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INTRODUCTION TO HILBERT SPACE

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CHAPTER I

NORMED LINEAR SPACES

1. METRIC SPACES

Throughout R will stand for the field of real numbers and R_+ the set of all nonnegative real numbers.

Definition 1. Let S be a set. A function $\rho : S \times S \to R_+$ is called a metric for S, if

(a) $\rho(x,y) = 0$ if and only if $x = y$

(b) $\rho(x,y) = \rho(y,x)$ for all $x,y \in S$

(c) $\rho(x,y) + \rho(y,z) \geq \rho(x,z)$ for all $x,y,z \in S$.

A metric space is a pair (S, ρ) where S is a set and ρ is a metric for S.

Examples:

1. For any set S, define

$$\rho(x,y) = 1 \quad \text{if } x \neq y$$
$$= 0 \quad \text{if } x = y$$

ρ is called a trivial metric

2. Let $S = R^n$, the set of all n-tuples of real numbers. If $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be any two elements of S define

$$\rho(x,y) = \left[\sum_{k=1}^{n} (x_k - y_k)^2\right]^{\frac{1}{2}}$$

3. Take S to be the set of all real valued continuous functions on $[a,b]$. A metric is defined by

$$\rho(f,g) = \max_{a \leq x \leq b} |f(x) - g(x)| \quad f,g \in S$$

4. Take S as in (3). Define now

$$\rho(f,g) = \int_a^b |f(x) - g(x)|dx$$

Notice that in (3) and (4), we have defined different metries so that the resulting metric spaces are also different.

Here after S itself will stand for the metric space (S,). There will be no confusion as the underlying metric will be understood from the context.

Definition 2. A sequence $\{x_n\}_{n=1}^{\infty}$ in S is said to converge or converges to a point x, if, given $\epsilon > 0$, there exists a positive integer $N = N(\epsilon)$ such that $n \geq N$ implies $\rho(x_n,x) < \epsilon$. We write $x_n \to x$. A sequence $\{x_n\}_{n=1}^{\infty}$ is said to be a fundamental sequence or cauchy sequence if, given $\epsilon > 0$, there exists a positive integer $N = N(\epsilon)$ such that

$$\rho(x_m,x_n) < \epsilon \text{ whenever } m,n \geq N(\epsilon)$$

i.e. $\rho(x_m,x_n) \to 0$ as $m,n \to \infty$.

Remarks

(i) A convergent sequence is cauchy

(ii) If a sequence converges, so does every subsequence

(iii) A subsequence of a cauchy sequence is cauchy

(iv) If a subsequence of a cauchy sequence converges, so does the original sequence

(v) if $x_n \to x$, $x_n \to y$ then $x = y$.

Definition 3. Let $A \subset S$ and $x \in S$. x is called a limit point of A if for each $\epsilon > 0$, there exists $y \in A$, $y \neq x$ such that $\rho(x,y) < \epsilon$. Let A' denote the set of all limit points of A. Then A' is called the derived set of A and the set $A \cup A'$ is called the closure of A, denoted by $\bar{A}$. A is closed if $A = \bar{A}$. A set B is open if $S \setminus B$ is closed.

Remark: $\bar{A}$ and A' are closed.

Definition 5. Let A be a subset of a metric space S. Then the diameter of A, denoted by diameter A, is defined by

$$\text{diam. } A = \underset{x,y \in A}{\text{l.u.b}} \left\{ \rho(x,y) \right\}$$

A is said to be bounded if diameter $A < \infty$; otherwise unbounded.

Definition 6. Let $x \in S$ and $r > 0$. Then the set $S_r(x) = \{ y \in S \mid \rho(x,y) < r \}$ is called the r-spherical neighborhood of x.

<u>Definition 6</u>. A metric space S is <u>complete</u> if every cauchy sequence in S converges to a point in S.

<u>Theorem 1</u>. <u>A subset A of a metric space S is open if and only if for each point</u> $x \in A$, <u>there exists</u> $\epsilon > 0$ <u>such that</u> $S_\epsilon(x) \subset A$

<u>Corollary</u>. $S_\epsilon(x)$ <u>is an open set</u>.

<u>Theorem 2.</u> <u>The union of an arbitrary number and the intersection of a finite number of open sets is open. The union of a finite number and the intersection of an arbitrary number of closed sets is closed.</u>

<u>Theorem 3</u>. <u>A metric space S is complete if and only if, for any sequence of closed and bounded sets</u>

$\{S_n\}$ <u>such that</u> $S_n \subset S_{n-1}$ and diam. $S_n \to 0$ <u>as</u> $n \to \infty$, $\bigcap_{n=1}^{\infty} S_n \neq \phi$

<u>Definition 7</u>. Let (S_1, ρ_1) and (S_2, ρ_2) be metric spaces and $f: S_1 \to S_2$ a mapping. f is said to be **<u>continuous at</u>** $x \in S_1$, if, given $\epsilon > 0$, there exists a $\delta > 0$ such that $\rho_1(y,x) < \delta$ implies $\rho_2(f(y), f(x)) < \epsilon$. f is <u>continuous on</u> S_1, if it is continuous at each point of S_1.

<u>Theorem 4</u>. <u>Let</u> (S_1, ρ_1) <u>and</u> (S_2, ρ_2) <u>be</u> metric spaces. <u>If</u> $f: S_1 \to S_2$ <u>and</u> $x \in S_1$, <u>the following are equivalent</u>.

(a) f *is continuous at* x

(b) *if* O_2 *is an open set in* S_2 *such that* $f(x) \in O_2$, *then there exists an open set* O_1 *such that* $x \in O_1$ *and* $f(O_1) \subset (O_2)$

(c) *if* $x_n \to x$ *in* S_1, *then* $f(x_n) \to f(x)$.

Theorem 5. *Let* S_1, S_2 *be metric spaces and* f: $S_1 \to S_2$ *be a mapping. Then the following are equivalent.*

(a) f *is continuous on* S_1

(b) *if* O_2 *is an open set in* S_2, *then* $f^{-1}(O_2)$ *is open in* S_1.

(c) *if* C_2 *is closed in* S_2, *then* $f^{-1}(C2)$ *is closed in* S_1.

Definition 8. Suppose f: $S_1 \to S_2$ is a mapping. If f is one-to-one and onto, we can define a function g: $S_2 \to S_1$ by $g(y) = x$ where $f(x) = y$. Notice that $f \cdot g$ is the identity mapping on S_2 and $g \circ f$ is the identity mapping on S_1. The function g is called the *inverse* of f and is denoted by f^{-1}

Definition 9. Let S_1, S_2 be metric spaces and f: $S_1 \to S_2$ be a mapping. If f is one-to-one and onto such that f and f^{-1} are both continuous, then f is said to be a *homeomorphism* between S_1 and S_2. Two metric spaces are said to be *homeomorphic* if there exists a homeomorphism between them.

2. Topological Spaces

Definition 10. A family J of subsets of a space (or set) S is called a topology for S if and only if

(a) $\emptyset$ and S are in J

(b) if $J_1 \subset J$, then $\cup \{A \mid A \in J_1\} \in J$; that is the union of sets of any subfamily of J is a member of J.

(c) the intersection of any finite number of sets in J is a set in J.

If a topology J is given for S, then S is called a topological space and the sets of J are called open subsets of S.

Definition 11. If J_1 and J_2 are two topologies for S, then J_1 is said to be weaker than J_2 if and only if $J_1 \subset J_2$.

Definition 12. If σ is any family of subsets of S, $J(\sigma)$ is the smallest topology for S which includes σ, if $J = J(\sigma)$ then σ is called a subbasis for J. Now every set in J is either $\emptyset$ or S or the union of finite intersections of elements in σ. If every set in J is a union of sets in σ then σ is called a basis for J.

Remark. A metric space is a topological space where a basis for the topology is the set of all spherical neighborhoods.

<u>Definition.13.</u> If A is a subset of S, then the union of all open subsets of A is called the <u>interior</u> of A, denoted by Int(A). If $p \in Int(A)$, then A is called a <u>neighbourhood</u> of p. A subset of S is <u>closed</u> if its complement is open. The <u>closure</u> $\bar{A}$ of A is the intersection of all closed sets containing A.

<u>Note.</u> A is open if and only if $A = Int(A)$ and A is closed if and only if $A = \bar{A}$.

<u>Definition 14.</u> If T is a subset of a topological space S, then a topology can be induced in T by taking as open subsets of T the intersections with T of open subsets of S. This is called the <u>relative topology</u> induced in T by the topology of S.

<u>Definition 15.</u> A family $\{D_\alpha\}_{\alpha \in \Lambda}$ of subsets of S is said to be a <u>covering</u> of $A \subset S$, if $A \subset \bigcup_\alpha D_\alpha$. A is said to be <u>compact</u> if every family of open sets which covers A contains a finite sub-family which covers A.

<u>Definition 16.</u> Let S_1, S_2 be topological spaces and $f: S_1 \to S_2$. Then f is said to be continuous at $x \in S_1$, if $f^{-1}(U) = \{y \mid f(y) \in U\}$ is a neighborhood of x when-ever U is a neighborhood of f(x). If f is one-to-one and onto and if both f and f^{-1} are continuous then f is called a homemorphism.

