MATSCIENCE REPORT 38

A MATHEMATICAL INTRODUCTION
TO
UNITARY SYMMETRIES

PROFESSOR M. GOURDIN
Visiting Scientist, Matscience

THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS-20 (India)

MATSCIENCE REPORT 38

THE INSTITUTE OF MATHEMATICAL SCIENCES

MADRAS-20 India)

A MATHEMATICAL INTRODUCTION
TO
UNITARY SYMMETRIES

Professor M.Gourdin*
Visiting Scientist, MATSCIENCE

---

* Visiting Scientist under the Council of Scientific and Industrial Research (India) scheme.

Permanent Address: Laboratoire De Physique Theorique Et
Hautes Energies, Faculte Des Sciences,
Universite De Paris, Orsay (France).

CONTENTS

CONTENTS

pter 1.

LIE GROUPS AND LIE ALGEBRA

I. Topological Groups:

1. Groups axioms

A set G is a group if the composition law, defined in G has the following properties:

- a Associativity: $a(bc) = (ab)c = abc$; $a, b, c \in G$.
- b Identity ~~unit~~ element e. $ea = ae = a$, $a \in G$.
- c Inverse ; $a^{-1} a = aa^{-1} = e$

2. Topological groups

The mapping $(a, b) \Rightarrow ab^{-1}$ of $G \times G$ into G is a continuous mapping. Such a condition is equivalent to the two following ones:

- a The mapping $a \Rightarrow a^{-1}$ of G into G is continuous.
- b The mapping $(a,b) \Rightarrow ab$ of $G \times G$ into G is continuous.

The mapping $a \Rightarrow a^{-1}$ of G into G coincides with its inverse because of the relation $(a^{-1})^{-1} = a$. Such a mapping, noted τ, is a homeomorphism of G.

3. Translations:

The mapping $a \Rightarrow am$ of G into G is one to one and continuous.

This homeomorphism of G is called a right translation ρ_m.

The mapping $a \Rightarrow na$ of G into G is one to one and continuous.

This homeomorphism of G is called a left translation λ_n. The right and left translations are related by

$$\lambda_m \tau \rho_m = \tau$$

4. Theorem

It is possible to show that the necessary and sufficient condition for a group G to be a topological group can be given in the following form:

a The translation ρ_m and λ_m are continuous ($m \in G$);

b The mapping $(a, b) \Rightarrow ab^{-1}$ of $G \times G$ into G is continuous at the point (e, e) of $G \times G$.

II. Lie Groups:

1. Definition

A group G is a Lie group if;

a G is an analytic manifold

b The mapping $(a, b) \Rightarrow ab$ of $G \times G$ into G is an analytic mapping

2. Composition functions

We choose a chart at the point e of G and we denote the coordinates of an element $a \in G$ by a^σ. The composition law can be written as

$$(ab)^\sigma = \phi^\sigma(a, b) \qquad a, b \in G$$

The compositions functions ϕ^σ are analytic functions of their arguments. We have the following evident properties:

$$\begin{aligned} \phi^\sigma(a,\ bc) &= \phi^\sigma(ab,\ c) \\ \phi^\sigma(a, e) &= \phi^\sigma(e, a) = a^\sigma \\ \phi^\sigma(a,\ a^{-1}) &= \phi^\sigma(a^{-1},\ a) = e^\sigma \end{aligned}$$

3. It can be easily shown that the mapping $a \Rightarrow a^{-1}$ of G into G is also an analytic mapping.

It follows that a Lie group is a topological group.

4. Structure constants

The identity transformation is described by the relation

$$a^\sigma = \phi^\sigma(a,\ e)$$

and we now consider an infinitesimal transformation in the neighbourhood of the identity

$$a^\sigma + da^\sigma = \phi^\sigma(a,\ e + \delta m) = \phi^\sigma(a, e) + \delta m^\rho \left[\frac{\partial}{\partial b^\rho} \phi^\sigma(a,b)\right]_{b=e}$$

The velocity field is defined by

$$\mu^\sigma_\rho(a) = \left[\frac{\partial}{\partial b^\rho}\ \phi^\sigma(a,\ b)\right]_{b\ =\ e}$$

and we obtain

$$da^\sigma = \mu^\sigma_\rho(a)\ \delta m^\rho$$

It is convenient to use the inverse matrix $\check{\mu}$ (a):

$$\delta^{\sigma}_{\tau} = \check{\mu}^{\sigma}_{\rho}(a)\,\mu^{\rho}_{\tau}(a)$$

The elimination of δm between the two relations:

$$da^{\sigma} = \mu^{\sigma}_{\rho}(a)\;\delta m^{\rho} \qquad db^{\sigma} = \mu^{\sigma}_{\rho}(b)\;\delta m^{\rho}$$

leads to the expression

$$\frac{\partial a^{\sigma}}{\partial b^{\tau}} = \mu^{\sigma}_{\rho}(a)\,\check{\mu}^{\rho}_{\tau}(b)$$

We now introduce the continuity condition $\dfrac{\partial^2 a^{\sigma}}{\partial b^{\rho}\partial b^{\tau}} = \dfrac{\partial^2 a^{\sigma}}{\partial b^{\tau}\partial b^{\rho}}$

By using the previous expression for the first derivative, we obtain:

$$\frac{\partial^2 a^{\sigma}}{\partial b^{\rho}\partial b^{\tau}} = \frac{\partial \mu^{\sigma}_{\alpha}(a)}{\partial a^{\lambda}} \cdot \mu^{\lambda}_{\beta}(a)\,\check{\mu}^{\beta}_{\rho}(b)\,\check{\mu}^{\alpha}_{\tau}(b) + \mu^{\sigma}_{\gamma}(a)\,\frac{\partial \check{\mu}^{\gamma}_{\tau}(b)}{\partial b^{\rho}}$$

$$\frac{\partial^2 a^{\sigma}}{\partial b^{\tau}\partial b^{\rho}} = \frac{\partial \mu^{\sigma}_{\beta}(a)}{\partial a^{\lambda}} \cdot \mu^{\lambda}_{\alpha}(a)\,\check{\mu}^{\alpha}_{\tau}(b)\,\check{\mu}^{\beta}_{\rho}(b) + \mu^{\sigma}_{\gamma}(a)\,\frac{\partial \check{\mu}^{\gamma}_{\rho}(b)}{\partial b^{\tau}}$$

Calculations are straight forward but tedious and we obtain the following equality:

$$\left[\frac{\partial \mu^{\sigma}_{\beta}(a)}{\partial a^{\lambda}} \cdot \mu^{\lambda}_{\alpha}(a) - \frac{\partial \mu^{\sigma}_{\alpha}(a)}{\partial a^{\lambda}} \mu^{\lambda}_{\beta}(a)\right] \check{\mu}^{\gamma}_{\sigma}(a) =$$

$$= \left[\frac{\partial \check{\mu}^{\gamma}_{\tau}(b)}{\partial b^{\rho}} - \frac{\partial \check{\mu}^{\gamma}_{\rho}(b)}{\partial b^{\tau}}\right] \cdot \mu^{\tau}_{\alpha}(b)\, \mu^{\rho}_{\beta}(b)$$

The LHS is function of a only and the RHS is function of b only. The two quantities a and b being independent variables, the two **sides** are constants. By definition, the structure constants $C^{\gamma}_{\alpha\beta}$ are given by the two equivalent expressions.

$$C^{\gamma}_{\alpha\beta} = \left[\frac{\partial \mu^{\sigma}_{\beta}(a)}{\partial a^{\lambda}} \cdot \mu^{\lambda}_{\alpha}(a) - \frac{\partial \mu^{\sigma}_{\alpha}(a)}{\partial a^{\lambda}} \mu^{\lambda}_{\beta}(a)\right] \check{\mu}^{\gamma}_{\sigma}(a)$$

$$C^{\gamma}_{\alpha\beta} = \left[\frac{\partial \check{\mu}^{\gamma}_{\tau}(a)}{\partial a^{\rho}} - \frac{\partial \check{\mu}^{\gamma}_{\rho}(a)}{\partial a^{\tau}}\right] \mu^{\tau}_{\alpha}(a)\, \mu^{\rho}_{\beta}(a)$$

An immediate property is:

$$C^{\gamma}_{\alpha\beta} + C^{\gamma}_{3\alpha} = 0$$

III. Lie Algebra:

1. Infinitesimal transformations

We are first working in an analytic manifold $\mathcal{E}$ of elements $a \in \mathcal{E}$. The set of the analytic functions f in $\mathcal{E}$ is denoted by $\mathcal{F}$ and the space of analytical infinitesimal transformations X by $\mathcal{T}$.

The elements X of $\mathcal{T}$ can be used to define the linear mapping

$$f \Rightarrow Xf$$

of $\mathcal{F}$ into $\mathcal{F}$. The quantities XY and YX allow also to define linear mappings of $\mathcal{F}$ into itself but, in general, XY and YX do not belong to the space $\mathcal{T}$.

Let us introduce a coordinate system:

$$X = \lambda^i \frac{\partial}{\partial a^i} \quad ; \quad Y = \nu^j \frac{\partial}{\partial a^j}$$

We have successively

$$xyf = \lambda^i \frac{\partial}{\partial a^i} \nu^j \frac{\partial}{\partial a^j} f = \lambda^i \frac{\partial \nu^j}{\partial a^i} \cdot \frac{\partial f}{\partial a^j} + \lambda^i \nu^j \frac{\partial^2 f}{\partial a^i \partial a^j}$$

$$yxf = \nu^j \frac{\partial}{\partial a^j} \lambda_i \frac{\partial}{\partial a^i} f = \nu^j \frac{\partial \lambda^i}{\partial a^j} \frac{\partial f}{\partial a^i} + \nu^j \lambda^i \frac{\partial^2 f}{\partial a^j \partial a^i}$$

The continuity condition $\frac{\partial^2 f}{\partial a^i \partial a^j} = \frac{\partial^2 f}{\partial a^j \partial a^i}$ allows us to write

$$[x, y] f = \left(\lambda_i \frac{\partial \nu^j}{\partial a^i} - \nu^i \frac{\partial \lambda^j}{\partial a^i} \right) \frac{\partial f}{\partial a^j}$$

and it follows that the commutator $[X, Y]$ is also an element of $\mathcal{E}$ which can be represented by:

$$[X, Y] = \left(\lambda^i \frac{\partial \nu^j}{\partial a^i} - \nu^i \frac{\partial \lambda^j}{\partial a^i}\right) \frac{\partial}{\partial a^j}$$

2. Lie algebra

The Lie product of two operators X and Y is the commutator $[X, Y]$. The space $\mathcal{E}$ can be considered as a linearalgebra on the field K where is defined $\mathcal{E}$ and we have the following properties:

a Linear algebra

$$[\alpha X + \beta Y, Z] = \alpha [X, Z] + \beta [Y, Z]$$

$$[X, \alpha Y + \beta Z] = \alpha [X, Y] + \beta [Y, Z]$$

for all $\alpha, \beta \in K$ and $X, Y, Z \in \mathcal{E}$

b Antisymmetry

$$[X, X] = 0$$

c Jacobi Identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

A Lie algebra is a linear algebra which satisfies the antisymmetry property and the Jacobi identity.

3. Lie algebra of a Lie group G.

A Lie group G is an analytic manifold and we consider the set $\mathcal{F}(G)$ of the analytic functions in G.

The right translations define completely the group G.

$$a \Rightarrow am \qquad\qquad a, m \in G$$

and induce, in $\mathcal{F}(G)$ a continuous mapping;

$$f \Rightarrow f_m \qquad\qquad f, f_m \in \mathcal{F}(G)$$

where

$$f_m(a) = f(am)$$

We now introduce a tangent vector L at the unit element ε of G. The infinitesimal right translations are defined by:

$$X(a)\, f(a) = \left[L(m) f_m\,(a)\right]_{m = e}$$

Let us precise these definitions with a coordinate system

$$L(m) = \lambda^{\sigma} \frac{\partial}{\partial m^{\sigma}} \qquad\qquad X(a) = \lambda^{\sigma} X_{\sigma}(a)$$

$$X_{\sigma}\,(a)\, f(a) = \left[\frac{\partial}{\partial m^{\sigma}} f_m\,(a)\right]_{m = e}$$

The right hand side can be evaluated using the relation given in a previous section:

$$\frac{\partial l^{\rho}}{\partial m^{\sigma}} = \mu^{\rho}_{\tau}(l)\, \check{\mu}^{\tau}_{\sigma}(m)$$

and we obtain:

$$\frac{\partial}{\partial m^\sigma} f_m(a) = \left[\frac{\partial}{\partial b^\rho} f(b) \mu^\rho_\tau(b) \check{\mu}^\tau_\sigma(m) \right]_{b=am}$$

In the limit $m = e$, we have $\check{\mu}^\tau_\sigma(e) = \delta^\tau_\sigma$ and the infinitesimal generators $X_\sigma(a)$ can be represented in terms of differential operators by:

$$X_\sigma(a) = \mu^\rho_\sigma(a) \frac{\partial}{\partial a^\rho}.$$

The Lie algebra of the generators $X_\sigma(a)$ is known from the Lie product of two operators as calculated in section 1.

$$[X_\rho, X_\sigma] = \left(\mu^\alpha_\rho(a) \frac{\partial \mu^\beta_\sigma(a)}{\partial a^\alpha} - \mu^\alpha_\sigma(a) \frac{\partial \mu^\beta_\rho(a)}{\partial a^\alpha} \right) \frac{\partial}{\partial a^\beta}$$

This expression can be simplified by using the structure constants introduced in Section II.

$$\mu^\alpha_\rho(a) \frac{\partial \mu^\beta_\sigma(a)}{\partial a^\alpha} - \mu^\alpha_\sigma(a) \frac{\partial \mu^\beta_\rho(a)}{\partial a^\alpha} = C^\tau_{\rho\sigma} \mu^\beta_\tau(a)$$

and we finally obtain the fundamental relation of a Lie algebra:

$$[X_\rho, X_\sigma] = C^\tau_{\rho\sigma} X_\tau$$

The antisymmetry property of the Lie algebra is contained in the antisymmetry character of the structure constants. The infinitesimal generators satisfy the Jacobi identity and it follows for the structure constants the relation

$$C^{\alpha}_{\rho\sigma} \quad C^{\beta}_{\alpha\tau} + C^{\alpha}_{\sigma\tau} \, C^{\beta}_{\alpha\rho} + C^{\alpha}_{\tau\rho} \quad C^{\beta}_{\alpha\sigma} = 0$$

IV. Simple and Semi Simple Lie Algebra

1. Definitions

We first give some classical definitions for the groups

a In an abelian group the multiplication law is commutative.

b A subgroup is a set of elements of a group which satisfies the group axioms. A trivial subgroup is the identity element itself.

c An invariant subgroup H of a group G is a subgroup of G such that:

$$a \; x \; a^{-1} \in H \text{ for all } x \in H \text{ and } a \in G$$

If we now consider the particular case of interest of Lie group it is easy to translate these properties in terms of Lie algebra.

a All the infinitesimal generators of the Lie algebra of an abelian group commute and all the structure constants are zero.

b The Lie algebra h of an analytic sub group H of a Lie group G is a sub-algebra of the Lie algebra g of G and the structure constants satisfy the relation:

$$C_{jk}^{\alpha} = 0 \quad \text{for all } X_j, X_k \in h \text{ if } X_\alpha \in g \text{ is not in } h.$$

c If now, H is an invariant sub-group of G, the structure constants verify the condition:

$$C_{j\alpha}^{\beta} = 0 \quad \text{for all } X_j \in h, \; X_\alpha, X_\beta \in g \quad \text{if } X_\beta \text{ is not in } h.$$

2. Simple group and simple algebra

A simple group has no invariant subgroups besides itself, the identity and perhaps discrete subgroups,

A simple algebra has no invariant subalgebra.

The Lie algebra of a simple Lie group is a simple algebra

3. Semi-simple group and Semi-simple algebra

A semi-simple group has no abelian invariant subgroup, besides itself, the identity and perhaps discrete subgroups

A semi-simple algebra has no abelian invariant subalgebra.

The Lie algebra of a semi simple Lie group is a semi simple algebra.

4. Cartan criterion for semi-simple algebra

We define the symmetrical Cartan tensor

$$g_{\rho\sigma} = c_{\rho\alpha}^{\beta} c_{\sigma\beta}^{\alpha}$$

The Cartan criterion is the following: a necessary and sufficient condition for a Lie algebra to be semi-simple is:

$$\det.(g_{\rho\sigma}) \neq 0.$$

For a semisimple algebra, the matrix $g_{\rho\sigma}$ is a regular matrix. This condition is obviously a necessary condition. If we suppose that the Lie algebra possesses an abelian invariant sub algebra h all the structure constants $c_{j\alpha}^{\beta}$ where $X_j \in h$ vanish and it follows that all elements $g_{j\gamma}$ of the row j of the Cartan tensor also vanish and $\det(g_{\rho\sigma}) = 0$

Cartan has proved that if $\det(g_{\rho\sigma}) \neq 0$ the Lie algebra is semi-simple.

5. Let us consider a semi simple Lie algebra. The Cartan tensor $g_{\rho\sigma}$ allows to define a symmetrical linear connexion in the Lie algebra. In particular, this tensor can be used to lower the indices. As an example, we have

$$c_{\rho\sigma\tau} = c_{\rho\sigma}^{\alpha} g_{\alpha\tau}$$

We replace $g_{\alpha\tau}$ by its definition and we apply the Jacobi identity

$$C_{\rho\sigma\tau} = C_{\beta\rho}^{\;\;\alpha}\, C_{\alpha\sigma}^{\;\;\gamma}\, C_{\gamma\tau}^{\;\;\beta} - C_{\beta\sigma}^{\;\;\alpha}\, C_{\alpha\rho}^{\;\;\gamma}\, C_{\gamma\tau}^{\;\;\beta}$$

The tensor $C_{\rho\sigma\tau}$ is invariant under a cyclic permutation of the indices and completely antisymmetric.

Chapter 2

LIE GROUPS OF TRANSFORMATIONS

I. Generalities

1. Definition:

G is a Lie group of transformations of an analytic manifold m if for each $x \in m$ and $a \in G$, one can find a $y \in m$ denoted $y = xa$ such that

a The mapping $(x, a) \Longrightarrow y$ of $m \times G$ into m is analytic

b $xe = e$ for each $x \in m$.

c Associativity $(xa)b = x(ab)$ for each $x \in m$ and $a, b \in G$.

If the unit element e of G is the only one element satisfying the condition b, the group is called an effective group,

2. Lie algebra

Let us define a chart in m and a chart in G and we use greek indices in G and latin indices in m. The mapping $xa \Longrightarrow y$ is written as

$$y^j = f^j(x, a)$$

where the composition functions f^j are analytic functions of their arguments. The velocity field is defined by

$$u^j_\sigma(x) = \left[\frac{\partial}{\partial a^\sigma} f^j(x, a) \right]_{a = e}$$

and the infinitesimal generators of the Lie algebra are given by

$$X_\sigma(x) = u^j_\sigma(x)\frac{\partial}{\partial x^j}.$$

For an effective group, the generators $X_\sigma(x)$ defined in this way are linearly independent and constitute a basis of the Lie algebra.

II. GROUP OF LINEAR TRANSFORMATIONS OF A VECTORSPACE ON THE FIELD OF REAL NUMBERS

We are working with a n-dimensional vector space on the field R of real numbers as analytic manifold; E(n,R). We define in E(n,R) a symmetrical linear connexion g, with a regular matrix which allows us to introduce a scalar product in E(n,R).

A. General Linear Group GL(n,R).

1. The regular nxn matrices with real coefficients generate the general linear group GL(n,R).

Any arbitrary nxn matrix with real coefficients $\mathfrak{M}(n,R)$, defines in the vector space E(n,R) a linear tranformation by

$$y^j = x^k S^j_k(a)$$

When the matrix S(a) is regular, the associated linear transformation is also called regular. It follows that the group of regular linear transformations in E(n,R) is isomorphic to the general linear group GL(n,R).

2. Let us now define, for the matrices $\mathfrak{M}(n,R)$, a basis E_{ij} by the matrix elements:

$$(E_{ij})_{kl} = g_{ki} g_{lj} .$$

The matrix g is regular and the E_{ij}'s span a complete basis. The matrix S(a) can be expanded onthis basis following:

$$S(a) = E_{ij} \ a^{ij}$$

The velocity field $u^j_\sigma(x)$ can then be written as

$$u^j_{[rs]}(x) = \frac{\partial}{\partial a^{rs}}\left[x^k s^j_k(a)\right]_{a=e} = x_r \delta^j_s$$

and the infinitesimal generators have a representation as differential operators

$$X_{rs} = x_r \frac{\partial}{\partial x^s}$$

We are now able to deduce the commutation rules of the Lie algebra

$$\left[X_{rs}, X_{tu}\right] = g_{st} X_{ru} - g_{ur} X_{ts}$$

The linear group GL(n,R) depends on n^2 independent real parameters and the Lie algebra has n^2 elements.

3. The product of two matrices E_{ij} is given by;

$$E_{ij} \quad E_{k\ell} \quad = \quad g_{jk} \quad E_{i\ell}$$

We can consider the matrices $\mathcal{M}n(n,R)$ as a Lie algebra on the real numbers with multiplication law given by the Lie product.

$$\left[E_{ij}, \; E_{k\ell}\right] \quad = \quad g_{jk} E_{i\ell} - g_{\ell i} E_{kj}$$

The previous equality shows clearly that the Lie algebra of the matrices $\mathcal{M}n(n,R)$ is isomorphic to the Lie algebra of the general linear group GL(n,R).

B. Special Linear Group SL(n,R):

1. The particular operator $X = g^{sr} X_{rs}$ commutes in an evident way with the n^2 infinitesimal generators X_{tu}. The transformation generated by X is given by:

$$dx^k = \varepsilon X x^k = \varepsilon x^k.$$

and is interpreted as a dilatation of center the origin . The group generated by X is an one parameter abelian group; subgroup of GL(n,R) and isomorphic to the additive group R of real numbers.

The factor group GL(n,R)/R is the special linear group SL(n,R). It can be defined as the set of unimodular linear tranformations in E(n,R) or, equalently, as the set of the nxn unimodular matrices with real coefficients. The number of independent real parameters is $n^2 - 1$.

2. The Lie algebra of SL(n,R) is immediately defined by the infinitesimal generators

$$X'_{rs} = X_{rs} - \frac{1}{n} g_{rs} X$$

The commutation laws are unchanged.

C. Pseudo-Orthogonal Groups: $O_s(n,R)$:

1. The scalar product, in E(n,R) is given by the symmetrical linear connection g :

$$(x,y) = g(x,y) = x^k g_{k\ell} y^\ell = (y, x)$$

Let us call as A an arbitrary linear transformation in

E(n,R). The conservation of the scalar product under the tranformation A is simply

$$(Ax,\ Ay) = (x,y)$$

This equality must be satisfied for all vectors x and y of E(n,R). The invariance propety takes then the simple form.

$$A^T\ g\ A = g$$

A matrix which verifies the previous relation is called a orthogonal matrix with respect to the connection g. The orthogonal matrices generate a subgroup of GL(n,R), the pseudo-orthogonal group.

2. The connexion g is a symmetrical bilinear regular form and can be diagonalised in the following way: $g_{ij} = \pm\delta_{ij}$. We choose in E(n,R) an orthonormalized basis such that:

$$g_{ij} = \delta_{ij} \quad i = 1,2,\ldots\ldots\ldots, n-s$$

$$g_{ij} = -\delta_{ij} \quad i = n-s+1,\ \ldots\ldots,\ n$$

The number s of time like vectors is called the signature. The pseudo-orthogonal groups are characterized by the signature s and noted $O_s(n,R)$. The two pseudo-orthogonal groups $O_s(n,R)$ and $O_{n-s}(n,R)$ are isomorphic.

In the particular case s = 0 (or s = n) the vector space is an euclidian space and the connexion g can always be chosen as the unit matrix I. The orthogonal group O(n,R) is the set of orthogonal matrices: $A^TA = I$.

