22

MATSCIENCE REPORT 22

LECTURES ON

RELATIVISTIC KINEMATICS AND POLARIZATION

BY

R. HAGEDORN,

Visiting Professor, MATSCIENCE

Notes prepared by

T. S. SANTHANAM and K. ANANTHANARAYANAN

THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS-4, INDIA.

MATSCIENCE REPORT 22

THE INSTITUTE OF MATHEMATICAL SCIENCES

MADRAS - 4 (India)

LECTURES+ ON RELATIVISTIC KINEMATICS AND POLARIZATION

By

R. Hagedorn++

Visiting Professor (November 1963 - January 1964)

MATSCIENCE, Madras.

Notes by

T.S.Santhanam and K.Ananthanarayanan

+These lectures are based on the contents of the book "Relativistic Kinematics" of the author, published by W.A.Benjamin INC, New York. 1963. In case of quotation, due reference should be made to this book.

++Permanent Address:

CERN, Geneva, Switzerland.

CONTENTS

$S(T)$ is equal to 1 when $F(X)$ is a scalar field and it is a matrix when $F(X)$ is a vector field.

We can write the same transformation in another form using the equation $X' = TX$ or $X = T^{-1}X'$ as

$$F'(X') = S(T) \quad F(T^{-1}X')$$

We can replace X' by X and write $F'(X) = S(T)F(T^{-1}X)$

Sec. 1.2
Lorentz Transformation:

Let K and K' be two reference systems moving with constant velocity with respect to each other. We call an 'event' or a world point the set of space-time coordinates

$$P = \{ x\, y\, z\, t \} \tag{1.4}$$

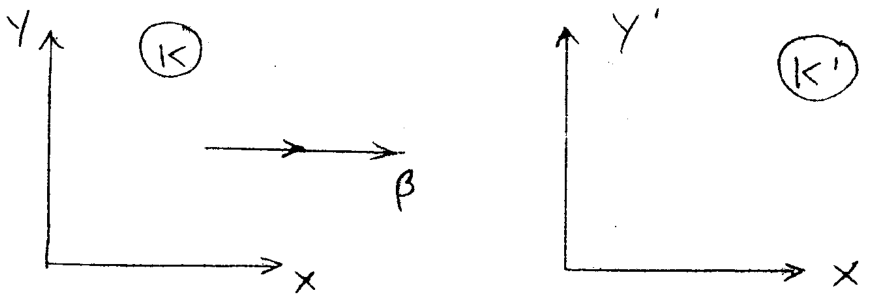


Fig. 1,2.

Let the directions of the axes of K and K' be parallel and such that x and x' axes coincide and are parallel to the relative velocity (see fig.1.2. Consider two points P and $\bar{P}$ connected by a light signal. Let P and $\bar{P}$ have coordinates $\{x\, y\, z\, t\}$ and $\{\bar{x}\, \bar{y}\, \bar{z}\, \bar{t}\}$ in the reference system K and $\{x'\, y'\, z'\, t'\}$ and $\{\bar{x}',\, \bar{y}',\, \bar{z}',\, \bar{t}'\}$ in the reference system K'.

The distance between P and $\bar{P}$ can be measured in K in two ways:

$$d^2 = (x-\bar{x})^2 + (y-\bar{y})^2 + (z-\bar{z})^2 \tag{1.5}$$

and
$$d^2 = (t-\bar{t})^2 \quad , \quad \because c = 1 \tag{1.6}$$

These two equations for d^2 (the square of the spatial-distance gives the equation
$$-\Delta x^2 - \Delta y^2 - \Delta z^2 + \Delta t^2 = 0$$
where $\Delta x^2 = (x-\bar{x})^2$

This equation is valid in K. In the reference system K' the corresponding equation is
$$-\Delta x'^2 - \Delta y'^2 - \Delta z^2 + \Delta t'^2 = 0$$

since the velocity of light is the same in the two inertial frames K and K' and equals unity in our unit system.

Let us denote these
$$ds^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$
and
$$ds'^2 = \Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2 \tag{1.7}$$

It is obvious that $ds \to 0$ implies $ds' \to 0$ as $P \to \bar{P}$ in both cases. Also the quantities ds and ds' are infinitesimal of the same order. So ds and ds' should be connected by the equation
$$ds = a\, ds'$$

Since K and K' are on an equal footing $ds' = a\, ds$ is also true. So $a = \pm 1$

Only $a = +1$ is valid as the group of transformation is continuous and connected.

Integrating the equations $ds = ds'$ we get

$$\int_{P_1}^{P_2} ds = \sqrt{-\Delta x^2 - \Delta y^2 - \Delta z^2 + \Delta t^2}$$

$$= \int_{P_1'}^{P_2'} ds' = \sqrt{-\Delta x'^2 - \Delta y'^2 - \Delta z'^2 + \Delta t'^2}$$

We conclude that 4 dimensional distance is a constant.

$$\Delta s^2 = -(\Delta x^2 + \Delta y^2 + \Delta z^2 - \Delta t^2), \quad \text{in } K$$

$$= -(\Delta x'^2 + \Delta y'^2 + \Delta z'^2 - \Delta t'^2), \quad \text{in } K' \tag{1.9}$$

It is possible to find a reference frame in which the two events P_1 and P_2 have <u>the same time</u> only if

$$\Delta s^2 \leqslant 0$$

as $$\Delta s^2 = -(\Delta x^2 + \Delta y^2 + \Delta z^2 - \Delta t^2)$$

$$= -(\Delta x^2 + \Delta y^2 + \Delta z^2) \text{ for } \Delta t = 0 \tag{1.9}$$

These two events are connected by space line distance.

Similarly it is possible to find a reference frame in which the two events P_1 and P_2 occur at the same place only if

$$\Delta s^2 \geqslant 0 \tag{1.11}$$

The two events are now connected by time like distances.

Let us take for one event the coordinates and consider all possible events with respect to this.

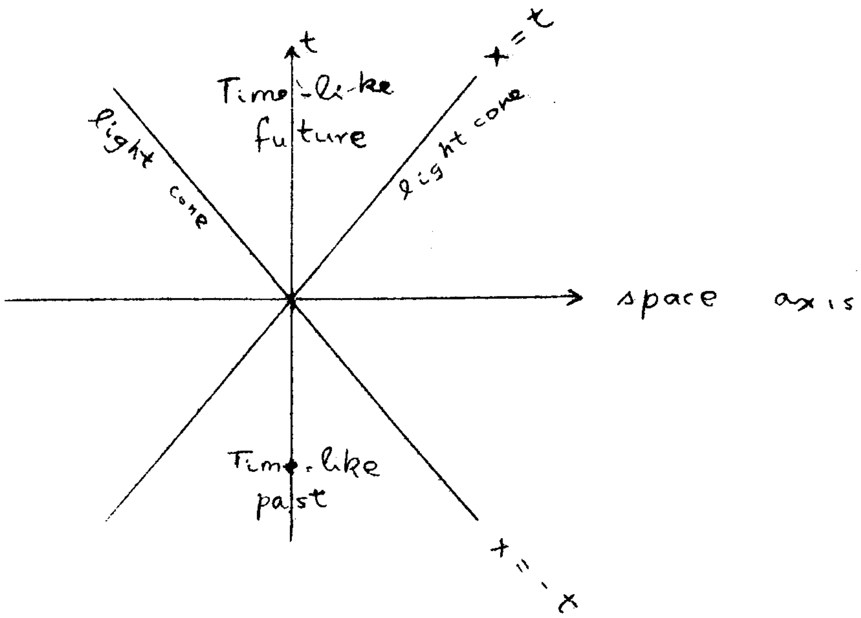


Fig. 1.3

The four dimensional distance from the origin is given by the invariant $s^2 = t^2 - x^2 - y^2 - z^2$

we can affix sign to the quantity

$$s = \sqrt{t^2 - x^2 - y^2 - z^2}$$

. We take the positive sign for the root whenever the point is in the upper half plane and negative sign for the root whenever the event is in the lower half plane.

Having done this we make the following statements.

(i) $s^2 = 0$ connects all those events with the origin which can be reached by a light signal. We call $s^2 = 0$ the light cone'

(ii) $s^2 > 0$ if $s > 0$, the event is in the forward light cone; if $s < 0$, the event is in the backward light cone.

Since all transformations form a connected continuous group they cannot transform an event from the forward light cone to the backward light cone.

The space-like points cannot be connected to the origin by light signal. So only those events in the backward light cone have an influence on O, and O can have influence on the events in the forward light cone. The space like points cannot interact with O. This is called the condition of causality.

<u>Problems:</u>

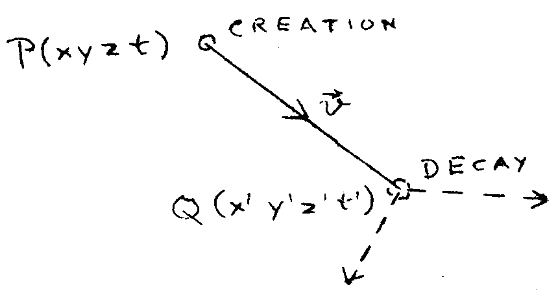
1. Define proper time of a moving body tobe the time shown by a clock which moves with that body. Use the invariance of Δs^2 to establish the lifetime of a particle measured by a clock in the lab system.

   (a) if the particle moves with constant velocity

   (b) if the particle moves arbitrarily

(2) Is the light quantum a stable particle?

Solution 1.



In the lab system we find the distance between creation and decay

$$\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

but in the particles system

$$\Delta s^2 = \Delta t'^2$$

since there it is at rest.

Hence
$$\Delta t' = \Delta t \sqrt{1 - \frac{\Delta x^2 + \Delta y^2 + \Delta z^2}{\Delta t^2}}$$

$$= \Delta t \sqrt{1 - \beta^2}$$

where β is the velocity of the particle in the lab system.

(a) $t_2' - t_1' = (t_2 - t_1)\sqrt{1 - \beta^2}$

(b) $t_2' - t_1' = \int_{t_1}^{t_2} dt \sqrt{1 - \beta^2}$

Solution 2:

Suppose γ quantum were instable then a lifetime should be definable and the only invariant way to define a lifetime is to measure it in the rest system of the particle (i.e. its proper time) Such a rest system does not exist by the constancy of the light

Even formally

lifetime $= t_2' - t_1'$

$$= (t_2 - t_1)\sqrt{1-\beta^2}$$

Since $\beta = 1$ and $(t_2 - t_1) = \infty$ (the observed lifetime in the lab), $(t_2' - t_1')$ is indeterminite.

Therefore the question is senseless for all particles of zero mass.

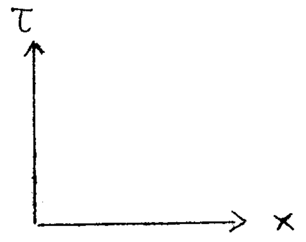
The Lorentz transformation from one inertial frame to another inertial frame

We define $\tau = it$ Now

$$\Delta s^2 = -(\Delta x^2 + \Delta y^2 + \Delta z^2 + \Delta \tau^2)$$

This is a quadratic form in four dimensional space $(x\ y\ z\ \tau)$ and is left invariant by the transformation sought.

Leaving out the trivial transformation viz. translations, the only continuous connected transformation leaving this quadratic form invariant is the rotation in the four dimensional space. Let us suppress the y and z coordinates in the discussion. This means that the relative velocity is parallel to the x axes which are themselves parallel and that we omit the three dimensional space rotations.