<u>Definition 17.</u> A <u>Hausdorff space</u> is a topological space in which every two distinct points have disjoint neighborhoods.

<u>Remark</u> A metric space is a Hausdorff space

For a detailed study of topological spaces, the reader is invited to General Topology by Kelly.

3. Normed Linear Spaces

Definition 18. Let K be a field and V an abelian group. Then V is called a vector space or linear space over K if to each element $\alpha \in K$ and each element $x \in V$, there corresponds an element αx satisfying the following conditions:

$$1) \quad \alpha(x + y) = \alpha x + \alpha y, \quad (\alpha + \beta)x = \alpha x + \beta x$$

$$2) \quad \alpha(\beta x) = (\alpha\beta)x \qquad 1.x = x$$

for all $\alpha, \beta \in K$ and $x, y \in V$ where 1 denotes the identity of the field K.

Elements of K and of V are respectively called scalars and vectors. Notice that $0 \cdot x$ = zero element of V, $-(\alpha x) = (-\alpha)x$ where 0 denotes the zero of the field K and $\alpha \in K$, $x \in V$.

Definition 19. Vectors $x_1, x_2, \ldots x_n$ are said to be linearly dependent if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all zero, such that $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$ is equal to the zero vector; otherwise $x_1 \ldots x_n$ are linearly independent. A subset B of a vector space V is said to be a linearly independent set if every finite number of vectors in B are linearly independent.

Definition 20. A vector space V is said to be finite dimensional if there exists a positive integer n such that any n + k vectors in V are linearly dependent where $k \geq 1$. If there exists no such integer, V is said to be infinite dimensional

<u>Definition 21.</u> A subset Bof a vector space V is called a <u>basis</u> for V if

(1) B is a linearly independent set

(2) every element in V can be expressed as linear combination of elements in B.

<u>Definition 22.</u> The number of vectors in a basis of a vector space is called the <u>linear dimension (or in short, dimension</u> of the vector space. The dimension of a finite dimensional vector space is finite.

<u>Theorem 6.</u> <u>Let V be a vector space. Then</u>

(a) <u>if</u> $x_1, x_2 \ldots x_n$ <u>are linearly independent, the equality</u> $\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} b_k x_k$ <u>implies</u> $a_k = b_k$ <u>for</u> $k = 1, 2 \ldots n$

(b) <u>each maximal linearly independent set of vectors forms a basis</u>

(c) <u>Any two bases have the same cardinality.</u>

<u>Corollary.</u> <u>If V is a finite dimensional vector space and if</u> $e_1, e_2, \ldots e_m$ <u>and</u> $x_1, x_2, \ldots x_n$ <u>are any two bases of</u> V, <u>then</u> $m = n$

<u>Definition 23.</u> Let E be a subset of a vector space V. E is called a <u>linear subspace</u> of V if for any two scalars a, b and any two vectors x, y in E, then $ax + by \in E$.

Remark. A linear subspace E of a vector space V over K form a vector space over K and the dimension of E $\leq$ dimension of V.

Definition 24. A normed linear space X is a vector space over the field of seal or complex numbers on which is defined a nonnegative function called the norm (norm of x being denoted by $\| x \|$) such that

$$\| x \| = 0 \text{ if and only if } x = 0$$

$$\| x + y \| \leq \| x \| + \| y \|$$

$$\| \alpha x \| = | \alpha | \ \| x \|$$

for all vectors x, y and scalar α.

X becomes a metric space if we define $\rho(x,y) = \| x - y \|$ for $x,y \in X$ and is called a Banach Space if it is complete in this metric.

Examples:

1. Let $X = R^n$. If α is a real number and $(x_1,x_2, \ldots x_n)$ $(y_1,y_2, \ldots y_n) \in R^n$, set

$$(x_1,x_2, \ldots, x_n) + (y_1,y_2, \ldots, y_n) = (x_1 + y_1, \ldots y_n) \in R^n$$

$$\alpha(x_1,x_2, \ldots x_n) = (\alpha x_1, \alpha x_2, \ldots \alpha x_n)$$

With these definitions of addition and scalar multiplication X becomes a vector space over R. Setting

$$\| x \| = \left(\sum_{k=1}^{n} x_k^2 \right)^{\frac{1}{2}}$$

We obtain a Banach space.

2. Let $C[a,b]$ = set of continuous real valued functions on $[a, b]$. If $f, f_1, f_2 \in C[a,b]$, define

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$
$$(\alpha f)(x) = \alpha . f(x)$$

for $x \in [a,b]$, then $C[a,b]$ becomes a vector space. A norm is defined by

$$\| f \| = \max_{[a,b]} |f(x)| \qquad f \in C[a,b]$$

and obtain a Banach space.

3. In the same vector space $C[a,b]$, if we define the norm by

$$\| f \| = \left(\int_a^b \{ f(t) \}^2 dt \right)^{\frac{1}{2}}$$

we get a normed linear space which is not complete.

Definition 25. Let M be a subset of a normed linear space X. M is called a linear manifold if $x, y \in M$ and α, β are scalars it follows that $\alpha x + \beta y \in M$. M is a subspace of X if M is a closed linear manifold.

Example:

1. In R^n, every linear manifold is also a subspace.

2. Consider ℓ_2, the set of all sequences $\{x_1, x_2, \ldots, x_n \}$ of real numbers such that $\sum_{k=1}^{\infty} x_k^2 < \infty$. The addition, scalar multiplication and norm in ℓ_2 are defined by

$$\{x_1, x_2, \ \} + \{y_1, y_2, \ldots \} = \{x_1 + y_1, x_2 + y_2, \}$$

$$\alpha \{x_1, x_2, \ldots \} = \{\alpha x_1, \alpha x_2, \ldots \}$$

and

$$\|x\| = \left(\sum_{k=1}^{\infty} x_k^2\right)^{\frac{1}{2}}$$

where $x = \{x_1, x_2, \ldots \}$ and α real number.

Let M be the set of all sequences in which all but a finite number of terms are zero. M is clearly a linear manifold. Consider the sequences of the form

$$\{1, 0, 0 \ \ldots\ldots \}$$

$$\{1, \tfrac{1}{2}, 0, \ldots\ldots \}$$

$$\{1, \tfrac{1}{2}, \tfrac{1}{2^2}, 0, \ldots\ldots \}$$

$$\{1, \tfrac{1}{2}, \ldots \tfrac{1}{2^n}, 0, \ldots \}$$

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This is a cauchy sequence and converges to $\{1, \frac{1}{2}, \frac{1}{2^2}, \ldots \frac{1}{2^n} \cdot\}$ which does not belong to M. M is not a subspace.

[4.] Factor Space

Let V be a vector space and M a linear subspace of V. Two elements x, y are said to be equivalent, $x \sim y$, if $x - y \in M$. If $x + M$, $y + M$ are two cosets, then the above equivalence relation tells us that either the two cosets are identical or disjoint. The set of all cosets is denoted by V/M. It is made a vector space by defining addition and scalar multiplication by

$$(x + M) + (y + M) = x + y + M$$
$$\alpha(x + M) = \alpha x + M$$

<u>Theorem 7.</u> <u>Let M be a subspace of a normed linear space X. Then X/M is a normed linear space if the norm is defined by</u> $\| y \| = \text{g.l.b.} \{ \| x \| \mid x \in y \}$ <u>for</u> $y \in X/M$. <u>If</u> X <u>is complete, then</u> X/M <u>is also complete</u>.

Proof: 1) $\| y \| = 0$ if and only if there exist $x_n \in y$ such that $\| x_n \| \to 0$. Since y is closed (why?) this will occur if and only if $0 \in y$ so that $\| y \| = 0$ if and only if $y = M$. The other axioms of the norm can be easily verified.

2) Suppose X is complete. If $\{y_n\}_{n=1}^{\infty}$ is a cauchy sequence in X/M, we can suppose, by passing on to a subsequence if necessary, that

$$\| y_{n+1} - y_n \| < \frac{1}{2^n}$$

We can then choose inductively a sequence $x_n \in y_n$ such that $\| x_{n+1} - x_n \| < \frac{1}{2^n}$ for $(x_n, y_{n+1}) = \rho(y_n, y_{n+1}) < \frac{1}{2^n}$ .. Then $\{x_n\}$ is a cauchy sequence in X. Since X is complete. there exists $x_0 \in X$ such that $x_n \to x_0$. Let y_0 be the coset containing x_0. Then $y_n \to y_0$ (check). By the property of cauchy sequence, X/M is complete.

5. Bounded linear transformation

By normed linear space, we shall generally understand to be over the field of complex numbers, the real case being explicitly labelled as real normed linear spaces.

Definition 26. Let X, Y be normed linear spaces. A function $T: X \to Y$ is called a transformation. T is said to be linear if

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$$

for $x_1, x_2 \in X$ and α_1, α_2 are scalars. T is said to be bounded if there exists M > 0 such that

$$\| T(x) \| \leq M. \| x \| \quad \text{for all } x \in X.$$

Theorem 8. Let X, Y be normed linear spaces and $T: X \to Y$ a linear transformation. Then

a) if T is continuous at x_0, then T is continuous on X

b) T is continuous if and only if it is bounded.

