THE INSTITUTE OF MATHEMATICAL SCIENCES

MADRAS-4 (India)

Lectures on

THE THEORY OF STRONG INTERACTIONS

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PRELIMINARY VERSION

Notes + by

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+ These notes have not been looked into by the lecturer.

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CHAPTER I

Introduction

In this series of lectures we shall mainly discuss the different possible ways of using dispersion relations in the description of the strong interactions of the elementary particles. I shall try to make these lectures complementary to Professor Roman's lectures*, where he gave an exhaustive discussion of dispersion relations for potential scattering.

At present, the method of dispersion relations seems to provide the most useful approach to the study of strong interactions. The older Lagrangian approach to strong interactions was found to have several limitations. The approach using dispersion relations is less ambitious than the Lagrangian approach. Here we do not hope to derive everything from a Lagrangian postulated at the outset; instead, we impose various general requirements and attempt to derive relations between scattering amplitudes.

In a theory using dispersion relations, we examine what restrictions can be obtained on the amplitude for a scattering or reaction process when we impose the requirements of (prescribed) analyticity, unitarity, and crossing symmetry. Prescription of the analyticity properties means that we must know the singularities of the scattering amplitude.

The main advantage of the technique of dispersion relations is that experimental information can be easily incorporated into the theory, and that we can derive relations between the amplitudes

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* Prof. Paul Roman: Lectures on Dispersion Relations; lectures given at the Institute of Mathematical Sciences, Madras; Sept.--Oct. 1963 -- Matscience Report No.20.

for different processes, which can be checked with experiment, e.g. we can relate the pion-pion interaction and nucleon form factors.

One of the most striking features of strong interactions is the recent discovery of a host of resonances. Low-energy phenomena seem to be dominated by the resonances. We can reasonabl hope that the contribution of the resonances gives the major part of scattering and reaction cross-sections and that the rest is either negligible or can be approximated by a few functions.

The occurrence of these resonances makes it simple to incorporate the experimental information, i.e., as the contribution of a few resonances. This makes dispersion relations particularly useful for describing low-energy phenomena.

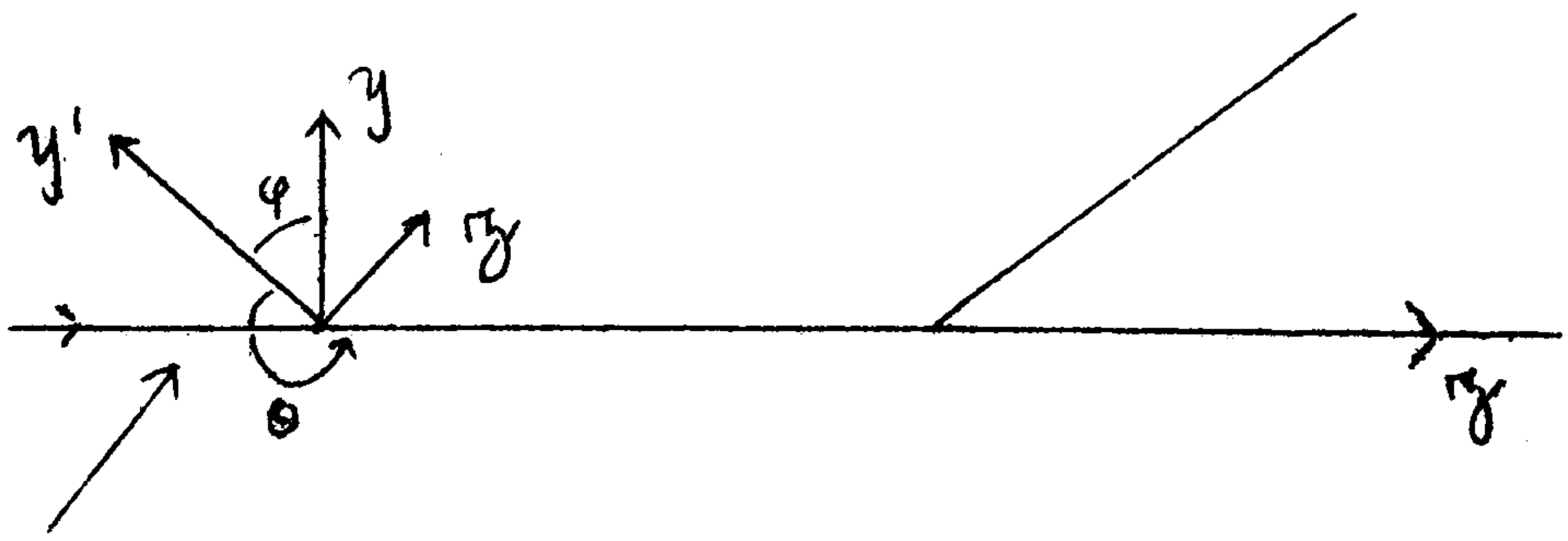
We shall begin this series of lectures by discussing kinematical generalities and some of the formalism of scattering theory We shall go on later to a detailed discussion of dispersion relations.

II. Kinematical Generalities.

Most of the information on strong interactions is obtained from scattering experiments.

1. General Expressions for Matrix Elements and Cross-sections.

Consider a typical scattering experiment, with coordinat chosen as illustrated in the diagram below:



Incident beam.

We summarise our notation briefly below: $(\hbar = c = 1)$

$$\frac{dW}{d\Omega} = N v n \frac{\partial\sigma}{\partial\Omega} \qquad (1)$$

Velocity $v = k/E$ $\qquad \lambda\!\!\!^{-} = 1/k$

$$\sigma = \int \frac{\partial\sigma}{\partial\Omega}\, d\Omega \qquad (2)$$

Assume that the different scattering centres are incoherent. To describe the scattering process, we define 2 complete sets of states at $t = -\infty$ and $t = +\infty$ which are free-particle states. Denote this by

$$\left.\begin{array}{ll} |\alpha\ in\rangle & t \to -\infty \\ \text{and} \quad |\alpha\ out\rangle & t \to +\infty \end{array}\right\} \qquad (3)$$

The transition from one basis to the other is effected by a <u>unitary operator</u>, the 'S-matrix' S, defined by

$$S|\alpha\ \text{out}\rangle = |\alpha\ \text{in}\rangle \tag{4}$$

$$S^{+} = S^{-1} \tag{4a}$$

In a scattering process we observe a transition from a state $|\alpha\ \text{in}\rangle$ to a state $|\alpha\ \text{out}\rangle$.

$|\alpha\ \text{in}\rangle$ and $|\alpha\ \text{out}\rangle$ each form a complete set, in the absence of bound states. If we know all the products

$$\langle\beta\ \text{out}|\alpha\ \text{in}\rangle$$

then we would know all matrix elements. $|\alpha\ \text{in}\rangle$ is an in state with quantum numbers α and $|\beta\ \text{out}\rangle$ is an out state with quantum numbers β. We have

$$\langle\beta\ \text{out}|\alpha\ \text{in}\rangle = \langle\beta\ \text{out}|S|\alpha\ \text{out}\rangle = S_{\beta\alpha} \tag{5}$$

where we have transformed to states in terms of the same basis. We could also write

$$\langle\beta\ \text{out}|\alpha\ \text{in}\rangle = \langle\beta\ \text{in}|S|\alpha\ \text{out}\rangle = S_{\beta\alpha} \tag{5a}$$

which gives the S-matrix in the in-basis. An important operator is the <u>transition operator</u> T, defined by

$$S = I + iT \tag{6}$$

where I is the identity operator. Before we can derive the cross-section in terms of T, we must define the normalization of our states.

Define the normalization

$$\langle \vec{k}', s' | \vec{k}, s \rangle = (2\pi)^3 \delta_{ss'} \delta(\vec{k} - \vec{k}') \tag{7}$$

[Note: This normalization is not invariant.]

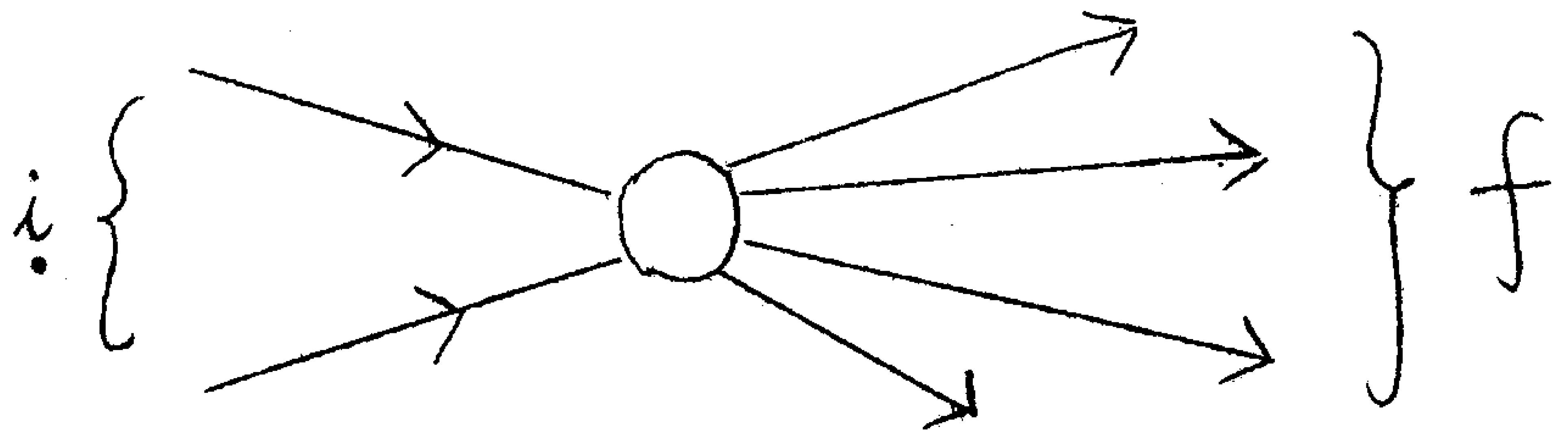
We also have the closure property:

$$\sum_{\vec{k}', s'} | \vec{k}', s' \rangle \langle \vec{k}', s' | k, s \rangle \frac{d^3 \vec{k}'}{(2\pi)^3} = | k, s \rangle \tag{8}$$

One way to write the T-matrix defined in equation (6) is the following:

$$T_{fi} = (2\pi)^4 \delta(P_f - P_i) \frac{F_{fi}}{(2E_1^i \, 2E_2^i \, 2E_1^f \, 2E_2^f \cdots)^{\frac{1}{2}}} \tag{9}$$

This is for a process with 2 initial particles i and any number of final particles f.



The operator F_{fi} defined thus is a Lorentz-invariant quantity.

To define the cross-section, take $|T|^2$ sum over all final states, and divide by the incident flux.

Example: Consider a particle incident on a stationary target particle. The transition probability is given by

$$\omega = \frac{1}{v}\int (T_{fi})^2 \prod_j \frac{d^3\vec{k}_j}{(2\pi)^3} \tag{10}$$

where v is the incident particle velocity. (The correct way to proceed would be to introduce wave packets, etc, and find the transition probability per unit time. But we know that the same results can be obtained from a δ - function normalization.)

Write (10) as

$$\sigma = \frac{1}{v}\int |F_{fi}|^2 \prod_j \frac{d^3\vec{k}_j}{(2\pi)^3}(2\pi)^4\delta(P_f - P_i)\frac{1}{(2E_1^i 2E_2^i\, 2E_1^f 2E_2^f \cdots)} \tag{11}$$

The incident flux is simply the incident velocity v . We have

$$v = q_1/E_1$$

$$\sigma = \frac{E_1^i}{q}\frac{(2\pi)^4}{4E_1^i E_2^i}\int |F_{fi}|^2\delta(P_f - P_i)\prod_j \frac{d^3\vec{k}_j}{(2\pi)^3\, 2E_j^f} \tag{12}$$

Note:

$$\frac{E_1^i(2\pi)^4}{q\,4E_1^iE_2^i} = \frac{(2\pi)^4}{4q\;E_2^i}$$

The total cross-section thus obtained is a Lorentz-invariant quantit

To see this, note that

$$\prod_j \frac{d^3k_j}{(2\pi)^3\, 2E_j^f} \tag{13}$$

is Lorentz-invariant, for

$$d^3\vec{k}_j/2E_j = \int d^4k_j\,\delta(k_j^2 + m_j^2)\,\theta(k_{j0}) \tag{14}$$

the integration being over the 4th component of k only.

Then equation (12) may be written in an explicitly invariant form.

The factor $(2\pi)^4 / 4\, q\, E_2^i$ is just the expression in the lab. system for the quantity.

$$(2\pi)^4 \Big/ \sqrt{(q_1 \cdot q_2)^2 - (m_1 m_2)^2} \qquad (15)$$

Substituting these into (12), we obtain

$$\sigma = \frac{(2\pi)^4}{4\left((q_1 \cdot q_2)^2 - (m_1 m_2)^2\right)^{1/2}} \int |F_{fi}|^2 \delta(P_f - P_i) \prod_j d^4 k_j \, \delta(k_j^2 + m_j^2)\, \theta(k_{j0}) \qquad (16)$$

which is explicitly covariant.

This may be specialised to any particular frame.

Example. <u>The centre-of-mass system:</u> (C.M.S)

Consider 2-body scattering in the c.m.s. Let $\vec{q}$ be the centre-of-mass momentum. (The c.m.s. is defined as the frame in which the total 3-momentum is zero.)

$$\begin{aligned}(q_1 \cdot q_2)^2 - (m_1 m_2)^2 &= (-q^2 - E_1 E_2)^2 - (m_1 m_2)^2 \\ &= q^4 + (E_1 E_2)^2 + 2 q^2 E_1 E_2 - m_1^2 m_2^2 \\ &= q^2 \{ 2q^2 + m_1^2 + m_2^2 + E_1 E_2) \\ &= q^2 (E_1 + E_2)^2 = q^2 W^2 \end{aligned} \qquad (17)$$

where W is the total energy in the c.m.s.

$$W = E_1 + E_2 \tag{17}$$

Therefore in the c.m.s., we have

$$\sigma = \frac{(2\pi)^4}{4qW}\int |F_{fi}|^2 \delta(P_f - P_0)\,\frac{d^3\vec{k}_1\, d^3\vec{k}_2}{(2\pi)^6 (2E_1)(2E_2)} \tag{18}$$

as the cross-section for a 2-body process, of the type A + B → C + D

$$\sigma = \frac{1}{16(2\pi)^2 qW}\int |F_{fi}|^2 \delta\left(\sqrt{k_1^2+m_1^2} + \sqrt{k_2^2+m_2^2} - W\right) \times \frac{d^3\vec{k}_2}{E_1E_2}\, k_1^2\, dk_1\, d\Omega \tag{18a}$$

$$dW = \left(\frac{k_1}{E_1} + \frac{k_2}{E_2}\right)\frac{dk_1}{2} = \frac{k_1 W}{E_1E_2}\, dk_1 \quad \text{or} \quad dk_1 = \frac{E_1E_2}{k_1W}\, dW$$

$$\sigma = \frac{1}{16(2\pi)^2 W^2}\,\frac{k}{q}\int |F_{fi}|^2\, d\Omega \tag{18b}$$

The differential cross-section in the c.m.s. is given by

$$\frac{d\sigma}{d\Omega} = \frac{k|F_{fi}|^2}{q(8\pi W)^2}\ ; \quad \frac{d\sigma}{d\Omega} = \frac{d\sigma}{\partial(\cos\theta)\,\partial\Omega} \tag{19}$$

$(\cos\theta)$ may be replaced by the momentum transfer variable.

e.g. For elastic scattering

$$a + b \longrightarrow a + b$$

$$t = -2q^2(1 - \cos\theta),$$

so that

$$\frac{d\sigma}{dt\, d\Omega} = \frac{d\sigma}{d(\cos\theta)\, d\varphi}\,\frac{1}{2q^2},$$

$$\frac{d\sigma}{dt\,d\varphi} = \frac{|F_{fi}|^2}{2q^2(8\pi W)^2} \tag{20}$$

Remark: If the asymptotic behaviour of F_{fi} for large energy is

$$F_{fi} \sim W^2$$

then the shape of the differential cross-section $\frac{d\sigma}{dt}$ is independent of the energy W.

This, together with the fact that

$$\mathrm{Im}\, F > \mathrm{Re}\, F$$

implies that we have a diffraction peak that has a constant width. While dealing with the cross-section, it is convenient to define a scattering amplitude $f(\theta, \varphi)$ by

$$\frac{d\sigma}{d\Omega} = |f(\theta,\varphi)|^2 = \frac{|F_{fi}|^2}{(8\pi W)^2} \tag{21}$$

for elastic scattering. $f(\theta,\varphi)$ will be different in different frames; it is only F_{fi} that is Lorentz-invariant: Define the phase of $f(\theta,\varphi)$ by defining $f(\theta,\varphi)$ in the c.m.s

$$f_{c.m.} = \frac{F}{8\pi W} \tag{22}$$

In the rest frame of the target particle, we have

$$f_{lab} = \frac{F}{8\pi M} \tag{23}$$

where M is the mass of the target particle.

2. Three-particle states and the Dalitz plot:

Consider a $2 \to 3$ production process. Write equation (12) as

$$\sigma = \frac{(2\pi)^4}{4qW}\int |F_{fi}|^2\,\delta(P_f - P_i)\,\frac{d^3k_1\,d^3k_2\,d^3k_3}{8(2\pi)\,E_1 E_2 E_3} \tag{24}$$

where E_1, E_2, E_3 are the momenta of the final particles. Write the δ-function as

$$\delta\left(\sqrt{k_1^2+m_1^2}+\sqrt{k_2^2+m_2^2}+\sqrt{k_1^2+k_2^2+2k_1k_2x+m_3^2}-W\right) \tag{25}$$

and use this to integrate over x, the cosine of the angle between $\vec{k}_1$ and $\vec{k}_2$

$$\int \frac{|F_{fi}|^2\,d^3k_1\;k_2^2\,dk_2\,dx\,d\varphi}{8(2\pi)^9\,E_1E_2E_3}\,\delta(\quad)$$

We have

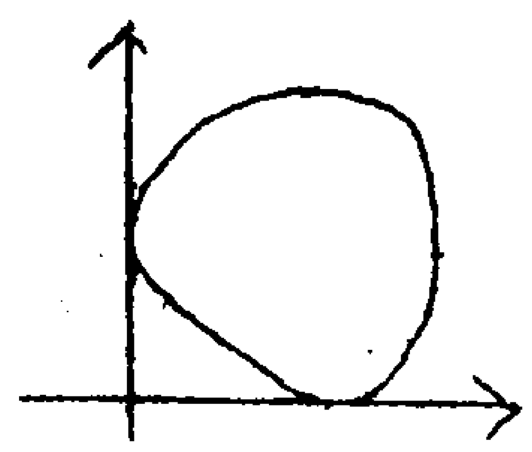
$$\sigma = \frac{1}{4qW}\int |F_{fi}|^2\,\frac{k_1\,dk_1\,k_2\,dk_2\,d\varphi\,d\Omega_1}{8(2\pi)^9\,E_1E_2}$$

$$\sigma = \frac{1}{4qW}\int |F_{fi}|^2\,\frac{d\varphi\,d\Omega_1\,dE_1\,dE_2}{8(2\pi)^9} \tag{26}$$

The kinematics of a 3-particle final state can be represented on the **Dalitz plot**.

Consider a diagram in which the number of events is plotted against T_1 and T_2.

Lorentz invariance requires that the points must lie within a closed curve tangent to the axis.



If $|F|^2 \approx$ a constant, the density of points on the Dalitz plot is a constant. Any variations in $|F|^2$ will show itself in corresponding variations in the density of points in the Dalitz plot.

The amplitude F_{fi}, if evaluated in perturbation theory, is the Feynman matrix element multiplied by a factor $\sqrt{2m}$ for each fermion line.

When perturbation theory is not applicable, information about it may be obtained from general invariance considerations.

3. The example of the 2π decay of the ρ meson:

ρ is a vector particle (1^-).

The most general form of the matrix element may be written

$$A(\varepsilon\cdot\overline{k_1+k_2}) + B(\varepsilon\cdot(k_1-k_2)) \qquad (27)$$

where ε is the polarization vector of the ρ meson, where A and B must be Lorentz invariant and hence functions of only k_1^2, k_2^2 and $k_1 \cdot k_2$. However

$$k_1\cdot k_2 = \tfrac{1}{2}\left[(k_1+k_2)^2 - k_1^2 - k_2^2\right] = \tfrac{1}{2}\left[k^2 - k_1^2 - k_2^2\right]$$

If all the particles are real, A, B can be functions only of the masses, as $k_1^2 = m^2$, etc.. . However, if this decay is part of a larger process, then one or more of the particles may be off the mass-shell and A, B would be functions of these momenta.

$$\varepsilon\cdot(k_1+k_2) = 0$$

as it is zero in the rest frame. Therefore the matrix element is only

$$B\,(\varepsilon\cdot\overline{k_1-k_2})$$

$$\therefore\; T = (2\pi)^4\,\delta(K-k_1-k_2)\,\frac{B\;\varepsilon\cdot(k_1-k_2)}{\sqrt{2M_p\,(2E_1)(2E_2)}} \qquad (28)$$

The transition probability per unit time, or the lifetime, or width, is given by

$$\Gamma = \frac{1}{3}\,\frac{B^2}{4\pi}\,(M^2-4\mu^2)^{3/2}\Big/64M^2 \qquad (29)$$

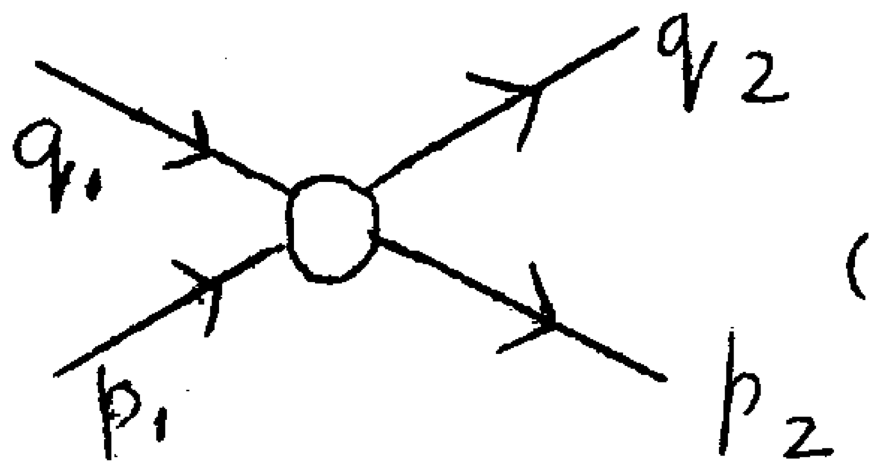
(This is the decay rate in the rest frame of the ρ meson).

III. <u>Elastic Scattering of 2 particles</u>:

Consider π N scattering.

The most general form of F is obtained as follows

$$\pi + N \rightarrow \pi + N$$
$$(q_1\,,\,p_1) \rightarrow (q_2\,,\,p_2)$$



(30)

The possible invariant quantities are

$$\left.\begin{aligned} &\bar{u}(p_2)\,u(p_1), \\ &\bar{u}(p_2)\,\gamma_\mu\,u(p_1)\,q_1^{\mu} \\ &\bar{u}(p_2)\,\gamma_\mu\,u(p_1)\,q_2^{\mu} \end{aligned}\right\} \qquad (31)$$

The most general Lorentz-invariant quantity is

$$A_1 \bar{u}(p_2) u(p_1) + A_2 \bar{u}(p_2) \gamma_\mu u(p_1) q_1^\mu + A_3 \bar{u}(p_2) \gamma_\mu u(p_1) q_2^\mu + \cdots \quad (32)$$

In this case we have only 2 independent amplitudes because of parity conservation. The usual method of choosing these 2 amplitudes is to write

$$F = 2m\, \bar{u}(p_2)\left[A + iB\, \gamma \cdot \frac{q_1+q_2}{2}\right] u(p_1) \quad (33)$$

(We have neglected the charge degrees of freedom ...).
A, B are functions of all the possible invariants; here there are 2 invariants, which may be chosen as s and t, where

$$s = (p_1 + q_1)^2 \quad \text{is the (total energy)}^2$$

$$t = (p_1 - q_1)^2 \quad \text{is the (4-momentum transfer)}^2 \text{ between the nucleons.} \quad (34)$$

The lowest order perturbation terms, the nucleon pole terms, contribute to B but not to A. Generally, it is assumed that A and B behave similarly, i.e., that spin introduces no essential complications.

Another set of amplitudes may be obtained by expressing $\bar{u}(p_2)$ and $u(p_1)$ in terms of spinors at rest.

$$\frac{2m\, \bar{\chi}_f\, (i\gamma \cdot p_1 - m)\left[A - \frac{iB\gamma}{2} \cdot (q_1+q_2)\right](i\gamma p_1 - m)\, \chi_i}{(E+m)} \quad (35)$$

where χ_f and χ_i are spinors in which the lower compo- nents are zero. (35) may be simplified into

$$F = 8\pi W \chi_f^{\dagger}[f_1 + f_2 \vec{\sigma}\cdot\hat{q}_2 \vec{\sigma}\cdot\hat{q}_1]\chi_i \qquad (36)$$

The scattering amplitude in the c.m.s. will be just

$$f = \chi_f^{\dagger}[f_1 + f_2(\vec{\sigma}\cdot\hat{q}_2 \vec{\sigma}\cdot\hat{q}_1)]\chi_i \qquad (37)$$

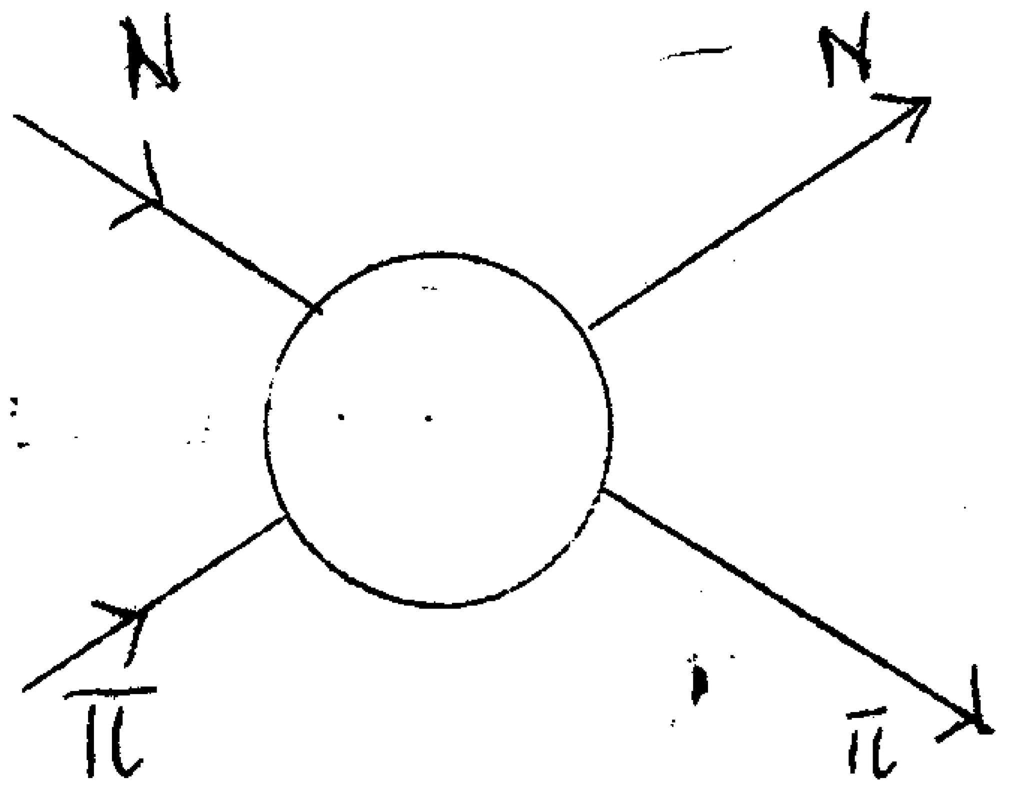
We may obtain

$$\left.\begin{aligned} f_1 &= \frac{E+m}{8\pi W}[A + B(W-m)] \\ f_2 &= (E-m)/8\pi W\,[-A + B(W+m)], \end{aligned}\right\} \qquad (38)$$

and the inverse

$$\left.\begin{aligned} A &= 4\pi\left[f_1 \frac{W+m}{E+m} - f_2 \frac{W-m}{E-m}\right] \\ B &= 4\pi\left[\frac{f_1}{E+m} + \frac{f_2}{E-m}\right] \end{aligned}\right\} \qquad (39)$$

We shall later see that in terms of f_1 and f_2 the amplitude for scattering without any change in the nucleon helicity is given by $(f_1 + f_2)$ while the amplitude for scattering with a change in the nucleon helicity is $(f_1 - f_2)$



$$F = g_m \bar{u}(p_2)\left[A - iB\gamma\cdot\frac{(q_1+q_2)}{2}\right]u(p_1)$$

A, B - are functions of only the kinematic invariants.

Introduce N spinors at rest.

Write $$u(p_1) = \frac{i\gamma\cdot p_1 - m}{(2m(E_1+m))^{1/2}}\chi_i$$

$$\bar{u}(p_2) = \frac{i\gamma\cdot p_2 - m}{(2m(E_2+m))}^{1/2} \qquad \bar{u}u = 1$$

$$\vec{\gamma} = -i\beta\vec{\alpha} \quad ; \quad \gamma_0 = \beta$$

Then $$\frac{i\gamma\cdot p_1 - m}{\sqrt{2m(E_1+m)}}u(p_1) = \frac{\vec{\alpha}\cdot\vec{p}_1 + \beta m + E_1}{\sqrt{2m(E+m)}}\chi_i$$

$$\chi_f^{\dagger}\frac{\vec{\alpha}\cdot\vec{p}_2 + \beta m + E_2}{(2m(E_2+m))^{1/2}}\beta\left[A - iB\gamma\frac{q_1+q_2}{2}\right]$$

$$\times\ \frac{\vec{\alpha}\cdot\vec{p}_1 + \beta m + E}{(2m(E_1+m))^{1/2}}\chi_i$$

Note: -- Only the even powers of the α operators will be non-zero between χ_f^+ and χ_i

We can reduce α's to σ's Thus the amplitude may be written in terms of the σ matrices between χ_f^+ and χ_i as $\bar{\chi}_f [f] \chi_i$ where f is a Lorentz-invariant quantity.

The only Lorentz invariants that could be constructed with the σ's are $(\vec{\sigma} \cdot \vec{p}_1) \cdot (\sigma \cdot \vec{p}_2)$ and 1 Thus the amplitude may be written as

$$\chi_f^+ \left[f_1 + f_2 (\vec{\sigma} \cdot \hat{p}_1)(\vec{\sigma} \cdot \vec{p}_2) \right] \chi_i$$

This form of writing the matrix element is convenient while dealing with the polarizations. Now one has to project out the spin states, as the projection over positive energy states has already been done.

The spin matrices take care of the spin complications. Matrix invariants can be constructed with the invariants at our disposal. $p_1 + q_1 = p_2 + q_2$ There are two invariants. We may choose them as s, t

$$s = -(p_1 + q_1)^2 \qquad p^2 = \vec{p}^2 - p_0^2$$

For a free particle, $p^2 = -m^2$

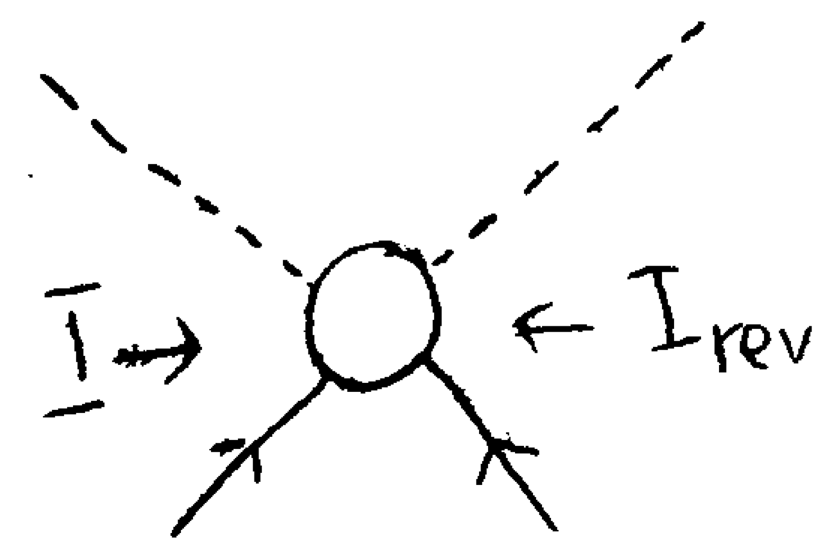
<u>Define</u>:

$$s = -(p_1 + q_1)^2 = -(p_2 + q_2)^2$$
$$t = -(q_2 - q_1)^2 = -(p_2 - p_1)^2$$
$$u = -(p_1 - q_2)^2 = -(p_2 - q_1)^2$$

These are related by $s + t + u = \sum_i m_i^2 = 2(m^2 + \mu^2)$ for π N scattering. Therefore A, B are functions of s, t, u -- we consider all these as variables for symmetry, although only two are independent.

The same invariant amplitude can be used for describing other reactions.

The same invariant amplitude can be used for describing other reactions also.



(I): π N Scattering

(I_{rev}): The time-reversed action.

We can also describe other related reactions by the same amplitude.

In general, consider the reaction

$$a + b \to c + d \tag{I}$$

with 4-momenta $\quad q_1 \; p_1 \to q_2 \; p_2$

The reaction obtained by replacing an incoming particle by an outgoing antiparticle with the opposite momentum.

$$\bar{c} + b \to \bar{a} + d \tag{II}$$

$$-q_2 \; p_1 \to -q_1 \; p_2$$

For this 'crossed' process, the <u>centre-of-mass energy squared</u> is given by

$$-(p_1 - q_2)^2 = u$$

The momentum transfer between the antiparticles $\bar{c}$ and $\bar{a}$ is $-[-q_1 - (-q_2)]^2$, which is still $= t$. Next consider the crossed process

$$\pi + \pi \to N + \bar{N}$$

or more generally

$$a + \bar{c} \to d + \bar{b} \tag{III}$$

$$a + \bar{c} \rightarrow \bar{b} + d$$
$$q_1 \quad -q_2 \qquad -p_1 \quad p_2$$

The c.m. energy squared is

$$-(q_1 - q_2)^2 = -(p_1 - p_2)^2 = t$$

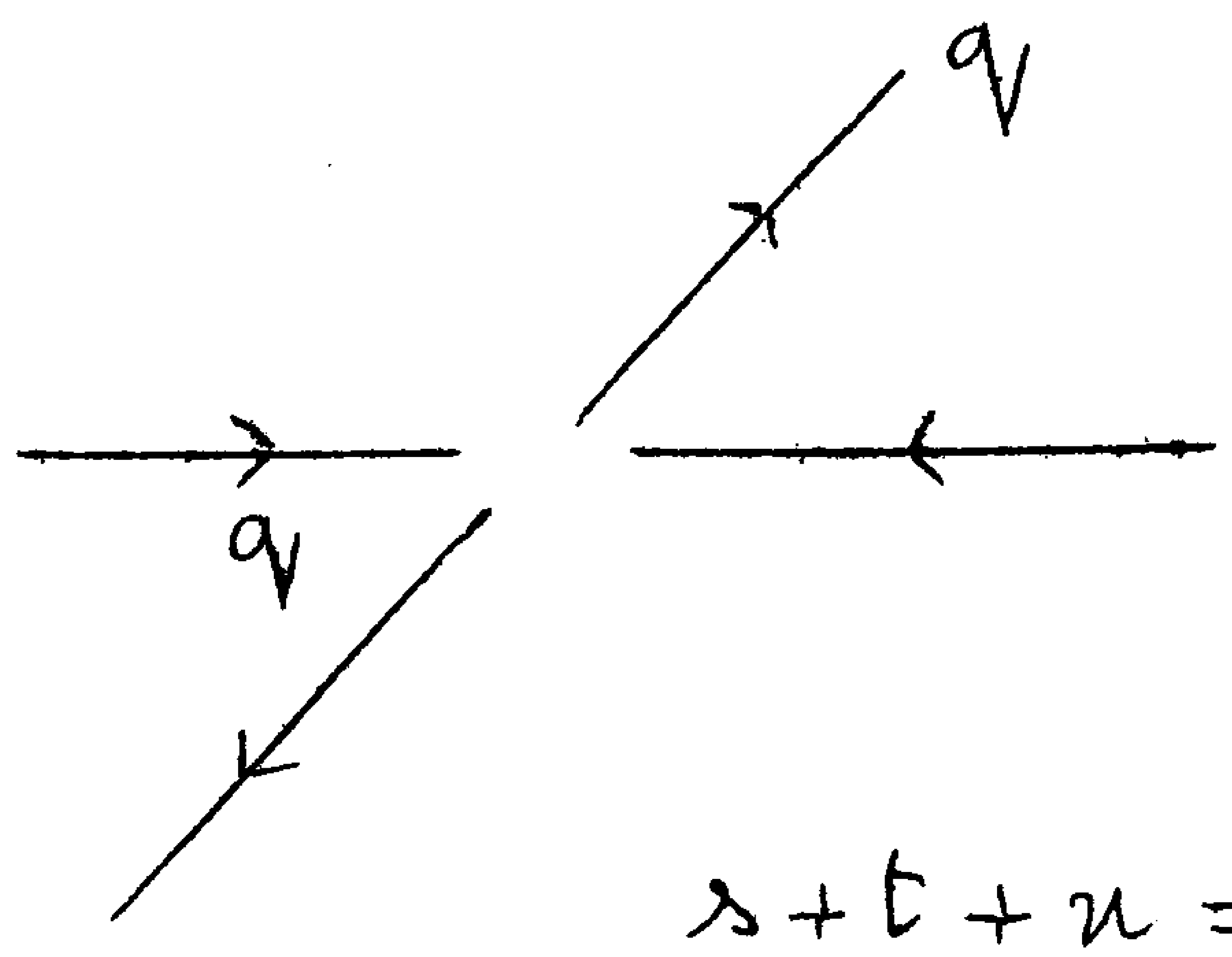
while the momentum transfer between a and $\bar{b}$ is

$$-(q_1 - (-p_1))^2 = -(q_1 + p_1)^2 = s$$

So the same invariants may be used to describe these different processes, I, II, and III.

However, the range of the variables s, t, u in each channel is such that the three channels are all disconnected. e.g. consider the scattering of 2 equal mass particles (each of mass m).

Process I.



In the c.m.s., we have

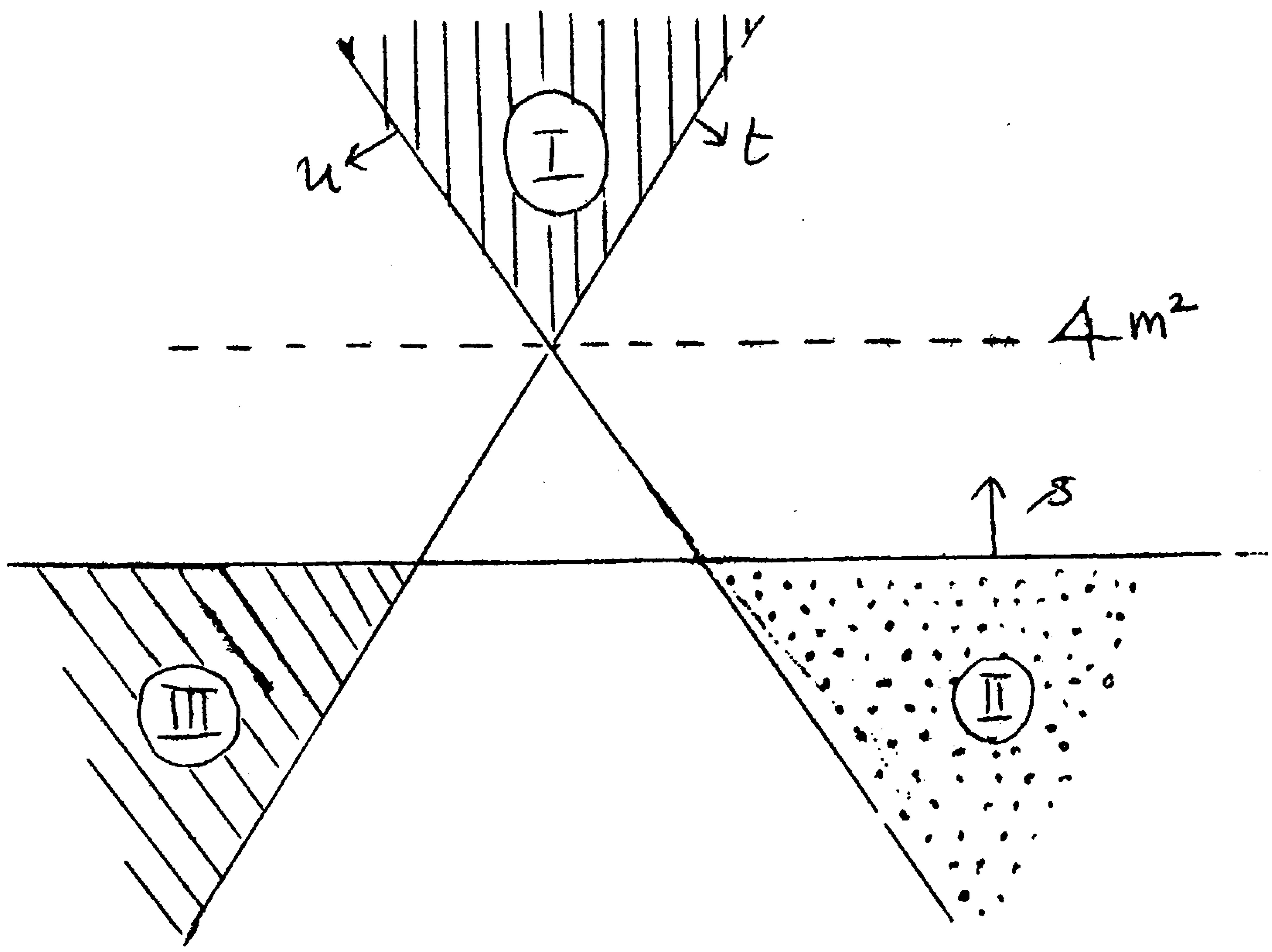
$$s = 4(q^2 + m^2)$$
$$t = -2q^2(1 - \cos\theta)$$
$$u = -2q^2(1 + \cos\theta)$$

$$s + t + u = 4m^2$$
$$s \geq 4m^2, \quad t \leq 0 \qquad u \leq 0$$

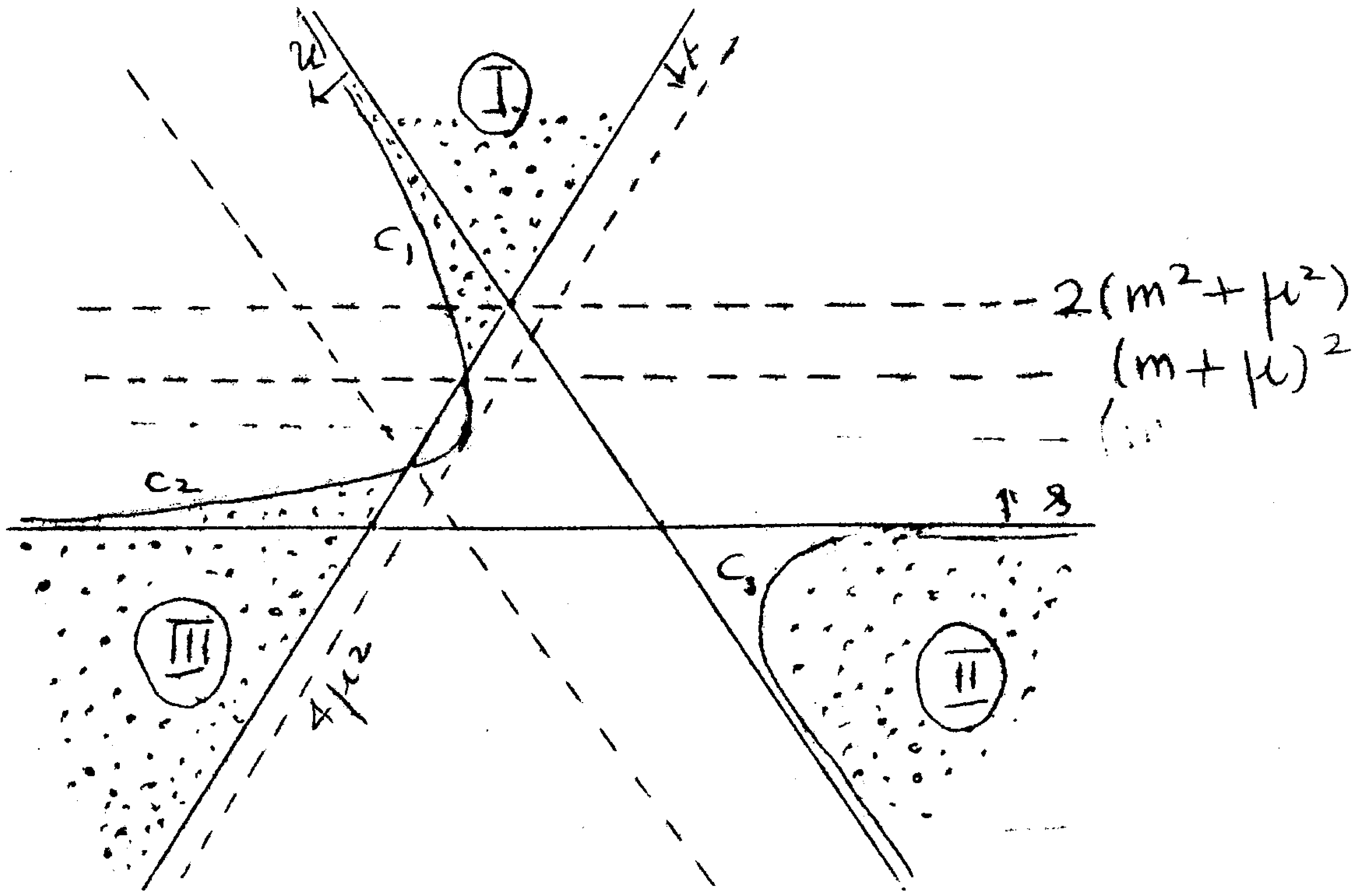
<u>Process II</u>. $s \leq 0$, $t \leq 4m^2$, $u \leq 0$

<u>Process III:</u> $s \leq 0$ $t \leq 0$ $u \geq 4m^2$

Representation of physical and unphysical regions; the Mandelstam diagram:- A convenient way to present this is using trilinear (homogeneous) coordinates with an equilateral triangle of reference



The ranges of the kinematical variables for the three processes are shown on the diagram; these ranges have no overlap when we represent them in the trilinear coordinate system shown above. For the **scattering** with particles of unequal mass, e.g., π N scattering, the situation is a little more complicated. We take the same trilinear system.



The threshold for process I is $s = (m+\mu)^2$ i.e.,

$s \geq (m+\mu)^2$ for process I.

$t \leq 0$

$$t = -2q^2(1-\cos\theta)_{\,\mathrm{for\ process\ I.}}$$

The limits for u are a little more complicated. The process I is given by

$$t = -2q^2(1-\cos\theta)$$
$$s+t+u = 2(m^2+\mu^2)$$

Therefore the largest value of u for given s, will be obtained when the value of t is the minimum possible.

t is negative in the physical region.

Therefore $\quad t_{min} = (t)_{\cos\theta=-1} = -4q^2$

The corresponding value of q is given by

$$q^2 = \tfrac{1}{4}t$$

$$s = \left(\sqrt{q^2+m^2}+\sqrt{q^2+\mu^2}\right)^2 = \left(\sqrt{m^2-\tfrac{t}{4}}+\sqrt{\mu^2-\tfrac{t}{4}}\right)^2$$

$$= \left(m^2+\mu^2-\tfrac{t}{2}\right) + 2\sqrt{m^2-t/4}\sqrt{\mu^2-\tfrac{t}{4}}$$

or $\left(s-m^2-\mu^2-t/2\right)^2 = 4(m^2-t/4)(\mu^2-t/4)$

On simplifying, this gives

$$(s-m^2-\mu^2)^2 + st - 4m^2\mu^2 = 0$$

or $\quad s^2+(m^2-\mu^2)^2-2(m^2+\mu^2)s+st=($

or $\quad$ using $\quad 2(m^2+\mu^2) = s+t+u,$

We obtain

$$-us = -(m^2+\mu^2)^2 + m^2\mu^2$$

which gives

$$us = (m^2-\mu^2)^2$$

which is just a hyperbola; the branch C_1 shown on the figure i the boundary of the physical region.

$$u \leq \frac{m^2-\mu^2}{s}$$

For $m = \mu$, this degenerates into the straignt line

For the process II, we obtain the boundary of the physica region by interchanging s and u in the above.

This gives the continuation of the same branch C_2 of the hyperbola

$$us = (m^2-\mu^2)^2$$

The physical region of the third crossed process (III) is given by the other branch C_2 of the hyperbola

$$us = (m^2-\mu^2)^2$$

(For unequal masses, the shape of the boundary is more complicated. When a decay is also possible,

$$a \rightarrow b + \bar{c} + d$$

then the physical region for the process lies inside the triangle.

The <u>same variables</u> can be used to describe three differe reactions.

We shall go further and say that all three processes are described by the <u>same function</u> $F(s, t, u)$

However, the ranges of s, t, u are different for the three processes I, II and III. Therefore this statement will be meaningful only if we can pass in a continuous way from one region to another.

IV. The Unitarity Relation;

Next, we consider the requirement of unitarity. Start with the unitarity property of the S operator:

$$S^{+} = S^{-1}$$

Define M_{fi} by

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta(P_f - P_i) M_{fi}$$

or

$$SS^{+} = I$$

$$(SS^{+})_{fi} = \delta_{fi} = \sum_n (S_{fn} S^{+}_{ni}) = \sum_n S_{fn} S^{*}_{in}$$

Substituting for S_{fi}, we obtain

$$\delta_{fi} = \sum \left(\delta_{fn} + i(2\pi)^4 \delta(P_f - P_n) M_{fn}\right) \left(\delta_{ni} - i(2\pi)^4 \delta(P_i - P_n) M^{*}_{in}\right)$$

On simplifying, this gives

$$i(M_{fi} - M_{if}^{*}) = -(2\pi)^4 \sum_n \delta(P_i - P_n) M_{fn} M$$

Take 2 examples.

Example 1 -- We restrict ourselves to 2-particle intermediate state

Consider the process

$$\pi\pi \longrightarrow \pi\pi$$

For $s < 16\mu^2$ the only allowed intermediate state is the 2-pion state, as $\pi\pi \longrightarrow 3\pi$ is forbidden by charge-independence (or conservation of G-parity.)

Then

$$M_{fi} = \frac{F_{fi}}{4\omega^2} = \frac{F_{fi}}{4W^2}$$

since $$W = 2w$$

The unitarity relation above now becomes

$$i\left(F_{fi} - F_{if}^*\right) = -\frac{(2\pi)^4}{W^2}\sum_n \delta(P_i - P_n) F_{fn} F_{in}^*$$

We now perform this summation, assuming $s < 16\mu^2$. The right hand side becomes

$$-(2\pi)^4 \int \delta(P_i - P_n) \frac{F_{fn} F_{in}^*}{W^2} \frac{d^3\vec{q}_1 d^3\vec{q}_2}{(2\pi)^6}$$

or

$$-i\left(F_{fi} - F_{if}^*\right) = \frac{1}{(2\pi)^2}\frac{1}{W^2}\int \delta\left(2\sqrt{q^2+\mu^2} - W\right) F_{fn} F_{in}^* q^2 \, dq \, d\Omega$$

Putting $dq = \frac{a}{2q} dw = w\,dw/4q$, we obtain

$$-i\left(F_{fi} - F_{if}^*\right) = \frac{1}{(2\pi)^2}\frac{\sqrt{\frac{W^2}{4} - \mu^2}}{4W} \int F_{fn} F_{in}^* \, d\Omega$$

This is known as the 'elastic unitarity condition'

If we define the phases such that

$$F_{if} = F_{fi}$$

then this may be written

$$2\,\mathcal{I}m\, F(W, \cos\theta_{12}) = \frac{1}{(2\pi)^2}\frac{\sqrt{\frac{W^2}{4} - \mu^2}}{4W}$$

$$\times \int F(W, \cos\theta_{13}) F(W, \cos\theta_{32}) \, d\Omega$$

where are the directions of the momenta in the initial, final, and the intermediate states.

Example 2 . - Elastic Scattering:

The unitarity relation becomes

$$-i(M_{ii}-M_{ii}^{*}) = -(2\pi)^4 \sum_n \delta(P_i-P_n) M_{in} M_{in}^{*}$$

or

$$2\,\mathrm{Im}\, M_{ii} = (2\pi)^4 \sum_n \delta(P_i-P_n)\,|M_{in}|^2$$

In terms of the F amplitude, we have

$$2\,\mathrm{Im}\, F_{ii} = (2\pi)^4\, 4E_1E_2 \sum_n \delta(P_i-P_f) \frac{|F_{in}|^2}{4E_1E_2\,2E_1}$$

The summation $\sum_n$ is over all possible intermediate states. When we restrict ourselves to particular intermediate states, we can do the integration explicitly.

$$\sum \int (2\pi)^4 \delta(P_i-P_f)\,|F_{in}|^2 \prod_j \frac{d^3k_j}{(2\pi)^3\,2E_j}$$

But this is just the total cross section, apart from certain facto i.e.

$$2\,\mathrm{Im}\, F_{ii} = 4qW\sigma_{total}$$

or

$$\mathrm{Im}\, F_{ii} = 2qW\sigma_{total}$$

i.e. the imaginary part of the forward scattering amplitude. This may be written in a more familiar form in terms of the f amplitude;

$$f(0,0) = \frac{F}{8\pi W}$$

F = Feynman amplitude
f = The CMS Scatt. ampli

$$\mathcal{I}m\, f_{ii} = \frac{q}{4\pi}\sigma_{total}$$

In the next lecture, we shall consider the complications introduced by charge and the restrictions from charge independence.

V. Charge Independence.

1. Charge independence and the concept of Isospin:

It is irrelevant to specify the charge of the particles in this respect

particles appear as multiplets.

Consider a multiplet as a basis vector in charge space.

Hence we introduce the isospin.

The commutation rules for isospin are

$$[T_i, T_j] = i\,\varepsilon_{ijk} T_k \tag{1}$$

$$[T_i, T^2] = 0$$

The rotation operator in isospin space is given by

$$R_\varphi = e^{-i\varphi \vec{T}\cdot\hat{n}} \tag{2}$$

The charge is related to the isospin by

$$Q = e\left(T_3 + \frac{N+S}{2}\right) \tag{3}$$

Example

The nucleon is an isospin doublet, with the components proton and neutron corresponding to projections $I_z = +\frac{1}{2}$ and $-1/2$ respectively.

$$|p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4}$$

The pion is an isospin triplet, with components π^+, π^-, π^0

$$\begin{array}{cccc} & |\pi^+\rangle & |\pi^-\rangle & |\pi^0\rangle \\ I_z & +1 & -1 & 0 \end{array} \tag{5}$$

It is useful to introduce

$$|\pi_1\rangle = \frac{1}{\sqrt{2}}\left[\,|\pi^+\rangle + |\pi^-\rangle\,\right] \tag{6}$$

$$|\pi_2\rangle = \frac{1}{\sqrt{2}}\left[\,|\pi^+\rangle - |\pi^-\rangle\,\right]$$

and

$$|\pi_3\rangle = |\pi^0\rangle$$

which transform like the Cartesian components of a vector in isospin space. $|\pi^+\rangle, |\pi^-\rangle, |\pi^0\rangle$ are the 'spherical' components.

The matrix elements of the isospin operator T_k between pion states are given by

$$\langle \pi_j | T_k | \pi_i \rangle = i\,\varepsilon_{ijk}$$

Consider the example of pion-nucleon scattering

$$\pi + N \longrightarrow \pi + N \tag{7}$$

The possible reactions are

$$\pi^+ p \to \pi^+ p$$
$$\pi^- p \to \pi^- p$$
$$\pi^- p \to \pi^0 n \text{ (charge exchange)} \tag{7a}$$

and

$$\pi^+ n \to \pi^+ n$$
$$\pi^+ n \to \pi^0 p \tag{7b}$$
$$\pi^- n \to \pi^- n$$

A priori we have these independentamplitudes to describe scattering; but charge independence reduces the number of amplitudes.

Because of charge independence, both T^2 and T_z will be conserved in the reaction.

Labelling the possible channels by the T values, there are just two independent amplitudes for $T = 3/2$ and $T = 1/2$ respectively.

Denote these by $F^{3/2}$ and $F^{1/2}$ respectively.

These are related to the πp amplitudes by

$$F_{\pi^+ p} = F_{3/2}$$
$$F_{\pi^- p} = \frac{1}{3}\left(F^{3/2} + F^{1/2}\right) \tag{8}$$
$$F_{\substack{\text{charge}\\\text{exchange}}} = \frac{\sqrt{2}}{3}\left(F^{3/2} - F^{1/2}\right)$$

These are related by

$$F_{\pi^+p} = F_{\pi^-p} + \sqrt{2}\, F_{\text{charge exchange}} \tag{9}$$

The differential cross sections are given by

$$\pi^+p \to \pi^+p\ ; \qquad \frac{d\sigma}{d\Omega} = |f^{3/2}|^2$$

$$\pi^-p \to \pi^-p\ ; \qquad \frac{d\sigma}{d\Omega} = \frac{1}{9}\Big\{|f^{\frac{3}{2}}|^2 + 4|f^{\frac{1}{2}}|^2 + 4\,\mathrm{Re}\, f^{3/2\,\times} f^{1/2}\Big)$$

$$\pi^-p \to \pi^0 n\ ; \qquad \frac{d\sigma}{d\Omega} = \frac{2}{9}\Big[|f^{\frac{3}{2}}|^2 + |f^{\frac{1}{2}}|^2 - 2\,\mathrm{Re}\, f^{3/2\,\times} f^{1/2}\Big] \tag{10}$$

where the $f^{(I)}$ are the c.m.s. scattering amplitudes in the isospin state I.

The total cross section for π^-p reactions (elastic sc + charge exchange) does not contain the interference term and is given by

$$\frac{d\sigma_{tot}}{d\Omega} = \frac{1}{3}\Big[\,|f^{3/2}|^2 + 2\,|f^{1/2}|^2\Big] \tag{11}$$

The total cross-sections for definite isospin states are given by

$$\sigma_{3/2} = \sigma_+\ ;\ \sigma_{1/2} = \frac{1}{2}\big[\sigma_- - \sigma_+\big] \tag{12}$$

The crosssections for π reactions on neutrons are given by charge symmetry alone.

$$(\pi^+ p \to \pi^+ p) = (\pi^- n \to \pi^- n)$$
$$(\pi^- p \to \pi^- p) = (\pi^+ n \to \pi^+ n) \qquad (13)$$
$$(\pi^- p \to \pi^- p) = (\pi^+ n \to \pi^0 p)$$

The total isospin is given by

$$\vec{T} = \vec{t} + \frac{\vec{\tau}}{2} \qquad (14)$$

where $\vec{t}$ and $\vec{\tau}$ are the isospin operators for the pion and nucleon respectively.

$$\therefore \vec{T}^2 = \vec{t}^2 + \frac{\vec{\tau}^2}{4} + \vec{t}\cdot\vec{\tau}$$

Therefore an eigenstate of $\vec{T}^2$, $\vec{t}^2$ and $\vec{\tau}^2$ is also an eigenstate of $\vec{t}\cdot\vec{\tau}$

$$\vec{t}\cdot\vec{\tau} = +1 \quad \text{for} \quad T = 3/2$$
$$\vec{t}\cdot\vec{\tau} = -2 \quad \text{for} \quad T = 1/2 \qquad (15)$$

The total amplitude may be written as a sum of $T = \frac{3}{2}$ and $T = \frac{1}{2}$ scattering amplitudes, given by the following

$$F = F^{3/2} P^{3/2} + F^{1/2} P^{1/2} \qquad (16)$$

where $P^{3/2}$ and $P^{1/2}$ are the projection operators for $T = 3/2$

and $T = \frac{1}{2}$ respectively, and are given by

$$\left.\begin{aligned} P_{3/2} &= \frac{2 + \vec{t}\cdot\vec{\tau}}{3} \\ P_{1/2} &= \frac{1 - \vec{t}\cdot\vec{\tau}}{3} \end{aligned}\right\} \tag{17}$$

We have thus written the scattering amplitude as an operator in the product charge space of the pion and the nucleon.

