tau}') \right| \subset \left| \bar{St}\,(a, Sd^{m'}\,(\bar{\tau}, \bar{\tau}_o) \right|$$

since $m' \leq m$. But

$$\left| \bar{St}\,(a, Sd^{m'}\,(\bar{\tau}, \bar{\tau}_o) \right| \subset \mathcal{U}_a$$

$$\left| \bar{St}\,(a, \bar{\tau}') \right| \subset \mathcal{U}_a \qquad (3)$$

It then follows from (1), (2) and (3) that

$$\left| \bar{S}t\,(a,\ {}_{n+1}P') \right| \subset \mathfrak{U}_a$$

Any vertex c of ${}_{n+1}P'$, which is not a vertex of ${}_nP'$ must be vertex of $\bar{\sigma}'$ interior to $\bar{\sigma}$, so that

$$\left| \bar{S}t\,(c,\ {}_{n+1}P') \right| = \left| \bar{S}t\,(c,\ \bar{\sigma}') \right| .$$

If further c is an interior point of some $\bar{\tau}$, then

$$\left| \bar{S}t\,(c,\ \bar{\sigma}') \right| = \left| \bar{S}t(c,\ \bar{\tau}') \right| .$$

Applying the lemma, there is a $\mathfrak{U}_c \in \mathfrak{U}$ such that

$$\left| \bar{S}t\,(c,\ \bar{\tau}') \right| \subset \mathfrak{U}_c$$

Hence

$$\left| \bar{S}t\,(c,\ {}_{n+1}P') \right| \subset \mathfrak{U}_c$$

If c is not an interior point of $\bar{\tau}$,

$$\left| \bar{S}t\,(c,\ \bar{\sigma}') \right| = \bigcup \left| \bar{S}t\,(c,\ \bar{\tau}') \right| ,$$

where the union ranges over all simplexes $\bar{\tau}$ of $\bar{\sigma}$ such that $c \in |\bar{\tau}|$ and each simplex has diameter less than ε, Hence, there is a set $\mathfrak{U}_c \in \mathfrak{U}$, containing c as well as every point of $\bar{\sigma}$ within distance ε from c. In particular,

$$\left| \bar{S}t(c,\ \bar{\sigma}') \right| \subset \mathfrak{U}_c .$$

6.30. Remark. Let $\mathcal{U}$ be an open covering for the polyhedron P and let Q be a sub-polyhedron of P such that the covering of Q by closed stars of Q refines the covering of Q induced by $\mathcal{U}$. Then there is a simplicial subdivision $(P,Q)'$ such that the covering of P by closed stars of $(P,Q)'$ refines Q.

Proof. As in the previous theorem, except that when $\bar{\sigma}$ is a simplex of Q, we take $\bar{\sigma}' = \bar{\sigma}$.

6.31. Exercise. Find an example to show that the result is not true if 'closed stars' are replaced by 'open stars'.

7. THE TOPOLOGY OF THE POLYHEDRON.

We know that a simplex is compact. A finite polyhedron is a union of finitely many compact sets and hence is compact.

7.1. THEOREM. Let A be a compact subset of a polyhedron P. We shall in fact show that $Cl\ A$ is a finite sub-polyhedron. For each simplex $\bar{\sigma}$ in $Cl\ A$, let $p_\sigma \in \sigma \cap A$. (This intersection is not empty). Let $\mathcal{P} = \{ p\sigma / \bar{\sigma} \in Cl\ A \}$. Then $\mathcal{P}$ intersects any simplex $\bar{\tau}$ of P in a finite number of points (possibly one for each face). Hence, $\mathcal{P} \cap \bar{\tau}$ is closed for each simplex $\bar{\tau}$ in $\mathcal{P}$, that is, $\mathcal{P}$ is closed in P. Hence $\mathcal{P}$ is closed in A. A being compact, $\mathcal{P}$ is compact.

By the same argument every subset of $\mathcal{P}$ is closed so that each point of $\mathcal{P}$ is open. Since $\mathcal{P}$ is compact, $\mathcal{P}$ is a finite set. Hence $Cl\ A$ is a finite subpolyhedron.

7.2. COROLLARY. If A is a compact subpolyhedron, then A is finite. In particular, if P is compact, then $\dot{P}$ is finite.

7.3. Remark. A polyhedron P contained in $\mathbb{R}^n$ inherits a topology from $\mathbb{R}^n$ and this topology need not agree with the polyhedron topology.

7.4. Example. Let $a_n = \langle \cos \frac{\pi}{n}, \sin \frac{\pi}{n} \rangle$, $n = 1,2,\ldots$.

$$b_n = \langle \frac{1}{n} \cos \frac{\pi}{n}, \frac{1}{n} \sin \frac{\pi}{n} \rangle .$$

Assume $a_0 = \langle 0,0 \rangle$. Let P_1 be the polyhedron consisting of simplexes $\{ a_0 a_n / n > 0 \}$ and P_2 consisting of simplexexes $\{ a_0 b_n / n > 0 \}$. As a set of $\mathbb{R}^2$, P_2 is compact while P_1 is not. As polyhedra, neighter P_1 or P_2 is compact, since they are not finite.

7.5. Remark. A subpolyhedron Q of a polyhedron P is compact if and only if it is finite.

7.6. LEMMA.. Let P be a polyhedron and $p \in P$. For any simplex σ of P containing p, every neighborhood of p intersects σ.

<u>Proof.</u> Let $\mathcal{U}$ be a neighborhood of p. Suppose $\mathcal{U} \cap \sigma = \phi$. Then $\bar{\sigma} \setminus \mathcal{U}$ is a closed set containing σ. Hence $\bar{\sigma} \setminus \mathcal{U} \supset \bar{\sigma}$ which is a contradiction. Hence the result follows.

7.7. THEOREM. A polyhedron is locally compact if and only if it is star-finite.

<u>Proof.</u> If the polyhedron P is star-finite, for each point $p \in P$, $St(p)$ is a neighborhood of p and $St(p)$ is finite and hence compact. Hence P is locally compact.

Conversely, assume that P is locally compact. Let N be an open set containing p such that $\bar{N}$ is compact. Then $N_0 = N \cap St(p)$ is also an open set containing p and $\bar{N}_0 \subset \bar{N} \cap \bar{St}(p)$, for any simplex $\bar{\sigma}$ containing p has interior points in N_0 and hence is a simplex of $Cl\ \bar{N}_0$. In particular, $\tilde{St}(p) \subset Cl\,\bar{N}_0$. Hence, it follows that the latter is finite and compact and hence the same is true of the former. Thus P is star-finite.

7.8. DEFINITION. Let P be a polyhedron. <u>A P-Barycentric function</u> p <u>on</u> P is a barycentric function defined on the vertices ${}_0P$ of P such that its support spans a simplex in P. We will call the simplex spanned by the support of p as the support of p.

7.9. <u>Remark.</u> We know that for any finite set of barycentric functions on ${}_0P$, say $p_1, \ldots, p_n$, the function

$$p = \sum_{i=1}^{n} \alpha_i p_i, \alpha_i \in [0,1] \quad , \quad \sum_{i=1}^{n} \alpha_i = 1$$

is also a barycentric function. But it need not be true for P-barycentric functions. If $p_1, \ldots, p_n$ are P-barycentric functions, and $p = \sum_{i=1}^{n} \alpha_i p_i$, then the support of p is the union of the supports of p_i, that is, $s(p) = \bigcup_{i=1}^{n} s(p_i)$. Then p is a P-barycentric function if and only if the supports of p_i are the faces of the same simplex.

7.10. THEOREM. Let $\mathcal{B}$ denote the set of P-barycentric functions on a polyhedron P. The set $\mathcal{B}_P$ is in 1:1 correspondence with the points of P, the correspondence being given by

$$\underline{p} \longrightarrow \sum_{v \in {}_0P} \underline{p}(v)\, v.$$

Proof. The sum on the right is well defined since p(v)=0 for all but a finite number of vertices. The support of $\underline{p}$ is

$$s(\underline{p}) \equiv \{ v_0, \ldots, v_r \}$$

Moreover, $v_0, \ldots, v_r$ are vertices of a simplex P_{s_0} that

$$\sum_{v \in V} \underline{p}(v)\, v = \sum_{i=0}^{r} \underline{p}(v_i)\, v_i \in v_0 \ldots v_r ,$$

For any point $p \in P$, there is a unique barycentric function $\underline{p}'$ defined on the vertices of $Cl_p = v_0 \ldots v_s$ such that

$$p = \sum_{i=0}^{s} \underline{p}'(v_i)v_i .$$

Extend $\underline{p}'$ to a P-barycentric function $\underline{p}$ on ${}_oP$ by setting $\underline{p}(v) = 0$ if v is not a vertex of $\mathcal{C}\iota_p$.

7.11. PROPOSITION. If p_1, p_2 are any two points of a polyhedron P, then

$$\| p_1 - p_2 \| = \left[\sum_{v \in {}_oP} \left(\underline{p}_1(v) - \underline{p}_2(v) \right)^2 \right]^{1/2}$$

defines a metric, called the natural metric of the polyhedron.

Proof. Exercise.

7.12. Remark. If the polyhedron is finite and the vertices are unit basis vectors of $\mathbb{R}^n$, the natural metric of the polyhedron is the restriction of the natural metric in $\mathbb{R}^n$. In particular, this metric agrees with the natural metric in P so that the topology of P defined by the natural metric is coarser than the polyhedral topology or Whitehead topology.

7.13. THEOREM. The topology of a polyhedron is Hausdorff.

Proof. Since a metric space is Hausdorff, the result follows from the fact that the open sets in the metric topology separating distinct points are also open in the polyhedral topology.

7.14. Exercises. a) Prove that every polyhedron is normal.

b) A polyhedron P is separable if and only if it has a countable number of simplexes.

7.15. Remark. In general, the metric topology of a polyhedron does not agree with the Whitehead topology. Let $a_0 = 0$ and $a_n = \langle \cos\frac{\pi}{n}, \sin\frac{\pi}{n} \rangle$. Let $P = \{a_0 a_n / n = 1,2,\dots\}$. We shall show that P fails to satisfy the first axiom of countability at a_0, and hence cannot be a metric space.

Suppose $\mathcal{U} = \{U_i / i = 1,2,\dots\}$ be any countable family of open neighborhoods at a_0. Then $U_n \cap a_0 a_n$ is an open set in $a_0 a_n$ and hence we can find $a_n' \in U_n \cap a_0 a_n$ with $a_n' \neq a_0$. Then $A_n = a_n' a_n$ is closed in $a_0 a_n$ and $\bigcup_n A_n$ is closed in P. The set $V = P \bigcup_n A_n$ is an open set containing a_0 and does not contain any of the U_n. However

7.16. THEOREM. The metric topology of a polyhedron agrees with the polyhedral topology if and only if the polyhedron is star-finite.

Proof. If the polyhedron is not star-finite, then a modification of the above argument will show that it is not a metric space.

Conversely, suppose that the polyhedron is star-finite and $A \subset P$ closed in the polyhedral topology. We have to show that A is closed in the metric topology also. From

$$\| p_1 - p_2 \| \geqslant | p_1(v) - p_2(v) |$$

where p_1, p_2 are any two points of P, and v is any vertex of P, it follows that the function

$$P \longrightarrow I$$

defined by

$$p \longrightarrow p(v)$$

for any vertex $v \in P$ is continuous in the metric topology. Then

$$St(v) \equiv \{ p \,/\, p(v) > 0 \}$$

is open in the metric topology, as is $St(p)$ for any $p \in P$.

Since the intersection of A with any simplex in P is closed in both topologies and $\overline{St}(p)$ is finite, it follows that $A \cap \overline{st}(p)$ is closed in both topologies. Thus, if p is an adherent point of A, in the metric topology, then p is also an adherent point of $A \cap \overline{St}(p)$. But the latter is closed in the metric topology so that $p \in A$. Hence A is closed in the metric topology.

8. CONNECTIVITY.

8.1. LEMMA. If A and B are connected and $A \cap B$ is nonempty, then $A \cup B$ is connected.

8.2. PROPERTY. Let A be a connected subset of a polyhedron P. Since any simplex $\bar{\sigma}$ is connected, $A \cup \bar{\sigma}$ is connected if $A \cap \bar{\sigma} \neq \phi$. It follows that St_A, $\overline{St}_A$, Cl_A, are all connected. Hence a connected component of P is a subpolyhedron.

8.3. THEOREM. A necessary and sufficient condition that a polyhedron P be connected is that it cannot be represented as a disjoint union of two nonempty sub-polyhedra.

8.4. Remark. If p is any point of the polyhedron P and $p' \in St_p$ then $p\,p' \subset St_p$ so that St_p is arcwise connected.

8.5. DEFINITION. Two vertices v_o, v_k are said to be connected, if there exists a sequence of vertices of P, $v_o, v_1, \ldots, v_k$ such that $v_{i-1}\,v_i$ is a 1-simplex of P, for $i = 1, 2, \ldots, k$. The 1-simplex $v_{i-1}\,v_i$ determines an arc or path $\widehat{v_{i-1}\,v_i}$ in P where

$$\widehat{v_{i-1}\,v_i}\,(t) = (1-t)\,v_{i-1} + t\,v_i, \quad 0 \leq t \leq 1.$$

The sequence of vertices give rise to a path $\widehat{v_o\,v_1}$, $\widehat{v_1\,v_2}, \ldots, \widehat{v_{k-1}\,v_k}$ joining v_o to v_k. Such a path, formed by the conjunction of 1-simplexes is called an edge path.

8.6. THEOREM. A polyhedron P is arcwise connected if and only if any two vertices of P can be connected by an edge path.

Proof. Suppose any two vertices can be connected by an edge path. Any point p_o of P lies in the open star of some vertex v_o of P. Then, $\widehat{p_o v_o}$ is the path joining p_o to v_o where

$$\widehat{p_o v_o} = \alpha p_o + (1-\alpha) v_o$$

If p_k is any other point of P, $p_k \in St\, v_k$ for some v_k and $\widehat{v_k p_k}$ is the arc from v_k to p_k. Then, a path from p_o to p_k is obtained by following the path $\widehat{p_o v_o}$, the edge path from v_o to v_k and the path $\widehat{v_k p_k}$. Thus P is arcwise connected.

Conversely, suppose P is arcwise connected. Let v be a fixed vertex of P. Let V_o be the set of vertices that can be joined to v; that is, there is an edge path from v to any vertex in V_o. Let V_1 be the set of remaining vertices. The union of all simplexes of P, all of whose vertices lie in V_i is a subpolyhedron P_i of P, $i = 0,1$. Clearly, P_o is disjoint from P_1. If there is a simplex with a vertex v_o in P_o and v_1 in P_1, then the conjunction of an edge path from v to v_o with $\widehat{v_o v_1}$ is an edge path from v to v_1, that is, $v_1 \in P_o$ which is a contradiction. Hence $P = P_o \cup P_1$.

Since an arcwise connected space is also connected, P_1 is empty, P_o is not empty, for $v \in P_o$. Thus, every vertex of P can be joined to v by an edge path. Since v is arbitrary, the result follows.

8.7. THEOREM. An n-polyhedron has dimension n.

8.8. COROLLARY. $\mathbb{R}^n$ has dimension n.

9. MAPS ON POLYHEDRA.

9.1. PROPOSITION. Let P be a polyhedron and X a topological space. Let

$$f : P \longrightarrow X$$

be a function. Then f is countinuous if and only if for each simplex $\bar{\sigma}$ of P, $f/\bar{\sigma}$ is continuous on $\bar{\sigma}$.