Proof: a) By hypothesis $x_n \to x_0$ so that $T(x_n) \to T(x_0)$. Now suppose $y_0 \in X$ and let $y_n \to y_0$. Then

$$T(y_n) = T(y_n - y_0 + x_0 + y_0 - x_0)$$

$$= T(y_n - y_0 + x_0) + T(y_0) - T(x_0)$$

by the linearity of T.

Since $y_n - y_0 + x_0 \to x_0$, $T(y_n - y_0 + x_0) \to T(x_0)$ so that

$$T(y_n) \to T(x_0) + T(y_0) - T(x_0) = T(y_0)$$

b) i) If T is bounded, there exists $M > 0$ such that

$$\| T(x) \| \leq M \| x \|$$

Hence $\| T(x) - T(x_0) \| = \| T(x - x_0) \| \leq M \| x - x_0 \|$ from which follows continuity.

ii) If T is not bounded, for each n, there exists x_n such that $\| T(x_n) \| > \| x_n \| n$. Set $y_n = \frac{x_n}{n \| x_n \|}$. Then $\| y_n \| = \frac{1}{n}$ and $\| T(y_n) \| > 1$. Hence $\| y_n \| \to 0$ but

$$T(y_n) \not\to T(o) = 0.$$ Hence not continuous at 0.

Notation: $B(X,Y)$ will denote the set of all bounded linear transformations of X into Y.

If T, T_1, $T_2 \in B(X,Y)$ and α is a complex number, we define $T_1 + T_2$, αT by

$$(T_1 + T_2)\ (x) = T_1\ (x)\ T_2(x)$$

$$(\alpha T)\ (x) = \alpha .\ T(x) \qquad x \in X.$$

Then $B(x,y)$ becomes a vector space. If $T \in B(x,y)$, notice that there exists $M > 0$ such that

$$\| T(x) \| \leq M. \| x \| \text{ for all } x \in X.$$

We define a norm by any one of the following

(i) $\| T \| = \text{glb} \quad M \mid \| T(x) \| \leq M \| x \|$

(ii) $\| T \| = \underset{x \neq 0}{\text{l.u.b}} \dfrac{\| T(x) \|}{\| x \|}$

(iii) $\| T \| = \underset{\| x \| = 1}{\text{l.u.b.}} \| T(x) \|$

It is easy to verify that (i), (ii) and (iii) are equivalent.

As a consequence of this definition, it follows that

$$\| T(x) \| \leq \| T \| \ \| x \|$$

<u>Theorem 9</u>. $B(X,Y)$ <u>is complete if Y is complete</u>.

<u>Proof:</u> Let $\{T_n\}$ be a cauchy sequence in $B(X,Y)$. Then $\| T_m - T_n \| < \epsilon$ for $m, n \geq n_\circ(\epsilon)$. Then for each $x \in X$, $\| T_m(x) - T_n(x) \| < \epsilon \| x \|$, so that $\{T_n(x)\}$ is a cauchy sequence in Y. Since Y is complete, there exists $T(x) \in Y$ such that $T_n(x) \to T(x)$. Thus we define a function $T: X \to Y$ by

$$T(x) = \lim_{n \to \infty} T_n(x)$$

1) $$T(\alpha_1 x_1 + \alpha_2 x_2) = \lim_{n \to \infty} T_n(\alpha_1 x_1 + \alpha_2 x_2)$$
$$= \alpha_1 \lim_{n \to \infty} T_n(x_1) + \alpha_2 \lim_{n \to \infty} T_n(x_2)$$
$$= \alpha_1 T(x_1) + \alpha_2 T(x_2)$$

2) Since $\| T_{n+p} - T_n \| < 1$ for all $n \geq N$ and all $p \geq 1$

$$\| T_{N+p} \| < \| T_N \| + 1 \text{ for all } p \geq 1$$

and

$$\| T \| = \lim_{p \to \infty} \| T_{N+p} \| < \| T_N \| + 1$$

and hence T is bounded.

3. To show that $\| T_n - T \| \to 0$ as $n \to \infty$. Now for $n \geq n_0(\epsilon)$, we have $\| T_{n+p} - T_n \| < \epsilon$ for $p = 1, 2 \ldots$ For $\| x \| = 1$,

$$\| T(x) - T_n(x) \| = \lim_{p \to \infty} \| T_{n+p} - T_n(x) \|$$

$$\leq \lim_{p \to \infty} \| T_{n+p} - T_n \| < \epsilon.$$

Hence $\| T_n - T \| < \epsilon$ if $n \geq n_0(\epsilon)$.

Thus $B(X,Y)$ is complete.

6. Hahn-Banach Extension Theorem

If Y = field of complex numbers, then Y is complete where the norm of $y \in Y$ is defined by $\| y \| = | y |$. We write $B(X,Y) = X^*$.

Definition 27. X^* is called the conjugate space or dual space of X. An element of X^* is called a bounded linear functional.

It follows immediately from Theorem 9 that X^* is complete whether or not X is.

Definition 28. Let S be a topological space. Let $A \subset B$ be subsets of S. Then A is said to be dense in B if $B \subset \bar{A}$ S is said to be separable if S contains a countable dense subset.

Theorem 10. (Hahn-Banach). Let M be a subspace of a normed linear space X. Then every bounded linear functional on M, can be extended to the whole of X with preservation of norm, that is, if $T \in M^*$ is given, there exists $S \in X^*$ such that

$$S(x) = T(x) \text{ for all } x \in M$$

and

$$\| S \|_X = \| T \|_M$$

Lemma: Let X be a real normed linear space and $T: X \to R$ a bounded linear functional. If $z', z'', z \in X$, then

$$T(z') - \|T\| \, \|z' + z\| \leq T(z'') + \|T\| \, \|z'' + z\|$$

Proof: Now

$$T(z') - T(z'') = T(z' - z'')$$

$$\leq \|T\| \, \|z' - z''\| = \|T\| \, \|(z' + z) - (z'' + z)\|$$

$$\leq \|T\| \, \|z' + z\| + \|T\| \, \|z'' + z\|$$

so that

$$T(z') - \|T\| \, \|z' + z\| \leq T(z'') + \|T\| \, \|z'' + z\|$$

Proof of Theorem [10] First suppose that X is a real normed linear space. Let $x_0 \notin M$. Set $M_1 = M + \{x_0\}$ i.e., the subspace generated by M and x_0. We first extend T to M_1. An element of M_1 can be represented as

$$y = tx_0 + x, \; x \in M$$

where t is a real number. If a desired functional S_1 exists, we must have

$$S_1(y) = tS_1(x_0) + T(x)$$

then

$$S_1(y) = T(x) - ct$$

In order that the norm of the functional is not increased, it is necessary that

$$|T(x)-ct| \leq \| T \| \; \| x + tx_0 \|$$

Dividing by t, we get

$$| T(x/t)-c | \leq \| \tfrac{x}{t} + x_0 \| \; \| T \|$$

Setting $x/t = \gamma$, we then would have

$$|T(\gamma) - c| \leq \| T \| \; \| \gamma + x_0 \|$$

or $$T(\gamma) - \| T \| \; \| \gamma + x_0 \| \leq c \leq T(\gamma) + \| T \| \; \| \gamma + x_0 \|$$

We have therefore to find c satisfying this inequality.

Set

$$c' = \inf \left\{ T(y) + \| T \| \| y + x_0 \| \mid y \in M \right\}$$

$$c'' = \sup \left\{ T(y) - \| T \| \| y + x_0 \| \mid y \in M \right\}$$

By lemma

Choose $c'' \leq c \leq c'$ and set

$$S_1(x) = T(x) - c^+$$

Then

$$\| S_1 \| = \| T \|$$

<u>Case (ii)</u>: <u>Complex Case</u>: A complex normed linear space becomes a real normed linear space if the scalar multiplication is restricted to real numbers. Further if T is a complex linear functional, set

$$T(x) = T_1(x) + iT_2(x)$$

where T_1 and T_2 are real and imaginary parts of T and T_1, T_2 are real functionals. Further

$$i\left[T_1(x) + iT_2(x)\right] = i\,T(x) = T(ix)$$
$$= T_1(ix) + iT_2(x)$$

so that

$$T_1(ix) = T_2(x)$$

and

$$T_2(ix) = T_1(x)$$

Moreover $T(x) = T_1(x) - iT_1(ix)$

Now $\| T_1 \| \leq \| T \|$. Then T_1 can be extended by case (i) to $M + \{ x_0 \}$, such that $\| T_1 \| \leq \| T \|$. If we add ix_0, we obtain the complex space generated by M and x_0 and the real functional S_1 defined on it. We now set

$$S(x) = S_1(x) - iS_1(ix)$$

on this subspace. We have noticed that this is correct on M. S is a real linear functional on the extended space and it needs to show it is complex linear, for that we notice

$$\begin{aligned} S(ix) &= S_1(ix) - iS_1(-x) \\ &= i \left[S_1(x) - iS_1(ix) \right] \\ &= iS(x) \end{aligned}$$

Finally, if x is given, we choose θ so that $i\theta$ $e^{i\theta}S(x)$ is real and nonnegative and we have

$$\begin{aligned} |S(x)| = |S(e^{i\theta}x)| = S_1(e^{i\theta}x) &\leq \| S_1 \| \, \| e^{i\theta}x \| \leq \\ &\leq \| T \| \| x \| \end{aligned}$$

This completes the proof for the complex case.