The general rotation of this space can be written as

$$\begin{aligned} x &= x' \cos\alpha - \tau' \sin\alpha \\ \tau &= x' \sin\alpha + \tau' \cos\alpha \end{aligned} \tag{1.12}$$

The origin in the K' frame has the coordinates $x' = 0$ and this is represented in K frame by the coordinates given by

$$\begin{aligned} x &= -\tau' \sin\alpha \\ \tau &= \tau' \cos\alpha \quad , \quad \frac{x}{\tau} = -\tan\alpha \end{aligned} \tag{1.13}$$

But as the origin of K' moves with constant velocity in the x direction

$$x = \beta t = \frac{\beta\tau}{i}$$

$$\frac{x}{\tau} = -i\beta \tag{1.14}$$

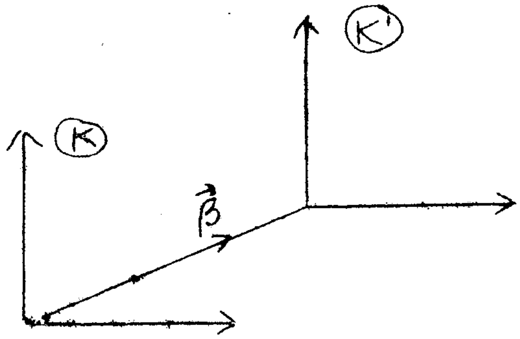
So we get from (1.13) and (1.14)

$$\begin{aligned} \tan\alpha &= i\beta \\ \cos\alpha &= \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{1-\beta^2}} = \gamma \\ \sin\alpha &= \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = i\beta\gamma \end{aligned} \tag{1.15}$$

Hence the transformation is

$$x = \gamma(x' + \beta t')$$
$$y = y' \quad ; \quad z = z' \qquad (1.16)$$
$$t = \gamma(t' + \beta x')$$

General transformation



The vector $\vec{x}'$ can be split into two parts $\vec{x}'_{\parallel}$ parallel to $\vec{\beta}$ and the vector $\vec{x}'_{\perp}$ perpendicular to $\vec{\beta}$ so that

$$\vec{x}' = \vec{x}'_{\parallel} + \vec{x}'_{\perp}$$
$$\vec{x}'_{\parallel} = \vec{\beta}\,\frac{\vec{\beta}\cdot\vec{x}}{\beta^2} \qquad (1.17)$$
$$\vec{x}'_{\perp} = \vec{x}' - \vec{\beta}\,\frac{\vec{\beta}\cdot\vec{x}}{\beta^2}$$

Only $\vec{x}'_{\parallel}$ is transformed and $\vec{x}'_{\perp}$ remains unaltered by the Lorentz transformation.

We write the equations

$$\vec{x}'_{\perp} = \vec{x}_{\perp}$$

and

$$\vec{x}_{\parallel} = \gamma (\vec{x}_{\parallel}' + \vec{\beta} t)$$

$$t = \gamma (t' + \vec{\beta} \cdot \vec{x}')$$

$$\vec{x} = \left(\vec{x}' - \vec{\beta} \frac{\vec{\beta} \cdot \vec{x}}{\beta^2} \right) + \gamma \vec{\beta} \frac{\vec{\beta} \cdot \vec{x}}{\beta^2} + \gamma \vec{\beta} t$$

$$= \vec{x}' + \beta \left[\gamma t' + \frac{\vec{\beta} \cdot \vec{x}}{\beta^2} (\gamma - 1) \right]$$

Now

$$\gamma^2 = \frac{1}{1 - \beta^2} ,$$

$$\beta^2 = \frac{\gamma^2 - 1}{\gamma^2} = \frac{(\gamma - 1)(\gamma + 1)}{\gamma^2}$$

Hence

$$\vec{x} = \vec{x}' + \vec{\beta} \left(\gamma t' + \frac{\gamma^2}{\gamma + 1} \vec{\beta} \cdot \vec{x}' \right)$$

$$t = \gamma \left[t + (\vec{\beta} \cdot \vec{x}') \right]$$

Inverse transformation

To get the inverse equation we interchange $\vec{X}'$ and $\vec{X}$ and change $\vec{\beta}$ to $-\vec{\beta}$, We have

$$\vec{x}' = \vec{x} + \vec{\beta}\gamma\left(\frac{\gamma}{\gamma+1}\vec{\beta}\cdot\vec{x} - t\right) \qquad (1.19)$$

$$t' = \gamma(t - \vec{\beta}\cdot\vec{x})$$

To verify this put $\beta = 0$, $\gamma = 1$. We get $\vec{x}' = \vec{x}$ and $t' = t$ which is true.

Sec.1.3

Lorentz Contraction

Lorentz contraction is based on the measurements of length as the space distance between two points at the same time.

$$\Delta\vec{x} = \Delta\vec{x}' + \vec{\beta}\gamma\left(\frac{\gamma}{\gamma+1}\vec{\beta}\cdot\Delta\vec{x}' + \Delta t'\right)$$

$$\Delta t = \gamma(\Delta t' + \vec{\beta}\cdot\Delta\vec{x}')$$

But as the two events are observed at the same time in the reference system K,

$$\Delta t = 0$$

So

$$\Delta t' = -\vec{\beta}\cdot\Delta\vec{x}'$$

Substituting this in the first equation we get

$$\Delta\vec{x} = \Delta\vec{x}' - \vec{\beta}\frac{\gamma}{\gamma+1}\vec{\beta}\cdot\Delta\vec{x} \qquad (1.20)$$

So the generalized formula for Lorentz contraction is

$$\Delta \vec{x} = \Delta \vec{x}' - \vec{\beta} \cdot \frac{\gamma}{\gamma+1}\left(\vec{\beta} \cdot \Delta \vec{x}'\right) \qquad (1.21)$$

If $\vec{\beta}$ is parallel to $\Delta \vec{x}'$ we find

$$\Delta \vec{x} = \Delta \vec{x}' \left(1 - \frac{\gamma}{\gamma+1} \beta^2\right)$$

$$= \Delta \vec{x}' \left(1 - \frac{\gamma}{\gamma+1} \frac{(\gamma-1)(\gamma+1)}{\gamma^2}\right)$$

$$= \frac{\Delta \vec{x}'}{\gamma} \qquad (1.22)$$

Time Dilatation

When the clock in the K' system is at rest $\Delta \vec{x}' = 0$, so that

$$\Delta t = \gamma(\Delta t') \qquad (1.23)$$

and

$$\Delta \vec{x} = \vec{\beta}(\gamma \Delta t') = \vec{\beta} \Delta t$$

Equation (a) expresses the time dilation since $\gamma \geqslant 1$ The second equation simply says that the clock in the K' system is moving with velocity $\vec{\beta}$ as seen from the system K which is obvious.

Transformation of velocities:

From the formula

$$\vec{x} = \vec{x}' + \vec{\beta}\gamma\left(\frac{\gamma}{\gamma+1}\vec{\beta}\cdot\vec{x} + t'\right)$$

$$t = \gamma\left(t' + \vec{\beta}\cdot\vec{x}'\right)$$

we have by taking differentials of both sides

$$d\vec{x} = d\vec{x}' + \vec{\beta}\gamma\left(\frac{\gamma}{\gamma+1}\vec{\beta}\cdot d\vec{x} + dt'\right) \tag{1.24}$$

$$dt = \gamma\left(dt' + \vec{\beta}\cdot d\vec{x}'\right)$$

By division we get

$$\vec{v} = \frac{d\vec{x}}{dt} = \frac{\frac{d\vec{x}'}{dt'} + \vec{\beta}\gamma\left(\frac{\gamma}{\gamma+1}\beta\cdot\frac{d\vec{x}'}{dt'} + 1\right)}{\gamma\left(1 + \beta\cdot\frac{d\vec{x}'}{dt'}\right)}$$

$$= \frac{\vec{v}' + \vec{\beta}\gamma\left(\frac{\gamma}{\gamma+1}\vec{\beta}\cdot\vec{v}' + 1\right)}{\gamma\left(1 + \vec{\beta}\cdot\vec{v}'\right)} \tag{1.25}$$

When $\vec{v}'$ is $\parallel$ to $\vec{\beta}$, $\vec{v}$ is $\parallel$ to $\vec{\beta}$ and

$$v = \frac{v' + \frac{\beta^2\gamma^2}{\gamma+1}v' + \beta\gamma}{\gamma(\beta v' + 1)}$$

Using $\beta^2 = \dfrac{(\gamma+1)(\gamma-1)}{\gamma^2}$

we have

$$v = \frac{v' + (\gamma-1)v' + \beta\gamma}{\gamma(\beta v' + 1)}$$

$$= \frac{v' + \beta}{1 + \beta v'} \qquad (1.26)$$

When the velocities are small we neglect $\beta v'$,

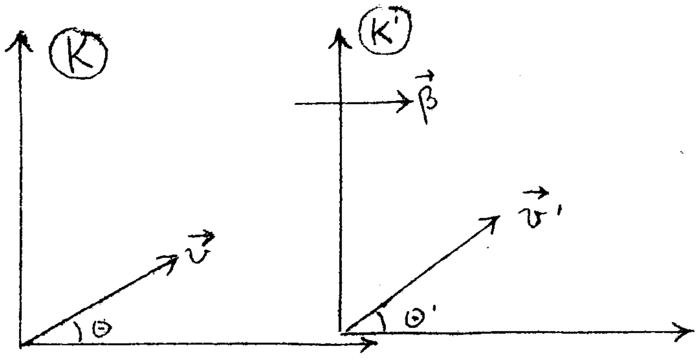
$v \sim v' + \beta$ in the nonrelativistic limit.

When $v' = \beta$,

$$v = \frac{v' + \beta}{1 + v'\beta} = \frac{2\beta}{(1+\beta^2)} \longrightarrow 1, \text{ as } \beta \to 1 \qquad (1.27)$$

so 1 is the maximum velocity possible physically.

Now choose the coordinates such that $\vec{\beta}$ points in the positive x direction and that v' lies in the $x'y'$ plane. Then $\vec{v}$ lies in the xy plane.



Transformation of angles

From the eqn.

$$\vec{v} = \frac{\vec{v}' + \vec{\beta}\gamma\left(\frac{\gamma}{\gamma+1}\vec{\beta}\cdot\vec{v}' + 1\right)}{\gamma\left(1+\vec{\beta}\cdot\vec{v}'\right)}$$

we get

$$v_x = v\cos\theta = \frac{v'\cos\theta' + \frac{\beta^2\gamma^2}{\gamma+1}v'\cos\theta' + \beta\gamma}{\gamma\left(1+\beta v'\cos\theta'\right)}$$

$$= \frac{\gamma\left(v'\cos\theta' + \beta\right)}{\gamma\left(1+\beta v'\cos\theta'\right)}$$

(1.28)

$$v_y = v\sin\theta = \frac{v'\sin\theta'}{\gamma\left(1+\beta v'\cos\theta'\right)}$$

$$\tan\theta = \frac{v'\sin\theta'}{\gamma\left(v'\cos\theta' + \beta\right)} \qquad (1.29)$$

This gives the transformation of the angles of a velocity.

Chapter 2.

Choice of units

Before actually going to the discussion of the choice of units, we observe that the quantities which made their frequent presence in the study of elementary particle physics are the velocity of light the Planck's constant and the mass of the nucleon . If , and denote the dimensionless numbers indicating the numerical value of these constants in a given system of units, then the elementary-particle-system of units is defined by

Hence we use the bracket to denote the measure which is dimensionless. Some people take the physical quantities and equal to unity. But let us take the numerical values equal to unity.

Let be the units of mass, length and time in our unit system which we have to distinguish from units.

We have

$$M_p = 1\ 1.m_o = (M)_{cgs}\ gm$$

$$\hbar = 1 \cdot \frac{m_o l_o^2}{t_o} = (\hbar)_{cgs} \quad \frac{gm.\ cm^2}{sec}$$

$$c = 1\ \frac{l_o}{t_o} = (c)_{cgs} \quad \frac{cm}{sec}$$

we find by solving for m_o, l_o, and t_o

$$m_o = M_p = (M_p)_{cgs}\ gm = \text{proton mass}$$
$$= (1.672) \cdot 10^{-24}\ gm$$

$$l_o = \frac{\hbar}{Mc} = \left(\frac{\hbar}{Mc}\right)_{cgs} cm = \text{proton Compton wave length}$$

$$= (0.211) \cdot 10^{-13}\ cm.$$

$$t_o = \frac{\hbar}{Mc^2} = \left(\frac{\hbar}{Mc^2}\right)_{cgs} sec$$ = the time in which light travels one proton compton wave length

$$= (0.07)\ 10^{-23}\ sec.$$

This has the consequence that new mass, length and time are numerically measured in multiples of the proton mass.