Proof. f is continuous if and only if for every closed set A of X, $f^{-1}(A)$ is closed in P, that is, $f^{-1}(A) \cap \bar{\sigma}$ is closed in $\bar{\sigma}$. This means $f/\bar{\sigma}$ is continuous.

9.2. PROPOSITION. Let P be a polyhedron. For each simplex $\bar{\sigma}$, suppose given a continuous function

$$f_{\bar{\sigma}} : \bar{\sigma} \longrightarrow X$$

such that whenever $\bar{\tau}$ is a face of $\bar{\sigma}$,

$$f_{\tau} = f_{\bar{\sigma}} / \bar{\tau}$$

Then, the maps $\{ f_{\bar{\sigma}} \}$ give rise to a continuous function

$$f : P \rightarrow X$$

such that $f/\bar{\sigma} = f_{\bar{\sigma}}$ for each simplex $\bar{\sigma}$.

Proof. Define

$$f : P \longrightarrow X$$

by $$f(p) = f_{\mathcal{C}\ell(p)}(p).$$

If $p \in \bar{\sigma}$, $\mathcal{C}\ell_p < \sigma$ and

$$f(p) = f_{\mathcal{C}\ell(p)}(p) = f_{\bar{\sigma}}(p)$$

or

$$f_{\bar{\sigma}} = f / \bar{\sigma} .$$

f is clearly continuous.

9,3. DEFINITION. Let P and Q be polyhedra. A function

$$f : P \longrightarrow Q$$

is <u>linear</u> if for each point $p \in P$

$$f(p) = \sum_{v \in_0 P} p(v)\ f(v).$$

That is, since the P-barycentric functions are in 1:1 correspondence with points of the polyhedron P, a function f from P to Q gives rise to a function from the P-barycentric functions β_P to the Q-barycentric functions β_Q. It is then linear, if this induced function is linear.

A linear map mapping vertices onto vertices is called <u>simplicial</u>.

9.4. PROPOSITION. A linear map

$$f : P \longrightarrow Q$$

is completely determined by its restriction to $_oP$. Moreover, a function

$$g : {}_oP \longrightarrow Q$$

has a unique linear extension

$$g : P \longrightarrow Q$$

if and only if the g-image of the set of vertices of any simplex of P is contained in some simplex of Q.

<u>Proof</u>. We shall prove the 'only if' part, asserting the rest as trivial. Let

$$f : P \longrightarrow Q$$

be linear. Then, for each point p, $f(p)$ regarded as a Q-barycentric function is given by

$$f(p) = \sum_{v \in {}_oP} p(v)\ f(v),$$

where $p(v)$ is the value of the P-barycentric function at v and $f(v)$ is the Q-barycentric function of the image of v under f.

For any $w \in {}_oQ$,

$$f(p)\ (w) = \sum_{v \in {}_oP} p(v)\ f(v)\ (w)$$

is different from zero only if for some $v \in {}_oP$,

$$p(v) \neq 0, \; f(v)\,(w) \neq 0.$$

That is,

$$v \in \mathcal{C}\ell_P(p) \quad \text{and} \quad w \in \mathcal{C}\ell_Q(f(v)).$$

Hence the support

$$s(f(p)) = \bigcup_{v \in {}_0Q_P(p)} s(f(v)).$$

But $f(p)$ is a Q-barycentric function so that $s(f(p))$ spans a simplex of Q containing $f(\mathcal{C}\ell_P(p))$. The result follows by taking p to be the barycenter (or interior point) of a given simplex.

9.5. PROPOSITION. The composition of linear (simplicial) maps is again linear (simplicial). Moreover, the identity map of a polyhedron is simplicial.

9.6. PROPOSITION. A linear map is continuous.

9.7. PROPOSITION. A simplicial map

$$f : P \longrightarrow Q$$

is 1:1 if its induced map from ${}_0P$ to ${}_0Q$ is 1:1

9.8. DEFINITION. Let P and Q be two polyhedra. A simplicial map

$$f : P \longrightarrow Q$$

is called an <u>isomorphism</u> if it is 1:1 and onto. P and Q are <u>isomorphic</u> if there exists an isomorphism between them.

9.9. PROPOSITION. a) The inverse of an isomorphism is an isomorphism.

b) Two polyhedra which are isomorphic are homeomorphic.

c) An isomorphism is an equivalence relation.

9.10. Examples. a) Consider $a_o = \langle 0,0 \rangle$, $a_n = \langle \cos \frac{\pi}{n}, \sin \frac{\pi}{n} \rangle$,

$$b_n = \langle \frac{1}{n} \cos \frac{\pi}{n}, \frac{1}{n} \sin \frac{\pi}{n} \rangle.$$

$P_1 = \{ a_o a_n / n = 1,2,\ldots \}$ and $P_2 = \{ a_o b_n / n = 1,2,\ldots \}$.

Then the correspondence $a_n \longleftrightarrow b_n$ extends an isomorphism.

b) Let $\mathcal{B}(A)$ denote the set of all barycentric functions defined on a set A. Then $\mathcal{B}(A)$ is a convex subset of $\mathcal{R}(A)$, a vector space generated by A. It is the union of all simplexes spanned by finite subsets of A (considered as a basis of $\mathcal{R}(A)$). Then $\mathcal{B}(A)$ with Whitehead topology is a polyhedron in $\mathcal{R}(A)$ For any polyhedron P, the P-barycentric functions constitute a subpolyhedron of $\mathcal{B}({}_oP)$ and the correspondence between the points of P and the P-barycentric functions is an isomorphism.

In particular, if P is finite, containing n vertices, then $\mathcal{R}({}_oP)$ can be identified with $\mathbb{R}^n$ and P is isomorphic to a subpolyhedron of the standard (n-1)-simplex $\sum^{n-1}$. The natural metric of P is just that inherited from $\sum^{n-1}$ (that is, the isomorphism is an isometry).

9.11. THEOREM. Let

$$f : P \longrightarrow Q$$

be a simplicial map. P_1 and P_2 are subpolyhedra of P. Q_1 is a subpolyhedron of Q. Then

a) $f(P_1)$ is a subpolyhedron of Q.

b) $f(P_1 \cup P_2) = f(P_1) \cup f(P_2)$

c) $P_1 \subset P_2 \Longrightarrow f(P_1) \subset f(P_2)$

d) $f^{-1}(Q_1)$ is a subpolyhedron of P.

e) If $\overline{\sigma}$ is a face of $\overline{\tau}$, then $f(\overline{\sigma})$ is a face of $f(\overline{\tau})$.

Proof. Exercise.

9.12. PROPOSITION. Let

$$f : P \longrightarrow Q$$

be a simplicial map. Q_o is a subpolyhedron of Q. $P_o = f^{-1}(Q_o)$ is a subpolyhedron of P. Then,

$$f : Sd(P, P_o) \longrightarrow Sd(Q, Q_o)$$

is also simplicial.

9.13. DEFINITION. Let $\mathcal{U}$ be a covering of a topological space Y. Suppose

$$f_o; f_1 : X \longrightarrow Y$$

are two maps of the topological space X into Y and are $\mathcal{U}$-approximate if for every $x \in X$, there is a $\mathcal{U}_x \in \mathcal{U}$ such that $f_o(x), f_1(x) \in \mathcal{U}_x$.

A $\mathcal{U}$-*homotopy* between the maps

$$f_0, f_1 : X \longrightarrow Y$$

is a homotopy

$$F : X \times I \longrightarrow Y, \quad I = [0,1]$$

such that for each $x \in X$

$$F(x,0) = f_0(x)$$

$$F(x,1) = f_1(x)$$

and there is a set $U_x \in \mathcal{U}$ with

$$F(x,t) \in U_x$$

for $t \in I$. This implies of course that f_0 and f_1 are $\mathcal{U}$-approximate. In particular, when Y is a metric space and $\mathcal{U}$ is the covering by open ε-neighborhoods, then we speak of ε-approximate maps and ε-homotopies. If Y is a polyhedron and $\mathcal{U}$ is a covering by simplexes, we speak of Y-approximate maps and Y-homotopies.

9.14. DEFINITION. A simplicial map between two polyhedra P and Q

$$s : P \longrightarrow Q$$

is a *simplicial approximation* to a map

$$f : P \longrightarrow Q$$

if for every point $p \in P$,

$$s(p) \in \mathcal{U} f(p)$$

that is, s and f are *Q-approximate*.

9.15. Example. Let

$$s. : V_n(\bar{\sigma}) \longrightarrow V_o(\bar{\sigma})$$

be a standard map. Then s extends linearly to a simplicial map

$$s : Sd^m(\bar{\sigma}) \longrightarrow \bar{\sigma}$$

Then, s is a simplicial approximation to the identity map on $|\bar{\sigma}|$.

9.16. THEOREM. Let P and Q be polyhedra and P_o is a subpolyhedron of P.

$$s : P \longrightarrow \hat{Q}$$

is a simplicial approximation of

$$f : P \longrightarrow Q$$

where f/P_o is simplicial. Then, s is Q-homotopic relative to P_o and

$$s/P_o = f/P_o .$$

Proof. If $f(v)$ is a vertex in Q for some vertex of P, then

$$s(v) \in Cl\ (f(v)) = \{f(v)\} .$$

Therefore

$$s(v) = f(v).$$

In particular, s and f agrees on the vertices of P_o. Hence

$$s/P_o = f/P_o.$$

For any point $p \in P$, $s(p)$ and $f(p)$ both belong to the simplex $\mathrm{Cl}\ f(p)$. Then for any $\alpha \in I$,

$$\alpha\, s(p) + (1 - \alpha)\, f(p) \in \mathrm{Cl}\ f(p).$$

Define a homotopy

$$H : |P| \times I \longrightarrow Q$$

by setting

$$H(p,\alpha) = \alpha\, s(p) + (1 - \alpha)\, f(p).$$

Then

$$H(p,1) = s(p)$$
$$H(p,0) = f(p)$$

and

$$H(P,\alpha) \in \mathrm{Cl}\ f(p) \quad \text{for fixed } p.$$

To complete the proof, we have to show that H is continuous. But for any simplex $\bar{\sigma}$ of P, the restriction of H to $|\bar{\sigma}| \times I$ is linear and therefore continuous, for we may consider $|\bar{\sigma}| \times I$ as a subset of $\mathbb{R}^{n+2}$, where $\bar{\sigma}$ is the representative of the simplex in $\bar{\Sigma}^n$ in $\mathbb{R}^{n+1}$ where I is the unit interval in $\mathbb{R}$. The proof is complete if we can show that a set $\mathcal{U}$ is open in $|P| \times I$ if and only if $(|\bar{\sigma}| \times I) \cap \mathcal{U}$ is open, in $|\bar{\sigma}| \times I$, for each simplex $\bar{\sigma}$ in P. The latter condition defines a topology on $|P| \times I$, the socalled Whitehead topology. Thus, to prove the continuity of H, it is enough to prove the following

9.17. LEMMA. The Whitehead topology agrees with the product topology on $|P| \times I$.

Proof. Since the Whitehead topology is finer than the product topology, we must show that a set $\mathcal{U}$ open in the Whitehead topology is open in the product topology. Let $\langle p_o, \alpha_o \rangle$ be any point in $\mathcal{U}$. We must find open sets $W \subset |P|$, $V \subset I$ such that

$$\langle p_o, \alpha_o \rangle \in W \times V \subset \mathcal{U}$$

which will establish that $\mathcal{U}$ is open in the product topology. First, we notice that $(\{p_o\} \times I) \cap \mathcal{U}$ is open in $\{p_o\} \times I$. Since I is normal, there is an open set V contained in I such that $\{p_o\} \times \overline{V} \subset \mathcal{U}$. Since I is compact, $\overline{V}$ and $\{p_o\} \times \overline{V}$ are compact. Set

$$W = \{p \in P \,/\, \{p\} \times \overline{V} \subset \mathcal{U}\}.$$

Then

$$\langle p_o, \alpha_o \rangle \in W \times V \subset \mathcal{U}.$$

It is sufficient now to show that W is an open set in P, that is, W intersects each simplex of P in a set open in that simplex. In other words, for any point $p \in W$ and any simplex $\overline{\sigma}$ containing p, there is a set W_p, open in $\overline{\sigma}$ such that

$$p \in W_p \subset \overline{\sigma} \cap W.$$

For each $\alpha \in \bar{V}$,

$$\langle p, \alpha \rangle \in \mathcal{U} \cap (|\bar{\sigma}| \times I).$$

Then, there are open sets W_α, V_α such that

$$W_\alpha \subset |\bar{\sigma}| \ , V_\alpha \subset I$$

with

$$\langle p, \alpha \rangle \in W_\alpha \times V_\alpha \subset \mathcal{U} \cap (|\bar{\sigma}| \times I).$$

Now $\{ V_\alpha / \alpha \in \bar{V} \}$ gives a covering of $\bar{V}$. Since $\bar{V}$ is compact, there exists a finite numbers of these, say $V_{\alpha_1}, \ldots, V_{\alpha_m}$ such that

$$\bar{V} \subset V_{\alpha_1} \cup V_{\alpha_2} \cup \ldots \cup V_{\alpha_m}.$$

Set

$$W_p = \bigcap_{j=1}^{m} W_{\alpha_j}.$$

This W_p does the job.

9.18. COROLLARY. The standard map

$$s : Sd^m \bar{\sigma} \longrightarrow \bar{\sigma}$$

is homotopic to the identity map of $\bar{\sigma}$.

9.19. Remark. A map

$$f : P \longrightarrow Q$$

in general cannot have simplicial approximation.

$$s : P \longrightarrow Q.$$

For example, the identity map

$$\mathrm{id} : \overline{\sigma} \longrightarrow \mathrm{Sd}\,\overline{\sigma}$$

has no simplicial approximation. Since this map is simplicial on the vertices, a simplicial approximation must agree with the identity on the vertices and hence a simplicial approximation cannot exist.

9.20. PROPOSITION. Let

$$s : P \longrightarrow Q$$

be a simplicial map and

$$f : P \longrightarrow Q$$

a map. Then s is a simplicial approximation to f if and only if

$$f\,(\mathrm{St}\,(v)\,) \subset \mathrm{St}(\,s(v)\,)$$

for each vertex v of P.

Proof. Suppose s is a simplicial approximation to f. Then, for each point $p \in P$

$$s(p) \in C\ell\ (f(p)).$$

Since s is simplicial, $s\,(\,C\ell\,(p)\,)$ must be a face of $C\ell\ f(p)$. Hence, if $p \in \mathrm{St}(v)$ for some vertex v, then $s(v)$ is a vertex in $C\ell\ f(p)$. Hence

$$f(p) \in \mathrm{St}\,(\,s(v)\,).$$

Conversely, suppose

$$f(\,\mathrm{St}(v)\,) \subset \mathrm{St}\,(\,s(v)\,)$$

for each vertex. If $p \in St(v)$, then

$$f(p) \in St(s(v))$$

and $s(v)$ is a vertex of $Cl\ f(p)$. Hence, it follows that $s(Cl\ (p))$ is a face of $Cl\ f(p)$. In particular,

$$s(p) \in Cl\ f(p).$$

9.21. COROLLARY. If

$$s : P \longrightarrow Q$$

$$t : Q \longrightarrow R$$

are simplicial approximations to

$$f : P \longrightarrow Q$$

and

$$g : Q \longrightarrow R$$

respectively, then

$$t_o : P \longrightarrow R$$

is a simplicial approximation to gf.