3. The pseudo orthogonal groups are sub-groups of GL(n,R). The Lie algebra of the pseudo-orthogonal group is a sub-algebra of that of the linear group. The infinitesimal generators Z_{ij} can be written as linear combinations of the X_{ij} previously defined:

$$Z_{ij} = \lambda_{ij}^{mn} X_{mn}$$

It is sufficient to impose the invariance of the norm of all vectors

$$Z_{ij} (x , x) \equiv 0 \quad \text{for all} \quad x \in E(n,R)$$

which can be transformed into:

$$\lambda_{ij}^{mn} x_m x_n = 0$$

The matrices λ_{ij} must be antisymmetrical : $\lambda_{ij}^{mn} + \lambda_{ij}^{nm} = 0$ and it is convenient to choose:

$$\lambda_{ij} = E_{ij} - E_{ji}$$

which gives for the Z_{ij}'s the explicit form:

$$Z_{ij} = X_{ij} - X_{ji}$$

It is possible to construct $\frac{n(n-1)}{2}$ linearly independent antisymmetric nxn matrices λ_{ij}. The Lie algebra of the pseudo-orthogonal groups is spanned by $\frac{n(n-1)}{2}$ infinitesimal generators Z_{ij} and the pseudo-orthogonal groups depend on $\frac{n(n-1)}{2}$ real independent parameters.

The commutation rules for the Z_{ij} are then given by:

$$[z_{ij}, z_{kl}] = g_{jk} z_{il} - g_{il} z_{kj} + g_{ik} z_{lj} - g_{jl} z_{ik}$$

4. For a given value of n, there exist only $\left[\frac{n}{2}\right] + 1$ non-equivalent pseudo-orthogonal groups.

Two particular sub-groups of $O_s(n,R)$ are the two orthogonal groups $O(S,R)$ and $O(n-s,R)$ and also the direct product.

$$O(S,R) \otimes O(n-s,R) \subset O_s(n,R)$$

D. Special Pseudo-Orthogonal Groups $SO_s(n,R)$:

1. As a consequence of the relation $A^T g\ A = g$, we obtain:

$$(\det A)^2 = (\det g)^2 = 1$$

It is then possible to define in the pseudo-orthogonal groups $O_s(n,R)$ an equivalence with respect to the sign of det A. only the coset det $A = +1$ is a sub-group called the special pseudo-orthogonal group $SC_s(n,R)$. This special group, of course is also a sub-group of the special linear group $SL(n,R)$ and more precisely

$$SO_s\ (n,R) = C_s(n,R) \cap SL(n,R)$$

2. In a euclidian space where $g \simeq I$, the group of unimodular orthogonal matrices is the special orthogonal group $SO(n,R)$.

3. The two groups $O_s(n,R)$ and $SO_s(n,R)$ have the same Lie algebra but they are not isomorphic.

E. Applications:

1. The signature of a three dimensional vector space can be $s = 0$ (or $s = 3$) and $s = 1$ (or $s=2$). We will have only two pseudo-orthogonal groups, $O(3,R)$ and $O_1(3,R)$. The infinitesimal generators can be represented by:

$$Z_{12} = x_1 \frac{\partial}{\partial x^2} - x_2 \frac{\partial}{\partial x^1}$$

$$Z_{23} = x_2 \frac{\partial}{\partial x^3} - x_3 \frac{\partial}{\partial x^2}$$

$$Z_{31} = x_3 \frac{\partial}{\partial x^1} - x_1 \frac{\partial}{\partial x^3}.$$

In the case of an euclidian space the connexion g can be taken as

$$g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

and the commutation rules are given by:

$$\left[Z_{23}, Z_{31}\right] = - Z_{12} ; \left[Z_{31}, Z_{12}\right] = -Z_{23} ; \left[Z_{12}, Z_{23}\right] = -Z_{31}$$

In the case of a pseudo euclidian space, the connexion g can be chosen so that

$$g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

and the commutation rules become:

$$\left[Z_{23}, Z_{31}\right] = + Z_{12} \qquad \left[Z_{31}, Z_{12}\right] = -Z_{23} \qquad \left[Z_{12}, Z_{23}\right] = - Z_{31}$$

Usually, for the orthogonal group O(3), the hermitic infinitesimal generators are defined following:

$$Z_{jk} = i \epsilon_{jkl} J_l$$

and the commutation rules take the familiar form $\vec{J} \times \vec{J} = i\vec{J}$.

2. Let us now consider a 2-dimensional vector space with a linear connexion g defined by: $g_{11} = 1$, $g_{22} = \varepsilon$ with $\varepsilon = \pm 1$. The general linear group GL(2 ,R) is a 4-parameter group and the Lie algebra is known from the commutation relations

$$\begin{aligned} \left[X_{11}, X_{12}\right] &= X_{12} & \qquad \left[X_{22}, X_{12}\right] &= -\varepsilon X_{12} \\ \left[X_{11}, X_{21}\right] &= -X_{21} & \qquad \left[X_{22}, X_{21}\right] &= \varepsilon X_{21} \\ \left[X_{11}, X_{22}\right] &= 0 & \qquad \left[X_{12}, X_{21}\right] &= \varepsilon X_{11} - X_{22} \end{aligned}$$

The generator $Y = X_{11} + \varepsilon X_{22}$ commutes with the four X_{jk}'s. The Lie algebra of the special linear group SL(2,R) can be conveniently defined by the following infinitesimal generators:

$$X^0 = \frac{1}{2}\left(X_{11} - \varepsilon X_{22} \right)$$

$$X^+ = \frac{1}{2}\left(X_{12} + \varepsilon X_{21} \right)$$

$$X^- = \frac{1}{2}\left(X_{12} - \varepsilon X_{21} \right)$$

which satisfy the commutation relations:

$$\left[X^0, X^+\right] = X^- \; ; \quad \left[X^+, X^-\right] = -X^0 \; ; \quad \left[X^-, X^0\right] = -X^+$$

If we compare these results with those obtained in the previous section, we immediately see that the special linear group SL(2,R) and the pseudo-orthogonal group $O_1(3,R)$ have two isomorphic Lie algebra. Of course the two groups are not isomorphic.

3. The Lie algebra of the orthogonal group in a 4-dimensional euclidian space is defined by six infinitesimal generators Z_{ij} and the commutation rules:

$$\left[Z_{ij}, Z_{kl}\right] = \delta_{jk} Z_{il} - \delta_{ik} Z_{jl} - \delta_{jl} Z_{ik} + \delta_{il} Z_{jk}$$

Let us define two sets of three generators by

$$Z_j^{\pm} = \frac{1}{2}\left(Z_{kl} \pm Z_{j4} \right)$$

where j, k, l is a cyclic permutation of 1,2,3. The following relations can be immediately verified :

$$\left[Z_j^{+}, Z_k^{-}\right] = 0$$

$$\left[Z_k^{\pm}, Z_l^{\pm}\right] = -\varepsilon_{klj} Z_j^{\pm}$$

The Lie algebra of the orthogonal group in a 4-dimensional euclidian space can be written as the direct sum of two Lie algebra, each of them being isomorphic to the Lie algebra of the orthogonal group in a 3-dimensional euclidian space.

As a consequence of this result, we have the evident isomorphism:

$$SO\ (4,R) \simeq SO(3,R) \otimes SO(3,R)$$

4. The homogeneous Lorentz group L is the pseudo-orthogonal group associated to a 4-dimensional vector space of signature S = 1, the Minkowski space. The connexion g is chosen so that $g_{11} = g_{22} = g_{33} = +1$ and $g_{00} = -1$. Under Lorentz transformations, the norm of each vector is invariant : $x_1^2 + x_2^2 + x_3^2 - x_0^2 = \text{Constant}.$

The Lie algebra of L is defined by the following commutation rules

$$[Z_{12}, Z_{23}] = -Z_{31} \ ; \quad [Z_{23}, Z_{31}] = -Z_{12} \ ; \quad [Z_{31}, Z_{12}] = -Z_{23}$$

$$[Z_{12}, Z_{01}] = -Z_{02} \ ; \quad [Z_{23}, Z_{02}] = -Z_{03} \ ; \quad [Z_{31}, Z_{03}] = -Z_{01}$$

$$[Z_{03}, Z_{23}] = -Z_{02} \ ; \quad [Z_{01}, Z_{31}] = -Z_{03} \ ; \quad [Z_{02}, Z_{12}] = -Z_{01}$$

$$[Z_{03}, Z_{01}] = +Z_{31} \ ; \quad [Z_{01}, Z_{02}] = +Z_{12} \ ; \quad [Z_{02}, Z_{03}] = +Z_{23}$$

Some particular sub-algebrae and sub-groups are evident from the previous equations and correspond to particular invariances:

a Z_{12}, Z_{23} , Z_{31} generate the Lie subalgebra of an orthogonal sub-group which leaves invariant the component x_0 and the space norm $x_1^2+x_2^2+x_3^2$.

b Z_{12}, Z_{01}, Z_{02}, generate the Lie sub algebra of a pseudo-orthogonal subgroup which leaves invariant the component x_3 and the quantity $x_1^2+x_2^2-x_0^2$.

c Z_{23}, Z_{02}, Z_{03} , in the same way generate a Lie subalgebra isomorphic to the previous one and the corresponding pseudo-orthogonal sub-group leaves invariant x_1 and $x_2^2+x_3^2-x_0^2$.

d Z_{31}, Z_{03}, Z_{01}; we have again the Lie subalgebra of a third pseudo-orthogonal sub-group and now time the invariant quantities are x_2 and $x_1^2+x_3^2-x_0^2$.

III. GROUP OF LINEAR TRANSFORMATIONS OF A VECTOR SPACE ON THE FIELD OF COMPLEX NUMBERS

We now introduce an n-dimensional vector space on the field of complex numbers E(n,C). A large part of the results previously obtained in a real vector space can easily be extended to a complex vector space. The hermitian product in E(n,C) is defined with a symmetrical antilinear connexion g which is a regular sesquilinear form in E(n,C)

Let us consider a Lie algebra, $\Lambda = \{X_\sigma\}$ on the real numbers with the commutation rules

$$[X_\sigma, X_\tau] = C^{\rho}_{\sigma\tau} X_\rho$$

If now the Lie algebra is defined on the complex numbers, it can be interesting to introduce its complex extension Λ^* as a new Lie algebra on the real numbers with infinitisimal generators X_σ and Y_σ satisfying.

$$[X_\sigma, X_\tau] = C^{\rho}_{\sigma\tau} X_\rho ; \; [X_\sigma, Y_\tau] = C^{\rho}_{\sigma\tau} Y_\rho , \; [Y_\sigma, Y_\tau] = -C^{\rho}_{\sigma\tau} X_\rho .$$

It is easy to verify that the complex extension of Λ^* is a direct sum of two Lie algebra isomorphic to Λ^*

$$(\Lambda^*)^* \simeq \Lambda^* \oplus \Lambda^*$$

A. General Linear Group GL(n,C):

1. The regular nxn matrices with complex coefficients generate the general linear group GL(n,C). The group of regular linear transformations in E(n,C) is isomorphic to

GL(n,C).

2. The Lie algebra of the general linear group GL(n,C) can be considered either as a Lie algebra on the complex numbers with the infinitesimal generators X_σ or a Lie algebra on the real number with the infinitesimal generators X_σ and Y_σ. The commutation laws of the complex extension of a real Lie algebra have been previously given and can also be directly obtained by using the method explained in the previous section for the real case

$$[X_{rs}, X_{tu}] = g_{st} X_{ru} - g_{ur} X_{tu}$$

$$[X_{rs}, Y_{tu}] = g_{st} Y_{ru} - g_{ur} Y_{tu}$$

$$[Y_{rs}, Y_{tu}] = -g_{st} X_{ru} + g_{ur} X_{tu}$$

3. The Lie algebra of the general linear group GL(n,C) is also the Lie algebra of the complex matrices m (n,C). The proof is identical to those obtained on the real case and a convenient basis will be

$$\left[E^{R}_{kl}\right]_{mn} = g_{mk} g_{nl} ; \quad \left[E^{I}_{kl}\right]_{mn} = i\, g_{mk} g_{nl} .$$

B. Special Linear Group SL(n,C):

1. The two operators $X = g^{sr} X_{rs}$ and $Y = g^{sr} Y_{rs}$ commute with all generators of the linear group GL(n,C). They generate a two parameter abelian group corresponding to complex dilatations of center the origin. This subgroup of GL(n,C) is isomorphic to the additive group C of complex numbers.

The factor group GL(n,C)/C is the special linear group SL(n,C). It can also be defined as the set of unimodular linear transformations in E(n,C) or, equivalently, as the set of the nxn unimodular matrices with complex coefficients. The number of independent real parameters is $2n^2 - 2$.

2. The Lie algebra of SL(n,C) is immediately defined by the infinitesimal generators

$$X'_{rs} = X_{rs} - \frac{1}{n} g_{rs} X$$

$$Y'_{rs} = Y_{rs} - \frac{1}{n} g_{rs} Y$$

The commutation laws are unchanged

C. Pseudo-Unitary Groups $U_s(n,C)$:

1. The hermitian product in E(n,C) is given by the antilinear connection

$$(x,y) = g(x,y) = \bar{x}^k g_{kl} y^l = \overline{(y,x)}$$

Let us call as A an arbitrary linear transformation in E(n,C). The conservation of the hermitian product under the transformation A is simply

$$(Ax, Ay) = (x,y)$$

This equality must be satisfied for all vectors in $E(n,C)$. The invariance property takes then the simple form

$$A^* g A = g$$

A matrix which satisfies the prévious equality is called an unitary matrix with respect to the connection g. The unitary matrices generate a subgroup of $GL(n,C)$, the pseudo-unitary group.

2. The connexion g is a symmetrical/sesquilinear regular form and can be diagonalized in the following way: $g_{ij} = \pm \delta_{ij}$ We will choose in $E(n,C)$, an orthogonalized basis such that:

$$g_{ij} = \delta_{ij} \qquad i = 1, 2, 3, ----- n-s$$

$$g_{ij} = -\delta_{ij} \qquad i = n-s+1, ------ n$$

The pseudo-unitary groups are characterized by the signature s and noted $U_s(n,C)$

In the particular case $s = 0$ (or $s = n$) the vector space is hermitian and the connection g can always be cho sen as the unit matrix $g \simeq I$. The unitary group $U(n,C)$ is the set of unitary nxn matrices $A^* A = I$.

3. The pseudo-unitary groups are subgroups of $GL(n,C)$. The Lie algebra of the pseudo-unitary group is a sub algebra of that of the complex linear group. The infinitesimal generators Z_{ij} can be written as linear combinations of the X_{ij} previously defined with complex coefficients.

$$Z_{ij} = \lambda_{ij}^{mn} X_{mn}$$

It is sufficient to impose the invariance of the norm of all vectors and the matrices λ_{ij} turn out to be antihermitian:
$\lambda^{mn}_{ij} + \bar{\lambda}^{nm}_{ij} = 0$; In F(n,C), it is possible to construct n^2 linearly independent nxn antihermitian matrices. The dimension of the Lie algebra of the pseudo-unitary groups is then n^2.

It is convenient to choose for the λ_{ij}

a $\frac{n(n-1)}{2}$ antisymmetric real matrices $E^R_{ij} - E^R_{ji}$

b $\frac{n(n+1)}{2}$ symmetric purely imaginary matrices $E^I_{ij} + E^I_{ji}$

and the infinitesimal generators can then be written as

$$Z_{ij} = -Z_{ji} = X_{ij} - X_{ji} .$$

$$Z^I_{ij} = +Z^I_{ji} = Y_{ij} + Y_{ji}$$

The commutation laws are the following:

$$[Z_{ij}, Z_{kl}] = g_{jk} Z_{il} - g_{ik} Z_{jl} - g_{jl} Z_{ik} + g_{il} Z_{jk} .$$

$$[Z_{ij}, Z^I_{kl}] = g_{jk} Z^I_{il} - g_{ik} Z^I_{jl} + g_{jl} Z^I_{ik} - g_{il} Z^I_{jk}$$

$$[Z^I_{ij}, Z^I_{kl}] = -g_{jk} Z_{il} - g_{ik} Z_{jl} - g_{jl} Z_{ik} - g_{il} Z_{jk} .$$

As a trivial consequence, the Z_{ij} generate a Lie sub-algebra isomorphic to the Lie algebra of the pseudo-orthogonal group $O_s(n,R)$.

4. The Lie algebra of the groups $GL(n,R)$ and $U_s(n,C)$ have the same complex extension which is the Lie algebra of $GL(n,C)$.

D. Special Pseudo-Unitary groups $SU_s(n,C)$

1. The operator $Z = g^{ij} Z^I_{ij}$ commutes with the n^2 infinitesimal generators Z_{ij} & Z^I_{ij}. It generates an one-parameter abelian group which is in fact a gauge group, all the components of a vector being multiplied by the same phase.

$$dx^k = \epsilon Z x^k = i \epsilon x^k \qquad \epsilon \text{ real}$$

This group is isomorphic to the one dimensional unitary group $U(1)$. The factor group.

$$U_s(n,C)/U(1) \qquad SU_s(n,C)$$

is the special pseudo-unitary group also defined as the set of unimodular nxn matrices with complex coefficients.

2. The Lie algebra of the unimodular pseudo-unitary groups is defined by n^2-1 infinitesimal generators

$$Z'_{ij} = Z_{ij} \;;\; Z'^I_{ij} = Z^I_{ij} - \frac{1}{n} g_{ij} Z$$

and the commutation laws are unchanged.

3. In an hermitian space where $g \simeq I$, the group of unimodular unitary matrices is the special unitary group $SU(n,C)$.

4. An inclusion which is a consequence of the explicit form given for the Lie algebra is the following

$$SO_s(n,R) \subset SU_s(n,C)$$

5. The Lie algebra of the groups $SL(n,R)$ and $SU_s(n,C)$ have the same complex extension which is the Lie algebra of the group $SL(n,C)$.

E. Complex Orthogonal Group $O(n,C)$

1. The pseudo-orthogonal group $O_s(n,R)$ is the group of linear transformations in $E(n,R)$ which leaves invariant the symmetrical bilinear form g.

In a complex vector space $E(n,c)$, the scalar product is now a complex number explicitly given by:

$$(x_1 + i x_2, y_1 + i y_2) = (x_1, y_1) - (x_2, y_2) + i(x_1, y_2) + i(x_2, y_1)$$

where each term is well defined in $E(n,R)$.

The group of linear transformations in $E(n,c)$ which leaves invariant this scalar product g, is the complex orthogonal group $O(n,C)$

2. This group can be considered as the complex extension of the pseudo-orthogonal groups $O_s(n,R)$. But with a convenient change of basis it is always possible now, in $E(n,C)$, to choose g as the unit matrix because of the definition of the scalar product. It follows that all pseudo-orthogonal groups $O_s(n,R)$ have the same complex extension $O(n,C)$.

3. The orthogonality condition can always be written as $A^T A = I$ and the infinitesimal generators of the Lie algebra are given by

$$Z_{ij} = X_{ij} - x_{ji} \; ; \quad Z^I_{ij} = Y_{ij} - Y_{ji}$$

Its dimensionality is simply n(n-1).

F. Applications:.

1. We first consider the unitary group U(2,C). The Lie algebra is spanned by four generators $Z_{12}, Z^I_{12}, Z^I_{11}, Z^I_{22}$, with the following commutation rules

$$[Z_{12}, Z^I_{11}] = -2 Z^I_{12} \; ; \quad [Z_{12}, Z^I_{22}] = 2 Z^I_{12} .$$

$$[Z^I_{12}, Z^I_{11}] = 2 Z_{12} \; ; \quad [Z^I_{12}, Z^I_{22}] = -2 Z_{12} .$$

$$[Z^I_{12}, Z_{12}] = Z^I_{22} - Z^I_{11} \; ; \quad [Z^I_{11}, Z^I_{22}] = 0 .$$

The linear combination $Z = Z^I_{11} + Z^I_{22}$ commutes with all the generators and can be associated to a gauge group U(1).

The Lie algebra of the special unitary group has therefore only three infinitesimal generators it is convenient to write in the form

$$M_1 = \frac{1}{2} Z^I_{12} \qquad M_2 = \frac{1}{2} Z_{12} \qquad M_3 = \frac{1}{4} (Z^I_{11} - Z^I_{22})$$

From the commutation relations given above, it is easy to deduce:

$$\left[M_i, M_j\right] = \varepsilon_{ijk} \quad M_k$$

The Lie algebra of SU(2,C) and the Lie algebra of the orthogonal group O(3,R) are two isomorphic three-parameters Lie algebra.

2. The groups SL(2,R) and $O_1(3,R)$ have two isomorphic Lie algebrae. Such an isomorphism remains true for the complex extensions and the groups SL(2,C) and the complex orthogonal group O(3,C) have also two isomorphic six-parameters Lie algebrae. By using the notations of the previous section the Lie algebra of SL(2,C) satisfies the following commutation rules:

$$[X^0, X^+] = X^- \ ; \quad [X^+, X^-] = -X^0 \ ; \quad [X^-, X^0] = -X^+$$

$$[X^0, Y^+] = Y^- \ ; \quad [X^+, Y^-] = -Y^0 \ ; \quad [X^-, Y^0] = -Y^+$$

$$[Y^0, X^+] = Y^- \ ; \quad [Y^+, X^-] = -Y^0 \ ; \quad [Y^-, X^0] = -Y^+$$

$$[Y^0, Y^+] = -X^- \ ; \quad [Y^+, Y^-] = X^0 \ ; \quad [Y^-, Y^0] = X^+$$

After comparison with relations written for the Lorentz group, we immediately see that the Lie algebra of SL(2,C), of O(4,C) and of the Lorentz group L are isomorphic.

3. The Lie algebra of the complex Lorentz group is isomorphic to the Lie algebra of the complex orthogonal group O(4,C). It has been shown in the previous section that the Lie algebra of the orthogonal group O(4,R). is the direct sum of two isomorphic Lie algebrae of the orthogonal group O(3, R). By using the results of the paragraph 3) it follows immediately that the Lie algebra of the complex Lorentz group is the direct sum of two isomorphic Lie algebrae of the real Lorentz group.

IV. GROUP OF LINEAR TRANSFORMATIONS OF A VECTOR SPACE ON THE FIELD OF QUATERNIONS

The complex numbers C can be considered as a 2-dimensional algebra on the field of real numbers R with the commutative multiplication law:

$$(a,b)\ (c,d) = (ac - bd\ ,\ ad + bc)$$

The quaternions Q can be defined as a 4-dimensional algebra on the field of real numbers R with the noncommutative multiplication law:

$$(a_0,\vec{a})\ (b_0,\vec{b}) = (a_0 b_0 - \vec{a}.\vec{b}\ ,\ a_0\vec{b} + b_0\vec{a} - \vec{a}x\vec{b})$$

A simple matrix representation of the quaternion $(a_0,\vec{a})$ can be realized with the help of the Pauli matrices: $(a_0,\vec{a}) = a_0 - i\vec{\sigma}\vec{a}$

The quaternions Q can also be considered as a 2-dimensional algebra on the field of complex numbers C with the multiplication law:

$$(x,y)(z,t) = (xz - \bar{y}t, \bar{x}t + yz)$$

An useful matrix representation of the quaternion (x, y) is then

$$(x,y) = \begin{vmatrix} x & \bar{y} \\ -y & \bar{x} \end{vmatrix}$$

In order to define the norm of a complex number, we first consider the complex conjugate $(a,b)^* = (a, -b)$ and the norm is simply

$$N^2 (a,b) = (a,b)^* (a,b) = (a^2 + b^2, 0)$$

For the quaternions, we proceed in the same way by introducing the hermitic conjugate $(a_0,\vec{a})^* = (a_0 . - \vec{a})$ and the norm is given by:

$$N^2 (a_0,\vec{a}) = (a_0,\vec{a})^* (a_0,\vec{a}) = (a_0^2 + \vec{a}^2 , 0)$$

In the language with the complex numbers, we obtain $(x,y) = (\bar{x} , -y)$ and the norm takes the simple form

$$N^2 (x,y) = (x,y)^* (x,y) = (x\bar{x} + y\bar{y}, 0)$$

The quaternionic product of two quaternions q_1 and q_2 will be defined by the quaternion $q_{12} = q_1^* q_2 = q_{21}^*$. By using the previous forms for the quaternions, we find

$$(a_0,\vec{a})^* (b_0,\vec{b}) = (a_0 b_0 + \vec{a}\, \vec{b}, a_0\vec{b} - b_0\vec{a} + \vec{a}x\vec{b})$$

$$(x,y)^*(z,t) = (\bar{x}z + \bar{y}t, xt - yz)$$

We now introduce a n-dimensional space on the field of the quaternions E(n, Q). The quaternion product of two vectors in E(n, Q) is defined with the **self conjugate** regular form g. $(g = g^*)$.