Let μ be the mass of a particle then $\mu = (\mu)\ M_p$

The compton wave length of this particle is then

$$\lambda_\mu = \frac{\hbar}{\mu c} = \frac{\hbar}{(\mu)\ M_p\ C} = \frac{1}{(\mu)}\ l_o$$

and therefore the numerical value of the compton wave length becomes

$$(\lambda\!\!\!^{-}_{\mu}) = \frac{1}{(\mu)}$$

Similarly, a time is attached to it

$$t_{\mu} = \frac{\hbar}{(\mu) m c^2} = \frac{1}{(\mu)} t_0$$

$$(t_{\mu}) = \frac{1}{(\mu)}$$

and any given length and time can be expressed by choosing the appropriate value of (μ)

<u>Constancy of the velocity of light</u>: It is a very important fact that the velocity of light in vacuum is constant in all the inertial frames. This fact leads to the formula $E = mc^2$. The correctness of the formula $E = mc^2$ in the low energy nuclear physics has been successfully verified with an accuracy of $1/1000$.

CHAPTER 3

Four Vectors and Invariants

Any set of four quantities transforming like

$ds = \{dt, d\vec{x}\}$ under Lorentz transformation are called four vectors. As the line element $ds^2 = dt^2 - d\vec{x}^2$ is left invariant by Lorentz transformation for any four vector

$$Q_t^2 - \vec{Q}^2$$

is a Lorentz invariant quantity. Consider two four vectors P and Q. Now $P+Q$ can be proved to be four vector as the Lorentz transformation is a linear transformation. So

$$(P+Q)^2 = P^2 + Q^2 + 2P \cdot Q \quad \text{is invariant.}$$

P^2, Q^2 are invariants.

So $P \cdot Q$ is also invariant (3.1)

$\vec{\beta} = \frac{d\vec{x}}{dt}$ is a three vector. The corresponding four vector is $dx/d\tau$ where $d\tau^2 = -ds^2$

Here X is $\{t, \vec{x}\}$. $dX/d\tau$ is four vector

dX is a four vector and $d\tau$ is an invariant quantity.

$$\frac{dX}{d\tau} = V = \{\gamma, \gamma\vec{\beta}\}$$

$$V^2 = \gamma^2 - \gamma^2\beta^2$$

$$= 1 = \text{invariant}$$

In the rest system of the particle $\vec{\beta} = 0$ and $\gamma = 1$, So $V^2 = 1$. Corresponding to the three momentum $m\vec{\beta}$ we have the four momentum mV, $mV = \{\gamma m, \gamma \vec{\beta} m\}$. Let us find what is the four component $m\gamma$

$$m\gamma = \frac{m}{\sqrt{1-\beta^2}} = m\left(1 + \frac{\beta^2}{2} + \cdots\right)$$

$$= m + E_{kinetic} + \cdots$$

$$= E$$

We recognize the fourth component to be the energy. m is the rest energy.

$$P = mV = \{m\gamma, m\vec{\beta}\gamma\}$$

$$= \{E, \vec{p}\}$$

CHAPTER 4

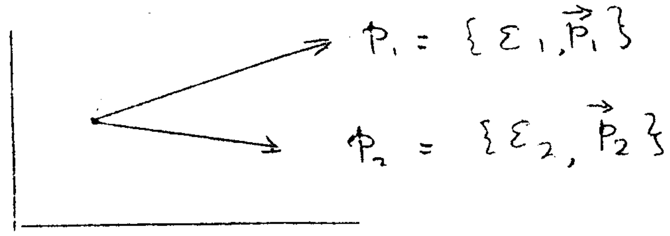
USE OF INVARIANTS

The concept of invariants wonderfully simplifies the calculation of quantities and some examples are cited below.

If a question is of such a nature that its answer will be the same, no matter in which Lorentz system one starts, then it must be possible to formulate the answer entirely with the help of these invariants which one can build with the available four vectors. One then finds the answer in a particular Lorentz system which one can choose freely and in such a way that the answer is there obvious or most easy. One looks then how the invariants appear in this particular system, expresses the answer to the problem by these invariants and one has found at the same time already the general answer.

Example 1: Centre of momentum energy and the velocity of the centre of momentum

Let two particles in some Lorentz system (eg. lab system) have four momenta p_1 and p_2 and masses m_1 and m_2 .



We ask the following question-what is the c.m. energy E. This question must have an answer independent of the Lorentz system in which p_1 and p_2 are given. It must be possible to be built from the invariants

$$p_1^2 = m_1^2$$

$$p_2^2 = m_2^2$$

$$p_1 p_2 \quad \text{or} \quad (p_1 + p_2)^2 \quad \text{or} \quad (p_1 - p_2)^2$$

Let us denote the centre of mass quantities with an arterisk. In C.M. system $\vec{p}_1^{\,*} + \vec{p}_2^{\,*} = 0$

$$E^* = \mathcal{E}_1^* + \mathcal{E}_2^*$$

$$p_1^* + p_2^* = \{ \mathcal{E}_1^* + \mathcal{E}_2^*, \; 0 \}$$

Hence

$$E^{*2} = (\mathcal{E}_1^* + \mathcal{E}_2^*)^2 = (p_1^* + p_2^*)^2$$

$$= (p_1 + p_2)^2$$

since $(p_1 + p_2)^2$ is invariant. We can define total mass of the system by the square of the total momentum

$$p^2 = (p_1 + p_2)^2 = M^2 = E^{*2}$$

$$= (\mathcal{E}_1 + \mathcal{E}_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 = \text{invariant} \qquad (4.1)$$

Kinematically the system can be replaced by a single particle with four momentum p and mass M. We can consider p_1 and p_2 themselves representing a system of particles.

The Velocity of the Centre of Momentum

$$P = \{ M\gamma, M\vec{\beta}\gamma \} = \{ E, \vec{P} \}$$

$$\vec{\beta}_{c.m} = \frac{\vec{P}}{E} = \frac{\vec{P}_1 + \vec{P}_2}{\mathcal{E}_1 + \mathcal{E}_2} \qquad (4.2)$$

We illustrate the above formulas by calculating the effect of the motion of the bound nucleons in nuclei (i) when it is parallel to an incoming beam of 25 Gev (ii) when it is antiparallel to the same beam.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the energies in the lab system of the proton from the beam and the proton in the nuclei.

$$E^{*2} = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2 p_1 p_2$$

$$= m_1^2 + m_2^2 + 2\mathcal{E}_1\mathcal{E}_2 \mp 2\sqrt{(\mathcal{E}_1^2 - m_1^2)(\mathcal{E}_2^2 - m_2^2)}$$

- sign for parallel motion and + sign for antiparallel motion.

We have $m_1 = m_2 = 1$

$\mathcal{E}_1 = 1.02$ (.02 = energy of bound nucleon)

$\mathcal{E}_2 = 1 + 25$

$$\mathcal{E}_1^2 - 1 \approx 0.04$$

$$\mathcal{E}_2^2 - 1 = 675$$

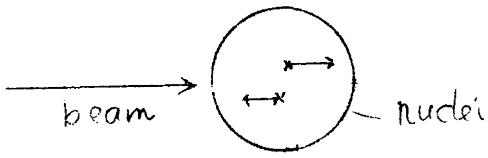
$$\sqrt{\mathcal{E}_1^2 - 1}\,\sqrt{\mathcal{E}_2^2 - 1} = \sqrt{.04}\,\sqrt{675} = 5.1$$

$$E^{*2} = 2(1 + 26 \pm 5.1) = \begin{cases} 65.2 & \text{antiparallel} \\ 44.8 & \text{parallel} \end{cases}$$

when the nucleon bound in the nuclei is at rest

$$E^{*2} = 2(1+26)$$

since $\quad \varepsilon_L^2 - 1 = 0$



$$E^{*} = \begin{cases} 8.08 & \text{for antiparallel motion} \\ 7.35 & \text{for nucleon at rest} \\ 6.70 & \text{for parallel motion} \end{cases}$$

Let us calculate the energies of incoming protons to produce the same centre of momentum energies on nucleons at rest. Now $\varepsilon_2 = 1$. If the energy of the incoming proton which produces there centre of mass energies on a nucleon at rest is denoted ε' then

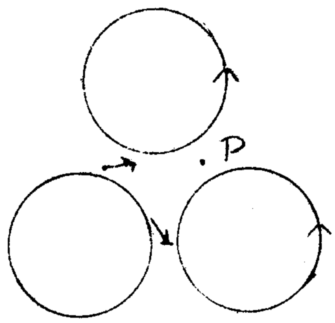
$$E^{*2} = 2 + 2\,\varepsilon_1 \varepsilon_2$$

$$E^{*2} = 2(\varepsilon' + 1) = \begin{cases} 65.2 \\ 54 \\ 44.8 \end{cases}$$

$$\varepsilon' = \begin{cases} 31.6 & \text{for antiparallel motion} \\ 26 & \text{for nucleon at rest} \\ 21.4 & \text{for parallel motion.} \end{cases}$$

This means that the 20 Mev nucleon motion is equivalent to about 5 Gev difference in primary energy!

Colliding beam experiment:



At the point P the protons meet in the opposite direction. Now the centre of momentum energy can be calculated as follows

$$\varepsilon_1 = \varepsilon_2 = \varepsilon$$

$$E_{cm}^2 = 2\left(1 + \varepsilon_1\varepsilon_2 + \sqrt{(\varepsilon_1^2-1)(\varepsilon_2^2-1)}\right)$$

$$= 2\left(1 + \varepsilon^2 + \varepsilon^2 - 1\right) = 4\varepsilon^2$$

$$E_{cm} = 2\varepsilon$$

$$= 52 \text{ GeV}$$ when we use the beam of 25 Gev kinetic energy protons.

To produce the same centre of momentum energy when one proton is at rest and a beam of protons hitting it let ε_1' be the energy of the proton of the beam.

Now

$$E_{cm}^2 = 2(1 + \varepsilon_1') \quad , \text{ since } \varepsilon_2 = 1$$

$$\sqrt{\varepsilon_1^2-1}\ \sqrt{\varepsilon_2^2-1} = 0$$

$$2\mathcal{E}_1' = 52^2 - 2$$

$$= 2702$$

$$\mathcal{E}_1' = 1351 \text{ Gev}$$

So we have to use two protons moving in the opposite direction to have great E_{cm}

Example

Suppose that a group of A nucleons (at rest as a whole would interact with an incoming proton of 25 Gev kinetic energy. What is the energy available for the production of particles and for kinetic energy? Put $M \sim 1$ Gev

We have the formula

$$E^{*2} = E_{cm}^2 = 2\mathcal{E}_1\mathcal{E}_2 + m_1^2 + m_2^2$$

Here

$$\mathcal{E}_1 = m_1 = A$$

$$m_2 = 1$$

$$\mathcal{E}_2 = 26$$

So

$$E_{cm}^2 = 2\mathcal{E}_1\mathcal{E}_2 + A^2 + 1 = (52A + A^2 + 1)$$

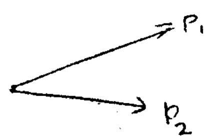
The available energy for particle production $E = E_{cm} - (A+1)$, as $(A+1)$ nucleons should be conserved.

$$E = \sqrt{52A + A^2 + 1} - (A+1)$$

we shall give below numerical example

A	1	2	5	10	20	40	100	∞
E	5.35	7.4	10.9	13.9	16.9	19.6	22.2	25

Example



What is the energy of particle 2 as seen from particle 1. First

we collect the invariants p_1^2, p_2^2 and $p_1 p_2$

In the rest frame of the particle 1.

$$p_1' = (m_1, 0) \qquad p_2' = (\mathcal{E}_2, \vec{p}_2)$$

$$p_1' p_2' = m_1 \mathcal{E}_2$$

$$\mathcal{E}_2 = \frac{p_1' \cdot p_2'}{m_1} \equiv E_{12}$$

$$= \frac{p_1 \cdot p_2}{m_1} \qquad (\, p_1' p_2' \text{ is invariant})$$

$$\vec{p}_{12}^{\,2} = E_{12}^2 - m_2^2$$

$$= \frac{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}{m_1^2}$$

The velocity v_{12} of the particle 2 as seen from 1 is

$$v_{12}^2 = \frac{|\vec{p}_{12}|^2}{E_{12}^2} = \frac{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}{(p_1 \cdot p_2)^2}$$

This relation is symmetric in the suffices 1 and 2, which is true as the velocity is only the relative velocity

The energy, momentum and velocity as seen from the centre of momentum system

We imagine a fictitious particle M whose four-momentum is

$$P = p_1 + p_2$$

where p_1 and p_2 are the four momenta of the two particles (m_1 and m_2) under consideration.