<u>Proof.</u> $(gf)\ (St(v)) = g(f(St(v))$

$$\subset g(St(s(v)) \subset St(t(s(v)) = St\ ((ts)\ (v)).$$

9.22. COROLLARY. A function

$$s : {}_oP \longrightarrow {}_oQ$$

extends to a simpocial approximation to

$$f : P \longrightarrow Q$$

if and only if

$$f^{-1} (St (s(v)) \supset St(v)$$

for each vertex of P.

<u>Proof</u>. If s extends to a simplicial approximation to f, then

$$St(s(v)) \supset f(St(v)).$$

that is,

$$f^{-1} (St(s(v))) \supset St(v).$$

Conversely, assume

$$f^{-1} \big(St (s(v)) \big) \supset St(v)$$

Then for every simplex $v_0, \ldots, v_k$ of P

$$f^{-1}\Big(\bigcap_{i=0}^{k} St(s(v_i))\Big) = \bigcap_{i=0}^{k} f^{-1}\big(St(s(v_i))\big) \supset \bigcap_{i=0}^{k} St(v_i) \neq \phi$$

since $s(v_0)$, ..., $s(v_k)$ are vertices of simplexes in Q. The result is now trivial.

9.23. DEFINITION. A map

$$f : P \longrightarrow Q$$

from a polyhedron P to a polyhedron Q is a star map if for every vertex v of P, there is a vertex w of Q such that

$$f\left(St(v)\right) \subset St(w)$$

Then we have the following

9.24. PROPOSITION. Let P and Q be two polyhedra. Then a map

$$f : P \longrightarrow Q$$

has a simplicial approximation if and only if it is a star map.

9.25. Remark. The identity map

$$id : \bar{\sigma} \longrightarrow Sd\ \bar{\sigma}$$

is not a star map.

9.26. PROPOSITION. Let

$$f : P \longrightarrow Q$$

be a map, where P and Q are polyhedra. Then, there is a simplicial subdivision P' on which f is a star map.

Proof. Since f is continuous, $\mathcal{U} = \left\{f^{-1}(St(w))/w \in {}_{o}Q\right\}$ is an open covering of P. By the refinement theorem, there is a simplicial subdivision P' of P sudh that the covering by stars of vertices of P' refines $\mathcal{U}$. Hence for every vertex $v \in P'$, there is a vertex w of Q such that

$$St(v) \subset f^{-1}(St(w)) \quad \text{or} \quad f(St(v)) \subset St(w)$$

Hence f is a star map.

9.27. SIMPLICIAL APPROXIMATION THEOREM. Let

$$f : P \longrightarrow Q$$

be a map, where P and Q are polyhedra. Then there exists a simplicial subdivision P' of P and a simplicial approximation

$$f : P' \longrightarrow Q.$$

Proof. Obvious.

9.28. COROLLARY. Let $\mathcal{U}$ be a covering of the polyhedron Q. Then, there exist simplicial subdivisions P' and Q' of P and Q respectively and a simplicial approximation

$$s : P' \longrightarrow Q'$$

such that f and s are $\mathcal{U}$-approximate.

Proof. Apply the refinement theorem to obtain a simplicial subdivision Q' of Q such that the covering by stars of vertices of Q' refines $\mathcal{U}$. Now apply the simplicial approximation to

$$f : P \longrightarrow Q'$$

This says that any map f can be approximated 'arbitrarily closely' by a simplicial map.

9.29. Remark. Let

$$f: P \longrightarrow Q$$

be a map between the polyhedra P and Q and P_o is a sub-polyhedron of P on which f is simplicial. That is, the restriction of f to P_o is simplicial. One would hope to find a simplicial subdivision $(P,P_o)'$ of P relative to P_o and a simplicial approximation to f which agrees with f on P_o. Unfortunately, this not in general possible. One can easily construct a map

$$f : a_o\, a_1\, a_2 \longrightarrow a_o\, a_1\, a_2$$

whose restriction to a_1a_2 is simplicial with the property that no simplicial approximation to f on any subdivision of $a_oa_1a_2$ is the identity on a_1a_2.

9.30. Exercise. Construct such a map.

9.31. Remark. The simplicial approximation to a map

$$f : P \longrightarrow Q$$

given by the simplicial approximation theorem is certainly not unique. In the first place, the appropriate subdivision P' of P is not uniquely determined. Secondly, once the P' has been selected there is still ambiguity in the selection of the vertex maps (c.f. simplicial approximation theorem). It is clear that the homotopyclass of a simplicial approximation to f is uniquely determined, being just the homotopy class of f. But so far, we have had no combinatorial formulation of homotopy. This will be our next goal. We must first show that $|P| \times I$ can be simplicially subdivided.

9.32. DEFINITION. A standard ordering of a polyhedron P is a partial ordering of the vertices of P, which linearly orders the vertices of any simplex of P. A polyhedron P together with a standard ordering is said to be standard ordered.

9.33. Remark. Any linear ordering of the vertices of P is certainly a standard ordering.

9.34. PROPOSITION. For any polyhedron P, $\mathrm{Sd}\, P$ has a natural standard ordering.

Proof. The vertices of $\mathrm{Sd}\, P$ are in 1:1 correspondence with the simplexes of P and the facial ordering of P induces a standard ordering in $\mathrm{Sd}\, P$.

9.35. Remark. Let P be a standard ordered polyhedron. We define a polyhedron $P \times I$ in $V \times R$ where P is in the vector space V. For each vertex v of P, there correspond two vertices $v^0 = \langle v, 0 \rangle$ and $v^1 = \langle v, 1 \rangle$ of $P \times I$. To each k-simplex $v_0, \ldots, v_k$ (in the standard order), there correspond $k+1$ $(k+1)$-simplexes defined by

$$v_0^0 v_1^0 \ldots v_i^0 v_i^1 \ldots v_k^1, \qquad i = 0, 1, \ldots, k$$

As topological spaces, $|P \times I|$ and $|P| \times I$ are homeomorphic.

The vertex function

$$v \longrightarrow v^{\varepsilon}, \qquad \varepsilon = 0, 1$$

extends to a simplicial map,

$$i_{\varepsilon} \cdot P \longrightarrow P \times I$$

which is an isomorphism into, so that P is isomorphic to the sub-polyhedron $P_{\varepsilon} = i_{\varepsilon}(P)$ of $P \times I$.

9.36. <u>Remark.</u> Let P and Q be two polyhedra. Suppose

$$f_0, f_1 : P \longrightarrow Q$$

are homotopic maps under a homotopy

$$F : |P| \times I \longrightarrow Q$$

where

$$F \circ i_{\varepsilon} = f_{\varepsilon}, \qquad \varepsilon = 0, 1.$$

Then F gives rise to map

$$F : |\mathrm{Sd}\, P| \times I \longrightarrow Q,$$

where $|\mathrm{Sd}\, P| \times I$ is the polyhedron defined above, such that $|(\mathrm{Sd}\, P) \times I|$ is homeomorphic to $|\mathrm{Sd}\, P| \times I = |P| \times I$

By the simplicial approximation theorem, we can find a subdivision $((\mathrm{Sd}\ P) \times I)'$ of $(\mathrm{Sd}\ P) \times I$. Furthermore, a simplicial approximation

$$S : ((\mathrm{Sd}\ P) \times I)' \longrightarrow Q$$

can be found to F. The simplicial subdivision $((\mathrm{Sd}\ P) \times I)'$ induces a subdivision P'_ε on $\mathrm{Sd}\ P_\varepsilon$. The restriction of S to P'_ε, s_ε is a simplicial approximation to F/P'_ε, which we can regard as f_ε. Under these circumstances, we say that S is a <u>simplicial homotopy</u> of s_0 and s_1 and s_0, s_1 are simplicially homotopic. Thus we have

9.37. PROPOSITION. Homotopic maps have simplcially homotopic, simplicial approximations.

9.38. PROPOSITION. Homotopic simplicial maps are simplcially homotopic.

9.39. PROPOSITION. Any two simplicial approximations arising from an application of the simplicial approximation theorem to a given mapsare simplically homotopic.

10. CARRIERS.

10.1. Let P be a polyhedron. Let $\underline{\mathcal{L}}(P)$ denote the set of all subpolyhedra of P. If Q is also a polyhedron, then a simplicial map

$$s : P \longrightarrow Q$$

induces a function

$$\underline{\mathcal{L}}(s) : \underline{\mathcal{L}}(P) \longrightarrow \underline{\mathcal{L}}(Q)$$

preserving order and unions. Explicitly, let $P_1, P_2 \in \underline{\mathcal{L}}(P)$. If $P_1 \subset P_2$, then $\underline{\mathcal{L}}(s)(P_1) \subset \underline{\mathcal{L}}(s)(P_2)$. Moreover,

$$\underline{\mathcal{L}}(s)(P_1 \cup P_2) = \underline{\mathcal{L}}(s)(P_1) \cup \underline{\mathcal{L}}(s)(P_2)$$

More generally, if $\{P_\lambda / \lambda \in \Lambda\} \subset \underline{\mathcal{L}}(P)$, then

$$\cup \{P_\lambda / \lambda \in \Lambda\} \in \underline{\mathcal{L}}(P)$$
$$\cap \{P_\lambda / \lambda \in \Lambda\} \in \underline{\mathcal{L}}(P)$$
$$\underline{\mathcal{L}}(s)(\cup\{P_\lambda / \lambda \in \Lambda\}) = \cup\{\underline{\mathcal{L}}(s)(P_\lambda) / \lambda \in \Lambda\}$$

10.2. Examples a) The operator which assigns to each subpolyhedron P_1, $\overline{St}\ P_1$, defines a function

$$\overline{St} : \underline{\mathcal{L}}(P) \longrightarrow \underline{\mathcal{L}}(P),$$

preserving order and unions.

b) If P' is a simplicial subdivision of P, (or more generally if $|P| = |P'|$) then the operator $Cl_{P'}$ defines a function

$$Cl_{P'} : \underline{\mathcal{L}}(P) \longrightarrow \underline{\mathcal{L}}(P')$$

with the same properties.

10.3. DEFINITION. Let L be a partially ordered set with the order relation '$<$' and M is a nonempty subset of L. Then $x \in L$ is a <u>greatest lower bound</u> of M (glb) if

a) $x \leq m$ for all $m \in M$

b) if $y < m$ for all $m \in M$ then $y < x$.

It follows from b) that the glb is unique. x is a <u>least upper bound</u> of M (lub) if

a') $m < x$ for all $m \in M$

b') $m < y$ for all $m \in M$ implies $x < y$.

The lub is also unique.

10.4. DEFINITION. A <u>lattice</u> is a partially ordered set in which any two elements have a glb and lub. If every subset has a glb and lub then the lattice is said to be <u>complete</u>.

10.5. <u>Notation</u>. For any polyhedron P, $\underline{\mathcal{L}}(P)$ will denote the lattice of all subpolyhedra of P and $\mathcal{L}(P)$ the lattice of finite subpolyhedra of P. $\mathcal{L}(P)$ is a <u>sublattice</u> of $\underline{\mathcal{L}}(P)$. That is, $\mathcal{L}(P) \subset \underline{\mathcal{L}}(P)$, the partial order in $\mathcal{L}(P)$ is that induced by $\underline{\mathcal{L}}(P)$ and the glb and lub of a set in $\mathcal{L}(P)$ are the same as its glb and lub in $\underline{\mathcal{L}}(P)$.

10.6. <u>Remark.</u> $\underline{\mathcal{L}}(P)$ is complete but $\mathcal{L}(P)$ is not (unless P is finite).

10.7. DEFINITION. Let P and Q be two polyhedra. A <u>carrier</u> from P to Q is an order preserving function

$$\Gamma : \underline{\mathcal{L}}(P) \to \underline{\mathcal{L}}(Q)$$

The carrier is a) <u>homomorphic</u> if

$$\Gamma(P_1 \cup P_2) = \Gamma(P_1) \cup \Gamma(P_2)$$

b) strongly homomorphic if

$$\Gamma\left(\bigcup\{P_\lambda / \lambda \in \Lambda\}\right) = \bigcup\{\Gamma(P_\lambda) / \lambda \in \Lambda\}$$

c) homogeneous if

$$\dim \Gamma(P_1) \leq \dim P_1 \text{ for all } P_1, P_2, P_\lambda \in \mathcal{L}(P).$$

d) finite if $\Gamma(P_1)$ is finite whenever P_1 is a finite subpolyhedron of P.

10.8. DEFINITION. If Γ_1 and Γ_2 are two carriers from P to Q, then Γ_1 dominates Γ_2 if for every $P_1 \in \mathcal{L}(P)$,

$$\Gamma_1(P_1) \supset \Gamma_2(P_1)$$

10.9. PROPOSITION. A strongly homomorphic carrier is completely determined by its action on the simplexes. In fact, any order preserving function from the simplexes of a polyhedron of P to $\mathcal{L}(Q)$ can be uniquely extended to a strongly homomorphic carrier from P to Q.

Proof. Each subpolyhedron is the union of simplexes and strongly homomorphic carriers preserve arbitrary unions.

10.10. COROLLARY. A strongly homomorphic Γ from P to Q is homogeneous (or finite) if and only if for each simplex $\bar{\sigma}$ of P, $\dim \Gamma(\bar{\sigma}) \leq \dim \bar{\sigma}$ (or $\Gamma(\bar{\sigma})$ is finite).

10.11. COROLLARY. Every carrier uniquely determines a strongly homomorphic carrier.

<u>Proof</u>. The restriction of the carrier to the simplexes uniquely extends to a strongly homomorphic carrier from P to Q.

10.12. DEFINITION. The unique strongly homomorphic carrier determined as in 10.11 by a given carrier Γ is called the <u>minimal carrier</u> of Γ .

10.13. PROPOSITION. Any carrier dominates its minimal carrier. Amy carrier which agrees with Γ on the simplexes dominates the minimal carrier of Γ .

10.14. PROPOSITION. The minimal carrier of a homogeneous (finite) carrier is homogeneous (finite).

10.15. PROPOSITION. The composition (as functions) of two(finite, homogeneous, homomorphic, strongly homomorphic) carriers is again a (finite, homogeneous, homomorphic, strongly homomorphic) carrier.

10.16. DEFINITION. Let

$$f : P \longrightarrow Q$$

be a map. Then f gives rise to a carrier

$$\underline{\mathcal{L}}(f) : \underline{\mathcal{L}}(P) \longrightarrow \underline{\mathcal{L}}(Q)$$

where

$$\underline{\mathcal{L}}(f)(P_1) = \mathrm{Cl}\, f(P_1)$$

$\underline{\mathcal{L}}$ (f) is <u>called the minimal carrier</u> of f.

10.17. PROPOSITION. The minimal carrier of a map is finite and strongly homomorphic.

10.18. <u>Examples</u>. a) The function $\underline{\mathcal{L}}$ (s) induced by a simplical map s is a finite, homogeneous, strongly homomorphic carrier, It is, in fact, the minimal carrier of s.

b) Let P and P' be two simplicial subdivisions of the some space, that is, $|P| = |P'|$. Then

$$\overline{St}_{P'} : \underline{\mathcal{L}}(P) \longrightarrow \underline{\mathcal{L}}(P')$$

$$\overline{Cl}_{P'} : \underline{\mathcal{L}}(P) \longrightarrow \underline{\mathcal{L}}(P')$$

are strongly homomorphic carriers. $\overline{Cl}_{P'}$, is finite, but $\overline{St}_{P'}$ is finite if and only if P' is star-finite. In general, neither of them is homogeneous.

c) $\overline{Cl}_{P'}$ is the minimal carrier of the identity map of $|P| \longrightarrow |P'|$.

d) The identity map of $\underline{\mathcal{L}}$ (P) is a finite, homogeneous and strongly homomorphic carrier. It is the minimal carrier induced by the identity map.