The quarternionic extension Λ^Q of a Lie algebra $\Lambda = \{X_\sigma\}$ defined on the real numbers can be also considered as a Lie algebra on the real numbers with the infinitesimal generators $X_\sigma, Y_\sigma^1, Y_\sigma^2, Y_\sigma^3$. Of course, the three complex extensions $\Lambda^* = \{X_\sigma, Y_\sigma^\alpha\}$ are isomorphic.

A. Linear Groups:

1. The regular nxn matrices with quarternionic coefficients generate the general linear group GL(n,Q). The group of regular linear transformations in F(n,Q) is isomorphic to GL(n,Q).

2. The Lie algebra of GL(n,Q) is the quaternionic extension of the Lie algebra of the real general linear group GL(n,R). It can also be regarded as the Lie algebra of the matrices $m(n,Q)$. The dimension of the Lie algebra is $4n^2$ and the commutation relations are given by:

$$[X_{jk}, X_{lm}] = g_{kl} X_{jm} - g_{mj} X_{lk}$$

$$[X_{jk}, Y_{lm}^\alpha] = g_{kl} Y_{jm}^\alpha - g_{mj} Y_{lk}^\alpha.$$

$$[Y_{jk}^\alpha, Y_{lm}^\alpha] = -g_{kl} X_{jm} + g_{mj} X_{lk}.$$

$$[Y_{jk}^\alpha, Y_{lm}^\beta] = -\varepsilon^{\alpha\beta}{}_\gamma \{g_{kl} Y_{jm}^\gamma + g_{mj} Y_{lk}^\gamma\}$$

3. The operator $X = g^{sr} X_{rs}$ commutes with all generators of the linear group and generates an one dimensional abelian sub-algebra.

B. Pseudo Symplectic Groups $Sp_s(n,Q)$:

1. The quaternionic product of two vectors in $E(n,Q)$ is a quaternion given by the connexion g:

$$(u,v) = g(u,v) = u^{*k} g_{kl} v^{l} = (v,u)^{*}$$

Let us call as A an arbitrary linear transformation in $E(n,Q)$. The conservation of the quaternionic product under the transformation A is simply,

$$(Au, Av) = (u,v)$$

The equality must be satisfied for all vectors in $E(n,Q)$ and the invariance property takes the simple form:

$$\tilde{A} \; g \; A = g$$

The matrix $\tilde{A}$ is defined by $(\tilde{A})_{ij} = (A_{ji})^{*}$. The matrices A which satisfy the previous equality are called symplectic matrices with respect to the connexion g. They generate a sub-group of $GL(n,Q)$, the pseudo symplectic group.

2. As previously we introduce the signature **s** of the vector space $E(n,Q)$ and the pseudo symplectic groups will be noted $Sp_s(n,Q)$.

3. The Lie algebra is a sub-algebra of the Lie algebra of the general linear group $GL(n,Q)$. The infinitesimal generators

Z_{ij} can be written as linear combinations with quaternionic coeffcients of the X_{ij} defined in section II,

$$Z_{ij} = \lambda_{ij}^{mn} X_{mn}.$$

It is sufficient to impose the invariance of the norm of all vectors and the matrix elements λ_{ij}^{mn} must satisfy the requirement

$$\lambda_{ij}^{mn} + \lambda_{ij}^{nm\,*} = 0$$

The infinitesimal generators can then be written as:

$$Z_{ij} = -Z_{ji} = X_{ij} - X_{ji}$$

$$Z^{\alpha}_{ij} = Z^{\alpha}_{ji} = Y^{\alpha}_{ij} + Y^{\alpha}_{ji} \qquad \alpha = 1,2,3.$$

The dimension of the Lie algebra of the pseudo-symplectic groups is then n(2n + 1)

The commutation laws can easily be written in the following form:

$$[Z_{ij}, Z_{kl}] = g_{jk} Z_{il} - g_{ik} Z_{jl} - g_{jl} Z_{ik} + g_{il} Z_{jk}.$$

$$[Z_{ij}, Z^{\alpha}_{kl}] = g_{jk} Z^{\alpha}_{il} - g_{ik} Z^{\alpha}_{jl} + g_{jl} Z^{\alpha}_{ik} - g_{il} Z^{\alpha}_{jk}$$

$$[Z^{\alpha}_{ij}, Z^{\alpha}_{kl}] = -g_{jk} Z_{il} - g_{ik} Z_{jl} - g_{jl} Z_{ik} - g_{il} Z_{jk}$$

$$[Z^{\alpha}_{ij}, Z^{\beta}_{kl}] = \varepsilon^{\alpha\beta}{}_{\gamma} \left\{ -g_{jk} Z^{\gamma}_{il} - g_{ik} Z^{\gamma}_{jl} - g_{jl} Z^{\gamma}_{ik} - g_{il} Z^{\gamma}_{jk} \right\}$$

As a trivial consequence, the Z_{ij}'s generate a Lie algebra isomorphic to the Lie algebra of the pseudo-orthogonal group $O_s(n,R)$ and the three isomorphic Lie algebrae $\{Z_{ij}, Z^{\alpha}_{ij}\}$ are isomorphic to the Lie algebra of the pseudo unitary group $U_s(n,C)$.

4. It is also extremely useful to represent the quaternion q by a set of two complex numbers (x, y). The components q^j of a vector q in $E(n,Q)$ can be considered as the components (x^j, y^j) of a vector X in $E(2n,\mathbf{C})$ and we define

$$X^j = x^j \; ; \quad X^{n+j} = y^j \qquad j = 1,2,3,\dots n$$

Let us now consider two vectors u and v of $E(n,Q)$; they can be associated to two vectors X and Y of $E(2n,C)$ by

$$u^j = (X^j, X^{n+j}) \; ; \quad v^k = (Y^k, Y^{n+k})$$

The quaternionic product $g(u,v)$ is defined by

$$g(u,v) = g_{jk}\, u^{j*} v^k$$

and in terms of X and Y we obtain for the quaternion (u,v) the form

$$g(u,v) = g_{jk}\left(\bar{X}^j Y^k + \bar{X}^{n+j} Y^{n+k},\; X^j Y^{n+k} - X^{n+j} Y^k\right)$$

We now introduce, in $E(2n,C)$, an antilinear symmetrical connection G^+ and a linear antisymmetrical connection G^- defined by the reduced form:

$$G^+ = \begin{vmatrix} g & 0 \\ 0 & g \end{vmatrix} \qquad G^- = \begin{vmatrix} 0 & g \\ -g & 0 \end{vmatrix}$$

and the quaternion $g(u,v)$, can then be written as:

$$g(u,v) = \left(G^+(x,y) \, , \, G^-(x,y) \right)$$

The linear group $GL(n,Q)$ is a subgroup of the linear group $GL(2n,C)$ in an evident way. The pseudo sympletic group $Sp_s(n,Q)$ can also be defined as the set of linear transformations in $E(2n,C)$ which leave invariant the two connections G^+ and G^-.

In an equivalent way, the group $Sp_s(n,Q)$ is the subgroup of the pseudo-unitary group $U_{2s}(2n,C)$ which conserves the antisymmetrical bilinear form G^-.

5. We first consider the 2n dimensional vector space $E(2n,R)$ with the connection $G^{\pm} = \begin{vmatrix} g & o \\ o & g \end{vmatrix}$; the components of a vector X are noted with the two sets on indices $j = 1,2, \ldots\ldots n$ and $n + j$.

The general linear group $GL(2n, R)$ acting in $E(2n,R)$ depends of $4n^2$ parameters. The infinitesimal generators of the Lie algebra will be divided into four sets of n^2 generators X_{ij}, $X_{i,n+j}$, $X_{n+i,j}$, $X_{n+i,n+j}$.

The sub-group of $GL(2n,R)$ which conserves the linear symmetrical connection G^+ is the pseudo-orthogonal group $O_{2s}(2n,R)$ which depends on $n(2n-1)$ parameters.

We will call real pseudo-sympletic group $Sp_{2s}(2n,R)$

the sub-group of GL(2n,R) which conserves the linear anti-symmetrical form G^{-}. The infinitesimal generators of the real pseudo sympletic group are given by:

$$A_{ij} = X_{ij} - X_{n+j\; n+i}$$

$$B_{ij} = X_{n+i\; j} + X_{n+j\; i} = B_{ji}$$

$$C_{ij} = X_{i\; n+j} + X_{j\; n+i} = C_{ji}$$

The commutations laws are given by

$$[A_{ij}, A_{kl}] = g_{jk} A_{il} - g_{li} A_{kj}$$

$$[B_{ij}, B_{kl}] = 0 = [C_{ij}, C_{kl}]$$

$$[A_{ij}, B_{kl}] = -g_{ik} B_{jl} - g_{il} B_{jk}$$

$$[A_{ij}, C_{kl}] = g_{jk} C_{il} + g_{jl} C_{ik}$$

$$[B_{ij}, C_{kl}] = -g_{jk} A_{il} - g_{ik} A_{jl} - g_{jl} A_{ik} - g_{il} A_{jk}$$

The n^2 generators A_{ij} define a Lie sub-algebra isomorphic to the Lie algebra of GL(n,R). The $\frac{n(n+1)}{2}$ generators B_{ij} and the $\frac{n(n+1)}{2}$ generators C_{ij} define two abelian Lie sub-algebras.

The dimension of the Lie algebra of the real pseudo-sympletic groups $Sp_{2s}(2n,R)$ is $\underline{n(2n + 1)}$.

The sub group of $GL(2n,R)$ which leaves invariant the connections G^+ and G^-, is the intersection of the groups $O_{2s}(2n,R)$ and $Sp_{2s}(2n,R)$. The infinitesimal generators are immediately known by the antisymmetry condition .

$$Z_{ij} = A_{ij} - A_{ji} \qquad \bar{Z}_{ij} = B_{ij} - C_{ij}$$

The dimension of the Lie algebra is $\underline{n}^2$ and the commutation relations are given from the previous expressions by:

$$[Z_{ij}, Z_{kl}] = g_{jk} Z_{il} - g_{ik} Z_{jl} - g_{jl} Z_{ik} + g_{il} Z_{jk}$$

$$[Z_{ij}, \bar{Z}_{kl}] = g_{jk} \bar{Z}_{il} - g_{ik} \bar{Z}_{jl} + g_{jl} \bar{Z}_{ik} - g_{il} \bar{Z}_{jk} .$$

$$[\bar{Z}_{ij}, \bar{Z}_{kl}] = - g_{jk} Z_{il} - g_{ik} Z_{jl} - g_{jl} Z_{ik} - g_{il} Z_{jk}$$

This Lie algebra is isomorphic to the Lie algebra of the pseudo unitary group $U_s(n,C)$.

6. We now introduce the 2n-dimensional complex space $E(2n,C)$ with the antilinear connexion G^*. The general linear group $GL(2n,C)$ acting on $E(2n,C)$ depends on $8n^2$ parameters and the Lie algebra is the complex extension $\{X, Y\}$ of the Lie algebra of $GL(2n,R)$.

The sub group of GL(2n,C) which conserves the antilinear symmetric connection G^+ is the pseudo unitary group $U_{2s}(2n,C)$ which depends on <u>$4n^2$</u> parameters.

We will call complex pseudo symplectic group $Sp_{2s}(2n,C)$ the sub group of GL(2n,C) which conserves the linear anti-symmetrical form G^-. The Lie algebra of $Sp_{2s}(2n,C)$ is the complex extension of the Lie algebra of $Sp_{2s}(2n,R)$ and is defined by the <u>$2n(2n+1)$</u> infinitesimal generators A, B, C, $\bar{A}$, $\bar{B}$, $\bar{C}$.

The sub group of GL(2n,C) which leaves invariant the connexions G^+ and G^- is the intersection of the pseudo-unitary group $U_{2s}(2n,C)$ and of the pseudo-symplectic group Sp $Sp_{2s}(2n,R)$. It is the pseudo-symplectic group $Sp_s(n,Q)$ previously defined. The infinitesimal generators are immediately given by the linear combinations:

$$A_{ij} - A_{ji} \ , \quad B_{ij} - C_{ij} \ , \quad \bar{A}_{ij} + \bar{A}_{ji} \ , \quad \bar{B}_{ij} \quad + \bar{C}_{ij}$$

The dimension of this Lie algebra is <u>$n(2n+1)$</u> and this value agrees with the previously obtained result.

7. The Lie algebra of the groups $Sp_{2s}(2n,R)$ and $Sp_s(n,Q)$ have the same complex extension which is the Lie algebra of the group $Sp_{2s}(2n,C)$.

Chapter 3

TOPOLOGICAL PROPERTIES

I. Compact Lie Groups:

1. Definition: In a compact space, any infinite sequence has its bound on the space

2. All the coefficients of an unitary matrix are bounded by the unity,

3. The unitary group U(n) is then a compact Lie group. It follows immediately that SU(n), O(n) and SO(n) are also compact Lie groups.

4. The symplectic group Sp(n,Q) is a closed subset of U(2n) and therefore it is a compact group.

II. Connected Lie Groups:

We give briefly some definitions and some properties in order to characterize a Lie group from a topological point of view.

1. Path:

Let us consider two points a and b in G. A path from a to b in G is described by a continuous function f(t) defined on the closed interval $0 \leq t \leq 1$ and such that

$$f(0) \Rightarrow a \quad f(1) \Rightarrow b$$

$$f(t) \Rightarrow r \quad \text{and} \quad r \in G \text{ for all } 0 \leq t \leq 1$$

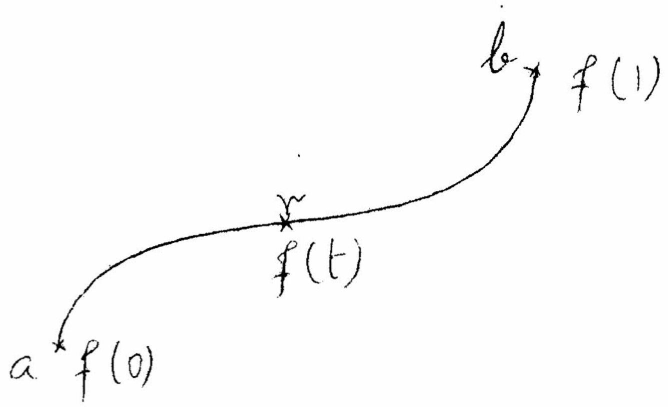


Figure 1
Path.

The existence of a path between a and b can be used to define an equivalence in G between a and b, such a property being reflexive, symmetric and transitive.

2. Connected Lie group:

A topological space is connected if it cannot be considered as the union of two non empty open subsets. We introduce a partition in G by using the equivalence defined above. If there exist a path joining two points a and b of G, these two points belong to the same equivalence class S_a which is also called the component of a.

For an analytic manifold, it is easy to see that the S_a's are open sets. From the previous definition, we obtain the sufficient and necessary condition: a Lie group is connected if and only if one can find a point a in G which can be joined to any arbitrary other point b of G by a path.

If a Lie group G is non-connected, only the identity component S_e can be a sub group. It is an invariant subgroup also called the connected component of G.

3. Homotopy:

Let us consider two paths $f_1(t)$ and $f_2(t)$ joining two points a and b of G

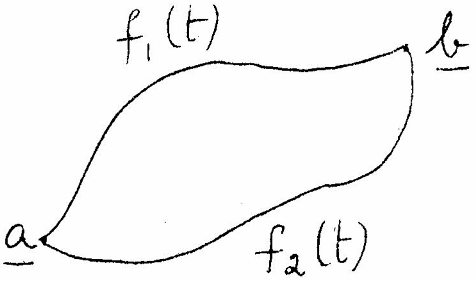


Figure 2.

The paths f_1 and f_2 are homotopic if f_1 can be continuously deformed into f_2 , the end points a and b remaining fixed.

The notion of homotopy allows to define an equivalence between two paths and to divide the paths into homotopy classes.

4. Simply Connected Lie groups:

A connected Lie group G is simply connected if the homotopy classes reduce to the identity. In a simply connected group, all the paths joining two points of G are equivalent.

5. Examples:

As an illustration of the previous definitions, we give, without proof the following important results.

a The real orthogonal group $O(n,R)$ is not connected; the two equivalence classes are characterized by $\det A = \pm 1$. The identity component is the connected special orthogonal group $SO(n,R)$.

b The Lorentz group L is not connected; the four equivalence classes are characterized by $\det A = \pm 1$ and $A_{oo} \geqslant 1$ or $A_{oo} \leqslant -1$. The identity component is $L^{\uparrow}_{+}$

The complex Lorentz group, isomorphic to the complex orthogonal group $O(4,C)$ is a 2-connected group

c The special unitary group $SU(n,C)$ is simply connected but the special orthogonal group $SO(n,R)$ is not simply connected.

III. Universal Covering Group:

1. The Lie algebra of a Lie group is uniquely defined but the converse is not true.

If the Lie algebrae g_1 and g_2 of two Lie groups G_1 and G_2 are isomorphic the Lie groups are only locally isomorphic.

2. To each Lie algebra of finite dimension on the real numbers, there corresponds an uniquely determined, connected, simply connected Lie group, called the universal covering group G^*.

3. All connected Lie groups G, locally isomorphic to G^* can be obtained from G^* with a covering homomorphism.

The kernel D of such a homomorphism is a discrete invariant subgroup of G^* and G^* being a connected group, D is a sub group of the center Z of G^*

$$G^*/D \simeq G \text{ with } D \subset Z(G^*)$$

4. Ado's theorem:

A Lie algebra of finite dimension on the real numbers is ismorphic to a sub-algebra of the Lie algebra of a general linear group GL(n,R) for a convenient value of n.

It follows that to each Lie algebra Λ of finite dimension on the real numbers corresponds a connected Lie group of Lie algebra Λ , which is an analytic subgroup of GL(n,R).

5. Let us consider the direct sum g of two Lie algebrae g_1 and g_2

$$g = g_1 \oplus g_2$$

The universal covering group of g is the direct product of the universal covering groups G_1^* and G_2^* of g_1 and g_2 :

$$G^* = G_1^* \otimes G_2^*$$

The center of G^* contains the direct product of the centers of G_1^* and G_2^* but in general, $Z(G^*)$ is much larger than this direct product

$$Z(G^*) \supset Z(G_1^*) \otimes Z(G_2^*)$$

6. Examples:

a We first consider the one parameter Lie algebra A_0. In an evident way, its universal covering group is the abelian additive group of the real numbers R. The mapping $\alpha \Rightarrow \exp(2i\pi\alpha)$ where $\alpha \in R$ is a covering homomorphism of R into the one dimensional unitary group $U(1)$ Due to the property $\exp(2i\pi n) = 1$ if n is an integer number, the kernel of the covering homomorphism is the discrete additive subgroup of the integer numbers N:

$$R/N \simeq U(1)$$

It can be easily seen that all the discrete subgroups of R are isomorphic to N and the only connected Lie groups of Lie algebra Λ_0 are R and $U(1)$.

b The unimodular unitary group $SU(n, C)$ is a connected simply connected group. It is therefore the universal covering group of its Lie algebra.

The center of $SU(n, C)$ is the set of all nxn unitary matrices. The general form is then ωI_n where I_n is the nxn unit matrix. The constant ω is restricted by the condition $\omega^n = 1$. It follows that the center Z_n of $SU(n, C)$ is isomorphic to the cyclic group of the roots of order n of the unity ; Z_n is also isomorphic to the integer numbers modulo n. If n is a prime number, Z_n has no subgroup besides the identity and itself. If now, n can be written as a product of two integers $n = pq$, the groups Z_p and Z_q are subgroups of Z_n.

For instance the connected groups associated to the SU(2,C) Lie algebra are SU(2,C) and the factor group SU(2,C)/Z_2 which, as it will be seen later, is isomorphic to SO(3,R).

In the case $n = 3$, we have the two connected locally isomorphic groups SU(3,C) and SU(3,C/Z_3. In the case $n = 6$ we have four connected Lie groups: SU(6,C), SU(6,C)/Z_2, SU(6,C)/Z_3 and SU(6,C)/Z_6.

Chapter 4.

LIE ALGEBRA OF THE SEMI SIMPLE GROUPS

I. Standard Form

1. The eigenvalue problem:

Let us call as X_σ the r infinitesimal generator of a Lie algebra Λ. We define as $A = a^\sigma X_\sigma$ an infinitesimal operator and we consider the eigenvalue problem defined by the equation

$$[A, X] = S \quad X$$

The eigenvector X associated to the eigenvalue S is an element of Λ: $X = x^\rho X_\rho$ and S is in general a complex number. The basic equation can then be written

$$a^\sigma x^\rho [X_\sigma, X_\rho] = S \; x^\tau X_\tau$$

Taking into account the communtation relations of the Lie algebra we obtain

$$[a^\sigma x^\rho c_{\sigma\rho}{}^\tau - S x^\tau] X_\tau = 0$$

The bracket is zero, because of the completeness of the X_τ basis in Λ.

$$(a^\sigma c_{\sigma\rho}{}^\tau - S \delta_\rho{}^\tau) x^\rho = 0$$

We have a system of r homogeneous linear equations with respect to the r quantities x^ρ. Besides the trival solution $x^\rho = 0$, we have a non zero solution if and only if the determinant of the coefficient vanishes:

$$\det - (a^{\sigma} c_{\sigma\rho}{}^{\tau} - S\, \delta_{\rho}{}^{\tau}) x^{\rho} = 0$$

This condition is an algebraic equation of degree r in the variable S and we have r roots, real or complex, degenerate or not. To each root corresponds an eigenvector. For a **semi** simple Lie algebra, Cartan has obtained extremely important results. If the operator A is choosen so that the equation in S has the maximum number of different roots:

<u>a</u> The root $S = 0$ is degenerate with the multiplicity ℓ and ℓ is called the rank of the semi-simple group

<u>b</u>. All the non zero roots are non degenerate.

2. Fundamental Relations

We first define our notations. A greek index ρ, σ, τ refers to an arbitrary component of the Lie algebra. For the generators E_α associated to non-zero roots we use the greek indices α, β, γ. and for the generators H_j associated to zero root we use the latin indices j, k .

We are now working with the two results obtained by Cartan for a rank ℓ semi simple group:

<u>a</u> The root zero is degenerate with the multiplicity ℓ

$$[A, H_j] = 0$$

b The non zero roots α are non degenerate

$$[A, E_{\alpha}] = \alpha E_{\alpha}$$

As an evident consequence of the first equality, A is an eigen-vector with the eigenvalue zero, it can then be written as a linear combination of the H_j ; $A = \lambda^{j} H_j$ and the generators H_j generate a abelian sub-algebra called the Cartan algebra which is maximal

$$[H_j, H_k] = 0 \quad \text{or} \quad C_{jk}{}^{\rho} = 0$$

We now use the Jacobi identity for the three operators, A, H E_{α} of the Lie algebra

$$[[A, H_j], E_{\alpha}] + [[H_j, E_{\alpha}], A] + [[E_{\alpha}, A], H_j] = 0$$

By using the properties a) and b) we obtain

$$[A, [H_j, E_{\alpha}]] = \alpha [H_j, E_{\alpha}]$$

which shows that $[H_j, E_{\alpha}]$ is an eigenvector corresponding to the non degenerate eigenvalue α. It follows that this vector must be propertional to E_{α}.

$$[H_j, E_{\alpha}] = \alpha_j E_{\alpha} \quad \text{or} \quad C_{j\alpha}{}^{\rho} = \alpha_j \quad \alpha \, \delta_{\alpha}^{\rho}$$

After comparison with the eigenvalue equation b), we deduce the relation

$$\alpha = \lambda^{j} \alpha_j$$

We now define a ℓ -dimensional vector space $\mathcal{E}_\ell$ associated to the Cartan sub-algebra. The quantity α can be considered as a vector in $\mathcal{E}_\ell$ with covariant components α_j, in the same way, λ can be considered as a vector in $\mathcal{E}_\ell$ with contravariant component λ^j. It will be also useful in the following to consider the H_j's as the covariant components of a vector H in $\mathcal{E}_\ell$.