We now go further, and write the scattering amplitude in the charge space of the nucleon alone, between definite charge states of the pion. Between π meson states with charge indices i and j, the matrix element of $(\vec{t}\cdot\vec{\tau})$ is

$$(t\cdot\vec{\tau})_{ji} = \sum_k \langle \pi_j | t_k | \pi_i \rangle \tau_k \tag{18}$$

since τ_k does not operate on the pion states

Note: $\langle \pi_j | t_k | \pi_i \rangle = i\, \epsilon_{ijk} \tau_k$

and that τ_k is given in terms of τ_i, τ_j by

$$[\tau_i, \tau_j] = 2i\, \tau_k \tag{19}$$

Thus, (18) may be written as

$$(t\cdot\vec{\tau})_{ji} = \frac{1}{2}[\tau_i, \tau_j] \tag{20}$$

(17) gives

$$(P_{3/2} - 2P_{1/2}) = (\vec{t}\cdot\vec{\tau})$$

so that

$$\langle j | (P_{3/2} - 2P_{1/2}) | i \rangle = \frac{1}{2}[\tau_i, \tau_j] \tag{21}$$

Also

$$\langle j | P_{3/2} + P_{1/2} | i \rangle = \delta_{ji} \tag{22}$$

by the definition of the projection operators. Therefore we may write $\langle j | F | i \rangle$ as

$$F_{ji} = \delta_{ji} F^{(+)} + \tfrac{1}{2}[\tau_j , \tau_i] F^{(-)} \tag{23}$$

where the amplitude is taken between definite pion states $|j\rangle$ $|i\rangle$ and is an operator only in the nucleon charge space. In the framework of dispersion theory, it is very useful to introduce the amplitudes $F^{(+)}$ and $F^{(-)}$. In terms of the older projection operator,

$$\begin{aligned} F_{ji} &= F^{3/2}(P_{3/2})_{ji} + F^{1/2}(P_{1/2})_{ji} \\ &= (P_{3/2} + P_{1/2})_{ji} F^{(+)} - (P_{3/2} - 2P_{1/2})_{ji} F^{(-)} \\ &= (P_{3/2})_{ji}[F^{(+)} - F^{(-)}] + (P_{1/2})_{ji}[F^{(+)} + 2F^{(-)}] \end{aligned}$$

so that

$$\left.\begin{aligned} F^{3/2} &= F^{(+)} - F^{(-)} \\ F^{1/2} &= F^{(+)} + 2F^{(-)} \end{aligned}\right\} \tag{24}$$

and

$$\left.\begin{aligned} F^{(+)} &= \tfrac{1}{3}(2F^{3/2} + F^{1/2}) \\ F^{(-)} &= \tfrac{1}{3}(F^{1/2} - F^{3/2}) \end{aligned}\right\} \tag{25}$$

The scattering amplitudes for definite particles is given by

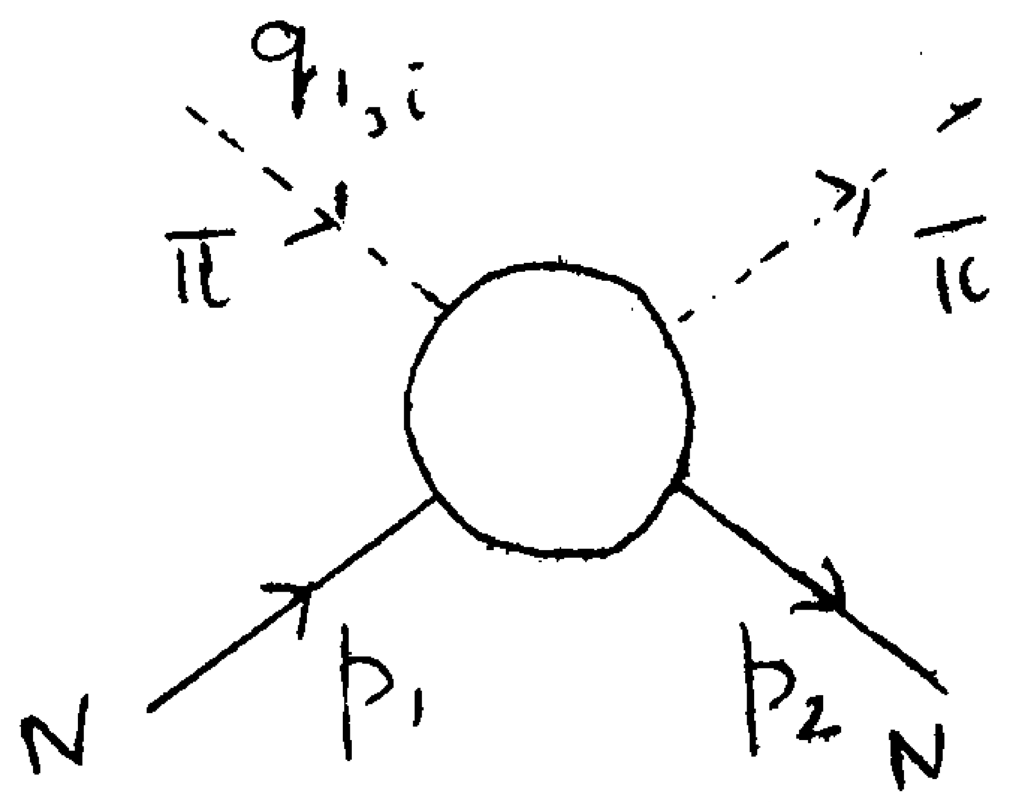
$$\left.\begin{aligned} F_{\pi^- p} &= F^{(+)} + F^{(-)} \\ F_{\pi^+ p} &= F^{(+)} - F^{(-)} \\ F_{ch.ex} &= -\sqrt{2}\, F^{(-)} \end{aligned}\right\} \qquad (26)$$

We have introduced the amplitudes $F^{(+)}$ and $F^{(-)}$ as these amplitudes have simple symmetry properties under exchange of the pion states.

The pion field can be described either by 2 complex fields π^+, π^- and a hermitian field π^0, or equally well by 3 hermitian fields $|\pi_1\rangle, |\pi_2\rangle, |\pi_3\rangle$. Denote

$$|\pi^i\rangle = \Phi_i |0\rangle$$

Consider pion-nucleon scattering



Suppose we describe this process by perturbation theory to all orders.

2. Crossing Relations.

If we replace the incoming π meson q_1, i by an outgoing meson with the opposite momentum, nothing is changed in the graph.

Similarly if the outgoing π meson $(q_2 j)$ is replaced by an incoming one nothing is changed. (Strictly, the real parts of these amplitudes are equal. This is because the amplitudes have branch points and are multi-valued.) Therefore

$$F(q_1 i, q_2 j) = F(-q_2 j, -q_1 i) \tag{27}$$

This is true in perturbation theory to all orders and can also be proved in field theory, independent of perturbation theory. By noting the definition of F in terms of $F^{(+)}$ and $F^{(-)}$, we obtain

$$F^{(\pm)}(q_1 q_2) = \pm F^{(\pm)}(-q_2, -q_1) \tag{28}$$

since δ_{ij} and $[\tau_j, \tau_i]$ are, respectively, symmetric and antisymmetric in i and j.

The interchange $(q_1 i) \to -(q_2 j)$ is equivalent to the replacement

$$s \longleftrightarrow u \tag{29}$$

where

$$s = -(p_1+q_1)^2 = -(p_2+q_2)^2$$
$$u = -(p_2-q_1)^2 = -(p_1-q_2)^2$$

as defined earlier.

Symmetry under this interchange

$$s \longleftrightarrow u$$

is known as crossing symmetry.

We have seen that the spin decomposition of the πN scattering amplitude is given by

$$F = 2m\bar{u}(p_2)\left[-A + \frac{iB}{2}\gamma\cdot(q_1+q_2)\right]u(p_1)$$

or, for $F^{(+)}$ and $F^{(-)}$

$$F^{(\pm)} = 2m\,\bar{u}(p_2)\left[-A^{(\pm)} + i\frac{B^{(\pm)}}{2}\gamma\cdot(q_1+q_2)\right]u(p_1) \tag{30}$$

The symmetry under the intercharge $(q_1 j) \rightarrow -(q_2 j)$ is given for the amplitudes $A^{\pm}$, $B^{\pm}$ by the following:

$$\left.\begin{aligned} A^{(+)}(s,t,u) &= A^{(+)}(u,t,s) \\ A^{(-)}(s,t,u) &= -A^{(-)}(u,t,s) \\ B^{(+)}(s,t,u) &= -B^{(+)}(u,t,s) \\ B^{(-)}(s,t,u) &= B^{(-)}(u,t,s) \end{aligned}\right\} \tag{31}$$

This crossing symmetry is simple only for the amplitude $F^{(+)}$ $F^{(-)}$. These may be used for giving the relations for (π^+p), (π^-p) scattering amplitudes, by replacing $F^{(\pm)}$ in terms of $F^{3/2}$ and $F^{1/2}$. On doing this, we obtain the crossing matrix

$$\begin{array}{cc} & u \downarrow \\ s \rightarrow & \begin{array}{|c|c|c|} \hline & 3/2 & 1/2 \\ \hline 3/2 & 1/3 & 4/3 \\ \hline 1/2 & 2/3 & -1/3 \\ \hline \end{array} \end{array} \tag{32}$$

[For the whole amplitude, not just the real part, the direct amplitude in the s channel taken ___ is related to the 'crossed' amplitude in the u channel, defined by the prescription ___ .]

For the t channel,

$$\pi + \pi \longrightarrow N + \bar{N}$$

the amplitudes $F^{(\pm)}$ are given by

$$F^{(+)} = \frac{1}{\sqrt{6}} F^{(0)} ; \quad F^{(-)} = \frac{1}{2} F^{1} \tag{33}$$

3. G-Invariance:

We ask how the amplitude transforms under rotations on the nucleon states in isospin space.

$$\left.\begin{aligned} |p\rangle &\rightarrow |p\rangle \cos\frac{\theta}{2} + |n\rangle \sin\frac{\theta}{2} \\ |n\rangle &\rightarrow -|p\rangle \sin\frac{\theta}{2} + |n\rangle \cos\frac{\theta}{2} \end{aligned}\right\} \tag{34}$$

We know that corresponding antiparticles $\bar{p}$ and $\bar{n}$ must also exist. The isospin projection of the $\bar{p}$ and $\bar{n}$ will be given by the following:

$$\begin{array}{llll} |p\rangle & \frac{1}{2} & |\bar{p}\rangle & -\frac{1}{2} \\ |n\rangle & -\frac{1}{2} & |\bar{n}\rangle & +\frac{1}{2} \end{array} \tag{35}$$

The effect of charge conjugation on the nucleon states is given by

$$\begin{aligned} C|p\rangle &= |\bar{p}\rangle \\ C|n\rangle &= |\bar{n}\rangle \end{aligned} \tag{36}$$

if we define the phases appropriately.

What are the transform properties of $|\bar{p}\rangle$ and $|\bar{n}\rangle$? We would like the $|\bar{p}\rangle$ and $|\bar{n}\rangle$ to transform like the $|n\rangle$ and $|p\rangle$ respectively, as they have $I_z = +1/2$ and -1/2 respectively.

This can be achieved by making the correspondence

$$|p\rangle \longrightarrow |\bar{n}\rangle$$
$$|n\rangle \longrightarrow |\bar{p}\rangle \qquad (37)$$

Then we would have the transformation properties

$$-|\bar{p}\rangle \rightarrow -|\bar{p}\rangle \cos\frac{\theta}{2} + |\bar{n}\rangle \sin\frac{\theta}{2}$$
$$|\bar{n}\rangle \rightarrow |\bar{p}\rangle \sin\frac{\theta}{2} + |\bar{n}\rangle \cos\frac{\theta}{2} \qquad (38)$$

This may be attained simply if we define the operator

$$G = C e^{i\pi I_2} \qquad (39)$$

which has the effect

$$G|p\rangle = |\bar{n}\rangle$$
$$G|n\rangle = |\bar{p}\rangle \qquad (40)$$

The effect of operating with G twice is given by

$$|p\rangle \xrightarrow{G} |\bar{n}\rangle \xrightarrow{G} -|p\rangle$$
$$|n\rangle \xrightarrow{G} |\bar{p}\rangle \xrightarrow{G} -|n\rangle \qquad (41)$$

This proves that G commutes with T_2,

$$[G, T_2] = 0$$

We can prove that G commutes with all components of $\vec{T}$

(42)

Thus G commutes with $\vec{T}$, whereas C does not (e.g. C anticommute with T_3).

This is the reason G is useful. G may be regarded as a generalization of C.

A possible application of the fact that G commutes with $\vec{T}$ is illustrated by the following:

Consider two-nucleon states

T_3	$T=1$	$T=0$
$+1$	$\vert pp\rangle$	
0	$\frac{1}{\sqrt{2}}[\vert pn\rangle - \vert np\rangle]$	$\frac{1}{\sqrt{2}}[\vert pn\rangle - \vert np\rangle]$
-1	$\vert nn\rangle$	

(43)

On applying G to one of the nucleons, the values of T_3 must not change, since $[G, T_3] = 0$. Thus the corresponding states for the $(N\bar{N})$ system are given by

I_3	$I=1$	$I=0$
$+1$	$\vert p\bar{n}\rangle$	
0	$\frac{1}{\sqrt{2}}[-\vert p\bar{p}\rangle + \vert n\bar{n}\rangle]$	$\frac{1}{\sqrt{2}}[-\vert p\bar{p}\rangle - \vert n\bar{n}\rangle]$
-1	$-\vert n\bar{p}\rangle$	

(44)

Suppose we operate the G operator on these $(N\bar{N})$ states.

This gives the following:

T_3	$T=1$	$T=0$
1	$-\lvert\bar{n}p\rangle$	
0	$\frac{1}{\sqrt{2}}[\,\lvert\bar{n}n\rangle - \lvert\bar{n}p\rangle]$	$\frac{1}{\sqrt{2}}[\,\lvert\bar{n}n\rangle + \lvert\bar{p}p\rangle]$
-1	$-\lvert n\bar{p}\rangle$	

Thus $\quad G\lvert N\bar{N}\rangle = (-1)^T \lvert\bar{N}N\rangle$. 2 fermion states.

A two Fermion. state must be totally antisymmetric under the exchange of baryonic number, isospin, space and spin.

This gives the condition

$$G P_s P_{spin}\lvert N\bar{N}\rangle = -\lvert N\bar{N}\rangle$$

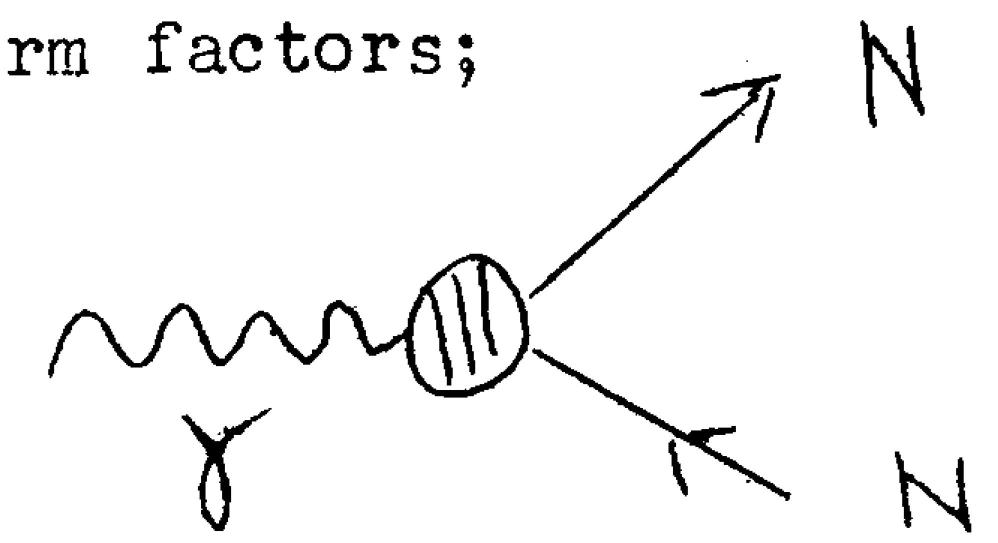
$$G\lvert N\bar{N}\rangle = (-1)^{T+\ell+s}\lvert N\bar{N}\rangle$$

$$G = (-1)^{T+\ell+s}$$

But $G = C e^{i\pi T_2}$; $G = C(-1)^T$; $C = (-1)^{\ell+s}$

Example -- <u>Application</u>.

Nucleon form factors;



The $(N\bar{N})$ and γ states are both eigenstates of the operator C. Therefore <u>G will be even if T is odd and odd if T is even</u>. This may be used in extending the notion of G parity to π mesons.

We do this by considering possible intermediate states in the $\gamma N\bar{N}$ vertex.

The effect of $e^{i\pi T_2}$ on the pion wave functions is given by

$$e^{i\pi T_2}|\pi_1\rangle = -|\pi_1\rangle$$
$$e^{i\pi T_2}|\pi_2\rangle = +|\pi_2\rangle$$
$$e^{i\pi T_2}|\pi_3\rangle = -|\pi_3\rangle$$

operating on this with C leaves the neutral π field unaltered.

Define the phases as

$$C|\pi^{\pm}\rangle = |\pi^{\mp}\rangle$$
$$C|\pi_3\rangle = |\pi_3\rangle$$

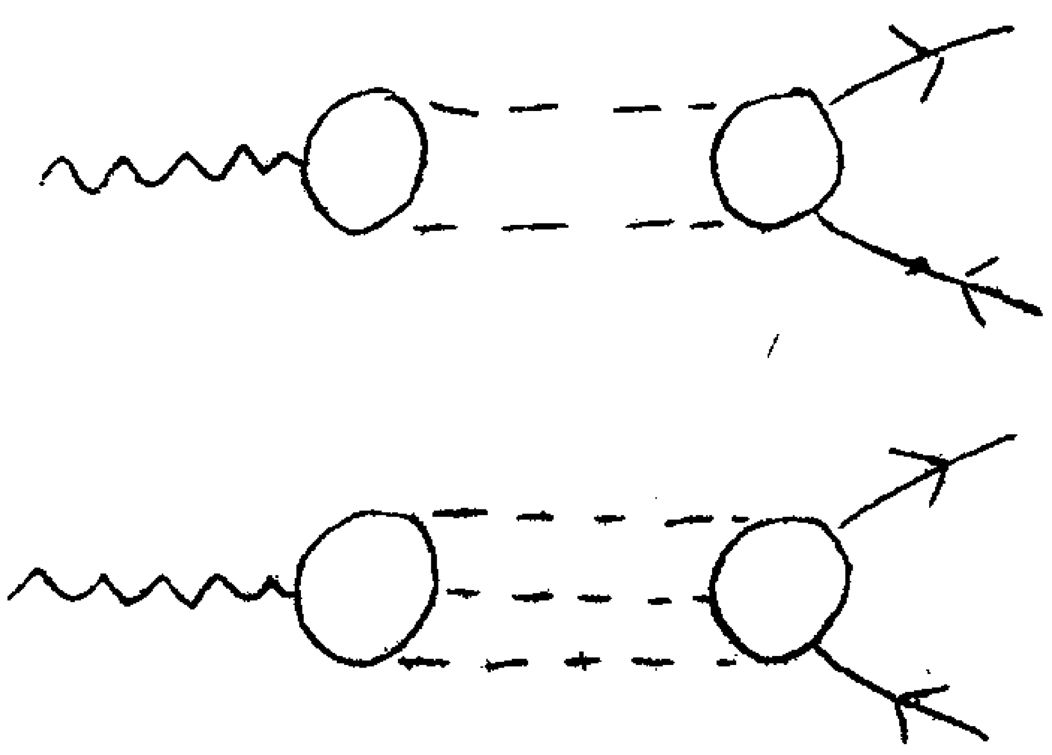
Then we have

$$|\pi_1\rangle \xrightarrow{G} -|\pi_1\rangle$$
$$|\pi_2\rangle \longrightarrow -|\pi_2\rangle$$
$$|\pi_3\rangle \longrightarrow -|\pi_3\rangle$$

Thus the pion field is odd under the G operation. This forbids a transition of an odd number of π mesons into an even number of π mesons.

$\lceil \Sigma \Lambda$ 3π vertex is already forbidden by parity conservation or reflection invariance.$\rfloor$

Consider 2π and 3π intermediate states in the N form factor.

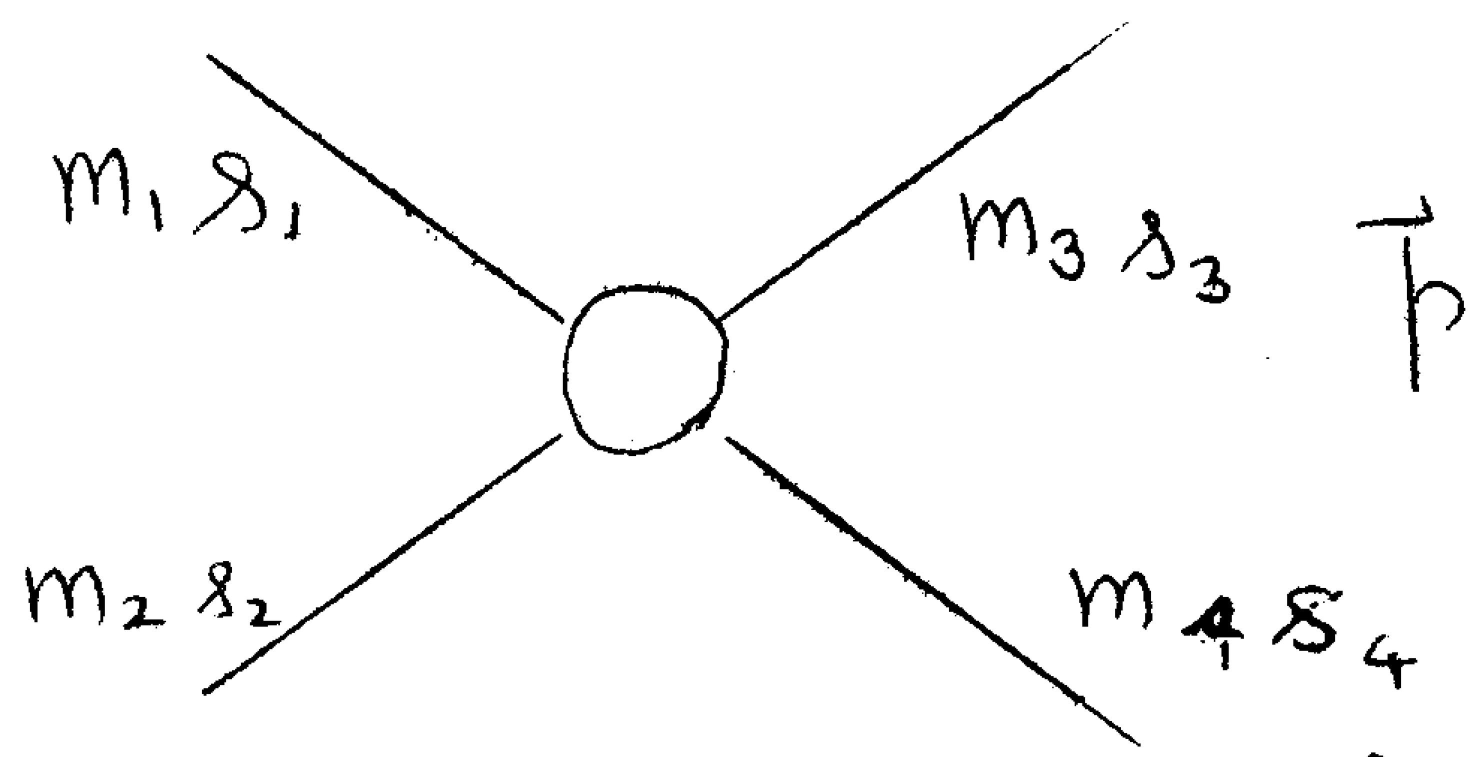


Therefore by G invariance the isoscalar form factor will receive contributions only from an odd number of pions whereas the isovector form factor will receive contributions only from intermediate states with an even number of pions.

These rules may be applied to the decays of multi-pion resonances.

VI. Partial Wave Analysis and the Helicity Formalism:

1. The Helicity formalism.



CMS: $\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4 =$

Define the c.m. scattering amplitude $f(W, \theta, \varphi)$

such that $|f(W, \theta, \varphi)|^2 = \partial\sigma/\partial\Omega$ (1)

First consider particles without spin. Then the scattering amplitude does not depend on φ. Therefore

$$f(W,\cos\theta) = \sum_{\ell} (2\ell+1) f_\ell(W) P_\ell(\cos\theta) \tag{2}$$

$f_\ell(W)$ can be obtained from $f(W,\cos\theta)$ by using the orthogc properties of $P_\ell(\cos\theta)$:

$$f_\ell(W) = \frac{1}{2}\int_{-1}^{+1} f(W,\cos\theta) P_\ell(\cos\theta)\, d(\cos\theta) \tag{3}$$

This partial wave analysis is useful when only a few partial waves make an important contribution, e.g., when there i a resonance in some particular partial wave, so that the partial wave expansion leads immediately to a good approximation.

Example - <u>Pion-nucleon scattering</u>:

We wrote the centre-of-mass amplitude as

$$\chi_f^\dagger [f_1 + f_2(\vec{\sigma}\cdot\hat{p}_2)(\vec{\sigma}\cdot\vec{p}_1)]\chi_i \tag{4}$$

The helicity is defined as the component of the spin along the momentum,

$$\mathcal{H} = \vec{s}\cdot\vec{p}/|\vec{p}| \tag{5}$$

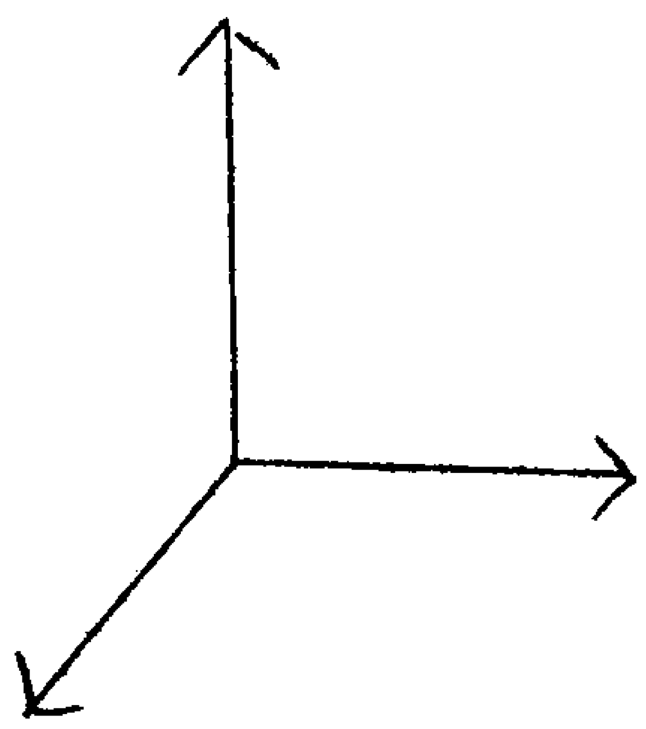
which is also equal to the component of the total angular momentı of the particle, since $\vec{L}\cdot\vec{p}=0$

$$\vec{J}\cdot\vec{p}/|\vec{p}| = \frac{(\vec{S}+\vec{L})\cdot\vec{p}}{|\vec{p}|} = \frac{\vec{S}\cdot\vec{p}}{|\vec{p}|} = \mathcal{H} \tag{6}$$

What we have to do is to construct eigenstates of the total angular momentum with a definite helicity.

To do this, we use the well-known results of group theory.

Start with a state with relative momentum along the z-direction, with helicities λ_1, λ_2 for the two particles



The rotated state

$$R_{\alpha\beta\gamma}\,|0,0,\lambda_1,\lambda_2\rangle \tag{7}$$

will have momentum along the α, β, γ direction and helicities λ_1, λ_2, since the helicities are invariant under rotation. A rotation of $(\varphi, \theta, 0)$ gives the state

$$R_{\varphi\theta 0}\,|0,0\,\lambda_1,\lambda_2\rangle = |\theta,\varphi;\lambda_1,\lambda_2\rangle \tag{8}$$

What we want to construct is a state

$$|j, m; \lambda_1, \lambda_2\rangle \tag{9}$$

with definite total angular momentum j and projection m

The states $|0,0;\lambda_1\lambda_2\rangle$ and $|jm;\lambda_1\lambda_2\rangle$ are related by

$$|00;\lambda_1\lambda_2\rangle = \sum_{j'm'} C_{j'm'}|j'm';\lambda_1\lambda_2\rangle \qquad (10)$$

On applying a rotation

$$R_{\varphi\Theta 0}|00;\lambda_1\lambda_2\rangle = \sum_{j'm'm''} C_{j'm'}|j'm'';\lambda_1\lambda_2\rangle \mathcal{D}^{j'}_{m''m'}(\varphi\Theta 0) \qquad (11)$$

Now let us use the orthonormality properties of the $\mathcal{D}^j$ matrices.

$$\int \mathcal{D}^{J*}_{MM'}(\alpha\beta\gamma)\mathcal{D}^{j}_{mm'}(\alpha\beta\gamma)\,dU = \frac{8\pi^2}{2j+1}\delta_{Jj}\delta_{Mm}\delta_{M'm'} \qquad (11)$$

where dU is the element in group space,

$$dU = d\alpha \sin\beta d\beta d\gamma$$

This gives

$$\mathcal{D}^{J*}_{M,M'}(\varphi\Theta 0)R_{\varphi\Theta 0}|00;\lambda_1\lambda_2\rangle\,d\Omega$$
$$= \delta_{M'\lambda} C_{J\lambda}|JM;\lambda_1\lambda_2\rangle\frac{4\pi}{2J+1} \quad \text{where } \lambda=\lambda_1-\lambda_2 \qquad (12)$$

This gives the method of constructing an angular momentum state with definite helicities.

2. Normalization.

We normalize our wave functions in the following way:

$$\langle \theta'\varphi';\lambda_1'\lambda_2' | \theta\varphi\lambda_1\lambda_2\rangle = \delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2} \times \delta(\cos\theta - \cos\theta')\,\delta(\varphi-\varphi') \tag{13}$$

$$\langle J'M'\lambda_1'\lambda_2' | JM\lambda_1\lambda_2\rangle = \delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}\delta_{JJ'} \times \delta_{MM'} \tag{14}$$

This normalization fixes the constant $C_{J\lambda}$ in (12) as

$$C_{J\lambda} = \sqrt{\frac{2J+1}{4\pi}} \tag{15}$$

so that

$$|JM;\lambda_1\lambda_2\rangle = \sqrt{\frac{2J+1}{4\pi}} \int \mathcal{D}^{J*}_{M\lambda}(\varphi\theta 0)\,|\theta,\varphi;\lambda_1\lambda_2\rangle \tag{16}$$

This gives

$$\langle \theta\varphi;\lambda_1\lambda_2 | JM;\lambda_1\lambda_2\rangle = \sqrt{\frac{2J+1}{4\pi}}\, \mathcal{D}^{J*}_{M\lambda}(\varphi\theta 0) \tag{17}$$

[Note: If we did not take helicity states, but states with definite m , the right hand side of (17) would have been spherical harmonics, multiplied by certain factors.]

Define:

$$T_{fi} = (2\pi)^4\,\delta(P_f - P_i)\,\frac{F_{fi}}{(2E_1^i\, 2E_2^i\, 2E_1^f\, 2E_2^f)} \tag{18}$$

In the c.m. system, we just have a collision between 2 particles resulting in a change of angle. In the c.m.s. we may write

$$T_{fi} = (2\pi)^4 \delta(P_f - P_i) \langle \theta_f, \varphi_f | T | \theta_i \varphi_i \rangle \tag{19}$$

Here we assume a normalization

$$\langle \vec{k}, s' | \vec{k}\, s \rangle = \delta_{ss'} \delta(\vec{k} - \vec{k}') (2\pi)^3 \tag{20}$$

To change to the normalization (13), (14), we must modify (19), multiplying it by a factor, R,

$$T_{fi} = (2\pi)^4 \delta(p_{f} - p_i) R \langle \theta_f, \varphi_f | T | \theta_i \varphi_i \rangle \tag{21}$$

where

$$R = (2\pi)^2 \frac{W}{\sqrt{p q E_1^i E_2^i E_1^f E_2^f}} \tag{22}$$

then

$$T_{fi} = (2\pi)^4 \delta(p_f - p_i) (2\pi)^2 \frac{W}{(p q E_1^i E_2^i E_1^f E_2^f)^{1/2}} \langle \theta_f \varphi_f | T | \theta_i$$

This gives

$$F_{fi} = 4 (2\pi)^2 \frac{W}{\sqrt{pq}} \langle \theta_f \varphi_f | T | \theta_i \varphi_i \rangle \tag{23}$$

The differential cross section is given, in terms of F, by

$$\frac{d\sigma}{d\Omega} = \frac{p}{q} \left| \frac{F}{8\pi W} \right|^2 \tag{24a}$$

$$= \frac{p}{q} \frac{1}{(8\pi W)^2} (2\pi)^4 \frac{W^2}{pq} |\langle \theta_f \varphi_f | T | \theta_i \varphi_i$$

$$= \left(\frac{2\pi}{q}\right)^2 |\langle \theta_f \varphi_f | T | \theta_i \varphi_i \rangle|^2 \tag{24b}$$

Introduce a complete set of helicity states $|J M \lambda_1' \lambda_2'\rangle$

(24b) becomes

$$\frac{d\sigma}{d\Omega} = \left(\frac{2\pi}{q}\right)^2 \sum_{JJ'MM'} \langle \theta_f \varphi_f \lambda_1' \lambda_2' | J'M' \lambda_1' \lambda_2' \rangle \cdot \langle J'M'\lambda_1'\lambda_2' | T | JM\lambda_1\lambda_2 \rangle \langle JM \lambda_1 \lambda_2 | 0 \, 0 \rangle$$

Rotational invariance implies conservation of angular momentum, which gives

$$\langle J'M'\lambda_1'\lambda_2' | T | JM\lambda_1\lambda_2 \rangle = \delta_{JJ'} \delta_{MM'} \langle \lambda_1'\lambda_2' | T^J(w) | \lambda_1 \lambda_2 \rangle \tag{25}$$

Using (17), we obtain

$$\frac{d\sigma}{d\Omega} = \left(\frac{2\pi}{q}\right)^2 \left| \sum_J \left(\frac{2J+1}{4\pi}\right) \langle \lambda_1'\lambda_2' | T^J(w) | \lambda_1\lambda_2 \rangle \mathcal{D}^{J*}_{M\lambda}(\varphi\,\theta\,0) \right|^2 \tag{26}$$

<u>Note:</u>

$$M = \lambda_1 - \lambda_2$$

comparing this with the definition of $f_{\lambda_1'\lambda_2';\lambda_1\lambda_2}(\theta,\varphi)$

$$\frac{d\sigma}{d\Omega} = \left| f_{\lambda_1'\lambda_2'\lambda_1\lambda_2}(\theta,\ \varphi) \right|^2 \tag{27}$$

We write

$$f_{\lambda_1'\lambda_2'\lambda_1\lambda_2}(\theta,\varphi) = \sum_J \frac{1}{q}\left(J+\frac{1}{2}\right) \langle \lambda_1'\lambda_2' | T^J(w) | \lambda_1\lambda_2 \rangle \mathcal{D}^{J*}_{M,} \tag{28}$$

3. <u>Transformation properties under space inversion and time reversal.</u>

<u>Space Inversion</u>:

$$P|JM\lambda_1\lambda_2\rangle = \eta_1\eta_2(-1)^{J-s_1-s_2}|JM;-\lambda_1,-\lambda_2\rangle \quad (29)$$

where s_1, s_2 are the spins and $\eta_1\eta_2$ are the intrinsic parity factors.

Example -- Take a 1-photon state:

$$P|JM\lambda\rangle = (-1)(-1)^{J-1}|JM;-\lambda\rangle \quad (30)$$

Consider the states

$$\frac{1}{\sqrt{2}}\{|JM;\lambda\rangle + |JM;-\lambda\rangle\} \quad (31a)$$

and

$$\frac{1}{\sqrt{2}}\{|JM;\lambda\rangle - |JM;-\lambda\rangle\} \quad (31b)$$

These have parities $(-1)^J$ and $(-1)^{J+1}$ respectively. The first is the electric multiple and the second is the magnetic radiation.

What is the angular distribution in such a state ?

The amplitudes for the states $|JM;\lambda\rangle$ and $|JM;-\lambda\rangle$ are

$$\mathcal{D}^{J^*}_{M\lambda}(\varphi\,\theta\,0) \text{ and } \mathcal{D}^{J^*}_{M,-\lambda}(\varphi\,\theta\,0) \quad (32)$$

The angular distribution is

$$|\mathcal{D}^{J^*}_{M\lambda}(\varphi\,\theta\,0)|^2 + |\mathcal{D}^{J^*}_{M,-\lambda}(\varphi\,\theta\,0)|^2 \quad (33)$$

as the 2 helicity states are orthogonal.

The transverse polarization is given by the interference between the two helicity amplitudes. Reflection invariance implies that

$$PSP^{-1} = S \tag{34}$$

This gives

$$\langle -\lambda_1', -\lambda_2' | T^J | -\lambda_1, -\lambda_2 \rangle = \frac{\eta_1'\eta_2'}{\eta_1\eta_2} (-1)^{s_1'+s_2'-s_1-s_2} \langle \lambda_1'\lambda_2' | T^J | \lambda_1 \lambda_2 \rangle \tag{35}$$

Time reversal.

$$T\,|JM\lambda_1\lambda_2\rangle = (-1)^{J-M}\,|J,-M,\lambda_1\lambda_2\rangle \tag{36}$$

Time reversal invariance requires

$$TST^{-1} = S^{\dagger} = S^{-1} \tag{37}$$

This implies

$$\langle \lambda_1'\lambda_2' | T^J | \lambda_1\lambda_2 \rangle = \langle \lambda_1\lambda_2 | T^J | \lambda_1'\lambda_2' \rangle \tag{38}$$

Under permutation of particles 1 and 2,

$$P_{12}\,|JM\lambda_1\lambda_2\rangle = (-1)^{J-2s}\,|JM\lambda_2\lambda_1\rangle \tag{39}$$

The even and odd states are given by

$$\frac{1}{\sqrt{2}}\left\{ |JM\lambda_1\lambda_2\rangle \pm (-1)^{J-2s} |JM\lambda_2\lambda_1\rangle \right\} \tag{40}$$

respectively.

Now use the connection between spin and statistics. Therefore for both fermions and bosons, the state will be

$$\frac{1}{\sqrt{2}}\left\{ |JM\lambda_1\lambda_2\rangle + (-1)^J |JM\lambda_2\lambda_1\rangle \right\} \tag{41}$$

since $(-1)^{2S} = -1$ for a fermion.

4. <u>The decay $\pi^0 \to 2\gamma$</u>:

The 2 photon state is given by

$$|JM\lambda_1\lambda_2\rangle + (-1)^J |JM\lambda_2\lambda_1\rangle \tag{42}$$

If the π^0 has spin 1 , the state must be

$$|JM\lambda_1\lambda_2\rangle - |JM\lambda_2\lambda_1\rangle$$

The other state with the ~~same~~ helicities is forbidden. But this will have m component $= \pm 2$ which is not possible, since $J = 1$. Therefore the π^0 cannot have spin 1.

This argument uses the fact that the photons are bosons. The helicities must be RR + LL .

<u>Identical particles:</u>

$$\langle \lambda_1'\lambda_2' | T^J | \lambda_1\lambda_2\rangle = \langle \lambda_2'\lambda_1' | T^J | \lambda_2\lambda_1\rangle \tag{43}$$

Number of independent scattering amplitudes:

N - N scattering

	$++$	$+-$	$-+$	$--$
$++$	f_1^J	f_5^J	f_6^J	f_2^J
$+-$		f_3^J	f_4^J	
$-+$		f_4^J	f_3^J	
$--$	f_2^J	f_6^J	f_5^J	f_1^J

(44)

$J = 0$ only $++$, $--$ not $= 0$.

$J \neq 0$ all 16 amplitudes are $\neq 0$

The restrictions arising from parity conservation and T invariance reduce the number of independent amplitudes.

T invariance $\Rightarrow$ symmetry about the principal diagonal, giving the final set of amplitudes as

$$\begin{matrix} f_1^J & f_5^J & f_6^J & f_2^J \\ f_5^J & f_3^J & f_4^J & f_6^J \\ f_6^J & f_4^J & f_3^J & f_5^J \\ f_2^J & f_6^J & f_5^J & \end{matrix} \tag{45}$$

In N N scattering there are only 5 independent amplitudes because of Pauli's principle,

$$f_5^J = f_6^J \tag{46}$$

We have found the number of independent amplitudes with a given value of J. But note:

$$f_{\lambda_1'\lambda_2'\lambda_1\lambda_2}(\theta\varphi) = \frac{1}{q}\sum_J (J+\tfrac{1}{2})\, f^J_{\lambda_1'\lambda_2'\lambda_1\lambda_2}\, \mathcal{D}^{J*}_{\lambda_1-\lambda_2,\,\lambda_1'-\lambda_2'}(\varphi\,\theta\,0)$$

$$= \frac{1}{q}\sum (J+\tfrac{1}{2})\, f^J_{\lambda_1'\lambda_2',\lambda_1\lambda_2}\, e^{-i\varphi(\lambda_1-\lambda_2)}\, d^J_{\lambda_1-\lambda_2,\lambda_1'-\lambda_2'}(\theta) \tag{47}$$

The symmetries of the $d^3(\theta)$ functions are

$$d^J_{\lambda\mu}(\theta) \longrightarrow d^J_{-\lambda-\mu}(\theta) \tag{48}$$

$$d^J_{\lambda\mu}(\theta) \longrightarrow d^J_{\mu\lambda}(\theta)$$

But we have

$$d^J_{\lambda\mu}(\theta) = (-1)^{\lambda-\mu}\, d^J_{-\lambda-\mu}(\theta)$$

$$d^J_{\lambda\mu}(\theta) = (-1)^{\lambda-\mu}\, d^J_{\lambda\mu}(\theta) \tag{49}$$

<u>This relationship does not depend on J,</u> so that it is also the relationship for the <u>total amplitude</u>, not just the partial-wave amplitudes with definite J.

The matrix for the total amplitudes is given by

	$++$	$+-$	$-+$	$--$
$++$	f_1	f_5	$-f_5$	f_2
$+-$	$-f_5$	f_3	f_4	$-f_5$
$-+$	f_5	f_4	f_3	f_5
$--$	f_2	f_5	$-f_5$	f_1

(50)

In N N scattering, the total spin is conserved; the parity is

$$(-1)^{\ell}(-1)^{S+1} \qquad (51)$$

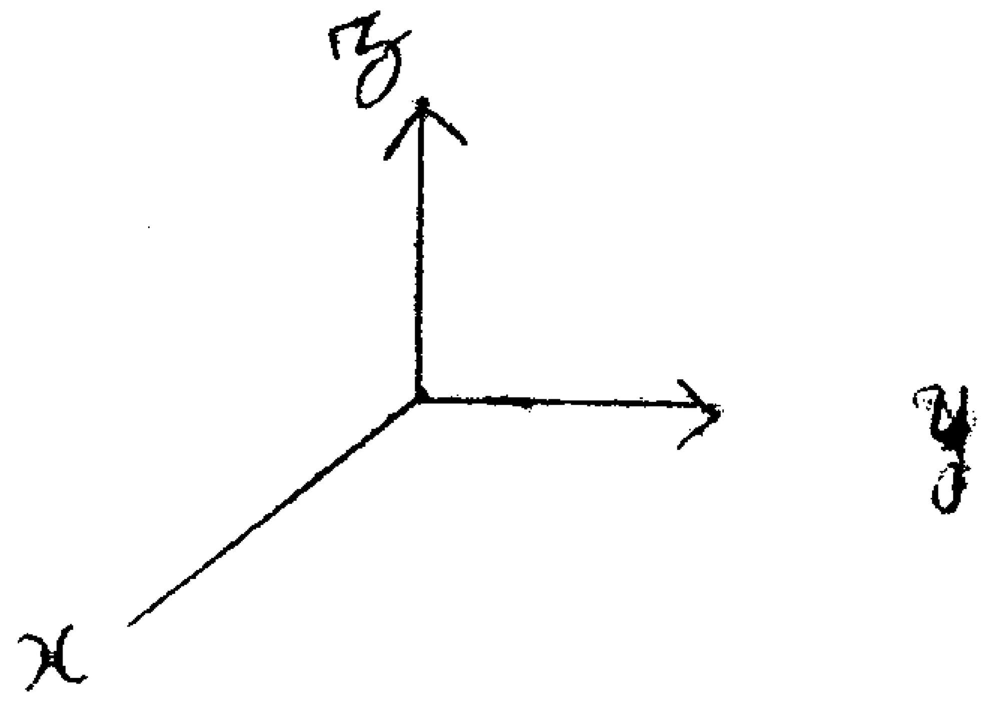
Next consider $N\bar{N} \longrightarrow N\bar{N}$

From charge conjugation, we can see that here also there are only 5 independent amplitudes.

6. Deviation of the parity property:

$$P|JM\lambda_1\lambda_2\rangle = \eta_1\eta_2(-1)^{J-s_1-s_2}|JM-\lambda_1-\lambda_2\rangle \qquad (52)$$

Start with a plane-wave state along the z axis.



Reflection in the xz plane effected by the operation

$$Y = e^{-i\pi J_y} P$$

has the result

$$Y|00\lambda_1\lambda_2\rangle = |00, -\lambda_1 - \lambda_2\rangle$$

Y commutes with a Lorentz transform along the z axis. But for a state at rest,

$$Y = e^{-i\pi J_y} P \tag{53}$$

the product of an inversion and a rotation by π about the y axis

But for a state at rest, we have

$$P|m\rangle = \eta|m\rangle$$
$$e^{-i\pi J_y} P|m\rangle = \eta \sum_{m'} d^s_{m'm}(\pi)|m'\rangle$$

This does not apply to a zero-mass particle; but for this we have

so that we have

$$Y|00\lambda_1\lambda_2\rangle = \eta_1\eta_2(-1)^{s_1+s_2-\lambda_1-\lambda_2}|00-\lambda_1-\lambda_2\rangle$$

For a photon, $\lambda = \pm 1$ $\quad (s-\lambda) = \text{even}$

$$Y|+1\rangle = \eta|-\rangle \qquad \eta = -1$$

The helicity states are

$$-\frac{1}{\sqrt{2}}(\varepsilon_x + i\varepsilon_y) \qquad \frac{1}{\sqrt{2}}(\varepsilon_x - i\varepsilon_y)$$
$$Y = e^{-i\pi J_y} P$$

$$P|JM\lambda_1\lambda_2\rangle = \sqrt{\frac{2J+1}{4\pi}} \int \mathcal{D}^{J*}_{M\lambda}(\alpha\beta\gamma) P R_{\alpha\beta\gamma} |00\lambda_1\lambda_2\rangle$$

P commut es with $R_{\alpha\beta\gamma}$

$$P\, R_{\alpha\beta\gamma} |00\lambda_1\lambda_2\rangle = R_{\alpha\beta\gamma}\, e^{i\pi J_y} Y |00\lambda_1\lambda_2\rangle$$

$$\therefore P|J\,M\,\lambda_1\,\lambda_2\rangle = \sqrt{\frac{2J+1}{4\pi}} \int \mathcal{D}^{J^*}_{M\lambda}(\alpha\beta\gamma)\, R_{\alpha\beta\gamma} R_{0\pi 0}\, \eta_1 \eta_2$$
$$\times (-1)^{s_1+s_2-\lambda_1-\lambda_2} |00-\lambda_1-\lambda_2\rangle$$

Note:

$$R_{\alpha\beta\gamma}\; R_{0+\pi 0} = R_{\alpha'\beta'\gamma'}$$
$$R_{\alpha\beta\gamma} = R_{\alpha'\beta'\gamma'}\, R_{\gamma\pi 0}$$

so that we have

$$\mathcal{D}^{J}_{M\lambda}(\alpha\beta\gamma) = \sum_{\mu} \mathcal{D}^{J}_{M\lambda}(\alpha'\beta\gamma')\, \mathcal{D}^{J}_{\mu\lambda}(0\pi 0)\, (-1)^{J-\lambda} \delta_{\mu}$$

so that we have

$$P|JM\lambda_1\lambda_2\rangle = \sqrt{\frac{2J+1}{4\pi}} \int \mathcal{D}^{J^*}_{M-\lambda}(\alpha',\beta',\gamma')$$
$$R_{\alpha'\beta'\gamma'}\, \eta_1 \eta_2 (-1)^{\lambda_1+\lambda_2-\lambda_1+\lambda_2} |00-\lambda_1-\lambda_2\rangle$$
$$\times (-1)^{-J+\lambda_1-\lambda_2}$$

$$= \sqrt{\frac{2J+1}{4\pi}} \int \mathcal{D}^{J^*}_{M-\lambda}(\alpha'\beta'\gamma')\, R_{\alpha'\beta'\gamma'}\, \eta_1 \eta_2 (-1)^{s_1+}$$
$$|00-\lambda_1-\lambda_2\rangle (-1)^{-J}$$

7. <u>Application of the Helicity formalism to Pion-nucleon Scattering.</u>

Consider pion-nucleon scattering.

$$\pi + N \longrightarrow \pi + N \tag{1}$$

Here only one of the particles, the nucleon, has a spin, and thus the helicity states are the states in which the nucleon has a definite helicity. The number of independent amplitudes is just two; this follows merely from parity conservation.

e.g. The helicity amplitudes for each J may be labelled as shown below:

Initial N helicity →
Final N helicity ↓

	+	—
+	++	+−
−	−+	−−

(2)

Define the partial wave amplitudes as

$$f^J_{++} = \langle + | T^J | + \rangle / 2q \tag{3}$$
$$f^J_{-+} = \langle - | T^J | + \rangle / 2q$$

The relations

$$f^J_{++} = f^J_{--} \; ; \qquad f^J_{+-} = - f^J_{-+} \tag{4}$$

follow from parity conservation. The helicity amplitudes may be written a

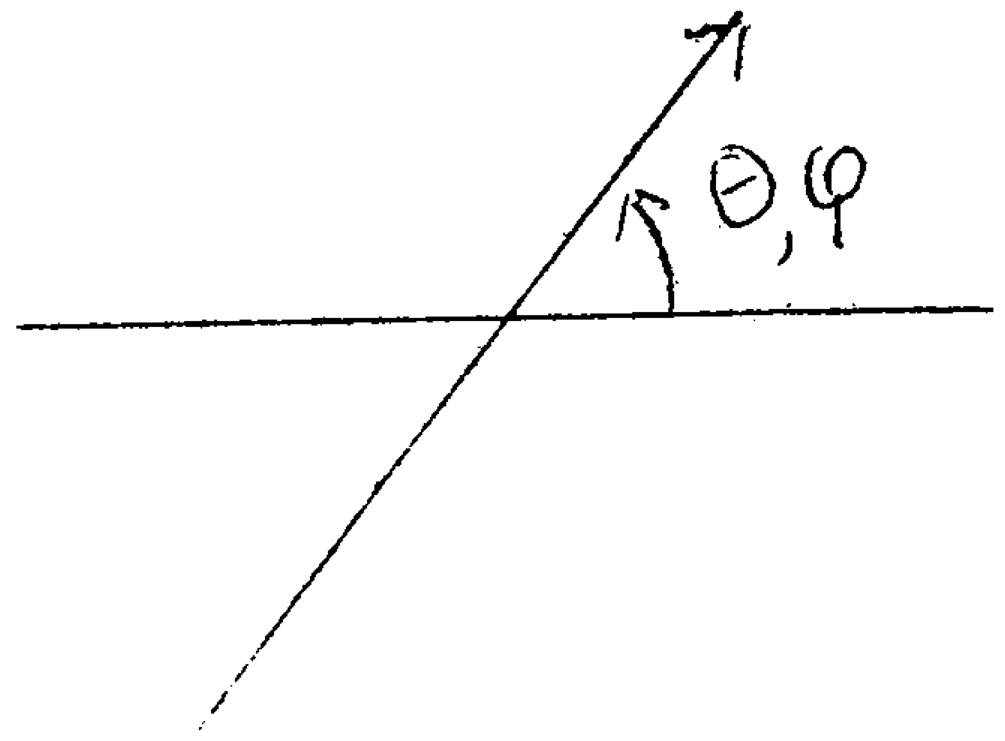
$$f_{++} = \sum_J (2J+1) f^J_{++} \mathcal{D}^{J*}_{\frac{1}{2}\frac{1}{2}}(\varphi,\theta,0) \tag{5}$$
$$f_{-+} = \sum_J (2J+1) f^J_{-+} \mathcal{D}^{J*}_{-\frac{1}{2}\frac{1}{2}}(\varphi,\theta,0)$$

and the inverse of this gives

$$f^J_{++} = \frac{1}{4\pi} \int f_{++} \mathcal{D}^J_{\frac{1}{2}\frac{1}{2}}(\varphi\,\theta\,0)\, d\Omega \tag{6}$$
$$f^J_{-+} = \frac{1}{4\pi} \int f_{-+} \mathcal{D}^J_{-\frac{1}{2}\frac{1}{2}}(\varphi\,\theta\,0)\, d\Omega$$

We can express the usual amplitudes $f_1\ f_2$ in terms of the helicity amplitudes. The amplitudes f_1, f_2 are given by

$$f = \chi_f^+ (f_1 + \vec{\sigma}\cdot\hat{p}_f)(\vec{\sigma}\cdot\hat{p}_i)\,\chi_i \tag{7}$$



The rotated wave functions are

$$\begin{aligned} \chi_f(+) &= \cos\tfrac{\theta}{2}\, e^{-i\frac{\varphi}{2}}\chi_+ + \sin\tfrac{\theta}{2}\, e^{i\frac{\varphi}{2}}\chi_- \\ \chi_f(-) &= -\sin\tfrac{\theta}{2}\, e^{-i\frac{\varphi}{2}}\chi_+ + \cos\tfrac{\theta}{2}\, e^{i\frac{\varphi}{2}}\chi_- \end{aligned} \tag{8}$$

This gives

$$\begin{aligned} f_{++} &= \cos\tfrac{\theta}{2}\, e^{+i\frac{\varphi}{2}}(f_1+f_2) \\ f_{-+} &= -\sin\tfrac{\theta}{2}\, e^{+i\frac{\varphi}{2}}(f_1-f_2) \end{aligned} \tag{9}$$

The partial wave amplitudes are given by

$$\begin{aligned} f^J_{++} &= \frac{1}{4\pi}\int (f_1+f_2)\, \mathcal{D}^J_{\frac12\frac12}(\varphi\,\theta\,0)\cos\tfrac{\theta}{2}\, e^{+i\frac{\varphi}{2}}\, d\Omega \\ f^J_{--} &= \frac{-1}{4\pi}\int (f_1-f_2)\, \mathcal{D}^J_{-\frac12\frac12}(\varphi\,\theta\,0)\sin\tfrac{\theta}{2}\, e^{+i\frac{\varphi}{2}} \\ &\qquad \times d\Omega \end{aligned} \tag{10}$$

The φ dependence of the $\mathcal{D}^J$ functions cancels the factors $e^{+i\varphi/2}$, and the result is, after integrating over φ ,

$$f^J_{++} = \frac{1}{2}\int (f_1+f_2)\, d^J_{\frac{1}{2}\frac{1}{2}}(\theta) \cos\frac{\theta}{2}\, d(\cos\theta)$$

$$f^J_{-+} = -\frac{1}{2}\int (f_1-f_2)\, d^J_{-\frac{1}{2}\frac{1}{2}}(\theta) \sin\frac{\theta}{2}\, d(\cos\theta) \qquad (11)$$

Substituting the values of $d^J_{\pm\frac{1}{2}\pm\frac{1}{2}}$ we obtain

$$f^J_{++} = \frac{1}{2}\int (f_1+f_2) \frac{1}{j+\frac{1}{2}}\left[P'_{j+\frac{1}{2}}(\cos\theta) - P'_{j-\frac{1}{2}}(\cos\theta)\right] \frac{1+\cos\theta}{2}\, d(\cos\theta)$$

$$f^J_{-+} = -\frac{1}{2}\int (f_1-f_2) \frac{1}{j+\frac{1}{2}}\left[P'_{j+\frac{1}{2}}(\cos\theta) + P'_{j-\frac{1}{2}}(\cos\theta)\right] \frac{1-\cos\theta}{2}\, d(\cos\theta) \qquad (12)$$

Using the properties of the Legendre polynomials, we obtain

$$f^J_{++} = \frac{1}{4}\int (f_1+f_2)\left[P_{j+\frac{1}{2}}(\cos\theta) + P_{j-\frac{1}{2}}(\cos\theta)\right] d(\cos\theta)$$

$$f^J_{-+} = \frac{1}{4}\int (f_1-f_2)\left[P_{j-\frac{1}{2}}(\cos\theta) - P_{j+\frac{1}{2}}(\cos\theta)\right] d(\cos\theta) \qquad (13)$$

The inverse of this is given by

$$f_1 = \sum_j \left\{ f^j_{++}\left(P'_{j+\frac{1}{2}} - P'_{j-\frac{1}{2}}\right) + f^j_{-+}\left(P'_{j+\frac{1}{2}} + P'_{j-\frac{1}{2}}\right)\right\}$$

$$f_2 = \sum_j \left\{ f^j_{++}\left(P'_{j+\frac{1}{2}} + P'_{j-\frac{1}{2}}\right) - f^j_{-+}\left(P'_{j+\frac{1}{2}} - P'_{j-\frac{1}{2}}\right)\right\} \qquad (14)$$

It is interesting to compare this with the usual approach, where one uses an orbital-angular momentum L , and couples it to the spin to give the total angular momentum J

However, in this special example of π-N scattering, because of parity conservation, there are just two states of definite parity, for a given value of J .

label

These have

$$\ell = j - 1/2 \qquad \Pi = -(-1)^{J-\frac{1}{2}} \qquad \ell_+$$

$$\ell = j + 1/2 \qquad \Pi = -(-1)^{J+\frac{1}{2}} \qquad (\ell+1)_- \tag{15}$$

These two states, with $j = (\ell + \frac{1}{2})$ and $j = (\ell - \frac{1}{2})$ respectively, are often denoted by the indices ℓ_+ and $(\ell+1)_-$ respectively.

$$f_{J\ell} = \frac{e^{2i\delta_{J\ell}} - 1}{2iq} = \frac{T^{j\ell}}{2q} = \frac{S^{j\ell} - 1}{2iq} = \frac{e^{i\delta_{j\ell}} \sin\delta_{j\ell}}{q} \tag{16}$$

where $S_{j\ell} = e^{2i\delta_{j\ell}}$

The states with definite parity are

$$\frac{1}{\sqrt{2}}\left\{ |j\, m+1/2\rangle \pm |j m - \tfrac{1}{2}\rangle \right\} \tag{17}$$

We can then obtain

$$f_{\ell+} = f^J_{++} + f^J_{-+}$$

$$f_{(\ell+1)-} = f^J_{++} - f^J_{-+}$$

We may use these to express f_1, f_2 in terms of $f_{\ell+}$ and $f_{\ell-}$

Equations (12) and (18) give

$$f_{\ell+} = \frac{1}{2}\int \left[f_1 P_{J-\frac{1}{2}}(\cos\theta) + f_2 P_{J+\frac{1}{2}}(\cos\theta) \right] d(\cos\theta)$$

$$= \frac{1}{2}\int \left[f_1 P_\ell(\cos\theta) + f_2 P_{\ell+1}(\cos\theta) \right] d(\cos\theta)$$

and

$$f_{\ell-} = \frac{1}{2}\int \left[f_1 P_{\ell+1}(\cos\theta) + f_2 P_\ell(\cos\theta) \right] d(\cos\theta) \tag{1}$$

Equations (19) may be written together as follows:

$$f_{\pm} = \frac{1}{2}\int [f_1 P_\ell(\cos\theta) + f_2 P_{\ell\pm 1}(\cos\theta)]\, d(\cos\theta) \tag{20}$$

The inverse relation can also be obtained directly:

$$f_1 = \sum_\ell f_{\ell+} P'_{\ell+1}(\cos\theta) - f_{\ell-} P'_{\ell-1}(\cos\theta)$$
$$f_2 = \sum_\ell (f_{\ell-} - f_{\ell+}) P'_\ell(\cos\theta) \tag{21}$$

These formulae are the basis of making approximations in the calculation of partial waves.

Note: The relation between the amplitude $f_{\lambda_1'\lambda_2'\lambda_1\lambda_2}$ and the differential cross section is obtained simply.

$$\frac{d\sigma}{d\Omega} = \Big[\sum_{J J'} (2J+1)(2J'+1) f^J_{\lambda'\lambda} f^{J'}_{\lambda'\lambda} \mathcal{D}^{J*}_{\lambda'\lambda}(\varphi\,\theta\,0)\, \mathcal{D}^{J'}_{\lambda'\lambda}(\varphi\,\theta\,0)\Big] \tag{22}$$

Two states with different helicities do not interfere. When the helicities are not observed, we must sum over λ, λ' and divide by $(2s+1)$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2s+1}\sum_{\lambda\lambda'} [\text{expression } 22] \tag{23}$$

We can express the product of $\mathcal{D}^{J*}_{\lambda'\lambda}$ and $\mathcal{D}^{J'}_{\lambda'\lambda}$ by using the Clebsch-Gordan series as

$$\mathcal{D}^*(\varphi\theta 0)\,\mathcal{D}(\varphi\theta 0) = \sum_\ell (-1)^{\lambda-\lambda'} C(JJ'\ell)(\lambda,-\lambda)\, C(JJ'\ell|\ldots)\, P_\ell(\cos\theta) \tag{24}$$

From the symmetry following from parity conservation, the imaginary part of f, f^{x} vanishes and we get

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\lambda+1} \sum_{(\lambda)} \sum_{jj'} (2j+1)(2j'+1) \, \mathrm{Re}\left\{ f^{j}_{\lambda'\lambda} \, f^{j'x}_{\lambda'\lambda} \right\}$$

$$\left[\sum_{(\ell)} (-1)^{\lambda-\lambda'} C(jj'\ell \mid \lambda, -\lambda) \, C(jj'\ell \mid \lambda', -\lambda') \right. \quad (25)$$

$$P_\ell (\cos\theta)$$

The Minami Ambiguity.