10.19. DEFINITION. A map f is carried by a carrier Γ if Γ dominates the minimal carrier of f. A homotopy F is <u>carried</u> by Γ if for each $\alpha \in I$, the map F($,\alpha$) is carried by Γ . We also say, then that the maps F(,o) and F(,1) are <u>homotopic in the carrier</u>.

10.20. Remark. a) All the carriers considered so far carry a map.

b) Consider the simplex $\bar{\sigma}^n$. Define Γ by

$$\Gamma : \mathcal{L}(\bar{\sigma}^n) \longrightarrow \mathcal{L}(\dot{\sigma}^n)$$

by letting Γ be the identity in $\dot{\sigma}^n$ and $\Gamma(\bar{\sigma}^n) = \dot{\sigma}^n$. Then Γ does not carry a map, for if it does, then we can show that is contractible and hence a contradiction.

11. THE CONE.

11.1. DEFINITION. Let X be a topological space. The cone of X, denoted by X is the space obtained from $X \times I$ by identifying all the points of $X \times \{1\}$. More precisely, $\hat{X}$ is the set of all equivalence classes of the equivalance relation R, defined by

$$\langle x,\alpha \rangle \; R \; \langle y,\beta \rangle \equiv \langle x,\alpha \rangle = \langle y,\beta \rangle \quad \text{for } \alpha = \beta = 1.$$

Thus $\hat{X} = X \times I/R$. All the equivalence classes except one, consist of a single point of $X \times I$ of the form $\langle x,\alpha \rangle$, $\alpha < 1$. The exceptional one consists of all points of the form $\{\langle x,1 \rangle / x \in X\}$ and is called the vertex of the cone.

Let

$$d_X : X \times I \longrightarrow \hat{X}$$

be the natural projection. The topology of $\hat{X}$ is the <u>identification topology</u>. It is the finest topology which makes d_X continuous. This means that a set $\mathcal{U}$ is open in $\hat{X}$ if and only if $d_X^{-1}(\mathcal{U})$ is open in $X \times I$. The function

$$i_X : X \longrightarrow \hat{X}$$

where $i_X(x) = \langle x, 0 \rangle$ embeds X in $\hat{X}$. The image $i_X(X)$ is called the <u>base of the cone</u>.

11.2. PROPOSITION. Let X and Y be topological spaces and

$$f : X \times I \longrightarrow Y$$

a map such that

$$f(x,1) = f(x',1)$$

for all $x, x' \in X$. Then f can be uniquely factored through d_X, that is, there exists a unique map

$$g : \hat{X} \longrightarrow Y$$

satisfying

$$g \circ d_X = f.$$

<u>Proof.</u> Since d_X is onto, set

$$g(d_X(x,\alpha)) = f(x,\alpha).$$

Then g is a well-defined function on $\hat{X}$. Let $\mathcal{U}$ be an open set in Y. Since f is continuous, $f^{-1}(\mathcal{U})$ is open in $X \times I$. But

$$f^{-1}(\mathcal{U}) = (g \circ d_X)^{-1}(\mathcal{U}) = d_X^{-1}(g^{-1}(\mathcal{U}))$$

Since $\hat{X}$ has identification topology and d_X is continuous, $g^{-1}(\mathcal{U})$ is open in $\hat{X}$. Hence g is continuous and that completes the proof.

11.3. COROLLARY. To every map

$$f : X \longrightarrow Y$$

there corresponds a unique map

$$\hat{f} : \hat{X} \longrightarrow \hat{Y}$$

such that

$$\hat{f} \circ i_X = i_Y \circ f.$$

<u>**Proof.**</u> Let I_I denote the identity on I. Then the function $d_Y \circ (f \times I_I)$ gives a map

$$X \times I \longrightarrow \hat{Y}$$

and satisfies the hypotheses of the proposition. Hence there exists

$$\hat{f} : \hat{X} \longrightarrow \hat{Y}$$

such that

$$\hat{f} \circ d_X = d_Y \circ (f \times I_I)$$

$$\begin{tikzcd}
X \times I \arrow[rrr, "f \times I_I"] \arrow[ddr, "d_X"'] & & & Y \times I \arrow[ddl, "d_Y"] \\
& X \arrow[r] \arrow[d, "i_X"] & Y & \\
& \hat{X} \arrow[r] & \hat{Y} \arrow[u, "i_Y"'] &
\end{tikzcd}$$

Then for $x \in X$,

$$(\hat{f} \circ i_X)(x) = \hat{f}(x,o) = \hat{f}\, d_X(x,o) = (\hat{f} \circ d_X)(x,o)$$

$$= (d_Y \circ f \times I_I)\ (x,o)$$

$$= d_Y((f \times I_I)(x,o)) = d_Y(f(x),o)$$

$$= (f(x),o) = i_Y(f(x)) = (i_Y \circ f)(x).$$

Hence

$$(\hat{f} \circ i_X) = (i_Y \circ f)\ (x), \text{ for all } x \in X$$

which implies

$$f \circ i_X = i_Y \circ f.$$

11.4. COROLLARY. The map

$$f : X \longrightarrow Y$$

is null homotopic, that is, homotopic to the constant map if and only if f can be extended to $\hat{X}$. That is, there is a map

$$g : \hat{X} \longrightarrow Y$$

with

$$g \circ I_X = f.$$

11.5. COROLLARY. X is contractible if and only if there is a map

$$r : \hat{X} \longrightarrow X$$

such that

$$r \circ i_X = i_X.$$

11.6. COROLLARY. If X is a contractible space, then any map

$$f \quad Y \longrightarrow X$$

can be extended over the cone over Y. That is, there is a map

$$g : \hat{Y} \longrightarrow X$$

such that

$$g \circ i_Y = f.$$

Proofs. Exercise.

11.7. PROPOSITION. If $v \in \mathbb{R}^n$ is independent of the compact set $A \subset \mathbb{R}^n$, then A is homeomorphic to $v\,A$.

Proof. The function

$$f : A \times I \longrightarrow v\,A$$

defined by

$$f(a,\alpha) = \alpha v + (1 - \alpha)a$$

is continuous. Moreover, f is 1:1 on $A \times I \setminus A \times \{1\}$ and $f(a,1) = v$ for all $a \in A$. Thus, there exists a map

$$g : \hat{A} \longrightarrow v\,A$$

which is 1:1 on the compact set $\hat{A}$. Hence g is a homeomorphism.

11.8. COROLLARY. For any simplex $\bar{\sigma}$, the cone over $\dot{\sigma}$, $\hat{\dot{\sigma}}$ is homeomorphic to $\bar{\sigma}$. Moreover, the cone over any simplex is homeomorphic to a simplex of one higher dimension.

11.9. DEFINITION. We now define the combinatorial analogue of the cone. Let P be a polyhedron in a vector space V. We can regard V as a subspace of $V \oplus \mathbb{R}$. The point $e = \langle 0,1 \rangle$ is then independent of V and hence P. For every simplex $\bar{\sigma}$ of P, e is independent of $\bar{\sigma}$ and hence $e\,\bar{\sigma}$ is a simplex. The collection of all simplexes of the form $e\bar{\sigma}$ defines a polyhedron eP or $\hat{P}$ called the cone over P. The vertex e of $\hat{P}$ is called the vertex of the cone. P is naturally a subpolyhedron of $\hat{P}$ and the vertices of $\hat{P}$ are just e and the vertices of P.

If $e_o \in V$ is independent of P, then we can construct the polyhedron e_oP called the cone over P with vertex e_o.

11.10. PROPOSITION. Regarding P, as a topological space, $|P|$, we can form the cone $|\hat{P}|$. As a polyhedron we form $e_o P$.

11.11. PROPOSITION. $|e_o P|$ and $|\hat{P}|$ are homeomorphic if e_o is independent of P.

11.12. PROPOSITION. Let P and Q be polyhedra,

$$s : P \longrightarrow Q$$

a simplicial map. Then, there exists a unique simplicial map

$$\hat{s} : \hat{P} \longrightarrow \hat{Q}$$

such that

$$\hat{s} \circ i_P = i_Q \circ s,$$

$$\begin{tikzcd} P \arrow[r, "s"] \arrow[d, "i_P"'] & Q \\ \hat{P} \arrow[r, "\hat{s}"] & \hat{Q} \arrow[u, "i_Q"'] \end{tikzcd}$$

where i_P (i_Q) is the inclusion of P (respectively Q) in $\hat{P}$ (respectively $\hat{Q}$).

11.13. DEFINITION. A polyhedron P is said to be conic if there exists a simplicial subdivision $(\hat{P}, P)^*$ of the cone over P, relative to P, and a simplicial map

$$C : (\hat{P}, P)^* \longrightarrow P$$

such that

$$C/P = i_P.$$

11.14. Remarks. a) If the polyhedron is conic, then it is certainly contractible. The converse is also true, that is, a contractible polyhedron is conic.

b) When P is finite, it is convenient to assume that $(\hat{P}, P)^*$ is $Sd^n(\hat{P}, P)$ for some n. In this case, we say P is finitely conic.

11.15. Convention. Unless otherwise specified, we shall assume that when P is finite, 'conic' means 'finitely conic', in what follows.

11.16. DEFINITION. A polyhedron P is star-like if there is some vertex v of P such that $P = \overline{St}(v)$. v is then called the star-vertex of P.

11.17. PROPOSITION. A star-like polyhedron is (finitely) conic.

11.18. COROLLARY. A simplex is conic (since it is star-like with any of its vertices).

11.19. Exercise. If P is conic and P^* is a simplicial subdivision of P, then P^* is conic.

12. CONIC CARRIERS.

12.1. DEFINITION. A carrier

$$\Gamma : \mathcal{L}(P) \to \mathcal{L}(Q)$$

is said to be conic, if for each simplex $\bar{\sigma} \in \mathcal{L}(P)$, $\Gamma(\bar{\sigma})$ is conic.

12.2. THEOREM. Every conic carrier carries a map. Any two maps carried by a conic carrier are homotopic under a homotopy carried by the carrier.

Proof. Let P and Q be polyhedra and

$$\Gamma : \mathcal{L}(P) \to \mathcal{L}(Q)$$

is a conic carrier. We construct a map

$$f : |P| \to |Q|$$

one simplex at a time, one dimension at a time. We define f first on the vertices ${}_0P$, then, one simplex at a time, extend this map to ${}_1P$. An induction on n, the dimension of ${}_nP$ will yield the desired function.

Now, we define f as follows. For each vertex v of P, choose a vertex $f(v) \in \Gamma(v)$. Then, for each 1-simplex $v_0 v_1$ of P, $f(v_0)$ and $f(v_1)$ are vertices of $\Gamma(v_0 v_1)$. Since $\Gamma(v_0 v_1)$ is conic, it is contractible. Hence, f can be extended to a map over the cone over $\{v_0, v_1\}$, that is, f can be extended over the 1-simplex $v_0 v_1$. Hence f can be extended over ${}_1P$. The continuity is trivial.

Suppose now that f has already been defined on ${}_nP$ and let $\bar{\sigma}$ be an (n+1)-simplex. Each n-face $\bar{\sigma}_i$, $i = 0,1,\ldots,n+1$ lies in ${}_nP$. Moreover

$$f(\bar{\sigma}_i) \subset \Gamma(\sigma_i) \subset \Gamma(\sigma_i)$$

so that f defines a map

$$\dot{\sigma} \longrightarrow \Gamma(\bar{\sigma}).$$

Using the same argument as before, f can be extended to a map over the cone of $\dot{\sigma}$, that is, $\bar{\sigma}$. Hence f can be extended to every (n+1)-simplex, so that f has now been defined on ${}_{n+1}P$. As before, the continuity follows trivially.

12.3. DEFINITION. Given two maps f_o, f_1 carried by Γ we shall define a homotopy

$$F : |P| \times I \longrightarrow Q$$

such that

$$F(\ , \varepsilon) = f_\varepsilon, \quad \varepsilon = 0,1$$

and, for every simplex $\bar{\sigma}$ of P,

$$(\bar{\sigma} \times I) \subset \Gamma(\bar{\sigma}).$$

As above, we define the homotopy, one simplex at a time, one dimension at a time. For each vertex $v \in {}_oP$, $f_o(v)$ and $f_1(v)$ both lie in $\Gamma(v)$. The function

$$F : \{\langle v,0\rangle, \langle v,1\rangle\} \longrightarrow \Gamma(v)$$

defined by

$$F(v, \varepsilon) = f_\varepsilon(v), \quad \varepsilon = 0,1$$

can be extended over the cone, since $\Gamma(v)$ is conic. That is, F can be extended to

$$F : \{v\} \times I \longrightarrow \Gamma(v)$$

This yields a homotopy of $f_0/_0P$ and $f_1/_0P$ carried by Γ. Suppose now that the homotopy F of $f_0/_nP$ and $f_1/_nP$ carried by Γ has already been defined and let $\overline{\sigma}$ be any $(n+1)$-simplex. Then, as before, F is defined on $\dot{\sigma} \times I$ and

$$F(\dot{\sigma} \times I) \subset \Gamma(\sigma).$$

Extend F as follows.

$$F(p, \varepsilon) = f_{\varepsilon}(p), \quad \varepsilon = 0,1, \ p \in \overline{\sigma}.$$

Then

$$F\Big((\dot{\sigma} \times I) \cup (\overline{\sigma} \times \{0\}) \cup (\overline{\sigma} \times \{1\})\Big) \subset \Gamma(\overline{\sigma})$$

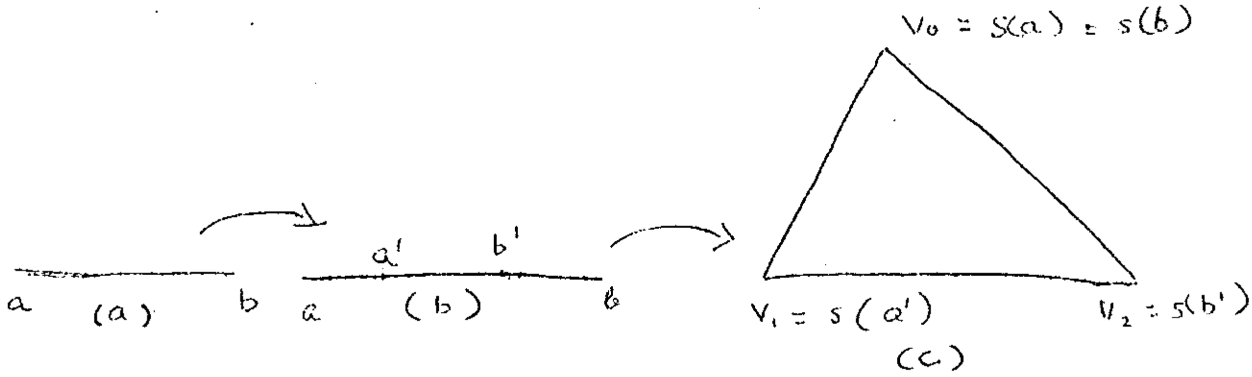
and since $\Gamma(\overline{\sigma})$ is conic, F can be extended over the cone of

$$(\dot{\sigma} \times I) \cup (\overline{\sigma} \times \{0\}) \cup (\overline{\sigma} \times \{1\})$$

which can be identified with $\overline{\sigma} \times I$. In this way the homotopy is extended over $_{n+1}P$. The continuity follows from Lemma 9.17. This completes the proof.