We apply again the Jacobi identity with the three generators A, E_α, E_β.

$$[[A, E_\alpha], E_\beta] + [[E_\alpha, E_\beta], A] + [[E_\beta, A], E_\alpha] = 0$$

We use the eigenvalue equation b) and obtain:

$$[A, [E_\alpha, E_\beta]] = (\alpha + \beta)[E_\alpha, E_\beta]$$

Three cases are possible:

<u>a</u> $(\alpha + \beta)$ is not a root and the operators E_α and E_β commute.

<u>b</u> $(\alpha + \beta) \neq 0$ is a root the commutation $[E_\alpha, E_\beta]$ is proportional to the operator $E_{\alpha+\beta}$

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} \, E_{\alpha+\beta} \qquad \text{or} \qquad C_{\alpha\beta}{}^{\rho} = N_{\alpha\beta}\,\delta^{\rho}_{\alpha+\beta}$$

<u>c</u> $\alpha + \beta = 0$ the commutator $[E_\alpha, E_{-\alpha}]$ is a eigenvector associated to the eigenvalue zero and can be written as a linear combination of the operators H_j

$$[E_\alpha , E_{-\alpha}] = C_{\alpha\,-\alpha}{}^{j} \, H_j$$

3. Theorem:

To each root α, it corresponds the root $-\alpha$. The proof of this theorem is based on the Cartan criterion for semi simple groups previously given.

We consider the element $g_{\alpha\tau}$ of the row α of the Cartan tensor

$$g_{\alpha\tau} = C_{\alpha\rho}{}^{\sigma} \; C_{\tau\sigma}{}^{\rho}$$

With the previous expressions obtained for the structure constants, $g_{\alpha\tau}$ becomes

$$g_{\alpha\tau} = -\alpha_j C_{\tau\alpha}{}^{j} + N_{\alpha\beta} C_{\tau\;\alpha+\beta}{}^{\beta} + C_{\alpha\;-\alpha}{}^{j} \; C_{\tau j}{}^{\alpha}$$

The three terms are non-vanishing if and only if τ can take the value $\tau = -\alpha$. The Cartan criterion is satisfied if and only if $-\alpha$ is a root and the only element of the row α different from zero is then $g_{\alpha\;-\alpha}$

For a Lie algebra of rank ℓ and dimension γ, there exists $\gamma - \ell$ non degenerate and non vanishing roots. From the previous result $\gamma - \ell$ is an even integer.

4. Cartan tensor

The normalization of the operators E_{α} can be choosen so that

$$g_{\alpha\;-\alpha} = 1$$

and the Cartan tensor takes the simple structure

$$g_{\rho\sigma} = \left|\begin{array}{c|cc|cc|cc}
g_{jk} & & & & 0 & & \\ \hline
 & 0 & 1 & & & & \\
 & 1 & 0 & & & 0 & \\ \cline{2-5}
 & & & 0 & 1 & & \\
0 & & & 1 & 0 & & \\ \cline{4-7}
 & & & & & 0 & 1 \\
 & & 0 & & & 1 & 0
\end{array}\right|$$

We have

$$\det g_{\rho\sigma} = \det g_{jk} (-1)^{\frac{r-\varphi}{2}}$$

From the certain criterion, it follows that g_{jk} is a regular matrix

$$\det g_{jk} \neq 0$$

Of course, such a result is independent of the normalization condition.

By using the definition of the Cartan tensor, we obtain an explicit expression for g_{jk}

$$g_{jk} = C_{j\alpha}^{\alpha} C_{k\alpha}^{\alpha} = \sum_{\alpha} \alpha_j \alpha_k = g_{kj}$$

The matrix g_{jk} is symmetrical and will be called simply g in the following.

5. Vector Space $\mathcal{E}_\ell{}'$:

The matrix g is used to define, in $\mathcal{E}_\ell$ a linear symmetrical connection. We introduce the inverse matrix g^{jk}

$$g^{jk} g_{km} = \delta_m^{\;j}$$

to write the scalar product into the form

$$(\alpha, \beta) = g(\alpha, \beta) = g^{jk} \alpha_j \beta_k = (\beta, \alpha)$$

The contravariant components of a vector α are given by

$$\alpha^j = g^{jk} \alpha_k$$

and the scalar product takes an equivalent form

$$(\alpha, \beta) = \alpha^j \beta_j = \alpha_k \beta^k$$

As an interesting consequence, we obtain

$$g^{jk} g_{jk} = \sum_\alpha g^{jk} \alpha_j \alpha_k = \sum_\alpha (\alpha, \alpha)$$

and

$$\sum_\alpha (\alpha, \alpha) = \ell$$

6. Commutation Relations

We now show that the contravariant components α^j are identical with the $C_{\alpha\,-\alpha}^{\;\;j}$. The proof uses essentially the antisymmetry property of the structure constants.

$$C_{\alpha\,-\alpha}^{\;\;j} = g^{jk} C_{\alpha\,-\alpha\,k} = g^{jk} C_{k\alpha\,-\alpha} = g^{jk} g_{-\alpha\beta} C_{k\alpha}^{\;\;\beta}$$

with the normalization condition $g_{-\alpha\beta} = \delta_{\beta\alpha}$, it follows immediately.

$$C_{\alpha -\alpha}{}^{j} = g^{jk}\alpha_{k} = \alpha^{j}$$

The commutation relation becomes

$$[E_{\alpha} , E_{-\alpha}] = \alpha^{j} H_{j}$$

It is extremely easy to deduce now a Lie sub-algebra, generated by E_{α} , $E_{-\alpha}$ and the linear combination $\alpha^{j} H_{j}$ one can write as a scalar product (α, H) ; we obtain

$$[E_{\alpha} , E_{-\alpha}] = (\alpha, H)$$

$$[(\alpha, H), E_{\alpha}] = (\alpha, \alpha) E_{\alpha}$$

This sub algebra is isomorphic to a SU(2) Lie algebra and corresponds to the sequence α, 0, $-\alpha$ of the roots

7. Lemme

If α, β, γ are three non zero roots such that $\alpha + \beta + \gamma = 0$ we have $N_{\alpha\beta} = N_{\beta\gamma} = N_{\gamma\alpha}$.

We use the Jacobi identity:

$$[E_{\alpha}, [E_{\beta} E_{\gamma}]] + [E_{\beta}, [E_{\gamma} , E_{\alpha}]] + [E_{\gamma} [E_{\alpha} , E_{\beta}]] = 0$$

and the commutation relations allow us to transform this equality into:

$$(\alpha, H) N_{\beta\gamma} + (\beta, H) N_{\gamma\alpha} + (\gamma , H) N_{\alpha\beta} = 0$$

The operators of the Cartan sub algebra are linearly independent and each component j of $\alpha\beta\gamma$ are solutions of the system:

$$\begin{cases} \alpha^j N_{\beta\gamma} + \beta^j N_{\gamma\alpha} + \gamma^j N_{\alpha\beta} = 0 \\ \alpha^j + \beta^j + \gamma^j = 0 \end{cases}$$

It can be easily seen that the only possibility to obtain non-zero roots α, β, γ is:

$$N_{\beta\gamma} = N_{\gamma\alpha} = N_{\alpha\beta}$$

8. Structure Constants

The structure constants $N_{\alpha\beta}$ are antisymmetric in the exchange of the two indices

$$N_{\alpha\beta} + N_{\beta\alpha} = 0$$

Let us apply the previous lemma for three non vanishing roots $-\alpha$, $\alpha + \beta$ and $-\beta$

$$N_{-\alpha, \alpha+\beta} = N_{\alpha+\beta, -\beta} = N_{-\beta, -\alpha}$$

Because of the symmetry $\alpha \longleftrightarrow -\alpha$ in the set of the roots, it is always possible to choose, for the operators E_α a normalization so that :

$$N_{-\beta, -\alpha} = N_{\alpha\beta}$$

Another relation between the structure constants is given by the normalization condition of the Cartan tensor. By using the previous symmetrics on the structure constants, we easily deduce

$$g_{\alpha\,-\alpha} = 1 = 2(\alpha,\alpha) + \sum_{\beta \neq -\alpha} N_{\alpha\beta}^{2}$$

II. Properties of the Roots

1. Theorem

If α β are two arbitrary roots

a the number $2\dfrac{(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer called a Cartan integer;

b the vector $\beta - 2\dfrac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha$ is also a root deduced from β by symmetry withrespect to the hyper plane through the origin perpendicular to α.

The proof of this fundamental theorem will be given into two steps. Let us first consider a root γ such that $\alpha + \gamma$ is not a root:

$$[E_\gamma, E_\alpha] = 0$$

We introduce the sequence

The only a finite number of generators E_{α} and the sequence of the X operators must also be finite:

$$X_{\gamma-(g+1)\alpha} = 0 = [X_{\gamma-g\alpha}, E_{-\alpha}]$$

These formulae can be inverted following:

$$\left\{\begin{array}{l} [X_{\gamma-(p+1)\alpha}, E_{\alpha}] = \mu_{p+1} X_{\gamma-p\alpha} \\ - - - - - - - - - - \rightarrow \\ [X_{\gamma-\alpha}, E_{\alpha}] = \mu_1 E_j \end{array}\right.$$

and, with the previous assumptions $\mu_0 = 0$

We now write the Jacobi identity for the three operators E_{α}, $E_{-\alpha}$ and $X_{\gamma-p\alpha}$:

$$[[E_{\alpha} . E_{-\alpha}], X_{\gamma-p\alpha}] + [[E_{-\alpha}, X_{\gamma-p\alpha}], E_{\alpha}] + [[X_{\gamma-p\alpha}, E_{\alpha}], E_{-\alpha}] = 0$$

By using the commutations relations this relation becomes:

$$\alpha^{j} [H_j, X_{\gamma-p\alpha}] - [X_{\gamma-(p+1)\alpha}, E_{\alpha}] + \mu_p [X_{\gamma-(p-1)\alpha}, E_{-\alpha}] = 0$$

and finally:

$$(\alpha . (\gamma-p\alpha)) X_{\gamma-p\alpha} - \mu_{p+1} X_{\gamma-p\alpha} + \mu_p X_{\gamma-p\alpha} = 0$$

We have obtained a recurrence formula for μ_p:

$$\mu_{p+1} = \mu_p + (\alpha,\gamma) - p(\alpha,\alpha)$$

Taking into account $\mu_0 = 0$, we deduce the explicit expression for μ_p

$$\mu_p = p(\alpha,\gamma) - \frac{p(p-1)}{2}(\alpha,\alpha)$$

The quantity g is defined by $\mu_{g+1} = 0$ and with the previous relation we find the values of g :

$$g = 2\frac{(\alpha,\gamma)}{(\alpha,\alpha)}$$

The theorem is now proved in the particular case where the sum $\alpha + \gamma$ of two roots is not a root. The quantity g is an integer and there exists a set of roots:

$$\gamma, \gamma-\alpha, \gamma-2\alpha, \ldots \quad \gamma - g\alpha = \gamma - 2\frac{(\gamma\cdot\alpha)}{(\alpha\cdot\alpha)}\alpha$$

We go back to the general case where $\alpha + \beta$ can be a root. We define as m and n , two positive integers such that $\beta + k\alpha$ is a root if and only if the algebraic integer k satisfies $-m \leq k \leq n$. The previous results can be used with the root $\gamma = \beta + n\alpha$. The value of g is simply $g = m+n$ and we obtain.

$$\underline{a} \quad 2(\alpha,\beta) = 2(\alpha,\gamma) -- 2n(\alpha,\alpha) = (m+n)(\alpha,\alpha) - 2n(\alpha,\alpha)$$

$$= (m-n)(\alpha,\alpha)$$

$$2\frac{(\alpha,\beta)}{(\alpha,\alpha)} = m - n$$

b the vector $\beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha = \beta + (n-m)\alpha$ is a root of the form $\beta + k\alpha$ because of the property $-m < n-m < n$

2. Consequences:

Let us consider the possible roots β proportional to a given root α: $\beta = k\alpha$. From the previous theorem, the quantity $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} = 2k$ is Cartan integer.

The operator E_α commutes with itself and both, 2α and $1/2\,\alpha$, cannot be roots.

As an immediate consequence, the only allowed values of k are $k = \pm 1, 0$. If a sequence contains zero as a root, this sequence must be simply $-\alpha$, 0, $+\alpha$. This case is realized in the Lie algebra of the SU(2) group.

3. Structure constants:

The operators $X_{\gamma-p\alpha}$ and $E_{\gamma-p\alpha}$ are related to each other by a product of structure constants:

$$X_{\gamma-p\alpha} = N_{\gamma,-\alpha}\, N_{\gamma-\alpha,-\alpha} \cdots E_{\gamma-(p-1)\alpha,-\alpha}, E_{\gamma-p\alpha},$$

and we immediately obtain an expression of μ_{p+1} in terms of these constants

$$\mu_{p+1} = N_{\gamma-p\alpha,-\alpha}\; N_{\gamma-(p+1)\alpha,\alpha}$$

We now use the notations introduce in § 1, $\gamma = \beta + n\alpha$,
$g = m + n$

$$\mu_{p+1} = N_{\beta+(n-p)\alpha, -\alpha} N_{\beta+(n-p-1)\alpha, \alpha}$$

and we can compare this expression with the explicit ones given in § 1.

$$\mu_{p+1} = \frac{1}{2} (\alpha,\alpha) (p+1) (m + n - p)$$

We consider the particular case $p = n - 1$. The properties of the structure constants allow us to write

$$\mu_n = N_{\beta+\alpha, -\alpha} N_{\beta,\alpha} = N_{-\alpha, \alpha+\beta} N_{\alpha,\beta} = N_{\alpha,\beta} N_{-\beta,\alpha} = N^2_{\alpha\beta}$$

and finally

$$N^2_{\alpha\beta} = \frac{1}{2} (\alpha,\alpha) \ n(\beta) \ [m(\beta) + 1]$$

where for a given root α, the positive integers m and n are functions of β.

The element $g_{\alpha-\alpha}$ of the Cartan tensor will then give the normalization of the root α

$$g_{\alpha-\alpha} = 1 = (\alpha,\alpha) \left\{ 2 + \frac{1}{2} \sum_{\beta\neq-\alpha} n(\beta) \ [m \ (\beta) + 1] \right\}$$

4. Root Diagram

The roots can be considered as vectors in the vector space $\mathcal{E}_\ell$. The root diagram is the graphical representation of the roots in $\mathcal{E}_\ell$.

Let us apply the previous theorem for the roots α and β:

$$p = 2 \frac{(\alpha, \beta)}{(\alpha, \alpha)} \qquad q = 2 \frac{(\alpha, \beta)}{(\beta, \beta)}$$

The two quantities p and q are algebric integers. We have

$$(\alpha, \beta)^2 = \frac{pq}{4} (\alpha, \alpha) (\beta, \beta)$$

and by using the Schwartz inequality

$$(\alpha, \beta)^2 \leq (\alpha, \alpha) (\beta, \beta)$$

we can define a real angle $\emptyset$ by:

$$\mathrm{Cos}^2 \emptyset = \frac{pq}{4}$$

Because of the symmetry $\alpha \longleftrightarrow -\alpha$ in the roots set, it is sufficient to study the angle $\emptyset$ between 0 and $\pi/2$. In order to simplify the discussion, we call as β, the root of larger norm $(\beta, \beta) \geq (\alpha, \alpha)$ and it follows immediately $p \geq q$. The numbers p and q being integers, the angle $\emptyset$ is restricted to the following values: 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$.

a Case $\phi = 0$ or $pq = 4$

The first evident solution is $p = q = 2$ corresponding to $\beta = \alpha$. A second possibility, $p = 4$ $q = 1$, leads to $\beta = 2\alpha$ and must be rejected.

b Case $\phi = \pi/6$ or $pq = 3$

We have only one solution $p = 3$, $q = 1$ $(\beta,\beta) = 3(\alpha,\alpha)$

c Case $\phi = \pi/4$ or $pq = 2$

We have only one solution $p = 2$, $q = 1$ and $(\beta,\beta) = 2(\alpha,\alpha)$

d Case $\phi = \pi/3$ or $pq = 1$

We have only one solution $p = 1$, $q = 1$ and $(\beta,\beta) = (\alpha,\alpha)$

e Case $\phi = \pi/2$ or $pq = 0$

The only physical possibility is $p = 0$ $q = 0$. Of course the ratio p/q is undetermined.

III. Simple Lie Algebrae

We first study in some details the particular cases of simple Lie algebrae of rank one and two. The results are generalized after to arbitrary rank simple Lie algebrae.

A. Simple Lie algebra of rank one

This Lie algebra is well known but it seems to us useful to deduce its properties in the general framework previously given. The simple Lie algebra of rank one corresponds to the three roots α, 0, $-\alpha$ and the one dimensional root diagram is simply

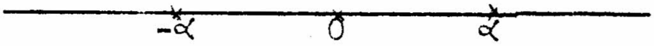


Fig. 1

The commutations relations are given by

$$[E_\alpha, E_{-\alpha}] = (\alpha, H) \qquad [(\alpha, H), E_\alpha] = (\alpha,\alpha)\, E_\alpha$$

The normalization condition gives $(\alpha, \alpha) = \frac{1}{2}$. The covariant and the contravariant components are both equal to $\frac{1}{\sqrt{2}}$ and the Cartan tensor can be written as:

$$g_{\rho\sigma} = \left|\begin{array}{c|cc} 1 & & \\ \hline & 0 & 1 \\ & 1 & 0 \end{array}\right|$$

With the convenient change of notations $E_{\pm} = E_\alpha$ $E_{\mp} = E_{-\alpha}$ $\alpha = \pm\frac{1}{\sqrt{2}}$, we obtain the commutation rules in a more familiar form

$$[E_{\pm}, E_{\mp}] = \pm 1/\sqrt{2} H \qquad [H, E_{\pm}] = \pm 1/\sqrt{2} E_{\pm}$$

The Lie algebra of the special unitary group SU(2) can be written in the form

$$\vec{J} \times \vec{J} = i \vec{J}$$

or equivalently

$$[J_1 \pm iJ_2, J_1 \mp iJ_2] = \pm 2 J_3 \cdot [J_3, J_1 \pm iJ_2] = \pm$$
$$\pm (J_1 \pm iJ_2)$$

The identification is obtained by:

$$E_{\pm} = \frac{J_1 \pm J_2}{2} \qquad H = \frac{1}{\sqrt{2}} J_3$$

B. Simple Lie algebrae of rank two

The root diagrams are two dimensional and, we explore all allowed possibilities for the angle $\emptyset$ in order to construct all rank two simple Lie algebrae. It is only necessary to consider two roots α and β with the angle $\emptyset$ to deduce all other roots simply by symmetry with respect to the straight line through the origin perpendicular to a root. All these reflections generate the Weyl group.

1. Diagram A_2

We consider two roots of equal norm α and γ, with the angle $\emptyset = \pi/3$. After application of the Weyl reflections, we obtain a regular hexagon and the Lie algebra is eight-dimensional.

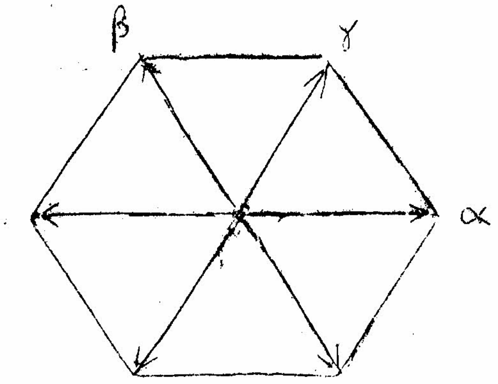


Figure 2
Root Diagram A_2

The six non vanishing roots have the same norm: $(\alpha,\alpha) = 1/3$. From figure 2, it can be easily seen that if α, β, γ are three non zero roots such that $\gamma = \beta + \alpha$, then $\beta - \alpha$ and $\beta + 2\alpha$ are not roots. In the previously defined language, $m = 0$ and $n = 1$. All the non vanishing structure constants have the same modulus: $N^2_{\alpha\beta} = 1/6$ and are known from one of them by using the symmetry properties.

2. Diagram B_2

We consider two roots α and β with the angle $\emptyset = \pi/4$ from the previous results $(\beta,\beta) = 2(\alpha,\alpha)$. After application of the Weyl reflection we obtain 4 roots of the type α and 4 roots of the type β. The Lie algebra is ten-dimensional.

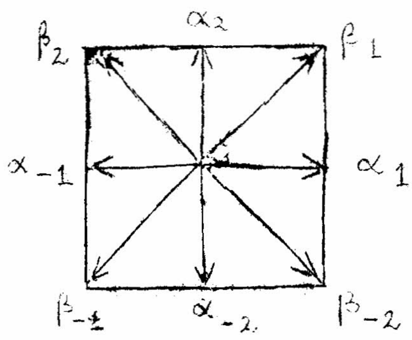


Figure 3
Root Diagram B_2

The norm of the roots is given by the normalization condition $\sum_{\alpha} (\alpha,\alpha) + \sum_{\beta} (\beta,\beta) = 2$ and we obtain

$$(\alpha, \alpha) = \frac{1}{6} \qquad (\beta, \beta) = \frac{1}{3}$$

In order to determine the non vanishing structure constants, we calculate the values of m and n associated to a given system of two roots:

a $\beta_i \; \beta_j$: $\beta_i + \beta_j$ can never be a root and $N_{\beta_i \beta_j} = 0$

b $\alpha_i \; \alpha_j$: there exists a sequence of three roots $\alpha_i - \alpha_j, \alpha_i, \alpha_i + \alpha_j$ with $i \neq j$ corresponding to $m = n = 1$ and $N^2_{\alpha_i \alpha_j} = 1/6$

c $\alpha_i \; \beta_j$: the two types of sequences of three roots are $\beta_j, \beta_j+\alpha_{-j}, \beta_j+2\alpha_{-j}$ and β_j , $\beta_j+\alpha_{-i}$, $\beta_j+2\alpha_{-i}$; in both cases $m = 0$ $n = 2$ and it follows immediately $N^2_{\beta_j, \alpha_{-j}} = N^2_{\beta_j, \alpha_{-i}} = 1/6$

All the non vanishing structure constants have the same magnitude and the phases are known from two of them. $N_{\alpha_i \alpha_j}$ and $N_{\beta_j, \alpha_{-j}}$ by using the symmetry properties .

3. Diagram G_2

We consider two roots α and β with the angle $\phi = \pi/6$ from the previous results $(\beta,\beta) = 3\ (\alpha,\alpha)$. After application of the Weyl reflections, we obtain 6 roots of the type α and 6 roots of the type β. The Lie algebra is 14-dimensional.