To get the energy of the two particles in centre of mass system we use the formula derived in the previous problem. Let us denote the centre of mass quantities with an asterisk. To find $\mathcal{E}_i^*$, etc. we consider the two particles M, M_i and calculate the quantities as seen by M. We have

$$\mathcal{E}_i^* = \frac{p_i \cdot P}{M}$$

$$|p_i^*|^2 = \frac{|P \cdot p_i|^2 - M^2 m_i^2}{M^2}$$

$$v_i^{*2} = \frac{(P \cdot p_i)^2 - M^2 m_i^2}{(P \cdot p_i)^2}$$

$$p_1 p_2 = \frac{1}{2}\left[(p_1 + p_2)^2 - p_1^2 - p_2^2 \right]$$

$$= \frac{1}{2}\left[M^2 - m_1^2 - m_2^2 \right]$$

We obtain

$$\mathcal{E}_1^* = \frac{M^2 + (m_1^2 - m_2^2)}{2M} \quad ; \quad \mathcal{E}_2^* = \frac{M^2 - (m_1^2 - m_2^2)}{2M}$$

$$\mathcal{E}_1^* + \mathcal{E}_2^* = M$$

$$|\vec{p}^*|^2 = |p_1^*|^2 = |p_2^*|^2$$

$$= \frac{M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}{4M^2}$$

$$= \frac{[(M^2 - (m_1 + m_2)^2][M^2 - (m_1 - m_2)^2]}{4M^2}$$

$$v^{*2} = \left(\frac{|\vec{p}|^2}{\mathcal{E}_1^*} \right)^2$$

CHAPTER 5

TRANSFORMATION OF DIFFERENTIAL CROSS-SECTION

We want to define cross-sections, in an invariant way. For this we have to analyse how cross-sections transform from one coordinate system to another.

Consider two sets of coordinates $(x_1 \ldots x_n)$ and $(y_1 \ldots y_n)$. The transformations from one set to another is defined by the sets of equations

$$x_i = x_i(y_1 \ldots y_n)$$

and its inverse

$$y_j = y_j(x_1 \ldots x_n) \tag{5.1}$$

Now the integral $\int\cdots\int_{R_x} f(x_1 \ldots x_n)\, dx_1 \ldots dx_n$ transform as

$$\int\cdots\int_{R_y} f\big(x_1(y_1 \ldots y_n),\, x_2(y_1 \ldots y_n) \ldots\big) \frac{\partial(x_1 \ldots x_n)}{\partial(y_1 \ldots y_n)}\, dy_1 \ldots dy_n \tag{5.2}$$

where R_x is certain boundary in the x space and is the corresponding boundary in the y space. The Jacobian $\dfrac{\partial(x_1 \cdots x_n)}{\partial(y_1 \cdots y_n)}$ has the following expression

$$\frac{\partial(x_1 \cdots x_n)}{\partial(y_1 \cdots y_n)} = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_2}{\partial y_1} & \cdots & \dfrac{\partial x_n}{\partial y_1} \\ \vdots & & & \\ \vdots & & & \\ \dfrac{\partial x_1}{\partial y_n} & \dfrac{\partial x_2}{\partial y_n} & \cdots & \dfrac{\partial x_n}{\partial y_n} \end{vmatrix} \tag{5.3}$$

In general transformations $x \rightarrow y$, $y \rightarrow z$ are carried out then the Jacobian is got by the rule

$$\frac{\partial(x_1 \cdots x_n)}{\partial(z_1 \cdots z_n)} = \frac{\partial(x_1 \cdots x_n)}{\partial(y_1 \cdots y_n)} \frac{\partial(y_1 \cdots y_n)}{\partial(z_1 \cdots z_n)} \tag{5.4}$$

Putting $x_i = z_i$ we get the identity

$$\frac{\partial(x_1 \cdots x_n)}{\partial(y_1 \cdots y_n)} \frac{\partial(y_1 \cdots y_n)}{\partial(x_1 \cdots x_n)} = 1 \tag{5.5}$$

The Jacobian is 1 if the volume is preserved by the transformation.

Examples:

1. Notations and translations

2. The canonical transformations.

If we take for R_x the volume element in x space , R_y becomes the volume element in the y space.

$$f(x_1 \dots x_n)\, dx_1 \dots dx_n$$

$$= f_1\big(x_1(y_1 \cdots y_n) \dots x_n(y_1 \dots y_n)\big)$$

$$\frac{\partial(x_1 \dots x_n)}{\partial(y_1 \dots y_n)}\, dy_1 \dots dy_n \tag{5.6}$$

We can look at this in two different ways -

1. by taking the special case f = constant we have

$$dx_1 \dots dx_n = \frac{\partial(x_1, x_2, \dots x_n)}{\partial(y_1, y_2 \dots y_n)}\, dy_1 \dots dy_n \tag{5.7}$$

We canconsider the Jacobian to belong to the volume element and interpret the equation

$$f(x_1 \dots x_n)\, dx_1 \dots dx_n = f(x_1(y) \dots x_n(y))\, \frac{\partial(x_1 \dots x_n)}{\partial(y_1 \dots y_n)}\, dy_1 \dots dy_n$$

as defining a new function

$$g(y_1 \dots y_n) \equiv f(x_1(y_1 \dots y_n) \dots x_n(y_1 \dots y_n)) \tag{5.8}$$

$$= f(x_1 \dots x_n)$$

It T is the transformation we can write

$$g(y) = f(\ \ x)$$

$$\text{or} \quad g(y) = f(T^{-1}y)$$

which is the transformation law for a scalar function. The function $g(y)$ has the same significance physically as $f(x)$ as they are related to one and the same volume element

$$dx_1 \cdots dx_n = \frac{\partial(x_1 \cdots x_n)}{\partial(y_1 \cdots y_n)} \, dy_1 \cdots dy_n$$

(ii) Now $dx_1 \cdots dx_n$ as well as $dy_1 \cdots dy_n$ might have convenient geometrical and/or physical interpretations. We then relate the function

$$f(x_1 \cdots x_n) \quad \text{to} \quad f(x_1(y_1 \cdots y_n) \cdots x_n(y_1 \cdots y_n)) \frac{\partial(x_1 \cdots x_n)}{\partial(y_1 \cdots y_n)}$$

We can now write the transformation equation as

$$f(x_1 \cdots x_n)\, dx_1 \cdots dx_n \equiv h(y_1 \cdots y_n) \frac{\partial(x_1 \cdots x_n)}{\partial(y_1 \cdots y_n)}$$

with

$$h(y_1 \cdots y_n) = f(x_1 \cdots x_n) \frac{\partial(x_1 \cdots x_n)}{\partial(y_1 \cdots y_n)} \tag{5.9}$$

where we express x_i in terms of y_i's in R.H.S. Now the transformation of scalar law function law does not hold. The functions $f(x)$, $g(y)$ and $h(y)$ have the properties of density functions.

Example 1. Transformation to Polar coordinates

We may define a differential cross-section either by the number of a definite kind of particles (per event) going into the volume element $dp_1\, dp_2\, dp_3$ in momentum space or by the number going into the solid angle element and having momentum bet p and $p+dp$. We call the first quantity $S(p_1\, p_2\, p_3)$.

We have now the transformation equation

$$\frac{\partial^3 S(p_1\, p_2\, p_3)}{\partial p_1\, \partial p_2\, \partial p_3}\, dp_1\, dp_2\, dp_3$$

$$= \frac{\partial^3 S(p_1(p\theta\phi), p_2(p,\theta\phi), p_3(p\theta\phi))}{\partial p_1\, \partial p_2\, \partial p_3}\, \frac{\partial(p_1\, p_2\, p_3)}{\partial(p\,\theta\,\phi)}\, dp\, d\theta\, d\phi \tag{5.9}$$

we know

$$dp_1\, dp_2\, dp_3 = p^2 \sin\theta\, dp\, d\theta\, d\phi$$

The Jacobian is

$$\frac{\partial(p_1, p_2, p_3)}{\partial(p, \theta, \phi)} = p^2 \sin\theta \tag{5.10}$$

We now define

$$\frac{\partial^3 S(p_1(p\theta\phi), p_2(p\theta\phi), p_3(p\theta\phi))}{\partial p_1\, \partial p_2\, \partial p_3}\, p^2$$

as the new cross-section and replace $\sin\theta\, d\theta d\phi$ by $d\Omega$

We have the eqn.

$$\frac{\partial^3 S(p_1 p_2 p_3)}{\partial p_1\, \partial p_2\, \partial p_3}\, dp_1\, dp_2\, dp_3 = \frac{\partial^2 \sigma(p\,\theta\,\phi)}{\partial p\, \partial \Omega}\, dp\, d\Omega$$

where

$$\frac{\partial^2 \sigma(p\, \theta\, \phi)}{\partial p\, \partial \Omega} = p^2 \frac{\partial^3 S(p_1 p_2 p_3)}{\partial p_1 \partial p_2 \partial p_3}$$

$$d\Omega = \sin\theta\, d\theta\, d\phi \tag{5.11}$$

This is an intermediate way of explaining the transformation equation.

-Example 2 Lorentz transformation

Let us choose the coordinates such that the axes of the system K' are parallel to the axes in K and their relative velocity is parallel to the x - axis.

The Lorentz transformation can be wirtten as

$$\begin{aligned} p_3 &= \gamma(p_3' + \beta E') & \qquad p_3' &= \gamma(p_3 - \beta E) \\ p_2 &= p_2' & p_2' &= p_2 \\ p_1 &= p_1' & p_1' &= p_1 \\ E &= \gamma(E' + \beta p_3') & E' &= \gamma(E - \beta p_3) \end{aligned} \tag{5.12}$$

We introduce now polar coordinates

$$\begin{aligned} p_1 &= p \sin\theta \cos\phi \\ p_2 &= p \sin\theta \sin\phi \\ p_3 &= p \cos\theta \end{aligned}$$

We find $p_2 = p_2'$ and $p_1 = p_1'$ imply

$$\phi = \phi'$$

Hence

$$p \cos \theta = \gamma (p' \cos \theta + \beta E')$$

$$p \sin \theta = p' \sin \theta'$$

$$\phi = \phi'$$

$$E = \gamma (E' + \beta p' \cos \theta') \tag{5.13}$$

and the inverse transformation

$$p' \cos \theta' = \gamma (p \cos \theta - \beta E)$$

$$p' \sin \theta' = p \sin \theta$$

$$\phi' = \phi$$

$$E' = \gamma (E - \beta p \cos \theta) \tag{5.13a}$$

are the other forms of the Lorentz transformation.

For the transformation of cross-section we require that the number of particlew going into the solid angle element $d\Omega$ and having momentum between p and $p+dp$ by the same as the number of particles going into the solid angle element $d\Omega'$ and with momentum but p' and $p'+dp'$.