12.4. <u>Remark</u>. It would seem from the above theorem that a likely candidate for the role of combinatorial analogue of continuous functions is the conic carrier. However, even though the composition of conic carriers carries a continuous function, namely the composition of the maps carried by each, the composition carrier, need not be conic. This is evident from

12.5. Example



P is the 1-simplex ab, Q is the subdivision of P obtained by adding two vertices, $a' = \frac{2}{3}a + \frac{1}{3}b$ and $b' = \frac{1}{3}a + \frac{2}{3}b$ and R is the frontier of a 2-simplex (refer figure (c)). The carrier from P to Q is just the closure carrier Cl_Q. The carrier from Q to R is just the carrier $\mathcal{L}(s)$ induced by the simplicial map

$$s(a) = s(b) = V_0$$
$$s(a') = V_1$$
$$s(b') = V_2.$$

Then

$$\mathcal{L}(s) \circ Cl_Q$$

carries the map

$$|s| : |P| = |Q| \longrightarrow |R|$$

and is not conic, since

$$\mathcal{L}(s) \circ Cl_Q (ab) = R$$

Moreover, the minimal carrier of a map need not be conic, as the above example also shows, so that there is no obvious way to get a conic carrier from a map. It would seem natural to consider compositions of conic carriers. This would however lead to other difficulties, since the map carried by such a carrier might depend upon its representation as a composition of conic carriers. We use a modification of the idea of 'composition'.

12.6. DEFINITION. An *acceptable carrier* is any finite sequence of conic carriers $\langle \Gamma_1, \Gamma_2, \ldots \Gamma_k \rangle$ such that the domain of Γ_i = range of Γ_{i+1}, $i = 1, \ldots, k-1$.

A map f is *carried* by the acceptable carrier $\langle \Gamma_1, \Gamma_2, \ldots, \Gamma_k \rangle$ if f can be represented as a compsotion

$$f = f_1 \circ f_2 \circ \ldots \circ f_k,$$

such that f_i is carried by Γ_i, $i = 1, 2, \ldots, k$. Similarly, for homotopies, f^0 is homotopic to f^1 in $\langle \Gamma_1, \Gamma_2, \ldots \Gamma_k \rangle$ if $f^\varepsilon = f_1^\varepsilon \circ f_2^\varepsilon \circ \ldots \circ f_k^\varepsilon$ and f_i^0 is homotopic to $\hat{f}_i$ in Γ_i, $i = 1, 2, \ldots, k$, $\varepsilon = 0, 1$.

If $\langle \Gamma_1, \Gamma_2, \ldots, \Gamma_k \rangle$, $\langle \Gamma_{k+1}, \ldots \Gamma_n \rangle$ are two acceptable carriers and the range of Γ_{k+1} is the domain of Γ_k, then their composition is the acceptable carrier

$$\langle \Gamma_1, \ldots, \Gamma_k \rangle \langle \Gamma_{k+1}, \ldots, \Gamma_n \rangle$$
$$= \langle \Gamma_1, \ldots, \Gamma_k, \Gamma_{k+1}, \ldots, \Gamma_n \rangle .$$

The connection between acceptable carriers and maps is provided by the following.

12.7. PROPOSITION. An acceptable carrier carries a map. Any two maps carried by an acceptable carrier are homotopic in the carrier.

A converse result deals with the minimal carrier of a map. But any map

$$f : P \longrightarrow Q$$

from a polyhedron P to a polyhedron Q is a star map on some simplicial subdivision P' of P. Thus, the minimal carrier of f relative to P, $\underline{\mathcal{L}}_P(f)$ can be factored as

$$\underline{\mathcal{L}}_P(f) = \underline{\mathcal{L}}_{P'}(f) \circ Cl_{P'}$$

where $\underline{\mathcal{L}}_{P'}(f)$ is the minimal carrier of f relative to P'. That $Cl_{P'}$ is conic follows from (a). If P is conic and P^* is a simplicial subdivision of P, then P^* is conic.

(b) A simplex is conic (since it is star-like with any vertex) and

(c) The properties of the closure function.

Thus, it remains only to show that $\underline{\mathcal{L}}_{P'}(f)$ is conic (which we shall show presently) to establish that $\langle \underline{\mathcal{L}}_{P'}(f), cl_{P'} \rangle$ is an acceptable carrier representating the minimal carrier of f, namely $\underline{\mathcal{L}}_P(f)$. This establishes the following

12.8. PROPOSITION. The minimal carrier of any map can be represented as an acceptable carrier.

12.9. LEMMA. The minimal carrier of a star map is conic.

__Proof.__ Consider any star map.

$$f : P \to Q$$

between the polyhedra P and Q, and let v be any vertex of P. Then for some vertex w of Q,

$$f(\, St(v)\,) \subset St(w).$$

It follows from this that for any simplex $\bar{\sigma}$ with vertex v,

$$f(\bar{\sigma}) \subset f(\,\overline{St}(v)\,) \subset \overline{St}(w)$$

so that

$$\underline{\mathcal{L}}(f)(\bar{\sigma}) = St(\, w, \underline{\mathcal{L}}(f)(\bar{\sigma})\,).$$

But w is a vertex of $cl\, f(v)$, a simplex in $\underline{\mathcal{L}}(f)(\bar{\sigma})$. Hence

$$\underline{\mathcal{L}}(f)(\bar{\sigma}) = St(\, w, \underline{\mathcal{L}}(f)(\bar{\sigma})\,)$$

and $\underline{\mathcal{L}}(f)(\bar{\sigma})$ is star-like and thus conic.

Observe that this take care of one of the shortcomings of simplicial maps, since, to every map there corresponds an acceptable carrier. However, the representation of the minimal carrier of a map as an acceptable carrier is certainly not unique. The existence of a simplicial approximation for a star map has already been stablished. This simplicial approximation is in fact carried by the minimal carrier of the given star map. It would be of interest to isolate that property of the carrier of which this is a consequence.

12.10. DEFINITION. Let

$$\Gamma : P \longrightarrow Q$$

be a carrier, where P, Q are polyhedra. Then Γ is a <u>star-carrier</u> if for every vertex v of P, there is a vertex w of Q such that (the value of) the carrier on every simplex $\bar{\sigma}$ of P with vertex v is star-like with vertex w. That is,

$$\Gamma(\bar{\sigma}) = \overline{St}(w, \Gamma(\bar{\sigma})).$$

12.11. PROPOSITION. The minimal carrier of a star map is a star-carrier.

12.12. PROPOSITION. A star-carrier carries a simplicial map.

12.13. PROPOSITION. The composition of two star-carriers is again a star-carrier.

12.14. <u>Remark</u>. One might expect Proposition to hold under the weaker condition that the carrier of every simplex is star-like. That this is not the case is shown by the following simple example, where P is a simplex and $Q = \text{Sd } P$

a_0 (a) a_1 $\qquad$ b_0 $b_{2(b)}$ b_1

Let

$$F : P \longrightarrow Q.$$

F is just $Cl(\quad, Q)$. Then Γ carries no simplicial map.

We are now ready to define the combinatorial analogue of homotopy. That is, we define an equivalence relation among acceptable carriers and show that it has all the right properties.

12.15. DEFINITION. Let $\langle \Gamma_1, ----, \Gamma_i, \Gamma_{i+1}, \Gamma_j, ---, \Gamma_k \rangle$ be an acceptable carrier and Γ, a conic carrier which dominates $\Gamma_i \circ \Gamma_{i+1} \circ -- \circ \Gamma_j$ $(j \geq i)$. Then we say that the acceptable carriers

$$\langle \Gamma_1, ---, \Gamma_{i-1}, \Gamma, \Gamma_{j+1}, ---, \Gamma_k \rangle$$

and $\langle \Gamma_1, ---, \Gamma_i, \Gamma_{i+1}, ---, \Gamma_j, ---, \Gamma_k \rangle$

are <u>related</u>. Two acceptable carriers are <u>contiguous</u> if there is a finite sequence of acceptable carriers starting with one of them and ending with the other, such that any consecutive pairs are related

$$\langle \Gamma_1, ---, \Gamma_k \rangle \sim \langle \Gamma_1', ---, \Gamma_j' \rangle$$

This clearly defines an equivalence relation.

12.16. PROPOSITION. The maps carried by contiguous acceptable carriers belong to the same homotopy class.

Proof. It is clear that if Γ is conic and dominates $\Gamma_i \circ \cdots \circ \Gamma_j$ where $\langle \Gamma_i, \ldots, \Gamma_j \rangle$ is an acceptable carrier, then any map carried by $\langle \Gamma_i, \ldots, \Gamma_j \rangle$ is also carried by Γ. It follows therefore that maps carried by related or even contiguous carriers are homotopic.

12.17. Remark. Let

$$f : P \longrightarrow Q$$

be a map, P, Q being polyhedra and let P_1, P_2 be simplicial subdivisions of P such that f is a star map on P_1, P_2. Finally, let $\langle \underline{\mathcal{L}}_1, Cl_1 \rangle, \langle \underline{\mathcal{L}}_2, Cl_2 \rangle$ denote the corresponding acceptable carrier representations of the minimal carrier of f, where

$$Cl_i : P \longrightarrow P_i, \quad i = 1,2,$$

is the closure carrier and

$$\underline{\mathcal{L}}_i : P_i \longrightarrow Q, \quad i = 1,2$$

is the minimal carrier of f.

12.18. PROPOSITION. $\langle \underline{\mathcal{L}}_1, Cl_1 \rangle$ is contiguous to $\langle \underline{\mathcal{L}}_2, Cl_2 \rangle$.

Proof. Let P_3 be a simplicial subdivision of P_1 such that the identity map

$$1 : P_3 \longrightarrow P_2$$

is a star map, and consider the following diagram

$$\begin{tikzcd}
 & P_1 \arrow[d, "cl_{13}"] \arrow[dr, "\underline{\mathcal{L}}_1"] & \\
P \arrow[ur, "cl_1"] \arrow[r, "cl_3"] \arrow[dr, "cl_2"'] & P_3 \arrow[r, "\underline{\mathcal{L}}_2"] \arrow[d, "cl_{32}"] & Q \\
 & P_2 \arrow[ur, "\underline{\mathcal{L}}_3"'] &
\end{tikzcd}$$

where the various carriers are closure carriers, except $\underline{\mathcal{L}}_i$, which are the minimal carriers of f on P_i, $i = 1,2,3$. The following relations hold.

a) $\underline{\mathcal{L}}_1 = \underline{\mathcal{L}}_3 \circ cl_{13}$ since $\underline{\mathcal{L}}_1$ and $\underline{\mathcal{L}}_3$ are minimal carriers.

b) $cl_3 = cl_{13} \circ cl_1$

c) $\underline{\mathcal{L}}_2 \circ cl_{32}$ is conic, since both are star carriers and dominate $\underline{\mathcal{L}}_3$.

d) $cl_2 = cl_{32} \circ cl_3$

Hence,

$$\langle \underline{\mathcal{L}}_1 , cl_1 \rangle \sim \langle \underline{\mathcal{L}}_3 , cl_{13}, cl_1 \rangle$$

$$\sim \langle \underline{\mathcal{L}}_3 , cl_3 \rangle \sim \langle \underline{\mathcal{L}}_2 , cl_{32}, cl_3 \rangle \sim \langle \underline{\mathcal{L}}_2 , cl_2 \rangle$$

12.19. PROPOSITION. Let

$$f : P \longrightarrow Q$$
$$g : Q \longrightarrow R$$

be maps, where P, Q, R are polyhedra. Then the minimal carriers of f, g and $f \circ g$ can be represented as acceptable carriers. We can represent the situation by the following diagram

$$\begin{tikzcd}
P \arrow[d, "Cl_{PP_1}"'] & & \\
P_1 \arrow[r, "\underline{\mathcal{L}}_{P_1}(f)"] \arrow[dd, "Cl_{P_1P_2}"'] & Q \arrow[d, shift left, "Cl_{QQ_1}"] & R \\
& Q_1 \arrow[u, shift left, "Cl_{Q_1Q}"] \arrow[ur, "\underline{\mathcal{L}}_{Q_1}(g)"'] & \\
P_2 \arrow[ur, "\underline{\mathcal{L}}_{P_2}(f)"'] & &
\end{tikzcd}$$

Then,

$$\langle \underline{\mathcal{L}}_{Q_1}(g), Cl_{QQ_1} \rangle \langle \underline{\mathcal{L}}_{P_1}(f), Cl_{PP_1} \rangle$$

$$\sim \langle \underline{\mathcal{L}}_{P_2}(g \circ f), Cl_{PP_2} \rangle$$

<u>Proof</u>. Clearly,
$$Cl_{Q_1Q} \circ \underline{\mathcal{L}}_{P_2}(f) \circ Cl_{P_1P_2} = \underline{\mathcal{L}}_{P_1}(f)$$
since all are minimal carriers. Hence

$$\langle \underline{\mathcal{L}}_{Q_1}(g), Cl_{QQ_1} \rangle \langle \underline{\mathcal{L}}_{P_1}(f), Cl_{PP_1} \rangle$$

$$= \langle \underline{\mathcal{L}}_{Q_1}(g), Cl_{QQ_1}, \underline{\mathcal{L}}_{P_1}(f), Cl_{PP_1} \rangle$$

$$\sim \langle \underline{\mathcal{L}}_{Q_1}(g), Cl_{QQ_1}, Cl_{Q_1Q}, \underline{\mathcal{L}}_{P_2}(f), Cl_{P_1P_2}, Cl_{PP_1} \rangle$$

$$\sim \langle \underline{\mathcal{L}}_{P_2}(g \circ f), Cl_{PP_2} \rangle,$$

since

$$\underline{\mathcal{L}}_{Q_1}(g) \circ \underline{\mathcal{L}}_{P_2}(f) = \underline{\mathcal{L}}_{P_2}(g \circ f)$$

$$\mathcal{C}\mathcal{L}_{P_1 P_2} \circ \mathcal{C}\mathcal{L}_{P P_1} = \mathcal{C}\mathcal{L}_{P P_2},$$

and $\mathcal{C}\mathcal{L}_{Q Q_1} \circ \mathcal{C}\mathcal{L}_{Q_1 Q}$ is conic and dominates the identity carrier.

12.20. PROPOSITION. To every map there is assigned an acceptable carrier which is unique upto contiguity.

12.21. PROPOSITION. The acceptable carriers corresponding to homotopic maps are contiguous.

Proof. Let

$$f_o, f_1 : P \longrightarrow Q$$

be maps, where P and Q are polyhhedra. There is a simplicial subdivision, P_1 of P on which f_o is a star map. Similarly, let P_2 be a subdivision of P_1 on which f_1 is a star map. Denoting Sd P_2 by P^*, P^* is standard ordered and both f_o and f_1 are star maps on P^*. Now suppose f_o, f_1 are homotopic and

$$F : P^* \times I \longrightarrow Q$$

is a map such that

$$f_\varepsilon = F \circ i_\varepsilon$$

where

$$i_\varepsilon : P^* \longrightarrow P^* \times I, \qquad \varepsilon = 0,1$$

are the isomorphisms of P^* into the 'base' and 'top' respectively of $P^* \times I$. Finally, let $(P^* \times I)^*$ be a simplicial subdivision of $P^* \times I$ on which F is a star map. We now have the following sequence of conic carriers.