When the polarization of the final N is not observed, then the Minami ambiguity is the statement that when, for a given value of J, we replace the phase shift for one value of ℓ by that of the other, then nothing is changed.

example -- If we replace the $P_{\frac{3}{2}}$ phase shifts by $D_{\frac{3}{2}}$ and $P_{\frac{1}{2}}$ by $S_{\frac{1}{2}}$ then

$$J = 3/2 \qquad \begin{matrix} 1 \\ 2 \end{matrix}$$

$$J = \tfrac{1}{2} \qquad \begin{matrix} 0 \\ 1 \end{matrix}$$

$$f^{J}_{++} = [f_{\ell +} + f_{(\ell+1)-}]/2 \qquad (26)$$

$$f^{J}_{-+} = [f_{\ell +} - f_{(\ell+1)-}]/2$$

when we make the replacement, nothing is changed.

Polarization.

A Lorentz transform along the direction of motion which brings a particle to rest does not change the helicity. When the particle is thus brought to rest, the helicity is just the spin projection.

The helicity amplitudes thus provide a convenient description.

Using the results

$$f_{++} = \cos\frac{\theta}{2}\,(f_1 + f_2)$$
$$f_{-+} = -\sin\frac{\theta}{2}\,(f_1 - f_2) \qquad (27)$$

we can obtain the transverse polarization as

$$(\vec{\sigma}\cdot\hat{n})\frac{d\sigma}{d\Omega} = (\sin\theta)\ \mathrm{Im}\{f_2 f_1^*\} \qquad (28)$$

<u>A remark</u>.

If we have only waves of a given parity, then only $f_{\ell+}$ or $f_{\ell-} \neq 0$ so that one of the ampl. $f_1 + f_2$ is an even function of $\cos\theta$ and the other odd. $f_2\ f_1^*$ is an odd function of $\cos\theta$. Therefore the polarization at $\theta = 90$ is automatically zero when only waves of one parity are present.

A strong polarization at 90° indicates that there is a good mixture of partial waves of both parities.

8. <u>The Unitarity Relation for Partial Waves</u>.

The S matrix is diagonal in the angular momentum representation. i.e., Angular momentum is conserved;

$$\begin{array}{ccc} J_1 & & \\ \boxed{S^{J_1}} & J_2 & \\ & \boxed{S^{J_2}} & J_3 \\ & & \boxed{S^{J_3}} \end{array} \qquad (29)$$

The unitarity relation can be written in terms of the diagonal elements, i.e., the partial wave amplitudes.

The ordinary unitarity relation thus immediately gives

$$-i\left(T^J_{fi} - T^J_{if}\right) = \sum_n T^J_{fn} T^{J*}_{in} \tag{30}$$

or

$$2\,\mathrm{Im}\,T_{fi} = \sum_n T^J_{fn} T^{J*}_{in}$$

Suppose only one of the partial waves is important.

Thus $$2\,\mathrm{Im}\,T_{fi} \sim T^J_{ff} T^{J*}_{if} \tag{31}$$

This would be valid only if the inelastic amplitudes are small. We can write the elastic amplitude T^J_{ff} in terms of real phase shift. We then get

$$2\,\mathrm{Im}\,T_{fi} \sim 2q\left(e^{i\delta_{ff}} \sin\delta_{ff}\right) T^{J*}_{if}$$

so that

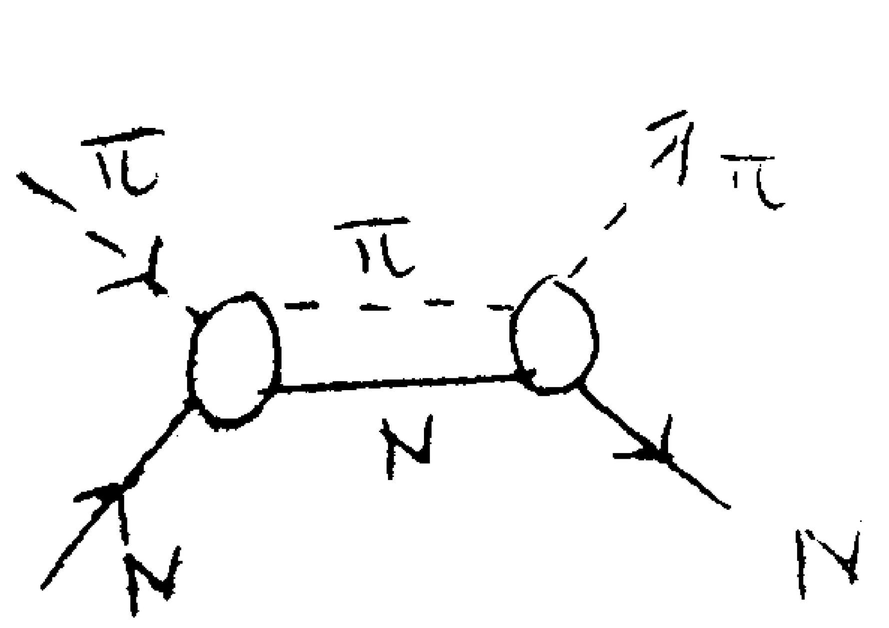
$$\mathrm{Im}\,T^J_{fi} \approx q\,e^{i\delta^J} \sin\delta_J\, T^{J*}_{if} \tag{32}$$

Example: <u>Photoproduction</u>.

Equation (32) implies that the phase of the production amplitude is equal to the pion-nucleon phase shift. This is known as the final-state interaction theorem.

For pion-nucleon scattering, this would give $\mathrm{Im}\,T$ in terms of the total cross-section; this gives the optical theorem. Keeping only the π N intermediate states in π N scattering, we obtain the following:

$$T^{J}_{fi} = 2q\, f^{J}_{fi}$$
$$2\, \mathrm{Im}\, T^{J}_{fi} \approx T^{\times}_{fi}\, T^{J}_{fi}$$
$$4q\, \mathrm{Im} f^{J} \approx 4q^{2} |f^{J}|^{2}$$



or
$$\mathrm{Im}\, f^{J} = q\, |f^{J}|^{2} \qquad (33)$$

The optical theorem for the total forward scattering amplitude is

$$\mathrm{Im} f(00) = \frac{q}{4\pi}\, \sigma_{tot} \qquad (34)$$

A similar optical theorem may be obtained for each partial wave also, as each partial wave is scattered independently of the others, giving

$$(2J+1)\, \mathrm{Im}\, f^{J}_{\lambda\lambda} = \frac{q}{4\pi}\, \sigma_{tot}^{J,\lambda} \qquad (35)$$

We can apply these to analyse pion-nucleon scattering. The analysis of the data is most convenient in terms of orbital angular momentum. The amplitudes with definite ℓ and J are

$$f_{\ell J} = \frac{e^{i\delta_{\ell J}} \sin\delta_{\ell J}}{q}, \quad \delta_{\ell J} \text{ real} \qquad (36)$$

in the region below all inelastic thresholds.

However, in the region of a few hundred Mev. pion lab. energy, inelastic phenomena are important and the above representation is no longer valid.

At energies above this region, the scattering amplitude may be written as follows:

For each value T of the isospin

$$S^{j,\ell T} = \eta^{j,\ell T} = \rho^{j\ell T} e^{2i\delta_{j\ell T}} \qquad \begin{matrix}[0 \leq \rho \leq 1] \\ [\delta_{j\ell} \text{ real}]\end{matrix} \tag{37}$$

ρ the inelasticity parameter

Elastic scattering: $\rho = 1$

For Complete absorption: $\rho = 0$

The scattering amplitude is pure imaginary

$$f_{\ell j} = \frac{\eta_{\ell j} - 1}{2iq} = \frac{\rho^{\ell j} e^{2i\delta_{\ell j}} - 1}{2iq} \tag{38}$$

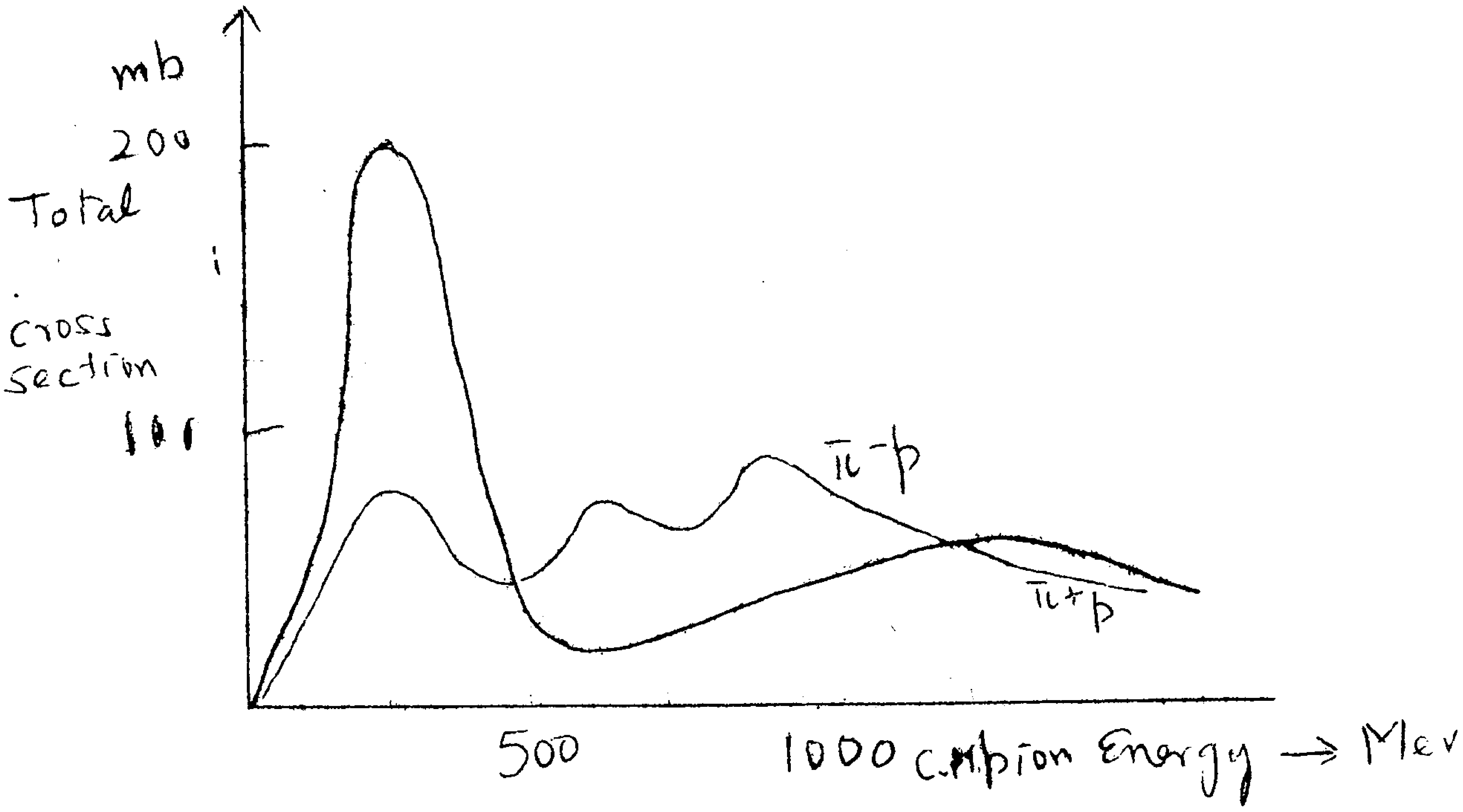
The elastic and inelastic crossections are given by

$$\left.\begin{aligned} \sigma_{el}^{j} &= \frac{\pi}{q^2}(j+\tfrac{1}{2})|1-\eta|^2 \\ \sigma_{inel}^{j} &= \frac{\pi}{q^2}(j+\tfrac{1}{2})(1-|\eta|^2) \end{aligned}\right\} \tag{39}$$

VII. Experimental Features of Pion-Nucleon Scattering.

We shall here discuss only the data between about 400 Mev and 1 Bev, as the low energy data are well-known. [In the low-energy region, the scattering amplitude is dominated by the resonances especially the $3/2$, $3/2$ resonance.]

The general features of pion-nucleon scattering are shown in the figure below, where the total cross-section is plotted against the energy.



It seems that we can distinguish broadly two regions, the region below about 2 Bev , dominated by the resonances, and the high energy region where diffraction scattering predominates.

The low energy region is marked by the occurrence of various resonances:-

(1) the $I = 3/2 \; J = 3/2^-$ resonance at 1238 Mev (total c.m. energy),

(2) a $J = 3/2^+$ resonance at 1512 Mev,

(3) a $J = 5/2^+$ resonance at 1680 Mev, etc.

The first equation is:

Are these peaks true resonances?

The peaks at 1512 Mev and 1683 Mev occur in $\pi^- p$ scattering but not in $\pi^+ p$. Therefore, if they are resonances, they must have an isospin of 1/2.

By subtracting the total σ'_λ from the background (smooth) curve, we can obtain the contribution of the resonance to the cross-section.

$$J = 5/2 \qquad \sim \sum a_n \cos^n\theta \qquad n \leq 5$$

$$J = 7/2 \qquad \sim \sum a_n \cos^n\theta \qquad n \leq$$

It is found that a good representation of the data can be obtain by keeping $n \leq 5$; this suggests that the third resonance has $J = 5/2$.

We expect the photoproduction processes also to be dominated by the second and third resonances in the region 1600 -- 1750 Mev.

The photoproduction data at 1512 Mev can be fitted by an expression

$$(5 - 3\cos^2\theta)$$

which can be explained by the assignment $J = 3/2$ to the 1512 Mev resonance.

Photoproduction data are not good at 1683 Mev.

This present data suggest

	J
1512	$3/2$
1688	$5/2$

The σ_{tot} does not tell us anything about the parity.

To obtain information about this, we examine the polarization in $\pi N \longrightarrow \pi N$ and photoproduction.

The polarization at 90° was found to be large in the region between the 2 resonances at 1238 Mev and 1512 Mev, and also between 1512 and 1688 Mev.

This indicates that the 1512 Mev resonance has a parit opposite to that of the first and third resonance.

If we can assume that only the three lowest partial wav dominate, then, as we know that the lowest resonance is $P_{3/2}$ we can deduce that the other two are $D_{3/2}$ and $F_{5/2}$ resonances.

We remark that the peaking may be not true resonances but cusp effects.

But if this were a cusp effect, it would be related to the presence of inelastic thresholds. The magnitude of the cusp effect could be related to the inelasticity parameter. The mean value of the inelastic parameter ρ turns out to be

$$\langle \rho \rangle \simeq 0.6$$

The inelasticity associated with the $J = 5/2$ wave,

$$\rho_{5/2} = 0.8 \pm 0.2$$

is higher than the average inelasticity, $\langle \rho \rangle \simeq 0.6$

If the resonance arises from the effect of the inelastic channels then ρ_λ is expected to be ≈ 0

This suggests that the 1688 Mev. is a resonance and not a cusp effect.

$$|S_{ej}| = 1$$

$$|S_{ej}| < 1$$

$$= \rho_{ej} e^{2i\delta_{ej}}$$

$$|S_{ej}| = \rho_{ej}$$

Define

$$R_{ej}(q^2) = \exp\left\{-\frac{i}{\pi}q \int_0^\infty \frac{\log \rho_{ej}(q^2)\, dq'^2}{q'(q'^2 - q^2 - i\varepsilon)}\right\}$$

The modulus of R would be related to the real part of the exponent and hence to the imaginary part of the integral

$$|R_{ej}(q^2)| = \exp\left\{-\frac{iq}{\pi}\, i\pi\, \frac{\log \rho_{ej}(q^2)}{q}\right\}$$
$$= \rho_{ej}(q^2)$$

Take the ratio S_{ej}/R_{ej}

This function has modulus unity, and hence can be written as

$$S_{ej}/R_{ej} = e^{2i\delta_{ej}} = S'_{ej}, \text{ say}$$

where δ_{ej} is a real phase.

$$\therefore \quad S_{\ell j} = R_{\ell j}\, e^{2i\delta_{\ell j}}$$

We can solve for $S'_{\ell j}$ by the N / D method, as $S'_{\ell j}$ is unimodular.

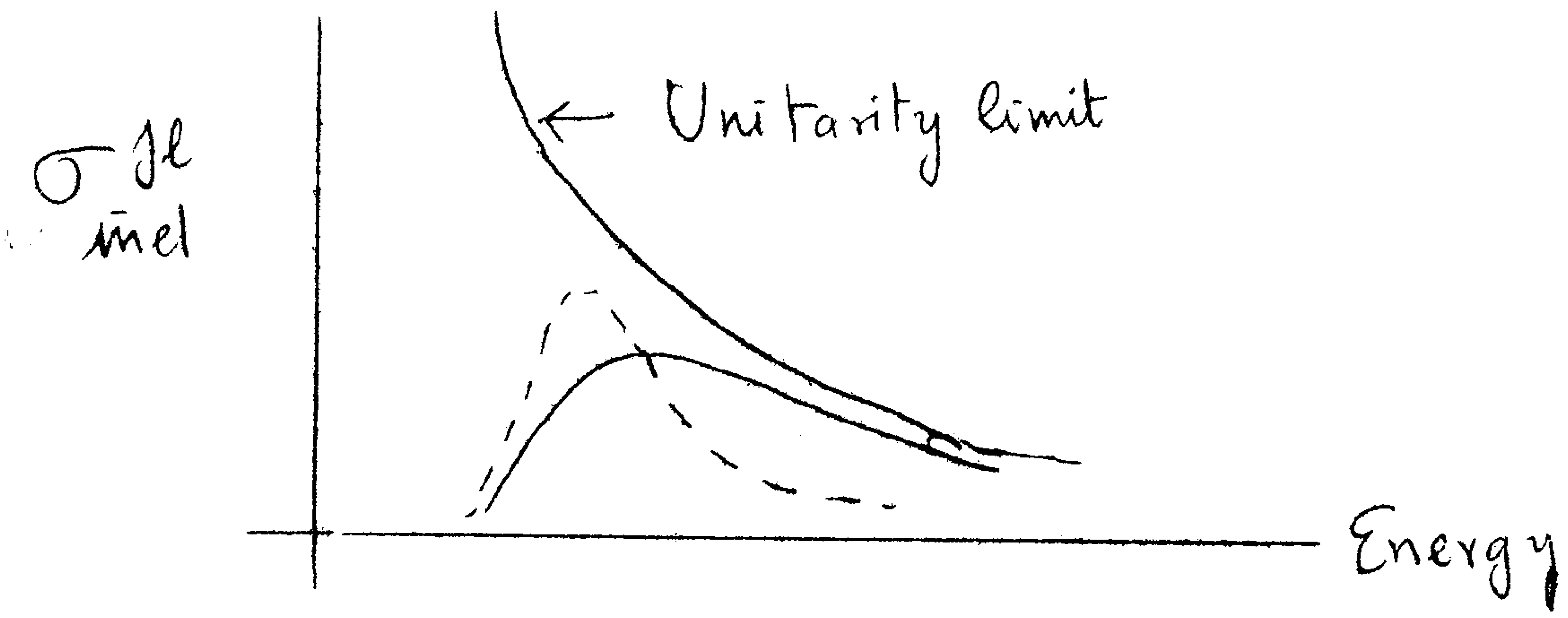
The phase $\delta_{\ell j}$ should not depend strongly on the behaviour of the inelastic channel.

Write

$$S_{\ell j} = R_{\ell j}\, e^{2i\delta_{\ell j}}$$
$$= \rho_{\ell j}\, e^{2i\delta_{\ell j}} \exp\left\{ -\frac{i}{\pi}\, \mathcal{I}m \int \frac{\log \rho(q'^2)\, d\ldots}{q'(q'^2 - q^2 - i\ldots}\right.$$

$S_{\ell j}$ is thus expressed as a product of two numbers, one that depends strongly on the inelastic effects and the other one weakly. Write this as

$$S_{\ell j} = e^{2i(\delta_{\ell j} + \delta^{I}_{\ell j})}$$



The inelastic σ is known to be bounded by unitarity, as expresse by

$$\sigma^{j}_{inel} = \frac{\pi}{q^2}(j+\tfrac{1}{2})(1-|\eta|^2)$$

We expect a peak in the elastic cross-section when there is a rapidly varying σ in the inelastic channel. The arguments have been given in a more complete way by Cook and Lee, and by Ball, Frazer and Nanenberg.

The above gives a simplified argument.

VIII. The Resonant States of Strongly Interacting Particles.

1. Nucleon and Hyperon Isobars.

We shall discuss the relation between the N^* and Y^*

S = 0			
	T	Jp	Mev
N^*	3/2	$P_{3/2}$	1238
N^{**}	1/2	$D_{3/2}$	1512
N^{***}	1/2	$F_{5/2}$	1688
N^{****}	3/2	7/2	1920(?)
S = -1			
	T	J	Mev
Y_0^*	0	1/2 (?)	1406
Y_0^{**}	0	$D_{3/2}$	1520
Y_0^{***}	0		1815
Y_1^*	1	$P_{3/2}$	1385
Y_1^{**}	1		1660

S = -2

Ξ^*	1/2	$P_{3/2}$	1532

It is customary to try to explain the approximate correspondence between the strange and non-strange resonances by postulating a new symmetry, or invariance under some group of transform.

A promising candidate for the correct symmetry group describing the strong interactions is SU_3. The octet model or the 8-representation of SU_3 was found useful in describing the baryons.

The baryon octet has the following properties.

	T	S
p	1/2	0
n	1/2	0
Λ^0	0	-1
Σ^+	1	-1
Σ^0	1	-1
Σ^-	1	-1
Ξ^0	1/2	-2
Ξ^-	1/2	-2

All have spin 1/2.

2. Meson resonances.

The same pattern seems to hold also for the pseudoscalar mesons and the vector mesons.

			Octet	Singlet
spin 0^-	T	S	spin 1^-	1^-
K^+	$1/2$	-1	$K^{*(+)}$	
K^0	$1/2$	-1	$K^{*(0)}$	
η	0	0	ω	φ
π^+	1	0	ρ^+	
π^0	1	0	ρ^0	
π^-	1	0	ρ^-	
$\bar{K}^0$	$1/2$	-1	$\bar{K}^{*(0)}$	
K^-	$1/2$	-1	$K^{*(-)}$	

Besides the 8 vector mesons, another $T = 0$, 1^{--} vector meson, the φ , at 1020 Mev, has been found.

We shall see later that this extra ninth vector meson need not give rise to any serious difficulties.

3. Resonances and the Representations of SU_3.

The meson-baryon resonances must belong to one of the irreducible representations into which the product of two 8-dimensional representations can be reduced.

If we wish to fit in a $T = 3/2$ resonance into our scheme, a 3-dimensional representation does not suffice. We can try either the 10-representation or the 27 representation. The 27 representation would require the existence of too many states with the same spin parity.

Let us try the 10-representation. This requires resonances with the following isospin and spin.

T	S	Observed particles.
3/2	0	N^* (1240 Mev.)
1	−1	Y_1^* (1385)
1/2	−2	Ξ^*(1532)
0	−3	†Ω^- (?) [$K\Xi$ bound states]

So we have another multiplet.

We could have a multiplet of $D_{3/2}$ resonances. The 8-representation would suffice here.

$T = 1/2$	$S = 0$	1512	N^{**}
$T = 0$	$S = -1$	1520	Y_0^{**}
$T = 1$	$S = -1$	(1660 ?)	Y_1^{**}
$T = 1/2$	$S = -2$	?	Ξ^{**}

† Recently, the Ω^- has been found at 1680 Mev in an experiment at Brookhaven.

The Mass.

If Strong interactions were actually invariant under then they should all have the same mass. However, we know that there are considerable mass-differences, which means that there must be a symmetry-breaking interaction. If the symmetry of SU_3 is broken only in the lowest order , then it leads to a mass formula:

For the baryon octet, this gives

$$(m_N + m_\Xi)/2 = (3m_\Lambda + m_\Sigma)/4$$

For the meson octet this does not give good agreement. But taking m^2 instead of m in the mass formula gives good agreement for the pseudoscalar meson octet $\pi, \eta, K, \bar{K}$

We know that the symmetry-breaking interactions must be quite strong. Higher order mass formulae have been given by Okubo. For the 10-representation, the members of the decuplet are equally spaced and hence the mass must be a linear function of the strangeness.

A linear spacing would predict an Ω^- at 1672 Mev., which would be a bound state.

Thus the members of the same decuplet would include bound states as well as resonances. Would this be correct?

We ask what is a resonance. Consider the N^*

If we assume it is a 1-channel resonance, then it should be represented by a pair of poles on the second sheet.

$$A^*(s^*) = A(s)$$

Therefore we always have 2 poles.

Consider the decuplet. Do they all have the same structure? The N^* is at 1238 Mev. which is about 160 Mev. above the ($N\pi$) threshold. The Ω^- would be far below the $\bar{K}\,\Xi$ threshold and therefore would not be observable from threshold effects in $\bar{K}\,\Xi$ scattering

We would say that the location of the poles is a function of the symmetry-breaking interaction.

$$\omega = F(\gamma) \pm \sqrt{\gamma_0 - \gamma}\, G(\gamma)$$

A virtual state.

A $P_{3/2}$ (virtual state) near threshold would not give a large effect, since the P wave is large near threshold.

The only way to observe the Ω^- would be, if it occurred as a bound state, which decayed into

$$\Omega^- \longrightarrow \Xi^0 + \pi^-$$

The mean value of the two roots $\dfrac{\omega_1 + \omega_2}{2} = F(\gamma)$ would be an analytic function of γ .

Next consider the pseudoscalar and vector mesons. In reactions producing mesons, the final mesons often occur correlated; from the effective mass distribution it is possible to find out whether the mesons have been produced by the decay of a resonance.

The ~~pseudo~~ scalar mesons are the following

	Mass	Width
K^+	494	0
K^0	494	0
η	550	Very small (e.m. decay)
π^+	139.6	0
π^0	139.6	Very small (e.m. decay)
π^-	139.6	0
$\bar{K}^0$	494	0
K^-	494	0

<u>The vector mesons</u> (1^-)

	Mass (MeV)	Width (MeV)
$K^{*(+)}$	885	~ 45
$K^{*(0)}$	885	~ 45
ω	785	~ 10
ρ^+	~ 750	~ 100
ρ^0	~ 750	~ 100
ρ^-	~ 750	~ 100
$\bar{K}^{*0}$	885	~ 45
K^{*-}	885	~ 45

The quantum mombers form a complete set with reference to strong interactions, since these are produced in strong interactions.

4. The ω meson:

The ω is observed in 3-pion correlations; $\pi + \pi + \pi$ in $T = 0$. If we assume that the ω decays via an (isospin-conserving) strong interaction. Then the only possible $T = 0$ 3π configuration is given by the wave function

$$\frac{1}{\sqrt{6}} \varepsilon_{\alpha\beta\gamma} | \pi_\alpha, \pi_\beta, \pi_\gamma \rangle$$

The amplitude must be totally antisymmetric (with reference to isospin) under interchange of any two pions, and similarly with reference to space coordinates. The overall spatial decay matrix element must be a pseudoscalar, linear in the ω polarization and a function of the momentum. Thus the matrix element must be

$$G = \varepsilon_{\mu\nu\sigma\rho} \Omega^\mu P_1^\nu P_2^\sigma P_3^\rho$$

where Ω is the polarization vector of the ω. Here, as in 2 body scattering, there are two invariants. Define

$$s = -(p_1+p_2)^2;\ t = -(p_1+p_3)^2;\ u = -(p_2+p$$
$$s+t+u = M_\omega^2 + 3\mu^2 = M^2+3\mu^2$$
$$G = G(s,t,u)$$

The requirement of total antisymmetry imposes a restriction on the amplitude.

The above is for a spin 1^- ω. Consider the most general matrix element for the decay of an ω of any spin.

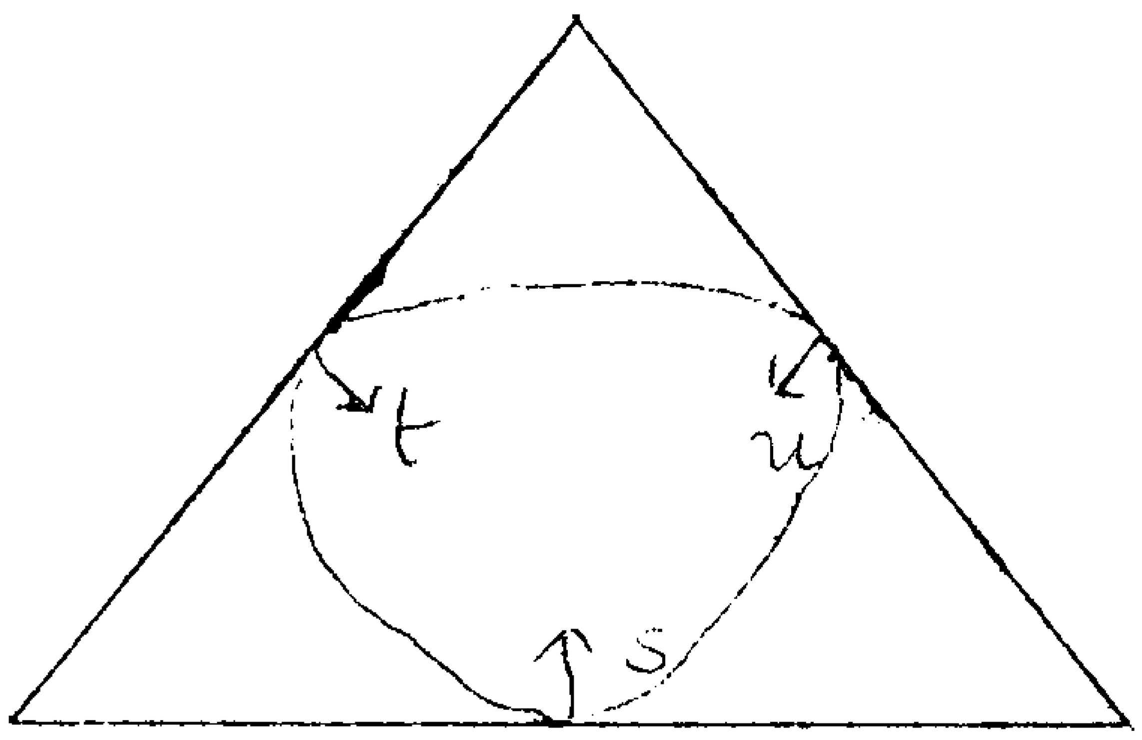
Consider the possibilities $1^+, 1^-, 0^-, 0^+$.
The requirements of Lorentz invariance and isospin conservation lead to the following possible decay matrix elements. A 0^+ particle cannot decay into 3 pions. A 1^+ particle would decay with a matrix element

$$G'(s,t,u)\,\Omega\left[p_1(u-t)+p_2(s-u)+p_3(t-s)\right]$$

which is totally antisymmetric. For a pseudoscalar ω, the decay matrix element would be

$$G''(s,t,u)\,(s-u)(t-u)(u-s)$$

We can take s, t, u as the homogeneous (trilin coordinates).



The Delitz plot.

The allowed region will be bounded by the region shown.

(i) For a 1^- ω , the density of points must vanish on the edge of the plot, when the momenta are collinear.

(ii) For a 1^+ ω , the density of points should be zero whenever $s = t = u$ (i.e. at the centre of the plot) and also when the momentum of one of the particles is maximum.

The experimental data favoured the assignment of 1^- to the ω meson.

We have assumed that isospin was conserved.

was in complete agreement with

Since G parity is conserved, the ω meson must have G parity ($-$). The notation for the ω meson, following A.M. Rosenfeld would then be $\overset{T}{0}\,(\overset{JPG}{1^{--}})$. The question arises why the ω-meson which decays strongly should be so narrow. This may be understood from the following considerations. As we have seen earlier, the decay matrix element of the ω may be written as $[G(s,t,u)\,\epsilon_{\mu\nu\lambda\rho}\,\Omega^{\mu} p_1^{\nu} p_2^{\lambda} p_3^{\rho}]$.

Write $G = g/(M)^3$

where g is a dimensional coupling constant. $M \equiv$ the inverse of the range of the interaction. Each π meson has a kinetic energy of $\left(\frac{780}{3} - 140\right)$ = 120 Mev. Therefore the decay rate $\sim \; g_x^2 \left(160\,\mathrm{Mev}/300\,\mathrm{Mev}\right)^6$

Replace P_3 by $K = P_1 + P_2 + P_3$

The matrix element is

$$G(s,t,u)\,\varepsilon_{\mu\nu\sigma\rho}\,P_1^{\nu}\,P_2^{\sigma}\,K^{\rho}$$

In the rest system of the ω, K has only a time component. Therefore the matrix element becomes

$$G(s,t,u)\;E\;(\vec{p}_1 \times \vec{p}_2)\,\Omega$$

Therefore if G is a constant, the pions will be produced in a p wave. Terms $\sim s, t, u$ in $G(s,t,u)$ will give ri[se] to d-wave, f- wave, etc. contributions.

Thus the centrifugal barrier of the p wave will strongly damp the decay.

Since the strong decay modes are suppressed by the centrifugal barrier, we may expect that other decay modes which are not suppressed so much by the centrifugal barrier, may be important, e.g., the electromagnetic decay modes.

Consider, for instance, the decay mode

$$\omega \longrightarrow \pi^0 + \gamma$$

We can write the matrix element as

$$G'\,\varepsilon_{\mu\nu\sigma\rho}\,\Omega^{\mu}A^{\nu}p^{\sigma}K^{\rho}$$

G' is now a constant, and $\sim \frac{1}{137}$ times the strong couplin[g] constant. However, the centrifugal barrier effect will not b[e so] strong, as there are only two powers of momentum.

This gives

$$R = \Gamma(\pi^0\gamma)/\Gamma(\pi^+\pi^-\pi^0) \sim 0.2$$

This was found to be experimentally true; the measured branching ratio was

$$R \simeq 15 \text{ to } 20\%$$

The width of the ω as recently measured is about 9 Mev., which is much larger than the earlier value of $\sim$ 1 Mev.

Another possible decay mode which can provide only via virtual electro-magnetic interactions, is the mode $\omega \to \pi^+\pi^-$; we expect a suppression off the decay by $(1/137)^2$

$$\omega \longrightarrow \pi^+ \pi^- \qquad 780 \text{ Mev.}$$

$$G \quad -1 \qquad +1 \qquad 750 \text{ Mev.} \quad \rho$$

However, the phase space is large; also the proximity of the meson will enhance the decay. The calculated ratio

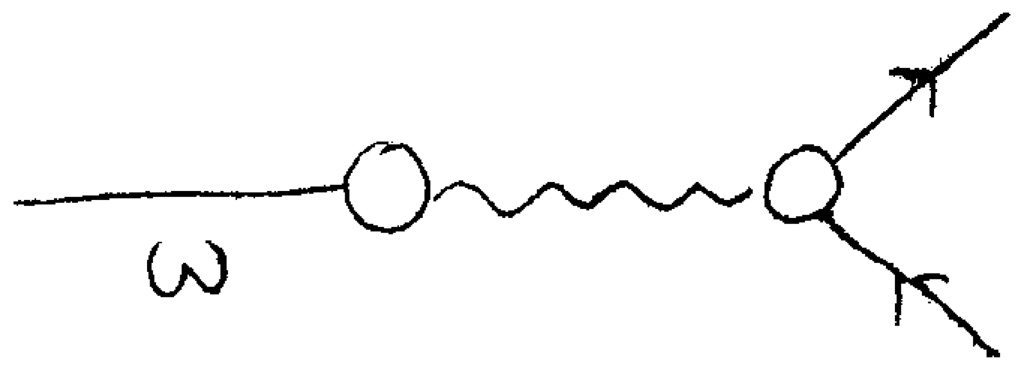
$\Gamma(\pi^+\pi^-)/\Gamma(\pi^+\pi^-\pi^0) \sim$ a few percent

agrees with the observed ratio of $\sim$ 4 per cent. $\omega \to \pi^0\pi^0$ is forbidden by charge conjugation invariance, since the ω must have $C = -1$, while a $(\pi^0\pi^0)$ state has $C = +1$. This may be obtained quickly from the relation

$$G = C(-1)^T$$

The ω decay is not a test of the boson character of the π meson. The above structure of the matrix element is already fixed by the vector character of the ω meson, <u>independently</u> of the boson nature (or otherwise) of the pion. Other possible decay modes:

$$\omega \rightarrow \pi^0 \pi^0 \gamma$$
$$\rightarrow \pi^+ \pi^- \gamma$$
$$\left.\begin{array}{l}\rightarrow e^+ e^- \\ \rightarrow \mu^+ \mu^-\end{array}\right\} \leftarrow \text{Are being looked for}$$



The ω-meson plays an important role in the nucleon form factors.

5. <u>The η meson (550 Mev).</u>

This was found in the configuration

$$\eta \rightarrow \pi^+ \pi^- \pi^0$$

its isospin was assigned to be $T = 0$.

A puzzling feature found that the η decayed into neutrals quite often

$$\Gamma(\text{neutrals}) / \Gamma(\pi^+ \pi^- \pi^0) \approx 3$$

How was this to be explained?

Remembering that the ω also decays into neutrals this would be quite normal if the η were a 1^- particle. Since the mass of the η is only 550 Mev, the result would be quite sensitive to the mechanism.

However, when we try to test the assignment 1^- by looking at the Dalitz plot, we meet with difficulties.

The density of points inside the Dalitz plot is found to be constant, which does not agree with any of the assignments $0^\pm$, $1^\pm$ etc.

This can be explained only if we assume that isospin is not conserved in the decay of the η

Note: It was only isospin conservation that fixed the matrix element for a 0^{--} ω meson as

$$(s-u)(t-s)(t-u)$$

If we assume that the η is a 0^- meson,

$$G \sim \alpha$$

and a constant density would be obtained on the Dalitz plot.

$$\eta \rightarrow \pi^+ \pi^- \pi^0$$

The uniform distribution on the Dalitz plot indicates that the $(\pi^+ \pi^- \pi^0)$ final state should be symmetrical under C ; Therefore $G = C(-1)^T = +$ by C conservation. This leads to the assignment (0^{-+})

We can see that often assignments would lead to a different distribution on the Dalitz plot. What are the other possible decay modes? The mode $(\pi^0\pi^0\pi^0)$ is now allowed, as the final state can have $T = 1^-$. The most favourable combination of the $T=1$ and $T=0$ states in the final state would give

$$\Gamma(\pi^0\pi^0\pi^0)/\Gamma(\pi^+\pi^-\pi^0) \leq 1.5$$

Another possible mode is

$$\eta \longrightarrow 2\gamma$$

with a matrix element

$$e^2 \varepsilon_{\mu\nu\sigma\rho} A^\mu A'^\nu k^\sigma k'^\rho$$

which gives $\Gamma(\gamma\gamma)/\Gamma(\pi^+\pi^-\pi^0) \sim 1.5$

The recently measured values of these ratios agree with these estimates. Another puzzling decay mode was the mode

$$\eta \longrightarrow \pi^+\pi^-\gamma$$

[The mode $\eta \rightarrow \pi^0\pi^0\gamma$ is forbidden by C invariance.]

As the mode

$$\eta \longrightarrow \pi^+\pi^-\gamma$$

would be a first order e.m. effect, and not a second order effect (as only the emission of 1 photon is involved)

Then why was the $(\pi^+\pi^-\gamma)$ mode not made a 100 times as large as the $\eta \rightarrow \gamma\gamma$ and $\eta \rightarrow 3\pi$ modes? This can be explained by the centrifugal barrier that is effective here, as in $\omega \longrightarrow 3\pi$ decay.

$$\varepsilon_{\mu\nu\sigma\rho}\, A^{\mu} p_1^{\nu} p_2^{\sigma} p^{\rho}$$

This leads to the result that the 20 % branching ratio observed

$$\frac{(\pi^+\pi^-\gamma)}{(\pi^+\pi^-\pi^0)} \sim 0{\cdot}20$$

is quite feasible. (because in the $\pi^+\pi^-\pi^0$ modes, the π's are emitted in an S-wave, while in the $\pi^+\pi^-\gamma$ mode, the $\pi^+\pi^-$ are in a p-wave.)

<u>Note:</u>

The mode

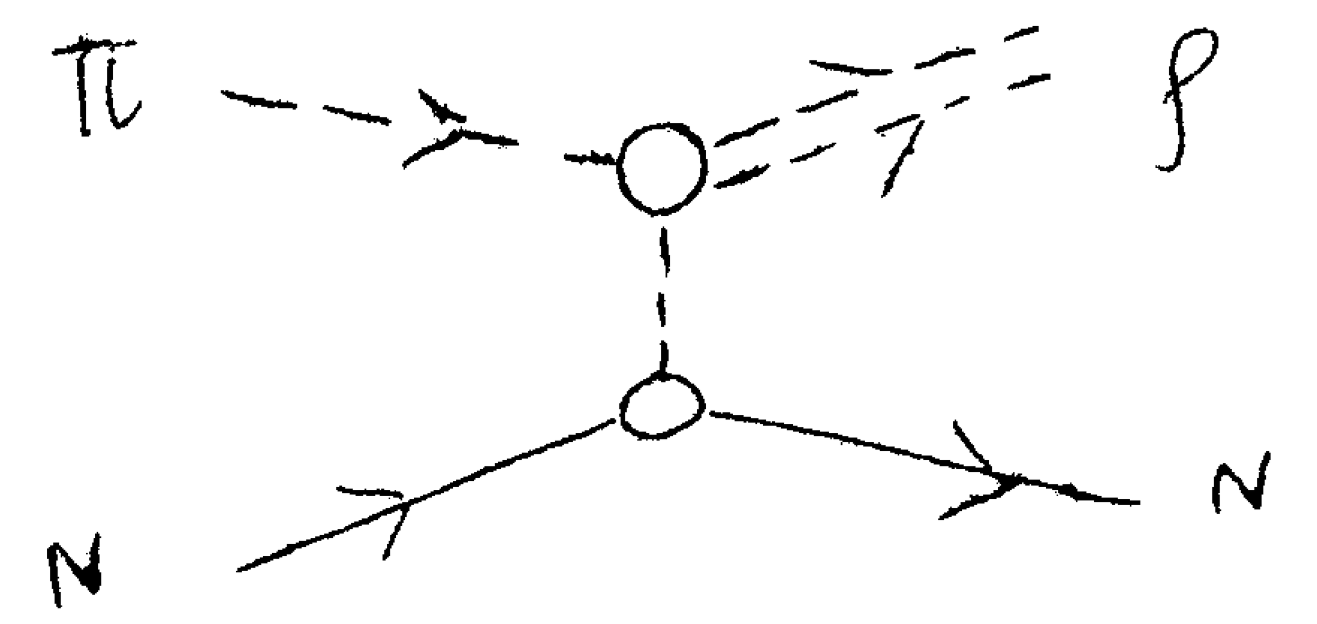
$$\omega^0 \longrightarrow \eta^0 + \pi^0$$

is <u>forbidden</u> by C-invariance.

6. <u>The ρ meson</u> $\left.\begin{array}{l}\pi^+\pi^0\\ \pi^-\pi^0\\ \pi^+\pi^-\end{array}\right\}$

	J	T	Mass	Γ
	1^-	1	~ 750 MeV	~ 100 MeV

There is strong evidence that the ρ is produced mainly by a peripheral interaction



Extrapolation to the pion pole gives the pion-pion cross secti as

$$\sigma_{\pi\pi}(\omega) = \lim_{t \to \mu^2} \left\{ \frac{2\pi q^2 (t-\mu^2)^2}{\frac{g^2}{4\pi} \frac{t^2}{4m^2} \omega \sqrt{\frac{\omega^2-\mu^2}{4}}} \cdot \frac{\partial^2 \sigma}{\partial(\partial \omega^2)} \right\}$$

Data are not accurate enough as yet to make a reliable extrapo tion. But the data gives the cross section at small negative values of t as $\sigma_{\pi\pi} \sim 12\pi \lambda^2$ which would be the value if $\sigma_{\pi\pi}$ were dominated by a p-wave resonance. Also the angular distribution is

$$\sim \cos^2\theta$$

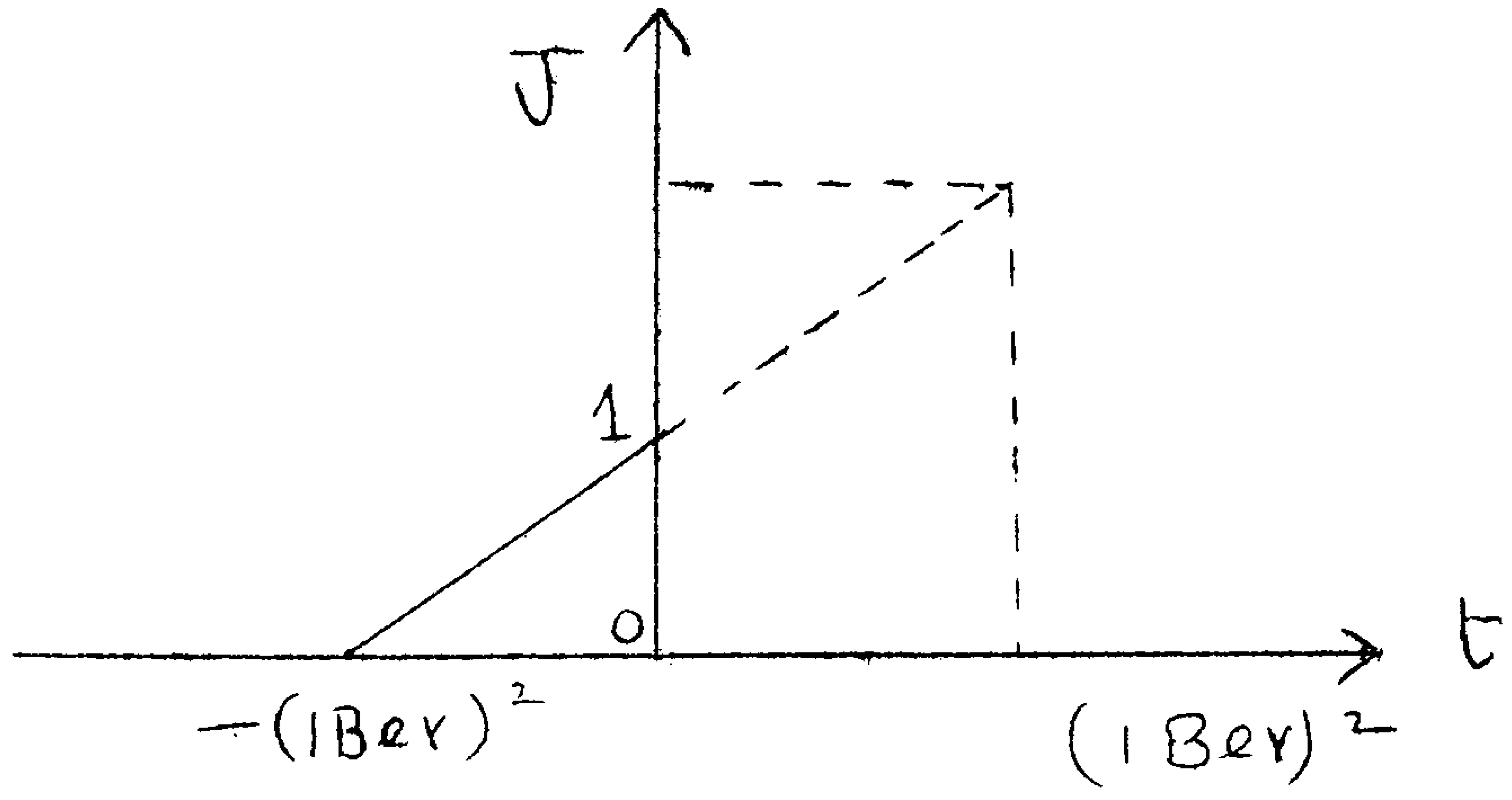
as would be expected for the decay of a 1^- ($\pi\pi$) resonance.

7. <u>The f^0 meson.</u> ($T=0$, $M=1250$ Mev, $J=2^+$

There is a strong correlation in the ($\pi\pi$) system, in $Q=0$, at $\sim$ 1250 Mev. We expect that the resonance has $T = 0$. This was interesting from the standpoint of the Reg pole hypothesis; the constancy of higher energy cross section would be explained if the scattering were dominated by a $T=0$ $C=+$, , Pomeranchuk trajectory.

$$\frac{d\sigma}{dt} \sim |f(t)|^2 e^{-2[1-\alpha(t)]\log\left(\frac{s}{s_0}\right)}$$

During the CERN conference, July 1963, pp scattering gave an estimate of $\alpha(t)$ that was nearly linear



A linear extrapolation to $+ve\ t$ would meet $J=2$ at $\sim$ 1250 Mev.

Such a resonance was found to at 1280 Mev. The angular distribution was far from isotropic; since (2π) in T = 0 cannot be in J = 1; the simplest assignment is J = 2 . This was later considered.

8. <u>The φ^0 meson</u> (1020 Mev, T = 0)

This was found in the $(K\bar{K})$ system, at 1020 Mev, with a width of $\sim$ 1 Mev. A $(K\bar{K})$ system can be in

$$(K_0\bar{K}),\qquad K^+K^-$$

We can never observe $K_0\bar{K}_0$; we must observe $K'K_0'$ or $K_0^1K_0^2$

$$K_0^{1,2} = (K_0 \pm \bar{K}_0)/\sqrt{2}$$

$K_0^1K_0^1$ or $K_0^2K_0^2$ is a symmetrical solution, while $K_0^1K_0^2$ is an antisymmetrical solution. Only the mode $K_0^1K_0^2$ was observed; Therefore it is angular in the charge index. Therefore it must be antisymmetric in the space indices also, since the K is a boson. Therefore the angular momentum must be odd.

How can J be determined ?

It can be determined at present as follows: The available $K.E.$ is very small. $K_0 \overline{K_0}$ is slightly heavier than $K^+ K^-$; this should make a large difference if the angular momentum is large, as the kinetic energy is very small.

The observed data support J = 1. The φ has not been found in the $(\pi\pi)$ system; this suggests that $G = (-)$ Thus we have $J^{PG} = 1^{--}$. The C value is $(-)$ if J is odd.

$$G = C(-1)^T$$

Therefore T is even. $T =$ only 0 or 1. Therefore $T = 0$. Therefore we finally obtain the assignment

$$T = 0, \quad J^P = 1^- \qquad G = (-)$$

Since the quantum numbers are the same as that of the ω, we expect the decay modes

$$\varphi \to 3\pi$$

$$\varphi \to \pi + \gamma$$

These have not been observed.

9. The K^*

$(K\pi)$ resonance $T=\frac{1}{2}$ 885 MeV $\Gamma \sim 50$ MeV

The 'best' determination of the spin of the K^* was made by Armenteros et al at CERN in the experiment

$$p\bar{p} \rightarrow K^0 \bar{K}^0 \pi$$

The allowed states are $^1S_0$ and $^3S_1$ (1^-)

The three mesons are produced in the combination $(K^* + K)$

$$p\bar{p} \rightarrow K^0 \bar{K}^0 \pi^0 \qquad K^* K$$

$$^1S_0 \qquad -(-1)^0$$

$$^3S_1 \qquad -(-1)^1$$

Charge conjugation.

For a spin 0 K^*
the state $^1S_0$ is forbidden by C inv
and the state $^3S_1$ is by parity.

It may be puzzling that there are two 1^- particles, the ω and ρ. This would lead to 9 vector mesons.

The extra particle could belong to a singlet representation of SU_3. Also, people have speculated that the φ and ω are mixed; and that the physically observed particles are mixture states.

IX. Dispersion Relations.

The pros and cons of dispersion relations as opposed to other approaches to elementary particle theory which are based on perturbation theory have often been discussed

First consider the scattering of a particle by a potential

$$H = H_0 + V$$

Let Ψ_k, φ_k be the eigenfunctions of H, H_0 respectively.

$$H\Psi_k = \omega_k \Psi_k$$

$$H_0\varphi_k = \omega_k\varphi_k \text{ : Denote } \varphi_k \equiv |k\rangle$$

[Assume that there are no bound states, else the spectrum for H and H_0 are different.]

Write these as

$$(H_0 + V)\,\Psi_k = \omega_k \Psi_k$$

$$H_0 \varphi_k = \omega_k \varphi_k$$

Subtracting, we obtain

$$(H_0 - \omega_k)\,(\Psi_k - \varphi_k) = -V\Psi_k$$

so that

$$\Psi_k - \varphi_k = -\frac{1}{H_0 - \omega_k} V \Psi_k$$

or

$$\Psi_k = \varphi_k + \frac{1}{\omega_k - H_0} V \Psi_k$$

Introducing a complete set of eigenstates of H_0,

$$H_0 |k'\rangle = \omega_{k'} |k'\rangle$$

write

$$\psi_k = \varphi_k + \frac{1}{\omega_k - H_0} \sum_{k'} |k'\rangle\langle k'| V \psi_k$$

$$= \varphi_k + \sum_{k'} \frac{1}{\omega_k - \omega_{k'}} (\varphi_{k'}^*, V\psi_k)\varphi_{k'}$$

$$\psi_k = \varphi_k + \frac{1}{\omega_k - H_0} V \left(\varphi_k + \frac{1}{\omega_k - H_0} V\psi_k\right),$$

We may write this as an expansion

$$\psi_k = \varphi_k + \frac{1}{\omega_k - H_0} V\varphi_k + \frac{1}{\omega_k - H_0} V \frac{1}{\omega_k - H_0}$$

which is an expansion in powers of the potential. Introduce, in each term, a complete set of states $|k'\rangle$, e.g.,

$$\frac{1}{\omega_k - H_0} |k'\rangle\langle k'| V \psi_k \cdots$$

giving

$$\psi_k = \varphi_k + \sum_{k'} \frac{1}{\omega_k - \omega_{k'}} \langle k'|V|k\rangle \varphi_{k'}$$

$$+ \sum_{k''} \frac{1}{\omega_k - \omega_{k'}} \langle k'|V|k''\rangle \frac{1}{\omega_k - \omega_{k''}} \langle k''|V|k\rangle \varphi_k + \cdots$$

This does not completely define ψ_k, as the denominators $(\omega_k - \omega_{k'})$ give rise to singularities. The expressions are defined by prescribing that the values at the singularities are to be taken as the limit

$$\lim_{\eta\to 0} \frac{1}{\omega_k - \omega_{k'} \pm i\eta} \rightarrow \varphi_k^{(\pm)}$$

We then obtain

$$\psi_k^{(\pm)} = \varphi_k + \lim_{\eta\to 0} \sum_{k'} \Big\{ \frac{1}{\omega_k - \omega_{k'} \pm i\eta} \langle k'|V|k\rangle$$

$$+ \sum_{k''} \frac{1}{\omega_k - \omega_{k''} \pm i\eta} \langle k'|V|k''\rangle \frac{1}{\omega_k - \omega_{k''}} \langle k''|V|k\rangle + \ldots \Big\}$$

ψ_k^{+} $\longrightarrow$ (plane wave + outgoing wave)

ψ_k^{-} $\longrightarrow$ (plane wave + incoming wave)

Ref: Chew and Goldberger: Phys. Rev. (1952).

Wick : Rev. Mod. Phys. (1952).

The scattering amplitude is described as the scalar product

$$\left(\psi_{k'}^{(-)}, \psi_k^{(+)} \right) = S_{k'k}$$

The important quantity is the T matrix, defined in terms of the matrix by

$$S_{k'k} = \delta_{k'k} + i(2\pi)\delta(\omega_{k'} - \omega_k) T_{k'k}$$

We have only a factor (2π) and not $(2\pi)^4$, since we are now considering scattering by a potential and not 2-particle scattering. Now suppose we use the definition of the states $\psi_k^{\pm}$

$$\psi_k^{(\pm)} = \varphi_k + \frac{1}{\omega_k - H_0 \pm i\eta} V \psi_k^{(\pm)}$$

[Note: We also have $\psi_k = \varphi_k + \frac{1}{\omega_k - H \pm i\eta} V\varphi_k$ leading to

$$\psi_{k'}^{(-)} = \psi_{k'}^{(+)} + 2\pi i\,\delta(\omega_k - \omega_{k'}) V\varphi_{k'}$$

$$\therefore \psi_{k'}^{(-)} = \psi_{k'}^{(+)} + 2\pi i\,\delta(\omega_k - \omega_{k'}) V\varphi_{k'}$$

$$\therefore \left(\psi_{k'}^{(-)}, \psi_k^{(+)}\right) \equiv \left(\psi_{k'}^{(+)}, \psi_k^{(+)}\right) - 2\pi i\,\delta(\omega_k - \omega_{k'})\, \varphi_{k'}^* V\psi_k^{(+)}$$

$$T_{k'k} = -\varphi_{k'}^* V\psi_k^{(+)}$$

or

$$\langle k'|T|k\rangle = -\left(\varphi_{k'}, V\psi_k^{(+)}\right)$$

$$\therefore T_{fi} = -\varphi_f^* V\psi_i^{(+)}$$

$$= -\varphi_f^* V \Big\{ \varphi_i + \sum_{k'} \frac{\langle k'|V|k_i\rangle}{\omega_i - \omega_{k'}} \varphi_{k'}$$

$$+ \sum_{k'k''} \frac{\langle k'|V|k''\rangle\langle k''|V|k_i\rangle}{(\omega_i - \omega_{k'})(\omega_i - \omega_{k''})} \varphi_{k'}$$

$$= -\varphi_f^* V\varphi_i - \sum_{k'} \frac{(\varphi_f^* V\varphi_{k'})\langle k'|V|k}{\omega_i - \omega_{k'}}$$

$$+ \sum_{k'k''} (\varphi_f^* V\varphi_{k'}) \frac{\langle k'|V|k''\rangle\langle k''|V|k_i\rangle}{(\omega_i - \omega_{k'})(\omega_i - \omega_{k''})}$$

$$= -\langle k_f|V|k_i\rangle - \sum_{k'} \frac{\langle k_f|V|k'\rangle}{\omega_i - \omega_{k'}} \Big\{ \langle k'|V|k_i\rangle + \sum_{k''} \frac{\langle k'|V|k''\rangle\langle k''|V|k_i\rangle}{\omega_i - \omega_{k''}} + \cdots \Big\}$$

$$= -\langle k_f|V|k_i\rangle + \sum_{k'} \frac{\langle k_f|V|k'\rangle\langle k'|T|k_i\rangle}{(\omega_i - \omega_{k'})}$$

or

$$\langle k_f|T|k_i\rangle = -\langle k_f|V|k_i\rangle + \sum_{k'} \frac{\langle k_f|V|k'\rangle\langle k'|T|k_i\rangle}{(\omega_i - \omega_{k'})}$$

The matrix element $\langle k_f|V|k'\rangle$, $\langle k'|T|k_i\rangle$ are off-the-mass shell.