So the transformation should be of the form

$$\frac{\partial^2 \sigma(p \theta \phi)}{\partial p \, \partial \Omega} dp \, d\Omega = \frac{\partial^2 \sigma'(p' \theta' \phi')}{\partial p' \partial \Omega'} dp' d\Omega' \tag{5.14}$$

This implies that

$$\frac{\partial^2 \sigma(p\,\theta\,\phi)}{\partial p\,\partial \Omega} = \frac{\partial^2 \sigma'(p'\,\theta'\,\phi')}{\partial p'\,\partial \Omega'} \quad \frac{\partial(p'\,\Omega')}{\partial(p\,\Omega)} \tag{5.15}$$

We have to calculate $\dfrac{\partial(p\,\Omega)}{\partial(p'\,\Omega')}$. From

$$(p\,\Omega) \to (p\,\theta\,\phi) \to (P_1\,P_2\,P_3) \to (P_1'\,P_2'\,P_3') \to (p'\,\theta'\,\phi') \to (p'\,\Omega')$$

we use the chain rule.

$$\frac{\partial(p\,\Omega)}{\partial(p'\,\Omega')} = \frac{\partial(p\,\Omega)}{\partial(p\,\theta\,\phi)} \; \frac{\partial(p\,\theta\,\phi)}{\partial(p_1\,p_2\,p_3)} \; \frac{\partial(p_1 P_2 P_3)}{\partial(P_1' P_2' P_3')} \; \frac{\partial(P_1' P_2' P_3')}{\partial(p'\,\theta'\,\phi')} \; \frac{\partial(p'\,\theta'\,\phi')}{\partial(p'\,\Omega')}$$

$$= \sin\theta \quad \frac{1}{p^2 \sin\theta} \quad \frac{\partial(p_1 P_2 P_3)}{\partial(p_1' P_2' P_3')} \quad p'^2 \sin\theta' \quad \frac{1}{\sin\theta'}$$

$$= \frac{p'^2}{p^2} \quad \frac{\partial(P_1\,P_2\,P_3)}{\partial(p_1'\,p_2'\,P_3')}$$

$$= \frac{p'^2}{p^2} \quad \frac{E}{E'}$$

$$= \frac{\sin^2\theta}{\sin^2\theta'} \quad \frac{E}{E'} \tag{5.16}$$

Here we have used the result

$$\frac{\partial(p_1 p_2 p_3)}{\partial(p_1' p_2' p_3')} = \frac{\partial p_3}{\partial p_3'} = \gamma + \beta \frac{\partial E'}{\partial p_3'}$$

$$= \gamma + \frac{p_3'}{E'}$$

$$= \frac{E}{E'} \qquad (5.17)$$

So

$$\frac{\partial^2 \sigma'(p'\Omega')}{\partial p' \partial \Omega'} = \left(\frac{\sin\theta}{\sin\theta'}\right)^2 \frac{E}{E'} \frac{\partial^2 \sigma(p\Omega)}{\partial p \partial \Omega} \qquad (5.18)$$

Also

$$\frac{\partial^2 \sigma(E'\Omega')}{\partial E' \partial \Omega'} = \frac{\partial^2 \sigma(p\Omega)}{\partial p \partial \Omega} \frac{\partial(p\Omega)}{\partial(E'\Omega')}$$

where

$$\frac{\partial(p\Omega)}{\partial(E'\Omega')} = \frac{\partial(p\Omega)}{\partial(p'\Omega')} \frac{\partial(p'\Omega')}{\partial(E'\Omega')}$$

$$= \frac{p'^2}{p^2} \frac{E}{E'} \frac{dp}{dE'}$$

$$= \frac{p'^2}{p^2} \frac{E}{E'} \frac{E'}{p'}$$

$$= \left(\frac{p' E}{p^2}\right)$$

$$\frac{\partial(p\,\Omega)}{\partial(\epsilon\,\Omega')} = \frac{p'\,\epsilon}{p^2}$$

and

$$\frac{\partial^2\sigma(\epsilon'\,\Omega')}{\partial\epsilon'\,\partial\Omega'} = \frac{\partial^2\sigma(p\,\Omega)}{\partial p\,\partial\Omega}\;\frac{p'\,\epsilon}{p^2} \tag{5.19}$$

Aliter:

Let $N(p_0\,p_1\,p_2\,p_3)$ be the number of particles with four momentum. Let $p = \{p_0\;\vec{p}\}$ and $p + dp = \{p_0 + dp_0\;,\vec{p} + d\vec{p}\}$

If we make a Lorentz transformation this number should be the same.

$$N(p)\,d^4p = N'(p')\,d^4p'$$

But as the Lorentz transformation is orthogonal transformation (rotation) in the four space,

$$\frac{\partial(p_0\,p_1\,p_2\,p_3)}{\partial(p_0'\,p_1'\,p_2'\,p_3')} = 1 \quad , \text{ie} \quad d^4p' = d^4p$$

So $N(p) = N'(p')$ behaves like a scalar under Lorentz transformation. We now impose the restriction that the particles has mass m :

$$N(p)\;\delta(p^2 - m^2)\;d^4p = N(p')\;\delta(p'^2 - m^2)\;d^4p'$$

as the rest mass is an invariant quantity. Integrate both sides of this equation with respect to p_0 from 0 to ∞

$$\int N(p)\,\delta(p^2-m^2)\,dp_0\,d\vec{p} = \int N'(p')\,\delta(p'^2-m^2)\,dp'_0\,d\vec{p}'$$

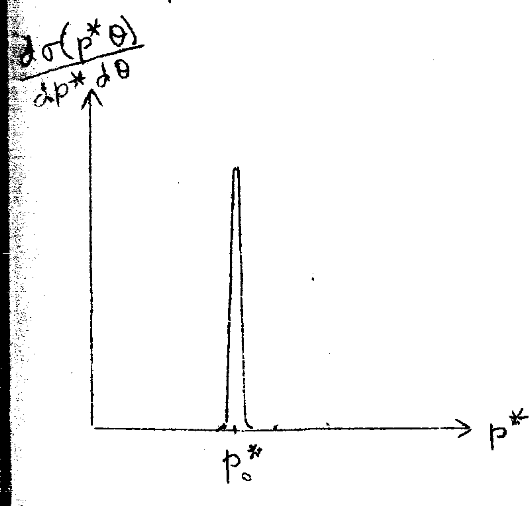
$$\delta(p^2-m^2) = \frac{\delta(p_0+E)+\delta(p_0-E)}{2E}, \qquad E=\sqrt{p^2+m^2} \tag{5.20}$$

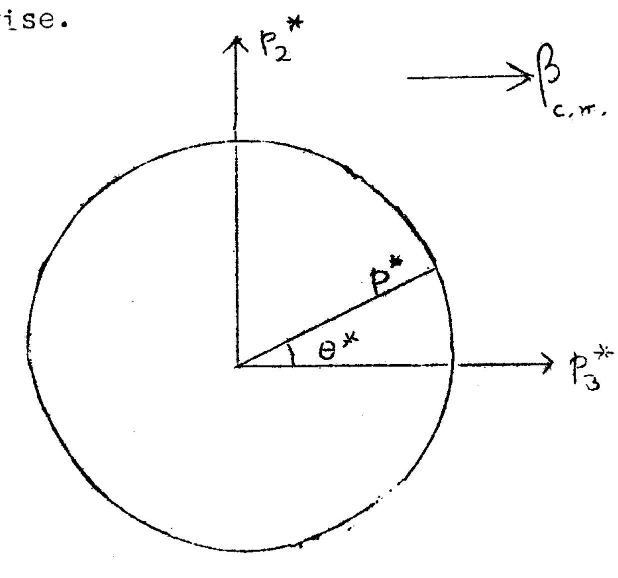
So we get after integration

$$N(E,p)\,\frac{d^3p}{E} = N'(E',p')\,\frac{d^3p'}{E'}$$

<u>The form of spectrum in different systems centre mass system</u>

Consider the model where the spectrum has a single peak $p^* = p_0^*$ and is zero otherwise.





In the polar coordinates $p^* \; \theta^*$ the spectrum is nonvanishing only in a circular ring at

$$p^* = p_0^*$$

$$v^* = \frac{p^*}{E^*} = \frac{p^*}{\sqrt{p^{*2} + m^2}}$$

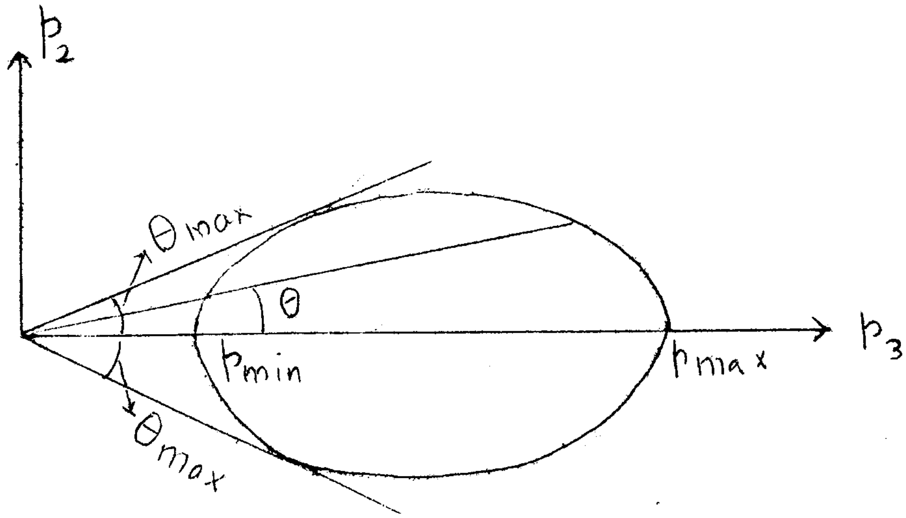
$$p^* = \sqrt{p_1^{*2} + p_2^{*2} + p_3^{*2}}$$

We have to consider three cases

(1) $v^* < \beta_{cm}$, (2) $v^* = \beta_{cm}$, (3) $v^* > \beta_{cm}$

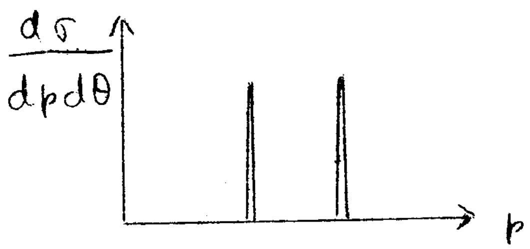
Case I

When $v^* < \beta_{cm}$ the distribution in the lab system has been drawn below



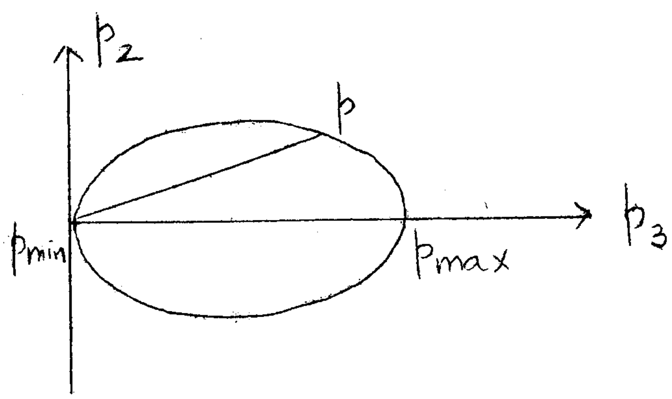
Since no particle goes backwards inthe lab system there is a maximum angle $\theta_{max} < \frac{\pi}{2}$. For any angle there correspond two momenta, one large and one small. The larger momentum obviously correspond to particles going forward in c.m. system and the small momentum corresponds to particles going backward.

Now for a given angle $\theta < \theta_{max}$

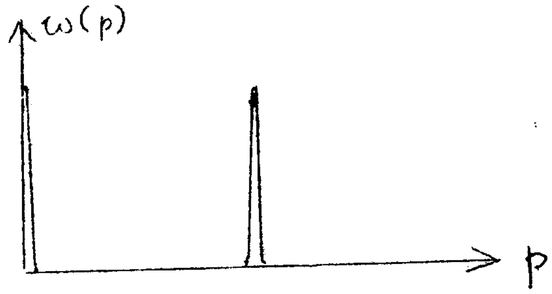


Case II When $\beta = v^*$

With decreasing β the maximum lab angle increases and reaches $\frac{\pi}{2}$ for $\beta = v^*$ and the ring reaches the origin.