Now if the space is separable, the proof can be completed by induction. Otherwise we appeal to Zorn's lemma.

<u>Corollary:</u> <u>Given $x_0 \in X$, $x_0 \neq o$, there exists $T \in X^*$ such that</u>

$$T(x_0) = \| x_0 \|, \ \| T \| = 1.$$

<u>Proof:</u> Given any point $x_0 \in X$, the set of elements $\{ \lambda x_0 \}$ forms a subspace (λ scalar). Then define

$$T(\lambda x_0) = \lambda \| x_0 \|$$

T is clearly a bounded linear functional on $\{ \lambda x_0 \}$

$$T(x_0) = \| x_0 \|, \ | T(\lambda x_0) | = | \lambda | \ \| x_0 \| = \| \lambda x_0 \|$$

This implies $\| T \| = 1$. T can be extended to the whole space by Hahn-Banach extension theorem.

7. Examples of Linear Functionals

1. Let $C[a,b]$ denote the set of all real valued continuous functions defined on $[a,b]$. Under the usual definition of addition and scalar multiplication for functions $C[a,b]$ is a real vector space which normed by

$$\| f \| = \max_{a \le t \le b} |f(t)| \qquad f \in C[a,b]$$

Then the integral

$$I(f) = \int_a^b f(t)dt \qquad f \in C[a,b]$$

represents a real linear functional on $C[a,b]$ whose norm is $b - a$.

2. Let $y_0 \in C[a,b]$. Now define

$$f(x) = \int_a^b x(t)y_0(t)dt \qquad x \in C[a,b]$$

Then f is a real linear functional on $C[a,b]$ and $\| f \| = \int_a^b |y_0(t)| dt$.

3. A linear functional of another type is defined on $C[a,b]$ by setting

$$\delta_{t_0} x(t) = x(t_0)$$

where δ is the Dirac δ-function. Notice that

$$\delta_{t_0} x(t) = \int_a^b x(t)\, \delta(t-t_0)\, dt.$$

8. Second Conjugate Space

Let X be a normed linear space and X* the conjugate space of X. [Since X* the conjugate space of X. Since X* itself is a normed linear space, we can speak of X** the space of bounded linear functionals on X*. X** is called the second conjugate space of X.

Definition 29. Let (S, ρ) and (S', ρ') be metric spaces. A one-to-one onto function $f: S \to S'$ is said to be an isometry if

$$\rho(x_1, x_2) = \rho'(f(x_1), f(x_2))$$

for $x_1, x_2 \in S$. Two metric spaces are said to be isometric if there exists an isometry between them.

Theorem 11. A normed linear space X is isometric to some linear manifold in X**.

Proof: We shall first show that each element in X defines a linear functional on X*. Let x be a fixed element in X. Define ψ_x by

$$\psi_x(f) = f(x) \qquad \text{for all } f \in X^*$$

Then ψ_x is a bounded linear functional on X*. Now, if α_1, α_2 are scalars and $f_1, f_2 \in X^*$, we have

$$\psi_x(\alpha_1 f_1 + \alpha_2 f_2) = (\alpha_1 f_1 + \alpha_2 f_2)(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x)$$

$$= \alpha_1 \psi_x(f_1) + \alpha_2 \psi_x(f_2)$$

and

$$|\Psi_x(f)| \leq \|f\| \cdot \|x\|$$

so that $\|\Psi_x\| \leq \|x\|$

To complete the proof of the theorem, it is enough to show now that $\|\Psi_x\| = \|x\|$. Now by the corollary to Hahn-Banach theorem, for each $x \in X$, we can find $f_0 \in X^*$ such that $|f_0(x)| = \|x\| \; \|f_0\|$. Then

$$|\Psi_x(f_0)| = \|f_0\| \, \|x\|$$

which implies $\|x\| \leq \|\Psi_x\|$.

This completes the proof.

<u>Definition 30</u> X is said to be reflexive if $X = X^{**}$

9. Weak and Strong Convergence.

Definition 31. Let X be a normed linear space. A sequence $\{x_n\}$ converges in norm to x or converges strongly to x if $\| x_n - x \| \to 0$ as $n \to \infty$. The sequence $\{x_n\}$ converges weakly to x if $f(x_n) \to f(x)$, for all $f \in X^*$

Theorem 12. Strong convergence implies weak convergence, but not conversely.

Proof: If $\| x_n - x \| \to 0$, then

$$|f(x_n) - f(x)| = |f(x_n - x)| \leq \|f\| \, \|x_n - x\| \to 0$$

for each $f \in X^*$.

Construct an example to show that the converse is false.

Definition 32. A sequence of functionals $\{f_n\}$ converges weakly to the linear functional f if

a) $\| f_n \|$ are uniformly bounded

b) $f_n(x) \to f(x)$ for each $x \in X$. This convergence is also known as weak* convergence.

CHAPTER II

HILBERT SPACE

10. Hilbert Space.

Definition 33. A Hilbert space H is a Banach space in which the norm satisfies an additional requirement viz.

$$\| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2 \| y \|^2$$

for all $x, y \in H$. An inner product (x,y) is then, defined by

$$4(x,y) = \| x + y \|^2 - \| x - y \|^2 + i \| x + iy \|^2 - i \| x - iy \|^2$$

Definition 34. A Euclidean space E is a vector space in which a function of two variables x,y denoted by (x,y), called inner product, is defined, satisfying.

(a) $(x,x) > 0$ if $x \neq 0$, $(x,x) = 0$ if $x = 0$

(b) $(y,x) = \overline{(x,y)}$

(c) $(x_1+x_2,y) = (x_1,y) + (x_2,y)$.

Define $\| x \| = (x,x)^{\frac{1}{2}}$. If E is complete in this norm, then E is called a Hilbert space.

Remark: If (b), (c), (d) are satisfied, we have a 'bilinear hermitian form'. (a) says that it is positive definite.

Theorem 13. Definitions (33) and (34) are equivalent.

We shall not attempt to prove this theorem. However, we obtain some properties necessary to prove the equivalence, which will also be used elsewhere.

Theorem 14. Let H be a Hilbert space. If $x, y \in H$ and λ is a scalar, then

(a) $(x, \lambda y) = \bar{\lambda}(x,y)$

(b) $|(x,y)| \leq \|x\| \; \|y\|$

Proof: a) $(x, \lambda y) = \overline{(\lambda y, x)} = \overline{\lambda (y,x)} = \bar{\lambda}\,\overline{(y,x)} = \bar{\lambda}(x,y)$

b) Let λ be any complex number.

Then

$$
\begin{aligned}
0 &\leq (x - \lambda y,\ x - \lambda y) \\
&= (x,x) + (x, -\lambda y) + (-\lambda y, x) + (-\lambda y, -\lambda y) \\
&= (x,x) - \bar{\lambda}(x,y) - \lambda \overline{(x,y)} + \lambda\bar{\lambda}(y,y)
\end{aligned}
$$

Assuming $y \neq 0$, set $\lambda = \dfrac{(x,y)}{(y,y)}$. Then

$$0 \leq (x,x) - \frac{\overline{(x,y)}\,(x,y)}{(y,y)}$$

which gives

$$(x,x)\,(y,y) - (x,y)\,\overline{(x,y)} \geq 0$$

or

$$|(x,y)|^2 \leq (x,x)\,(y,y) = \|x\|^2 \; \|y\|^2$$

Definition 35. Two elements x, y in a Hilbert space H are said to be orthogonal if $(x,y) = 0$ and write $x \perp y$. Two subsets $S_1, S_2 \subset H$ are said to be orthogonal, if $x_1 \perp x_2$ for $x_1 \in S_1$, $x_2 \in S_2$. If M is a subspace of H, the set of all elements of H that are orthogonal to M is denoted by $M^{\perp}$ and is called the orthogonal complement of M.

Theorem 15. $M^{\perp}$ is a subspace of H. Proof is immediate.