The equation may be written

$$T = -V + V\frac{1}{\omega_i - H_0}T$$

or

$$V = -T + V\frac{1}{\omega_i - H_0}T$$

$$= -\left[1 - V\frac{1}{\omega_i - H_0}\right]T$$

$$= -(\omega_i - H_0 - V)\frac{1}{\omega_i - H_0}T$$

$$= -(\omega_i - H)\frac{1}{\omega_i - H_0}T$$

$$\therefore \quad T = -(\omega_i - H_0)\frac{1}{\omega_i - H}V$$

$$= -(\omega_i - H + V)\frac{1}{\omega_i - H}V$$

or
$$T = -V - V\frac{1}{\omega_i - H}V$$

We originally had
$$T = -V + V\frac{1}{\omega_i - H_0}T$$

Take matrix elements and introduce a complete set of states in the second term on the right.

$$\langle k_f|T|k_i\rangle = -\langle k_f|V|k_i\rangle + \sum_k \frac{\langle k_f|T|k\rangle\langle k|T|k_i\rangle^*}{\omega_i - \omega_k}$$

This looks like a dispersion relation in that it involves off-the-mass-shell quantities. Consider forward scattering of a spinless particle by a potential.

$$T(\omega) = -V(0) + \sum_k \frac{|\langle k_i|T|k\rangle|^2}{\omega - \omega' + i\eta}$$

The imaginary part will come just from the $i\eta$

$$T(\omega) = -V(0) + \int \frac{d^3\bar{k}}{(2\pi)^3}\frac{|\langle k_i|T|k\rangle|^2}{\omega-\omega'+i\eta}$$

$$\therefore \operatorname{Im} T(\omega) = -\pi\int \delta(\omega-\omega')|\langle k_i|T|k\rangle|^2 \times d^3k/(2\pi)$$

$$= -\pi\int \delta(\omega-\omega')|\langle k_i|T|k\rangle|^2 \frac{k^2\,dk\,d\Omega_k}{(2\pi)^3}$$

$$= -\pi\frac{\omega k}{(2\pi)^3}\int |\langle k'|T|k\rangle|^2 d\Omega_k$$

$$= 2\operatorname{Im} T(\omega) = \frac{k\omega}{(2\pi)^2}\int |\langle k|T|k\rangle|^2 d\Omega_k$$

Thus the imaginary part of the amplitude is given by the matrix element on the mass shell. We have defined $T(\omega)$ for real ω as

$$T(\omega) = \lim_{z \to \omega + i\eta} T(z) \quad , \ \omega \text{ real}$$

It can be proved that

(i) $T(z)$ is analytic in z in the upper half z plane;

(ii) $T(z) \to -V(0)$ as $|z| \longrightarrow \infty$

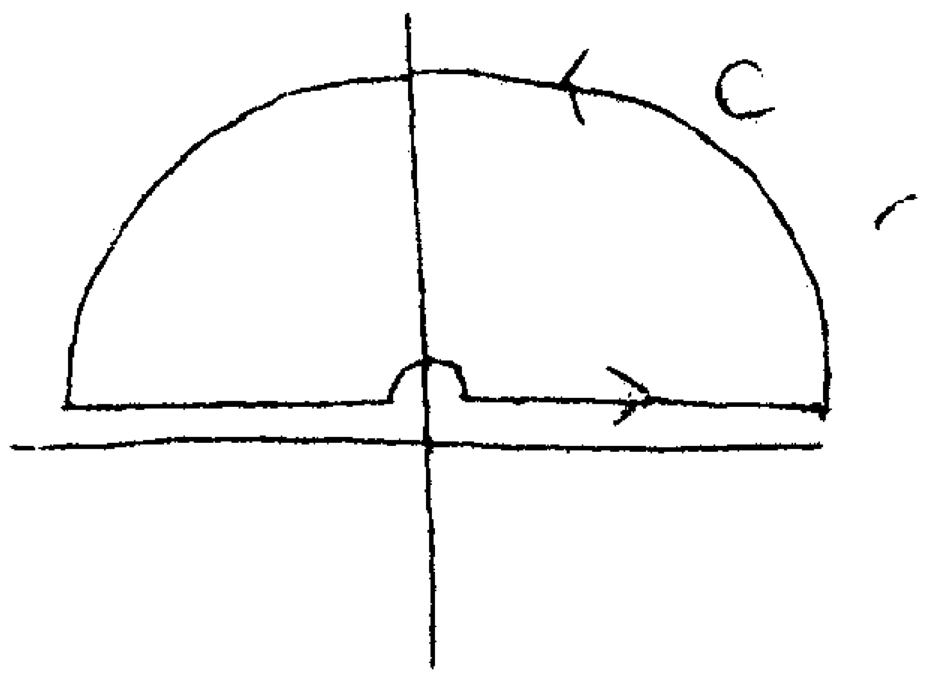
(i) can be proved from causality.

(ii) can be proved from the fact that the Born approximation is a valid approximation at very high energies.

$T(z)/z \longrightarrow 0$ as $\frac{1}{z}$, since $T(z) \to -V(0)$

Cauchy's theorem gives

$$T(z)/z = \frac{1}{2\pi i} \oint \frac{T(z')\,dz'}{z'(z'-z)}$$



$$P\int_{-\infty}^{+\infty} \frac{T(z')\,dz'}{z'(z'-z)} - i\pi \frac{T(0)}{(-z)}$$

$$T(\omega) = \lim_{\eta \to 0} T(\omega + i\eta) = \lim_{\eta \to 0} \omega \frac{1}{2\pi i} P\int_{-\infty}^{\infty} \frac{T(z')\,dz'}{z'(z'-\omega-i\eta)} + \frac{1}{2} T(0)$$

$$= \omega \frac{1}{2\pi i} P\int_{-\infty}^{\infty} \frac{T(z')\,dz'}{z'(z'-\omega)} + \frac{1}{2} T(\omega)$$

or $$T(\omega) = T(0) + \frac{\omega}{\pi i} P \int_{-\infty}^{\infty} \frac{T(z')\,dz'}{z'(z'-\omega)}, \text{ for } \omega \text{ real}$$

Take the real parts of each side. This gives

$$\operatorname{Re} T(\omega) = \operatorname{Re} T(0) + \frac{\omega}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Im} T(\omega')\,d\omega'}{\omega'(\omega'-\omega)} \quad (A)$$

Here, $\operatorname{Im} T(\omega)$ can be related to the matrix element $\langle k|T|k\rangle$ on the mass shell; however, we now require the matrix element for $\omega = -\infty$ to $\omega = +\infty$. $\langle k|T|k\rangle$ can be measured only for $\omega = 0$ to $\omega = +\infty$. But we can obtain the values for $\omega = -\infty$ to $\omega = 0$ by using crossing symmetry.

$$\operatorname{Re} T(\omega) = \operatorname{Re} T(0) + \frac{\omega}{\pi} P \int_{-\infty}^{-\omega_0} \frac{\operatorname{Im} T(\omega')\,d\omega'}{\omega'(\omega'-\omega)} + \int_{\omega_0}^{\infty} \frac{\operatorname{Im} T(\omega')(}{\omega'(\omega'}$$

$$= \operatorname{Re} T(0) + \frac{\omega}{\pi} P \int_{\omega_0}^{\infty} d\omega' \frac{\operatorname{Im} T(\omega')}{\omega'(\omega'-\omega)} + \int_{\omega_0}^{\infty} \frac{\operatorname{Im} T(\omega')\,d}{\omega'(\omega-\omega'}$$

In equation (A), take the limit $\omega \to \infty$. This gives

$$-V(0) = \operatorname{Re} T(0) - \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Im} T(\omega')\,d\omega'}{\omega'}$$

giving

$$\operatorname{Re} T(0) = -V(0) + \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Im} T(\omega')\,d\omega'}{\omega'}$$

This gives, substituting

$$\mathrm{Re}\,T(\omega) = -V(0) + \frac{1}{\pi} P \int_{-\infty}^{\infty} \left(\frac{\omega}{\omega'(\omega'-\omega)} + \frac{1}{\omega'} \right) \mathrm{Im}\,T(\omega') \, d\omega'$$

or

$$\mathrm{Re}\,T(\omega) = -V(0) + \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Im}\,T(\omega')\,d\omega'}{\omega'-\omega}$$

Using crossing symmetry, we obtain

$$\mathrm{Re}\,T(\omega) = -V(0) + \frac{1}{\pi} P \int_{\omega_0}^{\infty} \mathrm{Im}\,T(\omega') \left[\frac{1}{\omega'-\omega} + \frac{1}{\omega'+\omega} \right]$$

$$= -V(0) + \frac{1}{\pi} P \int_{\omega_0}^{\infty} \mathrm{Im}\,T(\omega') \frac{2\omega'}{\omega'^2 - \omega^2} \, d\omega'$$

$$\langle k_f | T | k_i \rangle = -\langle k_f | V | k_i \rangle + \sum_k \frac{\langle k_f | V | k \rangle \langle k | T | k_i \rangle}{\omega_i - \omega_k + i\eta}$$

This result obtained from dispersion relations and crossing symmetry is close to the result obtained in potential scattering.

Substituting

$$2\,\mathrm{Im}\,T(\omega) = \frac{k\omega}{(2\pi)^2} \int |\langle k | T | k \rangle|^2 \, d\Omega_k$$

we obtain

$$\mathrm{Re}\,T(\omega) = -V(0) + \frac{1}{\pi} P \int_{\omega_0}^{\omega} |\langle k' | T | k' \rangle|^2 \frac{2\omega' k' \omega}{(\omega'^2 - \omega^2)(2}$$

$$= -V(0) + P \int_{\omega_0}^{\infty} |\langle k' | T | k' \rangle|^2 \frac{4\omega' \, d^3k}{(\omega'^2 - \omega^2)(2\pi)^3} \, [d\omega'$$

This involves only on-the-mass-shell quantities. Compare this with the expression obtained from perturbation theory, which gives for forward scattering:

$$\mathrm{Re}\,T(\omega) = -V(0) + \int_0^\infty \frac{|\langle k|T|k\rangle|^2}{\omega-\omega'}\frac{d^3k}{(2\pi)^3}$$

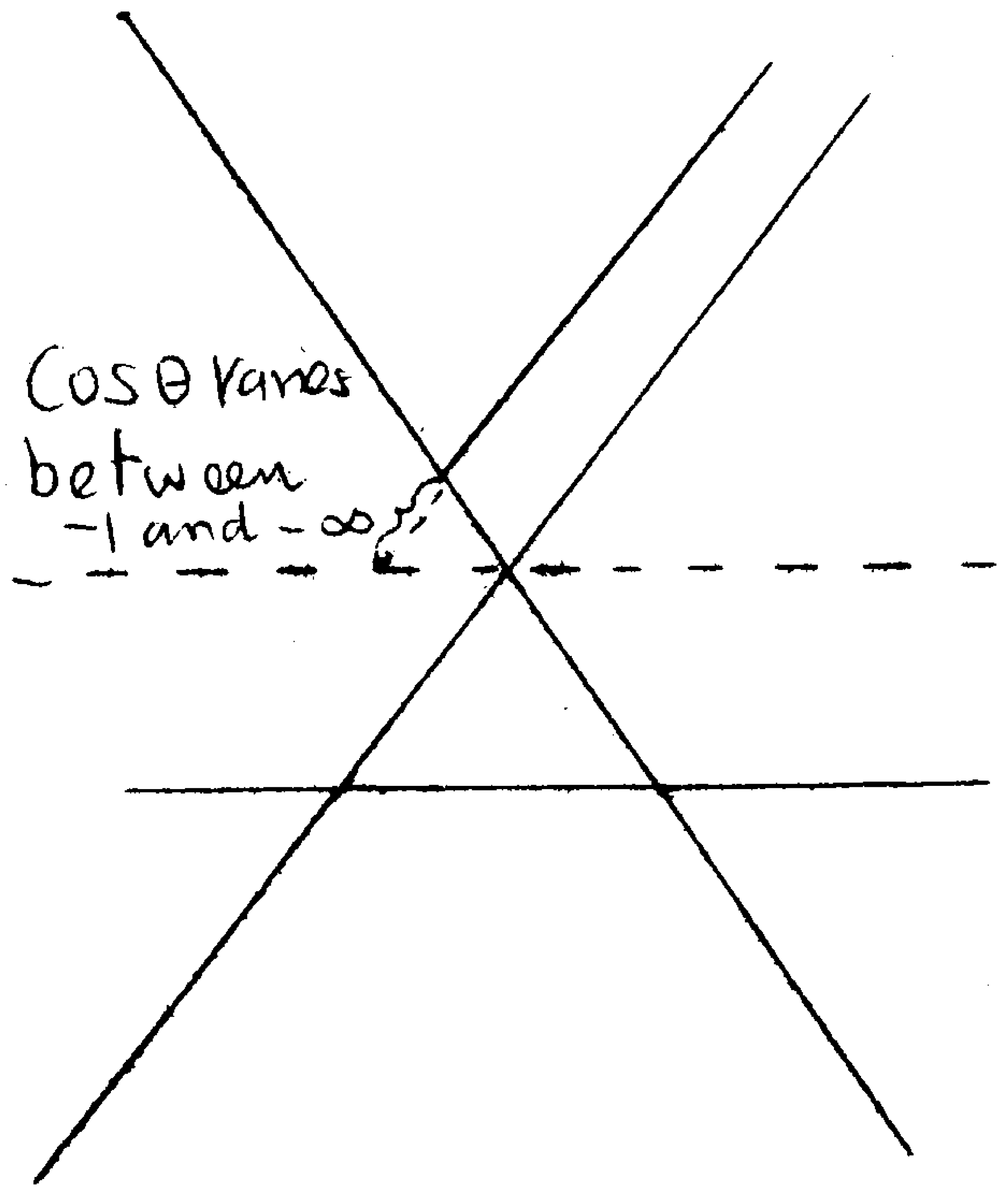
This is the result obtained from for mat approach to scattering; it involves off-the-mass-shell matrix elements.

The advantage of using dispersion relations is evident for forward scattering. But for scattering at non-forward angles, we have 2 variables, energy and momentum transfer.

Then, when we integrate over all energies, the momentum transfers involved become unphysical.

$$t = -2q^2(1-x)$$
$$x = 1 + t/2q^2$$
$$q^2 \geq -t/4$$

[for the physical region.]



Cos θ varies between - 1 and - ∞ and the amplitude cannot be obtained from experiment. But it could be obtained by extrapolating the partial-wave expansion (when it converges) to unphysical values of cos θ . Therefore the perturbation theory approach involves scattering amplitudes at unphysical energies while the dispersion theory approach involves unphysical scattering angles.

Also in the dispersion theory approach, we assumed crossing symmetry.

2. Different methods of approach to Dispersion Relations.

We have already derived two integral equations for the scattering amplitude -- one in terms of amplitudes off the mass shell, and another using some general analyticity properties of the amplitude, which led to an expression in terms of on-the-energy-shell quantities, although in the case of non-forward scattering, this involved the scattering amplitude for unphysical values of the scattering angle. During the last few years, dispersion relations have been in favour as they provide a direct and practical way of introducing experimental information in the analysis of scattering amplitudes.

Of course, it is a long way from the scattering of (spinless) particles by a potential to the full complexity of elementary particle interactions.

There are several possible ways of introducing analyticit properties into the scattering amplitudes for elementary particles. A great step found was made in 1955, when it was recognized that microcausality in field theory implied analyticity properties of the scattering amplitude. We shall later on see how it is possible to derive the analyticity properties of scattering amplitudes from the basic axioms of field theory, and the requirement of micro-causality.

Micro-causality in field theory is the requirement that the commutator

$$[\varphi_1(x), \varphi_2(x)]$$

vanishes when the separation $(x_1 - x_2)$ is space-like.

However, that the axioms of field theory and micro-causality did not take us very far. People are still working on this, but progress is very slow.

Attempts were made to extend the analyticity properties by different methods. Another approach that proved useful was to assume that the analyticity properties of the scattering amplitude as a whole was the analyticity properties of the infinite set of Feynman graphs that could contribute to the process.

This approach led to the Landau-Cutkosky rules for finding the location of the singularities and residues at these. In general, all these singularities are obtained from the intermediate states and can be related to the unitarity condition.

The next step was to establish that the S matrix was an analytic function with only those singularities associated with unitarity.-- <u>The postulate of maximal analyticity.</u>

Two other possible approaches must be mentioned:

1) We look at the analytic properties in particular soluble models, e.g. The Chew-Low Theory*.

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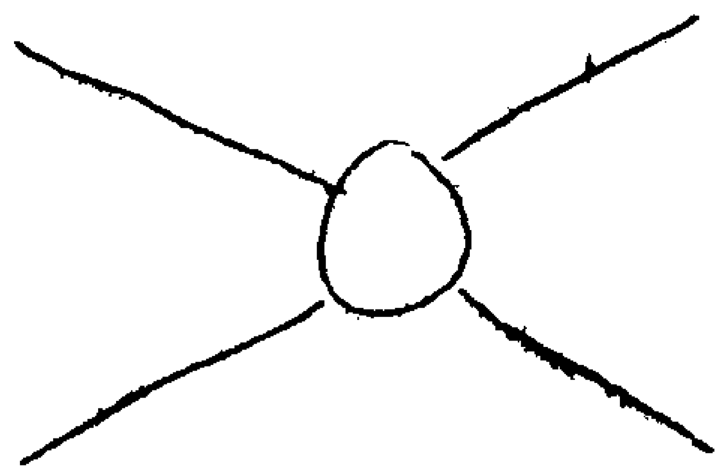
* G.C.Wick. Rev.Mod.Phys. (1955)

2) We consider the analytic properties in potential scattering.

We shall start directly with the S matrix approach, and later discuss an example in perturbation theory when the analyticity properties can be seen explicitly.

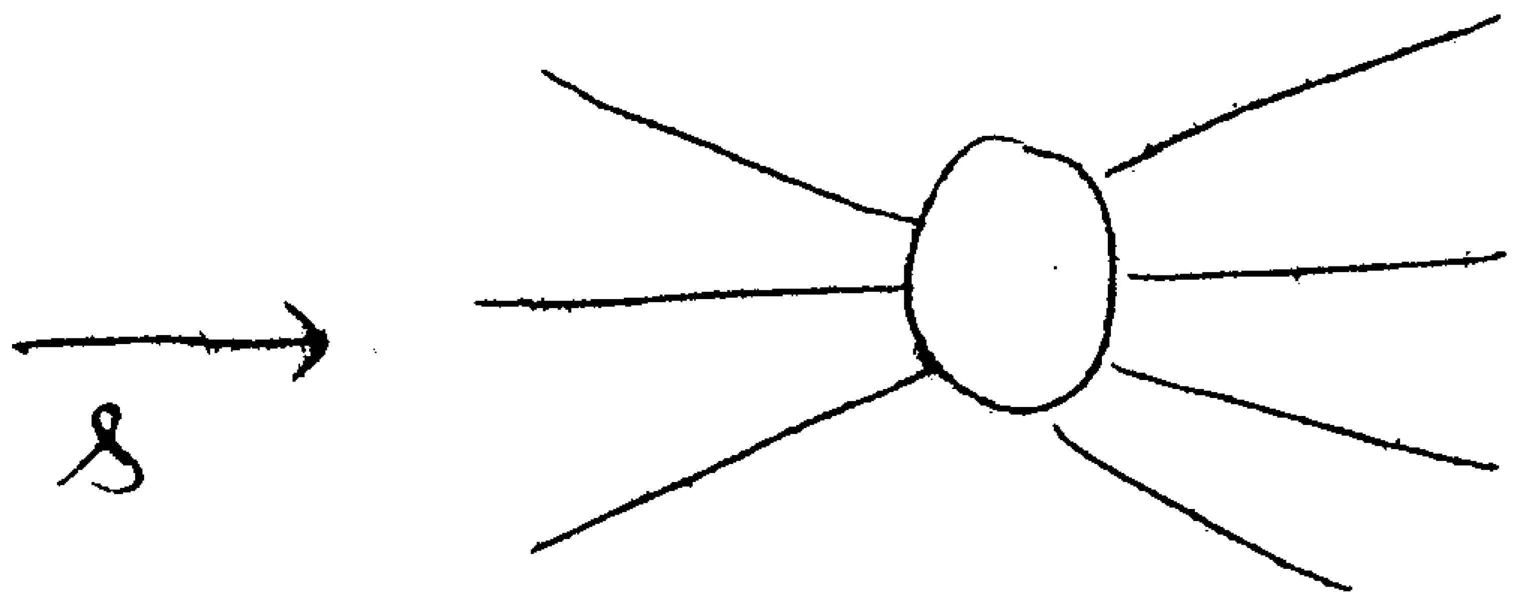
2 body scattering.



Consider the analyticity properties of the amplitude $A(s,t,u)$ as a function of the three variables s, t, u.

What are its singularities?

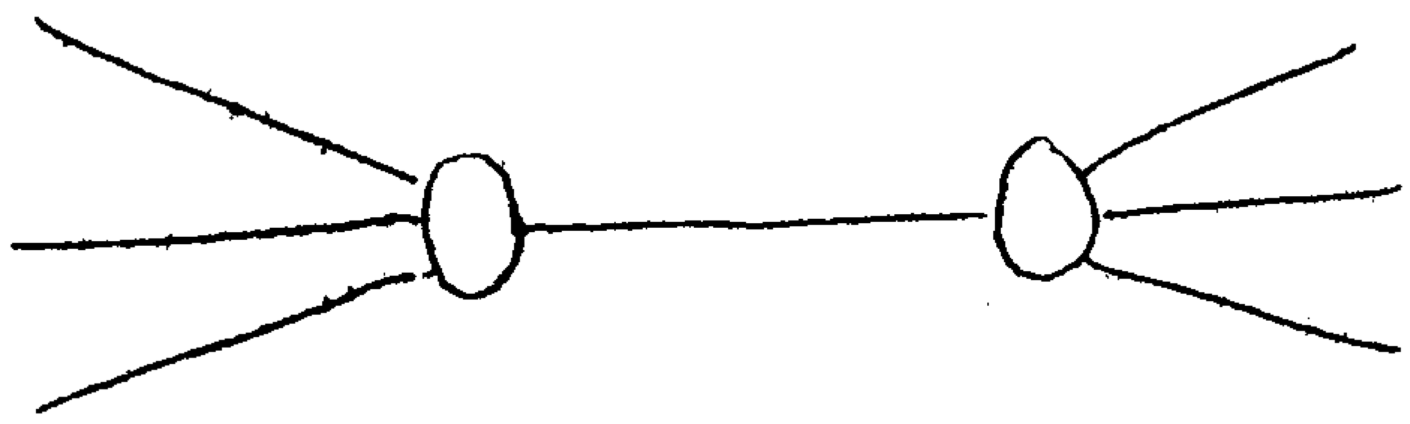
The simplest ones are poles. Consider a general process with any number of incoming and outgoing scattering amplitudes.



Suppose we split them in such a way that the particles on each side have a definite set of quantum numbers. Denote the $(\text{energy})^2$ variable for the total energy on one side by s.

[Note: This singularity does not follow from unitarity; however, if the bootstrap philosophy is correct, all poles would also follow from unitarity.]

[The unitarity relation here is 'Generalized unitarity', i.e., unitarity with all possible sets of intermediate states.]

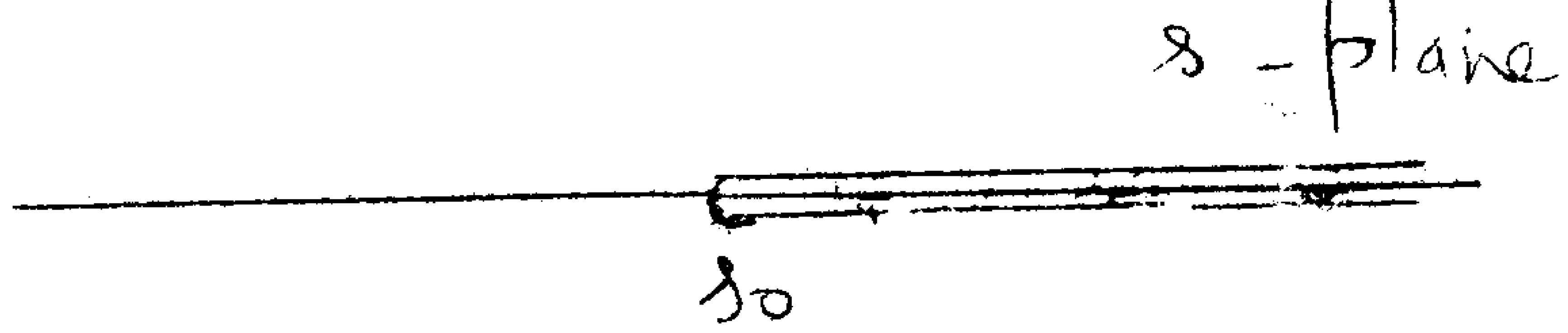


If we have a possible single-particle intermediate state with mass M , then this is assumed to give rise to a pole in the complete scattering amplitude also, i.e., the amplitude is of the form

$$A(s,t,u) = \frac{G}{s - M^2} + A'(s,t,u)$$

The scattering amplitudes will, in general, have branch points, whenever there is a threshold for the creation of real particle states.

That the scattering amplitude has a branch-point at a threshold $s = s_0$ may be seen simply from the fact that the amplitude $A(s)$ has zero imaginary part below the threshold s_0 but non-zero imaginary part above $s = s_0$. [This can be seen from the unitarity condition.]

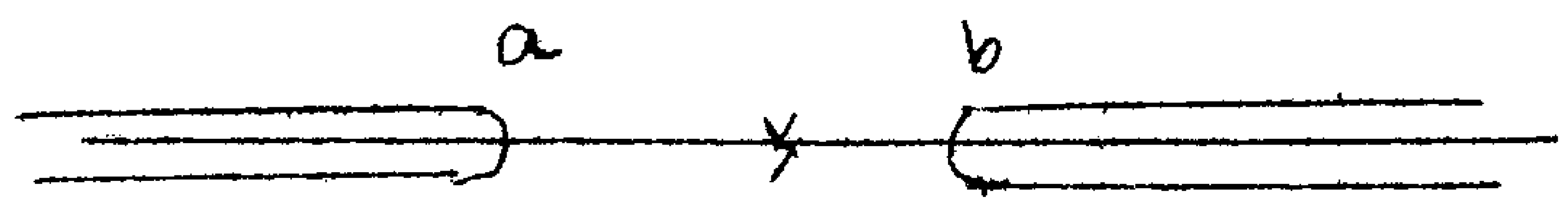


For real $s < s_0$, $A(s)$ is a real function, i.e.,
$A(s) = A^*(s^*)$, $s < s_0$, s real. <u>If the function $A(s)$ were analytic</u>, the fact that $\operatorname{Im} A(s) = 0$ for real $s < s_0$ would imply that $\operatorname{Im} A(s) = 0$ everywhere. However, we know that $\operatorname{Im} A(s) \neq 0$ for real $s > s_0$; hence the point $s = s_0$ must be a singularity of the amplitude. This singularity may be seen to be a branch point.

$$A(s) = A^*(s^*)$$
$$A(s + i\varepsilon) = A^*(s - i\varepsilon)$$
$$A(s - i\varepsilon) = A(s + i\varepsilon) - 2i \operatorname{Im} A(s + i\varepsilon)$$

<u>Spectral Representation:</u>

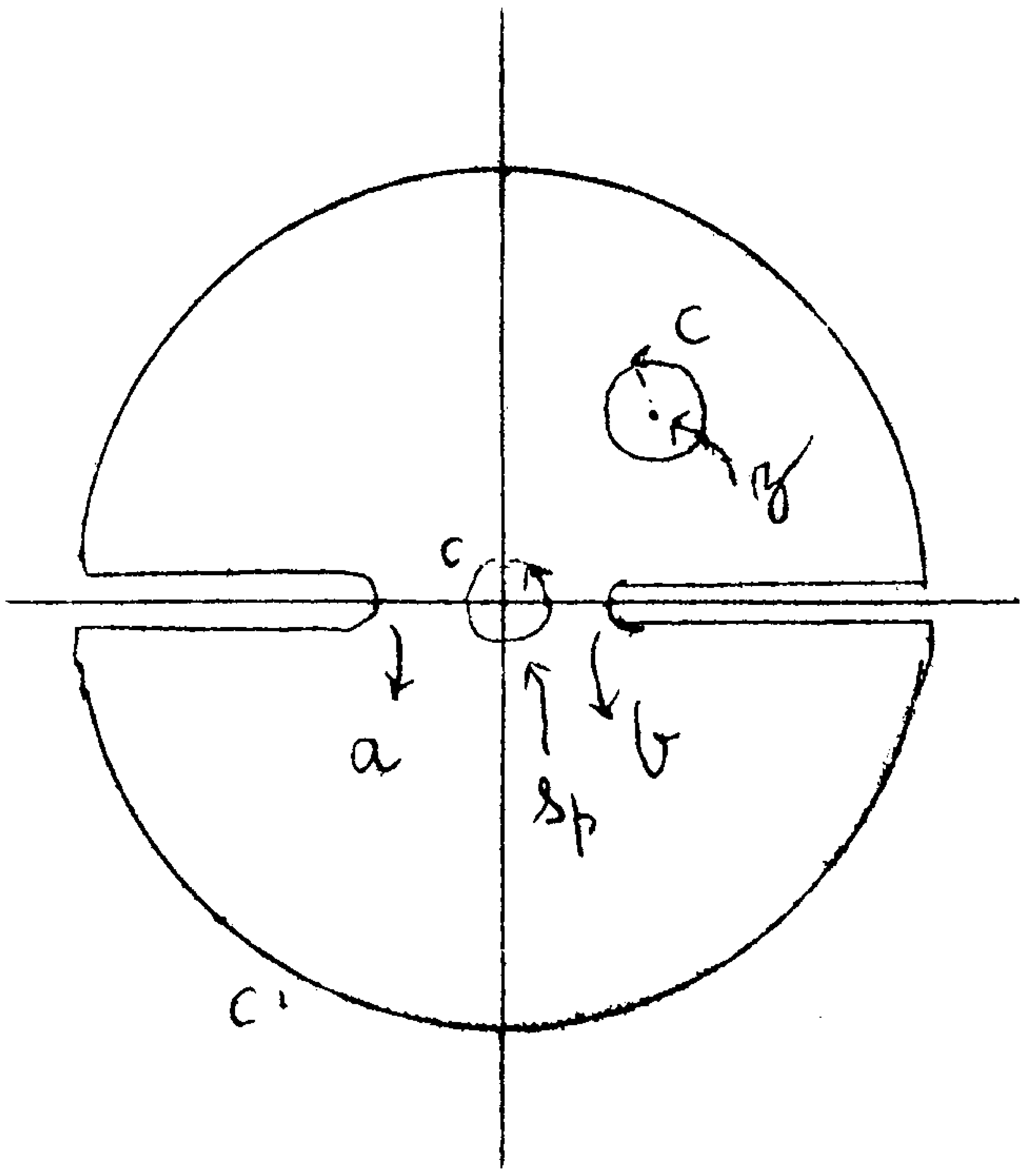
Consider a function $F(z)$ with 2 branch points on the real axis. Let the z - plane be cut along the real axis from b to $+\infty$ and a to $-\infty$.



By Cauchy's theorem, we obtain the relation

$$\oint_C \frac{F(z')}{z' - z} \, dz' = 2\pi i \, F(z)$$

The contour C is a circle round the point



The contour C can be distorted into the contour C' as shown, round the pole at $s = s_p$ and round the cuts and ... the infinite circle.

We assume that $F(z) \to 0$ as $z^{-\alpha}$, as $|z| \to \infty$ where $\alpha > 0$

We then obtain

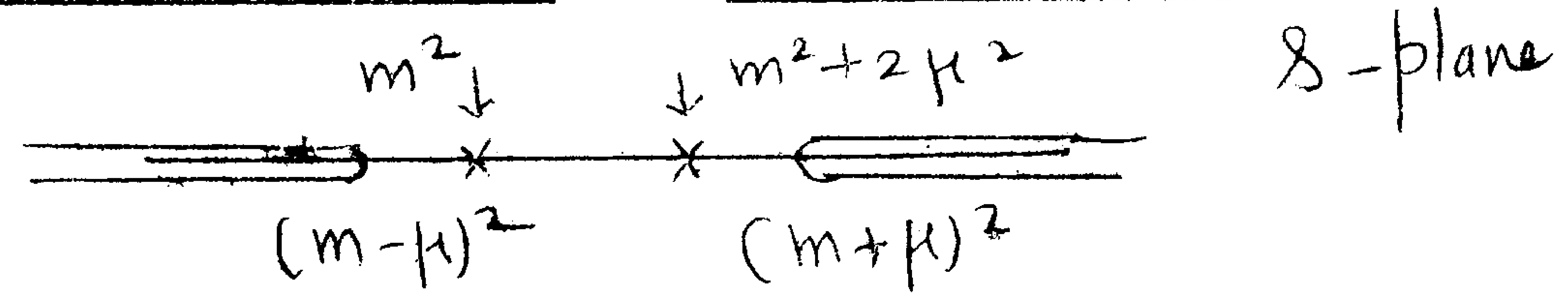
$$2\pi i \frac{G}{z - s_p} + \int_b^\infty \frac{\rho_B(z')}{z' - z} dz' + \int_{-\infty}^a \frac{\rho_a(z')}{z' - z} dz'$$

where $$\rho_B(z') = F(z'+i\epsilon) - F(z'-i\epsilon), \qquad -\infty < z' < b$$

or

$$F(z) = -\frac{G}{s_p - z} + \frac{1}{2\pi i}\int_b^{\infty} \frac{\rho_B(z')\,dz'}{z'-z} + \frac{1}{2\pi i}\int_{-\infty}^{a} \frac{\rho_a(z)}{z'-z}$$

4. <u>Pion-Nucleon Scattering</u> -- <u>FORWARD SCATTERING</u>.



In the s-plane, we have the following singularities (keeping t fixed at $t = 0$).

(i) A pole at $s = m^2$ on the real axis.

(ii) A pole at $s = m^2 + 2\mu^2$

(iii) A branch point at $s = (m+\mu)^2$

(iv) A branch point at $s = (m-\mu)^2$

In π N Scattering we have the important property of crossing symmetry, as we have seen in an earlier lecture; this implies

$$F(s, u) = F(u, s)$$

Therefore, since s has a branch point at $(m+\mu)^2$ it also has a branch point at $u = (m+\mu)^2$

$t = 0$

Therefore $u = (m+\mu)^2$ corresponds to

$$s = 2(m^2+\mu^2) - (m+\mu)^2 = (m-\mu)^2$$

Thus the cut in u from $(m+\mu)^2$ to ∞ is a cut in from $-\infty$ to $(m-\mu)^2$

Although a 1 - N intermediate state is not an intermediate state, we know from field theory that the analytic continuation (below threshold) of has a pole at $s = \mu^2$. The corresponding pole in u at $u = m^2$ gives a pole in the s plane at $s = m^2 + 2\mu^2$

These are the singularities forced on us from unitarity. Define a new variable

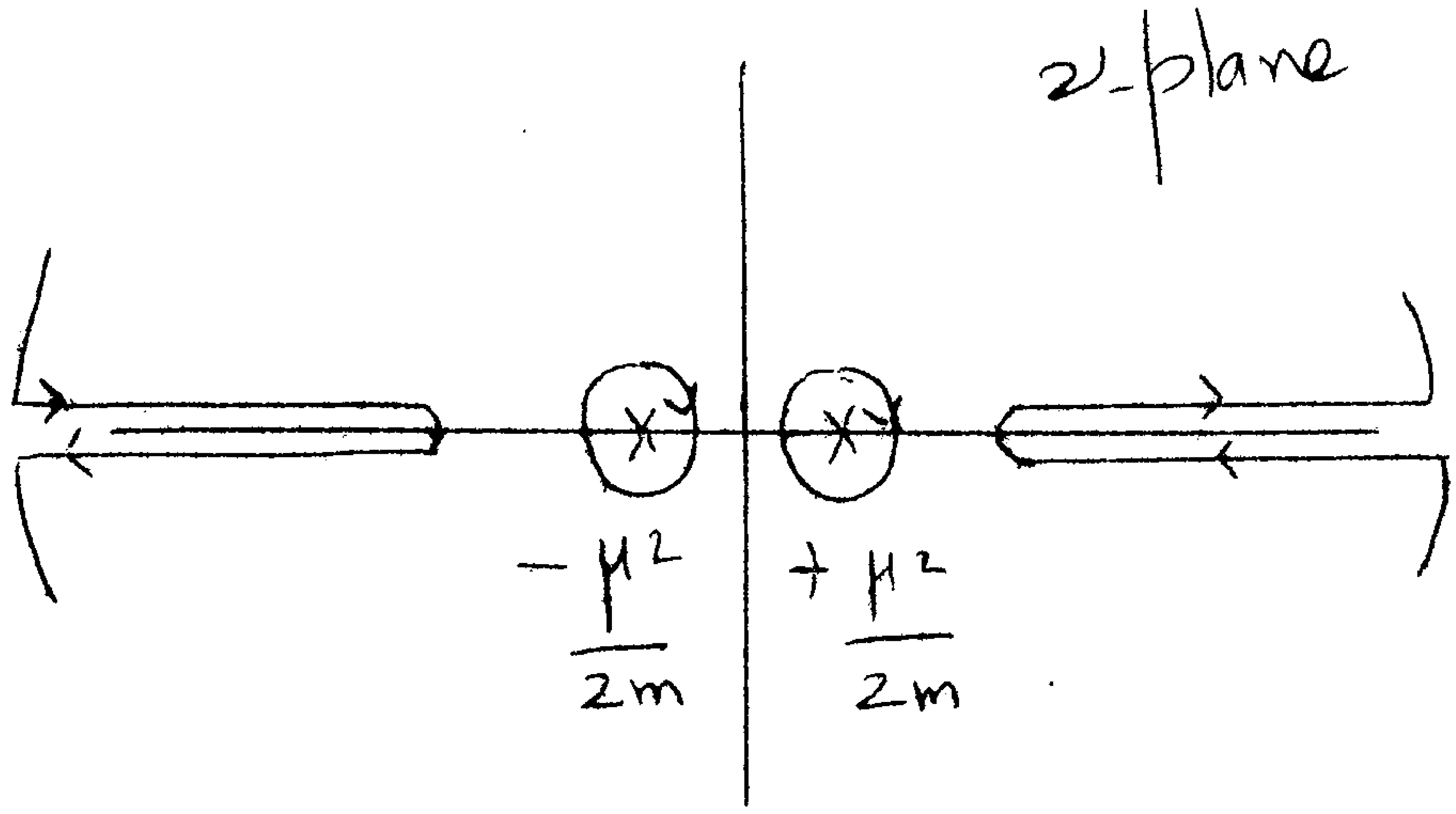
$$\nu = (s - m^2 - \mu^2)/2m$$

$$= ((p_1+q_1)^2 - m^2 - \mu^2)/2m$$

$$= (p_1 \cdot q_1)/m$$

= the meson laboratory energy.

In the ν plane, the branch points are at $+\mu$ and $-\mu$ and poles at $-\mu^2/2m$ and $+\mu^2/2m$



$$M(\nu) = \frac{1}{2\pi i}\oint_C \frac{M(\nu')d\nu'}{\nu'-\nu}$$

Assume that $M(\nu) \sim \nu^{-\alpha}$, $\alpha > 0$ where $|\nu| \to \infty$

$$M(\nu) = \frac{\Gamma}{-\frac{\mu^2}{2m} - \nu} + \frac{\Gamma}{\frac{\mu^2}{2m} - \nu}$$

$$+ \frac{1}{2\pi i}\int_{-\infty}^{\mu} 2i\,\frac{\mathrm{Im}\, M(\nu')\,d\nu'}{\nu'-\nu} + \frac{1}{2\pi i}\int_{\mu}^{\infty} 2i\,\frac{\mathrm{Im}\, M(\nu')\,d\nu'}{\nu'-\nu}$$

The physical scattering amplitude is defined as

$$M_{phys}(\nu) = \lim_{\eta\to 0} M(\nu + i\eta)$$

or

$$M_{phys}(\nu) = \left\{\frac{\Gamma}{-\frac{\mu^2}{2m} - \nu} + \frac{\Gamma'}{\frac{\mu^2}{2m} - \nu}\right.$$

$$+ \frac{1}{\pi}\int_{-\infty}^{-\mu} \frac{\mathrm{Im}\, M(\nu')\,d\nu'}{\nu'-\nu-i\eta} + \frac{1}{\pi}\int_{\mu}^{\infty} \frac{\mathrm{Im}\, T(\nu')}{\nu'-\nu \quad\;}$$

For $\nu' > 0$, we can put $\eta \to 0$ inside the integrand of the first integral but not in the second integral._7

Because of the branch points, we can safely define the crossing relations for $\mathrm{Re}\,M$

$$\mathrm{Re}\,M(s,u) = \mathrm{Re}\,M(u,s)$$

For the full amplitude, we can write the crossing relation only if we prescribe how the variables approach their values.

$$M(s+i\varepsilon, u) = M(u-i\varepsilon, s)$$

Note: when $s \longrightarrow u$ $\quad \nu \to -\nu$

$$\therefore \nu = (s - m^2 - \mu^2)/2m$$

Crossing symmetry thus gives

$$\mathrm{Re}\,M(\nu) = \mathrm{Re}\,M(-\nu)$$

Therefore

$$\int_{-\infty}^{-\mu} \frac{\mathrm{Im}\,M(\nu')\,d\nu'}{\nu'-\nu} = \int_{-\infty}^{-\mu} \frac{\mathrm{Im}\,M(\nu')\,d\nu'}{\nu'+\nu}$$

$$= \int_{\infty}^{\mu} \mathrm{Im}\,M(-\nu')(-d\nu') = \int_{\infty}^{\mu} \frac{\mathrm{Im}\,M(\nu')\,d\nu'}{\nu'-\nu}$$

$$\therefore \mathrm{Im}\,M(\nu) = -\,\mathrm{Im}\,M(-\nu)$$

$$\mathrm{Re}\,M(\nu) = \frac{\Gamma}{\frac{\mu^2}{2m}-\nu} + \frac{\Gamma}{\frac{\mu^2}{2m}+\nu} + \frac{P}{\pi}\int_{\mu}^{\infty} \mathrm{Im}\,M(\nu')\,d\nu' \left[\frac{1}{\nu'-\nu} + \frac{1}{\nu'+\nu}\right]$$

We can see explicitly that $\mathrm{Re}\,M(\nu) = \mathrm{Re}\,M(-\nu)$

The significance of the pole being at $\nu = \mu^2/2m$ is that if a real pion with lab. energy $\nu = \mu^2/2m$ is incident on a real nucleon, the resulting nucleon will be on its mass shell. [The scattering angle of the pion is, of course, virtual.]

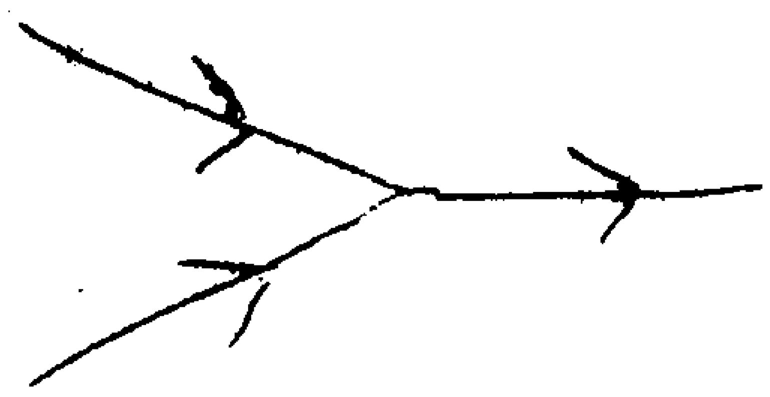
$$(\nu_0 - m)^2 - k^2 = m^2$$

$$k^2 + \mu^2 = \nu_0^2$$

$$(\nu_0 + m)^2 - \nu_0^2 + \mu^2 = m^2$$

$$\nu_0^2 + m^2 + 2\nu_0 m = -\nu^2$$

$$\nu_0 = -\frac{\mu^2}{2m}$$



The pole at $-\mu^2/2m$ is obtained when the pion is emitted instead of absorbed. These singularities are the only ones of the forward scattering amplitude.

Note: The residue at the pole may be obtained from perturbation theory; this is not very sensitive to the structure of the Lagrangian; e.g. it is well-known that the PS and PV theories give the same residue at the pole.

The amplitude $M(\nu)$ is defined as

$$M(\nu) = \lim_{\varepsilon \to +0} M(\nu + i\varepsilon)$$

We obtained

$$M(\nu) = \frac{\Gamma^2}{\frac{\mu^2}{2m} + \nu} - \frac{\Gamma'}{\frac{\mu^2}{2m} - \nu} + \frac{1}{\pi}\int_{-\infty}^{-\mu} \frac{\mathrm{Im}\, M(\nu')\, d\nu'}{\nu' - \nu} + \frac{1}{\pi}$$

$$\times \int \frac{\mathrm{Im}\, M(\nu')\, d\nu'}{\nu' - \nu}$$

This gives

$$\mathrm{Re}\,M(\nu) = \frac{\Gamma}{\frac{\mu^2}{2m}+\nu} - \frac{\Gamma'}{\frac{\mu^2}{2m}-\nu} + \frac{1}{\pi}\int_{-\infty}^{-\mu} \frac{\mathrm{Im}\,M(\nu')\,d\nu'}{\nu'-\nu} + \frac{P}{\pi}\int_{\mu}^{\infty} \frac{\mathrm{Im}\,M(\nu')}{\nu'-\nu}\,d\nu'$$

This is called a 'dispersion relation', as it relates to the real part of the amplitude to its imaginary part. When we cross, i.e. change $s \rightleftarrows u$ or $\nu \rightleftarrows -\nu$, we have a cut. For the real part of the amplitude, we have

$$\mathrm{Re}\,M(-\nu) = \mathrm{Re}\,M(\nu)$$

This implies that $\Gamma' = -\Gamma$ and that

$$\mathrm{Im}\,M(-\nu) = -\,\mathrm{Im}\,M(\nu)$$

The two crossing relations for $\mathrm{Re}\,M(\nu)$ and $\mathrm{Im}\,M(\nu)$ can be written together as

$$M(\nu+i\varepsilon) = M(-\nu-i\varepsilon)$$

$$\mathrm{Re}\,M(\nu) = \Gamma\left(\frac{1}{\frac{\mu^2}{2m}+\nu} + \frac{1}{\frac{\mu^2}{2m}-\nu}\right) + \frac{P}{\pi}\int d\nu'\,\mathrm{Im}\,M(\nu')\left(\frac{1}{\nu'-\nu} + \frac{1}{\nu'+\nu}\right)$$

Crossing gives $s \rightleftarrows u$ so that the first integral becomes

$$-\int_{\infty}^{\mu} \frac{\mathrm{Im}\,M(-\nu')}{-\nu'-\nu}\,d\nu' = \int_{\mu}^{\infty} \frac{\mathrm{Im}\,M(\nu')\,d\nu'}{\nu'+\nu}$$

so that we have the integral only along the positive real axis (μ to $+\infty$)

$\mathrm{Im}\, M(\nu)$ for forward scattering can be related to the total cross-section, by the optical theorem.

We earlier obtained the relation

$$\mathrm{Im}\, M(\nu) = 2 q W \sigma_{tot}$$

In terms of lab. variables,

$$\mathrm{Im}\, M(\nu) = \mathrm{Im} \sqrt{\nu^2 - \mu^2}\, \sigma_{tot}(\nu)$$

We have $\nu^2 - \mu^2 = \left[(s^2 - m^2 - \mu^2)^2 - 4 m^2 \mu^2 \right] / 4m^2$

$$= \frac{(W + m + \mu)(W + m - \mu)(W - m + \mu)(W - m - \mu)}{4m^2}$$

Note: $q^2 = \lambda(W, m, \mu) / 4W^2$

where $\lambda(W, m, \mu) = [W^2 - (m+\mu)^2][W^2 - (m-\mu)^2]$

$$\mathrm{Re}\, M(\nu) = \Gamma \left(\frac{1}{\frac{\mu^2}{2m} + \nu} + \frac{1}{\frac{\mu^2}{2m} - \nu} \right)$$

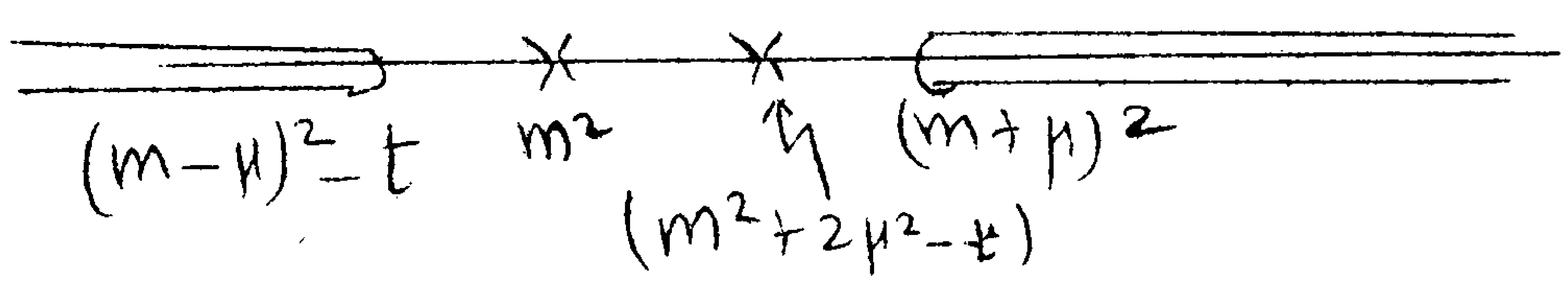
$$+ \frac{P}{\pi} \int_{\mu}^{\infty} d\nu' \, \mathrm{Im} \sqrt{\nu'^2 - \mu^2}\, \sigma_{tot}(\nu') \left[\frac{1}{\nu' - \nu} + \frac{1}{\nu' + \nu} \right]$$

5. <u>Dispersion relations for forward scattering</u>.

This is a very useful relation, as it involves only one parameter .

Before we can test this with experiment, we must take into account the complications arising from charge and spin also.

Before we go into that, we can write a dispersion relation for any fixed value of t (not necessarily $t = 0$). In the s- plane we have the same singularities arising from s; however, the singularities arising from u are at a different position; these are shown below:



The fact that only these singularities are present has been proved only in the region

$$0 < -t < \frac{32}{3}\mu^2\,\frac{2m+\mu}{2m-\mu}$$

However, this may have nothing to do with physics but may be only a limitation of the methods employed to prove the dispersion relations. In fact, it seems likely, and was conjectured to be generally valid, that these are the only singularities, for any value of t.

The dispersion relation now becomes

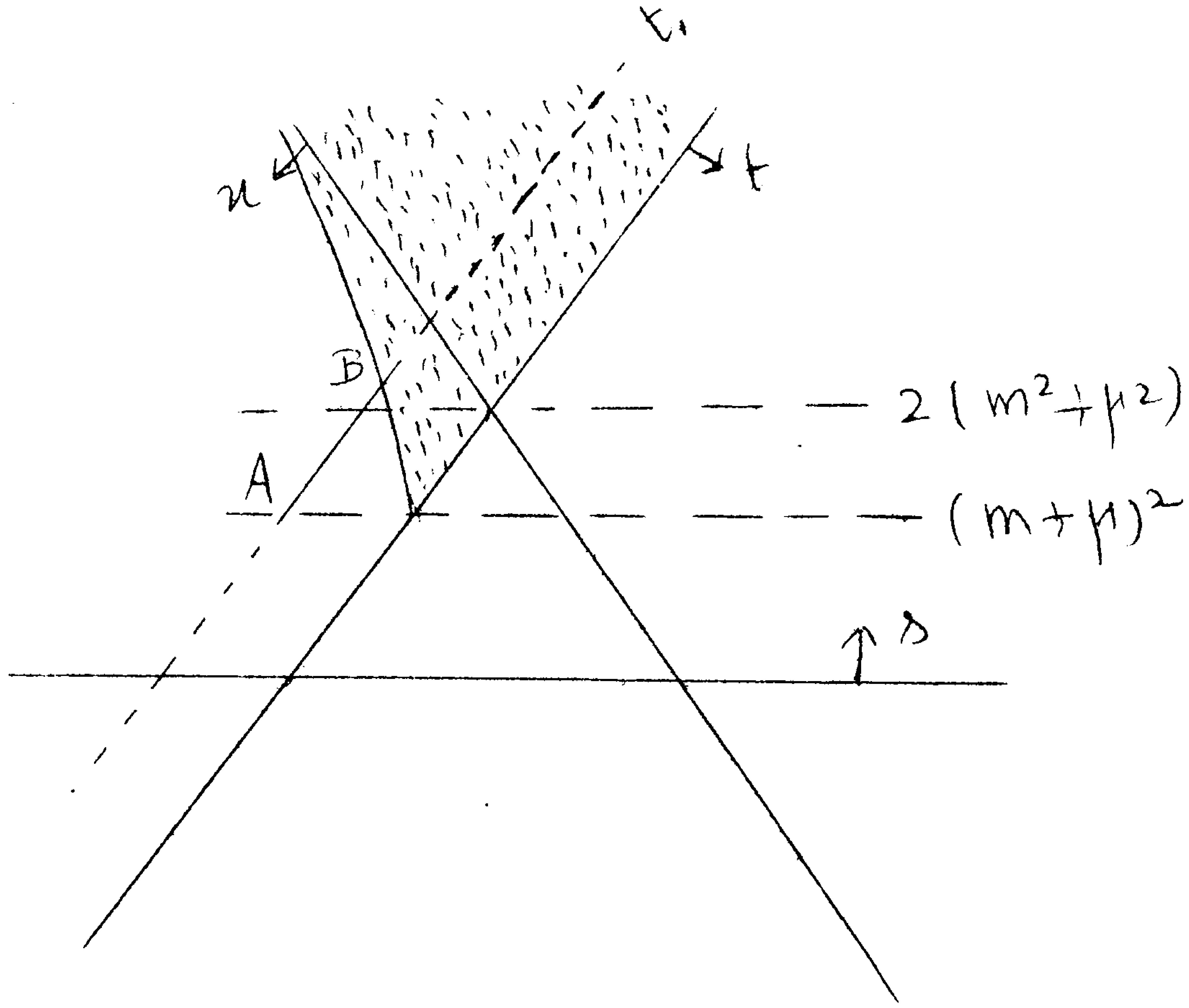
$$\mathrm{Re}\,M(s,t) = -\Gamma\left(\frac{2m}{m^2-s} + \frac{2m}{s-m^2-2\mu^2+t}\right) + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds'\,\mathrm{Im}\,M(s',t)\left[\frac{1}{s'-s} + \frac{1}{s'+s-2m^2-2\mu^2+t}\right]$$

The threshold for q^2 is now

$$\frac{s^2-2s(m^2+\mu^2)+(m^2-\mu^2)^2}{4s}$$

$$q^2 \geq -t/4$$

Consider the Mandelstam diagram.



For a fixed value t_0 of t, we need the scattering amplitu[illegible] for unphysical values of the scattering angle, corresponding to the region AB.

Next we consider the complications arising from charge and spin.

5. π N Scattering with Charge and Spin.

The π N scattering amplitude, decomposed in spin space and isospin space, may be written

$$M = -2m\bar{u}(p_f)\Big\{\Big[-A^{(+)} + i\frac{B^{(+)}}{2}\gamma\cdot(q_1+q_2)\Big]\delta_{\alpha\beta} + \Big[-A^{(-)} + i\frac{B^{(-)}}{2}\gamma\cdot(q_1+q_2)\Big]\tfrac{1}{2}[\tau_\beta,\tau_\alpha]\Big\}u(p_i)$$

In the forward direction, there is no spin flip, and the expression for the amplitude simplifies considerably. We have $p_i = p_f$ for forward scattering. Writing

$$\gamma\cdot(q_1+q_2) = \vec{\gamma}\cdot(\vec{q}_1+\vec{q}_2) + i\beta(\omega_1+\omega_2)$$
$$= 2(\vec{\gamma}\cdot\vec{q} + i\beta\omega)$$

We obtain

$$-2m\bar{u}(p)[-A + iB[\vec{\gamma}\cdot\vec{q} + i\beta\omega]u(p)$$
$$= -2m\bar{u}(p)[-A + iB(-\vec{\gamma}\cdot\vec{q} - i\beta E + i\beta W]u(p)$$

$$[\text{since } W = \omega + E]$$

$$= -2m\bar{u}(p)[-A + iB(-\gamma\cdot q) - BW\beta]u(p)$$

$$= -2m\bar{u}(p)[-A + B(m - \frac{EW}{m})u(p)$$

[if we use the Dirac equation.]

Note:- $m\nu = p - q$ in the lab. system. But $(p - q)$ is an invariant, hence it may be evaluated in the c.m.s. also. Therefore

$$m\nu = p - q = E\omega + q^2$$
$$= E(W - E) + q^2 = EW - m^2$$
$$= -m\left[m - \frac{EW}{m}\right]$$

$$\therefore M = -2m\,\bar{u}(p)\left[-A - \nu B\right]u(p)$$
$$= 2m\,\bar{u}(p)\left[A + \nu B\right]u(p)$$

Thus, instead of two amplitudes, we now have only one amplitude. The total amplitude is given by

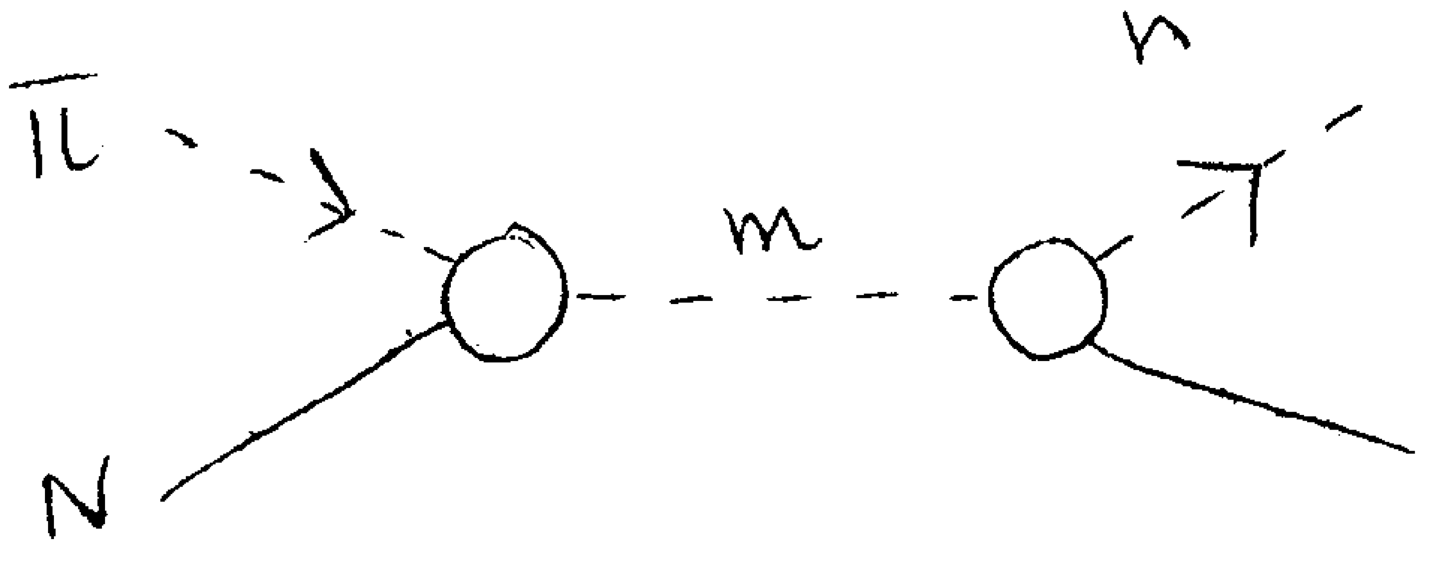
$$\left(A^{(+)} + \nu B^{(+)}\right)\delta_{\alpha\beta} - \left(A^{(-)} + \nu B^{(-)}\right)\frac{1}{2}\left[\tau_\alpha, \tau_\beta\right]$$

The crossing relations for the real parts are given by

$$\mathrm{Re}\, A^{(\pm)}(\nu) = \pm\, \mathrm{Re}\, A^{(\pm)}(-\nu)$$
$$\mathrm{Re}\, B^{(\pm)}(\nu) = \mp\, \mathrm{Re}\, B^{\pm}(-\nu)$$

We shall now obtain the residues at the nucleon pole. The unitarity relation may be written

$$i\left(M_{if} - M^*_{if}\right) = -(2\pi)^4 \sum_n \delta(p_i - p_n)\, M_{fn} M^*_{in}$$



The contribution of the pole to the amplitude may be written as

$$M_{fn} = G(p_n - p_f)^2 \frac{\bar{u}(p_n)\gamma_5 \tau_\alpha u(p_i)}{\sqrt{2E_f 2\omega 2E_n}} 2m$$

We have

$$(M_{in})^x = G^x(p_n - p_i)^2 \frac{(\bar{u}(p_i)\gamma_5 \tau_\alpha u(p_n))^*}{\sqrt{2E_i 2\omega 2E_n}} 2m$$

$$= G^x (p_n - p_i) \frac{(u^+(p_i)\beta\gamma_5\tau_\alpha u(p_n))^*}{\sqrt{2E_i 2\omega 2E_n}} 2m$$

$$= G^x (p_n - p_i) \frac{\bar{u}(p_n)\beta^{-1}\gamma_5^*\beta u(p_i)}{\sqrt{2E_i 2\omega 2E_n}} 2m$$

$$= -G^x (p_n - p_i) \frac{\bar{u}(p_n)\gamma_5 u(p_i)}{\sqrt{2E_i 2\omega 2E_n}} 2m$$

since $\beta^{-1}\gamma_5^*\beta = -\gamma_5$

Thus the residue here is of opposite sign, because the pion is pseudoscalar. For a scalar meson we would have a positive sign. Therefore the unitarity condition may be written

$$2 \mathcal{I}_m M = -(2\pi)^4 \; G((p_n - p_i)^2)$$

$$= -\sum_n \frac{4m^2}{2E_i 2E_n 2\omega} \delta(p_i - p_n)\bar{u}(p_f)\gamma_5\tau_\beta u(p_n) \bar{u}(p_n)\gamma_5\tau_\alpha u(p_i)$$

When we integrate over the energies of the intermediate state, we obtain the following result:

$$\mathrm{Im}\, F_{\beta\alpha} = \pi g^2 \nu_B \tau_\beta \tau_\alpha \delta(\nu - \nu_B) + \cdots$$

where $\nu_B = -\mu^2/2m$

$$\delta(m + \nu - E_n) = \frac{E_n}{m} \delta(\nu - \nu_B)$$

$G((p_n - p_i)^2)$ is a function of

$$\begin{aligned}
(p_n - p_i)^2 &= p_n^2 + p_i^2 - 2 p_n \cdot p_i \\
&= 2m^2 - 2(p_n \cdot p_i) \\
&= 2m^2 - 2 E_n m \quad \text{(in the lab system)} \\
&= 2m^2 - 2(\nu + m) m \\
&= -2\nu_B m
\end{aligned}$$

At the pole, $(p_n - p_i)^2 \to \mu^2$. Therefore the residue involves the factor

$$|G(\mu^2)|^2$$

Crossing symmetry tells us that we have another pole contribution also to $\mathrm{Im}\, F$. To obtain the crossed pole contribution, we must cross the isotopic spin indices also.