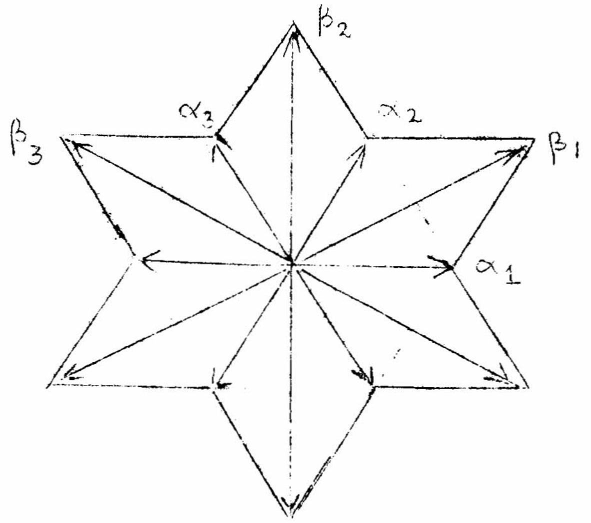


Figure 3
Root Diagram G_2

The norm of the roots is given by the normalization condition $\sum_{\alpha} (\alpha,\alpha) + \sum_{\beta} (\beta,\beta) = 2$ and we obtain:

$$(\alpha, \alpha) = 1/12 \qquad (\beta, \beta) = 1/4$$

It is easy to determine the structure constants by using the same method as in the previous section. All the non vanishing structure constants are known from three of them by using the symmetries properties and we have

$$N^2_{\alpha_1\alpha_3} = 1/6 \qquad N^2_{\alpha_1\beta_3} = 1/8 \qquad N^2_{\beta_1\beta_3} = 1/8$$

C. Simple Lie algebræ of rank ℓ

We try to extend the previous results to a ℓ dimensional space $\mathcal{E}_\ell$

1. Lie algebra A_ℓ

1. The root diagram A_2 exhibits an hexagonal symmetry. It is then convenient to introduce a three dimensional space and to represent the root diagram A_2 in the plane $X_1 + X_2 + X_3 = 0$. In this way, we define the triangular coordinates X_1, X_2, X_3 of sum zero and the non vanishing roots have the general form $\alpha_{ij} = e_i - e_j$

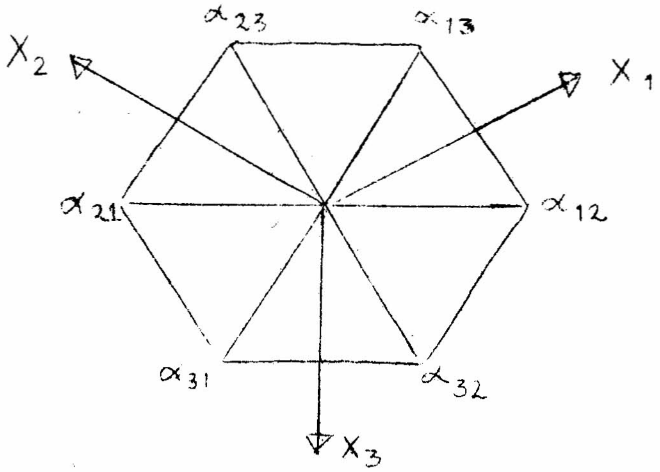


Figure 5
Root diagram A_2 and triangular coordinates

The natural generalization is to introduce a ($\ell+1$) dimensional space and ($\ell+1$) orthogonal vectors of equal norm e_j The $\ell(\ell+1)$ vectors

$$\alpha_{ij} = e_i - e_j$$

are located in the dimensional hyperplane $x_1+x_2+\cdot\ +x_{\ell+1}=0$. The Lie algebra A_ℓ of rank ℓ has the dimension $\nu = \ell(\ell+2)$

All the non vanishing roots α_{ij} have the same norm $(\alpha, \alpha) = 1/(\ell+1)$ and all the non-vanishing structure constants the same magnitude $N^2 = 1/2(\ell+1)$

2. Lie algebra B_ℓ

We introduce in the ℓ dimensional space $\mathcal{E}_\ell$ a system of ℓ orthogonal vectors of equal norm e_j

We first consider the following generalization of the Lie algebra B_2 by defining two sets of roots

a Roots of type α: 2ℓ roots given by $\pm e_j$

b Roots of type β : $2\ell(\ell-1)$ roots given by $\pm e_i \pm e_j$

As previously, we have of course, $(\beta,\beta) = 2(\alpha,\alpha)$ and the norms are given by

$$(\alpha,\alpha) = \frac{1}{2(2\ell-1)} \qquad (\beta,\beta) = \frac{1}{(2\ell-1)}$$

The Lie algebra B_ℓ of rank ℓ, has the dimension $\nu = \ell(2\ell+1)$ All the non-vanishing structure constants have the same magnitude $N^2 = 1/2(2\ell-1)$

3. Lie algebra C_ℓ

A different generalization of B_2 can be obtained by defining two sets of roots in the following way

a Roots of type α: $2\ell(\ell-1)$ roots given by $\pm e_i \quad \pm e_j$

b Roots of type β: 2ℓ roots given by $\pm 2e_j$

The values of the norms are given by:

$$(\alpha,\alpha) = \frac{1}{2(\ell+1)} \qquad (\beta,\beta) = \frac{1}{(\ell+1)}$$

The Lie algebra C_ℓ of rank ℓ has the dimension $r = \ell(2\ell+1)$ The structure constants can be divided into two classes following their magnitude:

$$\left(C_{\alpha_1\alpha_2}^{\alpha_3}\right)^2 = \frac{1}{4(\ell+1)} \qquad \left(C_{\alpha_1\alpha_2}^{\beta_3}\right)^2 = \left(C_{\alpha_1\beta_2}^{\alpha_3}\right)^2 = 1/2(\ell+1)$$

In an evident way, the Lie algebra C_2 is isomorphic to B_2

4. Lie algebra D_ℓ

A new Lie algebra of rank ℓ can be constructed with the following set of roots of equal norm $\pm e_i \quad \pm e_j$

We obtain in the way a Lie algebra of dimension $r = \ell(2\ell-1)$

The norm of the roots is given by: $(\alpha,\alpha) = 1/2(\ell-1)$ and all non vanishing structure constants have the same norm $N^2 = \frac{1}{4(\ell-1)}$

For $\ell = 2$, the Lie algebra D_2 is not simple and can be represented by the root diagram of the Figure 6

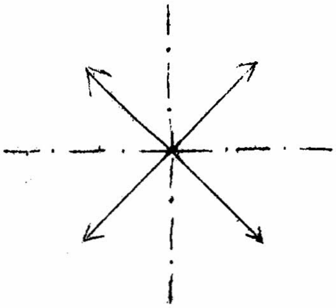


Figure 6

Root Diagram D_2

D_2 is a semi simple Lie algebra, direct sum of two rank one simple Lie algebra, A_1 :

$$D_2 \cong A_1 \oplus A_1$$

In the particular case $\ell = 3$, it can be easily shown that the two Lie algebra D_3 and A_3 are isomorphic by superposition of the root diagrams after rotation in $\mathcal{E}_3$

5. Exceptional Groups

The following results will be given without proof and we refer to the original papers of Cartan and to the subsequent works of Van der Waerden and Dyukin.

A_ℓ, B_ℓ, C_ℓ, D constitute the only four general classes of simple Lie algebras. To the four series, it can be added five exceptional groups characterized in Table 1.

Name	Rank	Dimension
G_2	2	14
F_4	4	52
E_6	6	78
E_7	7	133
E_8	8	248

Table 1
Exceptional Simple Lie algebras

IV. Realization by classical groups:

1. The determination of the standard for of a Lie algebra Λ on the real numbers R is obtained by resolving an eigenvalue problem which introduce the field of complex numbers. C. In reality, we are working with the complex extension Λ^* of the Lie algebra. Of course, the Lie algebra Λ^* is the complex extension of the non isomorphic Lie algebra Λ_α : for instance, all the Lie algebra of the pseudo-orthogonal groups $O_s(n,R)$ have the same complex extension which is the Lie algebra of the complex orthogonal group O(n,C). Cartan has shown that only one compact group can be associated to a standard form but of course, many non compact group can have the same standard form.

2. Lie algebra A_{n-1}

We consider the special linear group SL(n,R) acting in an euclidian space ($\mathfrak{g} = 1$). The Lie algebra is defined by $n^2 - 1$ infinitesimal generators X_{ij} such that $\sum_j X_{jj} = 0$

The general commutation laws are given by

$$[X_{ij}, X_{kl}] = \delta_{jk} X_{il} - \delta_{li} X_{kj}$$

It is convenient to work explicitely some particular relations

$$[X_{jj}, X_{kk}] = 0 \qquad j \neq k$$

$$[X_{jj}, X_{kl}] = (\delta_{jk} - \delta_{jl}) X_{kl}$$

$$[X_{kl}, X_{lk}] = \sum_j (\delta^j_k - \delta^j_l) X_{jj}$$

in order to exhibit clearly the standard form of the Lie algebra.

By putting $X_{jj} = \lambda_n H_j \qquad X_{ij} = \lambda_n E_{ij}$

the roots components are given by .

$$\alpha_{[kl]j} = \alpha_{[kl]}{}^{j} = \frac{1}{\lambda_n}(\delta_{jk} - \delta_{jl})$$

and the non vanishing structure constants by

$$N^2 = 1/\lambda_n$$

The normalization condition for the roots and the structure constants determines λ_n to be $\lambda_n = \sqrt{2n}$

The Lie algebra of the special linear group has the standard form A_{n-1} . From the previous results, this result is true for all unimodular pseudo unitary groups in an pseudo euclidian n-dimensional complex vector space. But from Cartan theorem, only one compact group can be associated to A_{n-1} and it is the unimodular unitary group SU(n,C).

3. Lie algebra B_ℓ and D_ℓ

We consider an euclidian n dimensional space on the real numbers. The Lie algebra of the orthogonal group is defined by $\frac{n(n-1)}{2}$ infinitesimal generators Z_{ij}

In order to study easily the standard form of the Lie algebra, it is convenient to use a complex basis in E(n,R) instead of the real one and a non-diagonal form for the connection g . We first define an index j with the following range of variation.

a $-\ell \leq j \leq \ell$ if $n = 2\ell+1$

b $-\ell \leq j \leq \ell$ excepted $\ell = 0$ if $n = 2\ell$

In such a basis g is represented by an anti-diagonal matrix with $g_{jk} = \delta_{j+k\,0}$ and the scalar product becomes

$$g(x,y) = \sum_j x_j y_{-j} = g(y,x)$$

The general form of the commutation laws is given by

$$[Z_{jk}, Z_{\ell m}] = g_{k\ell} Z_{jm} - g_{j\ell} Z_{km} - g_{km} Z_{j\ell} + g_{jm} Z_{k\ell}$$

It is convenient to write explicitely some particular relations:

$$[Z_{j\,-j}\,,\,Z_{k\,-k}] = 0$$

$$[Z_{j-j}\,,\,Z_{k\ell}] = (\delta_{jk} + \delta_{j\ell} - \delta_{-jk} - \delta_{-j\ell})\,Z_{k\ell}$$

$$[Z_{k\ell}\quad Z_{-\ell-k}] = \sum_{j} (\delta_{k}^{j} + \delta_{\ell}^{j})\,Z_{j-j}$$

in order to exhibit the standard form of the Lie algebra.

We now restrict j to positive values only $j = 1.2.\ldots,\ell$

The standard form is obtained by putting:

$$Z_{j-j} = \lambda_n H_j \qquad\qquad Z_{rs} = \lambda_n E_{rs}$$

The roots components are given by

$$\beta_{[rs]j} = \beta_{[rs]}^{\,j} = (\delta_{jr} + \delta_{js} - \delta_{j-r} - \delta_{j-s})\,\frac{1}{\lambda_n}$$

and, in the case $n = (2\ell+1)$ where r or s can take the value zero

$$\alpha_{[sr]j} = \alpha_{[sr]}^{\,j} = (\delta_{jr} - \delta_{j-r})\,\frac{1}{\lambda_n}$$

The non vanishing structure constants are all equal, in magnitude to $N^2 = 1/\lambda_n^2$

The normalization conditions for (α, α), (β, β), N^2 give the value of λ_n : $\lambda_n = \sqrt{2n-4}$.

The Lie algebra of the orthogonal group in an $(2\ell+1)$ vector space has the standard form B_ℓ and the Lie algebra of the orthogonal group in an 2ℓ vector space has the standard form D_ℓ . All the pseudo-orthogonal groups, irrespectively to the signature S have the same standard form.

4. Lie algebra C_n

We consider an 2n-dimensional eucludian space on the real number. The Lie algebra of the real symplectic group is defined by n(2n+1) infinitesimal generators A_{ij}, B_{ij}, C_{ij} with the commutation laws.

$$[A_{ij}, B_{k\ell}] = \delta_{jk} A_{i\ell} - \delta_{i\ell} A_{kj}$$

$$[B_{ij}, B_{k\ell}] = 0 = [C_{ij}, C_{k\ell}]$$

$$[B_{ij}, A_{k\ell}] = \delta_{ik} B_{j\ell} + \delta_{jk} B_{i\ell}$$

$$[A_{ij}, C_{k\ell}] = \delta_{jk} C_{i\ell} + \delta_{j\ell} C_{ik}$$

$$[C_{ij}, B_{k\ell}] = \delta_{ik} A_{j\ell} + \delta_{i\ell} A_{jk} + \delta_{jk} A_{i\ell} + \delta_{j\ell} A_{ik}$$

It is convenient to write explicitely some parituclar relations

$$[A_{jj}\,,\ A_{kk}] = 0$$
$$[A_{jj}\,,\ C_{kk}] = 2\ \delta_{jk}\ C_{kk}$$
$$[A_{jj}\,,\ B_{kk}] = -2\ \delta_{jk}\ B_{kk}$$
$$[A_{jj}\,,\ C_{kl}] = (\delta_{jk}+\delta_{jl})\ C_{kl}$$
$$[A_{jj}\,,\ A_{kl}] = (\delta_{jk}-\delta_{jl})\ A_{kl}$$
$$[A_{jj}\,,\ A_{lk}] = (-\delta_{jk}+\delta_{jl})\ A_{lk}$$
$$[A_{jj}\,,\ B_{kl}] = (-\delta_{jk}-\delta_{jl})\ B_{kl}$$

in order to exhibit clearly the standard form of the Lie algebra. By putting

$$A_{jj} = \lambda_n H_j \qquad B_{jj} = \mu_n E_{-j} \qquad C_{jj} = \mu_n E_j$$
$$C_{kl} = \lambda_n E_{kl} \qquad A_{kl} = \lambda_n\ E_{k-l} \qquad B_{kl} = \lambda_n E_{-k-l}$$

the roots components are given by

$$\alpha_{[\pm k\ \pm l]j} = \alpha_{[\pm k\ \pm l]}{}^{j} = \frac{1}{\lambda_n}(\pm\delta_{jk}\pm\delta_{jl})$$

$$\beta_{[\pm k]j} = \beta_{[\pm k]}{}^{j} = \frac{2}{\lambda_n}(\pm\delta_{jk})$$

The normalization condition for the roots give $\lambda_n = 2\sqrt{n+1}$ and for the structure constants $\mu_n = 2\sqrt{2(n+1)}$.

The Lie algebra of the real symplectic group $Sp(2n,R)$ has the standard form C_n . From the results of Chapter II, this result is also true for all pseudo symplectic group $Sp_s(n,Q)$ irrespectively to the signature S. But only the symplectic group $Sp(n,Q)$ is compact.

Chapter 5.

REPRESENTATIONS

I. Generalities

1. Definition:

Let us introduce a N dimensional vector space V_N and an abstract group G. We consider the group G_N of linear transformations of V_N represented by $N \times N$ matrices and such that G_N is homomorphic to G. By definition, G_N is a representation of dimension N of G. As a consequence

$$U(a)\,U(b) = U(ab)$$

for all $a, b \in G$ and $U(a),\ U(b) \in G_N$

If the homomorphism between G_N and G is an isomorphism the representation is said faithful. It can be shown that all representations of simple Lie algebras, except the identity, are faithful representations.

2. Equivalent representations:

Two representation $U_1(a)$ and $U_2(a)$ of G are equivalent if there exist a constant matrix A, independent of the group elements, and such that

$$U_2(a) = A\,U_1(a)\,A^{-1} \qquad \text{for all} \quad a \in G.$$

3. Reducibility:

A representation $U(a)$ of G in a vector space V_N is reducible if it leaves invariant a subspace V_1 of V_N. After a convenient change of basis, the matrix U can then be written in the form

$$U \doteq \begin{vmatrix} U_1 & 0 \\ U_3 & U_2 \end{vmatrix}$$

where the matrix U_1 has the same dimension as the vector space V_1.

If now $U_3 \equiv 0$, there exist two invariant sub spaces V_1 and V_2 of V_N such that the sum is precisely V_N, the representation U is said fully reducible into two representations U_1 and U_2

$$U = \begin{vmatrix} U_1 & 0 \\ 0 & U_2 \end{vmatrix}$$

4. Contragradient Representation:

We consider a N dimensional representation of G with the complex matrices U :

$$U(a)\, U(b) = U(ab) \qquad a, b \in G.$$

The complex conjugate matrices U^+ constitute a N dimensional representation of G

$$U^+(a)\, U^+(b) = U^+(ab)$$

The representation U and U^+ are called contragradient representations.

5. The Lie algebra of a Lie group G is defined by a set of r infinitesimal generators X_σ. It is possible to find, in the group G_N, a set of r $(N \times N)$ matrices also denoted X_σ, which have the commutations laws of the Lie algebra:

$$[X_\sigma, X_\rho] = C_{\sigma\rho}{}^{\tau} X_\tau \, .$$

6. Compact Semi-Simple Groups:

The following important theorem can be proved: all representations of a compact semi-simple group are equivalent to a representation with unitary matrices. We will consider only this case in the following.

The Lie algebra of a semi-simple group is defined, in its standard form by the infinitesimal generators H_j and E_α. In the unitary representation, the generators of the Cartan algebra can be represented by hermitian matrices; by using the commutation relation

$$[E_\alpha , E_{-\alpha}] = (\alpha , H)$$

it is possible to choose representation satisfying:

$$E_\alpha^* = E_{-\alpha}$$

II. Weights

We study the case of compact semi-simple groups for which the representations can be taken as unitary.

1. Definition:

We consider the ℓ -dimensional abelian Cartan sub algebra The operators H_j can be simultaneously diagonalized; in the vector space V_N we have the eigenvalue equation for each operator H_j ,

$$H_j |\Omega\rangle = m_j |\Omega\rangle$$

The numbers m_j can be considered as the covariant components of a vector m in the vector space $\mathcal{E}_\ell$ previously considered. The vector m is called a weight and $\mathcal{E}_\ell$ the weight space.

2. A simple weight is associated to one eigenvector only. For the rank $\ell > 1$ groups, the weights are not, in general simple.

3. Properties

We now give, without proof two elementary properties:

a There exist, at least one weight in each representation;

b The eigenvectors associated to different weights are linearly independent.

As a consequence, the maximum number of weights for a N-dimensional representation is precisely N .

4. Theorem 1.

If $|\Omega\rangle$ is an eigenvector associated to the weight m. the vector $E_\alpha|\Omega\rangle$ is either zero, or an eigenvector associated to the weight $m+\alpha$.

By definition, we have:

$$H_j |\Omega\rangle = m_j |\Omega\rangle$$

Let us consider the vector $E_\alpha |\Omega\rangle$. By using the relation:

$$H_j E_\alpha = E_\alpha H_j + [H_j, E_\alpha] = E_\alpha H_j + \alpha_j E_\alpha$$

we immediately obtain

$$H_j E_\alpha |\Omega\rangle = E_\alpha H_j |\Omega\rangle + \alpha_j E_\alpha |\Omega\rangle = (m+\alpha)_j E_\alpha |\Omega\rangle$$

If $E_\alpha |\Omega\rangle$ is not zero, it is an eigenvector associated to the weight $(m+\alpha)$.

5. Theorem 2

If m is a weight and α a root:

a the number $2\frac{(m,\alpha)}{(\alpha,\alpha)}$ is an integer

b the vector $m - \frac{2(m,\alpha)}{(\alpha,\alpha)}\alpha$ is also a weight, deduced from m by a reflection of the Weyl group.

The proof of this fundamental theorem is extremely similar to that given for the corresponding theorem with the roots in Section II of the previous chapter.

6. Equivalent Weights:

Two weights deduced from each other by an operation of the Weyl group are called equivalent. They have the same multiplicity.

III. Weyl Group

The Weyl group has been defined as the set of reflections with respect to the hyperplanes through the origin perpendicular to the roots α . We are now concerned with the determination of the Weyl group for the simple Lie groups by applying the fundamental theorem 2 -

1. Lie Algebra A_ℓ

The roots can be written $\alpha_{ij} = e_i - e_j$ and we expand the weight m on the basis of the vectors e_k

$$m = m_k e_k \qquad \text{with} \quad \sum_1^{\ell+1} m_k = 0$$

We immediately find

$$2\,\frac{(\alpha_{ij}, m)}{(\alpha, \alpha)} = m_i - m_j$$

We now use the theorem 2. From the part a , the differences $m_i - m_j$ are integer numbers. From the part b, the weight m' obtained by reflection from m is given by:

$$\sum_k m'_k e_k = \sum_k m_k e_k - (m_i - m_j)(e_i - e_j) =$$
$$= \sum_k m_k e_k - m_i e_i + m_j e_j + m_j e_i + m_i e_j$$

The Weyl group is the group of permutations of the components of the weights.

It follows that the maximum number of equivalent weights is $(\ell+1)!$

2- Lie Algebra B_ℓ

We have two series of roots

$$\begin{cases} \alpha_I = \varepsilon(I) e_i & I = \pm i \quad i = 1, 2, \dots, \ell \\ \beta_{IJ} = \varepsilon(I) e_i + \varepsilon(J) e_j & I = \pm i, \ J = \pm j \quad i, j = 1, 2, \dots, \ell \end{cases}$$

and $\varepsilon(I)$ is the sign of I.

It follows immediately

$$2\frac{(\alpha_I, m)}{(\alpha, \alpha)} = 2\varepsilon(I) m_i \qquad 2\frac{(\beta_{IJ}, m)}{(\beta, \beta)} = \varepsilon(I) m_i + \varepsilon(J) m_j$$

From the part a of the theorem 2, the components of a weight m must be either all integer numbers or all half integer numbers. The weights m' equivalent to m are defined from the part b by:

$$\sum_k m'_k e_k = \sum_k m_k e_k - [2\varepsilon(J) m_i][\varepsilon(I) e_i] = \sum_k m_k e_k - 2 m_i e_i$$

$$\sum_k m'_k e_k = \sum_k m_k e_k - [\varepsilon(I) m_i + \varepsilon(J) m_j][\varepsilon(I) e_i + \varepsilon(J) e_j]$$

$$\sum_k m'_k e_k = \sum_k m_k e_k - m_i e_i - m_j e_j - \varepsilon(I)\varepsilon(J)[m_i e_j + m_j e_i]$$

The Weyl group is the group of permutations of the components of the weights with an arbitrary number of changes of sign. It follows that the maximum number of equivalent weights is $2^{\ell}\,\ell\,!$

3. Lie Algebra C_ℓ

We have two series of roots:

$$\beta_I = 2\varepsilon(I)e_i \qquad\qquad \alpha_{IJ} = \varepsilon(I)e_i + \varepsilon(J)e_j$$

and it follows immediately :

$$2\frac{(\beta_I, m)}{(\beta,\beta)} = \varepsilon(I)m_i \qquad\qquad 2\frac{(\alpha_{IJ}, m)}{(\alpha,\alpha)} = \varepsilon(I)m_i + \varepsilon(J)m_j$$

From the part a of Theorem 2, the components of a weight m must be integer numbers.

The weights m' equivalent to m are defined from the part b by

$$\sum_k m'_k e_k = \sum_k m_k e_k - [\varepsilon(I)m_i][2\varepsilon(I)e_i] = \sum_k m_k e_k - 2m_i e_i$$

$$\sum_k m'_k e_k = \sum_k m_k e_k - m_i e_i - m_j e_j - \varepsilon(I)\varepsilon(J)[m_i e_j + m_j e_i]$$

The Weyl group is the same in B_ℓ and C_ℓ .

4. Lie Algebra D_ℓ

The roots of the Lie algebra D_ℓ have the general form

$$\alpha_{IJ} = \varepsilon(I)e_i + \varepsilon(J)e_j$$

and it follows immediately:

$$2\frac{(\alpha_{IJ}, m)}{(\alpha, \alpha)} = \varepsilon_I m_i + \varepsilon_J m_j$$

From the part a of theorem 2, the two quantities $m_i \pm m_j$ must be integer numbers.

The weights m' equivalent to m are given by:

$$\sum_k m'_k e_k = \sum_k m_k e_k - (m_i e_i + m_j e_j) - \varepsilon(I)\varepsilon(J)[m_i e_j + m_j e_i]$$

The Weyl group is the group of permutations of the components of the weights with an even number of changes of sign.

IV. Fundamental Weights:

1. We first introduce in the weight space $\mathcal{E}_\ell$ a relation. A vector is called a positive vector if its first non vanishing component is a positive number. We then have m_2 higher than m_1 if $m_2 - m_1$ is a positive vector. Of course, such a property depends of the basis in $\mathcal{E}_\ell$ but the consequences are intrinsically true by means of the Weyl group reflections.

For a semi-simple Lie algebra of rank ℓ and dimension r, these exist $(r - \ell)$ non vanishing, non degenerate roots α and $\frac{r-\ell}{2}$ positive roots symbolically denoted α^+.

2. Dominant Weight

In a set of equivalent weights, the dominant weight is higher than another weight of the set.

The highest weight of a representation is the highest dominant weight of the representation.