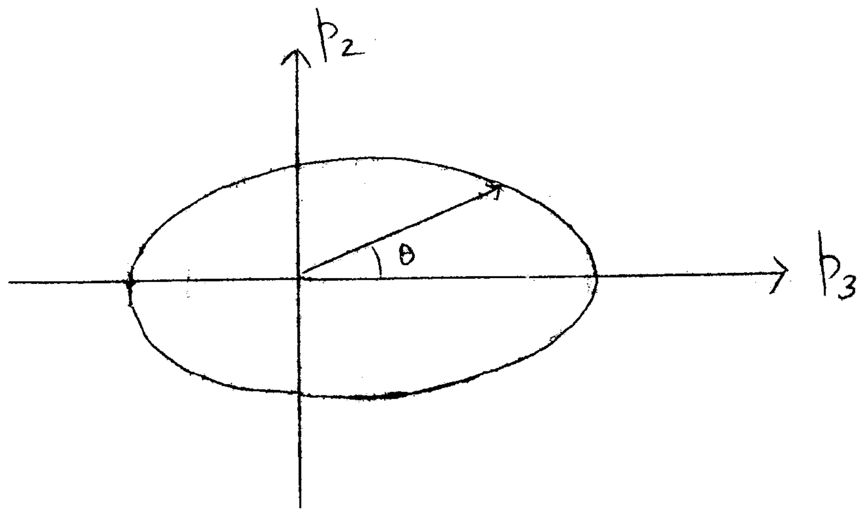


So we observe two separate peaks in the lab system for all $\Theta < \pi/2$. One is at the high energy and the other is at $p = 0$. As $\Theta \to \frac{\pi}{2}$ the high energy peak is shifted to zero



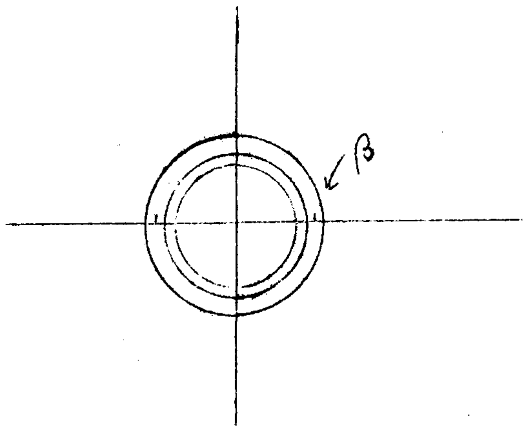
Case III $\beta < v^*$

The ring has crossed the origin $p = 0$ and $p = 0$ lies inside the ring. There corresponds only one momentum for every angle and there is only one place in the energy spectrum for every angle.

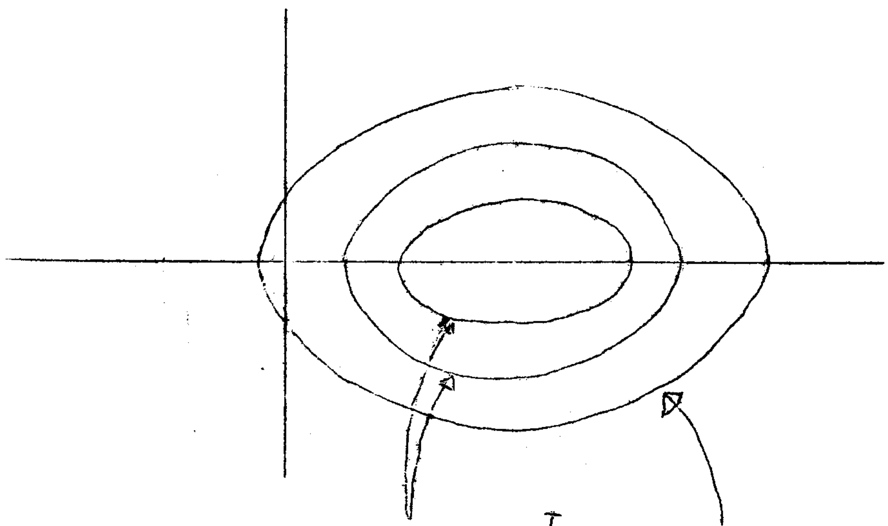


This last case occurs whenever the particles have zero mass. For $v = c$ and $\beta < c$

A general spectrum we may imagine 'to' 'be' composed of delta spectra which however are no longer isotropic in the c.m. system. We draw in the c.m. system ring shaped regions (not necessarily circular)

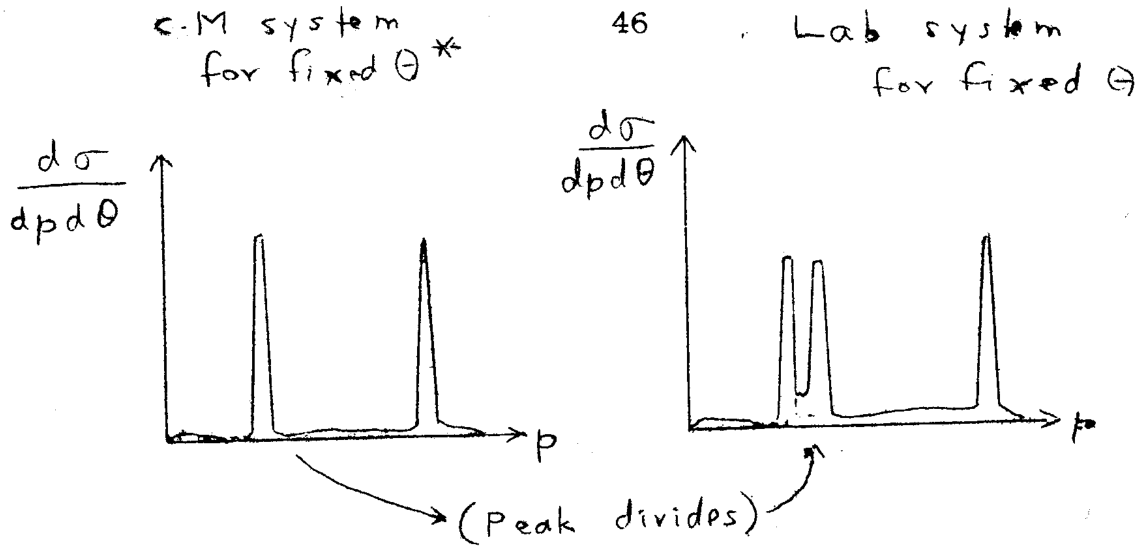


In the lab system the distribution looks as shown in the figure.



The peak is divided into two | peak is not divided.

The differential cross-section as a function of energy is plotted below



We now carry through the quantitative discussion of the transformation of the cross-sections assuming δ-shaped isotropic distribution. We describe in terms of momentum as it transform like a four vector under Lorentz transformation.

Consider the transformation

$$p_1 = p_1^*$$
$$p_2 = p_2^*$$
$$p_3 = \gamma(p_3^* + \beta E^*) \qquad (5.21)$$

$\frac{p_1^{*2} + p_2^{*2} + p_3^{*2}}{p^{*2}} = 1$ (where quantities with asterisk denote centre momentum quantities and those without asterisks denote lab quantities) defines a sphere redius p^*

In the lab system the eqn. $\frac{p_1^{*2} + p_2^{*2} + p_3^{*2}}{p^{*2}} = 1$

becomes

$$\frac{p_1^{*2} + p_2^{*2}}{p^{*2}} + \frac{(p_3 - \beta\gamma E^*)^2}{\gamma^2 p^{*2}} = 1,$$

$$E^* = \sqrt{p^{*2} + m^2}$$

So the sphere becomes a spheroid.

Consider the line $p_2 = p_3 \tan\theta$, $p_1 = 0$. We have two solutions $p_3^{\pm}$,

for a given θ

$$p_3^{\,2} \tan^2\theta + \frac{(p_3 - \gamma\beta E^*)^2}{\gamma^2} = p^{*2}$$

We have

$$p_3^{\,\pm} = \frac{\beta\gamma E^* \pm \sqrt{\beta^2\gamma^2 E^{*2} - (1+\gamma^2\tan^2\theta)(\beta^2\gamma^2 E^{*2} - \gamma^2\beta^2)}}{(1+\gamma^2\tan^2\theta)} \tag{5.23'}$$

The angle is got when $p_+ = p_-$, that is when

$$\tan^2\theta = \frac{v^{*2}}{\gamma^2(\beta^2 - v^{*2})} \;, \quad v^* = \frac{p^*}{E^*} \tag{5.24}$$

or $\tan^2\theta_{max} = \dfrac{v^{*2}}{\gamma^2(\beta^2 - v^{*2})}$

$\tan\theta = \infty$ when $\beta = v^*$ and $\theta = \pi/2$

There is no maximum angle when $v^* > \beta$

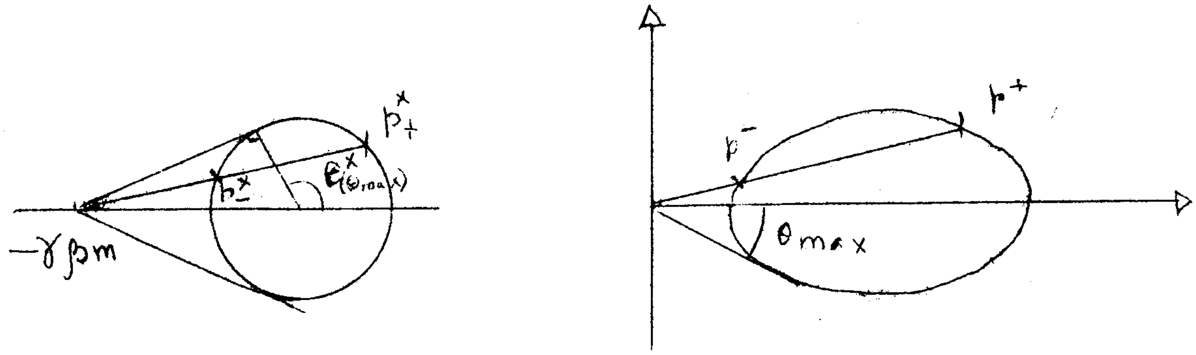
<u>We can get the expression for the angle in a simpler way as follows</u>

We attach to the lab system a fititious particle of the same mass. This particle is at rest in the lab system and has momentum $-\beta\gamma m$ in the c.m. system.

This is seen from the transformation $p_3^* = \gamma(p_3 - \beta E)$.

$p_1^* = p_2^* = p_1 = p_2 = 0$; $p_3 = 0$, $E = m$.

The maximum angle θ_{max} is given by the tangent cone from the origin $p_1 = p_2 = p_3 = 0$. to the ellipsoid and in the c.m. system the corresponding tange t cone from the point $-\beta\gamma m$ to the sphere



$$-\cos\theta^*(\theta_{max}) = \frac{p^*}{\gamma\beta m}$$

We write

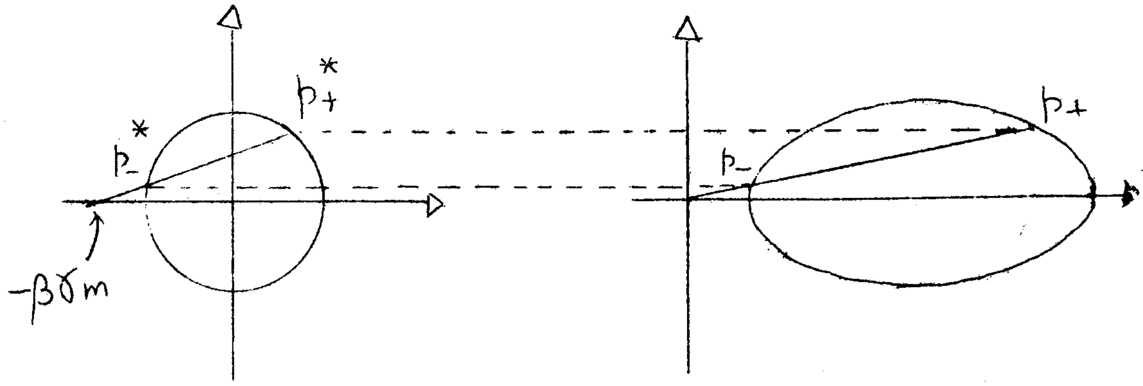
$$p^* = \frac{v^* m}{\sqrt{1-v^{*2}}} = m v^* \gamma^*,$$

$$\gamma^* = \frac{1}{\sqrt{1-v^{*2}}}$$

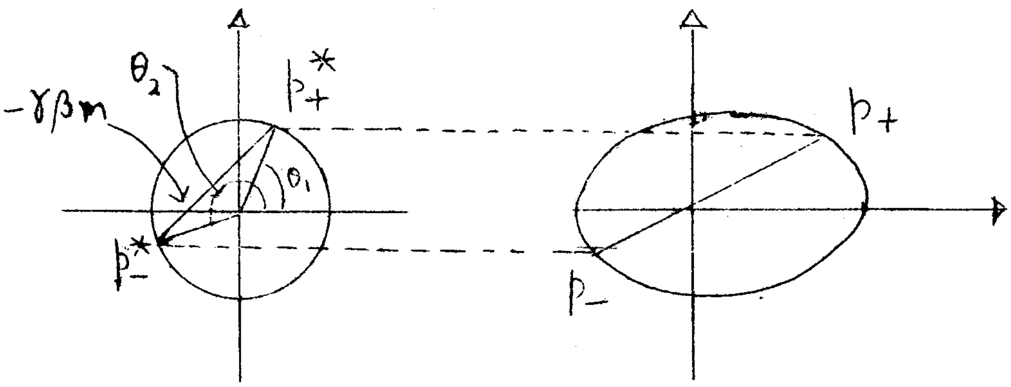
$$-\cos\theta^*(\theta_{max}) = \frac{m v^* \gamma^*}{\gamma\beta m} = \frac{v^*\gamma^*}{\gamma m} \tag{5.25}$$

This result leads to the following graphical contruction of the figures.