It follows that

$$\langle \underline{\mathcal{L}}(F), cl_2 \rangle \langle \underline{\mathcal{L}}(i_\varepsilon), cl_1 \rangle$$

$$= \langle \underline{\mathcal{L}}(F), cl_2, \underline{\mathcal{L}}(i_\varepsilon), cl_1 \rangle \sim \langle \underline{\mathcal{L}}(f_\varepsilon), cl_1 \rangle$$

Next, consider the carrier

$$\Gamma : P^* \longrightarrow P^* \times I$$

where for every simplex $\bar{\sigma}$ of P^*, $\Gamma(\bar{\sigma}) = \bar{\sigma} \times I$ regarded as a subpolyhedron of $P^* \times I$. Then, clearly Γ dominates both $\underline{\mathcal{L}}(i_0)$ and $\underline{\mathcal{L}}(i_1)$. To complete the proof, we need only show that Γ is conic, for then

$$\langle \underline{\mathcal{L}}(F), cl_2, \underline{\mathcal{L}}(i_0), cl_1 \rangle$$

$$\sim \langle \underline{\mathcal{L}}(F), cl_2, \Gamma', cl_1 \rangle$$

$$\sim \langle \underline{\mathcal{L}}(F), cl_2, \underline{\mathcal{L}}(i_1), cl_1 \rangle$$

But $\Gamma(\bar{\sigma})$ is star-like, since if $\bar{\sigma} \equiv v_0 \cdots v_k$, then v_0^0 is a vertex of every $(k+1)$-simplex of $\bar{\sigma} \times I$. This completes the proof.

13. ABSTRACT SIMPLICIAL COMPLEXES.

13.1. The last several sections were devoted to obtaining combinatorial conditions for topological properties of polyhedra and combinatorial analogues of topological notions. We now wish to separate out or abstract this combinatorial structure of polyhedra. Recall that a polyhedron is determined by its simplexes and the simplexes by their vertices. Hence,

13.2. DEFINITION. Let A be any set. An abstract simplicial complex K over A is a collection of non-empty finite subsets of A such that any non-empty subset of a set in K is also in K. The elements of K are called abstract simplexes. In particular, the one-element sets in K are called vertices of K. If α and β are abstract simplexes of K with $\alpha \supset \beta$, then β is said to be a face of α, or α is said to be a coface of β. The adjective 'proper', that is 'proper face' 'proper coface', is used to indicate that the inclusion is proper.

An abstract simplex with $n+1$ elements is called an abstract n-simplex. If for some integer n, there are n-simplexes in K, but no k-simplexes with $k > n$, then K is said to be an abstract simplicial n-complex. Otherwise, K is an abstract simplicial co-complex. K is finite if it contains only finitely many vertices.

It is sometimes convenient to consider the empty set ϕ as an abstract simplex of K. In this case, we say K is augmented and regard ϕ as a face of every simplex. When no confusion is likely to arise, we drop the adjectives 'abstract' and 'simplicial'. We will usually drop explicit reference to the set A also.

13.3. DEFINITION. Let P be a polyhedron. Then P determines a unique complex K(P) called the <u>vertex scheme</u> of P as follows: Taking for A the set of vertices of P,

$$K(P) \equiv \left\{ \{a_o \dots a_k\} \,/\, a_o \dots a_k \text{ is a simplex of } P \right\}.$$

It is clear that there is a 1:1 correspondence between the simplexes of K(P) and those of P with an n-simplex corresponding to an n-simplex. Moreover, the simplexes of K(P) correspond to <u>open</u> simplexes of P, in such a way that the face relation is preserved.

In view of this, any property of polyhedra given strictly in terms of the combinatorial structure can be transferred to complexes. Thus, if K is an abstract complex and α is a simplex of K, the <u>star of</u> α, St α, is the set of all co-faces of K. We can therefore speak of star-finite complexes. Observe that if σ is an open simplex in the polyhedron P, then St σ is just the union of all open simplexes of P, which have σ as a face, that is, the union of all open cofaces of σ. Similarly, we say that a complex K is connected if any two vertices are connected.

Henceforth, we shall often transfer terms and concepts dealing with the combinatorial structure of polyhedra, to complexes without explicit definition.

13.4. Examples of Complexes.

a). Let $A \equiv \{ a, b, c, d \}$. Then

$$K \equiv \{ \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}$$

and

$$K' \equiv \{ \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{b,c\}, \{a,c\}, \{b,d\}, \{a,b,c\} \}$$

are both finite complexes over A.

b) A complex over the set of integers is defined by admitting as simplexes, any finite set of integers with a common divisor > 1. For any $n > 1$, $\{ n, 2n \}$, $\{ n, 3n \}$, ... are simplexes of this complex, so it is not star-finite. Moreover, any integer k, $\{ n, 2n, 3n, \ldots, (k+1)n \}$ is a k-simplex in the complex, so it is a co-complex.

c) Another complex over the integers is defined by taking as simplexes any set of not more than $(n+1)$ integers. This is an n-complex which is not finite nor even star-finite.

13.5. <u>Remark</u>. Simplicial maps take vertices into vertices and are completely determined by their action on the vertices. Moreover, a vertex map is a simplicial map if and only if it takes a subset of vertices spanning a simplex into a subset of vertices spanning a simplex. We can therefore transfer the definition of simplicial map and isomorphism to complexes. A simplicial map on polyhedra has a natural image as a map on the vertex scheme of the polyhedra and this map is simplicial.

13.6. THEOREM. Every complex K is isomorphic to the vertex scheme of a linear polyhedron P unique upto isomorphism.

<u>Proof.</u> Let $\mathcal{B}_K$ be the set of barycentric functions on the vertices of K, whose supports are in K. Then $\mathcal{B}_K$, with the Whitehead topology, is a polyhedron. If we identify the vertex with the barycentric function which is on v , then K is precisely the vertex scheme of $\mathcal{B}_K$. Hence we may take $P = \mathcal{B}_K$. Clearly, if K is isomorphic to the vertex scheme of the polyhedron P', then this isomorphism extends to a simplicial isomorphism between P and P',

13.7. COROLLARY. If P and Q are polyhedra with isomorphic vertex schemes, then P and Q are isomorphic.

13.8. DEFINITION. A polyhedron P whose vertex scheme is isomorphic to a given complex K is called a <u>geometric realization</u> of K.

The previous theorem then not only asserts the existence of a geometric realization for any complex K, but gives an explicit construction of one also. When K is finite, the vector space $\mathbb{R}({}_0k)$ can be identified with $\mathbb{R}^n$ for some n and $\mathcal{B}_K$ is then a subpolyhedron of the representative simplex Σ^{n-1}. This is certainly not the most economical realisation as far as the dimension of the ambient space is concerned, for an (n-1)-simplex can be realised in $\mathbb{R}^{n-1}$. This raises the interesting question: What is the smallest value of k such that any finite n-complex can always be realized in $\mathbb{R}^k$? Clearly, k cannot be too small, for if $k = n$ say, then a complex containing 3n-simplexes with a common (n-1)-face cannot be realized in $\mathbb{R}^n$. To see this, let $\{v_0, \ldots, v_n\}$, $\{v_0', \ldots, v_n\}$ and $\{v_0^n \ldots v_n\}$ be the three n-simplexes, with the common face $\{v_1, \ldots v_n\}$. Then to each of the vertices v_i, assign the basis vector e_i of $\mathbb{R}^n$, $i = 1,2,\ldots,n$ to v_0 the origin and to v_0' the point $e_1 + e_2 + \ldots + e_n = e$. (If $n = 1$, see $e = 2e_1$). Since the points $e_0, e_1, \ldots, e_n, e$ and 0 span a compact convex set with an interior point, the union of the simplexes $e_0, \ldots, e_n, e$ is an n-cell in $\mathbb{R}^n$. The interior of this n-cell, the union of the two simplexes with their non-common proper faces removed, is open in $\mathbb{R}^n$ and hence in any geometric realization of the complex in $\mathbb{R}^n$. However, the complement of this sets does not contain the interior of the simplex $e_1 e_2 \dots e_n$ which is a face of an n-simplex in the complement, hence the complement cannot be closed. This contradiction shows that the complex cannot be geometrically realised in $\mathbb{R}^n$. What then is the smallest k ? The answer is provided in the following.

13.9. THEOREM. A finite n-complex can always be geometrically realised as a polyhedron in $\mathbb{R}^{2n+1}$.

Proof. Let K be an n-complex, with vertices $\{v_i \,/\, i = 0,1,\dots,k\}$. To construct a geometric realisation P of K in $\mathbb{R}^{2n+1}$, we must first select the set of vertices ${}_oP \equiv \{ p_o, \dots, p_k \}$ corresponding to the vertices of K. Then, P is the union of the simplexes spanned by the points in $\mathbb{R}^{2n+1}$ corresponding to the vertices of simplexes of K. Thus, for any simplex $v_{i_o} \dots v_{i_r}$ of K, the corresponding set $\{p_{i_o}, \dots, p_{i_r}\}$ must span a simplex and hence must be a.l.i. Since K is an arbitrary n-complex, a simplex of K may be spanned by any $(n+1)$ of the vertices of K, so that to cover all cases, any $(n+1)$ of the points of ${}_oP$ must be a.l.i. But this in itself is not sufficient for if $v_{i_o} \dots v_{i_r}, v_{j_o} \dots v_{j_s}$ are any two simplexes of K, we must be sure that the corresponding simplexes of P, $p_{i_o} \dots p_{i_r}, p_{j_o}, \dots, p_{j_s}$ intersect in a common face, (For example, any three of the set of points

$\{a_o = \langle 0,0,0 \rangle$, $a_1 = \langle 1,0,0 \rangle$, $a_2 = \langle 0,1,0 \rangle$,

$a_3 = \langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \rangle$, $a_4 = \langle \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle$ in $\mathbb{R}^3$ are a.l.i. However, the simplexes $a_o\, a_1\, a_2$ and $a_3\, a_4$ do not intersect in a common face) where $r \leq n$, $s \leq n$. For this, it is sufficient that the set $\{p_{i_o}, \ldots, p_{i_r}, p_{j_o}, \ldots, p_{j_s}\}$ be a.l.i., for then both simplexes of P will be faces of a simplex in $\mathbb{R}^{2n+1}$ and hence intersect in a common face. Again, to cover all cases, it is sufficient to select the set ${}_oP$ so that any set of $(r + s + 2)$ points is a.l.i. where $r \leq n$, $s \leq n$ so that any set of $(2n+2)$ points of ${}_oP$ is a.l.i.

A suitable set ${}_oP$ can be obtained from any set $\{b_1, \ldots, b_{2n+1}\}$ of linearly independent points by choosing

$$p_j = jb_1 + j^2 b_2 + \ldots + j^{2n+1} b_{2n+1}, \tag{1}$$

$j = 0,1,\ldots,k$. We must check that any set $\{p_{j_o}, \ldots, p_{j_{2n+1}}\}$ of points of ${}_oP$ is a.l.i. Let α be any real valued function on this set such that, writing α_i for $\alpha(p_{j_i})$

$$\alpha_o\, p_{j_o} + \ldots + \alpha_{2n+1}\, p_{j_{2n+1}} = 0 \tag{2}$$

and

$$\alpha_o + \ldots + \alpha_{2n+1} = 0 \tag{3}$$

We must show that $\alpha=0$. Substituting the representation (1) for p_{j_i} in equation (2) gives

$$\sum_{i=0}^{2n+1} \alpha_i \sum_{m=0}^{2n+1} j_i^m b_m = \sum_{m=0}^{2n+1} \left(\sum_{i=0}^{2n+1} \alpha_i \, j_i^m \right) b_m = 0 \qquad (4)$$

It follows from the linear independence of the set

$\{ b_m \,/\, m = 0,1,\ldots,2n+1 \}$ that

$$\sum_{i=0}^{2n+1} \alpha_i \, j_i^m = 0, \quad m = 0,1,\ldots,2n+1. \qquad (5)$$

The system of equations (5) together with (3) can be interpreted as a system of linear homogeneous equations in the α_i, [illegible] $i = 0,1,\ldots,2n+1$. This system will have a non-trivial solution only if the determinant

$$D = \begin{vmatrix} 1 & 1 & - & - & - & - & - & 1 \\ j_0 & j_1 & - & - & - & - & - & j_{2n+1} \\ j_0^2 & j_1^2 & - & - & - & - & - & j_{2n+1}^2 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ j_0^{2n+1} & j_1^{2n+1} & - & - & - & - & - & j_{2n+1}^{2n+1} \end{vmatrix}$$

is non-zero. But one can easily show that $D = \prod_{i_1 < i_2}^{2n+1} (j_{i_1} - j_{i_2})$ which is zero only if $j_{i_1} = j_{i_2}$. Hence, for any set of $(2n+2)$ points, the determinant is never zero, so the only solution is the trivial one and $\alpha = 0$. Thus any set of $(2n+2)$ points of ${}_0P_\lambda$ is a.l.i.

The above result is the 'best possible'. That is, for every integer n, there is a finite n-complex which cannot be realised in $\mathbb{R}^{2n}$. This is equivalent to the assertion that for every n there is an n-polyhedron which cannot be homeomorphic to a polyhedron in $\mathbb{R}^{2n}$. An example of such a polyhedron is the n-skeleton of a $(2n+2)$-simplex. This nontrivial result is due to Antonio Flores.

13.10. Let X be a topological space. To every covering $\mathcal{U} = \{U_\lambda / \lambda \in \Lambda\}$, we can associate an abstract complex $\mathcal{N}(\mathcal{U})$ as follows: The vertices of $\mathcal{N}(\mathcal{U})$ are just the sets of $\mathcal{U}$. A simplex of $\mathcal{N}(\mathcal{U})$ is a finite collection of sets of $\mathcal{U}$ with non-empty intersection. That is $\{U_{\lambda_0}, U_{\lambda_1}, \ldots, U_{\lambda_k}\}$ is a k-simplex if $\lambda_0, \ldots, \lambda_k$ are distinct and $\bigcap_{i=0}^{k} U_{\lambda_i} = \phi$. The complex $\mathcal{N}(\mathcal{U})$ is called the <u>nerve of the covering</u> $\mathcal{U}$.