3. Properties

We give now, without proof two important properties.

a The highest weight of an irreducible representation is simple. It follows that the set of equivalent weights to the highest weight of an irreducible representation is a set of simple weights.

b Two irreducible representations with the same highest weight are equivalent and conversly.

4. Fundamental dominant weight

Cartan has proved the following results:for a simple Lie group of rank ℓ , there exist ℓ fundamental dominant weights L^1 , L^2 , ... , L^ℓ , with the following properties

a Every dominant weight L can be written as a linear combination of the L^δ 's with non-negative integer coefficients.

$$L(\lambda_1, \lambda_2, \cdots, \lambda_\ell) = \sum_{\delta=1}^{\delta=\ell} \lambda_\delta \, L^\delta \qquad \lambda_\delta \geq 0$$

b To each L^δ corresponds a fundamental irreducible representation for which L^δ is the highest weight.

V. CHARACTER

1. Definition:

Let us consider the N dimensional vector space V_N of the representation $U(a)$ and two vectors $|\Omega_\alpha\rangle$ and $|\Omega_\beta\rangle$ of V_N . The trace of the $N \times N$ matrix

$$\langle \Omega_\beta | U(a) | \Omega_\alpha \rangle$$

is independent of the basis chosen in V_N and is called the character χ of the representation.

2. The theory of the characters has been studied by Weyl. We want to give here only some results but the notion of character is extremely useful because of the following theorem: two representations are equivalent if and only if they have the same characters.

3. We now introduce, in the wieght space $\mathcal{E}_\ell$, two vectors which are used in the calculation of the character and of the dimensionality of a representation. The first one is:

$$R = \frac{1}{2} \sum_{\alpha^+} \alpha$$

where the sum is extended over the positive roots only and the second depends of the representation as $L(\lambda_J)$

$$K(\lambda_J) = R + L(\lambda_J)$$

The elements of the Weyl group are noted by S and the vector SK is the result of the operation of S on K. For a compact semi-simple group, the character χ is given by the general formula

$$\chi(\lambda_J, \varphi) = \frac{\xi(\lambda_J, \varphi)}{\xi(0, \varphi)}$$

where

$$\xi(\lambda_J, \varphi) = \sum_S \delta_S \exp i (SK, \varphi)$$

δ_S is the parity of S and φ a vector of the weight space $\mathcal{E}_\ell$.

If all the weights m of a representation are known with their multiplicity γ_m, an extremely simple expression can be used

$$\chi(\lambda_\ell, \varphi) = \sum_m \gamma_m \exp i(m, \varphi)$$

4. Dimension

The dimension of the representation is given by :

$$N(\lambda_\ell) = \chi(\lambda_\ell, 0)$$

Weyl has shown the useful formula:

$$N(\lambda_\ell) = \prod_{\alpha^+} \frac{(\alpha, K(\lambda_\ell))}{(\alpha, R)} = \prod_{\alpha^+} \left(1 + \frac{(\alpha, L(\lambda_\ell))}{(\alpha, R)}\right)$$

where the product is extended to all the positive roots α^+

5. Contragradient representations

The characters of two contragradient representations are complex conjugate. This result is simply a consequence of the definition of the contragradient representations with complex conjugate matrices.

It follows that the weight diagrams of two contra-gradient representations can be deduced from each other by a symmetry with respect to the origin in the weight space.

The character of a representation equivalent to its contra-gradient is real and there is a necessary and sufficient condition. In an equivalent way the weight diagram is symmetric with respect to the origin in the weight space and there is also a necessary and sufficient condition.

VI. APPLICATION TO SIMPLE LIE GROUPS

A. Lie Algebra A_ℓ

1. The components m_j of any weight satisfy the two following requirements:all the differences $m_i - m_j$ are integer numbers and the sum of all the components vanish. The general structure of m_j is then a fraction with demominator $\ell + 1$, and all the numerators are equivalent modulo $\ell + 1$.

2. The ℓ fundamental dominant weights of A_ℓ can be written as

$$L^j = \frac{\lambda}{\ell+1}\left[(\ell+1-j)\sum_{k=1}^{k=j} e_k - j\sum_{k=j+1}^{k=\ell+1} e_k\right]$$

The number of weights equivalent to L^j is given by the number of independent permutations of the components of L^j , e.g. the number of combinations $C^j_{\ell+1}$:

$$C^j_{\ell+1} = C^{\ell+1-j}_{\ell+1} = \frac{(\ell+1)!}{j!\,(\ell+1-j)!}$$

3. All the weights of a fundamental representation F^j are equivalent to the highest weight which is the fundamental dominant weight L^j . It follows that all the weights are simple and the dimension of F^j is given by

$$\dim F^j = C^j_{\ell+1}$$

4. The two fundamental representations F^{j} and $F^{\ell+1-j}$ have the same dimension. It is easy to verify that the weights diagrams can be deduced to each other by a symmetry with respect to the origin in the weight space. The fundamental representations F^{j} and $F^{\ell+1-j}$ are contragradient representations.

In the case where ℓ is odd : $\ell+1 = 2\ell_1$, the fundamental representation F^{ℓ_1} is equivalent to its contragradient and can be chosen as real.

5. We define as S_α, a permutation, element of the Weyl group

$$S_\alpha = \frac{(1, 2, \cdots, \ell+1)}{(\alpha_1, \alpha_2, \cdots, \alpha_{\ell+1})}$$

All the equivalent simple weights of a fundamental representation F^{j} are deduced from L^{j} by an operation S_α of the Weyl group. We then obtain, for the character $\chi_j(\varphi)$ of F^{j} :

$$\chi_j(\varphi) = \sum_{S_\alpha} \exp i \left[\sum_{k=1}^{k=j} \varphi_{\alpha_k} \right]$$

where the vector φ satisfies the usual condition $\sum_{k=1}^{k=\ell+1} \varphi_k = 0$. It is then easy to verify on the explicit expression the relation

$$\chi_{\ell+1-j}(\varphi) = \chi_j^*(\varphi)$$

6. We now consider an irreducible representation, characterized by its highest weight

$$L = \sum_{j=1}^{j=\ell} \lambda_j L^j \qquad \lambda_j \geqslant 0$$

In order to calculate the dimension of the representation, we are first interested with the positive roots α_{mn} $(n > m)$ and the vector R, previously defined is given by :

$$R = \frac{1}{2} \sum_{k=0}^{k=[\ell/2]} (\ell - 2k)(e_{k+1} - e_{\ell+1-k})$$

We have successively

$$\left\{ \begin{array}{l} (\alpha_{mn}, R) = (n-m)(e,e) \\ (\alpha_{mn}, L) = \left(\sum_{j=m}^{j=n-1} \lambda_j \right)(e,e) \end{array} \right.$$

The dimension of the irreducible representation $D^N(\lambda_1, \lambda_2, \cdots, \lambda_\ell)$ is obtained by using the Weyl formula:

$$N(\lambda_1, \lambda_2, \cdots, \lambda_\ell) = \prod_{m<n} \left(1 + \frac{\sum_{j=m}^{j=n-1} \lambda_j}{n-m} \right)$$

7. A particular interesting case corresponds to all λ_j's equal to zero except two $\lambda_1 = \lambda_\ell = 1$. The highest weight is simply:

$$L(1, 0, \cdots, 0, 1) = L^1 + L^\ell = e_1 - e_{\ell+1} = \alpha_{1\,\ell+1}$$

All the equivalent weights of the highest weight are the $\ell(\ell+1)$ non vanishing roots α_{ij} of the A_ℓ Lie algebra.

The dimension of the representation is obtained from the previous formula

$$N(1,0,\cdots,0,1) = \ell(\ell+2)$$

and turns out to be equal to the dimension of the A_ℓ Lie algebra. Such a representation is called the adjoint representation of the Lie algebra and the weight diagram is simply the root diagram.

The character of the adjoint representation is given by

$$\chi_A(\varphi) = \ell + 2\sum_{m<n} \cos(\varphi_m - \varphi_n)$$

8. It is easy to show, by using the definition of the highest weights, that the representations $D^N(\lambda_1,\lambda_2,\cdots,\lambda_{\ell-1},\lambda_\ell)$ and $D^N(\lambda_\ell,\lambda_{\ell-1},\cdots,\lambda_2,\lambda_1)$ are contragradient representations.

It follows that only symmetric representations, defined by $\lambda_{\ell+1-j} = \lambda_j$ are equivalent to their contragradient representations.

9. Lie algebra A_1

From a pedagogical point of view, it is interesting to use the general language for the well known results of the A_1 Lie algebra.

We have one fundamental 2-dimensional representation, the spinor representation of the Lie algebra, and the fundamental weight is:

$$L^1 = \frac{1}{2}(e_1 - e_2)$$

In figure 1, we have represented the weight diagram of the fundamental representation and the three roots of the A_1 Lie algebra

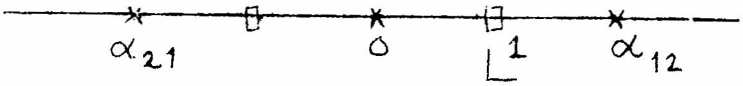


Figure 1
Fundamental representation
$D^2(1)$

The irreducible representation $D^N(\lambda)$ of highest weight $L = \lambda L^1$ has the dimension:

$$N(\lambda) = 1 + \lambda$$

All the weights are simple and of the general form $L = \mu L^1$ with $\mu = \lambda, \lambda-2, \dots, -\lambda$. The character of the representation $D^N(\lambda)$ is given by:

$$\chi(\lambda, \varphi) = \frac{\sin(\lambda+1)\varphi}{\sin \varphi}$$

In the usual language, $\lambda = 2J$ where J is the spin associated to the irreucible representation of the rotation group.

10. Lie algebra A_2

There exist two 3-dimensional contragradient fundamental representations. The fundamental dominant weights are given by

$$L^1 = \frac{1}{3}\left[2e_1 - (e_2 + e_3)\right]$$

$$L^2 = \frac{1}{3}\left[(e_1 + e_2) - 2e_3\right]$$

The corresponding two dimensional weight diagrams are drawn in Figures 2 and 3 and located with respect to the root diagram of the adjoint representation.

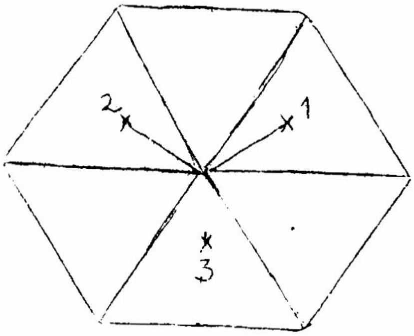


Figure 2

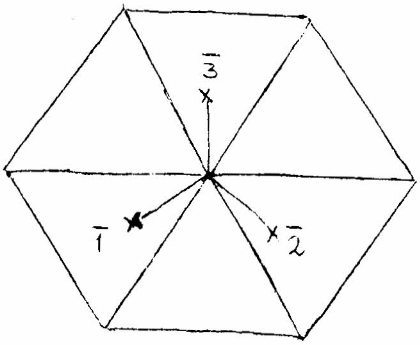


Figure 3

We will denote in the following the two fundamental representations by **3** and $\overline{3}$ and the adjoint representation by **8** . The characters of these three representations are given by

$$\chi(3,\varphi) = e^{i\varphi_1} + e^{i\varphi_2} + e^{i\varphi_3}$$

$$\chi(\overline{3},\varphi) = e^{-i\varphi_1} + e^{-i\varphi_2} + e^{-i\varphi_3}$$

$$(\chi)8,\varphi) = 2 + 2[\cos(\varphi_1-\varphi_2) + \cos(\varphi_2-\varphi_3) + \cos(\varphi_3-\varphi_1)]$$

with $\varphi_1 + \varphi_2 + \varphi_3 = 0$.

The dimension of the irreducible representation $D^N(\lambda_1,\lambda_2)$ is given by the symmetric formula :

$$N(\lambda_1,\lambda_2) = (1+\lambda_1)(1+\lambda_2)\left(1+\frac{\lambda_1+\lambda_2}{2}\right)$$

Only the representations $D(\lambda,\lambda)$ are equivalent to their contragradient and the dimension is then given by:

$$N(\lambda;\lambda) = (1+\lambda)^3$$

11. Lie Algebra A_3

There exist three fundamental representations associated to the following fundamental dominant weights :

$$\left\{\begin{aligned} L^1 &= \frac{1}{4}\left[3e_1-(e_2+e_3+e_4)\right] \\ L^2 &= \frac{1}{2}\left[(e_1+e_2)-(e_3+e_4)\right] \\ L^3 &= \frac{1}{4}\left[(e_1+e_2+e_3)-3e_4\right] \end{aligned}\right.$$

The representations F^1 and F^3 are two 4-dimensional contragradient representations and F^2 is a 6-dimensional representation equivalent to its contragradient.

The adjoint representation $D(1,0,1)$ is 15-dimensional as the Lie algebra A_3.

The dimension of the irreducible representation $D^N(\lambda_1,\lambda_2,\lambda_3)$ is given by:

$$N(\lambda_1,\lambda_2,\lambda_3)=(1+\lambda_1)(1+\lambda_2)(1+\lambda_3)\left(1+\frac{\lambda_1+\lambda_2}{2}\right)\left(1+\frac{\lambda_2+\lambda_3}{2}\right)\left(1+\frac{\lambda_1+\lambda_2+\lambda_3}{3}\right)$$

12. Lie algebra A_5

There exist five fundamental representations associated to the following fundamental dominant weights.

$$\left\{\begin{aligned} L^1 &= \frac{1}{6}\left[5e_1-(e_2+e_3+e_4+e_5+e_6)\right] \\ L^2 &= \frac{1}{3}\left[2(e_1+e_2)-(e_3+e_4+e_5+e_6)\right] \\ L^3 &= \frac{1}{2}\left[(e_1+e_2+e_3)-(e_4+e_5+e_6)\right] \\ L^4 &= \frac{1}{3}\left[(e_1+e_2+e_3+e_4)-2(e_5+e_6)\right] \\ L^5 &= \frac{1}{6}\left[(e_1+e_2+e_3+e_4+e_5)-6e_6\right] \end{aligned}\right.$$

The representations F^1 and F^5 are two 6-dimensional contragradient representations. The representations F^2 and F^4 are two 15-dimensional contragradient representations. The representation F^3 is a 20-dimensional representation equivalent to its contragradient. The adjoint representation $D(1,0,0,0,1)$ is 35-dimensional. The dimension of the irreducible representation $D^N(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ is given by

$$\begin{aligned} N(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5) = {} & (1+\lambda_1)(1+\lambda_2)(1+\lambda_3)(1+\lambda_4)(1+\lambda_5)\cdot \\ & \cdot\left(1+\frac{\lambda_1+\lambda_2}{2}\right)\left(1+\frac{\lambda_2+\lambda_3}{2}\right)\left(1+\frac{\lambda_3+\lambda_4}{2}\right)\left(1+\frac{\lambda_4+\lambda_5}{2}\right)\cdot \\ & \cdot\left(1+\frac{\lambda_1+\lambda_2+\lambda_3}{3}\right)\left(1+\frac{\lambda_2+\lambda_3+\lambda_4}{3}\right)\left(1+\frac{\lambda_3+\lambda_4+\lambda_5}{3}\right)\cdot \\ & \cdot\left(1+\frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{4}\right)\left(1+\frac{\lambda_2+\lambda_3+\lambda_4+\lambda_5}{4}\right)\left(1+\frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5}{5}\right) \end{aligned}$$

B. Lie Algebra B_ℓ

1. The ℓ fundamental dominant weights of B_ℓ can be written as

$$\left\{\begin{aligned} L^\delta &= \sum_{k=1}^{k=\delta} e_k \qquad \delta = 1,2,\cdots,\ell-1 \\ L^\ell &= \frac{1}{2}\sum_{k=1}^{k=\ell} e_k \end{aligned}\right.$$

We then have ℓ fundamental representations, F^δ, with the L^δ's as highest weights.

2. The Weyl group is the set of permutations of the components of a weight with an arbitrary number of changes of sign. It follows that all weight diagrams are invariant under

a symmetry with respect to the origin in the weight space and all representations are equivalent to their contragradient representation. As another consequence, all the characters are real numbers.

3. We now calculate the dimension of the irreducible representation $D^N(\lambda_1, \lambda_2, \dots, \lambda_\ell)$. The vector R is given by

$$R = \frac{1}{2} \sum_{k=1}^{k=\ell} (2\ell + 1 - 2k) e_k$$

The positive roots are α_j, β_{ij} and β_{i-j} with $0 < i < j$

We have successively:

$$\left\{ \begin{array}{l} (\alpha_j, L) = \frac{1}{2} \lambda_\ell + \sum_{j}^{\ell-1} \lambda_k \\ (\beta_{ij}, L) = \lambda_\ell + \sum_{i}^{\ell-1} \lambda_k + \sum_{j}^{\ell-1} \lambda_k \\ (\beta_{i-j}, L) = \sum_{i}^{j-1} \lambda_k \end{array} \right.$$

and the dimension $N(\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is finally given by

$$N(\lambda_1, \lambda_2, \dots, \lambda_\ell) =$$

$$= \prod_{m=1}^{m=\ell} \left\{ \left(1 + \frac{\lambda_\ell + 2\sum_{m}^{\ell-1} \lambda_k}{2\ell + 1 - 2m}\right) \prod_{n=m+1}^{n=\ell} \left[\left(1 + \frac{\lambda_\ell + \sum_{m}^{n-1} \lambda_k + 2\sum_{n}^{\ell-1} \lambda_k}{2\ell + 1 - m - n}\right) \left(1 + \frac{\sum_{m}^{n-1} \lambda_k}{n - m}\right) \right] \right\}$$

4. The fundamental representation F^1 is called the vector representation of the Lie algebra B_ℓ. The dimension of F^1 is given by the general formula:

$$\dim F^1 = 2\ell + 1$$

The weights of the vector representation are the 2ℓ simple weights equivalent to L^1 and the simple weight $m = 0$.

The character of the vector representation is then given by:

$$\chi_V(\varphi) = 1 + 2\sum_{k=1}^{k=\ell} \cos \varphi_k$$

5. The fundamental representation F^ℓ is called the spinor representation of the Lie algebra B_ℓ. The dimension of F^ℓ is given by the general formula

$$\dim F^\ell = 2^\ell$$

The weights of the spinor representation are the 2^ℓ simple weights equivalent to L^ℓ.

6. The fundamental representation F^2 has its weight diagram identical to the root diagram of the Lie algebra B_ℓ and it follows that F^2 is the adjoint representation. The dimension is given by the general formula:

$$\dim F^2 = \ell(2\ell + 1)$$

and the character of the adjoint representation is simply :

$$\chi_A(\varphi) = \ell + 2 \sum_{k=1}^{k=\ell} \cos \varphi_k + 4 \sum_{j<k} \cos \varphi_j \cos \varphi_k$$

7. Lie Algebra B_2

The fundamental weight of the vector and spinor representations are given by

$$\left\{ \begin{array}{l} L^1 = e_1 \\ L^2 = \frac{1}{2}(e_1 + e_2) \end{array} \right.$$

The corresponding two dimensional weight diagrams are drawn in Figures 4 and 5 and located with respect to the root diagram of the adjoint representation.

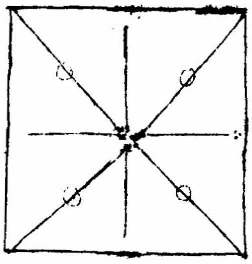


Figure 4

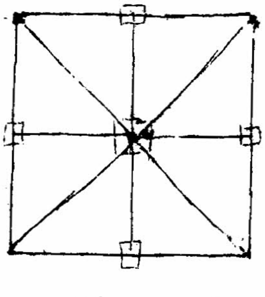


Figure 5

The characters of the fundamental and adjoint representations are given by :

$$\left\{\begin{aligned} \chi_V(\varphi) &= 1 + 2(\cos\varphi_1 + \cos\varphi_2) \\ \chi_S(\varphi) &= 4\cos\frac{\varphi_1}{2}\cos\frac{\varphi_2}{2} \\ \chi_A(\varphi) &= 2 + 2(\cos\varphi_1 + \cos\varphi_2) + 4\cos\varphi_1\cos\varphi_2 \end{aligned}\right.$$

The dimension of the irreducible representation $D^N(\lambda_1,\lambda_2)$ is given by the formula

$$N(\lambda_1,\lambda_2) = (1+\lambda_1)(1+\lambda_2)\left(1+\frac{\lambda_1+\lambda_2}{2}\right)\left(1+\frac{2\lambda_1+\lambda_2}{3}\right).$$

C. Lie Algebra C_ℓ

1. The ℓ fundamental dominant weights of C_ℓ can be written as

$$L^j = \sum_{k=1}^{k=j} e_k \qquad j = 1, 2, \dots, \ell$$

We have ℓ fundamental representations F^j with the L^j's as their highest weights.

2. As in the can of the B_ℓ Lie algebra, all the representations are equivalent to their contragradient representation and all the characters are real numbers.

3. We now calculate the dimension of the irreducible representation $D^N(\lambda_1,\lambda_2,\cdots,\lambda_\ell)$. The vector R is given by:

$$R = \sum_{k=1}^{k=\ell} (\ell+1-k)\, e_k$$

The positive roots are $\alpha_{ij} = e_i + e_j$, $\alpha_{i-j} = e_i - e_j$, $\beta_j = 2e_j$ with $0 < i < j \leq \ell$. We have successively:

$$\left\{ \begin{aligned} (\alpha_{ij}, L) &= \sum_{i}^{j-1} \lambda_k + 2\sum_{j}^{\ell} \lambda_k \\ (\alpha_{i-j}, L) &= \sum_{i}^{j-1} \lambda_k \\ (\beta_j, L) &= 2\sum_{j}^{\ell} \lambda_k \end{aligned} \right.$$

The dimension $N(\lambda_1, \lambda_2, \cdots, \lambda_\ell)$ is given by :

$$N(\lambda_1, \lambda_2, \cdots\cdots, \lambda_\ell) =$$

$$= \prod_{m=1}^{m=\ell} \left\{ \left(1 + \frac{\sum_{m}^{\ell} \lambda_k}{\ell + 1 - m}\right) \prod_{n=m+1}^{n=\ell} \left[\left(1 + \frac{\sum_{m}^{n-1} \lambda_k + 2\sum_{n}^{\ell} \lambda_k}{2\ell + 2 - m - n}\right)\left(1 + \frac{\sum_{m}^{n-1} \lambda_k}{n - m}\right) \right] \right\}$$

4. The dimension of the fundamental representation F^1 is given by the general formula to be

$$\dim F^1 = 2\ell$$

The weights are the 2ℓ simple weights equivalents to L^1

The character of F^1 is given by

$$\chi_1(\varphi) = 2\sum_{k=1}^{k-\ell} \cos \varphi_k$$

5. The dimension of the fundamental representation F^2 is obtained using the general formula

$$\dim F^2 = (\ell - 1)(2\ell + 1)$$

For the fundamental representation F^3, we have

$$\dim F^3 = \frac{1}{3}\ell(4\ell^2-1)$$

6. The adjoint representation has its highest weight given by $L_A = 2e_1$. It is the irreducible representation $D(2,0,0,\ldots\ldots,0)$ for which the weight diagram coincides with the root diagram of the Lie algebra c_ℓ. The dimension is calculated with the general formula:

$$N(2,0,0,\cdots\cdots,0) = \ell(2\ell+1)$$

The character of the adjoint representation is given by:

$$\chi_A(\varphi) = \ell + 2\sum_k \cos\varphi_k + 4\sum_{j<k}\cos\varphi_j\cos\varphi_k$$

7. <u>Lie algebra c_3</u>

The fundamental weights are

$$\left\{\begin{aligned} L^1 &= e_1 \\ L^2 &= e_1 + e_2 \\ L^3 &= e_1 + e_2 + e_3 \end{aligned}\right.$$

The first fundamental representation is six dimensional the second one 14-dimensional and the third one 35-dimensional. The adjoint representation $D(2,0,0)$ is 21-dimensional. The dimension of anirreducible representation $D^N(\lambda_1,\lambda_2,\lambda_3)$ is given by:

$$N(\lambda_1, \lambda_2, \lambda_3) = (1+\lambda_1)(1+\lambda_2)(1+\lambda_3)\left(1+\frac{\lambda_1+\lambda_2}{2}\right)\left(1+\frac{\lambda_2+\lambda_3}{2}\right) \cdot$$

$$\cdot \left(1+\frac{\lambda_1+\lambda_2+\lambda_3}{3}\right)\left(1+\frac{\lambda_2+2\lambda_3}{3}\right) \cdot$$

$$\cdot \left(1+\frac{\lambda_1+\lambda_2+2\lambda_3}{4}\right)\left(1+\frac{\lambda_1+2\lambda_2+2\lambda_3}{5}\right)$$

D-Lie Algebra $\mathfrak{D}_\ell$

1. The fundamental dominant weights of $\mathfrak{D}_\ell$ can be written as:

$$\left\{ \begin{aligned} L^j &= \sum_{k=1}^{k=j} e_k \qquad j = 1, 2, \ldots, \ell-2 \\ L^{\ell-1} &= \frac{1}{2}(e_1 + e_2 + \cdots + e_{\ell-1} + e_\ell) \\ L^{\ell} &= \frac{1}{2}(e_1 + e_2 + \cdots + e_{\ell-1} - e_\ell) \end{aligned} \right.$$

We have ℓ fundamental representations F^j with the L^j's as the highest weights.