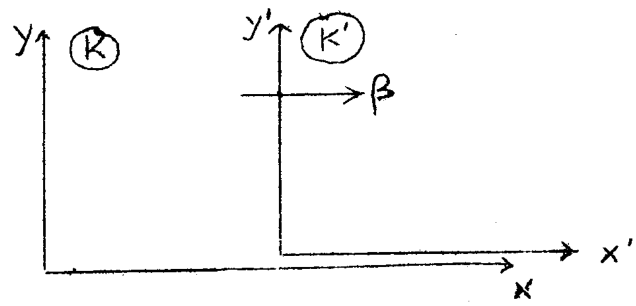
Draw the ellipse and the circle corresponding to the lab. and c.m. spectra. To one angle in the lab. (angle between p_3 axis and the straight line leaving the origin) and the corresponding two momenta (intersection of the above straight line with the ellipse), two angles and two momenta correspond such that for corresponding momenta the components $p_2 = p_2^*$ always.



The method is applicable for $v^* > \beta$ also.



Can we actually 'see' the Lorentz contraction?



We observe an event in two reference systems.

K' - the rest system of the object

K - the lab. system

The Lorentz transformation for an arbitrary four vector

$$
\begin{aligned}
x &= \gamma(x' + \beta t') \\
y &= y' \\
z &= z' \\
t &= \gamma(t' + \beta x')
\end{aligned}
\tag{5.26}
$$

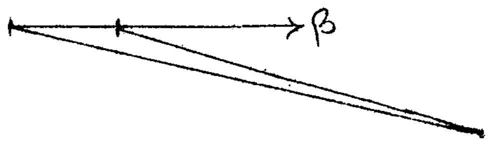
or

$$
\begin{aligned}
\Delta x &= \gamma(\Delta x' + \beta \Delta t') \\
\Delta y &= \Delta y' \\
\Delta z &= \Delta z' \\
\Delta t &= \gamma(\Delta t' + \beta \Delta x')
\end{aligned}
$$

Now the length Δx in the lab system is defined through $\Delta t = 0$

$$\Delta x = \gamma(\Delta x' + \beta(-\beta \Delta x'))$$
$$= \Delta x' \gamma(1-\beta^2)$$
$$= \frac{1}{\gamma} \Delta x'$$

This is Lorentz contraction.



However, when we 'observe' the object, the light pulses do not start at the same time, but are received at the same time. Therefore the condition $\Delta t = 0$ is not satisfied. Therefore, do we see anything other than the Lorentz contraction?

What we see

We consider a rectangular parallellopiped with length l_0 breadth b_0 and height h_0. How it is so becomes clear from the following. We need not worry ourself about h_0. We assume that the cube is in flight at an angle $\vartheta = \pi - \alpha$

We read off from the figure

$$b_{proj} = \sin\alpha \left(\beta \Delta t_b - \frac{b_0}{\tan\alpha}\right)$$

Calculate Δt_b

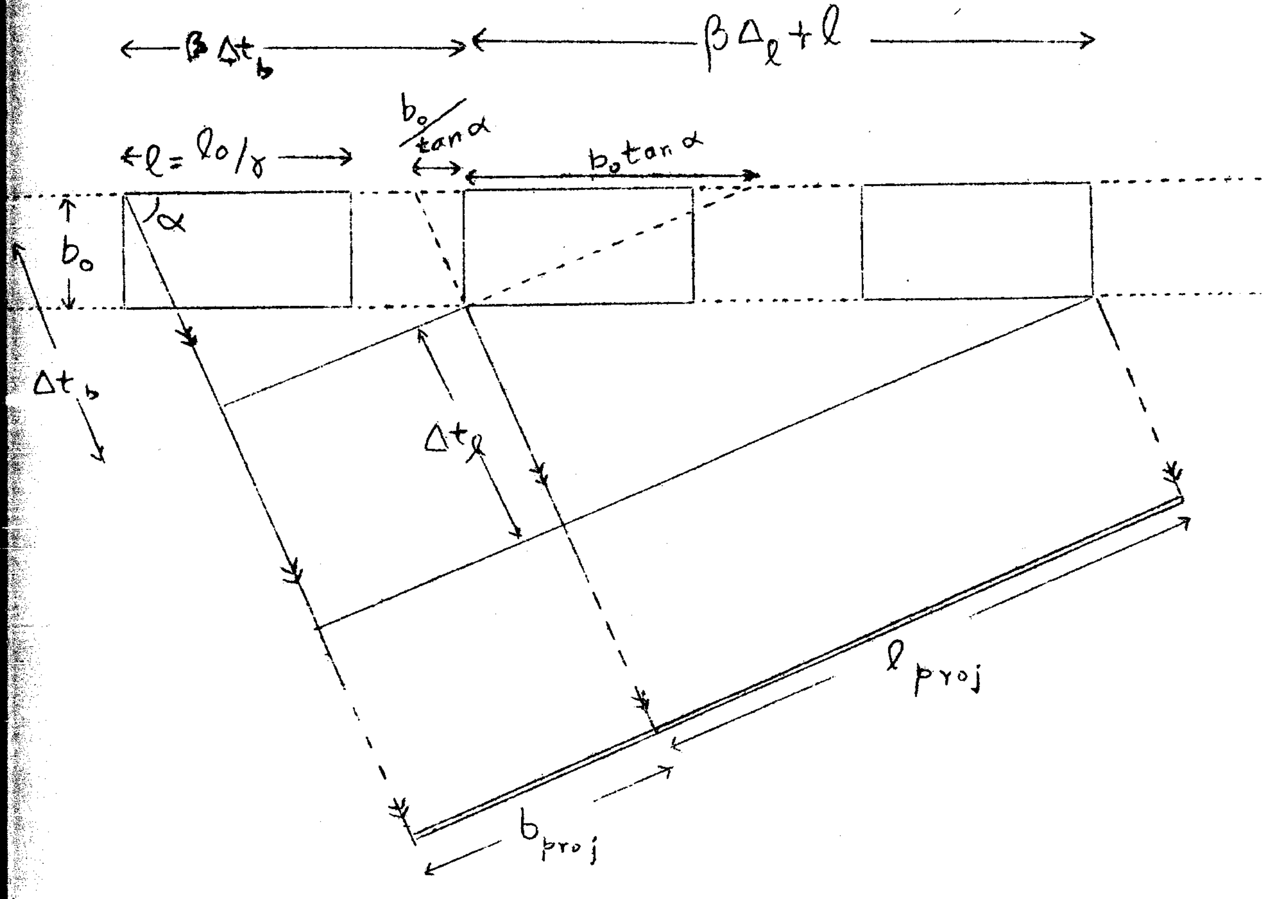
$$\Delta t_b = \cos\alpha(\beta \Delta t_b + b_0 \tan\alpha)$$
$$\Delta t_b = \frac{b_0 \sin\alpha}{1-\beta\cos\alpha}$$

Insert above

$$b_{proj} = b_o \left(\frac{\beta \sin^2\alpha}{1-\beta\cos\alpha} - \cos\alpha \right)$$

$$= b_o \frac{\beta \sin^2\alpha - \cos\alpha + \beta\cos^2\alpha}{1-\beta\cos\alpha}$$

$$= b_o \frac{\beta - \cos\alpha}{1-\beta\cos\alpha}$$



When $b_{proj} > 0$ the backside is visible (people see the backside)

When $b_{proj} < 0$ the frontface is visible

$b_{proj} > 0$ when $\beta > \cos\alpha$

Length wise (lengthside) ℓ:

$$\ell_p = \sin\alpha\,(\beta\Delta t_\ell + \ell)$$

$$\Delta t_\ell = \cos\alpha\,(\beta\Delta t_\ell + \ell)$$

$$= \frac{\ell\cos\alpha}{1-\beta\cos\alpha}$$

$$\ell_p = \frac{\sin\alpha\,(\beta\ell\cos\alpha + \ell - \beta\ell\cos\alpha)}{1-\beta\cos\alpha} = \frac{\ell\sin\alpha}{1-\beta\cos\alpha}$$

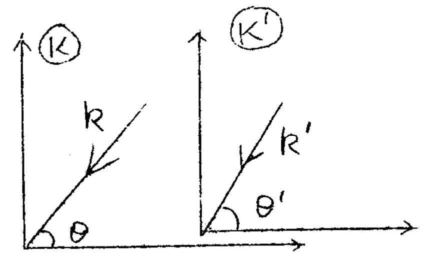
$$\ell = \frac{1}{\gamma}\ell_0$$

$$\ell_{proj} = \frac{\ell_0\sin\alpha}{\gamma(1-\beta\cos\alpha)} \quad \text{always} \geq 0$$

Proposition:

The projection of the moving cube is the same as the projection of a cube in the rest system. No Lorentz contraction?

For this purpose we consider the Aberration of light too (as our aim)



$$k = \{\omega, \vec{k}\} \quad \text{and} \quad k' = \{\omega', \vec{k}'\}$$

are th wave four-vector in K and K'. From 5.26 we get, when θ is the direction of observation,

$$k_x' = -\omega' \cos\theta' = \omega\gamma(-\cos\theta' - \beta)$$

$$k_y' = -\omega' \sin\theta' = -\omega \sin\theta$$

$$\omega' = \gamma\omega(1 + \beta\cos\theta)$$

The last equation gives Doppler Effect. We divide the I by II eqn. and obtain

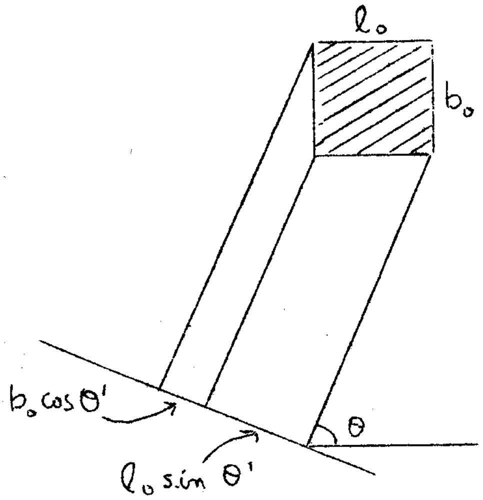
$$\cos\theta = \frac{-\beta + \cos\theta'}{1 - \beta\cos\theta'} \quad ; \quad \sin\theta = \frac{\sin\theta'}{\gamma(1 - \beta\cos\theta')}$$

$$\omega = \omega'\gamma(1 - \beta\cos\theta')$$

$$\cos\theta' = \frac{\beta + \cos\theta}{1 + \beta\cos\theta} \quad ; \quad \sin\theta' = \frac{\sin\theta}{\gamma(1 + \beta\cos\theta)}$$

$$\omega' = \omega\gamma(1 + \beta\cos\theta)$$

There we observe the body at an angle $\Theta = \pi - \alpha$ corresponding to an angle Θ' in the rest system



$$\cos\Theta' = \frac{\beta - \cos\alpha}{1 - \beta\cos\alpha}$$

$$\sin\Theta' = \frac{\sin\alpha}{\gamma(1 - \beta\cos\alpha)}$$

$$\omega' = \omega\gamma(1 - \beta\cos\alpha)$$

$$\left\{\begin{aligned} \cos(\pi - \alpha) &= -\cos\alpha \\ \sin(\pi - \alpha) &= \sin\alpha \end{aligned}\right\}$$