$\mathcal{N}(\mathcal{U})$, more presisely, its geometric realization may be thought of as some sort of polyhedral approximation to X. If $\mathcal{U}$ is a covering and $\mathcal{N}$ is not connected, that is, $\mathfrak{N}$ can be represented as the disjoint union of non-empty sub-complexes, then X is not connected, If X has dimension n, there is an open refinement $\mathcal{V}$ of $\mathcal{U}$ of order $\leq$ n+1, its nerve $\mathfrak{N}(\mathcal{V})$, is almost an n-complex. Further, if f is the function which effects the refinement, it induces a simplicial map

$$f : \mathfrak{N}(\mathcal{V}) \longrightarrow \mathfrak{N}(\mathcal{U})$$

Let (X,d) be a compact metric space of dimension n. Let $\mathcal{U} = U_1, \ldots, U_k$ be an open covering of X of order n+1. Then, for $i = 1,2,\ldots,k$,

$$f_i'(x) = d(x, X \setminus U_i)$$

is a continuous, real valued function which vanishes outside U_i. Let

$$f = f'_1 + f'_2 + \ldots + f'_k$$

Then, f is a real valued, continuous function which never vanishes. Set

$$f_i = \frac{f_i''}{f}, \; i = 1,2,\ldots,k.$$

For each point $x \in X$,

$$\underline{x}(\mathcal{U}_i) = f_i(x)$$

is a barycentric function on $\mathcal{U}$. Let N be a geometric realization of $\mathfrak{N}(\mathcal{U})$ in $\mathbb{R}^{2n+1}$ (notice that $\mathfrak{N}(\mathcal{U})$ is an n-complex). Denoting by u_i the vertices of N corresponding to $\mathcal{U}_i$, $\underline{x}$ can be regarded on the vertices of N. It is an N-barycentric function. The correspondence

$$x \longrightarrow \underline{x}$$

defines a function

$$j : X \longrightarrow N.$$

j is continuous. $x \in St(u_i)$ if and only if $\underline{x}(u_i) = f_i(x) \neq 0$ so that $j^{-1}(St(u_i)) = \mathcal{U}_i$. Then, if $j(x_1) = j(x_2)$, $x_1 \in \mathcal{U}_i$ if and only if $x_2 \in \mathcal{U}_i$, Thus, if the mesh of the covering $\mathcal{U}$ is ε, for each point $y \in N$, the diameterof $j^{-1}(y) < \varepsilon$.

13.11. DEFINITION. A map f of a metric space X into a space Y, is called an $\underline{\varepsilon\text{-map}}$, if for each $y \in Y$,

$$\operatorname{diam} f^{-1}(y) < \varepsilon .$$

13.12. THEOREM. If X is an n-dimensional, compact, metric space, then for any $\varepsilon > 0$, there exists an ε-map of X into $\mathbb{R}^{2n+1}$.

13.13. THEOREM. An n-dimensional compact metric space can be embedded in $\mathbb{R}^{2n+1}$.

13.14. DEFINITION. Let K be an abstract complex and 'a' an object not in K. Then, the <u>cone over</u> K is the abstract complex

$$aK = K \cup \left\{ \{a\} \cup \sigma \,/\, \sigma \in K \right\} \cup \{a\} .$$

13.15. PROPOSITION. A geometric realisation of the cone over K is isomorphic to the cone over the goemetric realisation of K.

13.16. PROPOSITION. A carrier Γ from K to L, where K, L are complexes extends to a carrier from the cone aK to the cone bL.

13.17. DEFINITION. Let K be a complex. The simplexes of K are partially ordered by the relation $\sigma < \tau \equiv \sigma$ a, a proper face of τ. We define a new complex Sd K over K by taking as simplexes all finite (possibly empty) linearly ordered sets of nonempty simplexes of K. Sd K is called the <u>barycentric subdivision</u> of K. If L is a subcomplex of K, then Sd(K,L) is the abstract complex over K defined as follows. A proper sequence in K is a linearly ordered sequence:

$$\sigma_o < \sigma_1 < \dots < \sigma_r$$

with σ_o and only σ_o in L (or σ_o may be the empty set). Then

$$Sd(K,L) = L \cup \{\sigma_o \cup \{\sigma_1 \dots \sigma_k\} \ / \ \sigma_o < \sigma_1 \dots < \sigma_r \text{ is proper}\} .$$

Sd(K,L) is called the <u>barycentric subdivision of K relative to L</u>. We define inductively $Sd^n(K,L)$ to be $Sd\,(Sd^{n-1}(K,L))$ and $Sd^o(K,L) = K$.

13.18. PROPOSITION. Let P be a geometric realisation of the complex K and Q the subpolyhedron corresponding to the subcomplex L. Then $Sd(P,Q)$ is a geometric realisation of $Sd(K,L)$.

Proof. Exercise.

13.19. Remark. To simplify matters, we define conic carriers for complexes to be carriers whose geometric realisations are conic, where the geometric realisation of a carrier is the carrier induced on the geometric realisations of the complexes for which the carrier is defined. Hence we define acceptable carriers for complexes as well.

14. CHAIN COMPLEXES.

14.1. DEFINITION. A chain complex $K = \{ C_n(K), \partial_n \}$ consists of the sequence

$$\longrightarrow C_{n+1}(K) \xrightarrow{\partial_{n+1}} C_n(K) \xrightarrow{\partial_n} C_{n-1}(k) \longrightarrow \cdots$$

such that for each integer n, positive, negative of equal to zero, there corresponds an abelian group $C_n(K)$ and a homomorphism ∂_n such that $\partial_{n-1} \circ \partial_n = 0$.

∂_n is called the boundary operator and $C_n(K)$ is the group of n-chains (n-dimensional chains).

Let

$$Z_n(K) = \ker \partial_n \quad (\text{subgroup of } C_n(K))$$
$$= \text{group of n-cycles.}$$

$$B_n(K) = \operatorname{Im} \partial_{n+1}$$
$$= \text{group of n-boundaries.}$$

Then $\partial_n \circ \partial_{n+1} = 0 \implies B_n(K) \subset Z_n(K)$

$$H_n(K) = \frac{Z_n(K)}{B_n(K)}$$

= n-dimensional homology group of the chain complex K.

Let K, K' be two chain complexes given by

$$\begin{array}{ccccccccc}
\longrightarrow & C_{n+1}(K) & \xrightarrow{\partial_{n+1}} & C_n(K) & \xrightarrow{\partial_n} & C_{n-1}(K) & \longrightarrow \cdots \\
& \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & \\
\longrightarrow & C_{n+1}(K') & \xrightarrow{\partial'_{n+1}} & C_n(K') & \xrightarrow{\partial'_n} & C_{n-1}(K') & \longrightarrow \cdots
\end{array}$$

respectively. We say that

$$f : K \longrightarrow K'$$

is a map if for each n, there is a homomorphism

$$f_n : C_n(K) \longrightarrow C_n(K')$$

such that

$$\partial'_n \circ f_n = f_{n-1} \circ \partial_n .$$

$f = \{f_n\}$ is called a chain-map or just a map.

14.2. Remarks. a) $f_n(Z_n(K)) \subset Z_n(K')$ for

$$\partial'_n(f_n(Z_n(K))) = (\partial'_n \circ f_n)(Z_n(K))$$
$$= (f_{n-1} \circ \partial_n)(Z_n(K)) = f_{n-1}(\partial_n Z_n(K))$$
$$= f_{n-1}(0) = 0 .$$

Hence $f_n\,(Z_n(K)) \subset Z_n(K')$.

b) $f_n(B_n(K)) \subset B_n(K')$ for

$$f_n(B_n(K)) = (f_n \circ \partial_{n+1})\,(C_{n+1}(K))$$

$$= (\partial'_{n+1} \circ f_{n+1})(C_{n+1}(K))$$

$$\subset \operatorname{Im} \partial'_{n+1} = B_n(K').$$

Thus cycles go to cycles and boundaries to boundaries. This induces a homomorphism.

$$f_* : \frac{Z_n(K)}{B_n(K)} \longrightarrow \frac{Z_n(K')}{B_n(K')},$$

that is,

$$f_* : H_n(K) \longrightarrow H_n(K').$$

If $z \in Z_n(K)$,

$$f_*\,(z + B_n(K)) = f_n(z) + B_n(K').$$

This independent of z, for if $z' \in B_n(K)$, then

$$f_*(z + z' + B_n(K)) = f_n(z + z') + B_n(K')$$

$$= f_n(z) + f_n(z') + B_n(K') \quad (\because f_n \text{ is a homomorphism})$$

$$= f_n(z) + B_n(K'), \quad (\because f_n(z') \in B_n(K')).$$

Let $K = \{ C_n(K), \partial_n \}$, $K' = \{ C_n(K'), \partial'_n \}$ be two chain complexes.

$$\begin{tikzcd}
\arrow[r] & C_{n+1}(K) \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n(K) \arrow[r, "\partial_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, "D^n"'] & C_{n-1}(K) \arrow[r] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, "D^{n-1}"'] & \cdots \\
\arrow[r] & C_{n+1}(K') \arrow[r, "\partial'_{n+1}"] & C_n(K') \arrow[r, "\partial'_n"] & C_{n-1}(K') \arrow[r] & \cdots
\end{tikzcd}$$

Let

$$\left.\begin{array}{l} f = \{ f_n \} \\ g = \{ g_n \} \end{array}\right\} : K \longrightarrow K'.$$

Two maps $f, g : K \longrightarrow K'$ are said to be <u>chain homotopic</u> if there exists a map $D = \{ D_n \}$, a sequence of homomorphisms

$$D_n : C_n(K) \longrightarrow C_{n+1}(K')$$

such that

$$\partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n = g_n - f_n .$$

We write

$$D : f \simeq g$$

if f and g are chain homotopic; $\simeq$ is an equivalence relation.

a) $f \simeq f$, that is, $\partial'_{n+1} \circ 0 + 0 \circ \partial_n = f_n - f_n$.

Take $D = \{0\}$.

b) If $f \simeq g$, then $\partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n = g_n - f_n$.

Take $D' = \{-D_n\}$.

Then, $\partial'_{n+1} \circ (-D_n) + (-D_{n-1}) \circ \partial_n = f_n - g_n$. Hence $g \simeq f$.

c) If $D: f \simeq g$, $D': g \simeq h$, then

$$\partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n = g_n - f_n$$

and

$$\partial'_{n+1} \circ D'_n + D'_{n-1} \circ \partial_n = h_n - g_n.$$

Let $D'' = D_n + D'_n$. Then

$$\partial'_{n+1} \circ (D_n + D'_n) + (D_{n-1} + D'_{n-1}) \circ \partial_n$$

$$= \partial'_{n+1} \circ D_n + \partial'_{n+1} \circ D'_n + D_{n-1} \circ \partial_n + D'_{n-1} \circ \partial_n$$

$$= \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n + \partial'_{n+1} D'_n + D'_{n-1} \circ \partial_n$$

$$= g_n - f_n + h_n - g_n = h_n - f_n.$$

Hence

$$D'': f \simeq h.$$

14.3. PROPOSITION. If $f \simeq g$, then $f_* = g_*$.

<u>Proof.</u> Let $f_*, g_* : H_n(K) \longrightarrow H_n(K')$. Let $\alpha \in H_n(K)$. Then

$$\alpha = z + B_n(K), \; z \in Z_n(K).$$

Now

$$g_n - f_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n.$$

$$g_*(\alpha) = g_*(z + B_n(K)) = g_n(z) + B_n(K')$$

$$= f_n(z) + \partial'_{n+1} \circ D_n(z) + D_{n-1} \circ \partial_n(z) + B_n(K')$$

$$(\because z \in Z_n(K) \;, \partial_n(z) = 0)$$

$$= f_n(z) + B_n(K') = f_*(\alpha).$$

Hence,

$$f_* = g_* .$$

14.4. <u>Remarks.</u> a) If

$$f : K \longrightarrow K$$

is the identity map, then

$$f_* - H_n(K) \longrightarrow H_n(K)$$

is the identity homomorphism.

b) If

$$f : K \longrightarrow K'$$
$$g : K' \longrightarrow K'',$$

then

$$(g \circ f)_* = g_* \circ f_*.$$

If $z \in Z_n(K)$,

$$f_* \,(z + B_n(K)\,) = f_n(z) + B_n(K').$$

If $z' \in Z_n(K')$,

$$g_* \,(z' + B_n(z')\,) = g_n(z') + B_n(K'').$$

$$\begin{aligned}
(g \circ f)_* \,(z + B_n(K)\,) &= (g \circ f)_n \,(z) + B_n(K'') \\
&= (g_n \circ f_n)\,(z) + B_n(K'') \\
&= g_n(f_n(z)) + B_n(K'') \\
&= g_*(f_n(z)) + B_n(K') \\
&= g_* \,(f_* \,(z + B_n \,(K))) \\
&= (g_* \circ f_*)\,((z + B_n(K)).
\end{aligned}$$

Hence

$$(g \circ f)_* = g_* \circ f_*.$$

We have only to check that $g \circ f$ is a chain map.

$$\begin{tikzcd}
C_n(K) \arrow[r, "\partial_n"] \arrow[d, "f_n"] \arrow[dd, bend right, "g_n o f_n"'] & C_{n-1}(K) \arrow[d, "f_{n-1}"] \arrow[dd, bend left, "g_{n-1} \circ f_{n-1}"] \\
C_n(K') \arrow[r, "\partial'_n"] \arrow[d, "g_n"] & C_{n-1}(K') \arrow[d, "g_{n-1}"] \\
C_n(K'') \arrow[r, "\partial''_n"] & C_{n-1}(K''')
\end{tikzcd}$$

$$\partial''_n \circ (g_n \circ f_n) = (\partial''_n \circ g_n) \circ f_n = (g_{n-1} \circ \partial'_n) \circ f_n$$

$$= g_{n-1} \circ (\partial'_n \circ f_n)$$

$$= g_{n-1} \circ (f_{n-1} \circ \partial_n)$$

$$= (g_{n-1} \circ f_{n-1}) \circ \partial_n .$$

$\{ D_n(x) \}$ is called the <u>deformation</u> chain of x.

$$\longrightarrow C_{n+1}(K) \xrightarrow{\partial_{n+1}} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \longrightarrow$$

$$\longrightarrow C_{n+1}(K') \xrightarrow{\partial'_{n+1}} C_n(K') \xrightarrow{\partial'_n} C_{n-1}(K') \longrightarrow$$

$$\longrightarrow C_{n+1}(K'') \xrightarrow{\partial''_{n+1}} C_n(K'') \xrightarrow{\partial''_n} C_{n-1}(K'') \longrightarrow$$

$$\begin{tikzcd}
{} \arrow[r] & C_{n+1}(K) \arrow[r, "\partial_{n+1}"] \arrow[d, shift right=2, "f_{n+1}"'] \arrow[d, shift left=2, "g_{n+1}"] & C_n(K) \arrow[r, "\partial_n"] \arrow[dl, "D^n"'] \arrow[d, shift right=2, "f_n"'] \arrow[d, shift left=2, "g_n"] & C_{n-1}(K) \arrow[r] \arrow[dl, "D^{n-1}"'] \arrow[d, shift right=2, "f_{n-1}"'] \arrow[d, shift left=2, "g_{n-1}"] & {} \\
{} \arrow[r] & C_{n+1}(K') \arrow[r, "\partial'_{n+1}"] \arrow[d, shift right=2, "f'_{n+1}"'] \arrow[d, shift left=2, "g'_{n+1}"] & C_n(K') \arrow[r, "\partial'_n"] \arrow[dl, "D'^n"'] \arrow[d, shift right=2, "f'_n"'] \arrow[d, shift left=2, "g'_n"] & C_{n-1}(K') \arrow[r] \arrow[dl, "D'^{n-1}"'] \arrow[d, shift right=2, "f'_{n-1}"'] \arrow[d, shift left=2, "g'_{n-1}"] & {} \\
{} \arrow[r] & C_{n+1}(K'') \arrow[r, "\partial''_{n+1}"] & C_n(K'') \arrow[r, "\partial''_n"] & C_{n-1}(K'') \arrow[r] & {}
\end{tikzcd}$$

14.5. THEOREM. If

$f, g : K \longrightarrow K'$, $f', g' : K' \longrightarrow K''$, and $f \simeq g$, $f' \simeq g'$

then

$$(f' \circ f) \simeq (g' \circ g) : K \longrightarrow K''$$

Proof.

$$g_n - f_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$$

$$g'_n - f'_n = \partial''_{n+1} \circ D'_n + D'_{n-1} \circ \partial'_n .$$

$$\begin{aligned}
g'_n\, g_n - f'_n\, f_n &= g'_n\, g_n - g'_n\, f_n + g'_n\, f_n - f'_n\, f_n \\
&= g'_n\,(g_n - f_n) + (g'_n - f'_n)\, f_n \\
&= g'_n\,(\partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n) + \\
&\qquad + \partial''_{n+1} \circ D'_n + D'_{n-1} \circ \partial'_n)\, f_n \\
&= g'_n\, \partial'_{n+1}\, D_n + g'_n\, D_{n-1}\, \partial_n + \partial''_{n+1}\, D'_n\, f_n + \\
&\qquad + D'_{n-1}\, \partial'_n\, f_n \\
&= \partial''_{n+1}\, g'_{n+1}\, D_n + g'_n\, D_{n-1}\, \partial_n + \partial''_{n+1}\, D'_n\, f_n + \\
&\qquad + D'_{n-1}\, f_{n-1}\, \partial_n \\
&= \partial''_{n+1}\,(g'_{n+1}\, D_n + D'_n\, f_n) + (g'_n\, D_{n-1} + D'_{n-1} f_{n-1})\, \partial_n
\end{aligned}$$

Define $D'' : K \longrightarrow K''$ by

$$D_n'' : C_n(K) \longrightarrow C_{n+1}(K'')$$

$$D_n'' = g_{n+1}' D_n + D_n' f_n.$$

Then,

$$D'' : f' \circ f \simeq g' \circ g.$$

14.6. DEFINITION. A map $f : K \to K'$ is called a <u>homotopy equivalence</u> if there exists a map

$$g : K' \longrightarrow K$$

such that

$g \circ f : K \longrightarrow K$ is homotopic to the identity of K.

and

$f \circ g : K \longrightarrow K$ is homotopic to the identity of K'.