2. The Weyl group is the set of all permutations of the components of the weights with an even number of changes of sign. The fundamental representations $F^1, F^2, \ldots, F^{\ell-2}$ are all equivalent to their contragradient representation. The same result is also true for $F^{\ell-1}$ and F^ℓ if ℓ is an even number. But if ℓ is an odd number, the fundamental representations $F^{\ell-1}$ and F^ℓ are contragradient representations.

3. We now calculate the dimension of the irreducible representation $\mathcal{D}^{M}(\lambda_1, \lambda_2, \dots, \lambda_\ell)$. The vector $\mathcal{R}$ is given by:

$$\mathcal{R} = \sum_{k=1}^{k=\ell} (\ell - k)\, e_k$$

The positive roots are $\alpha_{ij} = e_i + e_j$ and $\alpha_{ij} = e_i - e_j$ with $0 < i < j \leq \ell$. We have successively:

$$\left\{ \begin{aligned} (\alpha_{i+j}, L) &= \sum_{i}^{\ell-2} \lambda_k + \sum_{j}^{\ell-2} \lambda_k + \lambda_{\ell-1} + \tfrac{1}{2}(1 - \delta_{j\ell})\, \lambda_\ell \\ (\alpha_{i-j}, L) &= \sum_{i}^{j-1} \lambda_k + \delta_{j\ell} \lambda_\ell \end{aligned} \right.$$

and the dimension $N(\lambda_1, \lambda_2, \cdots, \lambda_\ell)$ is finally given by

$$N(\lambda_1, \lambda_2, \cdots, \lambda_\ell) =$$

$$= \prod_{m=1}^{m=\ell} \left\{ \left(1 + \frac{\lambda_\ell + \sum_m^{\ell-2} \lambda_k}{\ell - m}\right)\left(1 + \frac{\lambda_{\ell-1} + \sum_m^{\ell-2} \lambda_k}{\ell - m}\right) \cdot \prod_{n=m+1}^{n=\ell-1} \left[\left(1 + \frac{\sum_m^{n-1} \lambda_k + 2 \sum_n^{\ell-2} i \lambda_k + \lambda_{\ell-1} + \lambda_\ell}{(2\ell - m - n)}\right)\left(1 + \frac{\sum_m^{n-1} \lambda_k}{n - m}\right)\right] \right\}$$

4. The fundamental representation F^1 is called the vector representation of the Lie algebra $\mathcal{D}_\ell$. The dimension of F^1 is given by the general formula:

$$\dim F^1 = 2\ell$$

The weights of the vector representation are the 2ℓ simple weights equivalent to L^1.

The character of the vector representation is given by:

$$\chi_V(\varphi) = 2\sum_{k=1}^{k=\ell} \cos\varphi_k$$

5. The fundamental representations $F^{\ell-1}$ and F^{ℓ} are called the two spinor representations of the Lie algebra D_ℓ and they are inequivalent representations. The dimension of $F^{\ell-1}$ and F^{ℓ} is the same, due to the symmetry of $N(\lambda_1,\lambda_2,\cdots\lambda_{\ell-1},\lambda_\ell)$ in the exchange of $\lambda_{\ell-1}$ and λ_ℓ :

$$\dim F^{\ell-1} = \dim F^{\ell} = 2^{\ell}$$

The weights of the spinor representations are the simple weights equivalent to the highest weight $L^{\gamma-1}$ and L^{γ} .

6. The fundamental representation F^2 has a weight diagram identical to the root diagram of the Lie algebra D_ℓ and it follows that F^2 is the adjoint representation. The dimension is given by the general formula

$$\dim F^2 = \ell(2\ell-1)$$

The character of the adjoint representation is:

$$\chi_A(\varphi) = \ell + 4\sum_{\gamma<k} \cos\varphi_\gamma \cos\varphi_k$$

7. <u>Lie algebra D_4</u>

The fundamental representations are defined by the fundamental dominant weights:

$$\begin{aligned} L^1 &= e_1 \\ L^2 &= e_1 + e_2 \\ L^3 &= \tfrac{1}{2}(e_1 + e_2 + e_3 + e_4) \\ L^4 &= \tfrac{1}{2}(e_1 + e_2 + e_3 - e_4) \end{aligned}$$

The vector representation F^1 and the two spinor representations F^3 and F^4 are 8-dimensional representation. The adjoint representation is 28 -dimensional.

The characters of the fundamental representations are given by:

$$\chi_V(\varphi) = 2[\cos\varphi_1 + \cos\varphi_2 + \cos\varphi_3 + \cos\varphi_4]$$

$$\chi_A(\varphi) = 4[1 + \cos\varphi_1\cos\varphi_2 + \cos\varphi_1\cos\varphi_3 + \cos\varphi_1\cos\varphi_4 + \cos\varphi_2\cos\varphi_3 + \cos\varphi_2\cos\varphi_4 + \cos\varphi_3\cos\varphi_4]$$

$$\chi_S(\varphi) = 8[\cos\tfrac{\varphi_1}{2}\cos\tfrac{\varphi_2}{2}\cos\tfrac{\varphi_3}{2}\cos\tfrac{\varphi_4}{2} + \sin\tfrac{\varphi_1}{2}\sin\tfrac{\varphi_2}{2}\sin\tfrac{\varphi_3}{2}\sin\tfrac{\varphi_4}{2}]$$

$$\chi_{S'}(\varphi) = 8[\cos\tfrac{\varphi_1}{2}\cos\tfrac{\varphi_2}{2}\cos\tfrac{\varphi_3}{2}\cos\tfrac{\varphi_4}{2} - \sin\tfrac{\varphi_1}{2}\sin\tfrac{\varphi_2}{2}\sin\tfrac{\varphi_3}{2}\sin\tfrac{\varphi_4}{2}]$$

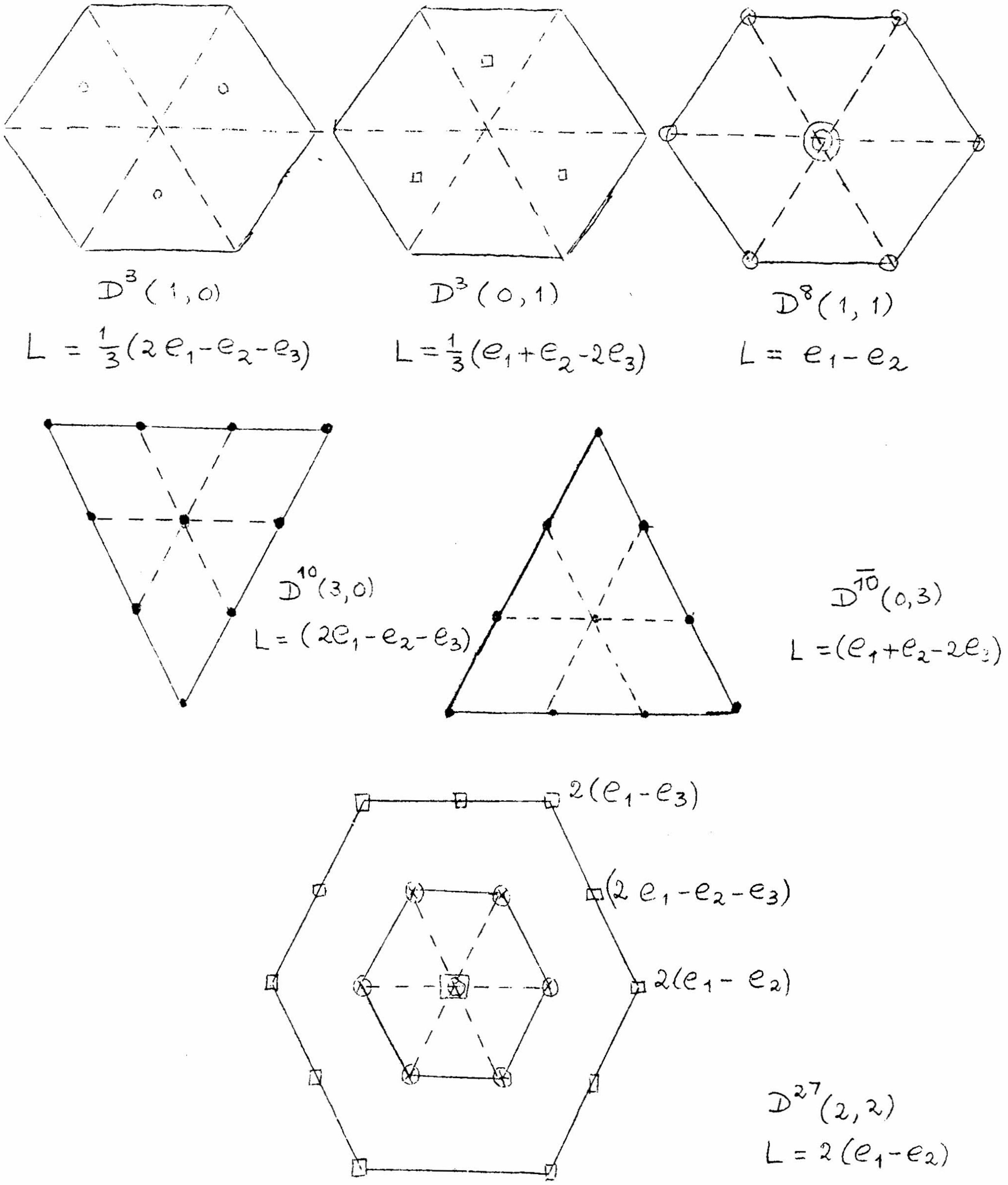
The representation $D^N(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ has the following dimension.

$$\begin{aligned} N(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = {} & (1+\lambda_1)(1+\lambda_2)(1+\lambda_3)(1+\lambda_4)\left(1+\frac{\lambda_1+\lambda_2}{2}\right)\left(1+\frac{\lambda_2+\lambda_3}{2}\right)\cdot \\ & \cdot\left(1+\frac{\lambda_2+\lambda_4}{2}\right)\left(1+\frac{\lambda_1+\lambda_2+\lambda_3}{3}\right)\left(1+\frac{\lambda_1+\lambda_2+\lambda_4}{3}\right)\left(1+\frac{\lambda_2+\lambda_3+\lambda_4}{3}\right)\cdot \\ & \cdot\left(1+\frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{4}\right)\left(1+\frac{\lambda_1+2\lambda_2+\lambda_3+\lambda_4}{5}\right) \end{aligned}$$

This formula exhibits a complete symmetry in the variables λ_1, λ_3, λ_4 associated to the three 8-dimensional fundamental representations.

VII. Examples of application of the theory of characters

1. Weight diagrams of A_2.



$D^3(1,0)$ $L = \frac{1}{3}(2e_1 - e_2 - e_3)$

$D^3(0,1)$ $L = \frac{1}{3}(e_1 + e_2 - 2e_3)$

$D^8(1,1)$ $L = e_1 - e_2$

$D^{10}(3,0)$ $L = (2e_1 - e_2 - e_3)$

$D^{\overline{10}}(0,3)$ $L = (e_1 + e_2 - 2e_3)$

$D^{27}(2,2)$ $L = 2(e_1 - e_2)$

Figure 6.

In Fig. 6 , we have given the weight diagrams of the representation of SU_3. The weight space is two dimensional. The highest weight L for each representation is indicated below the diagrams. It is seen that not all the weights of the representations 8 and 27 are simple. The zero weight in the '8' representation has multiplicity 2. While the zero weight of '27' has multiplicity 3. A rule about the multiplicity of the weights is that the multiplicity of weights on hexagons (boundary of weights) within hexagons goes on increasing by one till we reach a triangle, their it remains the same.

2. Characters of A_2

The characters of the different representation of A_2 can be found once the weights are given. In this case since the weights are given in a two dimensional plane we have the relation

$$\varphi_1 + \varphi_2 + \varphi_3 = 0$$

Then we have

$$\chi_3(\varphi) = \exp \frac{i}{3}(2\varphi_1 - \varphi_2 - \varphi_3) + \exp \frac{i}{3}(2\varphi_2 - \varphi_1 - \varphi_3) + \exp \frac{i}{3}(2\varphi_3 - \varphi_2 - \varphi_1)$$

making a change of variables

$$\varphi_2 - \varphi_1 = \Phi_3$$

$$\varphi_1 - \varphi_3 = \Phi_2$$

$$\varphi_3 - \varphi_2 = \Phi_1$$

Which still preserves the relation :

$$\phi_1 + \phi_2 + \phi_3 = 0$$

we have

$$\chi_3(\phi) = \exp \frac{i}{3}(\phi_2 - \phi_3) + \exp \frac{i}{3}(\phi_3 - \phi_1) + $$
$$+ \exp \frac{i}{3}(\phi_1 - \phi_2)$$

We also obtain

$$\chi_8(\phi) = 2\left[1 + \cos\phi_1 + \cos\phi_2 + \cos\phi_3\right]$$

$$\chi_{10}(\phi) = 1 + 2\left[\cos\phi_1 + \cos\phi_2 + \cos\phi_3\right] +$$
$$+ \exp i(\phi_2 - \phi_3) + \exp i(\phi_3 - \phi_1) + \exp i(\phi_1 - \phi_2)$$

$$\chi_{\overline{10}}(\phi) = 1 + 2\left[\cos\phi_1 + \cos\phi_2 + \cos\phi_3\right] +$$
$$+ \exp i(\phi_3 - \phi_2) + \exp i(\phi_1 - \phi_3) + \exp i(\phi_2 - \phi_1)$$

$$\chi_{27}(\phi) = 3 + 2\left[\cos\phi_1 + \cos\phi_2 + \cos\phi_3\right] +$$
$$+ 2\left[\cos 2\phi_1 + \cos 2\phi_2 + \cos 2\phi_3\right] +$$
$$+ 4\left[\cos\phi_1 \cos\phi_2 + \cos\phi_2 \cos\phi_3 + \cos\phi_3 \cos\phi_1\right]$$

3. Fundamental weight diagrams of D_4

The weight space is four dimensional with basis e_j $(j = 1, 2, 3, 4)$. The fundamental representations with their corresponding highest weights L are

8_{Sp} $\quad L = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$ $\quad D(1,0,0,0)$

$8_{Sp'}$ $\quad L = \frac{1}{2}(e_1 + e_2 + e_3 - e_4)$ $\quad D(0,1,0,0)$

8_V $\quad L = e_1$ $\quad D(0,0,1,0)$

28_A $\quad L = e_1 + e_2$ $\quad D(0,0,0,1)$

4. Fundamental characters of D_4

Knowing the highest weights the characters for the different representation can be immediately written down:

$$\chi_{Sp}(\phi) = 2\Big[\mathrm{Cos}\,\frac{\phi_1+\phi_2+\phi_3+\phi_4}{2} + \mathrm{Cos}\,\frac{\phi_1+\phi_2-\phi_3-\phi_4}{2}$$

$$\mathrm{Cos}\,\frac{\phi_1-\phi_2-\phi_3+\phi_4}{2} + \mathrm{Cos}\,\frac{\phi_1-\phi_2+\phi_3-\phi_4}{2}\Big]$$

The highest weights of the two spinor representations differ only in the sign of e_4 and hence their characters differ only in the sign of ϕ_4 :

$$\chi_{sp'}(\phi) = 2\left[\cos\frac{\phi_1+\phi_2+\phi_3-\phi_4}{2} + \cos\frac{\phi_1+\phi_2-\phi_3+\phi_4}{2} + \right.$$

$$\left. + \cos\frac{\phi_1-\phi_2-\phi_3-\phi_4}{2} + \cos\frac{\phi_1-\phi_2+\phi_3+\phi_4}{2}\right]$$

Also we have

$$\chi_V(\phi) = 2[\cos\phi_1 + \cos\phi_2 + \cos\phi_3 + \cos\phi_4]$$

$$\chi_A(\phi) = 4[1 + \cos\phi_1\cos\phi_2 + \cos\phi_2\cos\phi_3 +$$

$$+ \cos\phi_3\cos\phi_1 + \cos\phi_3\cos\phi_4 +$$

$$+ \cos\phi_2\cos\phi_4 + \cos\phi_1\cos\phi_4]$$

5. Inclusion $SU(3)/Z_3 \subset SO(8)$:

a) $A_2 \subset r$ D4

It is seen from section (4) that the representations 8_{Sp}, $8_{Sp'}$ and 8_V are inequivalent if one considers all orthogonal transformations. However using the characters one can show that if one restricts to the transformation contained in the A_2 sub-algebra of D4, the three eight dimensional representations 8_{Sp}, $8_{Sp'}$ and 8_V are equivalent and irreducible and that the adjoint representation 28_A of D_4 is reducible according to

$$28_R \Longrightarrow 8 \oplus 10 \oplus \overline{10}.$$

The inclusion $A_2 \subset D_4$ is realized by the projection of the four dimensional weight diagrams on the two dimensional plane defined by choosing any one of the four ϕ's to be zero and the sum of the other three equal to zero. For convenience, (to study the inclusion $A_2 \subset D_4$), we choose

$$\phi_4 = 0 \qquad \phi_1 + \phi_2 + \phi_3 = 0$$

Then we have

$$\chi_{sp}(\phi) = 2\left[1 + \cos\phi_1 + \cos\phi_2 + \cos\phi_3\right]$$

$$= \chi_8(\phi) \quad \text{of} \quad A_2$$

Similarly $\chi_{sp}(\phi) \Rightarrow \chi_8(\phi)$ and $\chi_v(\phi) \Rightarrow \chi_8(\phi)$

Thus the three 8 dimensional representation of D_4 are equivalent and irreducible with respect to the A_2 subalgebra. Further we have

$$\chi_A(\phi) \Rightarrow 4\left[1 + \cos\phi_1\cos\phi_2 + \cos\phi_2\cos\phi_3 + \right.$$

$$+ \cos\phi_3\cos\phi_1 + \cos\phi_1 + \cos\phi_2 +$$

$$\left. + \cos\phi_3\right]$$

$$= \chi_{10}(\phi) + \chi_{\overline{10}}(\phi) + \chi_8(\phi) \quad \text{of} \quad A_2.$$

Thus it is seen that for $A_2 \subset D_4$

$$8_{sp} \Longrightarrow 8 \; ; \; 8_{sp'} \Longrightarrow 8 \; ; \; 8_v \Longrightarrow 8 \; ;$$

$$28_A \Rightarrow 8 \oplus 10 \oplus \overline{10}$$

b) $SU_3 / Z_3 \subset D_4^* / Z_2$

If one defines D_4^* as the covering group of D_4 algebra their

$$D_4^* / Z_2 \simeq \Delta_4$$

One can realize three isomorphic but inequivalent groups of this type (Δ_{4sp} , $\Delta_{4sp'}$ and Δ_{4v}) by considering the tensorial powers of the eight dimensional representations 8_{sp}, $8_{sp'}$ and 8_v respectively. Further we have

$$\Delta_{4v} \simeq SO(8)$$

Where SO(8) is the orthogonal group on the eight dimensional space. Δ_4 contains SU_3 / Z_3 as a subgroup. The adjoint representation '8' of SU_3 / Z_3 is a self contragradient representation and can be described by 8 x 8 unitary unimodular **real matrices** which form a subset of 8 x 8 unimodular orthogonal matrices of SO(8).

VIII. THE SCHUR-FROBENIUS CLASSIFICATION:

1. We consider an unitary representation $U(a)$ of a compact semi-simple group G. If $U(a)$ is equivalent to its contragradient representation $U^+(a)$, there exist a constant matrix C such that

$$U^+(a) = C\,U(a)\,C^{-1} \quad \text{for all} \quad a \in G .$$

Taking into account the unitarity property written as $U^+ = U^{-1T}$, we also obtain,

$$C = U^T(a)\,C\,U(a)$$

The U transformations leave invariant a bilinear form in the N -dimensional representation space V_N.

2. By using a representation of the Lie algebra with $N \times N$ hermitian matrices, the transformation $U(a)$ can be written as

$$U(a) = \exp i\, a^\sigma X_\sigma$$

and the a^σ's are real parameters.

For the contragradient representation, we have :

$$U^+(a) = \exp i\, a^\sigma X'_\sigma$$

with

$$X'_\sigma = - X^T_\sigma$$

If the representations U and U^+ are equivalent, there exist a matrix C such that

$$X^T_\sigma = - C X_\sigma C^{-1}$$

for all the generators X_σ of the Lie algebra.

3. The properties of the matrix C can be obtained by iterating the basic relation. Without loss of generality, C can be chosen as unitary and using the Schur lemma, we can easily prove the following relations:

$$\left\{ \begin{array}{lll} C^T = \varepsilon_i C & & C^+ = \varepsilon_2 C \\ CC^* = I & CC^T = \varepsilon_2 I & C^2 = \varepsilon_1 \varepsilon_2 I \end{array} \right.$$

where $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$.

If a real representation can be used for $U(a)$, the matrix C can be chosen as real $(\varepsilon_2 = 1)$ and is an orthogonal matrix:

$$\left\{ \begin{array}{lll} C^T = \pm C & & C^+ = C \\ CC^* = I & CC^T = I & C^2 = \pm I \end{array} \right.$$

4. The Schur Frobenius classification:

An irreducible representation belongs to the class $\lambda = 1$, $\lambda = 0$ if it leaves invariant respectively:

a $\lambda = 1$ a symmetrical bilinear form $(\varepsilon_1 = +1)$

b $\lambda = 0$ no bilinear form

c $\lambda = -1$ an antisymmetrical bilinear form $(\varepsilon_2 = -1)$

5. Application:

As a consequence of the properties of the fundamental representations of the simple Lie groups obtained in the previous sections, all the irreducible representations of $B_\ell, C_\ell, D_{2\ell}$ belong to the classes $\lambda = \pm 1$.

Chapter 6

TENSOR ALGEBRA OF THE LINEAR GROUP.

I. Generalities

1. If H is a sub group of G , the irreducible representations of G can be either irreducible representations of H or reducible into a direct sum of irreducible representations of H .

2. The irreducible representations of a compact semi-simple group G can be taken as unitary. The unitary matrices of a N dimensional representation of G generate a sub-group of the unitary group .

3. It follows that the irreducible representations of a compact semi-simple group G can be studied from the irreducible representations of the unitary groups.

The importance of the tensor algebra of the unitary group is essentially due to this property.

4. It is convenient for simplicity to speak the language of the general linear group $GL(n,R)$ instead of that of the unitary group $U(n,R)$. As it has be shown in the previous chapter the two languages are equivalent from the point of view of irreducible representations.

II. Irreducible Representations of GL(n,R):

1. We consider a n -dimensional real vector space $E(n,R)$ and the dual real vector space $E^*(n,R)$, which is the space of the linear forms on E. The elements of E are called contravariant vectors and the elements of E^* covariant vectors

$$\overline{\eta} \in E \qquad x \in E^*$$

2. A contravariant tensor ξ_p is an element of the tensorial power of order p of E:

$$\xi_p \in E^{\otimes p} \qquad \xi_1 \in E \qquad \xi_0 \in R$$

In the same way, it is easy to introduce covariant tensors as the elements of $E^{*\otimes q}$ and mixed tensors as the elements of $E^{\otimes p} \otimes E^{*\otimes q}$

3. We now consider the general linear group $GL(n,R)$ and the unimodular linear transformations $SL(n,R)$.

The irreducible tensors can be associated in a one-to-one correspondance to the irreducible representations of the permutation group. We don't give the proof of this important result.