Theorem 16. Let M be a subspace of a Hilbert space H and $x \in H \setminus M$. Let

$$d = \text{glb}\left\{ \| x-y \| \;\middle|\; y \in M \right\}$$

There exists a unique $y_o \in M$, such that

$$d = \| x - y_o \|$$

Further $x - y_o \in M^{\perp}$

Proof. Choose $\{y_n\}$, $y_n \in M$, such that $\| y_n - x \| \to d$. If y_m, y_n are any two elements of this sequence, then $\frac{y_m + y_n}{2} \in M$ and

$$\left\| \frac{y_m + y_n}{2} - x \right\| \geq d$$

Now

$$\|y_m - y_n\|^2 = \|(y_m - x) - (y_n - x)\|^2$$

$$= 2\|y_m - x\|^2 + 2\|y_n - x\|^2 - \|y_m + y_n - 2x\|^2$$

$$= 2\|y_m - x\|^2 + 2\|y_n - x\|^2 - 4\left\|\frac{y_m + y_n}{2} - x\right\|^2$$

$$\leq 2\|y_m - x\|^2 + 2\|y_n - x\|^2 - 4d^2$$

$$\to 2d^2 + 2d^2 - 4d^2 = 0$$

as $m, n \to \infty$. Hence $\{y_n\}$ is a cauchy sequence. Since H is complete, there exists $y_0 \in H$ such that

$$\lim_{n \to \infty} y_n = y_0 \quad \text{(in the norm)}$$

and $y_0 \in M$ since M is closed. Then

$$d = \|x - y_0\|$$

To prove the uniqueness, let us suppose that there exists $z_0 \in M$ such that

$$\|x - z_0\| = d.$$

Then

$$\|y_0 - z_0\|^2 = \|(y_0 - x) - (z_0 - x)\|^2$$

$$= 2\|y_0 - x\|^2 + 2\|z_0 - x\|^2 - 4\left\|\frac{z_0 + y_0}{2} - x\right\|^2$$

$$\leq 2d^2 + 2d^2 - 4d^2 = 0$$

Hence $y_0 = z_0$

It remains to show that $x-y_0 \in M^{\perp}$, that is $(x-y_0, y) = 0$ for all $y \in M$. Let $x^1 = x-y_0$ and t a scalar. Since $y_0, y \in M$, the element $y_0 - ty \in M$.

Now
$$d^2 \leq \|x-(y_0-ty)\|^2 = (x-y_0+ty,\, x-y_0+ty)$$
$$= (x'+ty,\, x'+ty)$$
$$= (x',x') + t\overline{(x',y)} + \bar{t}(x',y) + t\bar{t}(y,y)$$

Set $t = -\dfrac{(x^1,y)}{(y,y)}$

Then

$$d^2 \leq (x',x') - \frac{(x',y)\overline{(x',y)}}{(y,y)} = d^2 - \frac{|(x',y)|^2}{(y,y)}$$

$$\therefore \quad -\frac{|(x',y)|^2}{(y,y)} \geq 0$$

That is, $|(x^1,y)| = 0$ or $(x^1,y) = 0$.

<u>Corollary 1:</u> <u>If M is a subspace of a Hilbert space H, then every element $x \in H$ can be uniquely represented as</u>

$$x = x_1 + x_2$$

<u>where</u> $x_1 \in M$, $x_2 \in M^{\perp}$

Corollary 2. If M is a subspace, then $M^{\perp\perp} = M$

11. Riesz Representation Theorem

Theorem 17. If y is a fixed element in a Hilbert space H, and T_y is defined by

$$T_y(x) = (x,y) \text{ for all } x \in H \text{ then } T_y \in H^*$$

Proof: Let α_1, α_2 be scalars and $x_1, x_2 \in H$.

Then

$$T_y(\alpha_1 x_1 + \alpha_2 x_2) = (\alpha_1 x_1 + \alpha_2 x_2, y)$$

$$= \alpha_1 (x_1, y) + \alpha_2 (x_2, y)$$

$$= \alpha_1 T_y(x_1) + \alpha_2 T_y(x_2)$$

Further

$$|T_y(x)| = |(x,y)| \leq \| x \| . \| y \|$$

so that

$$\| T_y \| \leq \| y \|$$

Theorem 18. Every bounded linear functional T on a Hilbert space H can be expressed uniquely in the form

$$T(x) = (x,y)$$

where y is fixed and $\| T \| = \| y \|$

<u>Proof:</u> Given a bounded linear functional T, let

$$M = \{ x \in H \mid T(x) = 0 \}$$

If M = H, take y = o. Suppose $M \neq H$. Then $M^{\perp} \neq 0$. There exists an element $z \neq 0$, $z \in M^{\perp}$, $z \notin M$. We notice that $T(z) \neq 0$. If x is any element in H, let

$$u = x - \frac{T(x)}{T(z)} z$$

Then T(u) = 0 so that $u \in M$ and $z \in M^{\perp}$. That is, $(u, z) = 0$ which gives

$$\left(x - \frac{T(x)}{T(z)} z \, , \, z \right) = 0$$

or

$$(x, z) - \frac{T(x)}{T(z)} (z, z) = 0$$

or

$$T(x) = \frac{T(z)}{(z, z)} (x, z) = \left(x, \frac{\overline{T(z)}}{(z, z)} z \right)$$

Take $y = \dfrac{\overline{T(z)}}{(z, z)} z$.

Then

$T(x) = (x, y)$ for all $x \in H$.

Now by Theorem 17, $\| T \| \leq \| y \|$

Also, from $T(y) = (y,y) = \| y \|^2$, if follows that $\| T \| \geq \| y \|$

Hence $\| T \| = \| y \|$

This completes the proof.

12. Orthonormal Sets

Definition 36. Let S be a subset of a Hilbert space H. Then the subspace generated by S is the smallest subspace of H containing S.

Definition 37. A set of elements $\{ e_\alpha \}_{\alpha \in \Lambda}$ in H is said to be orthonormal if

$$(e_\alpha, e_\beta) = 1 \quad \text{if } \alpha \neq \beta$$

and

$$\| e_\alpha \| = 1$$

If the subspace generated by the e_α's is the whole space H, then the orthonormal set $\{ e_\alpha \}$ is said to be complete in H.

<u>Theorem 19.</u> <u>Let</u> $e_1, e_2, \ldots e_n$ <u>be a finite orthonormal set in a Hilbert space H.</u> <u>If</u> $x \in H$, <u>then</u>

$$\| x - \sum_{j=1}^{n} \lambda_j e_j \|$$

<u>is minimum when</u> $\lambda_j = (x, e_j) \qquad j = 1, 2, \ldots n$

<u>Proof:</u> We have

$$\| x - \sum_{j=1}^{n} \lambda_j e_j \|^2 = \left(x - \sum_{j=1}^{n} \lambda_j e_j ,\ x - \sum_{j=1}^{n} \lambda_j e_j \right)$$

$$= (x, x) - \sum_{j=1}^{n} \lambda_j \overline{(x, e_j)} - \sum_{j=1}^{n} \overline{\lambda_j} (x, e_j) + \sum_{j=1}^{n} \lambda_j \overline{\lambda_j}$$

$$= (x, x) - \sum_{j=1}^{n} (x, e_j) \overline{(x, e_j)} + \sum_{j=1}^{n} (\lambda_j - (x, e_j))(\overline{\lambda_j} - \overline{(x, e_j)})$$

$$= \|x\|^2 - \sum_{j=1}^{n} |(x, e_j)|^2 + \sum_{j=1}^{n} |\lambda_j - (x, e_j)|^2$$

Now L.H.S. is minimum when R.H.S. is minimum that is, when

$$\sum_{j=1}^{n} | \lambda_j - (x, e_j) |^2 = 0$$

Hence $\lambda_j = (x, e_j)$ for $j = 1, 2, \ldots n$

Corollary 1. $\| x - \sum_{j=1}^{n} (x, e_j) e_j \|^2 = \| x \|^2 - \sum_{j=1}^{n} |(x, e_j)|^2$

Corollary 2. $\sum_{j=1}^{n} |(x, e_j)|^2 \leq \| x \|^2$ (Bessel's inequality)

Definition 38. Let $\{e_\alpha\}$ be an orthonormal set in a Hilbert space H and $x \in H$. Then the quantities (x, e_α) are called Fourier coefficients.

Remark: If $\{e_\alpha\}_{\alpha \in \Lambda}$ is an orthonormal set, it follows from corollary 2, that the number of nonzero Fourier coefficients is at most countable. Then the notation

$$\sum_{\alpha \in \Lambda} |(x, e_\alpha)|^2$$

is still valid and

$$\sum_{\alpha \in \Lambda} |(x, e_\alpha)|^2 \leq \| x \|^2$$

Theorem 20. Let $\{e_\alpha\}$ be an orthonormal system in a Hilbert space H. Then the following are equivalent.

(a) $\{e_\alpha\}$ is complete

(b) For $x \in H$, $(x, e_\alpha) = 0$ for all α implies $x = 0$

(c) $x = \sum_{\alpha} (x, e_\alpha) e_\alpha$ (Fourier expansion)

(d) if $x = \sum_{\alpha} (x, e_\alpha) e_\alpha$ and $y = \sum_{\alpha} (y, e_\alpha) e_\alpha$

then

$$(x,y) = \sum_{\alpha} (x, e_\alpha)\overline{(y, e_\alpha)} \qquad \text{(Passeval's relation)}$$

(e) $$\| x \|^2 = \sum_{\alpha} |(x, e_\alpha)|^2$$

<u>Proof</u>: a) $\Rightarrow$ c). Suppose $\{e_\alpha\}$ is complete in H. Then $(x,e_\alpha) = 0$ excepting at most countably many indices α. We shall therefore assume that we are dealing with a countable orthonormal set $\{e_n\}$ $n=1,2,$. Let

$$x_r = \sum_{j=1}^{r} (x, e_j) e_j.$$

For $s > r$,

$$\| x_s - x_r \|^2 = \| \sum_{j=r+1}^{s} (x, e_j) e_j \|^2$$

$$= \left(\sum_{j=r+1}^{s} (x, e_j) e_j , \sum_{j=r+1}^{s} (x, e_j) e_j \right)$$

$$= \sum_{j=r+1}^{s} |(x, e_j)|^2$$

and since $\sum_{j=1}^{\infty} |(x, e_j)|^2 \leq \|x\|^2$, it follows that $\{x_r\}_1^\infty$ is a cauchy sequence and hence converges to x'.