We have $\mathrm{Im}\, F_{\beta\alpha} = \pi g^2 \nu_B \tau_\beta \tau_\alpha \delta(\nu - \nu_B)$

$$= \nu_B \pi g^2 \left[\delta_{\alpha\beta} + \tfrac{1}{2}[\tau_\beta, \tau_\alpha]\right] \delta(\nu - \nu_B)$$

The crossed pole contributes an additional term; the total pole contribution is given by

$$\mathrm{Im}\, F_{\beta\alpha} = \nu_B \pi g^2 \delta_{\alpha\beta}\left[\delta(\nu - \nu_B) - \delta(\nu + \nu_B)\right.$$
$$\left. + \tfrac{1}{2}[\tau_\beta, \tau_\alpha]\left[\delta(\nu - \nu_B) + \delta(\nu + \nu_B)\right]\right]$$

The first term will contribute to $F(+)$ and the second term to $F(-)$. The dispersion relation for $F^{+}(\nu)$ and $F^{(-)}(\nu)$ may be directly written.

$$\mathrm{Im}\, F^{(+)}(\nu) = \nu_B \pi g^2 \left(\delta(\nu - \nu_B) - \delta(\nu + \nu_B)\right]$$

+ the Contributions from other intermediate states.

We may write the dispersion relation as

$$\mathrm{Re}\, F^{(+)}(\nu) = \nu_B g^2 \frac{1}{\nu_B - \nu} - \nu_B g^2 \frac{1}{-\nu_B - \nu}$$
$$+ \frac{P}{\pi} \int_\mu^\infty \mathrm{Im}\, F^{(+)}(\nu')\, d\nu' \left(\frac{1}{\nu' - \nu} + \frac{1}{\nu' + \nu}\right)$$

In the literature, one usually finds the dispersion relation written for the amplitude

$$\mathcal{F} = F / 8\pi m$$

We obtain

$$\mathrm{Re}\, \mathcal{F}_1^{(+)}(\nu) = \frac{\nu_B g^2}{8\pi m}\left(\frac{1}{\nu_B-\nu} + \frac{1}{\nu_B+\nu}\right)$$

$$+ \frac{P.}{\pi}\int_{\mu}^{\infty} \mathrm{Im}\, \mathcal{F}_1^{(+)}(\nu)\left[\frac{1}{\nu'-\nu} + \frac{1}{\nu'+\nu}\right\} d\nu'$$

Noting that $\nu_B = -\mu^2/2m$, we may write the residue at the pole as

$$-\frac{\mu^2}{2m}\,\frac{g^2}{8\pi m} = -f^2/4\pi$$

where f^2 is defined by

$$g^2\,\frac{\mu^2}{4m^2} = f^2$$

g is the coupling constant in the pseudoscalar theory and is the coupling constant in the static model of the Chew-Low theory (with pseudovector coupling). We finally obtain

$$\mathrm{Re}\, \mathcal{F}_1^{(\pm)}(\nu) = \pm\frac{f^2}{4\pi}\left(\frac{1}{\frac{\mu^2}{2m}-\nu} \pm \frac{1}{\frac{\mu^2}{4m}+\nu}\right)$$

Note: $\mathcal{F}_1^{(\pm)} = \dfrac{F^{(\pm)}}{8\pi m^2} + \dfrac{P}{\pi}\displaystyle\int_{\mu}^{\infty} \mathrm{Im}\, \mathcal{F}_1^{(\pm)}(\nu')\left(\frac{1}{\nu'-\nu} + \frac{1}{\nu'+\nu}\right) d\nu'$

We know that

$$\mathrm{Im}\, F^{(\pm)} = 2\left(\sqrt{\nu^2-\mu^2}\right) m\, \sigma_{tot}$$

[for forward scattering.

We also saw earlier that $F^{(+)}$ and $F^{(-)}$ are related to the $(\pi^+ p)$ and $(\pi^- p)$ scattering amplitudes by

$$2F^{(+)} = F(\pi^+ p) + F(\pi^- p)$$

and

$$2F^{(-)} = F(\pi^- p) - F(\pi^+ p)$$

so that the optical theorem in terms of $\mathcal{F}_1^{(\pm)}$ gives

$$\operatorname{Im} \mathcal{F}_1^{(\pm)}(\nu) = \frac{\sqrt{\nu^2-\mu^2}}{8\pi}\left[\sigma_{\pi^- p}(\nu) \pm \sigma_{\pi^+ p}(\nu)\right]$$

Substituting this into the dispersion relation for $\mathcal{F}_1^{(\pm)}(\nu)$ gives

$$\operatorname{Re} \mathcal{F}_1^{(\pm)}(\nu) = \frac{f^2}{4\pi}\left(\frac{1}{\frac{\mu^2}{2m} - \nu} \pm \frac{1}{\frac{\mu^2}{2m}+\nu}\right)$$

$$+ \frac{1}{8\pi^2} P\int_\mu^\infty d\nu' \sqrt{\nu'^2-\mu^2}\left[\sigma_{\pi^- p}(\nu') \pm \sigma_{\pi^+ p}(\nu')\right]\left[\frac{1}{\nu'-\nu} \pm \frac{1}{\nu'}\right]$$

This relates $\operatorname{Re} \mathcal{F}_1^{(\pm)}(\nu)$ directly to the $\pi^+ p$ and $\pi^- p$ (total) scattering cross-sections.

Combining the two pole terms, and writing the last term inside the integrand as one factor, we may write $\operatorname{Re} \mathcal{F}^{(-)}(\nu)$ as

$$\operatorname{Re} \mathcal{F}_1^{(-)}(\nu) = -\frac{f^2}{4\pi}\left[\frac{2\nu}{\left(\frac{\mu^2}{2m}\right)^2 - \nu^2}\right]$$

$$+ \frac{2\nu}{8\pi^2} P\int_\mu^\infty d\nu' \sqrt{\nu'^2-\mu^2}\left[\sigma_+(\nu') - \sigma_-(\nu')\right]\frac{1}{\nu'^2-\nu^2}\, d\nu'.$$

The integral will converge only if $\left[\sigma_+(\nu) - \sigma_-(\nu)\right] \to 0$ at least as $\nu^{-\alpha}$ as $\nu \to \infty$, α being > 0

It seems reasonable to assume that the cross-section does, in fact, have such a behaviour at large energies.

Note that

$$\frac{\nu^2}{\nu'^2(\nu'^2-\nu^2)}+\frac{1}{\nu'^2}=\frac{\nu'^2}{\nu'^2(\nu'^2-\nu^2)}=\frac{1}{\nu'^2-\nu^2}$$

We can then write the relation for $Re\,\mathcal{F}^{(-)}$ in the form

$$Re\,\mathcal{F}^{(-)}(\nu)=-\frac{b^2}{4\pi^2}\frac{2\nu}{\left(\frac{\mu^2}{2m}\right)^2-\nu^2}+\frac{2\nu^3}{8\pi^2}P\int_\mu^\infty[\sigma_+(\nu')-\sigma_-(\nu')]\,d\nu'\times\sqrt{\nu'^2-\mu^2}\Big/\nu'^2(\nu'^2-\nu^2)$$

$$+\frac{2\nu}{8\pi^2}\int_\mu^\infty\sqrt{\nu'^2-\mu^2}\,[\sigma_+(\nu')-\sigma_-(\nu')]\,d\nu'/\nu'^2$$

We have separated the integral into two parts, the first of which converges more rapidly than the original integral, the second part being independent of ν.

We write the second part as a number (multiplying ν). We then obtain

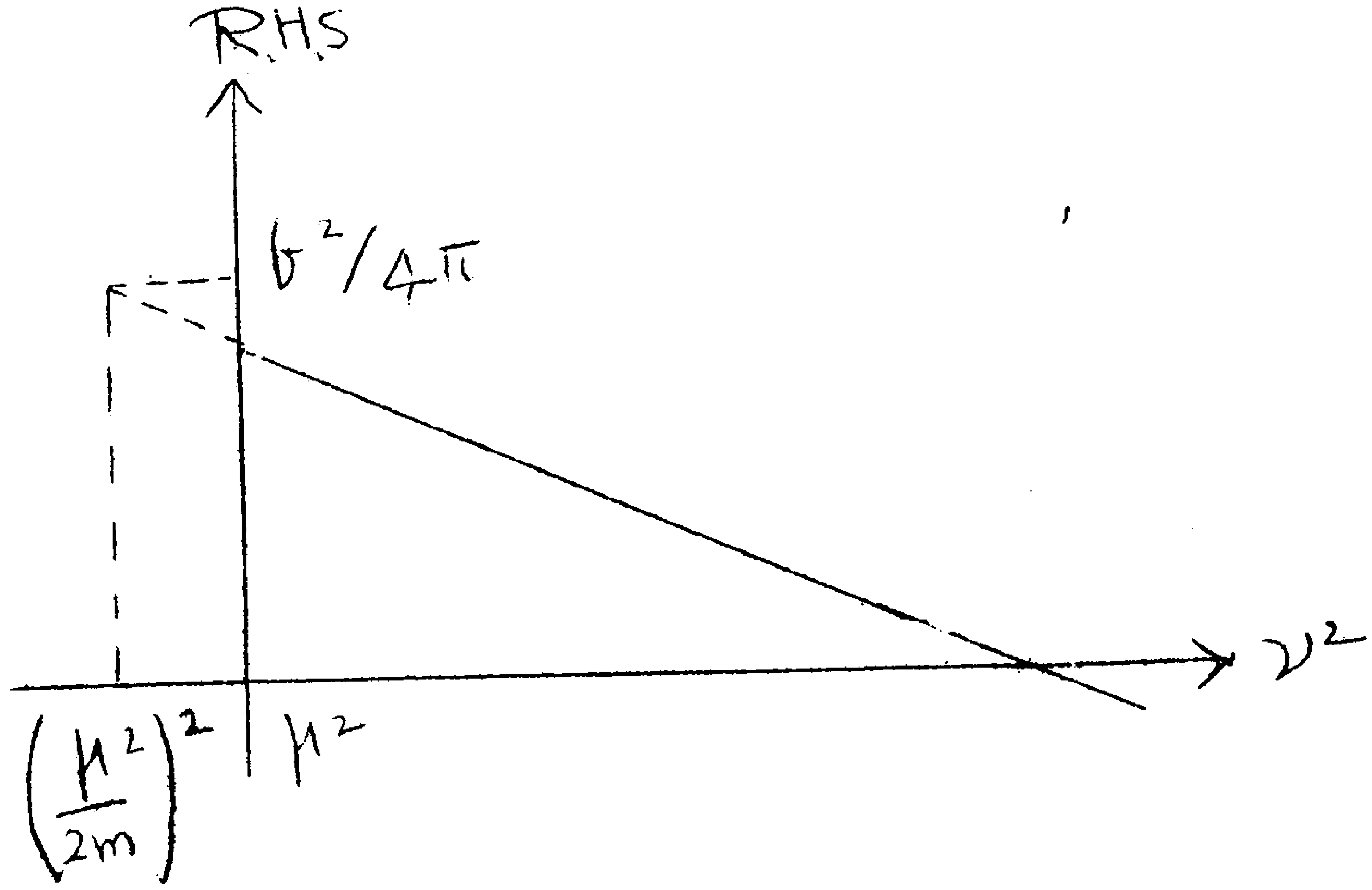
$$\left[Re\,\mathcal{F}^{(-)}(\nu)-\frac{2\nu^3}{8\pi^2}P\int_\mu^\infty(\sigma_+-\sigma_-)\,d\nu'\sqrt{\nu'^2-\mu^2}\Big/\nu'^2(\nu'^2-\nu^2)\right]\times\left[\left(\frac{\mu^2}{2m}\right)^2-\nu^2\right]\Big/2\nu$$

$$=-\frac{b^2}{4\pi}+\frac{\left(\frac{\mu^2}{2m}\right)^2-\nu^2}{8\pi^2}\lambda \qquad \text{(a)}$$

This is a powerful statement; the left hand side, which can be obtained from the π^+p and π^-p cross-sections, must vary linearly with ν

This provides a test of the dispersion relations we have written for forward scattering.

If we can extrapolate the relation (a) to the unphysical point $\left(\frac{\mu^2}{2m}\right)^2$, the second term on the right hand side vanishes, and we obtain the coupling constant $f^2/4\pi$



This gives $f^2/4\pi \approx 0.1$ or $g^2/4\pi \approx 14$.

This was one of the earliest examples of polology.

For $F^{(+)}(\nu)$, the integral will diverge; and we must write the dispersion relation in a different form. [This is known as making a subtraction.]

$$\operatorname{Re} F^{(\pm)}(\nu) = \pm \frac{f^2}{4\pi}\left(\frac{1}{\nu_0-\nu} \pm \frac{1}{\nu_0+\nu}\right)$$
$$+ \frac{1}{8\pi^2} P\int_{\mu}^{\infty} d\nu' \sqrt{\nu'^2-\mu^2}\left(\sigma_-(\nu') \pm \sigma_+(\nu')\right)\left(\frac{1}{\nu'-\nu} \pm \frac{1}{\nu' \cdots}\right.$$

where

$$\nu_0 = \mu^2/2m$$

and f^2 is related to g^2 by

$$f^2 = g^2 \mu^2/4m^2$$

Previously we obtained a relation for $\mathrm{Re}\, F^{(-)}(\nu)$ where $F^{(-)}(\nu)$ is the amplitude for charge-exchange scattering, in terms of the cross-sections.

$$\mathrm{Re}\, F^{(-)}(\nu) = -\frac{f^2}{4\pi}\frac{2\nu}{\nu_0^2-\nu^2} + \frac{2\nu}{8\pi^2} P\int_{\mu}^{\infty} \sqrt{\nu'^2-\mu^2}\,[\sigma_-(\nu') - \sigma_+(\nu')]\frac{d\nu'}{\nu'^2-\nu}$$

We had the crossing relation

$$\mathrm{Re}\, F(-\nu) = -\mathrm{Re}\, F(\nu)$$

This integral will make sense if $(\sigma_-(\nu') - \sigma_+(\nu'))$ decreases as $\nu^{-\alpha}$, $\alpha > 0$, when $\nu \to \infty$. We also saw how the pion-nucleon coupling constant could be obtained by extrapolating to the nucleon pole.

Suppose we write the dispersion relation for $\mathrm{Re}\, F^{(+)}(\nu)$. We now have

$$\mathrm{Re}\, F^{(+)}(\nu) = +\frac{f^2}{4\pi}\frac{2\nu}{\nu_0^2-\nu^2} + \frac{1}{8\pi^2} P\int_{\mu}^{\infty} \sqrt{\nu'^2-\mu^2}\,[\sigma_- + \sigma_+]\frac{2\nu'}{(\nu'^2-\nu^2)}$$

This integral may diverge, if $\sigma_-(\nu') + \sigma_+(\nu')$ does not vanish fast enough as $\nu \to \infty$.

Then we must write the dispersion relation in a different form. In writing the above dispersion relation we assumed that $F(\nu) \to 0$ as $|\nu| \to \infty$. Suppose $F(\nu) \longrightarrow$ a constant, C, as $|\nu| \to \infty$. What would be the form of the dispersion relation ?

We may now write the Cauchy integral relation, not for the scattering amplitude $F(\nu)$, but for the function $\frac{F(\nu)}{\nu}$:

$$\frac{F(\nu)}{\nu} = \frac{1}{2\pi i} \oint \frac{F(\nu')\,d\nu'}{\nu'(\nu'-\nu)} \tag{1}$$

The dispersion relation for $F^{(\pm)}(\nu)$ was obtained as

$$\operatorname{Re} F^{(\pm)}(\nu) = \pm\frac{f^2}{4\pi}\left(\frac{1}{\nu_0-\nu} \pm \frac{1}{\nu_0+\nu}\right) + \frac{1}{\pi} P\int_\mu^\infty d\nu' \operatorname{Im} F^{(\pm)}(\nu')\left[\frac{1}{\nu'-\nu} \pm \frac{1}{\nu'+\nu}\right] \tag{2}$$

The dispersion relation for $\frac{F^\pm(\nu)}{\nu}$ would be obtained as

$$\operatorname{Re}\left[\frac{F^{(\pm)}(\nu)}{\nu}\right] = -\operatorname{Re}\frac{F^\pm(0)}{\nu} \pm \frac{f^2}{4\pi}\left[\frac{1}{\nu_0(\nu_0-\nu)} \pm \frac{1}{\nu_0(\nu_0+\nu)}\right] + \frac{1}{\pi} P\int_\mu^\infty \operatorname{Im} F^{(\pm)}(\nu')\,d\nu'\left[\frac{1}{\nu'(\nu'-\nu)} \mp \frac{1}{\nu'(\nu'+1)}\right] \tag{3}$$

The term $-\operatorname{Re}\frac{F^{(\mp)}(0)}{\nu}$ comes from the additional pole at $\nu = 0$.

The second dispersion relation does not require as good a knowledge of $\operatorname{Im} F(\nu)$ at $\nu \to \infty$ as the first one does. However, in the second dispersion relation, we need to know an additional parameter .

The second dispersion relation is known as a 'subtracted' dispersion relation, because the second dispersion relation can be obtained by taking the first relation and formally subtracting from it $\operatorname{Re} F^{(\mp)}(0)$ as obtained from the first relation.

$$\operatorname{Re} F^{(+)}(0) = \frac{f^2}{4\pi}\left(\frac{1}{\nu_0} + \frac{1}{\nu}\right) + \frac{1}{\pi}\int_\mu^\infty \operatorname{Im} F^-(\nu')\,d\nu'\left(\frac{1}{\nu'} \cdots\right) \tag{4}$$

Subtracting (4) from (2) gives

$$\mathrm{Re}\, F^{(+)}(\nu) - \mathrm{Re}\, F^{(+)}(0) = \frac{f^2}{4\pi}\left(\frac{2\nu_0}{\nu_0^2-\nu^2} - \frac{2}{\nu_0}\right)$$

$$+ \frac{1}{\pi} P \int_{\mu}^{\infty} \mathrm{Im}\, F^{(+)}(\nu') \left[\frac{2\nu'}{\nu'^2-\nu^2} - \frac{2}{\nu'} \right] d\nu' \qquad (5)$$

which is exactly the relation (3).

We can use this method of subtraction to get rid of divergences as long as $F(\nu)$ behaves as a polynomian at infinity.

$$\mathrm{Im}\, F(\nu) \xrightarrow[|\nu|\to\infty]{} \nu^{N-1} \qquad (6)$$

then we can write a dispersion relation with N subtractions, obtaining

$$\mathrm{Re}\, F(\nu) = \left\{ \begin{matrix} \text{pole} \\ \text{term} \end{matrix} \right\} + \sum_{n=0}^{N-1} F^{(n)}(0) + \frac{(\nu-\nu_0)^N}{\pi} \int_{\mu}^{\infty} \frac{\mathrm{Im}\, F(\nu')\, d\nu'}{(\nu'-\nu)(\nu'-\nu_0)}$$

$$+ \frac{(\nu-\nu_0)^N}{\pi} \int_{-\infty}^{\mu} \frac{\mathrm{Im}\, F(\nu')}{(\nu'-\nu)(\nu'-\nu_0)^N}\, d\nu' \qquad (7)$$

This can be obtained by writing the Cauchy relation for $\frac{F(\nu)}{\nu^N}$: the same relation can also be obtained formally by subtracting, provided we make each subtraction at a different point ν_α , and only finally let each $\nu_\alpha \to 0$.

Note: If we introduce one subtraction at ν_0 , we get a constant $\mathrm{Re}\, F^{+}(\nu_0)$.

However,

$$\mathrm{Re}\, F^{(+)}(\nu_0) = \mathrm{Re}\, F^{(+)}(-\nu_0)$$

So we might as well do another subtraction at $\nu = -\nu_0$.

A convenient subtraction point is the point $\nu = \mu$, or zero kinetic energy. The dispersion relation with two subtractions, one at $\nu = \mu$ and the other at $\nu = -\mu$, is given by writing the Cauchy relation for the function

$$F(\nu)/(\nu-\mu)(\nu+\mu)$$

i.e.,

$$F(\nu) = \oint \frac{F(\nu')}{(\nu'-\nu)(\nu'+\mu)(\nu'-\mu)} d\nu' \tag{8}$$

We then obtain the dispersion relation

$$\mathrm{Re}\left[\frac{F^{(+)}(\nu)}{(\nu-\mu)(\nu+\mu)}\right] = \frac{\mathrm{Re}\,F^{(+)}(\nu)}{2\mu(\mu-\nu)} + \frac{\mathrm{Re}\,F^{(+)}(\nu)}{(\mu+\nu)2\mu}$$

$$= \frac{f^2}{4\pi}\left[\frac{1}{(\nu_0-\nu)(\nu_0^2-\mu^2)} + \frac{1}{(\nu_0+\nu)(\nu_0^2-\mu^2)}\right]$$

$$+ \frac{1}{\pi} P\int_\mu^\infty \mathrm{Im}\,F(\nu')\,d\nu' \frac{1}{\nu'^2-\mu^2}\left[\frac{1}{\nu'-\nu} + \frac{1}{\nu'+\nu}\right] d\nu' \tag{9}$$

<u>or</u> for real ,

$$\frac{\mathrm{Re}\,F^{(+)}(\nu)}{\nu'^2-\mu^2} - \frac{\mathrm{Re}\,F^{(+)}(\mu)}{\nu^2-\mu^2} = \frac{f^2}{4\pi}\frac{2\nu_0}{(\nu_0^2-\mu^2)(\nu_0^2-\nu^2)}$$

$$+ \frac{P}{\pi}\int_\mu^\infty \frac{2\nu'\,\mathrm{Im}\,F(\nu')\,d\nu'}{(\nu'^2-\mu^2)(\nu'^2-\nu^2)} \tag{10}$$

$$\mathrm{Re}\,F^{(+)}(\nu) - \mathrm{Re}\,F^{(+)}(\mu) = 2(\nu^2-\mu^2)\left\{\frac{f^2}{4\pi}\frac{\nu_0}{(\nu_0^2+\mu^2)(\nu_0^2}\right.$$

$$+ \frac{1}{8\pi^2} P\int_\mu^\infty \frac{\nu}{\sqrt{\nu'^2-\mu^2}}\,\frac{\sigma_+(\nu')+\sigma_-(\nu')}{\nu'^2-\nu^2}\,d\nu' \tag{11}$$

$$\operatorname{Im} F(\nu) = \frac{\sqrt{\nu'^2 - \mu^2}}{8\pi} \left[\sigma_+(\nu) + \sigma_-(\nu) \right] \qquad (12)$$

This dispersion relation has been written in the nucleon rest system. In the centre-of-mass system, we obtain

$$\frac{\operatorname{Re} \mathcal{F}^{(+)}(\nu)}{\operatorname{Re} f(\nu)} = q \Big/ \sqrt{\nu^2 - \mu^2} \qquad (13)$$

In the c.m.s. we know the amplitude from the phase shifts and the scattering length; the amplitude in the lab. system can be obtained using (13). We shall discuss one other example of subtraction in deriving the Pomeranchuk theorem.

If

$$\begin{aligned} \sigma_+(\nu) &\rightarrow C_+ , \quad |\nu| \rightarrow \infty \\ \sigma_-(\nu) &\rightarrow C_- , \quad |\nu| \rightarrow \infty \end{aligned} \qquad (14)$$

and if $C_+ \neq C_-$, then we must write subtracted dispersion relations for both $f^{(+)}(\nu)$ and $f^{(-)}(\nu)$. Making two subtractions for $f^{(-)}$ at $\nu = 0$, we obtain

$$\operatorname{Re} F^{(-)}(\nu) = \operatorname{Re} F^{(-)}(0) + \nu \operatorname{Re} F'^{(-)}(0) - \frac{f^2}{4\pi} \frac{2\nu^3}{\nu_0^2(\nu_0^2 - }$$
$$+ \frac{\nu^3}{\pi} \int_\mu^\infty 2 \frac{\operatorname{Im} F^{(-)}(\nu')}{\nu'^2 - \nu^2} d\nu' \qquad (15)$$

But $\operatorname{Re} F^{(-)}(0) = 0$, from crossing symmetry. Therefore we have a dispersion relation with only one parameter $\operatorname{Re} F'^{(-)}(0$

$$\operatorname{Re} F^{(-)}(\nu) = \nu \operatorname{Re} F'^{(-)}(0) - \frac{f^2}{4\pi} \frac{2\nu^3}{\nu_0^2(\nu_0^2 - \nu^2)} \qquad (16)$$
$$+ \nu^3/\pi \int_\mu^\infty \frac{2 \operatorname{Im} F^{(-)}(\nu') d\nu'}{\nu'^2 - \nu^2}$$

Replacing $\mathcal{I}m\, F(\nu)$ by the value in terms of the total cross-section, we obtain

$$\mathrm{Re}\, F^{(-)}(\nu) = \nu \Big\{ \mathrm{Re}\, F'^{(-)}(0) - \frac{b^2}{4\pi} \frac{2\nu^2}{\nu_0^2(\nu_0^2-\nu^2)} + \frac{\nu}{4\pi^2} P \int_{\mu}^{\infty} \frac{\sqrt{\nu'^2-\mu^2}\,[\sigma_-(\nu') - \sigma_+(\nu')]}{\nu'^2(\nu'^2-\nu^2)} d\nu' \qquad (17)$$

For $|\nu| \to \infty$, we can prove that the second principal value integral in (17) behaves as

$$(\log \nu)(\sigma_-(\infty) - \sigma_+(\infty)) \qquad (18)$$

so that

$$\mathrm{Re}\, F^{(-)}(\nu) \sim \nu C + (\nu \log \nu)[\sigma_-(\infty) - \sigma_+(\infty)] \qquad (19)$$

For the imaginary part of $F^{(-)}(\nu)$, we have

$$\mathcal{I}m\, F^{(-)}(\nu) \sim \nu[\sigma_-(\nu) - \sigma_+(\nu)] \qquad (20)$$

We have thus obtained the peculiar result that the ratio

$$\mathrm{Re}\, F(\nu)/\mathcal{I}m\, F(\nu) \sim \log \nu \quad \text{as } |\nu| \to \infty \qquad (21)$$

i.e., that $\mathrm{Re}\, F(\nu) \gg \mathcal{I}m\, F(\nu)$ (22)

This is contrary to what we expect from the fact that a large number of channels are open at high energies, so that we expect almost complete absorption.

$$\text{e.g.} \quad f_\ell = \frac{\eta_\ell - 1}{2iq} = \frac{\rho_\ell e^{2i\delta} - 1}{2iq} \qquad (23)$$

For pure absorption, $\delta = 0$, and

$$\sigma_{el} = \sigma_{inel} \tag{23}$$

$$Re\, F / g_m F \rightarrow 0 \tag{23}$$

The only way to reconcile this picture with the result obtained from dispersion relations is to assume that

$$\left[\sigma_-(\infty) - \sigma_+(\infty) \right] = 0 \tag{24}$$

But then $\dfrac{Re\, F^{(-)}(\nu)}{g_m F^{(-)}(\nu)} \rightarrow \infty$, as $|\nu| \rightarrow \infty$. But we note that when $\sigma_-(\infty) - \sigma_+(\infty) = 0$, when we can writ an unsubtracted dispersion relation.

The result

$$\sigma_-(\infty) = \sigma_+(\infty) \tag{24}$$

is one of Pomeranchuk's theorems.

There are other ways of proving this relation also.

6. <u>Fixed momentum-transfer Dispersion Relations</u>.

For forward scattering, we had essentially only one amplitude F given by

$$F = A + \nu B \tag{25}$$

where
$$A = A(s = s, t = 0)$$

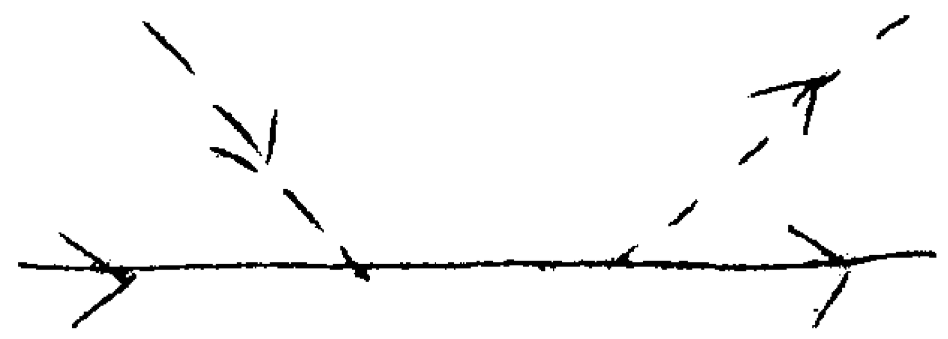
and
$$B = B(s = s, t = 0)$$

For a fixed value of t, the dispersion relation obtained is

$$\mathrm{Re}\, A^{(\pm)}(s,t) = \frac{P}{\pi}\int_{\mu}^{\infty} \mathrm{Im}\, A^{\pm}(s',t)\left[\frac{1}{s'-s} \pm \frac{1}{s'+s-2m^2-2\mu^2+t}\right] ds' \tag{26a}$$

$$\mathrm{Re}\, B^{(\pm)}(s,t) = \frac{g^2}{m^2-s} \mp \frac{g^2}{s-m^2-2\mu^2+t} + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \mathrm{Im}\, B^{(\pm)}(s',t) \times \left[\frac{1}{s'-s} \mp \frac{1}{s'+s-2m^2-2\mu^2+t}\right] \tag{26b}$$

Note that the nucleon pole contributes only to the B amplitude and not to A. This may be seen by writing the amplitude for the pole diagrams.



Direct pole

Crossed pole

What Chew, Goldberger, Low & Nambu did was that they assumed the fixed t dispersion relations and expanded $\operatorname{Im} A^{(\pm)}(s,t)$ in term of partial-wave amplitudes, and assumed that only a few partial waves gave an appreciable contribution to $\operatorname{Im} A(s,t)$.

For π-N Scattering at low energies, the $(3\,3)$ resonance dominates, what was done by CGLN was to replace $\operatorname{Im} A$ by the contribution of the $(\frac{3}{2},\frac{3}{2})$ resonance.

Note: We require the scattering amplitude for unphysica values for part of the range.

7. <u>Dispersion Relations for the Spin-flip Amplitude.</u>

We saw that in the c.m.s. we have two amplitudes f_1 and f_2. f_2 is the spin-flip amplitude, and f_1 is the amplitude without spin-flip. At low energies, the spin-flip amplitude is given approximately by B . In the forward directio we have

$$\operatorname{Re} B^{(\pm)}(s,0) = \frac{g^2}{m^2-s} - \frac{g^2}{s-m^2-2\mu^2} + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \operatorname{Im} B^{(\pm)}(s',0)\left[\frac{1}{s'-s} \mp \frac{1}{s'-s-2m^2-2\mu^2}\right] ds' \tag{27}$$

We have

$$\left.\begin{aligned} B^{3/2} &= B^{(+)} - B^{(-)} \\ \frac{1}{3}\left(B^{3/2} + 2B^{1/2}\right) &= B^{(+)} + B^{(-)} \end{aligned}\right\} \tag{28}$$

The dispersion relation in terms of $B^{3/2}$, $B^{1/2}$ may be written

$$\mathrm{Re}\, B^{3/2}(s,t) = \frac{2g^2}{s-m^2-2\mu^2} + \frac{P}{\pi}\int^{\infty} ds' \frac{\mathrm{Im}\, B^{3/2}(s',0)}{s'-s}$$

$$-\frac{1}{3\pi}\int_{(m+\mu)^2}^{\infty} \frac{B^{3/2}(s',0) + 2B^{1/2}(s',0)}{s'+s-2m^2-2\mu^2}\, ds' \qquad (29)$$

Note: $B^{1/2}(s,0)$ occurs only in the non-singular integral; as the $T=\frac{1}{2}$ amplitude is negligible, the $B^{1/2}$ term can be neglected.

Then we can write a dispersion relation (in the forward direction) for the amplitude $B^{3/2}$, in terms of the variable

$$\nu = (s-m^2-\mu^2)/2m$$

We finally obtain

$$m\left(\nu-\frac{\mu^2}{2m}\right)\left[-\mathrm{Re}\, B^{3/2}(\nu) + \frac{2}{3}\nu\frac{P}{\pi}\int_{\mu} \frac{2\nu'+\nu}{\nu'(\nu'^2-\nu^2)}\, \mathrm{Im}\, B^{3/2}(\nu')\, d\nu'\right.$$

$$= g^2 - m\left(\nu-\frac{\mu^2}{2m}\right)\frac{2}{3}\frac{1}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{Im}\, B^{3/2}(\nu')\, d\nu'}{\nu'} \qquad (30)$$

where we have, as we did earlier, split the integral into two part one that converges better and the other being a constant independe of ν. To do this we have used

$$\int \frac{\mathrm{Im}\, B^{3/2}(\nu')\, d\nu'}{(\nu'-\nu)} = \nu\int_{\mu}^{\infty}\frac{2\nu'+\nu}{\nu'(\nu'-\nu)}\, \mathrm{Im}\, B^{3/2}(\nu')\, d\nu'$$

$$+\int_{\mu}^{\infty}\frac{\mathrm{Im}\, B^{3/2}(\nu')\, d\nu'}{\nu} \qquad (31)$$

(30) can be directly compared with experiment. We can plot the left hand side of (30) as a function of ν ; we should obtain a linear relation; the intercept on the y axis gives the coupling constant g^2 .

This relation was useful in distinguishing the Fermi and Yang phase shifts. The two sets of phase shifts gave an equally good fit to the angular distribution and cross section. The Yang phase shifts gave, in addition to the $(\frac{3}{2}, \frac{3}{2})$ resonance, a $P_{1/2}$ resonance at 136 Mev, which would be masked by the rapid residue near the $(3/2, 3/2)$ resonance λ. However the Yang set of phase shifts does not give a linear relation;

This is like making a polarization experiment, as we are measuring the spin-flip amplitude.

The spin flip amplitude is zero at $\theta = 0$, but we have written a relation for the coefficient of the factor that goes to zero.

8. Dispersion Relations from Field Theory.

In the last few lectures we saw how dispersion relations could be written down by assuming that the scattering amplitude had just the singularities required by unitarity and crossing symmetry.

We also saw that these dispersion relations, which were just integral relations between the real and imaginary parts of the scatt. amplitude were very useful, e.g. in providing a simple method of determining the π-N coupling constant, in providing a deduction of the Pomeranchuk theorem, and in distinguishing between the Fermi and Yang sets of phase shifts for π N scattering

In this lecture we shall consider how far these dispersion relations can be proved starting with the axioms of field theory.

This question has been well discussed in the review talks by Goldberger and by Jackson *. Some details of the proof are not given in these but may be found in the book by Bogoliubov and Shirkov**

Consider the elastic scattering on a nucleon of a spinless (scalar or pseudoscalar) meson, described by the 'free' wave equation

$$(\mu^2 - \Box)\,\phi(x) = 0 \tag{1}$$

where the d'Alambertean $\Box$ is given by

$$\Box = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x_0^2} \tag{1a}$$

The mass in this K.G. equation is the physical mass; at the very outset we have all physical quantities. The concept of 'bare' states and 'bare' operators is not introduced at all. For an interacting field (e.g a field interacting with a source current $j(x)$); the equation is

$$(\mu^2 - \Box)\,\phi(x) = j(x) \tag{2}$$

Let $\{ \sigma_\alpha(x) \}$ be a set of bounded wave functions that are

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* Relations de dispersion et particules élémentaires, M.L. Gold Dispersion Relations, Ed. by Screaton G.R, Oliver & Boyd [bc

** Introduction to the theory of quantized fields, Interscience solutions of the Klein-Gordon wave equation:

$$(\mu^2 - \Box) f_\alpha(x) = 0 \tag{3}$$

Choose a normalization that is independent of time, such a normalization is defined by the equation

$$i \int \left\{ f_\beta^*(x) \frac{\partial}{\partial x_0} f_\alpha(x) - \frac{\partial}{\partial x_0} f_\beta^*(x) f_\alpha(x) \right\} d^3x = \delta_{\alpha\beta} \tag{4}$$

On taking the derivative of this with reference to time, we obtain an equation which can be reduced to

$$i \int \left[f_\beta^*(x) \left\{ \frac{\partial^2}{\partial x_0^2} f_\alpha(x) \right\} - \left\{ \frac{\partial^2}{\partial x_0^2} f_\beta^*(x) \right\} f_\alpha(x) \right] d^3x = \delta_{\alpha\beta}$$

Using the wave equation, $\partial^2/\partial x_0^2$ may be replaced by

$$\left(\partial^2/\partial x^2 - \mu^2 \right)$$

Then, on doing an integration by parts, and using the fact that the functions $f_\alpha(x)$ are bounded, the derivation of the left hand side of (4) reduces to zero.

i.e., $$\frac{\partial}{\partial x_0}\left[i \int \left\{ f_\beta^*(x) \frac{\partial}{\partial x} f_\alpha(x) - \left[\frac{\partial}{\partial x_0} f_\beta^*(x) \right] f_\alpha(x) \right\} \right] = 0 \tag{5}$$

so that the normalization is independent of time.

The $f_\alpha(x)$ can be taken as wave packets or as waves in a box. A simpler procedure is to use δ- function normalization (which may be obtained by taking the volume of box V to tend to infinity).

$$f_k(x) = \frac{1}{\sqrt{2\omega_k}} e^{i\bar{k}.\bar{x} - k_0 x_0} \tag{6}$$

Any field operator $\phi(x)$ may be expanded in the form

$$\phi(x) = \sum_\alpha f_\alpha(x)\phi_\alpha + f_\alpha^*(x)\,\phi_\alpha^\dagger \tag{7}$$

Using the orthogonality relation between $f_\alpha(x)$, we obtain

$$\begin{aligned} \phi_\alpha^\dagger &= i\int \phi(x) \overleftrightarrow{\frac{\partial}{\partial x_0}} f_\alpha(x)\, d^3x \\ \phi_\alpha &= -i\int \phi(x) \overleftrightarrow{\frac{\partial}{\partial x_0}} f_\alpha^*(x)\, d^3x \end{aligned} \tag{8}$$

For a free field, if we use wave functions which are solutions of the wave equation, we have time-independent operators ϕ_α. If we use functions that are not solutions of the K.G. equation, then the operators are, in general, functions of time.

For interacting fields, we obtain, in general, time-dependant operators $\phi_\alpha(x)$, if we use wave functions which are solutions of the K.G. equation.

We now assume that we have a field which describes the system. Assume that $\phi_\alpha^\dagger$, ϕ_α are the limits, for infinite time, of operator functions $\phi_\alpha(x)$

Define

$$\phi^\dagger_{\alpha\,\substack{in\\out}} = \lim_{t \to \mp\infty} \phi_\alpha^\dagger(x_0) \tag{9}$$

Define this as the weak-limit:

$$\langle m' | \phi_{\alpha\,\substack{in\\out}} | m \rangle = \lim_{t \to \mp\infty} \langle m' | \phi_\alpha(x_0) | m \rangle \tag{10}$$

The operators $\phi_{\alpha\, in}$ and $\phi_{\alpha\, out}$ define two sets of basis wave functions in Hilbert space, $\phi^\dagger_{\alpha\, in} |0\rangle$ and $\phi^\dagger_{\alpha\, out} |0\rangle$. We obtain

$$\langle m' | \phi_{out} | m \rangle - \langle m' | \phi_{\alpha\, in} | m \rangle = \int_{-\infty}^{\infty} \frac{\partial}{\partial x_0} \langle m' | \phi_\alpha(x_0) | m \rangle dx_0 \tag{11}$$

The S-matrix element is given by

$$S_{\beta\alpha} = \langle out\, q_2 p_2 | q_1 p_1, in \rangle$$

Note: $|p_2\rangle$, $|q_2\rangle$ etc. are physical one-particle states.

$$\langle p_2 | \phi_{q_2\, out} | q_1 p_1, in \rangle \tag{12}$$

$$= \langle p_2 | \phi_{q, in} | q_1 p_1\, in \rangle + \int_{-\infty}^{\infty} \frac{\partial}{\partial x_0} \langle p_2 | \phi_{q_2}(x_0) | q_1 p_1 \rangle\, dx_0$$

$$= \delta_{\beta\alpha} - i \int d^4x \frac{\partial}{\partial x_0} \langle p_2 | \phi(x) | q_1 p_1\, in \rangle \overleftrightarrow{\frac{\partial}{\partial x_0}} f^*_{q_2}(x) \tag{13}$$

$$= \delta_{\beta\alpha} - i \int d^4x \int \Big[\langle p_2 | \phi(x) | q_1 p_1\, in \rangle \frac{\partial^2}{\partial x_0^2} f^*_{q_2}(x) \tag{14}$$

$$- \langle p_2 | \frac{\partial^2}{\partial x_0^2} \phi(x) | p_1 q_1\, in \rangle f^*_{q_2}(x) \tag{15}$$

Write $\partial^2/\partial x_0^2 = \frac{\partial^2}{\partial x^2} - \mu^2$

The operation with $\partial^2/\partial x^2$ can be performed.

This finally gives

$$\delta_{\beta\alpha} - i\int d^4x \left[\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x_0^2} - \mu^2 \right) \langle p_2 | \phi(x) | q_1 p \right.$$

$$= \delta_{\beta\alpha} - i\int d^4x \langle p_2 | j(x) | q_1 p_1 in \rangle f^*_{q_2}(x)$$

$$\text{where } j = -(\Box - \mu^2)\phi(x) \tag{16}$$

We have thus obtained

$$S_{\beta\alpha} = \delta_{\beta\alpha} + i\int d^4x \langle p_2 | j(x) | p_1 q_1 in \rangle f^*_{q_2}(x) \tag{17}$$

This is the reduction formula. We have reduced the matrix element of S between two 2-particle states to the matrix element of a current operator between a 2-particle state and a one-particle state. We can repeat the above procedure with the other meson operator $\phi^\dagger_{q_1 in}$

We write

$$\langle p_2 | j(x) | p_1 q_1 in \rangle = \langle p_2 | j(x) \phi^\dagger_{q_1, in} | p_1 \rangle$$

$$= \langle p_2 | \left([j(x), \phi^\dagger_{q_1, in}] + \phi^\dagger_{q_1, in} j(x) \right) | p_1 \rangle$$

The second term gives zero, since we have

$$\langle p_2 | \phi^\dagger_{q_1 in} = 0$$

We can write this as

$$\lim_{y_0 \to -\infty} \langle p_2 | j(x)\, \phi^{\dagger}_{q_1}(y_0) | p_1 \rangle$$

Then one has to express the requirement that $j(x)$ operates only after $\phi_{q_1}{}^{\dagger}(y_0)$ has operated. This can be done in different ways; we shall do this by writing a factor $\theta(x_0 - y_0)$. We then obtain

$$\langle p_2 | j(x) | p_1 q\, in \rangle = \langle p_2 | j(x)\, \phi^{\dagger}_{q\, in} | p_1 \rangle$$

$$= \langle p_2 | [\, j(x), \phi^{\dagger}_{q, in}\,] | p_1 \rangle$$

$$= \lim_{y_0 \to -\infty} \langle p_2 | \theta(x_0 - y_0) [\, j(x), \phi^{\dagger}_{q, in}\,] | p_2 \rangle \qquad (18)$$

$$= \lim_{y_0 \to -\infty} i \langle p_2 | R(j(x), \phi^{\dagger}_{q_1}(y_0) | p_1 \rangle \qquad (19)$$

where $$R(j(x), \phi^{\dagger}_{q_1}(y_0)) = \frac{1}{i}\, \theta(x_0 - y_0) \times [\, j(x), \phi^{\dagger}{}_{q_1}(y_0)\,] \qquad (20)$$

$$= \lim_{y \to +\infty} i \langle p_1 | R(j(x), \phi^{\dagger}_{q_1}(y_0)) | p_1 \rangle - \int_{-\infty}^{\infty} \frac{\partial}{\partial y_0} \langle p_2 | R(j(x), \phi^{\dagger}_{q_1}(y_0)) | p_1 \rangle\, dy_0 \qquad (21)$$

The first term vanishes.

We thus obtain

$$\langle p_2 | j(x) | p_1 q_1\, in \rangle = -i \int_{-\infty}^{\infty} \frac{\partial}{\partial y_0} \langle p_2 | R(j(x), \phi^{\dagger}_{q_1}(y_0))$$

$$= \int \frac{\partial}{\partial y_0} \langle p_2 | R(j(x), \phi(y)) | p_1 \rangle \frac{\overleftrightarrow{\partial}}{\partial y_0} f_{q_1}(y)\, d^4 y \qquad (22)$$

on writing

$$\phi_{q_1}^{\dagger}(y_0) = i \int \phi(y) \frac{\overleftrightarrow{\partial}}{\partial x_0} f_\alpha(y)\, d^3 y$$

This can be reduced to the following

$$\int \frac{\partial}{\partial y_0}\Big(-i\,\theta(x_0-y_0)[j(x),\phi(y)]\frac{\partial}{\partial y_0}f_{q_1}(y)$$
$$-i\,\theta(x_0-y_0)[j(x)\frac{\partial}{\partial y_0}\phi(y_0)]f_{q_1}(y)\Big)\,d^4y$$
$$=\int\Big\{i\,\delta(x_0-y_0)\Big([j(x),\phi(y)]\frac{\partial}{\partial y_0}f_{q_1}(y)-f_{q_1}(y)$$
$$\times[j(x),\frac{\partial}{\partial y_0}\phi(y)]\Big)-i\theta(x_0-y_0)[j(x),\big(j(x),$$
$$\big(\phi(y)\frac{\partial^2}{\partial y_0^2}f_{q_1}(y)-\frac{\partial^2}{\partial y_0^2}\phi(y)\big)f_{q_1}(y)\big)$$

Write

$\frac{\partial^2}{\partial y_0^2}$ as $\frac{\partial^2}{\partial y^2}-\mu^2$

Integrating by parts, we obtain $\left(\frac{\partial^2}{\partial x^2}-\mu^2\right)\phi(y)=j(y)$

which gives a factor

$$-i\,\theta(x_0-y_0)[j(x),j(y)]=R(j(x),j(y))f_{q_1}(y)$$

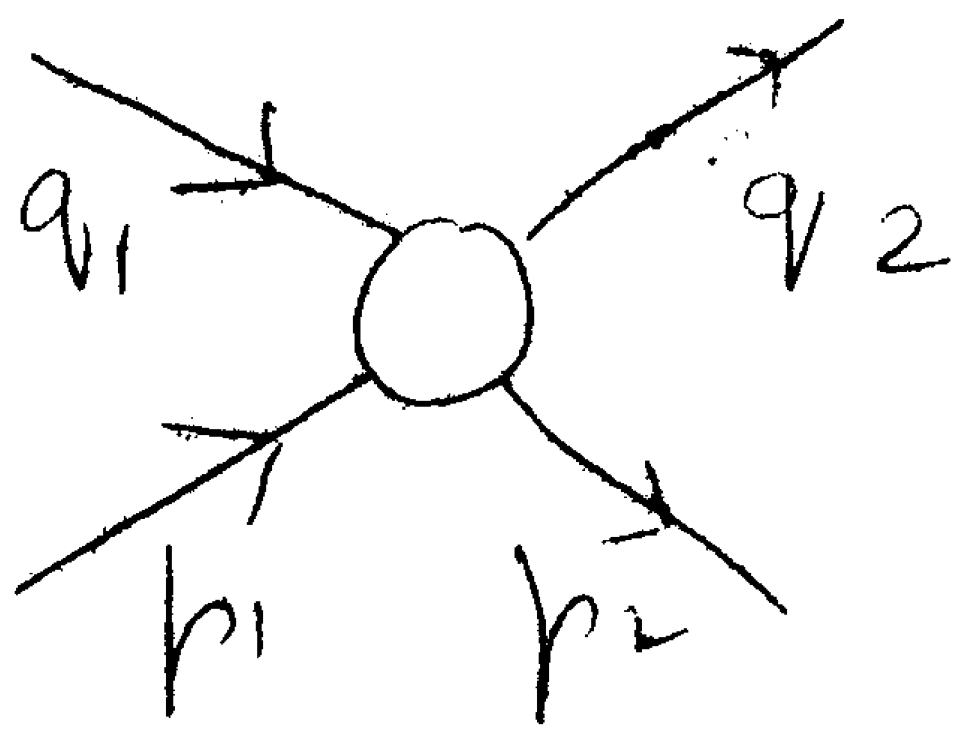
Finally we have

$$\langle p_2|\,j(x)\,|\,p_1q_1\,in\rangle=$$
$$=\int d^4y\,\langle p_1|\Big\{R(j(x),j(y))\,f_{q_1}(y)$$
$$+i\,\delta(x_0-y_0)\Big([j(x),\phi(y)]\frac{\partial}{\partial y_0}f_{q_1}(y)$$
$$-f_{q_1}(y)[j(x),\frac{\partial}{\partial y_0}\phi(y)]\Big\}\qquad (23)$$

The overall result is the following:

$$S_{\beta\alpha} = \delta_{\beta\alpha} - i\int d^4x\, d^4y \langle p_2 | R(j(x), j(y)) \, f^*_{q_2}(x)\, f_{q_1}(y) + \delta \text{ terms}; \tag{24}$$

We have thus reduced the S-matrix element between 2-particle states as the matrix element of the retarded product of 2-currents between one-particle nucleon states.



We could have reduced the S-matrix element into the matrix element of the retarded product of nucleon-current operators between one-particle meson states.

Ref: M.L. Goldberger: loc.cit.

J.D. Jackson: loc.cit.

The $\delta(x_0 - y_0)$ term will give a term in terms proportional to $e^{i(q_1-q_2)x}$ which is just $e^{i(p_2-p_1)^2 x}$ if the commutator $[j(x), \frac{\partial}{\partial y_0}\phi(y)]$ is a δ function. If the latter is a derivation of a δ function, we would obtain

This is the reason the δ function is neglected in a discussion of the analyticity properties.

Taking

$$f_q(x) = e^{-iqx}/\sqrt{2q_0}$$

we obtain

$$S_{\beta\alpha} - \delta_{\beta\alpha} = i\frac{1}{\sqrt{2q_{10}\,2q_{20}}}\int d^4x\,d^4y\; e^{-iq_2x}\, e^{-iq_1y} \times \langle p_2 | R(j(x), j(y)) | p_1\rangle$$

+ polynomials in p_1, p_2 coming from the δ function terms. (25)

For pion-nucleon scattering, we have

$$S_{\beta\alpha} = \delta_{\beta\alpha} + i(2\pi)^4 \frac{F_{\beta\alpha}\, 2m}{\sqrt{2p_{10}\,2p_{20}\,2q_{10}\,2q_{20}}} \tag{26}$$

Take any field $\phi(x)$; the operator obtained by a translation $\underline{a}$ will be

$$\phi(x) = e^{-i(P\cdot a)}\,\phi(x-a)\,e^{i(P\cdot a)} \tag{27}$$

Keeping integral term alone in (25), we may write (25) as

$$e^{-iP_2y}\, R(j(x-y), j(0))\, e^{iP_1y} = e^{i(p_1+q_1-p_2-q_2)y}\, e^{-iq_2(x-y)}$$

We can then write the expression for the Feynman amplitude

$$F_{\beta\alpha} = -\sqrt{p_{10}p_{20}}/m \int d^4x\, e^{-iq_2x} \langle p_2 | R(j(x), j(0)) | p_1\rangle \tag{28}$$

We could have performed a more symmetrical translation obtain $R(j(\frac{x}{2}), j(-\frac{x}{2}))$ instead of the form obtained above.

First consider forward pion-nucleon scattering.

In the nucleon frame, we may write the forward scattering amplitude as

$$F_{\alpha\alpha} = -\int d^4x\, e^{-iq_2 x} \langle p | R(j(x), j(0)) | p \rangle$$

It is easy to see that this cannot be a function of the pion momentum.

[The most general vector is $f_1 + \vec{\sigma} \cdot \vec{f}_2$ where $\vec{f}_2$ must be a pseudovector. However, we have only 1 vector, the meson momentum, at our disposal, and we cannot construct a pseudovector.]

The matrix element will depend only on the meson energy ν ; $F_{\beta\alpha} = F_{\beta\alpha}(\nu)$ We obtain for the amplitude at an energy $-\nu$,

$$F_{\alpha\alpha}(-\nu) = -\int d^4x\, e^{+i(-\bar{q}_2 \cdot \bar{x} - q_{20} x_0)} \langle p | R(j(x), j(0)) | p \rangle$$

Since this does not depend on the direction of $\bar{q}_2$, we can change the sign of $\vec{q}_2$, obtaining

$$F_{\alpha\alpha}(-\nu) = -\int d^4x\, e^{+i(\bar{q}_2 \bar{x} - q_{20} x_0)} \langle p | R(j(x), j(0)) | p \rangle$$

i.e.,

$$F_{\alpha\alpha}(-q_{20}) = F^*_{\alpha\alpha}(q_{20})$$
$$\mathrm{Re}\, F(\nu) = \mathrm{Re}\, F(\nu)$$
$$\mathrm{Im}\, F(-\nu) = -\mathrm{Im}\, F(-\nu)$$

For forward scattering with charge-exchange, we must specify the charge-indices say (i, j)

$$\therefore\ F_{\alpha\alpha\,ij}(-\nu) = F^{*}_{\alpha\alpha\,ji}(\nu) = F^{\dagger}_{\alpha\alpha\,ij}(\nu)$$

if we include the charge indices also.

We have already obtained the reduction formula for pion-nucleon scattering, obtaining

$$F = -\sqrt{E_1E_2}/m \int d^4x\, e^{-iq_2x} \langle p_2 | R(j(x), j(0)) | p_1\rangle \tag{1}$$

or, for forward scattering,

$$F = -\int d^4x\, e^{-iq_2x} \langle p | R(j(x), j(0)) | p\rangle \tag{1a}$$

We saw that the forward scattering amplitude thus obtained obeyed the crossing relations. We can write this as

$$F = -\int e^{-iq_{20}x_0} dx_0 \int r^2 dr\, d\varphi\, d(\cos\theta)\, e^{-iq_2 r\cos\theta} \langle p | R(j(x), j(0)) | p\rangle$$

$$= -4\pi \int e^{iq_{20}x_0} dx_0 \int dr\, \frac{r}{q_2} \sin q_2 r \langle p | R(j(x), j(0)) | p\rangle \tag{2}$$

$$F^{*}(-q_{20}) = -4\pi \int dx_0\, e^{+iq_{20}x_0} dr\, \frac{r}{q_2} \sin q_2 r\, e^{+iq_{20}x_0} \langle p | R(j(x), j(0)) | p\rangle \tag{3}$$

which is just the crossing symmetry when we ignore the charge index. If we include the charge index, we will obtain

$$F_{\beta\alpha} = -4\pi i \int dx_0 \int dr \, . \, \frac{r}{q_2} \sin q_2 r \, \langle p | R(j_\beta(x), j_\alpha(0)) \, e^{i q_{20} x_0} \tag{4}$$

so that

$$F^{\times}_{\beta\alpha}(-q_{20}) = -4\pi \int dx_0 dr \, \frac{r}{q_2} \sin(q_2 r) \, e^{+i q_{20} x_0} \langle p | R(j(x), j \qquad \tag{5}$$
$$= F_{\alpha\beta}(q_{20})$$

Writing

$$F = F^{(+)} \delta_{\beta\alpha} + \frac{1}{2} [\tau_\beta, \tau_\alpha] F^{(-)} \tag{6}$$

we obtain

$$\begin{aligned} F^{(+)}(-q_2) &= F^{(+)*}(q_{20}) \\ F^{(-)}(-q_{20}) &= -F^{(-)}(q_{20}) \end{aligned} \tag{7}$$

For the real and imaginary parts, we obtain

$$\left.\begin{aligned} \mathrm{Re}\, F^{(\pm)}(-q_{20}) &= \pm \mathrm{Re}\, F^{(\pm)}(q_{20}) \\ \mathrm{Im}\, F^{(\pm)}(-q_{20}) &= \mp F^{(\pm)}(q_{20}) \end{aligned}\right\} \tag{8}$$

Write

$$F(\nu) = D(\nu) + i A(\nu) \tag{9}$$

where $D(\nu)$ and $A(\nu)$ are the real and imaginary parts of $F(\nu)$ when ν is real. For complex ν, they are the continuations of these, and are called the dispersive (D) and absorptive (A) parts, respectively, of the amplitudes.

We can obtain the expressions for $D(\nu)$ and $A(\nu)$ quite easily. We may write

$$F(\nu) = -\int d^4x\, e^{-iq_2x} \langle p| \tfrac{1}{i}\, \theta(x_0)\, [j(x), j(0)]|p\rangle$$

$$= i\int d^4x\, e^{-iq_2x} \langle p| \{ \tfrac{1}{2}\, \epsilon(x_0)\, [j(x), j(0)] + \tfrac{1}{2}[j(x), j(0)] \}|p\rangle \quad (1$$

$$= \frac{i}{2}\int d^4x\, e^{-iq_2x} \langle p|\, \epsilon(x_0)\, [j(x), j(0)]|p\rangle + \frac{i}{2}\int d^4x\, e^{-iq_2x} \langle p|[j(x), j(0)]|p\rangle \quad (11)$$

We shall prove that the first part is pure real and the second part pure imaginary.

<u>Proof:</u> Let the first term $\equiv D(\nu)$; the second term $= iA$

Write down $A^*(\nu)$

$$A^*(\nu) = \frac{1}{2}\int d^4x\, e^{iq_2x} \langle p|[j(x), j(0)]|p\rangle^*$$

$$= \frac{1}{2}\int d^4x\, e^{-iq_2x} \langle p|[j(-x), j(0)]|p\rangle \quad (12)$$

$$= A(\nu)$$

since the adjoint of the commutator $= (-1)$ the commutator of the adjoint. <u>Therefore $A(\nu)$ is real.</u>

Similarly we may obtain

$$D^*(\nu) = D(\nu) \text{ for real } \nu \quad (13)$$

$$D^*(\nu) = -\frac{i}{2}\int d^3x \int dx_0\, e^{iq_2x} \langle p|\, \epsilon(x_0)[j(x), j(0)]$$

$$= -\frac{i}{2}\int d^3x \int d(-x_0)\, e^{-iq_2x} \langle p|\, \epsilon(x_0)\, [j(x), j(0)] \quad (13a)$$

i.e. <u>$D(\nu)$ is real.</u>

Note: $$j(x) = e^{-iP\cdot a}\, j(x-a)\, e^{iP\cdot a} \tag{14}$$

We now write

$$\begin{aligned} A(\gamma) &= \frac{1}{2}\int d^4x\, e^{-iq_2x} \langle p|[j(x), j(0)]|p\rangle \\ &= \frac{1}{2}\sum_n \int d^4x\, e^{-iq_2x}\big(\langle p|j(x)|n\rangle\langle n|j(0)|p\rangle \\ &\qquad - \langle p|j(0)|n\rangle\langle n|j(0)|p\rangle \end{aligned} \tag{15}$$

Write

$$\langle a|j(x)|b\rangle = e^{-iP_ax}\, j(0)\, e^{iP_bx} \tag{16}$$

We then obtain

$$\begin{aligned} A(\gamma) &= \frac{1}{2}\sum_n \int d^4x\Big(e^{-i(q_2+p_2-p_n)x} \langle p_2|j(0)|n\rangle\langle n|j(0)|p \\ &\quad - e^{-i(q_2+p_n-p_1)x} \langle p_2|j(0)|n\rangle\langle n|j(0)|p_1\rangle \\ &= \frac{1}{2}\sum_n (2\pi)^4\big[\delta(q_2+p_2-p_n)\langle p_2|j(0)|n\rangle\langle n|j(0)|p \\ &\quad - \delta(q_2+p_n-p_1)\langle p_2|j(0)|n\rangle\langle n|j(0)|p_1\rangle \end{aligned} \tag{17}$$

The first term is just what would be obtained by unitarity in the direct channel; the second part corresponds to the term obtained by crossing.

<u>Note</u>: This is the general expression for $p_1 \neq p_2$. We obtain the expression for forward scattering by putting $p_1 = p_2$.

In the following, we consider forward scattering, with $p_1 = p_2 = p$

$$F(q_{20}) = 4\pi \int dr \int dx_0 \, r \frac{\sin(q_2 r)}{q_2} \langle p | R | j(x), j(0) | p \rangle e^{i q_{20} x_0}$$
$$= \int_0^\infty F_r(q_{20}) \, dr, \qquad (18)$$

where $F_r(q_{20}) = 4\pi \int dx_0 \, e^{i q_{20} x} \frac{r \sin(q_2 r)}{q_2}$
$$\times \langle p | R(j(x), j(0)) | p \rangle \qquad (18a)$$

Therefore

$$F_r(q_{20}) = 4\pi r \frac{\sin\left(\sqrt{q_{20}^2 - \mu^2}\, r\right)}{\sqrt{q_{20}^2 - \mu^2}} \int_0^\infty dx_0 \, e^{i q_{20} x_0} \langle p | R(j(x), j(0)) \qquad (19)$$

We have assumed till now that there was a free pion field at $t = +\infty$ and $t = -\infty$ which was the limit as $t \to \pm\infty$ of an interpolating pion field.

Till now we have not used the idea of micro-causality. We now introduce this concept.