4. The irreducible representation of the permutation group of p elements G_p are easily described by using the Young tables and the Young diagrams.

A Young table is a set of n non negative integer numbers such that:

$$f_1 \geqslant f_2 \geqslant \cdots\cdots \geqslant f_n \geqslant 0$$

with the restriction

$$\sum_{j=1}^{j=n} f_j = p$$

In other terms, it is a partition of the number p

The associated Young diagram is a set of p boxes divided in n rows with f_j boxes in the j^{th} row

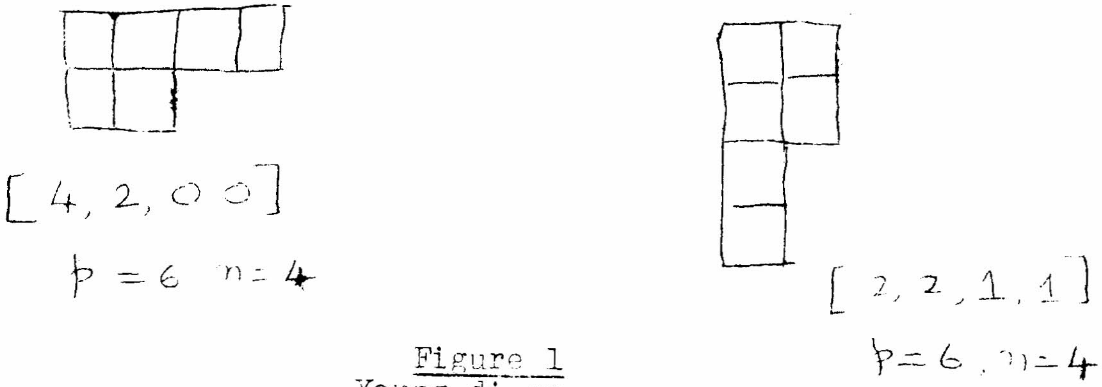


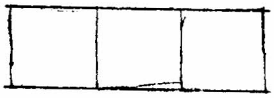
Figure 1
Young diagrams

5. We now go back to a contravariant tensor of order p, the dimension of E being precisely n. The Young table describes a symmetry of the tensor and the properties are the following:

a the indices associated to each box of a horizontal row are symmetrized

b the indices associated to each box of a vertical column are antisymmetrized.

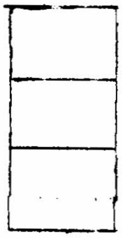
For instance, a completely symmetrized tensor of rank p is associated to the partition $f_1 = p \quad f_2 = f_3 = \cdots = f_n = 0$ and the corresponding Young diagram has only one row.



[3,0,0,0]
$p = 3 \quad n = 4$
Figure 2

Such a tensor is an element of the vector sub-space $SE^{\otimes p}$ of the completely symmetrical tensors. The dimension of $SE^{\otimes p}$ is the combination number C^{p}_{n+p-1}

A completely antisymmetrized tensor of rank p is associated to the partition $f_1 = f_2 = \cdots = f_p = 1$ $f_{p+1} = f_{p+2} = \cdots f_n = 0$ and the corresponding Young diagram has only one column



[1,1,1,0]

$p = 3 \quad n = 4$

Figure 3.

Such a tensor is an element of the vector sub-space $AE^{\otimes p}$ of the completely antisymmetrical tensors. The dimension of $AE^{\otimes p}$ is the combination number C^{p}_{n}. Of course, it is not possible to construct a completely antisymmetrical tensor of order $p > n$ and the maximum number of rows of a Young diagram is precisely n.

6. For a covariant tensor of order ν, element of $E^{*\otimes\nu}$, the previous results can be extended in the following way. To each partition of the number ν, we associate a set of non positive integer numbers

$$0 \geq f_1 \geq f_2 \geq \cdots \geq f_n$$

with the restriction

$$\sum_{j=1}^{j=n} f_j = -q$$

The corresponding Young diagram is a set of q boxes divided into n rows with $-f_j$ boxes in the j^{th} row


$[-1,-1,-1,-3]$
$q=6$ $n=4$

$[0,0,-3,-3]$
$q=6$ $n=4$

Figure 4

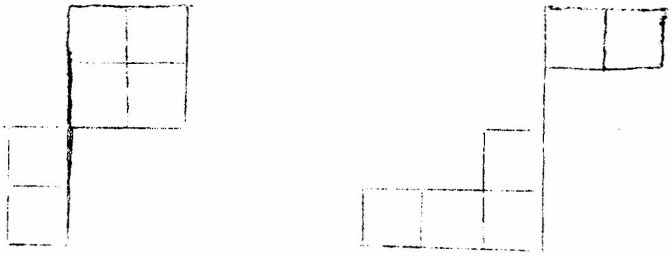
For a mixed tensor, element of $E^* \otimes q$ one associates to each partition of p and q, a set of n algebraic integer numbers

$$f_1 \geqslant f_2 \geqslant \cdots \geqslant f_j \geqslant 0 \geqslant f_{j+1} \geqslant \cdots \geqslant f_k$$

such that

$$\sum_{k=1}^{k=j} f_k = p \qquad \sum_{k=j+1}^{k=n} f_k = -q$$

The corresponding Young diagram is then immediately drawn by using the previous results.



[2, 2, -1, -1]
$p = 4,\ q = 3,\ n = 4$

[2, 0, -1, -3]
$p = 2,\ q = 4\ \ n = 4$

Figure 6
Young diagrams for mixed tensors

7. We are now interested with the completely antisymmetrized one component tensor of order n .

Let us introduce in E a basis e_j the corresponding basis in $A\ E^{\otimes n}$ can be written with exterior tensorial products as:

$$\overline{e}_1 \wedge \overline{e}_2 \wedge \cdots \cdots \wedge \overline{e}_n$$

We consider n linearly independent vectors $\overline{x}_{(j)}$ of E the completely antisymmetrized product

$$\overline{x}_{(1)} \wedge \overline{x}_{(2)} \wedge \cdots \cdots \wedge \overline{x}_{(n)}$$

is the only linearly independent element of $A\ E^{\otimes n}$.

By using the coordinates

$$\overline{x}_{(j)} = x^{k}_{(j)}\ \overline{e}_k$$

We immediately obtain

$$\overline{x}_{(1)} \wedge \overline{x}_{(2)} \wedge \cdots \cdots \overline{x}_{(n)} = \left[\sum_{[\sigma]} \chi_{(\sigma)} x^{\sigma_1}_{(1)} \cdots \cdots x^{\sigma_n}_{(n)} \right]$$
$$\times\ \overline{e}_1 \wedge \overline{e}_2 \wedge \cdots \cdots \wedge \overline{e}_n$$

where $[\sigma]$ is the permutation $\begin{vmatrix} 1 & 2, \ldots, & n \\ \sigma_1 & \sigma_2 \ldots & \sigma_n \end{vmatrix}$ of parity $\chi(\sigma)$. The bracket is simply the determinant $D(\overline{x}_{(1)}, \overline{x}_{(2)}, \ldots, \overline{x}_{(n}$

$$\overline{x}_{(1)} \wedge \overline{x}_{(2)} \ldots, \overline{x}_{(n)} = D(\overline{x}_{(1)}, \overline{x}_{(2)}, \ldots, \overline{x}_{(n)})$$
$$= \overline{e}_1 \wedge \overline{e}_2 \wedge \ldots \wedge \overline{e}_n$$

We now perform a regular linear transformation in , represented by the element A of $GL(n, R)$:

$$\overline{x}'_{(k)} = \overline{x}_{(j)} A^j_k$$

By using the previous results

$$\overline{x}'_{(1)} \wedge \overline{x}'_{(2)} \wedge \ldots \wedge \overline{x}'_{(n)} = \operatorname{Det} A \; \overline{x}_{(1)} \wedge \overline{x}_{(2)} \wedge \ldots \overline{x}_{(n)}$$

and finally,

$$\operatorname{Det}(\overline{x}'_{(1)}, \overline{x}'_{(2)}, \ldots, \overline{x}'_{(n)}) = \operatorname{Det} A \operatorname{Det}(\overline{x}_{(1)} \ldots \overline{x}_{(n)})$$

If now, A is an unimodular matrix, element of the special linear group $SL(n, R)$ the quantity $\operatorname{Det}(\overline{x}_{(1)}, \overline{x}_{(2)}, \ldots, \overline{x}_{(n)})$ is an invariant.

III. IRREDUCIBLE REPRESENTATIONS OF SL(n,R):

1. We have just seen that for the unimodular linear transformations of E(n,R), the one component representation $[1,1,\dots,1]$ is invariant. In other terms, the two representations $[1,1,\dots,1]$ and $[0,0,0,\dots,0]$ are equivalent. In a more general way the two inequivalent representations of GL(n,R),

$$[f_1\ f_2 \dots f_n] \quad \text{and} \quad [f_1'\ f_2' \dots f_n']$$

are equivalent in SL(n,R) if and only if

$$f_j' = f_j + S \qquad j = 1, 2 \dots n$$

where S is an algebraic integer number independent of j.

In particular, in SL(n,R), the two representations

$$[f_1,\ f_2 \dots f_{n-1}\ f_n] \quad \text{and} \quad [f_1 - f_n,\ f_2 - f_n \dots f_{n-1} - f_n\ 0]$$

are equivalent and with respect to unimodular transformations, any tensor is equivalent to a contravariant tensor with $f_n = 0$.

2. It is convenient to introduce the completely anti-symmetrical tensor of order n, element of $\wedge\ E^{*\otimes n}$, $\varepsilon_{\sigma_1 \sigma_2 \dots \sigma_n}$, defined by

$$\varepsilon_{1\,2 \dots n} = 1 \qquad \varepsilon_{\sigma_1 \sigma_2 \dots \sigma_n} = \chi(\sigma)$$

In this language, we have:

$$\mathrm{Det}\left(\bar{x}_{(1)}\ \bar{x}_{(2)}, \dots, \bar{x}_{(n)}\right) = \sum_{[\sigma]} \varepsilon_{\sigma_1 \sigma_2 \dots \sigma_n} x^{\sigma_1}_{(1)} x^{\sigma_2}_{(2)} \dots x^{\sigma_n}$$

The determinant is invariant under unimodular transformations but transforms with a factor $\mathrm{Det}\, A$ for all $A \in GL(n,R)$.

In the same way, the quantity:

$$\varepsilon_{\sigma_1 \sigma_2 \cdots \sigma_n} \; x^{\sigma_2}_{(2)} \cdots \; x^{\sigma_n}_{(n)}$$

transforms like a covariant vector of E^{*} under unimodular transformations. A straightforward generalization of this result is the following: the two irreducible representations of $GL(n,R)$:

$$[\, f_1, f_2, \ldots f_n \,] \quad \text{and} \quad [\, f_1 + S,\; f_2 + S, \cdots, f_n + S \,]$$

are equivalent for unimodular transformations but we have to add an extra factor $(\mathrm{Det}\, A)^{S}$ if now A is an element of the general linear group.

3. As a consequence, an irreducible representation of $GL(n,R)$ is also an irreducible representation of $SL(n,R)$. It is then sufficient to study the irreducible representation of $SL(n,R)$ and with the previous statement we are able to deduce all the irreducible representations of $GL(n,R)$.

Such a result was expected because the homomorphism from $GL(n,R)$ into $SL(n,R)$ is a central homomorphism.

Of course, the representations $[\, f_1 + S,\; f_2 + S, \cdots f_n + S \,]$ have the same dimension independent of S.

4. The irreducible representations of $SL(n,R)$ can then be characterized by a set of $(n-1)$ non negative integer numbers.

It is convenient to work with the representation $[f_1, f_2, \dots f_{n-1}, 0]$ and to define

$$\lambda_j = f_j - f_{j+1} \qquad j = 1, 2, \dots n-1 .$$

and conversely

$$f_j = \sum_{k=j}^{k=n-1} \lambda_k$$

The representation $(\lambda_1, \lambda_2, \dots \lambda_{n-1})$ can be associated to a contravariant tensor of order p given by

$$p = \sum_{j=1}^{j=n-1} f_j = \sum_{k=1}^{k=n-1} k \lambda_k$$

The representations R^k with all the λ_j's equal to zero, except $\lambda_k = 1$, corresponds to a completely antisymmetrical tensor of order k .

$$f_1 = f_2 = \dots = f_k = 1, \quad f_{k+1} = f_{k+2} = \dots = f_{n-1} = f_n = 0$$

The dimension of this representation is C_n^k as the dimension of the fundamental representation F^k of the A_{n-1} Lie algebra discussed in Chapter IV. It can be shown that R^k and F^k are isomorphic and more generaly, the highest weight of the irreducible representation $[f_1\ f_2, \dots f_{n-1}, 0]$ is simply given by

$$L = \sum_{j=1}^{j=n-1} \left(f_j - \frac{p}{n} \right) e_j$$

For instance, the n dimensional fundamental representation of A_{n-1}, F^1 is associated to the vectors of E and the contragradient representation to the vectors of [illegible] considered also as completely antisymmetrized contravariant tensors of order $n-1$.

IV. ADJOINT REPRESENTATION

1. The Lie algebra of the general linear group $GL(n,R)$ is the set of n^2 infinitesimal generators X_σ. It has been shown thus the linear combination $X = \sigma_2 X_\sigma$ commutes with all the generators.

The $n^2 - 1$ generators X_σ of trace zero, generate the simple Lie algebra of type A_{n-1} of the special linear group $SL(n,R)$.

We also consider a sub-algebra of A_{n-1}, with infinitesimal generators L_σ and the commutation laws

$$[L_\rho, L_\sigma] = C_{\rho\sigma}^{\tau} L_\tau$$

Of course, this sub-algebra can be A_{n-1} itself.

2. The adjoint representation of the Lie algebra A_{n-1} $D(1, 0, \ldots 0, 1)$, can be associated, from the previous results of section III, to an irreducible mixed tensor of order 2, or equivalently, to a covariant tensor of order $n+1$. We now study the mixed tensor of order two.

3. The second order mixed tensor ξ^{R}_{σ} are the elements of $E \otimes E^*$. Let us now consider the representation with unitary matrices of the Lie algebra A_{n-1}

$$UU^* = I$$

The infinitesimal generators can be represented by hermitic matrices following

$$U = I + i\varepsilon^\rho X_\rho \qquad U^* = I - i\varepsilon^\rho X_\rho$$

where ε^ρ is a set of real infinitesimal parameters and

$$X_\rho^* = X_\rho$$

The tensor ξ is transformed, according to

$$\xi' = U^* \xi U$$

and, for infinitesimal transformation, we obtain

$$\xi'^{k}_{j} = \left\{ \delta^m_j - i\varepsilon^\rho [X_\rho]^m_j \right\} \xi^n_m \left\{ \delta^k_n + i\varepsilon^\sigma [X_\sigma]^k_n \right\}$$

and after reduction

$$\xi'^{k}_{j} = \xi^k_j - i\,\varepsilon^\rho \left\{ [X_\rho]^m_j\, \delta^k_n - [X_\rho]^k_n\, \delta^m_j \right\} \xi^n_m$$

From the previous expression we immediately verify the invariance of the trace of ξ

$$\mathrm{Tr}\,\xi = \delta^j_k\, \xi^k_j$$

4. We now consider the quantities

$$\phi_\sigma = [L_\sigma]^j_k\, \xi^k_j$$

The transformation laws of the ϕ_σ' s are deduced from those of the $\xi^{k'}_{j}$ s , taking into account the commutation laws of the Lie sub-algebra we obtain,

$$\phi_\sigma' = \phi_\sigma + i\,\varepsilon^{\rho}\, C_{\rho\sigma}^{\tau}\,\phi_\tau .$$

In the basis of the ϕ_σ 's , the infinitesimal generators of the Lie algebra $[L_\sigma]$ are represented by the structure constants of this Lie algebra

$$[L_\sigma]^{\tau}_{\rho} = C_{\rho\sigma}^{\tau}$$

This result can be also interpreted as a consequence of the Jacobi identify satisfied by the structure constants. The dimension of the representation is the dimension of the Lie algebra and we have extracted the adjoint representation of the Lie algebra.

The adjoint representation is irreducible if and only if the Lie algebra is simple. For instance, the n^2-1 quantities

$$\Phi_\sigma = [X_\sigma]^{j}_{k}\,\xi^{k}_{j}$$

are a basis of the adjoint representation of SL(n,R)

5. The n^2 components of the second order mixed tensor , have been reduced in the following way:

<u>a</u> The invariant trace $\delta^{j}_{k}\,\xi^{k}_{j}$

<u>b</u> The n^2-1 components of trace zero $\xi^{k}_{j} - \frac{1}{n}\delta^{k}_{j}\,\mathrm{Tr}\,\xi$

V. PRODUCT OF REPRESENTATION.

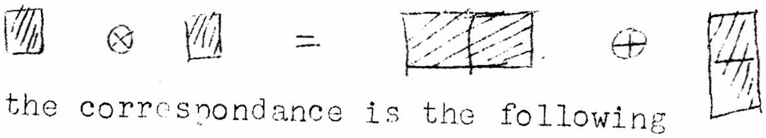
The reduction of a product of representation is the determination of the irreducible components of a tensor.

1. Second Order tensors.

We first consider the case of a contravariant tensor. The indices can be symmetrized and antisymmetrized following the decomposition of the tensorial product into a symmetrical and an exterior product

$$\overline{x}_1 \otimes \overline{x}_2 = \overline{x}_1 \vee \overline{x}_2 \oplus \overline{x}_1 \wedge \overline{x}_2$$

In terms of Young diagrams, we have:



and the correspondance is the following

 corresponds to $\overline{x}_1 \vee \overline{x}_2$ with $\frac{n(n+1)}{2}$ components

 corresponds to $\overline{x}_1 \wedge \overline{x}_2$ with $\frac{n(n-1)}{2}$ components

In the general linear group the product of representations is written as

$$[1,0,0,\dots 0] \otimes [1,0,\dots\dots 0] = [2,0,0,\dots 0] \oplus [1,1,0\dots 0]$$

and in the special linear group, the corresponding expression is

$$D(1,0,0,\dots,0) \otimes D(1,0,0\dots 0) = D(2,0,0,0,\dots) \oplus D(0,1,0,\dots,0)$$

The same results can easily be obtained for covariant second order tensors using

$$\underline{x}_1 \otimes \underline{x}_2 = \underline{x}_1 \vee \underline{x}_2 \oplus \underline{x}_1 \wedge \underline{x}_2$$

and we obtain the same expressions for the product of the contragradient representations

$$[0,\cdots,0,-1] \otimes [0,\cdots,0,0,-1] = [0,\cdots,0,0,-2] \oplus [0,\cdots,0,-1,-1]$$

in GL(n,R) and for SL(n,R)

$$D(0,\cdots,0,1) \otimes D(0,\cdots,0,0,1) = D(\cdots,0,0,2) \oplus D(0,\cdots,0,1,0)$$

The case of a mixed second order tensor has been studied with some details in the previous section. In terms of product of representations we obtain simply in GL(n,R).

$$[1,0,\cdots,0,0] \otimes [0,0,\cdots,0,-1] = [0,0,\cdots,0,0] \oplus [1,\cdots,0,-1]$$

and in SL(n,R)

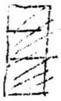
$$D(1,0\cdots,0,0) \otimes D(0,0\cdots,0,1) = D(0,0,\cdots,0,0) \oplus D(1,0\cdots,0$$

2. Third order contravariant tensor:

We use the method of the Young diagrams and we have only three possibilities

with $\frac{n(n+1)\ (n+2)}{6}$ components

with $\frac{n(n^2-1)}{3}$ components



with $\frac{n(n-1)\ (n-2)}{6}$ components

The second possibility can be reached in two different ways and we obtain the following reduction in GL(n,R)

$$[1,0\cdots 0]^{\otimes 3} = [3,0,0,\cdots 0] \oplus 2[2,1,0\cdots 0] \oplus [1,1,1,0\cdots 0]$$

and in $SL(n,\mathbb{R})$ we have

$$D(1,0,0\cdots 0) \otimes D(1,0,0\cdots 0) \otimes D(1,0\cdots 0) = D(3,0\cdots 0) \oplus 2D(1,1,0\cdots 0)$$
$$\oplus D(0,0,1,0\cdots 0)$$

3. General Case

Let us consider two irreducible representations $[f]$ and $[\bar{f}']$ of $GL(n,R)$. It is always possible to introduce the representations $[f]$ and $[f']$ equivalent in $SL(n,\mathbb{R})$ respectively to $[\bar{f}]$ and $[\bar{f}']$ and such that $f_n \cdot f'_n = 0$

We are then working with representations $[f]$ and $[f']$ associated to contravariant tensors where all the f_j's and f'_j's are positive. The best way to reduce the product $[f] \otimes [f']$ is to use the Young diagrams following the Littlewood method.

The $[f]$ diagram has f_1 boxes α, f_2 boxes β , f_3 boxes γ , etc. The boxes of the diagram $[f]$ are added to the diagram $[f']$ in the following way.

a With the α's , we form a new Young diagram, excluding the case where two boxes α are in the same column.

b With the β's , we form a new Young diagram, excluding the first row and the case where two boxes β are in the same column.

c With the γ's, we form a new Young diagram, excluding the first and the second rows and the case where two boxes γ are in the same column.

and so an with all the boxes of the diagram $[f]$.

4. As an example, the Littlewood method can be used to reduce the product of two adjoint representations of the Lie algebra A_{n-1}. The result, written in GL(n,R) is the following

$$[1,0,\cdots,0,-1] \otimes [1,0,\cdots,0,-1]$$

$$= [0,\cdots,0] \oplus 2[1,0,\cdots,0,-1]$$

$$\oplus [2,0,\cdots,0,-1,-1] + [1,1,0\cdots 0,-2] \oplus [1,1,0,\cdots,0,-1,-1]$$

$$\oplus [2,0 \cdots 0,-2]$$

and in SL(n,R), we obtain

$$D(1,0,0\cdots 0,1) \otimes D(1,0,\cdots 0,1) = D(0\cdots 0) \oplus 2D(1,0\cdots 0,1)$$

$$\oplus D(2,0\cdots 0,1,0) \oplus D(0,1,0\cdots 0,2) \oplus D(0,1,0\cdots 0,1,0)$$

$$\oplus D(2,0\cdots 0,2)$$

The dimension of these irreducible representations can be calculated using the general formula given in the Chapter XV. We add the symbol S or A according as the representation enters in the symmetrical or in the antisymmetrical part of the product

$$\left\{\begin{array}{lll} N(0\cdots 0) = 1 & & S \\ N(1,0\cdots 0,1) = n^2-1 & & S \text{ and } A \\ N(2,0\cdots 0,1,0) = \frac{1}{4}(n^2-4)(n^2-1) & & A \\ N(0,1,0\cdots 0,2) = \frac{1}{4}(n^2-4)(n^2-1) & & A \\ N(0,1,0\cdots 0,1,0) = \frac{1}{4}(n-3)n^2(n+1) & \cdots & S \\ N(2,0\cdots 0,2) = \frac{1}{4}(n-1)n^2(n+3) & \cdots & S \end{array}\right.$$

The representations being denoted by their dimensionality, we obtain

$$\left\{\begin{array}{ll} \ell=1 & 3\otimes 3 = 1\oplus 3\oplus 5 \\ \ell=2 & 8\otimes 8 = 1\oplus 8_s\oplus 8_a\oplus 10\oplus \overline{10}\oplus 27 \\ \ell=3 & 15\otimes 15 = 1\oplus 15_s\oplus 15_a\oplus 45\oplus \overline{45}\oplus 20\oplus 84 \\ \ell=4 & 24\otimes 24 = 1\oplus 24_s\oplus 24_a\oplus 126\oplus \overline{126}\oplus 75\oplus 200 \\ \ell=5 & 35\otimes 35 = 1\oplus 35_s\oplus 35_a\oplus 280\oplus \overline{280}\oplus 189\oplus 405 \end{array}\right.$$

Excepted the case $\ell = 1$, the adjoint representation is present in both the symmetrical and the antisymmetrical part of the product of two adjoint representations.