Further $(x',e_n) = (x,e_n)$ since $(x_r,e_n) = (x,e_n)$ for $r \geq n$ and since $x_r \to x'$. Thus $x - x' \perp e_n$ for each n so that $x - x' \perp e_\alpha$ for each α and hence $x - x'$ is $\perp$ to the subspace generated by $\{e_\alpha\}$. Since $\{e_\alpha\}$ is complete, $(x-x',x-x') = 0$

which implies $x = x'$.

(c) $\Rightarrow$ (d) is easily verified if we notice that (x,y) is continuous in x and y

(d) $\Rightarrow$ (e) take $y = x$ in d)

(e) $\Rightarrow$ b) If there exists $x \in H$, such that $x \neq 0$, $(x, e_\alpha) = 0$ for all α, then by e), it follows that

$$\| x \|^2 = \Sigma |(x, e_\alpha)|^2 = 0$$

which gives a contradiction.

b) $\Rightarrow$ a). Suppose the subspace generated by $\{e_\alpha\}$ is M. Assume $M \neq H$. Let $y \in H$, $y \notin M$.

Let N be the linear manifold of all vectors of the form $x + \alpha y$, $x \in M$ and α scalar. Let T be a functional on defined by $T(x+\alpha y) = \alpha$. Then T is a bounded linear functional on N and can be extended to all of H by Hahn-Banach extension theorem. Thus there exists a nontrivial functional T $(\neq 0)$ such that $T \perp M$. We assume that $\| T \| = 1$. Further $T(x) = (x, y)$ for some fixed $y \in H$ and $\| y \| = 1$. Thus $(y, e_\alpha) = 0$ since $y \perp M$. This gives a contradiction.

Remarks:

1. Every nonzero Hilbert space contains a complete orthonormal system.

2. In a separable Hilbert space, any orthonormal system has either a finite or countably infinite number of elements.

3. All complete orthonormal systems in a Hilbert space are equivalent i.e., have the same cardinality.

4. The cardinal number of a complete orthonormal system is defined to be the dimension of the Hilbert space.

13. Orthogonal Sum of Subspaces

Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of pairwise orthogonal subspaces of a Hilbert space H. The smallest subspace containing M_λ's is called the orthogonal sum of the subspaces M_λ and we write $\sum_\lambda \oplus M_\lambda$. If the M_λ's are finite or countably infinite in number, we write

$$M_1 \oplus M_2 \oplus \cdots \oplus M_n \qquad \text{(finite case)}$$

and

$$M_1 \oplus M_2 \oplus \cdots$$

1. The sum $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ of a finite number of subspaces is the set of all elements of the form

$$x = x_1 + x_2 + \cdots + x_n \quad \text{where } x_k \in M_k,\ k = 1, 2, \ldots n$$

2. The sum $M = M_1 \oplus M_2 \oplus \cdots$ of a countable number of subspaces is the set of all elements of the form

$$x = x_1 + x_2 + \cdots$$

where $x_k \in M_k$ and the series converges.

3. The sum $M = \sum_{\lambda} \oplus M_{\lambda}$ is the set of all elements of the form $x = \sum_{\lambda} x_{\lambda}$ such that

(i) $x_{\lambda} \in M_{\lambda}$

(ii) all but a finite number or at most a countable number of $x_{\lambda}'s$ are zero

(iii) the series converges.

14. Direct Sum of Hilbert Spaces

Let $H_1, H_2, \ldots H_n$ be Hilbert spaces. Let $H_1 \oplus H_2 \oplus \cdots \oplus H_n$ denote the set of all elements of the form

$$x = \{x_1, x_2, \ldots x_n\}, \quad x_k \in H_k.$$

Define

$$x + y = \{x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n\}$$
$$\alpha x = \{\alpha x_1, \alpha x_2, \ldots, \alpha x_n\}$$
$$(x,y) = \sum_{k=1}^{n} (x_k, y_k)$$

where $x = \{x_1, x_2, \ \ x_n\}$ and $y = \{y_1, y_2, \ldots y_n\}$

Then $H_1 \oplus \ldots \oplus H_n$ is a Hilbert space called the direct sum of $H_1, H_2, \ldots H_n$.

Let now $\{H_n\}_1^{\infty}$ be a sequence of Hilbert spaces. Consider the set of all elements of the form

$$x = \{x_1, x_2, \ldots x_n, \ldots\} \qquad x_k \in H_k$$

such that

$$\sum_{k=1}^{\infty} \|x_k\|^2 < \infty$$

Define

$$x + y = \{x_1 + y_1,\ x_2 + y_2 \ldots\ \}$$

$$\alpha x = \{\alpha x_1,\ \alpha x_2,\ \ldots\ \}$$

$$(x,y) = \sum_{k=1}^{\infty} (x_k, y_k)$$

The resulting Hilbert space is called the direct sum of the spaces $\{H_n\}$

Let $\{H_\lambda\}_{\lambda \in \Lambda}$ be a family of Hilbert spaces. Consider the set consisting of all complexes $x = \{x_\lambda\}_{\lambda \in \Lambda}$ such that

a) the number of nonzero elements in x is at most countable.

b) $\sum_\lambda \|x_\lambda\|^2 < \infty$

Define

$$x + y = \{x_\lambda + y_\lambda\}$$

$$\alpha x = \{\alpha x_\lambda\}$$

$$(x,y) = \sum_\lambda (x_\lambda, y_\lambda)$$

where $x = \{x_\lambda\}$, $y = \{y_\lambda\}$

The resulting Hilbert space is called the direct sum $\sum_\lambda \oplus H_\lambda$ of the spaces $\{H_\lambda\}$.

15. Inverse Transformation

Definition 39. Let X,Y be Banach spaces and $T: X \to Y$ a bounded linear transformation. T is said to be invertible if for all $y \in Y$, the equation $Tx = y$ has a unique solution. If T is invertible, we can find a transformation $T': Y \to X$ by $T'(y) = x$, when $Tx = y$. Then T' is called the inverse of T, denoted by T^{-1}.

Theory 21. Let $T: X \to Y$ be a bounded linear transformation If T^{-1} exists, then

a) T^{-1} is linear

b) T^{-1} is bounded.

Proof of (a) is simple, but that of (b) is hard and we shall omit it.

Notice that if T is one-to-one and onto, T^{-1} exists.

16. Closed Transformation

Definition 40. Let X,Y be Banach spaces and Δ a linear manifold in X. Let T be a transformation defined on Δ. T is said to be closed if it has the following property. Given a sequence $\{x_n\}$ such that $x_n \in \Delta$ and $x_n \to x$, $Tx_n \to y$ then $x \in \Delta$ and $Tx = y$.

Example: Bounded linear transformations are closed.

Theorem 22. If T is closed and T^{-1} exists then T^{-1} is closed.

17. Adjoint Transformation

Let X,Y be Banach spaces. Given a bounded linear transformation T from X to Y, we will find that there exists a bounded linear transformation T* from Y* to X* such that the norm is unaltered.

We define $T^*: Y^* \to X^*$ as follows:

Let $y^* \in Y^*$ be fixed, but arbitrary. Define

$$(T^*y^*)\ (x) = y^*\ (Tx) \text{ for all } x \in X.$$

and $T^*y^* \in X^*$. Since y^* is arbitrary, T^* is defined on Y^*.

Is T^* a bounded linear transformation? Let us check.

Let $y_1^*, y_2^* \in Y^*$ and α_1, α_2 scalars. For each $x \in X$, we have

$$\begin{aligned}(T^*(\alpha_1 y_1^* + \alpha_2 y_2^*))(x) &= (\alpha_1 y_1^* + \alpha_2 y_2^*)(Tx) \\ &= \alpha_1 y_1^*(Tx) + \alpha_2 y_2^*(Tx) \\ &= \alpha_1 (T^* y_1^*)(x) + \alpha_2 (T^* y_2^*)(x)\end{aligned}$$

This implies

$$T^*(\alpha_1 y_1^* + \alpha_2 y_2^*) = \alpha_1 T^* y_1^* + \alpha_2 T^* Y_2^*$$

which shows T^* is linear.

Now,

$$|T^* y^*(x)| = |y^*(Tx)| \leq \|y^*\| \, \|Tx\|$$
$$\leq \|y^*\| \, \|T\| \, \|x\|$$

so that

$$\|T^*y^*\| \leq \|T\| \; \|y^*\|$$

Hence $\|T^*\| \leq \|T\|$

Since T is a bounded linear transformation, T^* is also a bounded linear transformation.