<u>Micro causality</u> is the postulate that $[j(x), j(0)]$ is $= 0$ for $r^2 > x_0^2$

[i.e. space like separations] (20)

where $r^2 = \vec{x}^2$ i.e. $x = (0, \vec{r})$ (20a)

We require the properties of the expression (19) as a function of q_{20}. Behaviour at $q_{20} \to \infty$. When $\mathrm{Im}\, q_{20} > 0$, the factor $\sin \sqrt{q_{20}^2 - \mu^2}\, r$ has an increasing exponential,

$e^{+\operatorname{Im} q_{20} r}$; however, the integrand has a decreasing exponential factor $e^{-\operatorname{Im} q_{20} r}$. <u>When we assume micro causality, the lower limit in (19) becomes r and not zero (since the integrand is $= 0$ for $x_0 < r$</u>). Therefore the decreasing exponential factor $e^{-\operatorname{Im} q_{20} x_0}$ always damps the divergence from the factor $e^{\operatorname{Im} q_{20} r}$, since $x_0 > r$, and the expression (19) is well-behaved.

On the contrary, for $\operatorname{Im} q_{20} < 0$, the expression (19) diverges.

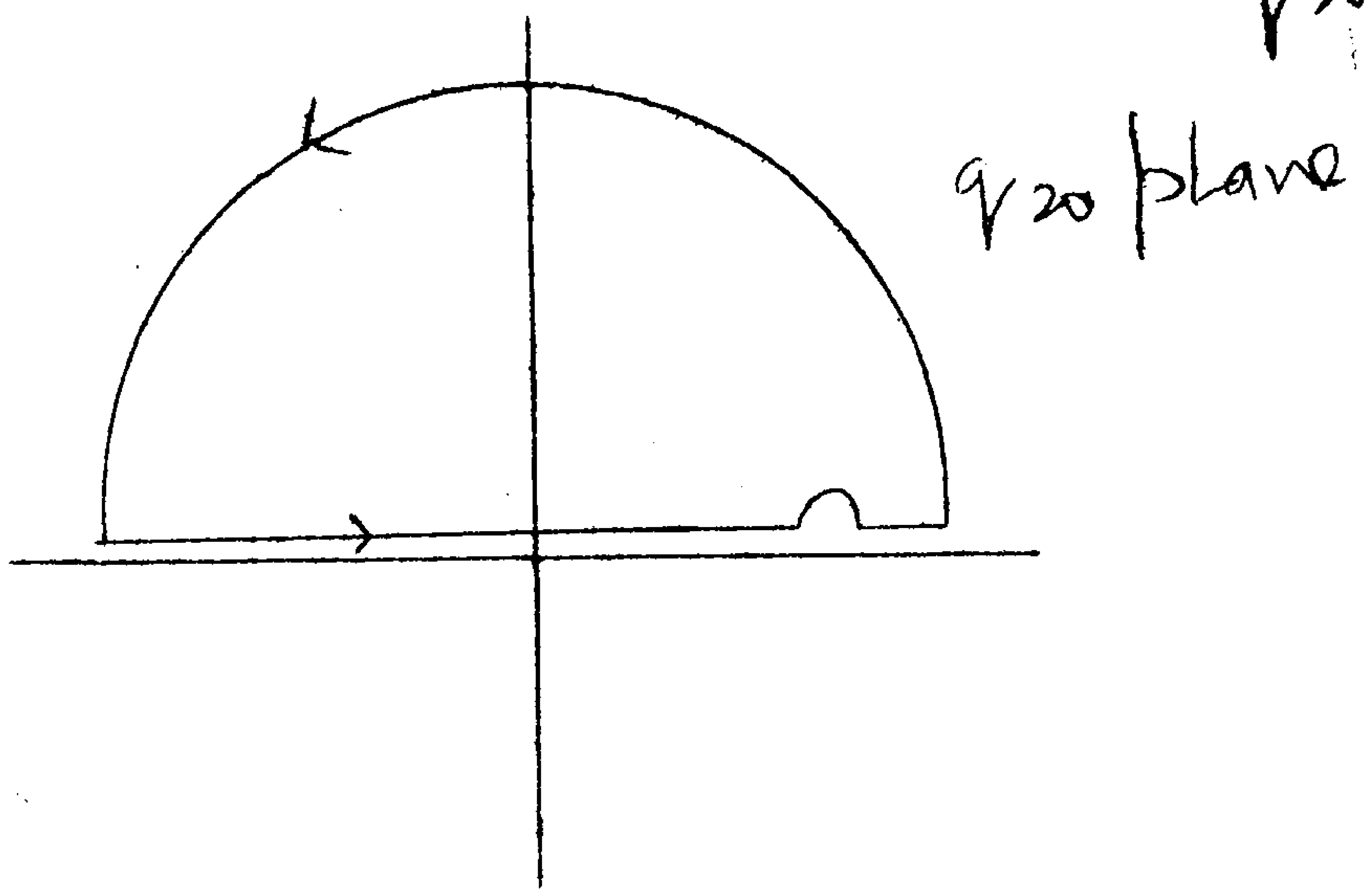
We have thus proved that microcausality implies that

$$F(q_{20}) = \int_r^\infty F_r(q_{20})\, dr$$

is well-behaved at $|q_{20}| \to \infty$ in the upper-half q_{20} plane,

$$q(20) \to q_{20} \;,\; |q_{20}| \to \text{ with } \operatorname{Im} q_{20} > 0 \tag{21}$$

To obtain a dispersion relation, we write the Cauchy integral relation with a contour closed in the upper half q_{20} plane.



Noting that the integral on the semi-circle vanishes, we obtain

$$F_\gamma(\nu) = \lim_{\varepsilon\to 0}\left[\int_{-\infty}^{\nu-\varepsilon}\frac{F_\gamma(\nu')d\nu'}{\nu'-\nu} + \int_{\nu+\varepsilon}^{\infty}\frac{F_\gamma(\nu')d\nu'}{\nu'-\nu}\right]$$

$$= -\frac{i}{\pi}\lim_{\varepsilon\to 0}\int_{-\infty}^{\nu-\varepsilon}\frac{F_\gamma(\nu')d\nu'}{\nu'-\nu} + \int_{\nu+\varepsilon}^{\infty}\frac{F_\gamma(\nu')d\nu'}{\nu'-\nu} \tag{22}$$

Taking the real part of each side, we obtain

$$\mathrm{Re}\,F_\gamma(\nu) = \frac{1}{\pi}\,P\int_{-\infty}^{\infty}\frac{\mathrm{Im}\,F_\gamma(\nu')d\nu'}{\nu'-\nu} \tag{23}$$

We thus have

$$\mathrm{Re}\,F(\nu) = \frac{1}{\pi}\int_0^\infty d\gamma\,P\int_{-\infty}^{\infty}\frac{\mathrm{Im}\,F_\gamma(\nu')d\nu'}{\nu'-\nu} \tag{24}$$

In order to be able to interchange the order of integration, the integral $P\int \frac{d\nu'\,\mathrm{Im}\,F_\gamma(\nu')}{\nu'-\nu}$ must converge uniformly in γ, $\int \frac{d\gamma\,F_\gamma(\nu')d\nu'}{\nu'-\nu}$ must converge uniformly in ν, and the double integral must also converge. We define the integral by the prescription $q_{20} \to q_{20} + i\varepsilon$ and take the limit $\varepsilon \to 0$

$$F(q_{20}) = \lim_{\varepsilon\to 0}\int_0^\infty d\gamma\; F_\gamma(q_{20}+i\varepsilon)\,d\gamma$$

The convergence properties are then satisfied. Assuming that these are fulfilled, we obtain

$$\mathrm{Re}\,F(\nu) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\,P\int_{-\infty}^{\infty}\frac{\mathrm{Im}\,F(\nu'+i\varepsilon)d\nu'}{\nu'-\nu} \tag{25}$$

For the integral in r to converge, we must have a sine function $\sin(\sqrt{q_{20}^2 - \mu^2}\, r)$ which gives an imaginary exponential. For this we require $q_{20}^2 > \mu^2$, else we will have $\sinh(\sqrt{q_{20}^2 - \mu^2})r$ which will give a real exponential that always diverges. This condition will be satisfied provided we have no intermediate state below threshold.

$$\mathrm{Re}\,F(\nu) = \frac{1}{\pi}\int_{-\infty}^{-\mu} \frac{\mathrm{Im}\,F(\nu')\,d\nu'}{\nu - \nu'} + \frac{1}{\pi}\int_{\mu}^{\infty} \frac{\mathrm{Im}\,F(\nu')\,d\nu'}{\nu' - \nu}$$

$$+ \frac{1}{\pi}\int_0^{\infty} dr \int_{-\mu}^{\mu} \frac{\mathrm{Im}\,F_r(\nu')\,d\nu'}{\nu' - \nu} \qquad (26)$$

This is true for $\pi\pi$ scattering, but not for πN scattering, where we have a pole at $q_{20}^2 =$ below the threshold.

When this happens, the integral in r no longer converges. We now write the dispersion integral as a sum of three parts, in which the ranges of integration is from $-\infty$ to $-\mu$, $-\mu$ to $+\mu$, and $+\mu$ to $+\infty$ respectively.

For the integrals from μ to ∞ and $-\infty$ to $-\mu$, we have no difficulty; however, we cannot say anything about the integral from $-\mu$ to μ, because of the divergence of the term $\sin(\sqrt{q_{20}^2 - \mu^2})r$

What may be done now is to consider the amplitude as a function of μ also, in addition to ν, and first write a dispersion relation for $\mu^2 < q_{20}^2$, for a particular value of q_{20}. This gives

$$\mathrm{Re}\,F(\nu, \mu^2) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Im}\,F(\nu', \mu^2)\,d\nu'}{\nu' - \nu} \qquad (27)$$

One then continues this, in μ^2, to the physical value of μ^2.

This can be done only for restricted values of t: this is what gives rise to the restriction on t for which fixed momentum-transfer dispersion relations can be written.

9. Dispersion relations for Arbitrary Momentum Transfer; The Mandelstam Representation.

We proved the dispersion relations for forward scattering but only under strong limitations. For fixed, non-zero momentum transfer, we earlier assume that the amplitude had the same singularities as for forward scattering. This assumption would mean that we can write the dispersion relation

$$\mathrm{Re}\, B^{(-)}(s,t) = \frac{g^2}{m^2-s} + \frac{g^2}{s-m^2-2\mu^2+t} + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \frac{\mathrm{Im}\, B^{(-)}(s',t)}{s'-s} + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{\mathrm{Im}\, B^{(-)}(s',t)}{s'+s-2m^2-2\mu^2+t}\, ds' \qquad (28)$$

This can be proved for values of t below a certain limit,

$$t =$$

However, this limitation is probably due only to the methods used and is not a physical limitation of the analyticity properties of the amplitude. Crossing symmetry gives the relation

$$\mathrm{Im}\, B^{(-)}(s,t) = -\mathrm{Im}\, B^{(-)}(u,t) \qquad (29)$$

We shall define

$$\left.\begin{aligned} B_s^{(+)} &= \mathrm{Im}\, B^{(-)}(s,t), & s &> (m+\mu)^2 \\ B_u^{(-)} &= \mathrm{Im}\, B^{(-)}(u,t) & u &> (m+\mu)^2 \end{aligned}\right\} \quad (30)$$

We have

$$B_s^{(-)}(s,t) = 0 \quad \text{for} \quad s < (m+\mu)^2 \qquad (31)$$

Therefore it cannot be an analytic function of s. But it could be an analytic function of t (and of u). Assuming that as a function of t and u, $B_s(s,t)$ has only the singularities required by unitarity, we write

$$B_s(s,t) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{\rho_{st}(s,t')}{t'-t}\,dt' + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{\rho_{su}(s,u')}{u'-u}\,du' \qquad (32)$$

and similarly

$$B_u(s,t) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{\rho_{ut}(u,t')\,dt'}{t'-t} + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{\rho_{su}(u,s')}{s'-s}\,ds' \qquad (33)$$

There could also be additional pole terms.

We can now write, for the complete scattering amplitude, the following representation:

$$\mathrm{Re}\, B(s,t) = \frac{g^2}{m^2-s} + \frac{g^2}{m^2-u} + \frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} ds' \int_{4\mu^2}^{\infty} dt' \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)}$$

$$+ \frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} ds' \int_{(m+\mu)^2}^{\infty} du' \frac{\rho_{su}(s',u')}{(s'-s)(u'+2m^2+2\mu^2-s'-t)}$$

$$+ \frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} du' \int_{4\mu^2}^{\infty} dt' \frac{\rho_{ut}(u',t')}{(s'-s)(u'-u)} + \frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} du' \int_{(m+\mu)^2}^{\infty} ds' \frac{\rho_{su}(u',s')}{(u'-u)(s'+2m^2+ - t-u}$$

(34)

Note that the u in the denominator in (32) is related to s in $B_s(s,t)$ by $u = 2m^2+2\mu^2 - s - t$ therefore, when we integrate $B_s(s',t)$ over s', the dependence of the denominator of the second term on s' must also be taken into account.

We may re-write equation (34) by coupling second and fourth integrals, obtaining

$$Re\, B(s,t) = \frac{g^2}{m^2 s} + g^2/m^2-u + \frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} ds' \int dt' \frac{\rho_{st}(s}{(s'-s)(t'-t}$$

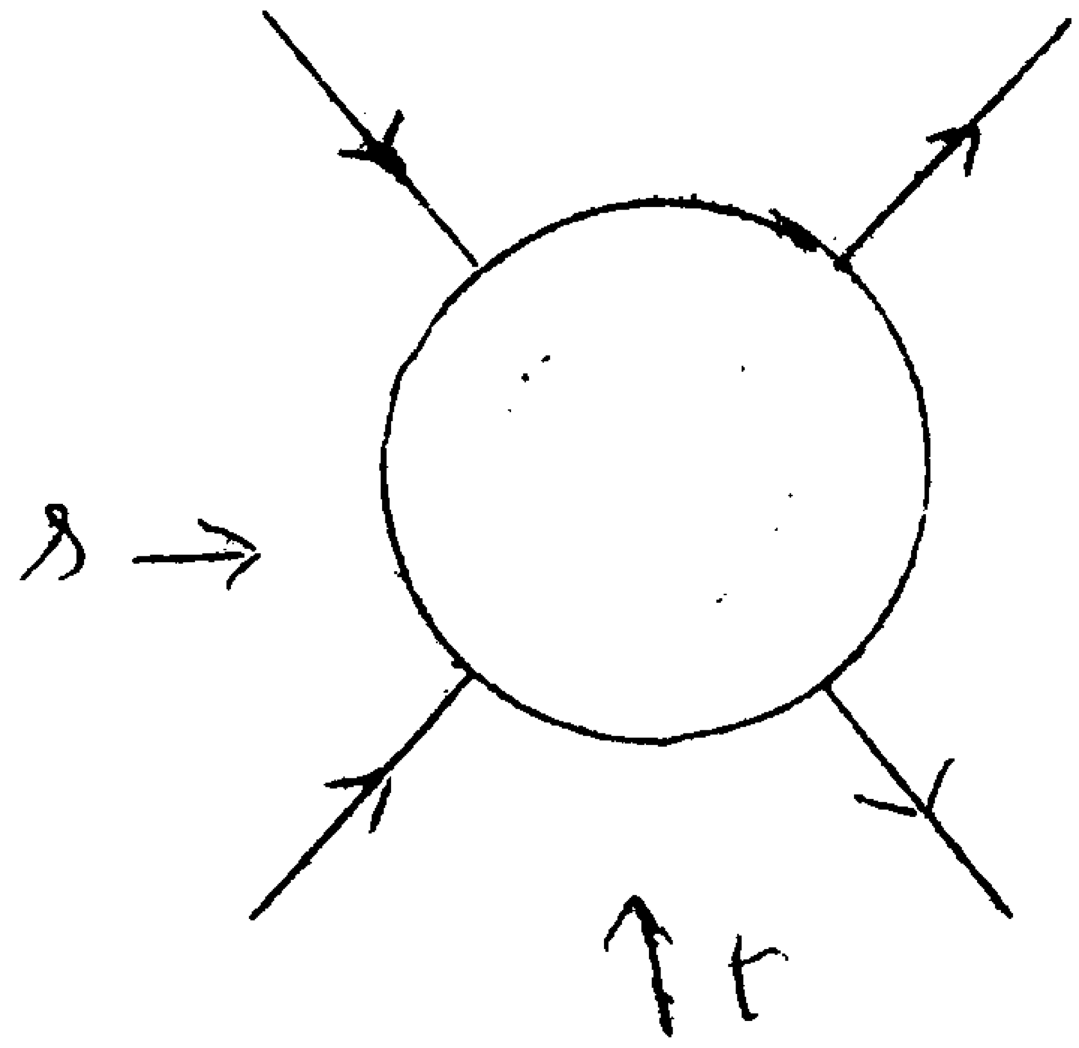
$$+ \frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} ds' \int_{(m+\mu)^2}^{\infty} du' \frac{\rho_{su}(s',u')}{(s'-s)(u'-u)}$$

$$+ \frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} du' \int_{4\mu^2}^{\infty} dt'\, \rho_{ut}(u',t')$$

We saw how in extending the dispersion relation to take into account analyticity in momentum transfer also, we arrive naturally at the Mandelstam representation:

$$F(s,t,u) = \frac{1}{\pi}\int_{s_0}^{\infty}\frac{\sigma_1(s')\,ds'}{s'-s} + \frac{1}{\pi^2}\int_{s_0}^{\infty}\int_{t_0}^{\infty}\frac{\rho_{12}(s',t')\,ds'\,dt'}{(s'-s)(t'-t)} + P_{st}$$

where P_{stu} denotes all the terms obtained by permuting s, t and u



$$s+t+u = \sum_{i=1}^{4} m_i^2$$

This 'unsubtracted' representation is valid only if the spectral functions go to zero at infinite values of their arguments. In general, we would have to write a representation with one or more subtractions.

The Mandelstam representation tells us that the scattering amplitude, as a function of each of the variables s, t, u has a branch cut on the real axis, from threshold to infinity.

Starting from these double dispersion relations, we may easily obtain single-variable dispersion relations by keeping the momentum transfer fixed.

The imaginary part of the amplitude in the physical region is given by

$$\mathrm{Im}\, F(s,t,u) = \sigma_1(s) + \frac{1}{\pi}\int_{t_0}^{\infty} \frac{\rho_{12}(s,t')}{t'-t}\,dt' + \frac{1}{\pi}\int_{u_0}^{\infty} \frac{\rho_{23}(s,u')}{u'-u}\,du' \qquad s > (m_1+m_2)^2$$

We define the 'absorptive part' A_I of A as the analytic continuation of $\mathrm{Im}\, F(s,t,u)$ on the whole s plane.

$$A_I(s,t,u) = \sigma_1(s) + \frac{1}{\pi}\int_{t_0}^{\infty} \frac{\rho_{12}(s,t')\,dt'}{t'-t} + \frac{1}{\pi}\int_{u_0}^{\infty} \frac{\rho_{13}(s,u')\,du'}{u'-u}$$

Similarly, we may write the absorptive parts in the t and u channels:

$$A_{III}(s,t,u) = \sigma_3(s) + \frac{1}{\pi}\int_{t_0}^{\infty} \frac{\rho_{13}(u,t')\,dt'}{t'-t} + \frac{1}{\pi}\int_{s_0}^{\infty} \frac{\rho_{23}(s',u)\,ds'}{s'-s}$$

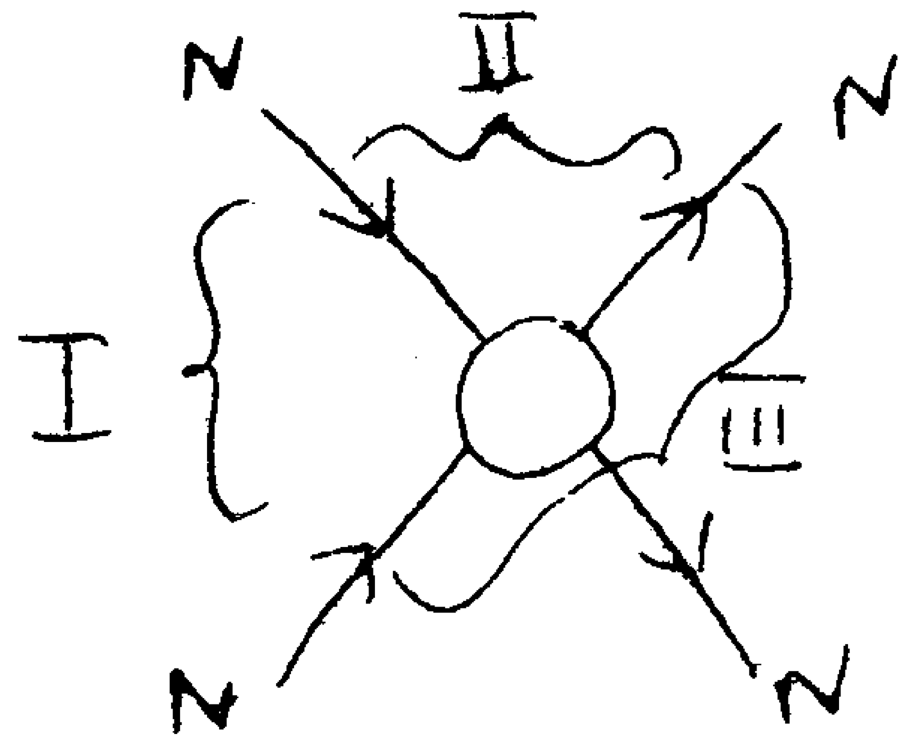
The dispersion relation for fixed momentum-transfer is given by

$$F(s,t,u) = \frac{1}{\pi}\int_{s_0}^{\infty} \frac{A_I(s',t)\,ds'}{s'-s} + \frac{1}{\pi}\int_{u_0}^{\infty} \frac{A_{III}(u',t)\,du'}{u'-u}$$

We may similarly write dispersion relations keeping s or u fixed.

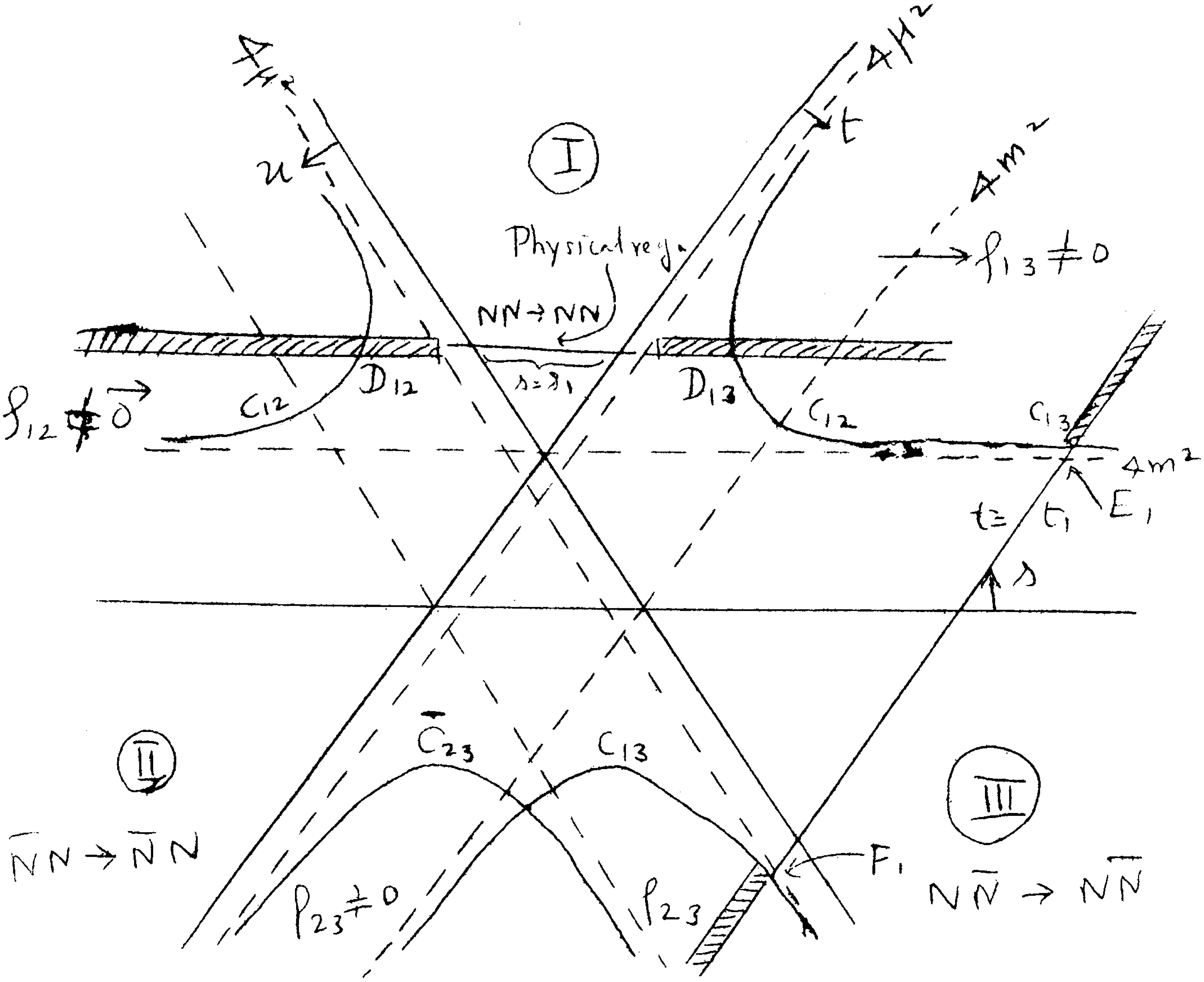
The Mandelstam representation is the most analytic function compatible with crossing symmetry and unitarity. Unitarity imposes strong restrictions on the spectral functions.

We shall again introduce the Mandelstam diagram. We shall draw it for NN scattering, ignoring charge and spin.



The three reactions to be considered are:

I	$N + N \rightarrow N + N$	(s)
II	$N + \bar{N} \rightarrow N + \bar{N}$	(u)
III	$\bar{N} + N \rightarrow \bar{N} + N$	(t)



The physical regions for the processes I, II and III are shown by the shaded regions. As we emphasized earlier, there is no overlap between these three regions. The scattering amplitudes for the 3 variables are the different elements of a single analytic function; this function has the singularities given by the Mandelstam representation.

The cut in s starts from $t = 4m^2$ where m is the nucleon mass; the lowest-mass intermediate state is the two-nucleon state. The cut in t and u starts from $4\mu^2$ where μ is the pion mass, because the lowest-mass intermediate state for the process $N + \bar{N} \rightarrow N + \bar{N}$ is the two-pion state. These cuts are shown in the diagram.

The spectral functions can be non-zero only in the regions indicated (covered by crossed lines). Actually, it is found that $\rho_{ij} \neq 0$ only in regions whose boundary curve is asymptotic to the lines $s = 4m^2$ and $t = 4\mu^2$ (for ρ_{13}) and $s = 4m^2$, $u = 4\mu^2$ (for ρ_{12}). These curves are shown by heavy lines in the diagram - C_{12}, C_{13} etc.

[These boundary curves are obtained by considering the lowest-mass intermediate states; the contributions of higher-mass intermediate states to the spectral functions will be non-zero only in a smaller region, included in the regions bounded by C_{12}, C_{13}, ...]

For a fixed value s_0 of s, the cuts in t and u start, not at thresholds, but at the points D_{12}, D_{13} respectively shown on the figure.

Similar considerations apply to the other two channels. The spectral function ρ_{23} is non-zero in the regions bounded by C_{23} and $\overline{C}_{23}$. C_{23} is asymptotic to the lines $t = 4m^2$, $u = 4\mu^2$ while $\overline{C}_{13}$ is asymptotic to the lines $t = 4\mu^2$, $u = 4m^2$. For a fixed value $t = t_1$ of t, the cuts in s and u start from the points E_1, F_1 respectively shown in the figure.

We have imposed the requirement of maximum analyticity on the amplitude. A small change in the analyticity properties can entirely change the shape of the function. However, the assumption of maximum analyticity serves as a very useful guide to the method of approximation to be used.

At very high energies, the scattering amplitude will be almost purely imaginary.

10. Methods of approximation.

a) The Strip Approximation:

It is a standard approximation*. This is discussed in the papers of Chew and Frautschi, and of Virendra Singh and Udgaonker

b) A Low-energy polynomial approximation (The Cine-Fubini Approximation).

At low energies, the spectral function is non-zero only in a region that is fairly removed from the physical region, i.e. at fairly large positive values of t and u. Note that in the physical region, t and u are both negative (for equal-mass scattering), $t \leq 0, u \leq 0$. So we might hope that the imaginary part of the amplitude may be approximated by a polynomial in t and u, i.e., the double spectral function is entirely ignored.

For nucleon-nucleon scattering, we also have poles in each channel: In the $N-N$ channel we have the deuteron pole, and in the $N\bar{N}$ channel we have the pion pole.

11. Polology:

We shall start by discussing the poles in NN and $N\bar{N}$ scattering.

(i) The pion pole in $N + \bar{N} \rightarrow N + \bar{N}$ (t channel).

The scattering amplitude may be written

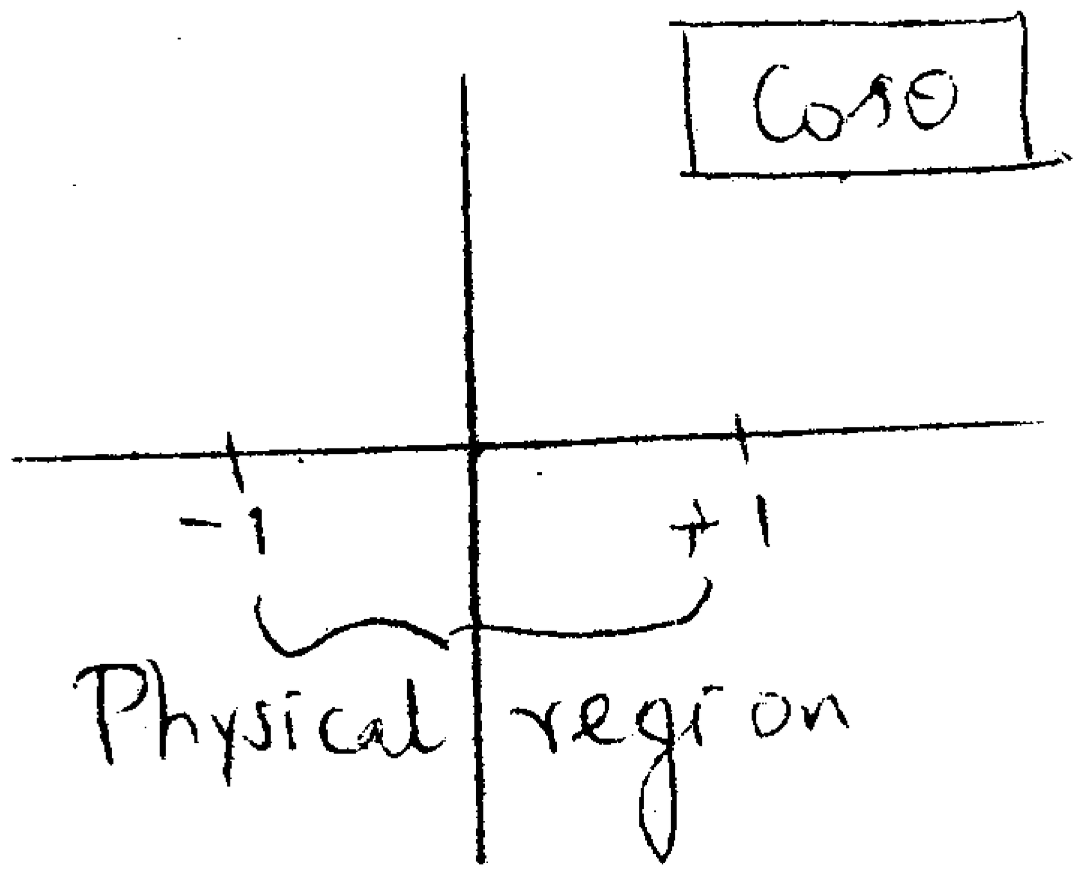
$$F(s,t,u) = \frac{G}{(t-\mu^2)} + \frac{G}{u-\mu^2} + F'(s,t,u)$$

We have a constant G because the pion has spin zero. In general, we would have a factor $P_J(\cos\theta_t)$ for a pole of spin J; this gives an s- dependence. The pole in u has the same residue G when we neglect charge and spin.

We may extrapolate to the pole (which lies in the unphysical region).

$$\lim_{t \to \mu^2} |F(s,t,u)|^2 (t-\mu^2)^2 \longrightarrow G^2$$

This extrapolation to the pole is important in connection with the Peripheral model. Consider nucleon-nucleon scattering. The physical region in the $\cos\theta$ plane is along the real axis from -1 to $+1$.



In the t plane, the pole occurs at $t = +\mu^2$.

This corresponds to

$$\cos\theta = 1 + \mu^2/2q^2$$

as $$t = -2q^2(1-\cos\theta)$$

Similarly, the pole in u gives a pole in $\cos\theta$ at

$$\cos\theta = 1 + 2\mu^2/(s-4m^2)$$

since we have

$$u = -2q^2(1+\cos\theta)$$

and $$s = 4(q^2 + m^2)$$

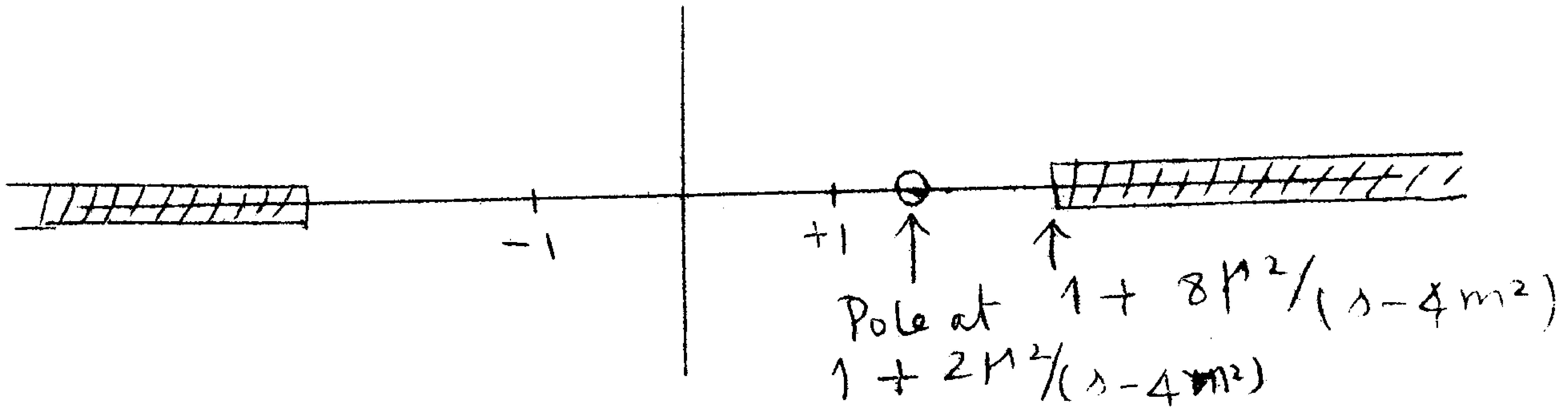
or $$q^2 = s/4 - m^2$$

The branch points in t and u give rise to branch points in $\cos\theta$ at

$$\cos\theta = 1 + 8\mu^2/(s-4m^2)$$

and $$\cos\theta = -1 - 8\mu^2/(s-4m^2)$$

respectively.



The cuts are far away compared to the poles, when we are in the low-energy region.

We may then expect at first sight that a pole approximation may be a good approximation. However, because the π meson is pseudoscalar, the amplitude vanishes at $\cos\theta = +1$, i.e., in the forward direction. Because of the zero at $+1$, it turns out that the pole at $1 + \frac{2\mu^2}{s - 4m^2}$ does not provide a good approximation.

However, we may except that a polynomial approximation

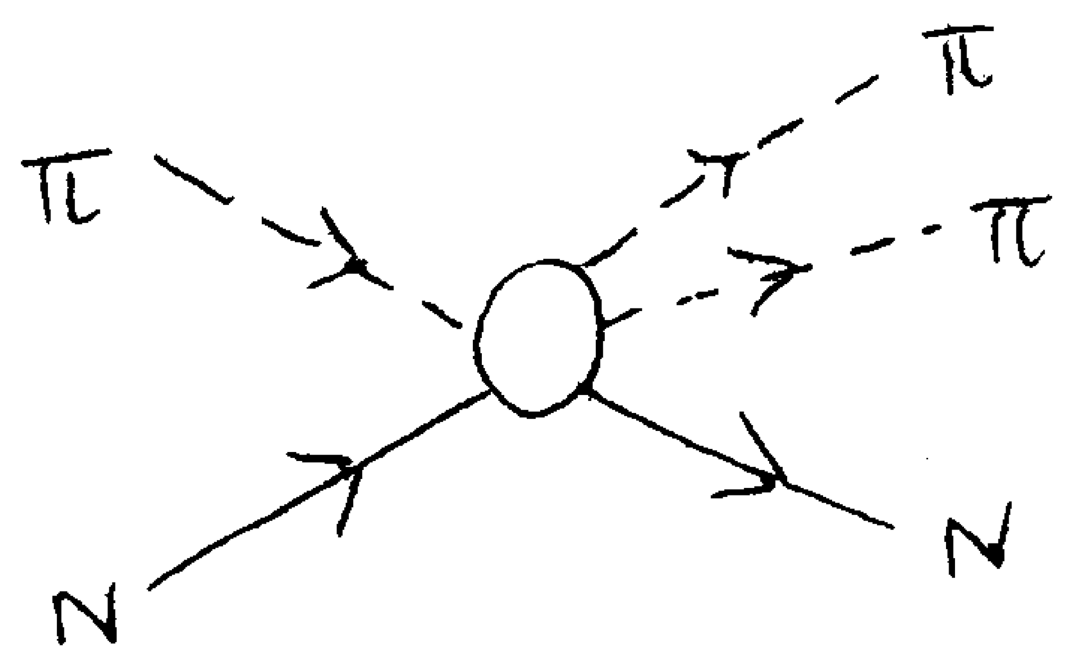
$$(t - \mu^2)\, F(s, t) \sim \sum_{i=0}^{N} c_i t^i$$

should be a good approximation at least up to $t \approx \mu^2$

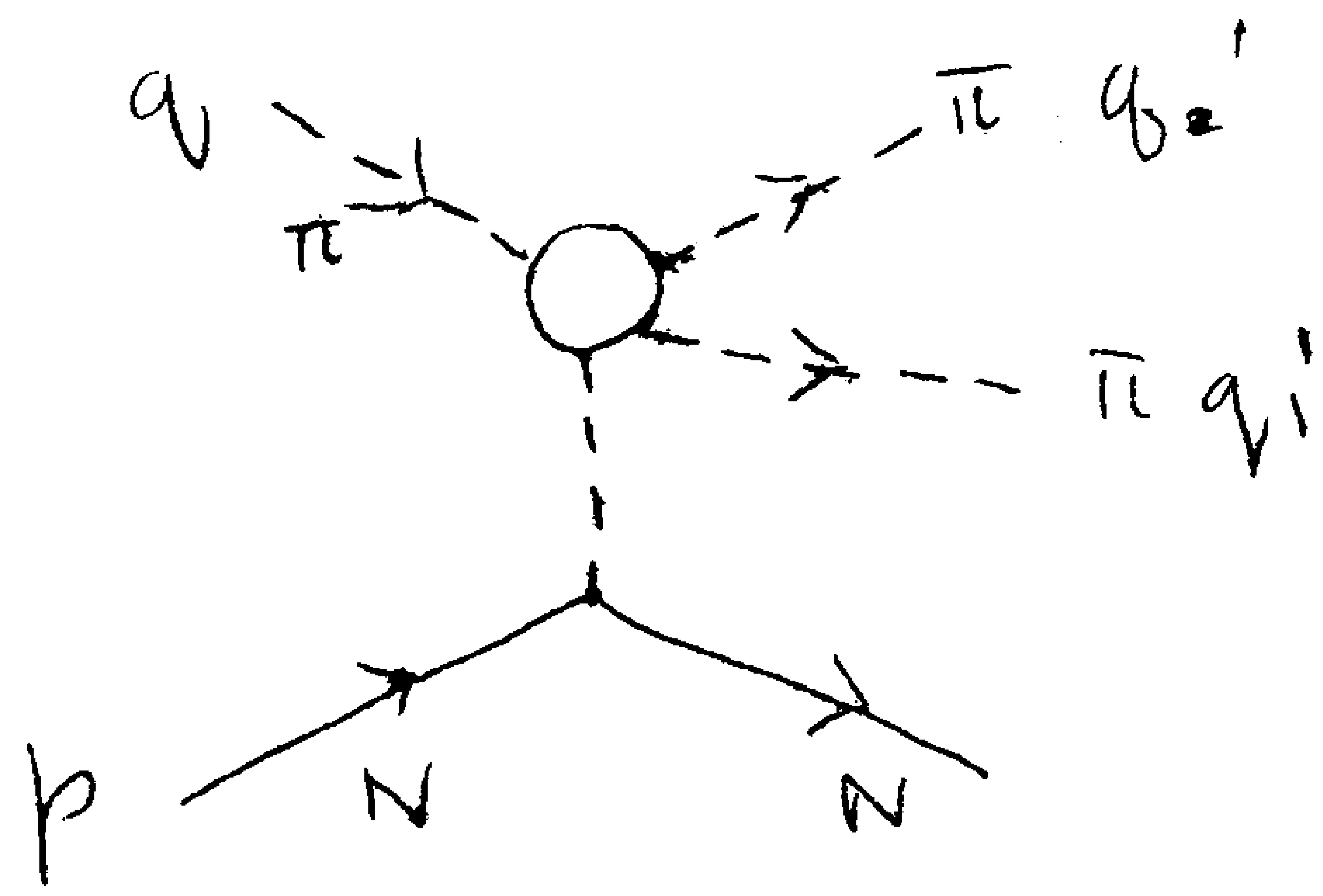
One-pion-exchange gives a good part of the higher phase shifts. As the 1-pion contribution can be calculated, the number of phase-shifts, i.e., the number of parameters to be filled, may be reduced if the one-pion-exchange contribution is taken to represent the higher phase shifts. This has been done for N - N scattering.

Application of polology in the peripheral model:

Consider inelastic pion-nucleon scattering.



The simplest pole contribution would be the one-pion pole contribution.



The contribution of the 1-pion exchange is given by

$$i g \bar{u}(p') \gamma_5 u(p) \frac{1}{\mu^2 - t} A(\omega, \cos\theta)$$

The contribution of this to the total cross section, on summing over all polarization is obtained as

$$\sigma_{tot} = \frac{q^2}{4(2\pi)^5 q m} \int \delta(p + q - p' - q_1' - q_2') \frac{t}{(t - \mu^2)^2} \cdot |A|^2 \frac{d^3\vec{p}\,' \, d^3\vec{q}_1{}' \, d^3\vec{q}_2}{8E'\omega_1'\omega_2'}$$

The cross-section contains other terms also, terms with a simple pole $1/(t-\mu^2)$ and others with no pole at $t=\mu^2$

As we are going to extrapolate to the pole at $t=\mu^2$ by multiplying by $(t-\mu^2)^2$ we neglect the other terms.

The above expression is, of course, valid only at $t=\mu^2$ as we have neglected all the unknown dependence on t contained in the form factors.

We evaluate the integral as follows:

$$\int \delta(p+q-p_1'-q_1'-q_2')\frac{t}{(t-\mu^2)^2}\delta(p_1'^2-m^2)\,\theta(p_0'$$
$$\delta(q_1'^2-\mu^2)\,\theta(q_{10}')\,\delta(q_2'^2-\mu^2)\,\theta(q_{20}')\,|A|^2$$
$$d^4p'\,d^4q_1'\,d^4q$$

$$=\int\frac{t}{(t-\mu^2)^2}\delta(p_1'^2-m^2)\,\theta(p_0')\,\delta(q_1'^2-\mu^2)$$
$$\theta(q_{10}')\,d^4p'\,d^4q_1'\,\delta((p+q-p'-q_1')-\mu$$

The expression is Lorentz-invariant. It is convenient to compute it in the frame in which the 4-vector $(p+q-p')$ is along the time-axis.

$$p+q-p'=(\omega,0)$$
$$\delta(q_1^2-\mu^2)=\delta(\omega^2+\mu^2-2\omega_1\omega_1'-\mu^2)$$
$$=\frac{1}{2\omega}\delta(\omega_1'-\frac{\omega}{2})$$

ω is the energy of 2 final π mesons in their c.m. frame. We then obtain

$$\int |A|^2 \frac{t}{(t-\mu^2)^2} \delta(p'^2 - m^2)\,\theta(p_0')\,\delta(q_1^2 - \mu^2)$$
$$d^4p'\, dq_1' q_1^2\, d\Omega' \delta\left(\frac{\omega^2}{4} - q_1'^2 - \mu^2\right) \frac{1}{2\omega} \frac{1}{2q_1'} \delta\left(q_1' - \sqrt{\frac{\omega^2}{4}}\right.$$

so that

$$\sigma = \frac{g^2}{4(2\pi)^5 qm} \int |A|^2 \frac{t}{(t-\mu^2)^2} \frac{d^3\bar{p}'}{2E'} \frac{1}{4\omega} \sqrt{\frac{\omega^2}{4} - \mu^2}\, d\Omega'$$

We know that the $\pi\pi$ scattering cross section $\sigma_{\pi\pi}(\omega)$ is given by

$$\sigma_{\pi\pi}(\omega) = \int \left|\frac{A}{8\pi\omega}\right|^2 d\Omega'$$

Therefore we may write

$$\sigma = \frac{g^2}{4(2\pi)^5 qm} \int \frac{d^3\bar{p}'}{2E'} \frac{t}{(t-\mu^2)^2} (2\pi)^2 \omega \frac{t}{(t-\mu^2)^2} \sqrt{\frac{\omega^2}{4} - \mu^2} \times \sigma_{\pi\pi}(\omega)$$

We have

$$d^3\bar{p}' = p'^2\, dp\, d(\cos\theta')\, d\varphi'$$

It is convenient to transform to the variables ω, t instead of p', $\cos\theta'$. Note that

$$t = (p-p')^2 = p^2 + p'^2 - 2pp'$$
$$= 2m(m - E'_{lab})$$
$$\omega^2 = (q_1' + q_2')^2 = (p+q-p')^2$$
$$= (p+q)^2 - p'^2 - 2p'(p+q)$$
$$= (\omega_0 + m)^2 - \omega_0^2 + \mu^2 + m^2 - 2E'(\omega_0 + m)$$
$$+ 2p'q\cos\theta$$

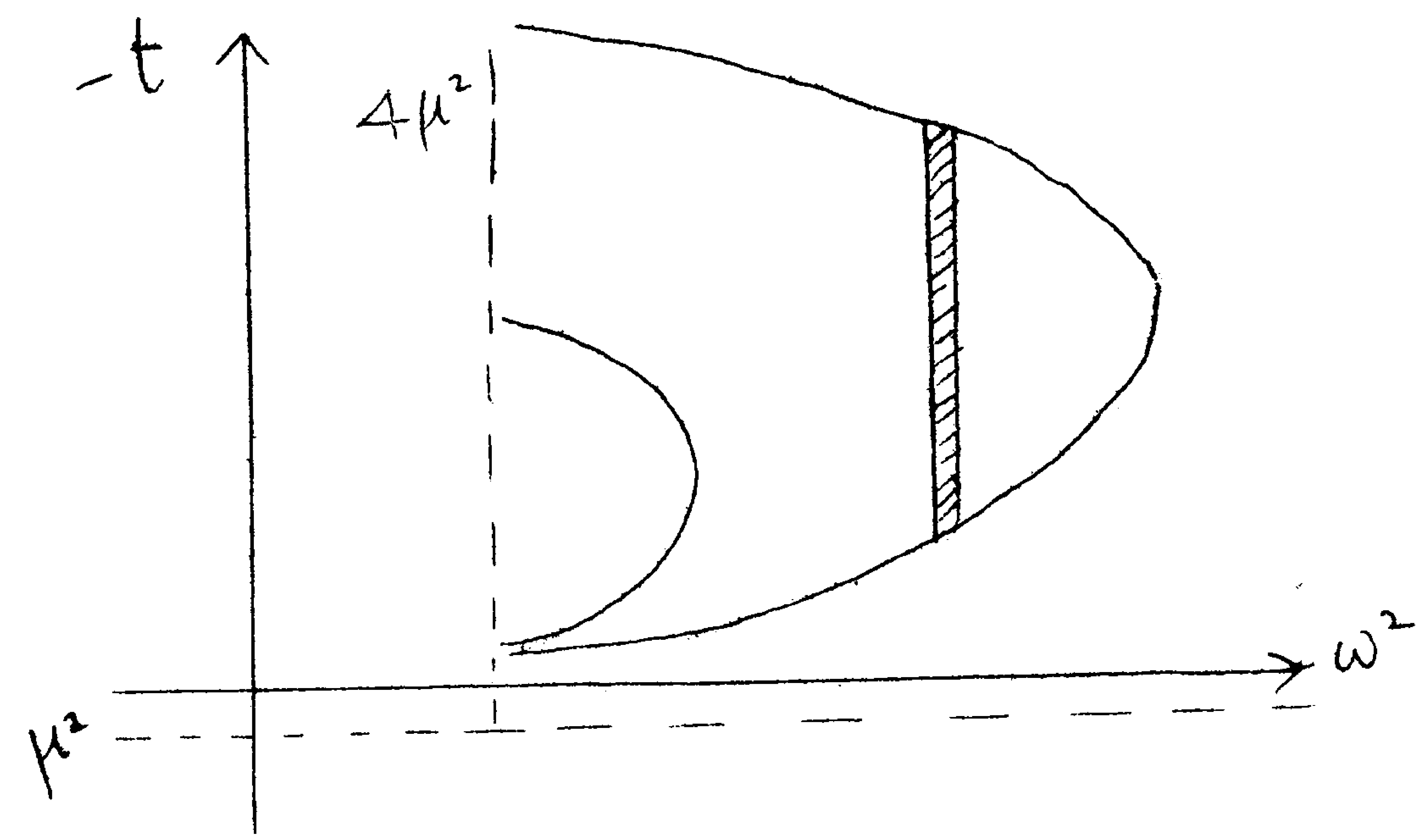
making these transformations, we obtain

$$\frac{\partial^2\sigma}{\partial t\,\partial(\omega^2)} = \frac{1}{2\pi q^2}\,\frac{g^2}{4\pi}\,\frac{t}{4m^2}\,\omega\sqrt{\frac{\omega^2}{4}-\mu^2}\cdot\frac{\sigma_{\pi\pi}(\omega)}{(t-\mu^2)^2}$$

This is approximate as all other contributions to the amplitude have been ignored. However we have the exact result

$$\lim_{t\to\mu^2}\left\{(t-\mu^2)^2\frac{\partial^2\sigma}{\partial t\,\partial(\omega^2)}\;\frac{2\pi q^2\,4m^2}{\frac{g^2}{4\pi}\,t\,\omega\sqrt{\frac{\omega^2}{4}-\mu^2}}\right\} = \sigma_{\pi\pi}(\omega)$$

We must obtain $\partial^2\sigma/\partial t\,\partial(\omega^2)$ as a function of t, for different fixed values of ω, and then extrapolate to $t = \mu^2$ for each value of ω.



Till now, reliable data for the extrapolation have not been obtained. Especially because of the factor t in the expression on the left hand side.

People have been tempted to use the peripheral model for small values of t, where one may expect that the one-pion pole term is still important. However, now there is the difficulty that there are unknown functions of t involved in the amplitude.

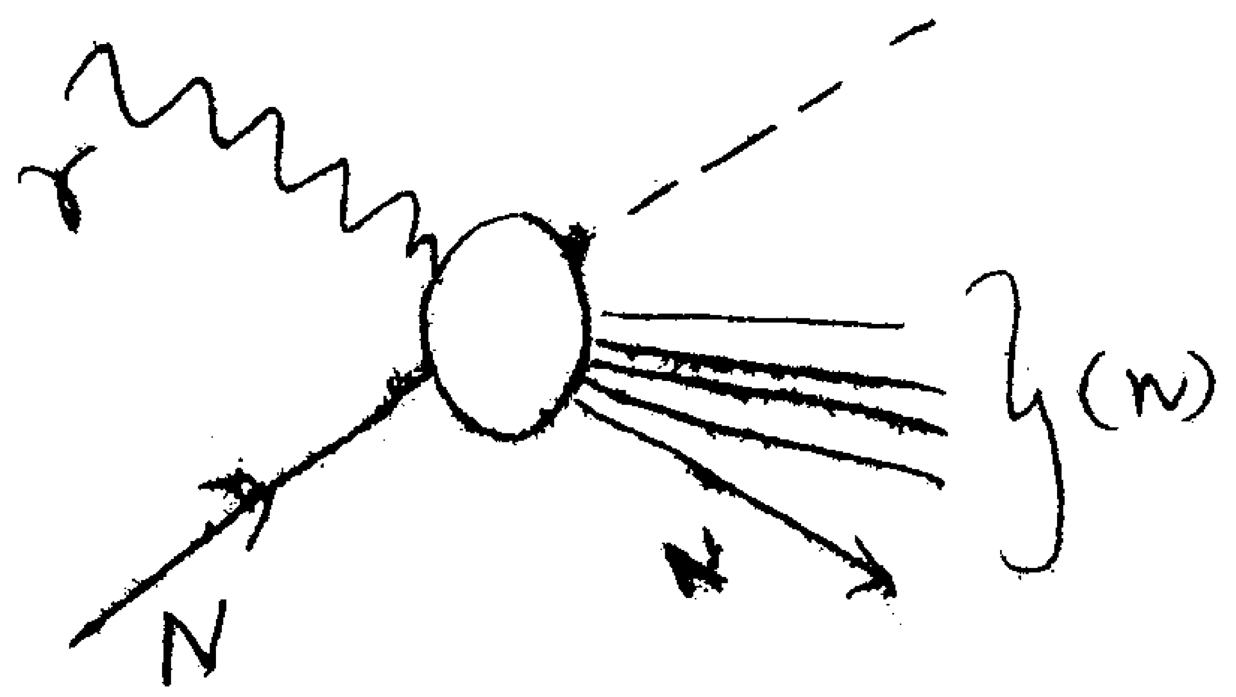
The pion-pole term will be important when $\sigma_{\pi\pi}$ is large, e.g., when there is ω resonance. We have already seen that there are at least 2 $\pi\pi$ resonances, the ρ and f^0.

Note that we have made 2 hypotheses.

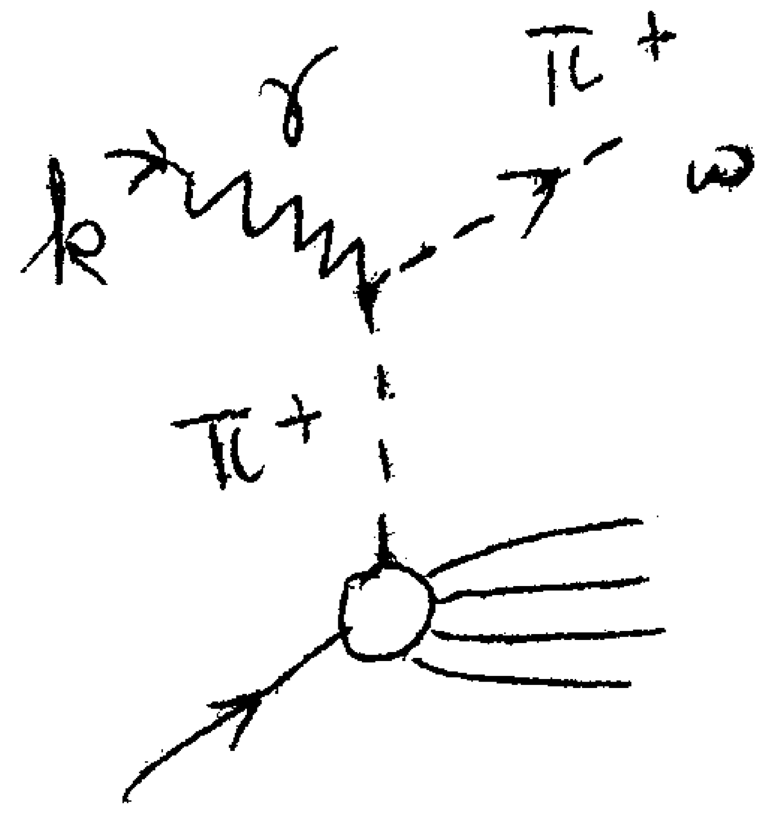
(i) <u>On analyticity</u>, that there is a pion pole.

(ii) <u>A Dynamical assumption</u>: This pole is dominant.

An example of a process in which a one-pion pole may be important is the photoproduction of several pions.



The one-pion pole contribution shown in the diagram below gives the following contribution to the cross-section:



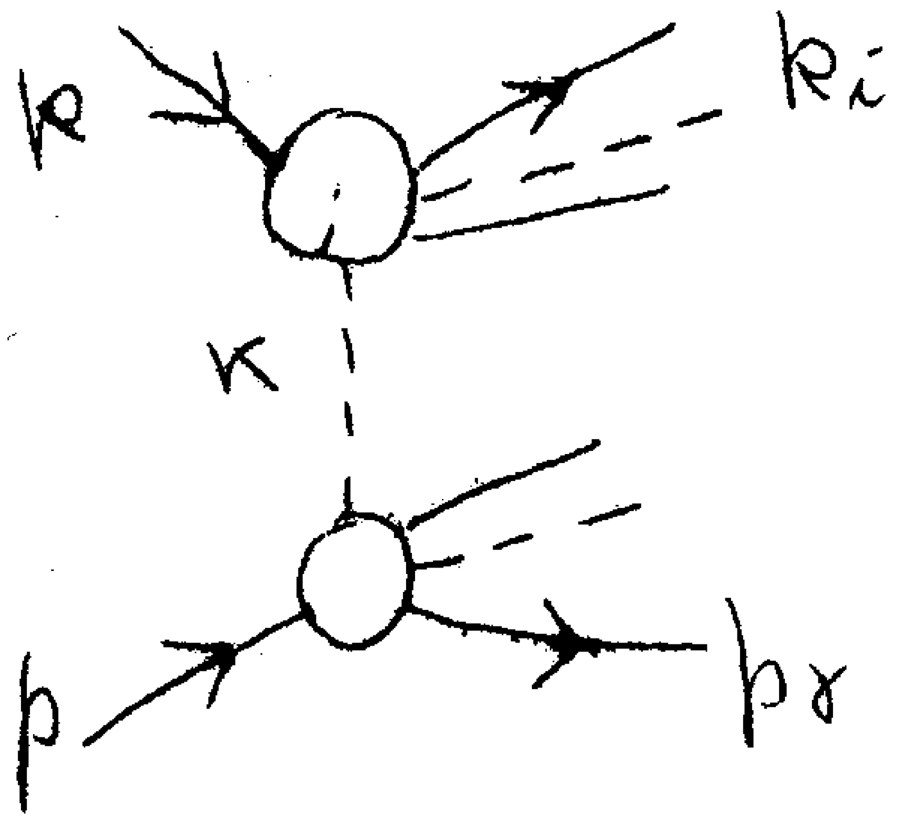
$$\frac{\partial^2\sigma}{\partial t\,\partial\omega} = \frac{\alpha}{2\pi}\,\frac{\sin^2\theta}{\left(1-\frac{v}{\omega}\cos\theta\right)^2}\,\frac{\omega(k-\omega)}{4\pi k^3} \times \sigma_{tot}(k-\omega)$$

For production in the forward direction at high energies, the factor $\left(1-\frac{v}{\omega}\cos\theta\right)^2$ will be very small, and the cross-section will be large.

The factor $\sin^2\theta$ comes from the fact that the photon has spin one, and hence the production of a spin 0 pion in the forward direction is forbidden.

<u>Tests of the one-pion Exchange Model</u>:

Consider the peripheral model shown in the diagram below.



The fact that the exchanged particle has spin 0 has an important consequence: the matrix element is a product of two Lorentz invariants.

$$\sigma = \int \frac{|A(k,k_i)|^2\,|B(p,p_\gamma)|^2}{(t-\mu^2)^2}$$

$$\delta\Big(k+p-\sum_i k_i-\sum_j k_j\Big)\prod_i\frac{d^3k_i}{2E_i}\prod_j\frac{d^3p_j}{2E_j}$$

In the rest system of the particle k, if this particle is unpolarized, the factor $|A(k,k_i)|^2$ will be invariant if we rotate all the vectors $\vec{k_i}$ together, keeping $\sum k_i$ fixed, in order to keep the cross section $|A(k,k_i)|^2$ fixed. Since $\vec{k}=0$, $\Sigma\vec{k_i}=\vec{K}$, where $\vec{K}$ is the momentum transfer vector.

Thus in the rest system of one of the initial particles (k), the distribution of the particles k_i must be symmetrical about the direction $\vec{K}$.