If f and g are homotopic equivalents and

$$f_* : H_n(K) \longrightarrow H_n(K')$$

$$g_* : H_n(K') \longrightarrow H_n(K)$$

then

$f \circ g \simeq$ identity gives $f_* \circ g_* = (f \circ g)_* =$ identity

$g \circ f \simeq$ identity gives $g_* \circ f_* = (g \circ f)_* =$ identity.

Hence the homology groups are isomorphic.

14.7. DEFINITION. Let $K = \{C_n(K), \partial_n\}$ be a chain complex. Then, a chain complex $L = \{C_n(L), \partial'_n\}$ is a <u>subcomplex</u> of K, if $C_n(L) \subset C_n(K)$ and $\partial_n / C_n(L) = \partial'_n$.

14.8. DEFINITION. Let $\dfrac{C_n(K)}{C_n(L)} = C_n(M)$

$$\partial_n : C_n(K) \longrightarrow C_{n-1}(K)$$

$$\partial_n \cdot \left[C_n(L) \right] \subset C_{n-1}(L).$$

We get the natural map

$$\partial_n : \frac{C_n(K)}{C_n(L)} \longrightarrow \frac{C_{n-1}(K)}{C_{n-1}(L)} = C_{n-1}(M)$$

$$C_n(L) + x \longrightarrow \partial_n x + C_{n-1}(L).$$

We get a chain complex $\{C_n(M), \partial_n\}$, which is called a <u>quotient complex</u>.

14.9. DEFINITION. The sequence

$$\longrightarrow C_{n+1}(K) \xrightarrow{\partial_{n+1}} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \longrightarrow \cdots$$

is <u>exact</u> if $\operatorname{Ker} \partial_n = \operatorname{Im} \partial_{n+1}$ for every n.

14.10. THEOREM. The sequence

$$0 \longrightarrow C_n(L) \xrightarrow[\text{inclusion}]{i} C_n(K) \xrightarrow[\text{canonical map}]{j} C_n(M) \longrightarrow 0$$

is exact.

<u>Proof.</u> ker i = 0 = Im 0. j is trivially onto.
ker j = $C_n(L)$ = Im i.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\arrow[r] & C_{n+1}(L) \arrow[r, "\partial_{n+1}"] \arrow[d, "i_{n+1}"] & C_n(L) \arrow[r, "\partial_n"] \arrow[d, "i_n"] & C_{n-1}(L) \arrow[r] \arrow[d, "i_{n-1}"] & \cdots \\
\arrow[r] & C_{n+1}(K) \arrow[r, "\partial_{n+1}"] \arrow[d, "j_{n+1}"] & C_n(K) \arrow[r, "\partial_n"] \arrow[d, "j_n"] & C_{n-1}(K) \arrow[r] \arrow[d, "j_{n-1}"] & \cdots \\
\arrow[r] & C_{n+1}(M) \arrow[r] \arrow[d] & C_n(M) \arrow[r, "\partial_n"] \arrow[d] & C_{n-1}(M) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

This diagram is commutative and $\{i_n\}$, $\{j_n\}$ are chain-maps.

a) $i_n \circ \partial_{n+1} = \partial_{n+1}\, i_{n+1}$.

Suppose $x \in C_{n+1}(L)$.

$x \longrightarrow \partial_{n+1}(x) \longrightarrow \partial_{n+1}(x) \in C_n(K)$ for $i_n \circ \partial_{n+1}$

$x \longrightarrow x \longrightarrow \partial_{n+1}(x) \in C_n(K)$ for $\partial_{n+1} \circ i_{n+1}$.

b) $j_n\, \partial_{n+1} = \partial_{n+1}\, j_{n+1}$.

Suppose $x \in C_{n+1}(K)$.

$x \longrightarrow \partial_{n+1}(x) \longrightarrow \partial_{n+1}(x) + C_n(L)$ for $j_n\, \partial_{n+1}$

$x \longrightarrow x + C_{n+1}(L) \longrightarrow \partial_{n+1}(x) + C_n(L)$ for $\partial_{n+1}\, j_{n+1}$.

Therefore, i_n and j_n are chain maps.

$$\longrightarrow H_n(L) \xrightarrow{i_*} H_n(K) \xrightarrow{j_*} H_n(M) \longrightarrow H_{n-1}(L)$$
$$\xrightarrow{i_*} H_{n-1}(K) \xrightarrow{j_*} H_{n-1}(M)$$

This is called the homology sequence for (K,L).

14.11. LEMMA. Let $x \in C_n(K)$.

a) $j_x \in Z_n(M) \Longleftrightarrow \partial_x \in i(Z_{n-1}(L))$

b) $j_x \in B_n(M) \Longleftrightarrow x \in B_{n-1}(K) \cup i(C_n(L))$.

Proof. a) $j_x \in Z_n(M) \Longleftrightarrow \partial_{j_x} = 0 \Longleftrightarrow \partial_x \in i(C_{n-1}(L))$

There exists $y \in C_{n-1}(L)$ such that $\partial x = iy$. Now ∂_{iy} $= \partial\partial_x = 0 \Rightarrow \partial_{iy} = 0$, that is, $i\partial y = 0 \Rightarrow \partial y = 0$ since i is 1:1.

Hence, $\partial_x \in i(Z_{n-1}(L))$.

b) $jx \in B_n(M) \Longleftrightarrow jx = \partial y$, $y \in C_{n+1}(M)$. Hence $y = jz$, $z \in C_{n+1}(K)$, since j is onto.

$$jx = \partial jz = j\partial z.$$

Hence

$$j(x - \partial z) = 0$$

Therefore,

$$x - \partial z \in \ker j = \operatorname{Im} i = i(C_n(L))$$

Thus,

$$x - \partial z = iu, \quad u \in C_n(L)$$

or

$$x = \partial z + iu \in B_n(K) \cup i\, C_n(L).$$

Conversely, if $x = \partial z + iu$, $z \in C_{n+1}(K)$, $u \in C_n(L)$, then

$$j(x - \partial z) = j\,i\,u = 0 \text{ and } jx = j\,\partial z = \partial j z.$$

If $y = jz$, then $y \in C_{n+1}(M)$. Hence $j\,x \in B_n(M)$.

14.12. Remark. Let

$$\longrightarrow H_n(L) \xrightarrow{i_*} H_n(K) \xrightarrow{j_*} H_n(M) \xrightarrow{\partial_*} H_{n-1}(L) \longrightarrow \cdots$$

be a chain map. Then, we define

$$\partial_* : H_n(M) \longrightarrow H_{n-1}(L).$$

Let $\alpha \in H_n(M)$. Then, $\alpha = B_n(M) + x$, $x \in Z_n(M)$. Take $y \in C_n(K)$ with $j\,y = x$. By Lemma, $\partial y \in i\,(Z_{n-1}(L))$, that is, $\partial y = iz$ where $z \in Z_{n-1}(L)$. Let $\beta = z + B_{n-1}(L)$. Set

$$\partial_* \alpha = \beta.$$

Suppose $\alpha = B_n(M) + x = B_n(M)\; x'$, $x, x' \in Z_n(M)$. Then $x - x' \in B_n(M)$. Moreover,

$$jy = x, \; jy' = x' \;, \text{ where } y, y' \in C_n(K)$$

$$\partial y = iz, \; \partial y' = iz', \text{ where } z, z' \in Z_{n-1}(L).$$

We claim that $z - z' \in B_{n-1}(L)$. For

$$j(y-y') = x-x' \in B_n(M), \; y-y' \in B_n(K) \cup i(C_n(L)),$$

or

$$y - y' = \partial u + i v \;, \; u \in C_{n+1}(K), \; v \in C_n(L).$$

Hence

$$i\,(z - z' - \partial v) = 0$$

or

$$z - z' = \partial v \in B_{n-1}(L).$$

14.13. Remark. ∂_* is a homomorphism.

Proof. If $\alpha = B_n(M) + x$, $\alpha' = B_n(M) + x'$, then

$$\partial_*(\alpha) = B_{n-1}(L) + z, \; \partial_*(\alpha') = B_{n-1}(L) + z'.$$

Then

$$\alpha + \alpha' = B_n(M) + x + x'.$$

But

$$j(y + y') = x + x', \quad (y + y') = i\,(z + z').$$

Therefore

$$(\alpha + \alpha') = B_{n-1}(L) + z + z' = (B_{n-1}(L) + z) \; \omega \; (B_{n-1}(L) + z')$$

14.14. THEOREM. The homology sequence of (K,L)

$$\longrightarrow H_n(L) \xrightarrow{i_*} H_n(K) \xrightarrow{j_*} H_n(M) \xrightarrow{\partial_*} H_{n-1}(L) \longrightarrow \cdots$$

is exact.

Proof. By definition, we must prove three propositions.

a) kernel $j_* = \mathrm{im}\; i_*$

b) kernel $\partial_* = \mathrm{im}\; j_*$

c) kernel $i_* = \mathrm{im}\; \partial_*$

a) Let $\alpha \in H_n(L)$. Then $\alpha = B_n(L) + x$ where $x \in Z_n(L)$.

$$i_*(\alpha) = i(x) + B_n(K)$$

$$j_*\, i_*(\alpha) = j\, i(x) + B_n(M) = B_n(M).$$

Hence $j_*\, i_* = 0$ and so $\mathrm{im}\; i_* \subset \ker j_*$, Conversely, if $\alpha \in H_n(K)$, with $j_*\,(\alpha) = 0$,

$$\alpha = B_n(K) + x,\ x \in Z_n(K),$$

$$j_*(\alpha) = B_n(M) + j(x).$$

But

$$j_*(\alpha) = 0 \implies j_* \in B_n(M).$$

By Lemma

$$x \in B_n(K) \cup i(C_n(L))$$

that is

$$x = \partial u + i v \quad , \; u \in C_{n+1}(K), \; v \in C_n(L).$$

$$0 = \partial x = \partial(\partial u + i v) = \partial\partial u + \partial i v = i\partial v \Rightarrow \partial v = 0$$

(since i is 1:1).

Hence

$$v \in Z_n(L)$$

$$\begin{array}{ccccc} \longrightarrow & H_n(L) & \xrightarrow{i_*} & H_n(K) & \longrightarrow \\ & v + B_n(L) & & iv + B_n(K). & \end{array}$$

We only need to show that $iv - x \in B_n(K)$. But

$$iv - x = \partial u \in B_n(K).$$

Hence, $\ker j_* \subset \operatorname{im} i_*$. Combining the two inclusions, we get

$\ker j_* = \operatorname{im} i_*$.

b) Let $\alpha \in H_n(K)$. Then, $\alpha = B_n(K) + x$, $x \in Z_n(K)$.

$$j_*(\alpha) = j(x) + B_n(M)$$

$$\partial_* j_*(\alpha) = B_{n-1}(L) + z, \; z \in Z_{n-1}(L)$$

$$iz = \partial x = 0 \quad (\text{since } x \in Z_n(K)).$$

Hence

$$\Rightarrow z = 0 .$$

$$\partial_* j_* (z) = B_{n-1}(L)$$

which gives

$$\partial_* j_* = 0.$$

So, im $j_* \subset \ker \partial_*$. Conversely, if $\alpha \in H_n(M)$, with $\partial_*(\alpha) = 0$, then

$$\alpha = x + B_n(M), \quad x \in Z_n(M).$$

Then

$$jy = x, \quad \partial y = iz, \; z \in C_{n-1}(L).$$

$$\partial_* \alpha = z + B_{n-1}(L) = 0 \implies z \in B_{n-1}(L).$$

Then

$$z = \partial u, \; u \in C_n(L)$$

$$\partial y = iz = i\,\partial u = \partial iu = y - iu = v \in \ker \partial = Z_n(K)$$

Let $\beta = v + B_n(K)$

Then

$$j_* \beta = jv + B_n(M) = jy + B_n(M) = x + B_n(M) = \alpha$$

$$(\because j\,i = 0).$$

Hence,

$$\ker \partial_* \subset \operatorname{im} j_*$$

and so

$$\ker \partial_* = \operatorname{im} j_*$$

c) Let $\alpha \in H_n(M)$. Then, $\alpha = B_n(M) + x$, $x \in Z_n(M)$

$$\partial_*(\alpha) = z + B_{n-1}(L), \; z \in Z_{n-1}(L),$$

$$iz = \partial y, \; jy = x.$$

$$i_* \partial_* \alpha = i(z) + B_{n-1}(K) = \partial y + B_{n-1}(K)$$

$$= B_{n-1}(K) \quad (\because \partial y \in B_{n-1}(K)).$$

Hence $i_* \partial_{*}\pi = 0$ and so $\operatorname{im} \partial_* \subset \ker i_*$. Conversely, if $\alpha \in H_n(L)$, with $i_* \alpha = 0$, then

$$\alpha = x + B_n(L), \; x \in Z_n(M)$$

$$ix \in B_n(K) \text{ and } ix = \partial y \text{ for some } y \in C_{n+1}(K).$$

Then

$$\partial j\, y = - j \partial y = j\, ix = 0.$$

Thus

$$jy \in Z_{n+1}(M).$$

If $z \in H_{n+1}(M)$ is the coset of jy, then it follows that $\partial_* z = y$. This gives $\ker i_* \subset \operatorname{im} \partial_*$ and thus $\ker i_* = \operatorname{im} \partial_*$.

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