<u>Definition 41</u>. T^* is called the <u>adjoint</u> of T.

<u>Definition 42</u>. If $S,T \in B(X,Y)$, then the <u>product</u> ST is defined to be their composition.

<u>Theorem 23</u>. <u>Let X,Y be Banach spaces and $S,T \in B(X,Y)$. Then</u>

(a) $\|T^*\| = \|T\|$

(b) $(S + T)^* = S^* + T^*$

(c) $(\alpha S)^* = \alpha S^*$ α-scalar

<u>If</u> X = Y

(d) $(ST)^* = T^*S^*$

Proof: We noticed that $\|T^*\| \leq \|T\|$. Thus to prove (a), it is enough to show $\|T^*\| \geq \|T\|$. Given $x \in X$, there exists $y^* \in Y^*$ such that $\|y^*\| = 1$ and $y^*(Tx) = \|y^*\| \; \|Tx\|$, by corollary to Hahn-Banach theorem. Then we have

$$\|y^*\| \; \|Tx\| = |y^*Tx| = |T^*y^*x| \leq \|T^*\| \; \|y^*\| \; \|x\|$$

Hence

$\| Tx \| \leq \| T^* \| \; \| x \|$ so that $\| T \| \leq \| T^* \|$

(b) and (c) are easy to verify.

We shall now prove (d).

If $y^* \in Y^* = X^*$, and $x \in X$, we have

$$((ST)^* y^*) x = y^* ((ST)x) = y^* (S(Tx))$$

$$= S^* (y^* Tx) = (S^* y^*) (Tx)$$

$$= T^* (S^* y^*) x = ((T^* S^*) y^*) x$$

This implies $(ST)^* + T^*S^*$.

<u>Remark</u>: In case $X = Y = H$ (Hilbert space) then adjoint is defined by

$(Tx, y) = (x, T^*y)$ for all $x, y \in H$.

We shall now consider the transformations T which are more general than the bounded ones defined on the whole of H.

Let H_1, H_2 be Hilbert spaces. We shall now define adjoint of a transformation from H_1 to H_2, but assume that the domain of definition is a subset of H_1.

<u>Notation</u>: domT will denote the domain of definition of T.

Let T be a transformation defined on a linear manifold Δ which is dense in H_1. Consider for some $y \in H_2$, a representation of the form

$$(Tx,y) = (x,z)$$

for all $x \in \Delta$. Let Δ^* denote the set of all such vectors y in H_2 and define

$$T^*: H_2 \rightarrow H_1$$

with dom $T^* = \Delta^*$ by

$$T^*y = z$$

First we notice that z is uniquely defined. In fact, if $(Tx,y) = (x,z')$ also, then $(x, z) = (x,z')$ for all $x \in \Delta$ which implies $z - z' \perp \Delta$. Since Δ is dense in H_1, $z - z' \perp H_1$. We then have $z - z' = 0$ or $z = z'$.

It can be easily verified that T^* is linear and closed. Further one notices that T^* is defined by

$$(Tx,y) = (x,T^*y)$$

for all $x \in \Delta$, $y \in \Delta^*$.

Theorem 24. T* is linear and closed.

Proof: i) T* is linear. Let $x \in \Delta$, $y_1, y_2 \in \Delta^*$ and c a complex number.

Now,
$$(x, T^*(y_1 + y_2)) = (Tx, y_1 + y_2) = (Tx, y_1) + (Tx, y_2)$$
$$= (x, T^* y_1) + (x, T^* y_2)$$
$$= (x, T^* y_1 + T^* y_2) \text{ for all } x \in \Delta$$

so that
$$T^*(y_1 + y_2) = T^* y_1 + T^* y_2$$

Also
$$(x, T^* c y) = (Tx, cy) = \bar{c}\,(Tx, y) = \bar{c}\,(x, T^* y)$$
$$= (x, c\, T^* y) \qquad \text{for all } x \in \Delta$$

so that
$$T^*(c\, y) = c\,(T^* y)$$

The fact that T* is closed can be easily verified.

Definition 43. Two transformations S, T are said to be equal if dom$T$ = dom$S$ and $T(x) = S(x)$ for all $x \in$ dom S.

Theorem 25. If T has an inverse T^{-1} and dom$T$ and dom T^{-1} are dense in H_1, H_2 respectively, then $(T^*)^{-1} = (T^*)^{-1}$.

Proof: Let $x \in$ dom$T$, $y \in$ dom $(T^{-1})^*$

Then

$$(x,y) = (T^{-1}Tx,y) = (Tx,(T^{-1})^*y)$$

implies

$$(T^{-1})^*\, y \in \text{dom } T^*$$

and

$$(T^*)\ (T^{-1})^*\ y = y \qquad (1)$$

Now let $x \in \text{dom } T^{-1}$ and $y \in \text{dom} T^*$. Then

$$(x,y) = (TT^{-1}x,y) = (T^{-1}x,T^*y)$$

implies,

$$T^*y \in \text{dom } (T^{-1})^*$$

and

$$(T^{-1})^*\ T^*\ y = y$$

Hence

$$(T^{-1})^* = (T^*)^{-1}$$

Definition 44. If T, S are two transformations such that dom $T \subset$ dom S and $T(x) = S(x)$ for all $x \in$ dom T, then S is said to be an extension of T and we write $T \subset S$.

Theorem 26. Let H_1, H_2 be Hilbert spaces. Suppose $T, S: H_1 \to H_2$ such that dom T, dom S are dense in H_1. Then

(a) $(\lambda T)^* = \bar{\lambda} T^*$

(b) if $T \subset S$, then $T^* \supset S^*$

(c) $(T+S)^* \supset T^* + S^*$

If $H_1 = H_2$

(d) $(TS)^* \supset S^*T^*$

(e) $(T + \lambda I)^* = T^* + \bar{\lambda} I$

where λ is a scalar.

Proof: a) From

$$(x, (\lambda T)^* y) = ((\lambda T) x, y) = \lambda (Tx, y)$$

$$= \lambda (x, T^* y) = (x, \bar{\lambda}(T^* y))$$

$$= (x, (\bar{\lambda} T^*) y)$$

it follows that $(\lambda T)^* = \bar{\lambda} T^*$

b) Let $y \in \operatorname{dom} S^*$. If $x \in \operatorname{dom} T$, then

$$T(x) = S(x)$$

and

$$(Tx, y) = (Sx, y) = (x, S^*y)$$

which implies $y \in \operatorname{dom} T^*$ and $T^*y = S^*y$

Hence $T^* \supset S^*$

Other results can be proved similarly.

18. Self-adjoint Operator

Hereafter all our transformations are in a Hilbert space H, that is T: H $\rightarrow$ H and assume dom T need not be equal to H.

Definition 45. An operator is a linear transformation of H into itself.

Definition 46. Let H be a Hilbert space and T an operator (not necessarily linear) in H. T is said to be Hermitian if

$$(Tx,y) = (x,Ty)$$

for all $x,y \in$ dom T. If dom T is dense in H, then a Hermitian operator is called a symmetric operator. An operator T with dom T dense in H is said to be self adjoint if $T^* = T$.

Notice that for a symmetric operator $T \subset T^*$.

Clearly a self adjoint operator is closed.

Theorem 27. If T is self-adjoint, for arbitrary real $\alpha \neq 0$ and β, the operator $\alpha T + \beta T$ is also self-adjoint.

Proof: $(\alpha T + \beta I)^* = \alpha T^* + \beta I = \alpha T + \beta I$

Theorem 28. If T is symmetric and range of T = H, then T is self-adjoint.

Proof: Enough to prove dom T = dom T^*. Let $y \in$ dom T^* Let $y \in$ dom T^* and $z = T^*y$. Since T is onto H, there exists $y' \in H$ such that $Ty' = z$. Then if $x \in$ dom T, then

$$(Tx,y) = (x,T^*y) = (x,Ty') = (Tx,y')$$

and hence $(Tx, y-y') = 0$. Since $\{Tx\} = H$, it follows that $y-y' = 0$ or $y = y'$

Theorem 29. Let T be a self adjoint operator with dom $T = \Delta$ and let $c = \lambda + i\mu$ with $\mu \neq 0$. Then $cI - T$ maps Δ onto H in a one-to-one manner. Then the operator $(cI - T)^{-1}$ is bounded and

$$\| (cT - T)^{-1} \| \leq |\mu|^{-1}$$

Proof: Let $x \in \Delta$, $y = (cI - T)x$.