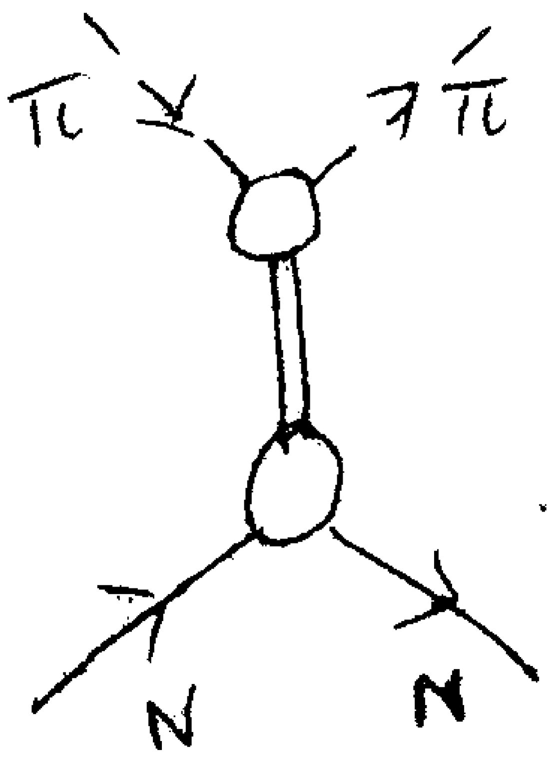
Similarly, in the rest system of the particle p, the distribution of the particles p_i must be symmetrical about the direction of $\vec{K}$.

When the exchanged particle has spin 1, then one can calculate the correlation between the polarization of the particles coming out from the different vertices.

The unknown t dependence at the vertices cancels in the polarization correlation, as it occurs both in the polarization and in the cross-section.

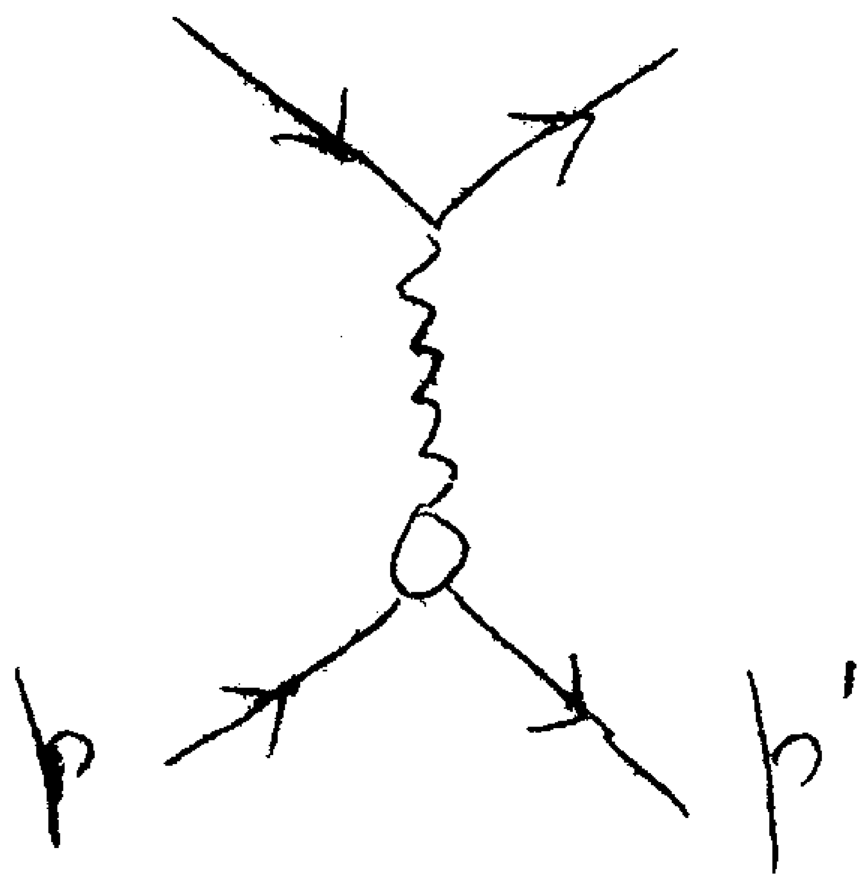
One may expect that the peripheral model may be extended to include the new meson resonances.

e.g. In pion-nucleon scattering, the lowest-mass state that can be exchanged is a 2-pion state. We know that there is a $T=1$, $J=1$ $(\pi\pi)$ resonance, the ρ meson (as well as a $T=0, J=2$ (f^0) resonance.)



We may expect that the peripheral contribution from ρ exchange may be important for some values of the momentum transfer.

We shall now see how polology may be applied usefully to form factors. Consider electron-nucleon scattering. The electromagnetic nucleon current may be written as



$$j = \bar{u}(p')\left[\gamma_\mu F_1(t) + \sigma_{\mu\nu}\frac{q_0^2}{2m}F_2(t)\right]u(p)$$

One may introduce 2 form factors for the proton and 2 for the neutron. Instead, it is usual to define the isoscalar and isovector form factors

$$F_i^S = (F_i^p + F_i^n)/2$$

$$F_i^V = (F_i^p - F_i^n)/2$$

We may write dispersion relations for the form factors:

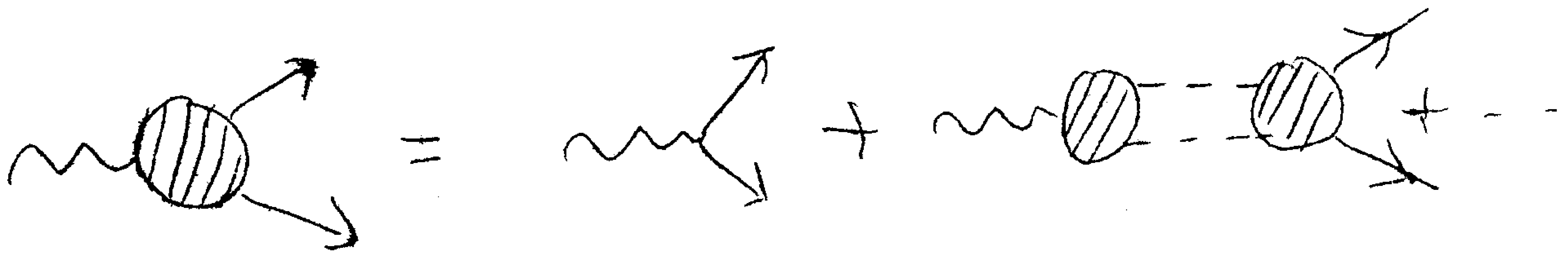
$$F_1^{S,V}(t) = \frac{e}{2} - \frac{t}{\pi}\int_{t_0}^{\infty}\frac{\mathrm{Im}\, F_1^{S,V}(t')\,dt'}{t'(t'-t)}$$

$$F_2^{S,V}(t) = \frac{e(\mu_p \pm \mu_n)}{2} - \frac{t}{\pi}\int_{t_0}^{\infty}\frac{\mathrm{Im}\, F_2^{S,V}(t')\,dt'}{t'(t'-t)}$$

The Isovector Form factor:

$$F_1^V(t) = \frac{e}{2} - \frac{t}{\pi}\int_{4\mu^2}^{\infty} \frac{\mathrm{Im}\, F_1^V(t')\, dt'}{t'(t'-t)}$$

The lowest mass intermediate state that contributes here is the 2-pion state.



To find the contribution of the 2-pion intermediate state, we require the $\gamma \to \pi\pi$ amplitude and the $\pi\bar{\pi} \to N\bar{N}$ amplitude.

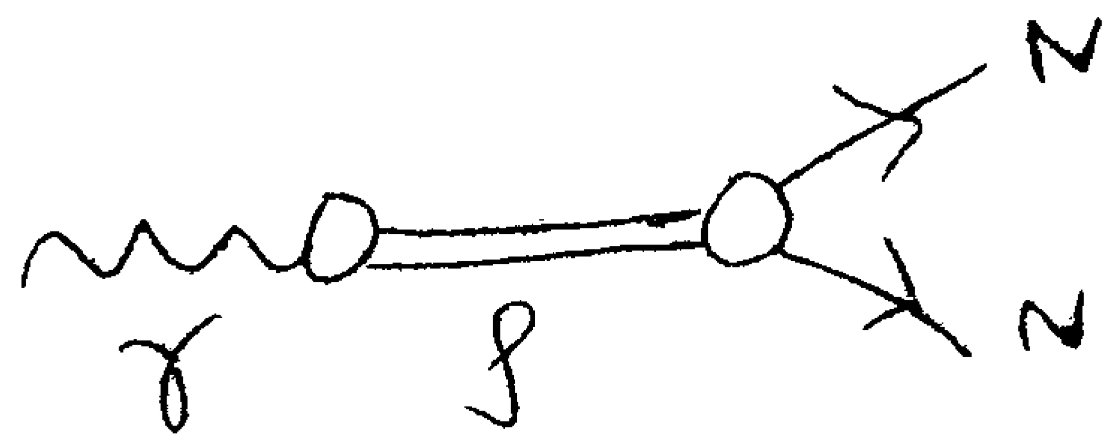
We may write a dispersion relation for each of these amplitudes:

Frazer and Fulco deduced from this that the $\pi\pi$ scattering in the $J = 1$, $T = 1$ state had a peculiar structure; they predicted that there was a resonance in this state. Such a resonance was discovered later; this was the ρ meson.

If we work the other way and put in the observed ρ meson, we get good agreement for the isovector form factor. We obtain

$$F_1^V(t) \simeq \frac{e}{2}\left[1 - \frac{at}{1-m_\rho^2}\right]$$

One may approximate all this by the simple-minded picture in which the isovector form factor is assumed to be dominated by the ρ meson pole.



This gives

$$F_1^V(t) \approx \frac{A(t)}{t - m_\rho^2} + C$$

The lowest intermediate state here is the 3π state.

A direct calculation of the 3-pion contribution to the amplitude for $\gamma \to N\bar{N}$ is very difficult.

However, we may expect that the $I = 0$, $J = 1$, ω meson and φ meson may dominate the isoscalar form factor.

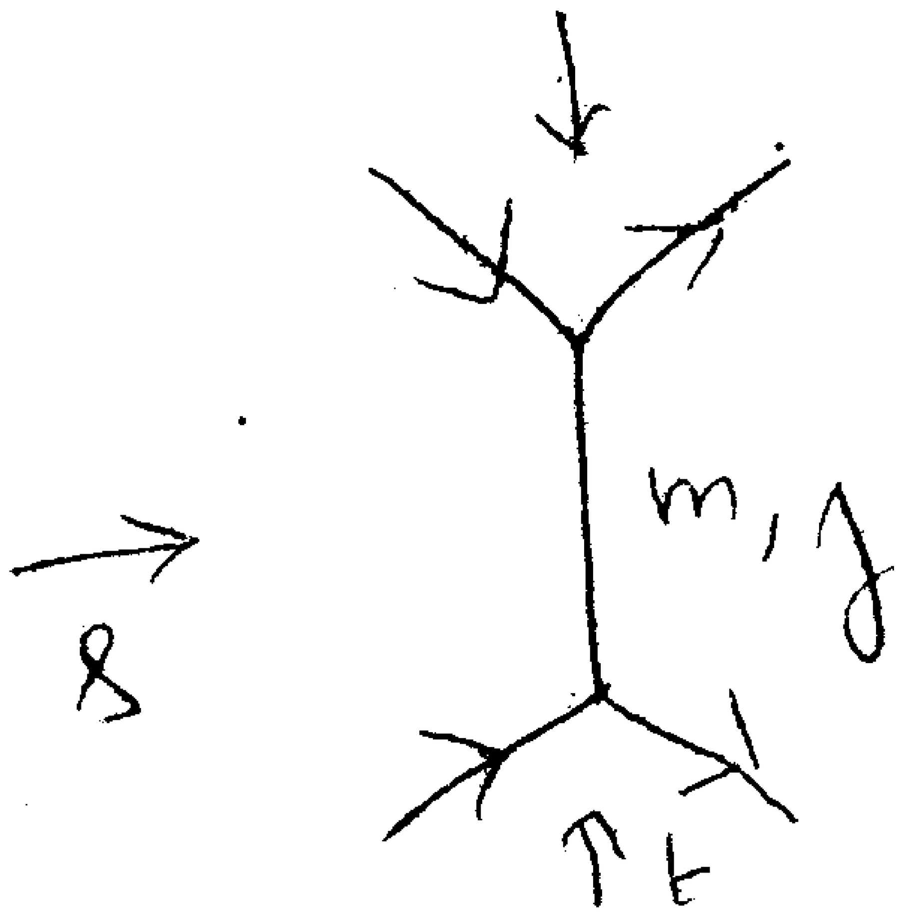
We then obtain

$$F_1^S(t) \approx \left[\frac{A'(t)}{t - m_\omega^2} + C'\right]$$

a similar contribution from the φ meson.

Since the masses of the ρ and ω are nearly equal, F_1^S and F_1^V are expected to be similar. If A and A' are also nearly equal, then this would explain the very small charge form factor of the neutron.

Although this simple pole technique is useful, it leads to difficulties if applied generally. For instance, consider the exchange of a particle with spin J in the scattering of 2 equal-mass spinless particles.



The contribution of this pole to the amplitude is of the form

$$\frac{P_j(\cos\theta_t)}{t-m^2} + \cdots$$

In the s channel, the scattering angle is given by

$$\cos\theta_s = 1 + \frac{t}{2q^2} = 1 + \frac{t}{\frac{s}{2} - 2\mu^2} = 1 + \frac{2t}{s - 4\mu^2}$$

Similarly, the scattering angle in the t channel is given by

$$\cos\theta_t = 1 + \frac{2s}{t - 4\mu^2}$$

At high energies (in the s-channel), i.e., for large s, the pole term

$$P_j\left(1 + \frac{2s}{t-4\mu^2}\right) \Big/ t - m^2 + \cdots$$

behaves as s^j for fixed t, so that the cross-section behaves as

$$s^{2(j-1)}$$

This would lead to a badly divergent asymptotic behaviour when the exchanged particle has a high spin. This was one of the difficulties that motivated the introduction of Regge poles.

X. Partial Wave Dispersion Relations.

1. Analyticity properties of partial waves:

In an earlier lecture we discussed the Mandelstam representation for the scattering amplitude, from which one could obtain dispersion relations for fixed momentum transfer:

$$F(s,t,u) = \text{pole terms} + \frac{1}{\pi}\int_{t=0}^{\infty} \frac{F_t(s,t')\,dt'}{t'-t} + \frac{1}{\pi}\int_{u_0}^{\infty} F_u(s,u')\,du'/(u'-u) \qquad (1)$$

The partial wave amplitude $F_\ell(s)$ is given by

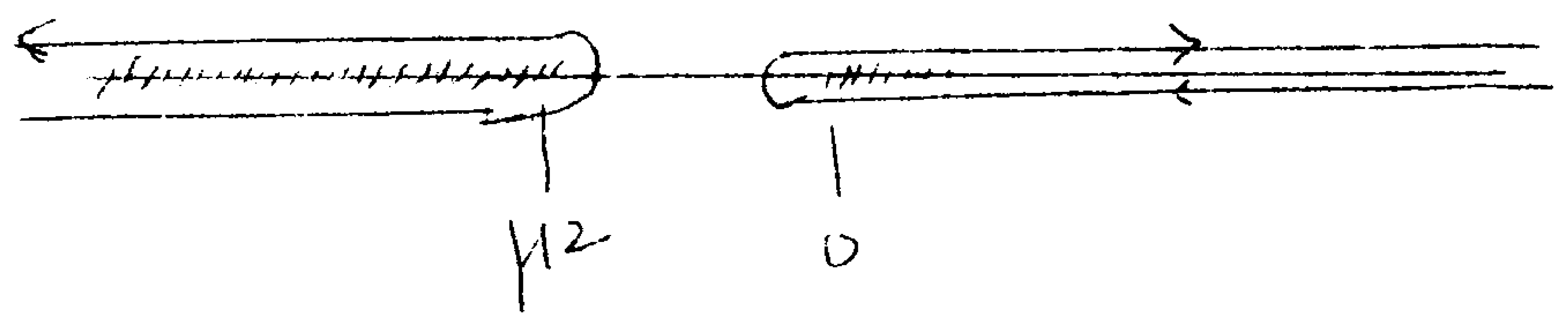
$$F_\ell(s) = \frac{1}{2}\int_{-1}^{+1} F(s,t,u)\,P_\ell(\cos\theta)\,d(\cos\theta) \qquad (2)$$

We may replace t and u in the integrals on the right hand side of (1) by their expressions in terms of $\cos\theta$. For simplicity, consider the scattering of particles of equal mass μ, so that

$$\begin{aligned} t &= -2q^2(1-\cos\theta) \\ u &= -2q^2(1+\cos\theta) \end{aligned} \qquad (3)$$

The scattering amplitude has a cut in q^2 from 0 to ∞, and from $-\mu^2$ to ∞, the latter arising from the vanishing of the denominator of the integrand of the first integral in (1)

q^2-plane



We may write a dispersion relation for the partial wave amplitude:

$$F_\ell(q^2) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\,F_\ell(q'^2)\,dq'^2}{q'^2-q^2} + \frac{1}{\pi}\int_{-\infty}^{-\mu^2} \frac{\mathrm{Im}\,F_\ell(q'^2)\,dq'^2}{q'^2-q^2} \qquad (4)$$

Consider elastic scattering.

The partial-wave amplitude $f_\ell(q^2)$ is given by

$$f_\ell(q^2) = \frac{F_\ell(q^2)}{8\pi W} = \frac{F_\ell(q^2)}{16\pi\sqrt{q^2+\mu^2}} \qquad (5)$$

$$= e^{i\delta_\ell}\sin\delta_\ell/q. \qquad (6)$$

which defines the phase shift δ_ℓ . Therefore

$$F_\ell(q^2) = 16\pi\left(\sqrt{q^2+\mu^2}/q\right)e^{i\delta_\ell}\sin\delta_\ell \qquad (7)$$

This gives, for the imaginary part,

$$\mathop{\mathrm{Im}\,F_\ell(q^2)}_{q^2>0} = \frac{16\pi\sqrt{q^2+\mu^2}}{q}\sin^2\delta_\ell \qquad (8)$$

But we have

$$|F_\ell(q^2)|^2 = (16\pi)^2(q^2+\mu^2)/q^2\,\sin^2\delta_\ell \qquad (9)$$

Therefore,

$$\mathrm{Im}\,F_\ell(q^2) = \frac{q}{16\pi\sqrt{q^2+\mu^2}}\,|F_\ell(q^2)|^2,\ \text{when } q^2>0 \qquad (10)$$

This is true above $q^2 = 0$ and below the first inelastic threshold.

We must obtain $\Im m F_\ell$ for unphysical values of also, in order to evaluate the second integral in equation (4). To obtain $\Im m F_\ell$ in the unphysical region, we use the expression for $F(s,t,u)$ in terms of its absorptive parts:

$$F(s,t,u) = \frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{F_t(s,t')dt'}{t'-t} + \frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{F_u(s,u')du'}{u'-u} \tag{11}$$

$$\Im m F(s,t,u) = - F_t(s,t) - F_u(s,u) \tag{12}$$

or

$$\Im m F(q^2,\cos\theta) = - \Im m F(q_t^2,\cos\theta_t) - \Im m F(q^2_u,\cos\theta_u) \tag{12a}$$

$$t = -2q^2(1-\cos\theta) \; ; \; \cos\theta_s = 1 + t/2q^2$$
$$u = -2q^2(1+\cos\theta) \qquad\qquad = -1 - u/2q^2 \tag{13}$$

Similarly, $\cos\theta_t = 1 + \dfrac{s}{2q_t^2} \; ; \; q_t^2 = \dfrac{t}{4} - \mu^2$

and $\cos\theta_u = -1 - \dfrac{s}{2q_u^2} \; ; \; q_u^2 = \dfrac{u}{4} - \mu^2$

Therefore (12a) may be written

$$\Im m F(q^2,\cos\theta) = - \Im m F\left(q_t^2, 1 + \frac{s}{2q_t^2}\right) - \Im m F\left(q_u^2, -1 - \frac{s}{2q_u^2}\right) \tag{15}$$

Therefore

$$\underset{q^2<-\mu^2}{\mathrm{Im}\, F_\ell(q^2, \cos\theta)} = -\frac{1}{2}\int_{-1}^{+1} \mathrm{Im}\, F\left(q_t^2, 1+\frac{\Delta}{2q_t^2}\right) P_\ell\left(1+\frac{t}{2q^2}\right) d(\cos\theta)$$

$$-\frac{1}{2}\int_{-1}^{+1} \mathrm{Im}\, F\left(q_u^2, -1-\frac{\Delta}{2q_u^2}\right) P_\ell\left(1+\frac{t}{2q^2}\right) d(\cos\theta) \qquad (16)$$

$$= +\int_{-\mu^2}^{-q^2-\mu^2} \mathrm{Im}\, F\left(q_t^2, 1+\frac{\Delta}{2q_t^2}\right) P_\ell\left(1+\frac{t}{2q^2}\right) dq_t^2/q^2$$

$$+\int_{-\mu^2}^{-q^2-\mu^2} \mathrm{Im}\, F\left(q_u^2, -1-\frac{\Delta}{2q_u^2}\right) P_\ell\left(1+\frac{t}{2q^2}\right) dq_u^2/q^2 \qquad (17)$$

We have made the following change of variables:

$$\cos\theta = 1 + 2(q_t^2+\mu^2)/q^2$$
$$d(\cos\theta) = 2/q^2\, d(q_t^2) \qquad (17a)$$

$$\cos\theta = -1 - 4(q_u^2+\mu^2)/2q^2$$
$$d(\cos\theta) = -2dq_u^2/q^2 \qquad (17b)$$

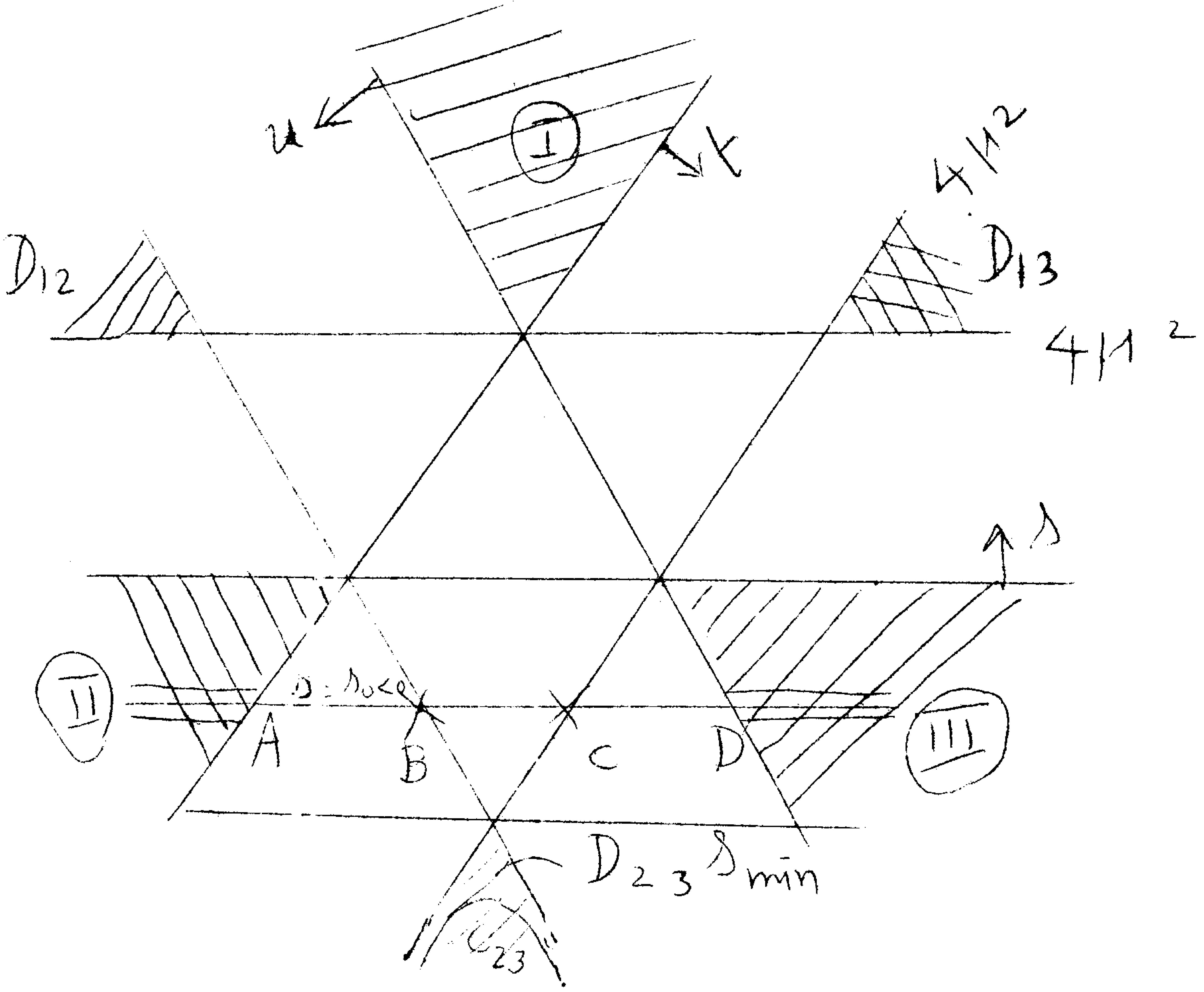
We know that below the threshold $q_t^2=0$, $q_u^2=0$, the absorptive parts are zero. Hence we may take the lower limits of integration in (17) as 0 instead of $-\mu^2$. The imaginary parts of the amplitudes that occur in the integrals are for unphysical values of the scattering angle.

Last time we obtained the expressions for the imaginary part of the scattering amplitude:

$$\mathrm{Im}\, F_\ell(q^2) = 16\pi \frac{\sqrt{q^2+\mu^2}}{q} \sin^2\delta_\ell$$
$$= \frac{q}{16\pi\sqrt{q^2+\mu^2}} \left| F_\ell(q^2) \right|^2 ,$$

which is valid for $q^2 \geq 0$ and below the inelastic threshold.

The thresholds for the s, t and u channels are each $4\mu^2$. The physical regions of the channels are shown in the Mandelstam diagram below; the regions where the double spectral functions are non-zero are also shown in the diagram:



When we are in the region

$$s \leq 0$$

$$\text{i.e.} \quad q^2 \leq -\mu^2$$

i.e., in the physical region of the t channel, we have

$$\operatorname{Im} F(s,t,u) = -F_t(s,t,u) - F_u(s,t,u$$
$$= -\operatorname{Im} F(t,s,u) - \operatorname{Im} F(u,t,$$

as we are in the physical region of the t channel. We then obtain, as we have seen earlier,

$$\operatorname{Im} F_\ell(q^2) = \int_0^{-q^2-\mu^2} \operatorname{Im} F\left(q_t^2, 1+\frac{s}{2q_t^2}\right) P_\ell\left(1+2\frac{q_t}{q}\right.$$
$$q^2 < \mu^2 \qquad dq_t^2/q^2$$
$$+ \int_0^{-q^2-\mu^2} \operatorname{Im} F\left(q_u^2, -1-s/2q_u^2\right) P_\ell\left(-1-2\left(\frac{q_u^2+\mu^2}{q^2}\right)\right) dq$$

For a given value s_0 of s, we will have to integrate over the lines AB and CD for the 2 integrals, as shown in the diagram. These regions are entirely unphysical.

The scattering angle is given by

$$\cos\theta = 1 + t/2q^2$$

The values of $\cos\theta$ involved are unphysical.

We have

$$\operatorname{Im} F(q^2, \cos\theta) = \sum_\ell \operatorname{Im} F_\ell(q^2) P_\ell(\cos$$

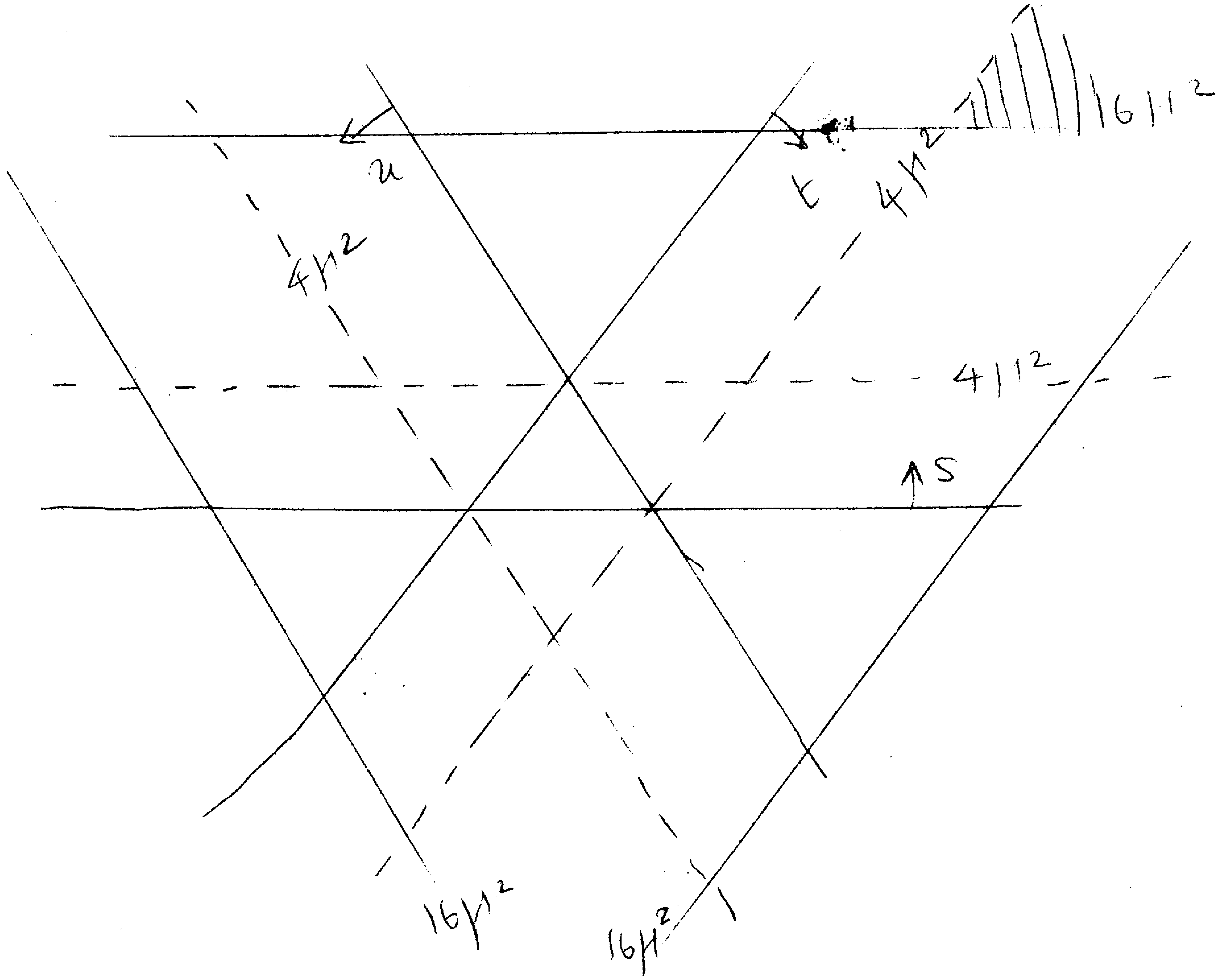
when θ is real.

$$\mathrm{Im}\,F_\ell(q^2) = \int_0^{-q^2-\mu^2} \sum_{\ell'} \mathrm{Im}\,F_{\ell'}(q_t^2)\, P_{\ell'}\left(1+\frac{s}{2q_t^2}\right) P_\ell\left(1+\frac{2(q_t^2+\mu^2)}{q^2}\right) dq_t^2/q^2$$
$$+ \int_0^{-q^2-\mu^2} \sum_{\ell'} \mathrm{Im}\,F_{\ell'}(q_u^2)\, P_{\ell'}\left(-1-\frac{s}{2q_u^2}\right) P_\ell\left(-1-\frac{2(q_u^2+}{q^2}\right) dq_u^2/q^2$$

The Legendre polynomial expansion will be valid so long as $\mathrm{Im}\,F(q^2, \cos\theta)$ has no singularity. Below the value $s = s_{min}$ given by $t = u = 4\mu^2$ the amplitude $F(q^2, \cos\theta)$ has a singularity, as the double spectral function ρ_{23} is non-zero.

However, we have seen that the boundary curves are only asymptotic to the lines $t = 4\mu^2$, $u = 4\mu^2$, so that the double spectral function is non-zero only well below $s = s_{min}$

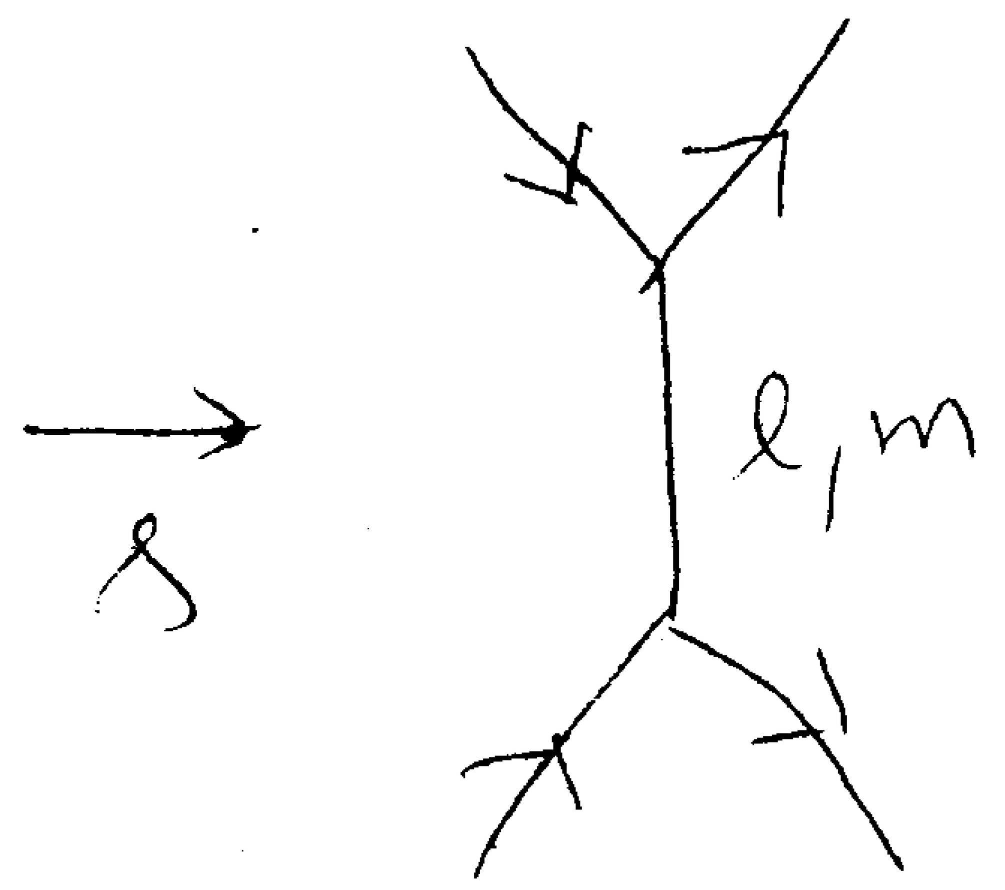
In $\pi\pi$ scattering, the lowest mass intermediate states are shown below.



The double spectral functions will be non-zero in the region $s > 4\mu^3$, $t > 16\mu^2$, and in the region $s > 16\mu^2$, $t > 4\mu^2$.

It may be dangerous to extrapolate the physical amplitude for large unphysical $\cos\theta$, as $P_\ell(\cos\theta) \sim (\cos\theta)$ for large $|\cos\theta|$, and the Legendre expansion may not converge.

Consider the contribution of a single-particle exchange to the imaginary part of the partial-wave amplitude.



We have

$$F(q^2, \cos\theta) = \frac{g^2}{m^2 - t} + \cdots$$

$$F_\ell(q^2) = \frac{1}{2}\int_{-1}^{+1} \frac{g^2}{(m^2-t)} P_\ell(\cos\theta)\, d(\cos\theta)$$

$$= \frac{1}{2}\int_{-1}^{+1} \frac{g^2}{m^2-t} P_\ell\left(1+\frac{t}{2q^2}\right) d(\cos\theta)$$

$$= \frac{1}{2} g^2 \int_{-4q^2}^{0} \frac{1}{m^2-t} P_\ell\left(1+\frac{t}{2q^2}\right)\frac{dt}{2q^2}$$

Then (as t is approached from below, $t - i\varepsilon$)

$$\operatorname{Im} F_\ell(q^2) = -\frac{\pi}{2} g^2 P_\ell\left(1+\frac{m^2}{2q^2}\right)\frac{1}{2q^2},$$

or

$$\operatorname{Im} F_\ell(q^2)_{q^2 < -\mu^2} = \frac{-\pi}{4q^2} g^2 P_\ell\left(1+\frac{m^2}{2q^2}\right)$$

This is the procedure used in making a bootstrap calculation for, say, a $\pi\,\pi$ resonance.*

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* Prof. C.Zemach: Lectures at the Summer School at Kodaikanal, June 1963.

2. Methods of solving for partial-wave Amplitudes.

We shall discuss 2 methods of solving for the partial-wave amplitude, which differ in the way the right-hand cut is treated.

We may distinguish two cases,

(i) where the most likely final state is the same as the initial state;

(ii) where the most likely final state is not the same as the initial state, i.e. the important intermediate state in the unitarity relation is different from the initial state, e.g. the $(\pi + N)$ state in photoproduction, or the $(\pi + \pi)$ state in .

Appropriate methods for dealing with these two situations are

(i) the N/D method; and

(ii) the Omires method.

(i) The N/D Method. This is a method of solving for the partial-wave amplitude. We shall consider the example of the scattering of 2 neutral equal-mass spinless particles.

The partial-wave amplitude $F_\ell(q^2)$ has 2 branch cuts:

(i) the right-hand cut, from $q^2 = 0$ to ∞

(ii) the left-hand cut, from $q^2 = -\infty$ to -0

Define the variable $\nu = q^2$.

We have

$$F_\ell(\nu) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\, F_\ell(\nu')\,d\nu'}{\nu'-\nu} + \frac{1}{\pi}\int_{-\infty}^{-\mu^2} \frac{T_\ell(\nu')\,d\nu'}{\nu'-\nu} \qquad (1)$$

We can show that we can always write $F_\ell(\nu)$ as a ratio of 2 functions, such that one of these has only the left-hand cut and one has only the right-hand cut.

$$F_\ell(\nu) = 16\pi\,\frac{N_\ell(\nu)}{D_\ell(\nu)} \qquad (2)$$

has only the left hand cut and $D_\ell(\nu)$ has only the right hand cut.

$$\left.\begin{aligned} \mathrm{Im}\, N_\ell(\nu) &= 0 \qquad \nu > 0 \\ \mathrm{Im}\, N_\ell(\nu) &= D_\ell(\nu)\,\frac{T_\ell(\nu)}{16\pi} \qquad \nu < -\mu^2 \end{aligned}\right\} \quad (3)$$

The dispersion relation for $N_\ell(\nu)$ is given by

$$N_\ell(\nu) = \int_{-\infty}^{-\mu^2} \frac{\mathrm{Im}\, N_\ell(\nu')\,d\nu'}{\nu'-\nu-i\varepsilon} = \frac{1}{16\pi}\int_{-\infty}^{-\mu^2} \frac{D_\ell(\nu')T_\ell(\nu')\,d\nu'}{\nu'-\nu-i\varepsilon} \quad (4)$$

where $D_\ell(\nu)$ is given by

$$\mathrm{Im}\, D_\ell(\nu) = 0 \quad , \quad \nu < 0$$

$$\mathrm{Im}\, D_\ell(\nu) = N_\ell\, \mathrm{Im}\frac{1}{F_\ell} = -N_\ell\sqrt{\frac{\nu}{\nu+\mu^2}} \qquad \nu > 0 \quad (5)$$

We can write a dispersion relation for $D_\ell(\nu)$ also; we shall write a dispersion relation for $D_\ell(\nu)$ with one subtraction

subtraction:

$$D_\ell(\nu) = D_\ell(0) - \frac{\nu}{\pi}\int_0^\infty N_\ell(\nu')\sqrt{\frac{\nu'}{\nu'+\mu^2}}\,\frac{d\nu'}{\nu'(\nu'-\nu-i\varepsilon)} \qquad (6)$$

We may choose $D_\ell(0) = 1$. (4) and (5) provide a pair of coupled equations which we may solve for $N(\nu)$ and $D(\nu)$, e.g. by iteration.

For π-π scattering, such a procedure was used by Chew and Mandelstam. For π-π scattering, because of charge complications, crossing symmetry leads to more complicated equations; we now have to solve for 3 amplitudes.

For unequal-mass particles, there are more complications, as there are more amplitudes to solve for, the 'crossed' processes being different from the direct process.

When we approximate the imaginary part of the amplitude in the crossed channel by a few partial waves, i.e., by a polynomial in t, we completely neglect the double spectral function. [The real part of the amplitude is treated exactly.]

To solve the coupled equations for the N and D functions, we proceed as follows:

$$A_\ell(\nu) = 16\pi\, N_\ell(\nu)/D_\ell(\nu)$$

$$N_\ell(\nu) = \frac{1}{\pi}\int_{-\infty}^{-\mu^2} \frac{T_\ell(\nu')\,D_\ell(\nu')\,d\nu'}{16\pi\,(\nu'-\nu-i\varepsilon)};$$

$$D_\ell(\nu) = 1 - \frac{\nu}{\pi}\int_0^\infty \frac{N_\ell(\nu')}{\nu'(\nu'-\nu-i\varepsilon)}\sqrt{\frac{\nu'}{\nu'+\mu^2}}\,d\nu'$$

Subtracting the expression for $N_\ell(\nu)$ in terms of $D_\ell(\nu)$ we obtain

$$D_\ell(\nu) = 1 - \frac{\nu}{\pi} \int_0^\infty \sqrt{\frac{\nu'}{\nu'+\mu^2}} \frac{1}{\pi} \int_{-\infty}^{-\mu^2} \frac{T_\ell(\nu'') D_\ell(\nu'') d\nu''}{16\pi(\nu''-\nu'-i\varepsilon)}$$

or

$$D_\ell(\nu) = 1 + \frac{\nu}{\pi} \int_{-\infty}^{-\mu^2} K_\ell(\nu',\nu) D_\ell(\nu') d\nu'$$

where

$$K_\ell(\nu,\nu') = \frac{2}{\pi} \frac{T_\ell(\nu')}{16\pi} \left[\frac{\tan^{-1}\sqrt{\frac{\nu'+\mu^2}{-\nu'}}}{\sqrt{-\nu'(\nu'+\mu^2)}} - \frac{\tan^{-1}\sqrt{\frac{\nu+\mu^2}{-\nu}}}{\sqrt{-\nu(\nu+\mu^2)}} \right]$$

We thus have an integral equation for $D_\ell(\nu,\nu')$ <u>provided</u> we know $T_\ell(\nu,\nu')$, the discontinuity on the left-hand cut.

To find this, we need to do some extrapolation.

One may try to solve by successive approximations, such that in the first approximation, we take $T_\ell \approx$ a constant.

We get

$$N_\ell(\nu) \equiv c, \quad \text{a constant.}$$

Subtracting this into the integral for $D_\ell(\nu)$ we obtain

$$D_\ell(\nu) = 1 - c\frac{\nu}{\pi} \int_0^\infty \sqrt{\frac{\nu'}{\nu'+\mu^2}} d\nu' \frac{1}{\nu'(\nu'-\nu-i\xi)}$$

Taking the real and imaginary parts of this equation gives

$$\mathrm{Re}\, D_\ell(\nu) = 1 - c\frac{\nu}{\pi} P \int_0^\infty d\nu' \sqrt{\frac{\nu'}{\nu'+\mu^2}} \frac{1}{\nu'(\nu'-\nu)}$$

$$= A - c\sqrt{\frac{\nu}{\nu+1}} \log(\sqrt{\nu} + \sqrt{\nu+1})$$

and

$$\mathrm{Im}\, D_\ell(\nu) = -c\sqrt{\frac{\nu}{\nu+\mu^2}}$$

$\mathrm{Im}\, D_\ell(\nu)$ is closely related to $\cot\delta_\ell$ where δ_ℓ is the phase shift. We have

$$\cot\delta_\ell = \frac{\mathrm{Re}\, A_\ell}{\mathrm{Im}\, A_\ell} = \frac{\mathrm{Re}\left(\frac{N_\ell}{D_\ell}\right)}{\mathrm{Im}\left(\frac{N_\ell}{D_\ell}\right)} = \frac{\mathrm{Re}\, N_\ell\ \mathrm{Re}\frac{1}{D_\ell}}{\mathrm{Re}\, N_\ell\ \mathrm{Im}\frac{1}{D_\ell}}$$

[since $\mathrm{Im}\, N = 0$ to the right of $\nu = -\mu^2$

$$= \frac{\mathrm{Re}\, D_\ell}{\mathrm{Im}\, D_\ell} = \frac{\mathrm{Re}\, D_\ell}{\left[-c\sqrt{\frac{\nu}{\nu+\mu^2}}\right]}$$

or

$$\sqrt{\frac{\nu}{\nu+\mu^2}}\cot\delta_\ell = -\frac{1}{c}\mathrm{Re}\, D$$

$$= -\frac{A}{c} + \sqrt{\frac{\nu}{\nu+\mu^2}}\log\left(\sqrt{\nu}+\sqrt{\nu+1}\right)$$

$$= -\frac{1}{a} + \sqrt{\frac{\nu}{\nu+\mu^2}}\log\left(\sqrt{\nu}+\sqrt{\nu+1}\right) \quad (a)$$

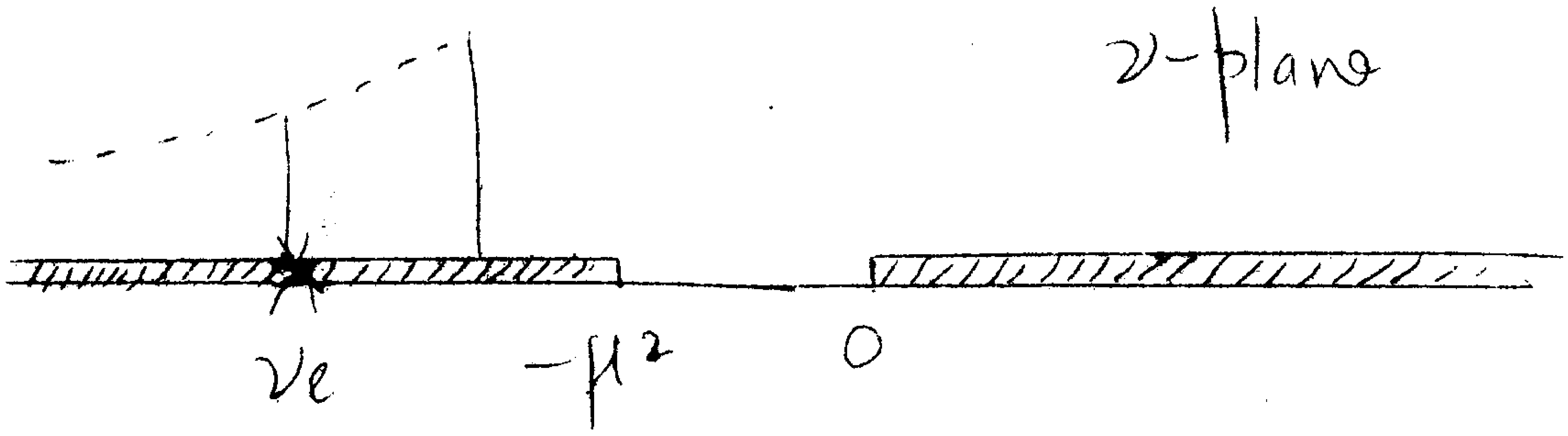
This is a modified effective-range formula, where $a = c/A$ is the effective scattering length. Equation (a) is known as the Chew-Mandelstam effective-range formula.

It can be expected to be valid for $\nu \ll \mu^2$, and only for the S-wave. For higher partial waves, taking $N \approx$ a constant would be a bad approximation.

[Note: for $\nu \ll \mu^2$, $\sqrt{\frac{\nu}{\nu+\mu^2}} \approx \frac{\sqrt{\nu}}{\mu} = q/\mu$]

In the next approximation, we may try to approximate the discontinuity on the cut by a pole.

For instance, if there is a resonance in the crossed channel, the discontinuity along the left hand cut would be of the shape indicated in the figure.



We may be able to get a gord representation of the discontinuity by replacing N_e by a pole,

$$N_e \approx \Gamma/(\nu+\nu_i)\quad ,$$

with the residue Γ chosen suitably. Substituting this into the integral for $D_e(\nu)$ we obtain

$$D_e(\nu) = 1 - \frac{\nu+\nu_i}{\pi}\,\Gamma \int_0^\infty \frac{\sqrt{\nu'}}{\sqrt{\nu'+\mu^2}}\,\frac{d\nu'}{(\nu+\nu_i)^2(\nu'-\nu-i\varepsilon)}$$

For $\nu \ll \mu^2$ we may write

$$\mathrm{Im}\, D_e \approx -\frac{\Gamma\sqrt{\nu}}{\mu(\nu+\nu_i)}\,, \quad \text{putting} \sqrt{\frac{\nu'}{\nu'+\mu^2}} \approx \sqrt{\frac{\nu'}{\mu}}$$

$$\mathrm{Re}\, D_e \approx 1 - \frac{\Gamma}{2\mu\sqrt{\nu_i}}\left(\frac{\nu_i+\nu}{\nu_i-\nu}\right)$$

We then obtain

$$\cot\delta \approx \frac{\mathrm{Re}\, D_e}{\mathrm{Im}\, D_e} \approx -\mu\,\frac{(\nu+\nu_i)}{\Gamma(\nu)} + \frac{\nu_i-\nu}{2\sqrt{\nu_i}\sqrt{\nu}}$$

or

$$\sqrt{\nu}\,\cot\delta = \left(-\frac{\mu\nu_i}{\Gamma} + \frac{1}{2\sqrt{\nu_i}}\right) - \nu\left[\frac{\mu}{\Gamma} + \frac{1}{2\sqrt{\nu_i}}\right]$$

All the above is true for the scattering of equal-mass particles, e.g. nucleon-nucleon scattering neglecting spin.

For the $P-$ wave, we would need at least 2 poles.

The S-wave scattering length a_0 willbe given by

$$-\frac{1}{a_0} = \left[-\frac{\mu\nu_i}{\Gamma} + \sqrt{\nu_i}/2\right]$$

Suppose Γ is positive, i.e., the 'potential' is attractive.

<u>Note</u>: $\quad \frac{1}{a_0} = 0 \quad \text{if} \quad \frac{\mu\nu_i}{\Gamma} = \sqrt{\nu_i}/2$

or $\quad \Gamma = 2\mu\sqrt{\nu}$

When Γ has this value, we obtain an infinite scattering length, which corresponds to a bound state at threshold. For this value of Γ, $\mathrm{Re}\,D_\ell = 0$ for $\nu = 0$. For $\Gamma > 2\mu\sqrt{\nu_i}$, $\mathrm{Re}\,D_\ell$ can have a zero for a negative value of ν, i.e., a bound state. [The S-matrix has a pole at this point.]

The value of ν_i corresponds roughly to the range of the interaction.

When the potential is repulsive, Γ is negative, and the scattering length is always positive. We may work backward and deduce Γ and ν_i from the experimental values of the scattering length and the effective range.

Note: In nucleon-nucleon scattering, the largest mass state that can be exchanged is a pion. Therefore, the left-hand-cut starts at $\nu = -\mu^2/4$, if μ is the pion-mass.

[For pion-pion scattering, the cut starts at $\nu = -\mu^2$.]

For nucleon-nucleon scattering, the value of ν_1 obtained from the scattering length and effective range lies to the left of $\nu = -\mu^2$ so that 2-pion exchange must also be important in low energy scattering.

In $\pi\pi$ scattering, the nearby part of the left-hand-cut will be dominated by the ρ- resonance. The farther part of the left-hand-cut may be approximated by a phenomenological pole at $\nu = \nu_2$ where residue and position are fitted to give the best value.

A discouraging feature is that the S-wave amplitude depends strongly on this pole.

A different approach is the bootstrap approach, where one approximates the whole of the left-hand cut by a ρ exchange and fits the width and position of the ρ such that this gives a ρ resonance in the direct channel with the same values for the ρ width and mass.

(ii) The Omne's method. The N/D method was most appropriate with elastic uniterity, when we had the relation

$$\operatorname{Im} F_\ell = \frac{q}{8\pi W} |F_\ell|^2$$

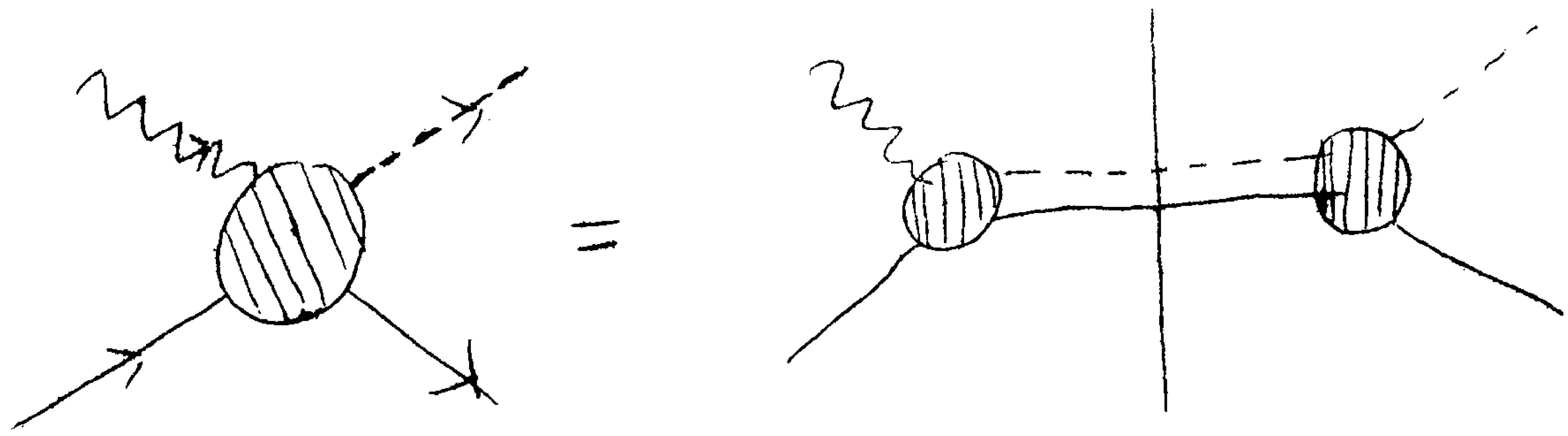
We have other situations, e.g.

$$\gamma + N \longrightarrow \pi + N$$

where the unitarity condition may be written in the form

$$\mathrm{Im}\, F_\ell = \frac{q}{8\pi W}\, G_\ell^*(q^2)\, F_\ell(q^2)$$

where $G_\ell(q^2)$ is the scattering amplitude for π N scattering (i.e. the final state amplitude in this case).



The $(\pi + N)$ intermediate state gives a much more important contribution than the $\gamma + N$ intermediate state. We write

$$F_\ell(\nu) = \varphi_\ell(\nu) + \frac{1}{\pi}\int_0^\infty \frac{q'}{8\pi W'}\, \frac{G_\ell^*(\nu')\, F_\ell(\nu')\, d\nu'}{\nu' - \nu - i\varepsilon}$$

Keeping to the low energy region where the final-state interaction (i.e. π N scattering) can be described by a real phase shift, we may write

$$G_\ell(\nu') = \frac{8\pi W'}{q'}\, e^{i\delta} \sin\delta$$

Substituting this into the original equation, we obtain

$$F_\ell(\nu) = \varphi_\ell(\nu) + \frac{1}{\pi}\int_0^\infty e^{-i\delta}\frac{\sin\delta\, F_\ell(\nu')d\nu'}{\nu'-\nu-i\varepsilon}$$

The solution of this has been given by Omnes; it is just

$$F_\ell(\nu) = \left[\varphi_\ell(\nu) + \frac{1}{\pi}e^{\rho(\nu)+i\delta(\nu)}\int_0^\infty \frac{\varphi_\ell(\nu')\sin\delta(\nu')e^{-\rho(\nu')}}{\nu'-\nu-i\xi}\right.$$

where

$$\rho(\nu) = \frac{1}{\pi}P\int_0^\infty \frac{\delta(\nu')}{\nu'-\nu}d\nu'$$

This is known as the Omnes-Muskhelishvili equation:

Ref: Omnis: Nuovo Cim.8, (1958)

Application of the Omnes Equation:

I) Final-State Interaction.

Consider a reaction where a pair of particles occurring in the final state interaction strongly.

Suppose that only this final state interaction is important, i.e., apart from this, the S-matrix = a constant. Then we have the equation

$$F_\ell(\nu) = c + \frac{1}{\pi}\int_0^\infty e^{-i\delta}\frac{\sin\delta\, F_\ell(\nu')d\nu'}{\nu'-\nu-i\varepsilon}$$

so that we have the solution

$$F_\ell(\nu) = c\left[1 + \frac{1}{\pi}e^{\rho(\nu)+i\delta(\nu)}\int_0^\infty \frac{\sin\delta(\nu')e^{-\rho(\nu')}d\nu'}{\nu'-\nu-i\xi}\right.$$

Define

$$f(\nu) = \exp\left(-\frac{1}{\pi}\int_0^\infty \frac{\delta(\nu')\, d\nu'}{\nu' - \nu + i\varepsilon}\right)$$

Therefore

$$f(\nu) = e^{i\delta}\, e^{-\frac{1}{\pi} P\int_0^\infty \frac{\delta(\nu')}{\nu'-\nu} d\nu'}$$

$$= e^{i\delta}\, e^{-[\rho(\nu)]} = e^{-[\rho(\nu) - i\delta(\nu)]}$$

$$\therefore\ f^*(\nu) = e^{-[\rho(\nu) + i\delta(\nu)]}$$

$$\therefore\ F_e(\nu) = c\left[1 + \frac{1}{\pi}\frac{1}{f^*(\nu)}\int_0^\infty d\nu' \frac{\sin\delta(\nu')}{\nu'-\nu-i\varepsilon} f(\nu') e^{-i}\right]$$

$$= c\left[1 + \frac{1}{\pi}\frac{1}{f^*(\nu)}\int_0^\infty \frac{\mathrm{Im}\, f(\nu')\, d\nu'}{\nu'-\nu-i\varepsilon}\right]$$

The dispersion relation for $f^*(\nu)$ is

$$f^*(\nu) = 1 - \frac{1}{\pi}\int_0^\infty \mathrm{Im} \frac{f(\nu')\, d\nu'}{\nu'-\nu-i\varepsilon}$$

so that the expression for $F_e(\nu)$ becomes

$$F_e(\nu) = C\left[1 + \frac{1}{\pi}\frac{1}{f^*(\nu)}\int_0^\infty \frac{\mathrm{Im}\, f(\nu')\, d\nu'}{\nu'-\nu-i\varepsilon}\right]$$

or

$$F_e(\nu) = c\left[1 + \frac{1}{f^*(\nu)}\{-f^*(\nu) + 1\}\right]$$

or

$$F_e(\nu) = c / f^*(\nu)$$

We can ask: When will final state interaction be important?

We have

$$\frac{1}{|f(\nu)|} = \exp \frac{1}{\pi} P \int_0^\infty \frac{\delta(\nu')}{\nu' - \nu} d\nu'$$

This will be large when $\delta(\nu)$ varies rapidly in the neighbourhood of the values of ν considered, and if $\delta(\nu)$ is large.

At a threshold for an inelastic channel, $\delta(\nu)$ varies rapidly. One may get a peak or a bump near threshold because of this .

Also, when there is an isobar found, final state interaction may be important.

How does this compare with the Watson approach to final-state interaction, in which

$$F_\ell(\nu) = C A_\ell(\nu) = C e^{i\delta} \sin\delta \sqrt{\frac{\nu+1}{\nu}}$$

The two approaches will give identical results when

$$A_\ell(\nu) = 1/f^*(\nu)$$

$A_\ell(\nu)$ has 2 cuts, but $f(\nu)$ has only the right-hand cut.

The two approaches will give the same result if the left-hand cut in $A_\ell(\nu)$ can be neglected.

This will be certainly true if the scattering amplitude is given by the Chew-Mandelstam effective range formula.

Now we shall consider an approach in which we try to get as much as possible of the double spectral function, in contrast to some of the approximation methods considered earlier in which we neglected the double spectral function.

Consider again the Mandelstam diagram for $\pi\pi$ scattering; we choose this process as it has the simplest features possible. The reason we were able to take a polynomial expansion or some such rough approximation for the double spectral function was that the regions in which the double spectral functions were non-zero were far from the region of s, t, u under consideration. We had the double dispersion representation

$$F(s,t,u) = \frac{1}{\pi^2}\int_{4\mu^2}^{\infty}\int_{4\mu^2}^{\infty} \frac{\rho_{12}(s',t')}{(s'-s)(t'+t)}\,ds'\,dt' + P_{stu} \tag{1}$$

or

$$F(s,t,u) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{A_\ell(s\,t')\,dt'}{t'-t} + \frac{1}{\pi}\int_{4\mu^2} \frac{A_u(s,u')\,d}{u'-u} \tag{2}$$

Let us try to calculate the double spectral function.

First note that the absorption parts are easily expressed in terms of the double spectral functions:

$$A_t(s,t,u) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{\rho_{12}(t,s')\,ds'}{s'-s} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{\rho_{13}(t,u')}{u'-u}\,du' \qquad (3)$$
$t > 4\mu^2$

i.e., for fixed t with $t > 4\mu^2$, $A_t(s,t)$ is a function of s with a cut from $4\mu^2$ to ∞ and a function of u with a cut from $4\mu^2$ to ∞.