Then

$$(y,y) = ((cI - T)x, (cI - T)x)$$

$$= ((\lambda I - T + i\mu I)x, (\lambda I - T + i\mu I)x)$$

$$= ((\lambda I - T)x, (\lambda I - T)x) + (i\mu x, i\mu x)$$

$$+ ((\lambda I - T)x, i\mu x) + (i\mu x, (\lambda I - T)x)$$

$$= \|(\lambda I - T)x\|^2 + |\mu|^2 \|x\|^2$$

for

$$((\lambda I - T)x, i\mu x) = (\lambda x, i\mu x) - (Tx, i\mu x)$$

$$= -(i\mu x, \lambda x) + (i\mu x, Tx)$$

$$= -(i\mu x, (\lambda I - T)x)$$

Then, from $(y,y) = \| (\lambda I - T)\, x \|^2 + |\mu|^2 \, \| x \|^2$ it follows that

$$x \neq 0 \text{ implies } y \neq 0.$$

Hence $cI - T$ is one-to-one and the inverse transformation is defined on its range. Also $(cI - T)^{-1} y = x$ and

$$\| x \|^2 \leq |\mu|^{-2} \, \| y \|^2$$

so that

$$\| (cI - T)^{-1} y \| \leq |\mu|^{-1} \, \| y \|$$

which implies

$$\| (cI - T)^{-1} \| \leq |\mu|^{-1}$$

It remains to show that range is H. We first show range is dense in H and later to show it is H.

Suppose z is orthogonal to all elements $(cI - T)x$, $x \in \Delta$. Then

$$((cI - T)x, z) = 0 \text{ or } (cx, z) = (Tx, z)$$

Thus

$$(Tx, z) = (x, \bar{c} z)$$

which implies $z \in \operatorname{dom} T^*$ and $T^* z = \bar{c} z$.

Since T is self-adjoint, we have

$$Tz = \bar{c} z$$

Hence

$$\bar{c}(z, z) = (\bar{c} z, z) = (Tz, z) = (z, Tz) = (z, \bar{c} z)$$

$$= c(z, z).$$

Since $c \neq \bar{c}$, $z = 0$

Hence range of $cI - T$ is dense in H. Let y be an arbitrary element in H. Let $\{y_n\}$ be a sequence in the range of T such that $y_n \to y$. Let x_n be the solution of

$$(cI - T)\, x_n = y_n$$

Then

$$\| y_n - y_m \|^2 = \| (\lambda I - T)(x_n - x_m) \|^2 + |\mu|^2 \| x_n - x_m \|^2$$

$$\geq |\mu|^2 \| x_n - x_m \|^2$$

This implies, $\{x_n\}$ is a cauchy sequence. Let $x_n \to x$. Since T is closed, $x \in$ dom T and $(cI-T)\,x = y$. Thus $cI - T$ is onto.

<u>Definition 47</u>. An operator U on a Hilbert space H is said to be unitary if dom U = range of U = H and $(x,y) = (Ux, Uy)$ for all $x, y \in H$.

<u>Theorem 30</u>. <u>If T is a self-adjoint operator in a Hilbert space H, then</u> $U = (T + iI)\,(T - iI)^{-1}$ <u>is a unitary operator in H.</u>

<u>Proof</u>: <u>Exercise</u>

19. Projections

Let M be a subspace of a Hilbert space H. Then we have seen that $H = M \oplus M^{\perp}$ and $x \in H$ implies $x = x_1 + x_2$ uniquely where $x_1 \in M$ and $x_2 \in M^{\perp}$.

Definition 48. The operator P defined by

$$Px = x_1$$

is called the projection operator onto M. We also write P_M for P

Theorem 31. The operator P is bounded, linear, Hermitian and idem potent i.e. $P^2 = P$

Proof: Since $\| Px \| \leq \| x \|$, P is bounded. If $x = x_1 + x_2$, $y = y_1 + y_2$, then

$$Px = x_1, Py = y_1 \quad \text{and}$$

$$(x_1, x_2) = (y_1, y_2) = (x_1, y_2) = (x_2, y_1) = 0$$

Then

$$(Px, y) = (x_1, y_1 + y_2) = (x_1, y_1)$$

$$(x, Py) = (x_1 + x_2, y_1) = (x_1, y_1)$$

so that $(Px, y) = (x, Py)$ and P is Hermitian and hence linear

Next, $x_1 \in M$ implies $Px_1 = x_1$ so that $P^2 x = P(Px) = Px_1 = x_1 = Px$ for all $x \in H$. This means $P^2 = P$

<u>Theorem 32. If</u> P <u>is a Hermitian operator in H with</u> dom P = H <u>such that</u> $P^2 = P$, <u>then</u> P <u>is a projection operator</u>

<u>Proof</u>: Since $P^2 = P$, we have

$$\|Px\|^2 = (Px, Px) = (P^2 x, x) = (Px, x) \le \|x\| \, \|Px\|$$

so that $\|Px\| \le \|x\|$. Hence P is bounded and hence continuous.

Let $M = \{ x \in H \mid Px = x \}$

Then M is a subspace of H since P is continuous. If x∈M, let $x_1 = Px$, $x_2 = (I-P)x$ so that

$$x = x_1 + x_2 \quad \text{and} \quad Px_1 = P^2 x = Px = x_1$$

Hence $x_1 \in M$. If $y \in M$, then $Py = y$.

Then

$$(y, x_2) = (y, (I-P)x) = ((I-P)y, x) = (y - Py, x) = 0$$

so that $x_2 \in M^\perp$.

Hence P is a projection operator on M.

Exercises:

1. Let P_{M_1}, P_{M_2} be projection operator. Then $P_{M_1} P_{M_2}$ is a projection operator if and only if

$$P_{M_1} P_{M_2} = P_{M_2} P_{M_1}$$

Moreover, if $P_M = P_{M_1} P_{M_2}$, then $M = M_1 \cap M_2$

Hint: Use $P = P^*$

2. Show that the subspaces M_1, M_2 are orthogonal if and only if

$$P_{M_1} P_{M_2} = 0$$

Theorem 33. Let $\{P_{M_j}\}_{j=1}^{n}$ be projection operators on on a Hilbert space H. Then $P_{M_1} + P_{M_2} + \cdots + P_{M_n}$ is a projection operator if and only if

$$(*) \qquad P_{M_j} P_{M_k} = 0 \text{ for } j \neq k$$

Moreover, if $P_{M_1} + P_{M_2} + \cdots + P_{M_n} = P_M$, then

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$$

Proof: Suppose $(*)$ holds. Then

$$(P_{M_1} + P_{M_2} + \cdots + P_{M_n})^2 = P_{M_1}^2 + P_{M_2}^2 + \cdots + P_{M_n}^2 = P_{M_1} + \cdots + P_{M_n}$$

so that $P_{M_1} + P_{M_2} + \cdots + P_{M_n}$ is a projection operator.

Conversely, suppose that $P_{M_1} + \quad + P_{M_n}$ is a projection operator. Suppose x∈H, then

$$\|x\|^2 \geq ((P_{M_1} + P_{M_2} + \quad + P_{M_n})x, x)$$

$$= \sum_{j=1}^{n} (P_{M_j} x, x) \geq (P_{M_j} x, x) + (P_{M_k} x, x)$$

so that

$$(**) \qquad \|P_{M_j} x\|^2 + \|P_{M_k} x\|^2 \leq \|x\|^2$$

set $\quad x = P_{M_k} y \quad$ in $\quad (**)$

Then

$$\|P_{M_j} \cdot P_{M_k} y\|^2 + \|P_{M_k} y\|^2 \leq \|P_{M_k} y\|^2$$

which implies

$$\|P_{M_j} \cdot P_{M_k} y\|^2 = 0 \quad \text{and} \quad P_{M_j} \cdot P_{M_k} = 0$$

Now suppose $(*)$ is satisfied and let

$P_M = P_{M_1} \oplus \quad + P_{M_n}$. Then x∈H,

$$P_M x = P_{M_1} x + \quad + P_{M_n} x \in M_1 \oplus M_2 \oplus \cdot \oplus M_n$$

so that

$$M \subset M_1 \oplus M_2 \oplus \cdot \cdot \oplus M_n .$$

If $x \in M_1 \oplus \cdots \oplus M_n$ then $x = x_1 + \cdots + x_n$, where $x_k \in M_k$ so that by $(*)$

$$\begin{aligned} P_{M_j} x_k &= 0 \quad \text{if } j \neq k \\ &= x_k \quad \text{if } j = k \end{aligned}$$

Then

$$P_M x = P_{M_1} x + \cdots + P_{M_n} x = x_1 + \cdots + x_n = x$$

and hence $x \in M$. Thus

$$M \supset M_1 \oplus M_2 \oplus \cdots \oplus M_n$$

This completes the proof.

Theorem 34. If $\{M_n\}$ is a sequence of pair-wise orthogonal subspaces of a Hilbert space H and if the series

$$P_{M_1} x + P_{M_2} x + \cdots + P_{M_n} x + \cdots$$

converges to $P_M x$ for each $x \in H$, then

$$M = M_1 \oplus M_2 \oplus \cdots$$

Proof: Exercise

Exercises:

1. Show that the following are equivalent.

(a) $M_2 \subset M_1$

(b) $P_{M_1} P_{M_2} = P_{M_2} P_{M_1} = P_{M_2}$

(c) $\| P_{M_2} x \| \leq \| P_{M_1} x \|$ for all $x \in H$.

2. $P_{M_1} - P_{M_2}$ is a projection operator if and only if $M_2 \subset M_1$. Then

$$P_{M_1} - P_{M_2} = P_{M_1 \sim M_2}$$

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