Note that for $4\mu^2 < t < 16\mu^2$ the only possible intermediate state will be the (2π) state.

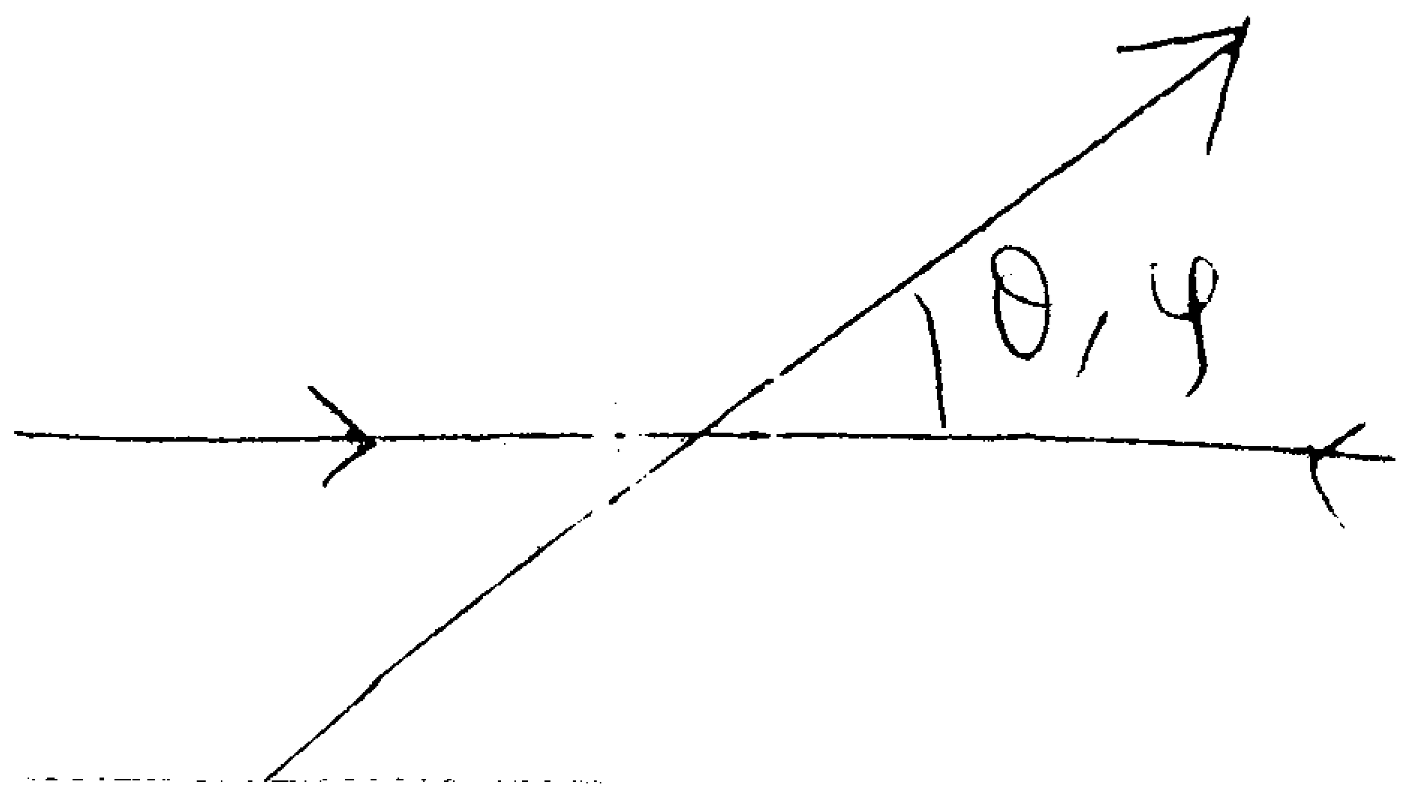
For $4\mu^2 < s < 16\mu^2$, the unitarity condition gives the relation

$$\mathrm{Im}\,F(q^2,\cos\theta) = A_s(u,t) = \frac{q}{32\pi^2 W}\int F^*(q^2,\cos\ldots)\,F(q^2,\cos\theta_3\ldots$$

[We have

$$\mathrm{Im}\,F \sim \sum F^*_{i\to m}\,F_{f\to n} \qquad (5)$$

as the intermediate state is a 2-particle state, the summation over intermediate states, for a given total energy, reduces to an integration over the angles.]



We have $\cos\theta_{32} = \cos\theta_{12}\cos\theta_{13} + \sin\theta_{12}\sin\theta_{13}$

(6)

We now substitute equation (2) into equation (4), and obtain

$$A_s(u,t) = \frac{q}{32\pi^2 w}\int d\Omega_3 \frac{1}{\pi}\int_{4M^2}^{\infty}\int_{4M^2}^{\infty}\frac{A_t^*(s,t')dt'}{t'+2q^2(1-\cos\theta_{13})} \times \frac{dt'' A_t(s,t'')}{t''+2q^2(1-\cos\theta_{32})} + \text{other terms} \quad (7)$$

The 3 other terms contain integrals of products of the form

$$A_u^*(s,u')A_u(s,u'')\,;\; A_u^*(s,u')A_t(s,t'')$$

and $A_t^*(s,t')A_u(s,u'')$

We must substitute the expression (6) for $\cos\theta_{23}$ into the denominators of (7), and carry out the integration over $d\Omega_3$ We note that

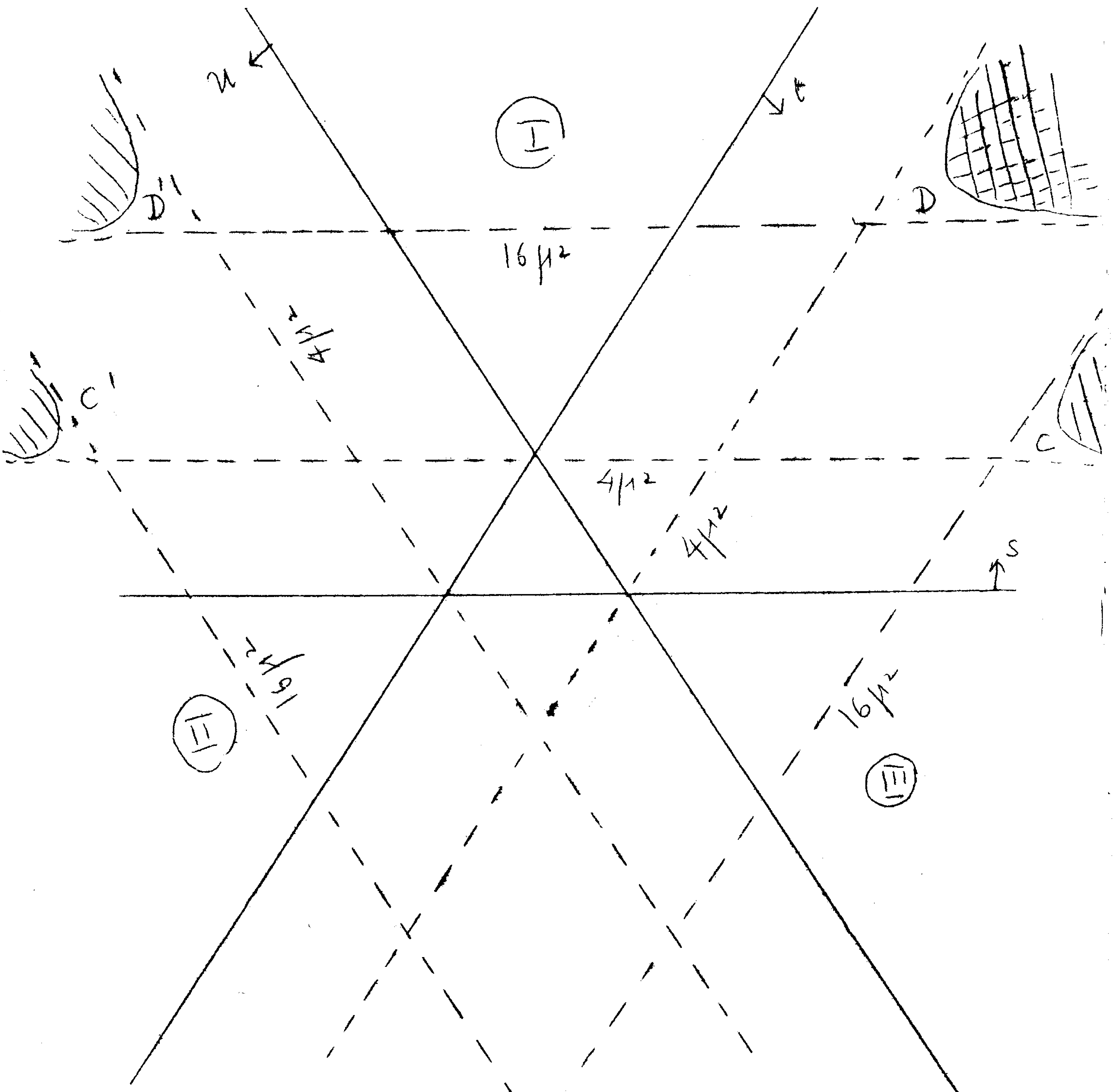
$$\frac{q}{\omega}\int\frac{d\Omega_3}{(t''+2q^2(1-\cos\theta_{13}))(t'''+2q^2(1-\cos\theta_{32}))} \quad (8)$$

$$= 2\pi\int K(s,t',t'',t''')/(t'-t)\,dt'$$

where $K(s,t',t'',t''') = \theta(b)/\sqrt{s_b}$, (9)

where $\theta(t)$ is a step function.

Note: $t = -2q^2(1-\cos\theta_{12})$


u
t
I
D''
D
16μ²
4μ²
C'
C
4μ²
4μ²
s
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16μ²
II
III

For $\pi\pi$ scattering, the function b is given by

$$b = (s-4\mu^2)[t'^2 + t''^2 + t'''^2 - 2t't'' - 2t't''' - 2t''t''' - 4t't''t''' \tag{10}$$

When $b = 0$, the minimum value of t' is obtained when

$$t'' = 4\mu^2$$

and $t''' = 4\mu^2$

When we put $t'' = 4\mu^2,\ t''' = 4\mu^2$, the curve $b = 0$ becomes

$$(s-4\mu^2)(t'-16\mu^2) - 64\mu^2 = 0 \tag{11}$$

This is the curve giving the boundary of the region in which the contribution to ρ_{12} of intermediate states such that $t > 16$, $s > 4\mu^2$ is non-vanishing.

Equation (11) has 2 branches, $C,\ C'$, as shown in the figure. We may similarly obtain the boundary curves (D, D') corresponding to the contribution of intermediate states with $s > 16\mu^2,\ t < 4\mu^2$

Substituting (8), (9), and (10) into (7)m we obtain

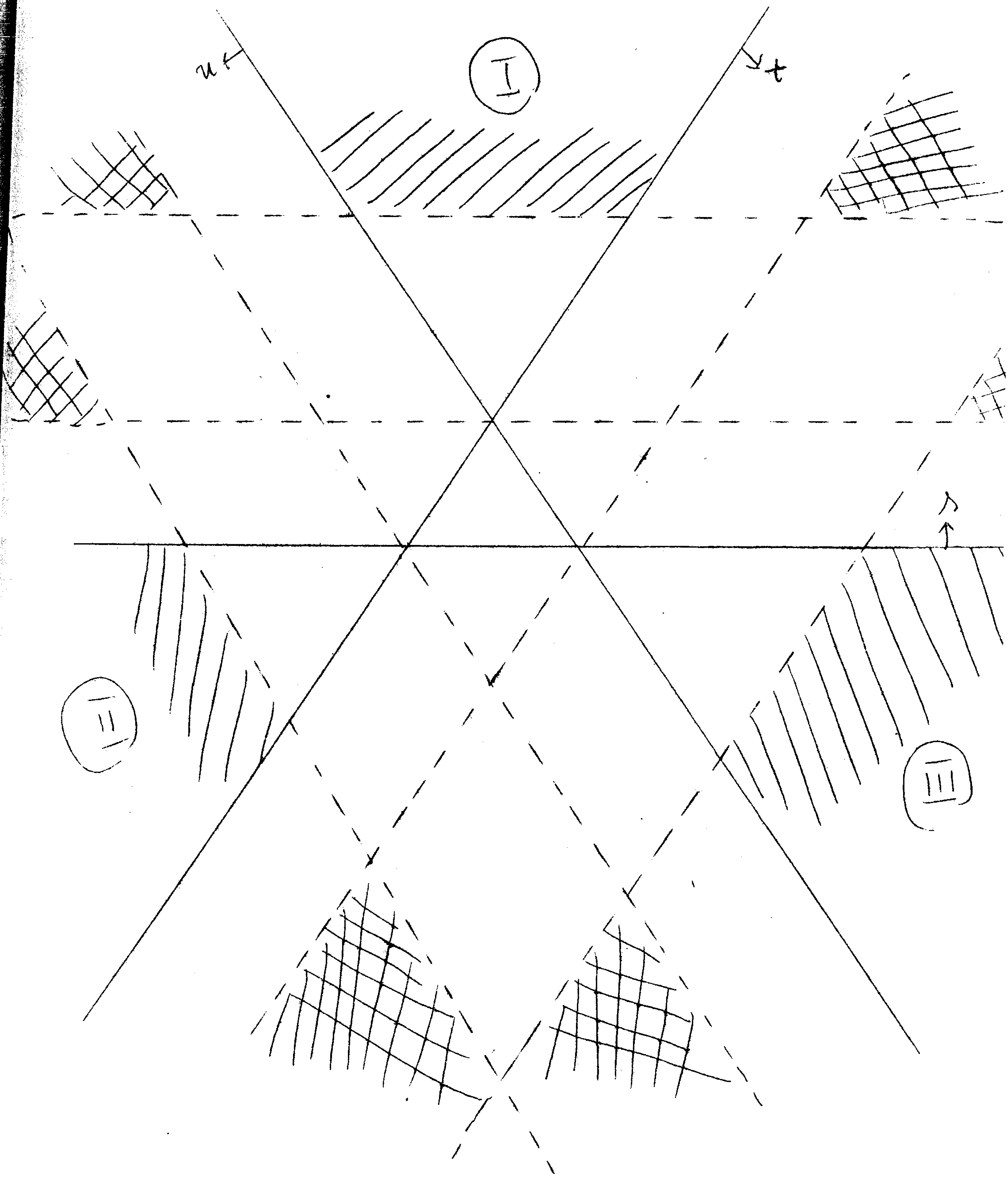
$$A_s(s,t,u) = \frac{1}{16\pi^2}\frac{1}{\pi}\int dt''\,dt'''\,dt'\,A_t^*(s,t'')\,A_t(s, \ldots)\,\frac{2\pi}{\pi}\,\frac{k(s,t',t'',t''')}{t'-t} \tag{12}$$

$+$ other terms

Comparing with the equation

$$A_s(s,t,u) = \frac{1}{\pi}\int \frac{\rho_{12}(s,t')\,dt'}{t'-t} + \frac{1}{\pi}\int \frac{\rho_{13}(s,u')\,du'}{u'-u} \qquad (13)$$

we obtain the following expressions for the double spectral functions in terms of the absorptive parts in the t and u channels.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$$\rho_{12}(s,t) = \frac{1}{16\pi^2}\int dt''dt'''\left[A_t^*(s,t'')A_t(s,t''') + A_u^*(s,t'')A_u(s,t''')\right]K(s,t,t'',t''') \qquad (14a)$$

and

$$\rho_{13}(s,u) = \frac{1}{16\pi^2}\int dt''dt'''\left[A_t^*(s,t'')A_u(s,t''') + A_u^*(s,t'')A_t(s,t''')\right]K(s,t,t'',t''') \qquad (14b)$$

These are known as the strip equations. They give the double spectral functions in the region $s < 16\mu^2$, $t > 4\mu^2$ and the region $u < 16\mu^2$, $s, t > 4\mu^2$ by crossing symmetry.

The regions between $s = 4\mu^2$ and $s = 16\mu^2$, $t = 4\mu^2$ and $t = 16\mu^2$, $u = 4\mu^2$ and $u = 16\mu^2$ are known as the 'strip' regions.

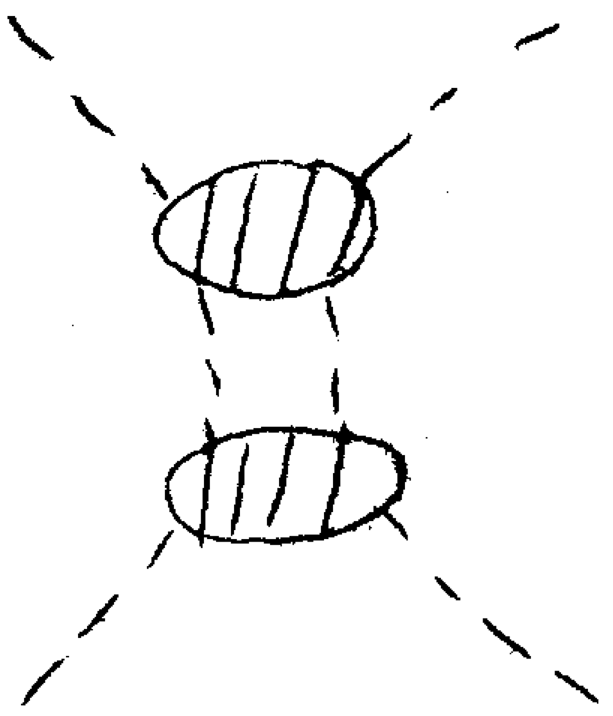
Our approximation of elastic unitarity gives the double spectral functions in these regions, but not in the other regions. [I II III in the diagram are the physical regions, while the strip regions are indicated by crossed lines.]

The strip equations will give the imaginary part of the scattering amplitude in the high-energy, low-momentum-transfer region, and the low-energy, high momentum-transfer region.

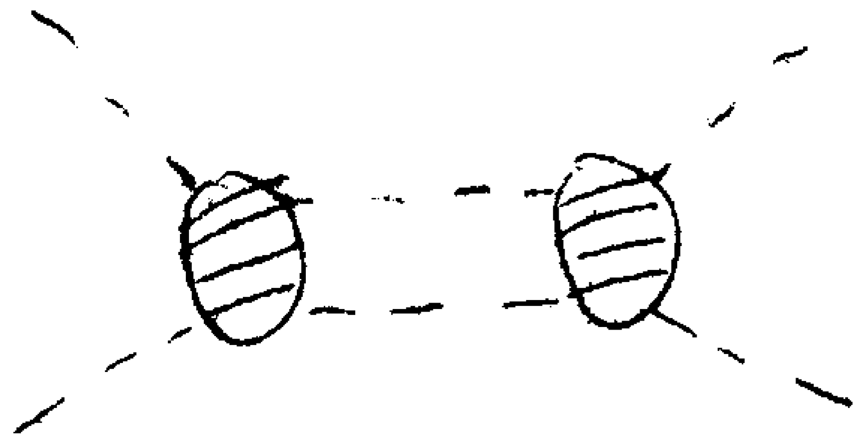
The strip equations can be made the basis of a step-by-step approximation; this is the basis of the approximation method of Cini, Fubini, and Stanghellini which led to the multiperipheral model.

The equations (14) can be used to give the spectral function in the strip region, <u>provided we know the scattering amplitude in the elastic region</u>. e.g., Using the known low-energy scattering amplitude, we can derive the scattering amplitude for high energies but low momentum transfers.

The type of diagrams corresponding to the strip approximation are the ones shown below.



and



We may write

$$F(s,t,u) = \frac{1}{\pi^2}\int_{4\mu^2}^{\infty} dt' \int_{4\mu^2}^{\infty} ds' \frac{\rho_{12}(s,t')}{(t'-t)(s'-s)} + P_{stu}($$

$$4\mu^2 < s' < 16\mu^2;\ t' > 16\mu^2 \qquad 4\mu^2 < t' < \ldots\ s' > 16\mu^2$$

$$\left.\begin{aligned} &\approx \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{ds'}{s'-s}\left(a_0 + a_1 t + a_2 t^2 + \cdots\right) + \\ &+ \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{dt'}{t'-t}\left(b_0 + b_1 s + b_2 s^2 + \cdots\right) \\ &+ P_{stu} \end{aligned}\right\} \quad (16)$$

where we have separated the region of integration in (15) into two regions, one in which $t' > 16\mu^2$ and one in which $s' > 16\mu^2$. These correspond, respectively, to the first and second term in (16).

In the first region, the factor $\int dt' \frac{\rho_{12}(s',t')}{t'-t}$ is approximated by a polynomial in t, in the second region, the factor $\left\{ \int ds' \frac{\rho_{12}(s',t')}{s'-s} \right.$ is approximated by a polynomial in s:

$$\int_{4\mu^2}^{\infty} dt' \frac{\rho(s,t')}{t'-t} \simeq a_0 + a_1 t + a_2 t^2 + \cdots, \; t \text{ small}$$

$$\int ds' \frac{\rho(s,t')}{s'-s} \simeq b_0 + b_1 s + b_2 s^2 + \cdots, \; s \text{ small} \tag{17}$$

This approximation by low-order polynomials is expected to be valid for small s and small t.

This is known as the <u>Cini-Fubini approximation</u>.

Suppose we have approximated the absorptive part by a polynomial as follows:

$$F(s,t,u) \simeq \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{ds'}{s'-s} \left[a_0(s) + a_1(s) t + \cdots \right] + P_{\text{ot}}\, u \tag{18}$$

Note: $$t = -2q^2(1-\cos\theta) = \left(s/2 - 2\mu^2\right)(1-\cos\theta)$$

Suppose we have a pole in the scattering amplitude at $s = m^2$ which contributes an amplitude

$$P_J(\cos\theta) / (s - m^2)$$

For a P- wave pole, this gives

$$F(s,t,u) \simeq \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{ds'}{s'-s}\left[a_0(s) + \delta(s-m^2)t\right] + P_{stu} \tag{19}$$

Suppose we have a resonance in s , say a P- wave resonance. Then

$$F(s,t,u) \simeq \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{ds'}{(s'-s)}\left[a_0(s) + a_1(s)t + \cdots\right] + P_{stu} \tag{20}$$

where $a_1(s) \approx \sin^2\delta$

Thus a bound state pole and a resonance give similar expressions for the amplitude.

The single-integral approximation of Cini and Fubini thus gives some justification for the polology techniques. So far we have considered only spinless particles with equal mass.

When the particles involved have spin, there are more amplitudes to be considered and the algebraic complications are much greater.

Let us summarise all the assumptions we have made till now.

1) The scattering amplitude $F(s,t,u)$ is an analytic function of s, t, u with cuts on the positive s, t, u real axis, from $4\mu^2$ to ∞

$F(s,t,u)$ expressed as a function of the external momenta, has the same analytic properties as the formal sum of all Feynman graphs including all known particles.

The method of obtaining the singularities of these graphs are known as the Landau rules.

2) The discontinuity across the cuts are given by the unitarity relation.

2') The discontinuities across the cuts are given as non-linear functionals of the amplitudes; these are given by the Cutkosky rules.

3) $F(s,t,u)$ is founded by a polynomial in s, t, u (as each of these variables $\to \infty$).

All these properties are summarized by the Mandelstam representation, which was the last word in strong interactions a few years ago.

However, such an approach soon faced difficulties.

1) The first of these arises when we consider particles of high spin. The exchange of a particle with spin J, contributes a pole to the scattering amplitude.

$$F(s,t,u) = \frac{P_J(\cos\theta_t)}{t-m^2} + F'(s,u) \qquad (21)$$

$$= P_J\left(1+\frac{s}{\frac{t}{2}-2m^2}\right)\Big/(t-m^2) + F'(s,u) \qquad (21a)$$

as

$$\cos\theta_t = 1+\frac{s}{2q_t^2} = 1+\frac{s}{\frac{t}{2}-2m^2} \qquad (22)$$

The first term in (21) thus has the asymptotic behaviour

$$F(s, t, u) \sim s^{j} \quad , \quad s \to \infty \tag{23}$$

This would suggest that the exchange of a system with high spin j would give an amplitude that $\sim s^{j}$ at high energies.

However, experiments at the highest energies seem to show that total cross sections go to a constant.

This is expressed by the Pomeranchuk rules.

In the Pomeranchuk model, the scattering amplitude at high enerties is almost pure imaginary, as would be expected if there was almost pure absorption.

$$F(s, t, u) \simeq i\, s\, f(t) \tag{24}$$

$$\left.\begin{aligned} f_{cM} &= F(s, t, u) / 8\pi\sqrt{s} \\ \sigma_{tot} &= \frac{1}{q}\, \mathrm{Im}\, f(0, 0) \\ \sigma_{tot} &\approx F(s, t, u)/s \end{aligned}\right\} \tag{25}$$

2) A more subtle difficulty arises when we examine whether the Pomeranchuk model is compatible with the analyticity property implied by the Mandelstam representation. The latter had been examined by Froissart, and have been studied more recently by A. Martin under much less restrictive assumptions. The result is that the Pomeranchuk model of 'diffraction' scattering, which explains the observed constancy of total cross-sections, is not compatible with the analyticity implied by the Mandelstam representation.

Another paradox is the so called Gribov paradox, which arises when we try to use the Pomeranchuk model with the Mandelstam representation simultaneously. We can get $\rho_{12}(s,t)$ either from the equation

$$\operatorname{Im} F(s,t) \approx i s f(t)$$

$$= \frac{1}{\pi}\int \frac{\rho_{12}(s,t')\,dt'}{t'-t} + \frac{1}{\pi}\int \frac{\rho_{13}(s,u')\,du}{u'-u} \qquad (26)$$

or from the unitarity relation as given by the strip equation.

The expressions for $\rho_{12}(s,t)$ obtained in these two ways are strongly different.

This arises because we extend the Pomeranchuk expression

$$F(s,t,u) = i s f(t)$$

into the unphysical region in t . The paradox would be resolved if the amplitude in the unphysical region were <u>not</u> of the Pomeranchuk form, but of an oscillatory form

$$F(s,t,u) = i\, s^{\alpha(t)}\, f(t) \tag{27}$$

such that $\alpha(t) \leq 1$ when $t < 0$ and $\alpha(t)$ is complex when $t > 0$.

(27) was postulated by Chew et al in order to resolve the Gribov paradox. Later they found that this behaviour could be obtained in potential scattering in Regge's formulation.

XI. Regge Poles.

1. Analytic continuation in ℓ :

We have seen that the Gribov paradox would be resolved if the scattering amplitude had the asymptotic behaviour

$$A \rightarrow i\, s^{\alpha(t)} f(t) , \qquad s \rightarrow \infty$$

where $\alpha(t)$ is real and ≤ 1 for $t < 0$, and is complex when $t > 0$.

This was suggested by Chew. It was then found that Regge had obtained, in potential scattering, the behaviour

$$A(s,t) \rightarrow \beta(t)\, t^{\alpha(s)} , \quad t \rightarrow \infty \text{ for fixed}$$

If we assume crossing symmetry, then large t at fixed s would mean large energy in the crossed channel, for a fixed value of the momentum transfer.

We shall briefly consider Regge's work in potential scattering.

We consider the case of the scattering of 2 equal mass, neutral, spinless particles by a superposition of Yukawa potentials.

We may write the partial wave expansion

$$A(q^2, \cos\theta) = \sum_{\ell} a_{\ell}(s)\, P_{\ell}(\cos\theta) \tag{1}$$

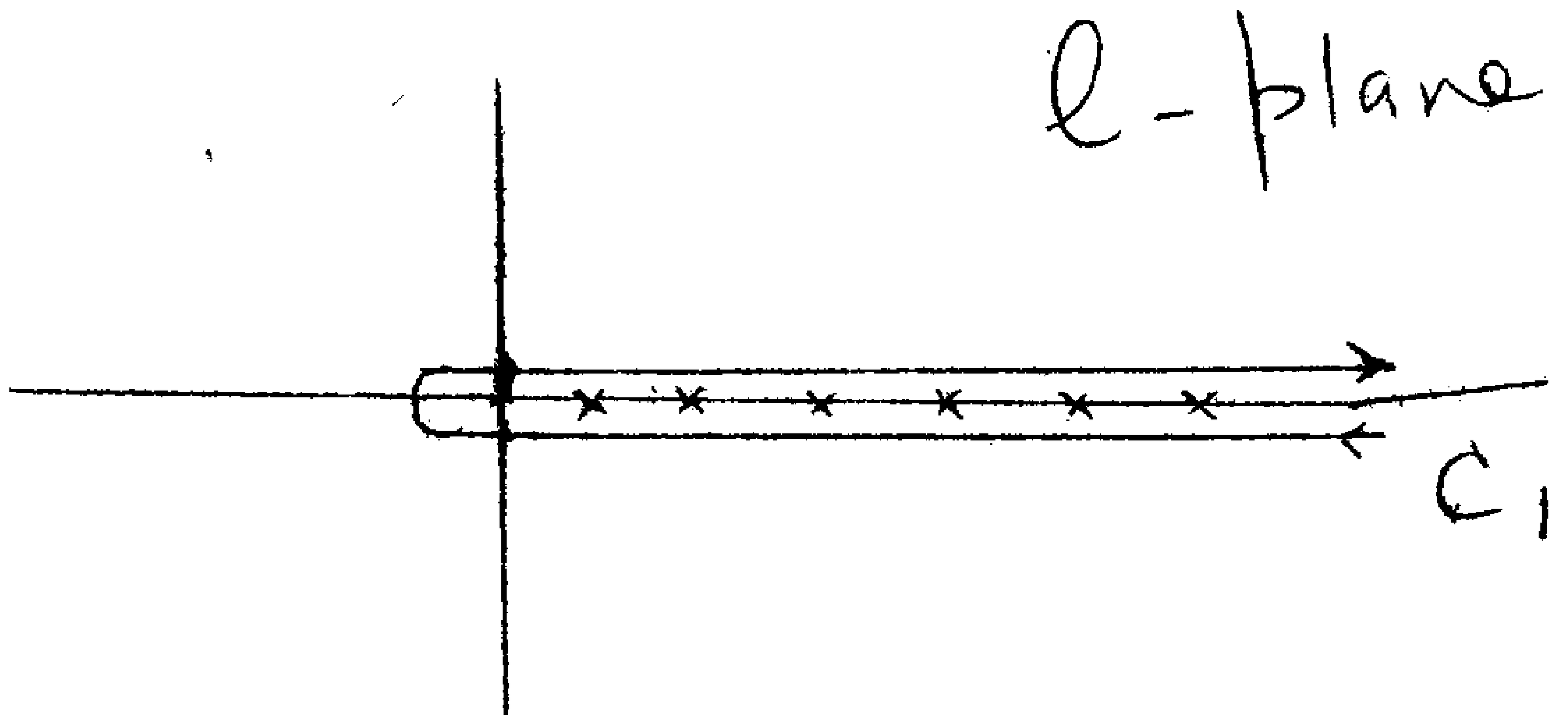
where $a_{\ell}(s)$ may be obtained by inversion,

$$a_{\ell}(s) = \frac{1}{2}\int_{-1}^{+1} A(s, \cos\theta)\, P_{\ell}(\cos\theta)\, d(\cos\theta) \tag{2}$$

We may replace (1) by a Watson-Sommerfeld transform:

$$A(q^2, \cos\theta) = -\frac{1}{2\pi i}\oint_{C_1} \frac{a(\ell, s)\, P_{\ell}(-\cos\theta)\, d\ell}{\frac{1}{\pi}\sin \pi\ell} \tag{3}$$

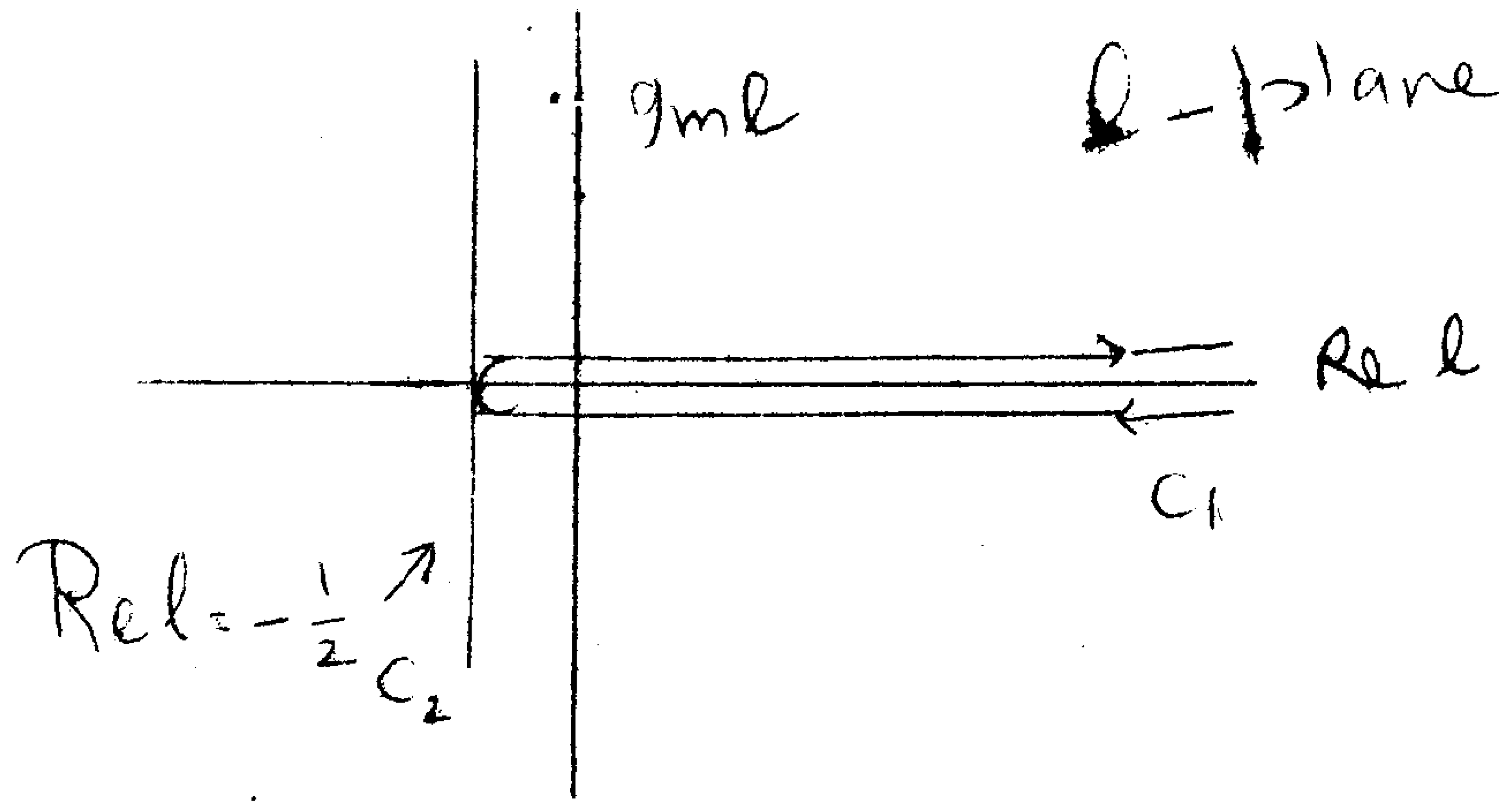
where C_1 is a contour encircling all the positive integers, as shown in the figure.



We assume that $a(\ell, s)$ has no poles on the positive real ℓ axis, so that the sum of the residues at the poles of $\sin \pi\ell$, at $\ell = 0, 1, 2, \ldots$ gives back the partial-wave expansion (1).

Note: $P_\ell(-\cos\theta) = (-1)^\ell P_\ell(\cos\theta)$ (4)

Regge proved that, for a superposition of Yukawa potentials, the partial-wave amplitude $a(\ell, s)$ is meromorphic in the region $\mathrm{Im}\,\ell > 0$, $\mathrm{Re}\,\ell > -\frac{1}{2}$ of the ℓ- plane.



We may deform the contour C_1 into the position C_2, if we can assume that the integral vanishes at infinity in the right-half ℓ plane. We must then write (3) as the sum of a line integral, along C_2 and the contribution of the poles crossed by the contour during the deformation from the position C_1 to the position C_2. We then write

$$A(q^2, \cos\theta) = -\frac{1}{2i}\oint \frac{a(\ell, s)\, P_\ell(\cos\theta)\, d\ell}{\sin\pi\ell} - \pi \sum_\alpha \beta_{\alpha(s)}(s)\, P_{\alpha(s)}(-\cos\theta) \Big/ \sin\pi\alpha(s) \qquad (4)$$

The pole terms in (4) are known as the ''Regge Poles''.

For large $\cos\theta$, the integral in (4) goes as $(\cos\theta)^{-1/2}$ and we know that

$$P_\alpha(-\cos\theta) \sim (\cos\theta)^\alpha \qquad |\cos\theta| \to \infty$$

Then for large $\cos\theta$, the pole terms in (4) will dominate in (4):

$$A(q^2, \cos\theta) \underset{|\cos\theta| \gg 1}{\sim} -\pi \sum_{\alpha} \frac{\beta_{\alpha(s)} P_{\alpha(s)}(-\cos\theta)}{\sin[\pi\alpha(s)]} \tag{5}$$

In particular, for very large $\cos\theta$, the Regge pole with the largest value of $\mathrm{Re}\,\alpha$ dominates in (4), and we have

$$A(q^2, \cos\theta) \sim -\pi \frac{\beta_{\alpha_0}(s)\, P_{\alpha_0}(-\cos\theta)}{\sin(\pi\alpha_0)} \tag{6}$$

What we have done is to interpolate the partial-wave amplitude $a_\ell(s)$ known for physical values of ℓ, $\ell = 0, 1, 2 \ldots$ by a function $a(\ell, s)$ for all values of ℓ, by using the Sommerfeld-Watson transform.

It can be proved that this interpolation is unique provided we have a suitable asymptotic behaviour for $|s| \to \infty$ This can be proved using Carlson's Theorem.

The advantage of the Regge representation over the partial-wave expansion is that the latter converges only inside the Lehmann ellipse in the $\cos\theta$ plane.

If we note that

$$\cos\theta = 1 + t/2q^2 \tag{7}$$

we have the asymptotic behaviour

$$A(s,t) \sim \beta(s)\, t^{\alpha_0(s)} \qquad \begin{matrix} t \to \infty \\ s \text{ fixed} \end{matrix} \tag{8}$$

where $\alpha_0(s)$ is the Regge trajectory with the largest value of

Re α. With crossing symmetry, this would imply that

$$A(s,t) \sim s^{\alpha_0(t)} f(t), \quad \begin{matrix} s \to \infty \\ t \text{ fixed} \end{matrix} \tag{9}$$

One may conjecture that such a behaviour is true not only for potential scattering but also in a complete theory of elementary particles. The contribution of a Regge pole in the crossed channel, to the scattering amplitude at high energies would be given by

$$-\pi \beta_\alpha(t)\, s^{\alpha(t)} / \sin[\pi\alpha(t)] \tag{10}$$

To give a constant total crosssection at high energies, we would then require that

$$\alpha(0) = 1$$

2. <u>The concept of signature:</u>

The Mandelstam Representation requires that

$$A(s,t,u) = \frac{1}{\pi^2} \iint \frac{\rho_{12}(s',t')\, ds'\, dt'}{(t'-t)(s'-s)} + P_{stu}$$

$$= \frac{1}{\pi} \int \frac{A_t(s,t')\, dt'}{t'-t} + \frac{1}{\pi} \int \frac{A_u(s,u')\, du'}{u'-u} \tag{11}$$

Writing t and u in terms of $\cos\theta$,

$$\left.\begin{aligned} t &= -2q^2(1-\cos\theta) \\ u &= -2q^2(1+\cos\theta) \end{aligned}\right\} \tag{12}$$

We may write the partial-wave projection of $A(s,t,u)$ as

$$a_\ell(s) = \frac{1}{2}\int_{-1}^{+1} A(s,t,u)\, P_\ell(\cos\theta)\, d(\cos\theta)$$

$$= \frac{1}{2\pi}\int dt' \int_{-1}^{+1} \frac{d(\cos\theta)\, A_t(s,t')}{t' + 2q^2(1 - \cos\theta)}$$

$$+ \frac{1}{2\pi}\int du' \int_{-1}^{+1} d(\cos\theta)\, \frac{A_u(s,u')}{u' + 2q^2(1+\cos\theta)} \qquad (13)$$

$$= \frac{1}{2\pi q^2}\int dt'\, A_t(s,t')\, Q_\ell\left(1 + \frac{t'}{2q^2}\right)$$

$$- \frac{1}{2\pi q^2}\int du'\, A_u(s,u')\, Q_\ell\left(-1 - u'/2q^2\right)$$

$$= \frac{1}{2\pi q^2}\int dt'\, A_t(s,t')\, Q_\ell(1 + t'/2q^2)$$

$$+ (-1)^\ell/2\pi q^2 \int du'\, A_u(s,u')\, Q_\ell\left(1 + \frac{u'}{2q^2}\right) \qquad (14)$$

since $$Q_\ell(-z) = (-1)^{\ell-1} Q_\ell(z) \qquad (15)$$

We know that

$$Q_\ell(z) \sim \left(\frac{1}{z + \sqrt{z^2+1}}\right)^\ell, \quad |z| \to \infty \qquad (16)$$

The integrals will converge for

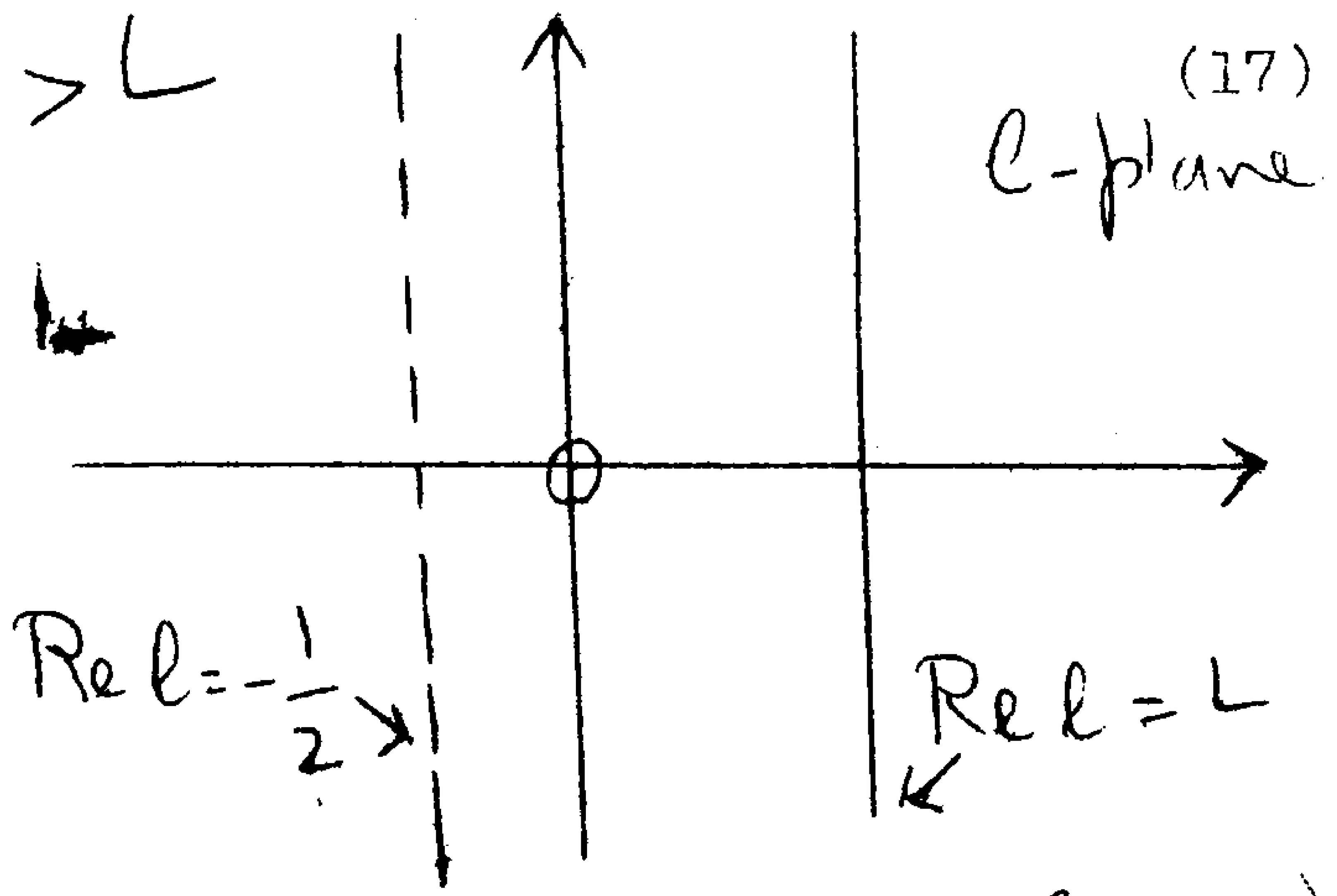
$$\ell > L$$

where L is related to the number of subtractions N by

$$L = (N+1)$$

We can therefore continue $a_\ell(s)$ to complex ℓ, provided

$$\mathrm{Re}\,\ell > L \tag{17}$$



Thus to the right of $\mathrm{Re}\,\ell = L$ in the ℓ- plane, $a(\ell, s)$ is an analytic function. The conjecture is that it is a meromorphic function of ℓ at least between $\mathrm{Re}\,\ell = -\frac{1}{2}$ and $\mathrm{Re}\,\ell = L$. The factor $(-1)^\ell$ causes a difficulty, [as it oscillates at $|\ell| \to \infty$. To get rid of this difficulty we define two new amplitudes:

$$a_\ell^{(+)}(s) \equiv a_\ell(s) \qquad \text{for even } \ell,$$

and

$$a_\ell^{(-)}(s) \equiv a_\ell(s) \qquad \text{for odd } \ell,$$

i.e.,

$$a_\ell^{(\pm)}(s) = \frac{1}{2}\int dz' \left[A_t(s, z') \pm A_u(s, z')\right] Q_\ell(z') \tag{18}$$

Thus, in general, the continuations to complex ℓ away from even and odd physical values of ℓ will be differemt. The two amplitudes $a^{(+)}(\ell, s)$ and $a^{(-)}(\ell, s)$ are said to have positive and negative signatures respectively. The positive signature amplitude $a^{(+)}(\ell, s)$ will have Regge poles corresponding to physical states only for even integral ℓ, and $a^{(-)}(\ell, s)$ only for odd integral ℓ.

The total scattering amplitude may be written

$$A(s,t) = \frac{1}{2}\left[A^{(+)}(s,+\cos\theta) + A^{(+)}(s,-\cos\theta) + A^{(-)}(s,\cos\theta) - A^{(-)}(s,-\cos\theta)\right] \tag{19}$$

A Regge pole with positive or negative signature will contribute to the amplitude a term

$$A_1(s,t) = -\frac{\pi}{2}\left[\frac{\beta_\alpha^{\pm}(s)P_\alpha(-\cos\theta)}{\sin\pi\alpha} \pm \frac{\beta_\alpha^{(\pm)}(s)\,P_\alpha(\cos\theta)}{\sin\pi\alpha}\right]$$

$$= -\frac{\pi}{2}\,\beta_\alpha^{\pm}(s)\frac{\left[P_\alpha(-\cos\theta) \pm P_\alpha(+\cos\theta)\right]}{[\sin\pi\alpha(s)]} \tag{20}$$

For non-integral values of l, $P_\alpha(\cos\theta)$ has a cut from -1 to $-\infty$, and $P_\alpha(-\cos\theta)$ has a cut from $+1$ to $+\infty$.

For a fixed value of t, $\cos\theta$ is related to s and t. The cut in $\cos\theta$ from $+1$ to $+\infty$ runs from $t = 0$ to $t = \infty$, while the cut in $\cos\theta$ from 1 to ∞ runs from $u = 0$ to $u = +\infty$.

For small values of s, these cuts in t and u are different from the cuts implied by the Mandelstam representation, which run from $t = 4\mu^2$ to $t = \infty$ and from $u = 4\mu^2$ to $u = +\infty$.

Similarly, we have, for a Regge pole in the crossed channel,

$$A(s,t) \simeq -\frac{\pi}{2}\frac{F(t)}{\sin\pi\alpha(t)}\left[P_\alpha(-\cos\theta_t) \pm P_\alpha(\cos\theta_t)\right] \tag{21}$$

$$\left.\begin{aligned} \cos\theta_t &= 1 + \frac{s}{2q^2} \\ P_\alpha(-\cos\theta_t) &= P_\alpha\left(-\left[1 + \frac{s}{2q_t^2}\right]\right) \end{aligned}\right\} \qquad (22)$$

We have

$$P_\alpha(-(z \pm i\varepsilon)) = 2\pi \sin\pi\alpha \, Q_\alpha(z) + e^{\mp i\pi\alpha} P_\alpha(z) \qquad (23)$$

[The first term vanishes at large z.]

Since we are interested in the physical sheet defined by $s = \lim_{\epsilon \to 0}(s + i\epsilon)$ we must take $e^{-i\pi\alpha}$ in the expression. We finally obtain

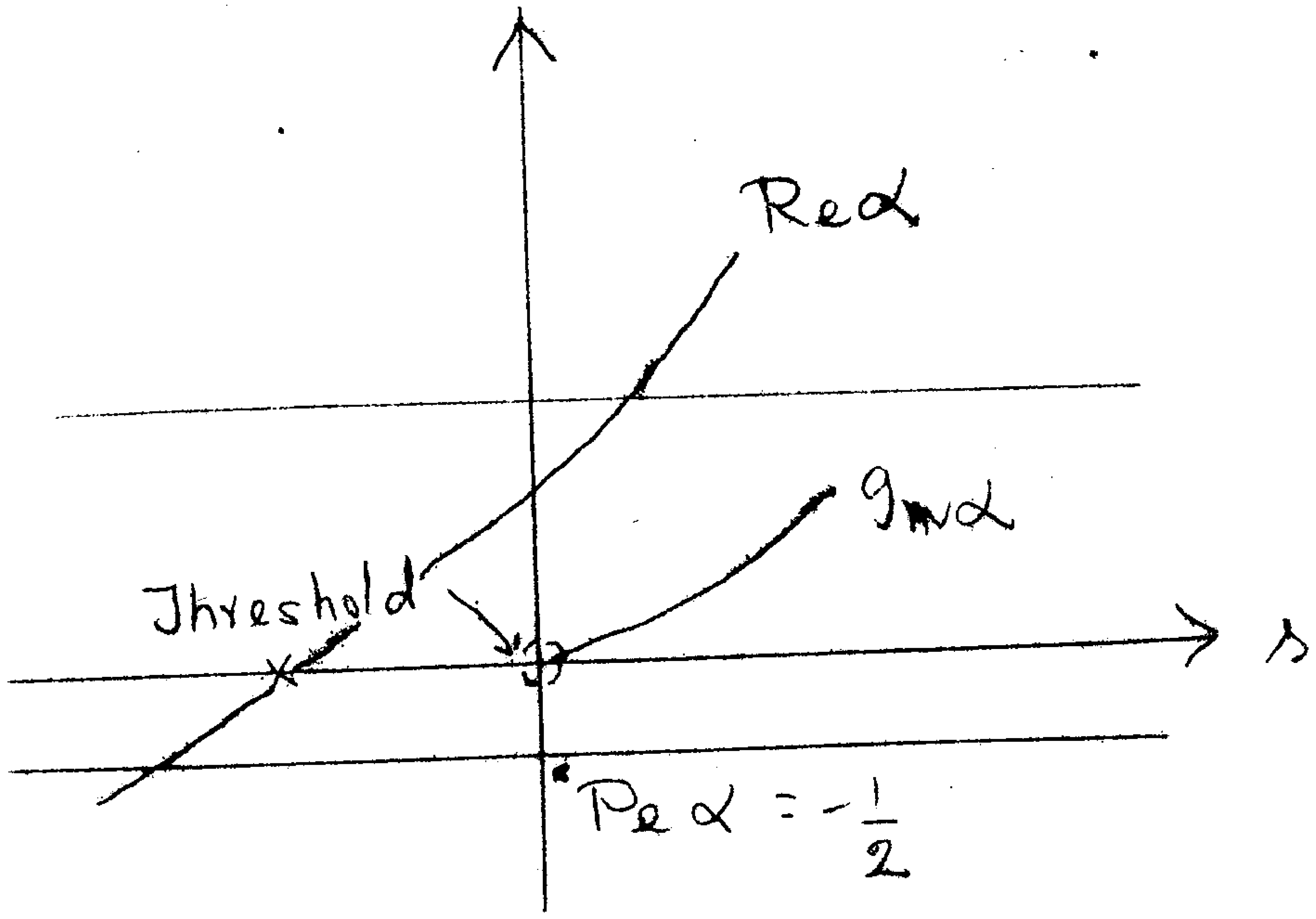
$$A(s,t) \simeq \frac{F(t)}{\sin\pi\alpha(t)}\left[1 \pm e^{-i\pi\alpha(t)}\right] s^{\alpha(t)} \qquad (24)$$

The dominant pole that determines the high-energy elastic cross-sections must have

$$\alpha(0) = 1$$

3. <u>Behaviour of the poles as a function of the energy:</u>

Consider scattering by a potential



If $\mathrm{Re}\,\alpha(s)$ crosses an integral value of ℓ , $\ell = 0, 1, 2, \ldots$ at a value of s below threshold, there is a pole at the corresponding value of s , with 'spin' ℓ . This will give a bound state if the signature is correct. When s rises above threshold, then $\mathrm{Im}\,\alpha(s) > 0$. If $\mathrm{Re}\,\alpha(s)$ now crosses an integral value of s , the corresponding pole in $A(s,t)$ will give a resonance if $\mathrm{Im}\,\alpha$ is small, e.g. If $\mathrm{Re}\,\alpha(s)$ crosses $\alpha = 1$ at $s = s_0$, the amplitude in the neighbourhood of $s = s_0$ is of the form

$$\frac{F(s)\; P_1(\cos\theta)}{[\mathrm{Re}\,\alpha'(s)](s - s_0) + i\,\mathrm{Im}\,\alpha / \mathrm{Re}\,\alpha'}$$

Let us now go from potential scattering to a relativistic theory.

Because of the signature factor, only poles at alternate integral values of   will correspond to physical states. The contribution of a pole to the differential cross-section will be

$$\frac{1}{8\pi s}\left|\frac{F(t)}{\sin\pi\alpha(t)}\right|^2\left|1+e^{-\pi i\,\alpha(t)}\right|^2 s^{2\alpha(t)}$$
$$= |G(t)|^2 s^{(2\alpha(t)-1)}$$

If $\alpha(t)$ is approximately linear in the region of t under consideration, then we have

$$\alpha(t) \simeq 1-\gamma t$$

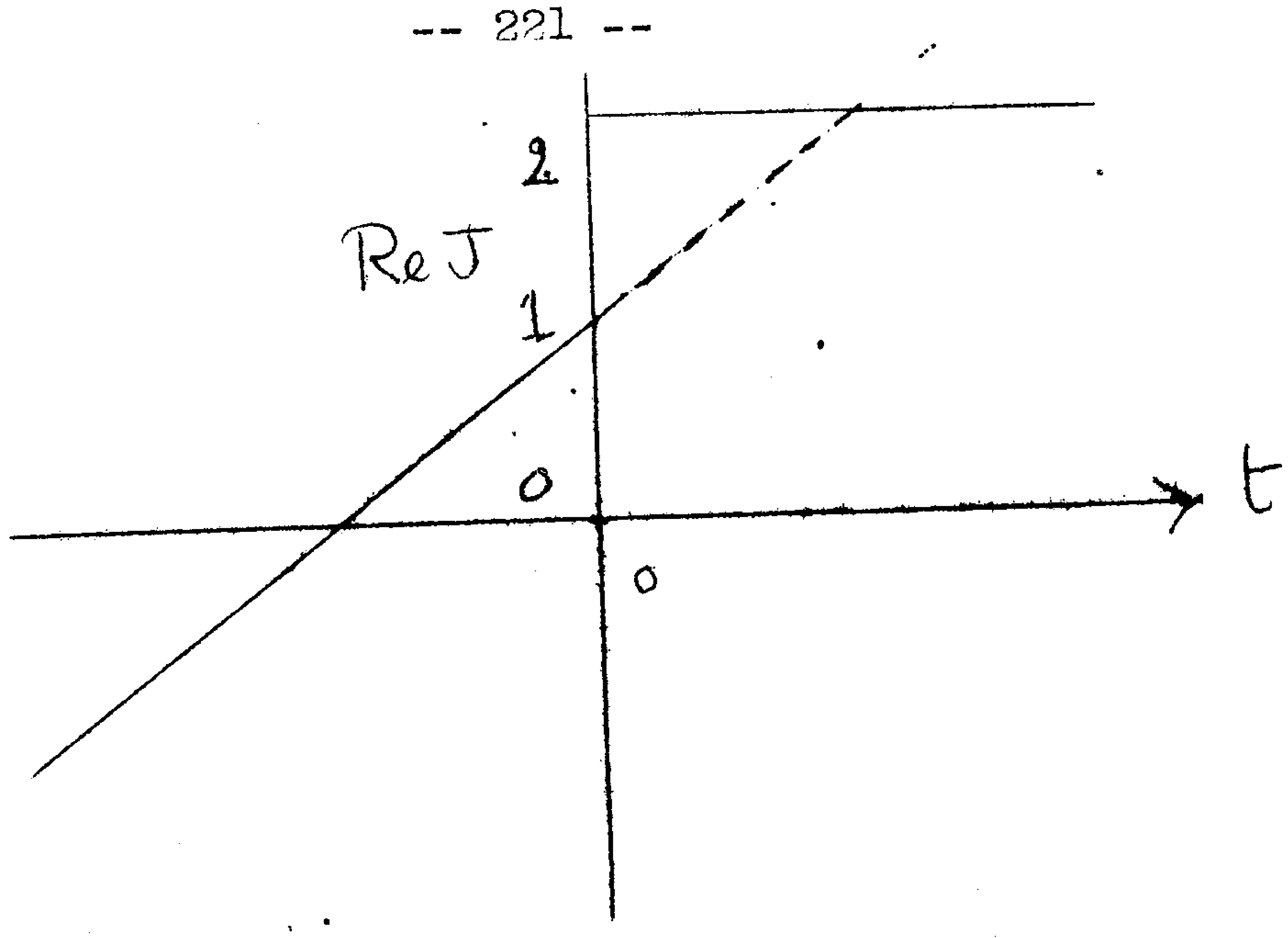
for small t, if $\alpha(t)$ is the trajectory that denominates high energy scattering. Then we have

$$\frac{d\sigma}{dt} \sim \frac{|A|^2}{s^2} = |G(t)|^2 s^{(2\alpha(t)-2)}$$
$$= |G(t)|^2 s^{-2\gamma t}$$

This predicts a shrinking of the diffraction peak; this was found to be true in p-p scattering from 5 to 20 Gev. and gave good support to the Regge pole hypothesis.

Such a shrinking would not be obtained from the model of diffraction scattering by an optical sphere.

The experiments on CERN gave for $\alpha(t)$ the shape indicated in the figure.



The slope was roughly $1/(\mathrm{GeV})^2$. A straight-line extrapolation of the trajectory to positive t would meet the point $J = 2$ at $t \simeq (1.25\ \mathrm{Bev})^2$. We may then expect a $J = 0$ $I = 0$, $S = 0$ particle with a mass $m =$ 1.~~2~~5 Bev. The f^0 meson was later observed at about this energy.

However, experiments at Brookhaven showed that the π-p and K-p diffraction peaks did not shrink with increasing energy.

It is quite possible that the f^0 meson is quite unrelated to the vacuum or Pomeranchuk trajectory.

Chew and Frautschi assumed that all elementary particles lie at the intersections of Regge trajectories. The contribution to the cross section of the exchange of the various poles would be

$$\sigma_{tot}(pp) = \sigma_\infty + R_\omega\, s^{(\alpha_\omega(0)-1)} + R_\rho\, s^{(\alpha_\rho(0)-1)} + \cdots$$

We similarly obtain

$$\sigma_{tot}(p\bar{p}) = \sigma_\infty - R_\omega\, s^{(\alpha_\omega(0)-1)} + R_\rho\, s^{(\alpha_\rho(0)-1)} + \cdots$$

and

$$\sigma_{tot}(pn) = \sigma_{\infty} + R_{\omega}\, s^{(\alpha_{\omega}(0)-1)} - R_{\rho}\, s^{(\alpha_{\rho}(0)-1)} +$$

By comparing $\sigma_{tot}(pp)$, $\sigma_{tot}(p\bar{p})$ and $\sigma_{tot}(pn)$ we can estimate R_{ω} and R_{ρ}. It is observed, however, that $\sigma_{tot}(\bar{p}p)$ goes to a constant at large energies, while $\sigma_{tot}(pp)$ increases slowly.

To explain this, one may introduce a new trajectory which cancels the contribution of the ω in pp scattering but adds to it in p $\bar{p}$ scattering.

Recent experiments have shown that Regge poles may not be the whole story behind the complexities of strong interactions. The occurrence of branch points in the l plane is quite possible.

Even then, the dominant features of forward scattering at high energies seem to be obtainable from low energy data, just as with the strip approximation.

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