Therefore one obtains with (2) and (3)

$$b_{proj} = b_0 \cos\Theta'$$

$$l_{proj} = l_0 \sin\Theta'$$

We see the cube at an angle $\pi - \alpha$, but the figure which we see is the same as the one which would be observed in the rest system at an angle θ'. One

$$\cos\theta' = 0 \quad \text{when} \quad \beta = \cos\alpha_0$$

When β is very near 1, then α_0 is very small

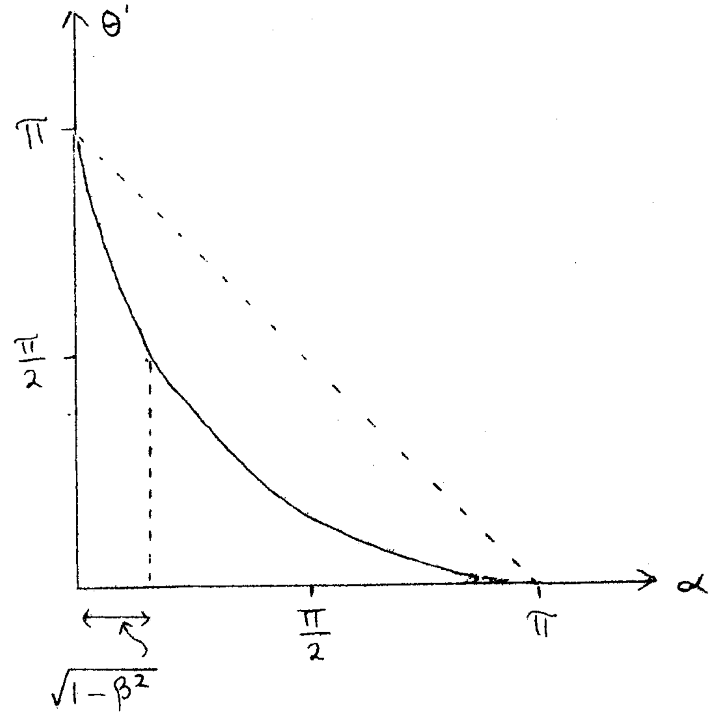
$$\alpha_0 \approx \sqrt{1-\cos^2\alpha} = \sqrt{1-\beta^2}$$

Then

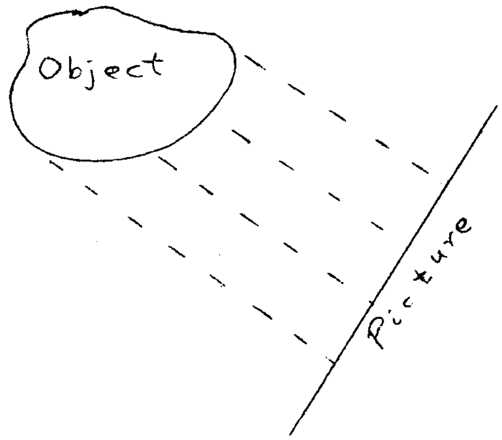
$$1 - \beta\cos\alpha = 1-\beta^2$$

$$\omega' = \omega\sqrt{1-\beta^2} \quad \text{or} \quad \omega = \frac{\omega'}{\sqrt{1-\beta^2}}$$

At the angle the cube has its length turned off. From there we see more and more of the backside until it diappears



II <u>One can understand this also</u>



The 'figure' travels with an electromagnetic wave

with wave vector

$$k = \{\omega, \vec{k}\}$$

and in every Lorentz system the wavefronts lie perpendicular to $\vec{k}$

Now (we) get for any two worldpoints x_1, x_2

$$\Delta s^2 = \Delta t^2 - \Delta \vec{x}^2$$

We take the x-y plane on the wavesurface; therefore our condition for the 'picture'

$$\Delta z = 0 \;, \; \Delta t = 0 \quad \text{so}$$

$$\Delta s^2 = -(\Delta x^2 + \Delta y^2)$$

However we could have obtained in the rest system the same figure-condition in the primed coordinate system, since the wave-points are perpendicular to the wave vector in every system.

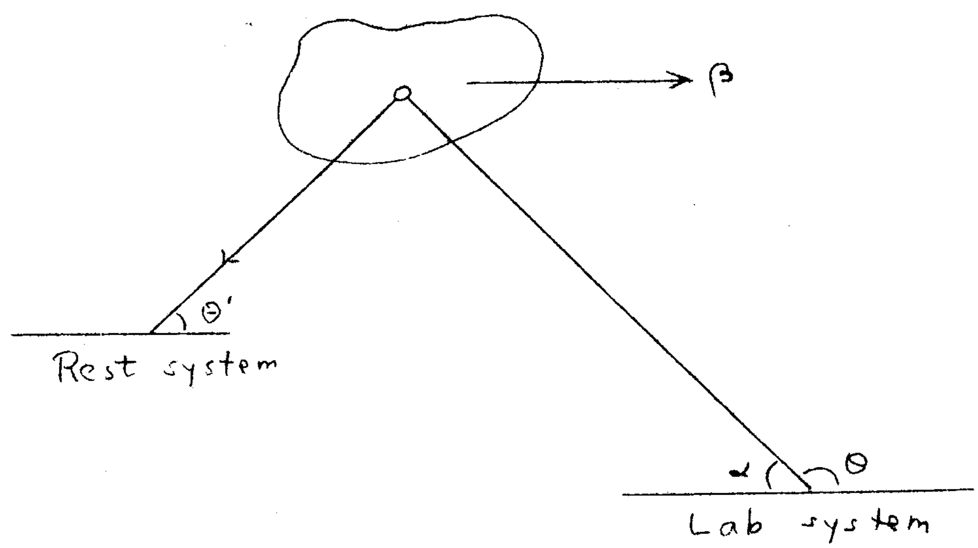
$$\Delta s^2 = -(\Delta x'^2 + \Delta y'^2) = -(\Delta x^2 + \Delta y^2)$$

But Δs^2 is an invariant

Indeed the 'figure' is invariant, however under the condition, we compare it essentially (with) a Lorentz covt. form.

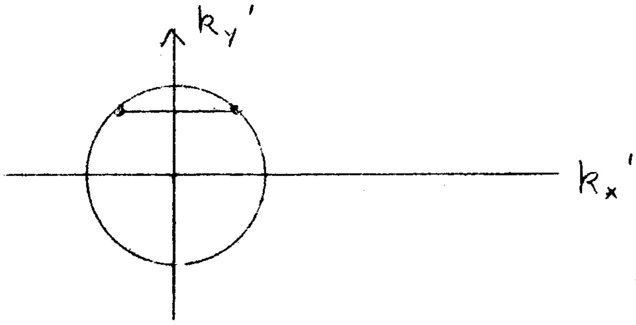
The figure which we observe in the lab. system at angle $\Theta = \pi - \alpha$ is the same as that which will be observed in the rest system at an angle Θ' where $\Theta = \pi - \alpha$ and Θ' are connected by the aberration formula.

The light which will be radiated in the rest system in the direction of observation Θ' comes at angle $\Theta = \pi - \alpha$ in the lab. system



III. Finally we consider the effect of the transformation on a differential cross-section:

We assume in the rest system, a body emitting isotropically nonchromatic radiation ω'



we have then according to (5.26)

$$k_x = \gamma(k_x' + \beta\omega')$$

$$k_y = k_y'$$

$$\Delta k_x = \gamma \, \Delta k_{x'}$$

Therefore the sphere will contract by a factor γ ellipsoid

$$a = \gamma\omega'$$

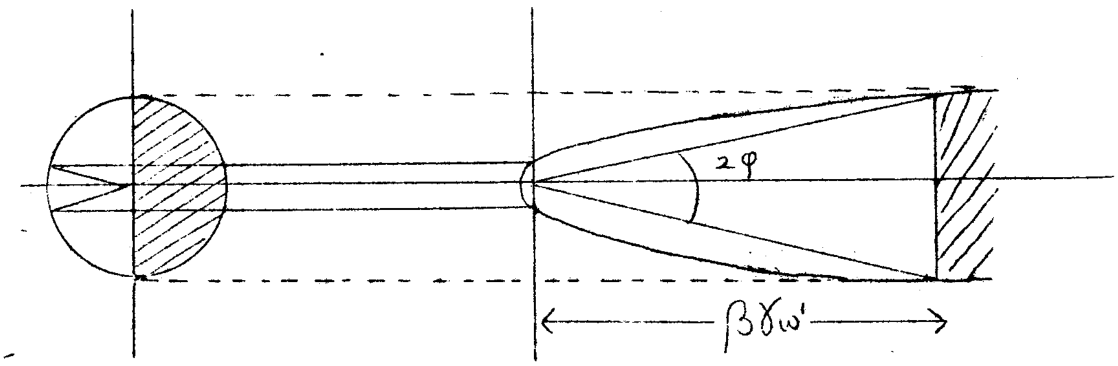
$$b = c = \omega'$$

Middle point

$k_{x,max} = \omega'\gamma(1+\beta)$ forward in both system

$k_{x,min} = \omega'\gamma(-1+\beta)$ backward in both system

$\frac{k_{x,max} + k_{x,min}}{2} = \beta\gamma\omega'$ therefore



$$\varphi \sim \frac{1}{\sqrt{\beta\gamma}}$$

Part of the radiation (number of quanta-not energy) goes in an angle $\lesssim \varphi \sim \frac{1}{\beta\gamma} \sim \frac{1}{\gamma} = \sqrt{1-\beta^2}$ according to the above

The radiation which started out at angle $\pi/2$ appears in the lab. system at an angle $- \frac{1}{\gamma} = \sqrt{1-\beta^2}$

The source of light which radiated isotropically in the rest system appears in the lab. system as reflector radiating all the light strongly in a sharply defined cone in the forward direction (with higher frequency) and radiating very little in the backward direction (with lower frequency).

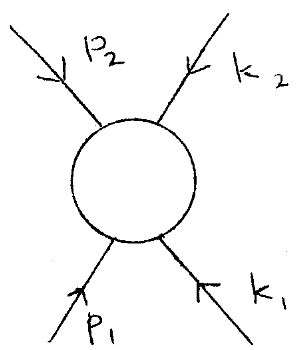
If we consider the fast moving body to be a distribution of light points radiating isotropically in the rest system and distributed over the surface of the body then the above considerations hold for each there points.

Therefore we . happen to see also a large part of the radiation emitted in the backward direction of the object since in the lab. system it moves forward direction.

CHAPTER 6

THEORY OF SCATTERING

Consider the two particles of four momentum p_1 and k_1 being scattered to two particles of four momentum $-p_2$ and $-k_2$. We use the convention all the four momenta p_1, k_1, p_2 and k_2 are ingoing. This has the advantage of taking into account the cross reactions also as we can take any two of the four to be incoming and the other two outgoing with change of the sign of four momenta to represent physical particles.



In the case we first told we have p_1 and k_1 being scattered to p_2' and k_2' where $p_2' = -p_2$, $k_2' = k_2$. p_1, k_1, p_2' and k_2' are all physical.

The independent variables of scattering:

Out of the four momenta we can form 10 Lorentz invariants. These are p_1^2, p_2^2, k_1^2, k_2^2, $p_1 k_1$, $p_1 k_2$, $p_1 p_2$, $p_2 k_1$, $p_2 k_2$, $k_1 k_2$. Out of these 10 invariants $p_1^2 = p_2^2 = m^2$, $k_1^2 = k_2^2 = \mu^2$ are not useful as they are fixed parameters and so are not variables. The remaining six invariants are indeed variable which can be used to describe the scattering process. They are not all independent as we have the four momentum conservation requirement

$$p_1 + p_2 + k_1 + k_2 = 0$$

This four-vector equation is equivalent to four simple equations. So the number of independent variables reduces to two only.

We cannot select these two out of the six arbitrarily. Multiplying the conservation equation

$$p_1 + p_2 + k_1 + k_2 = 0 \tag{6.1}$$

by p_1, p_2, k_1, k_2 successively, we have the four equations

$$\begin{aligned}
p_1 (k_1 + k_2) &= -p_1 p_1 - p_1 p_2 = -m^2 - p_1 p_2 \\
p_2 (k_1 + k_2) &= -p_2 p_2 - p_2 p_1 = -m^2 - p_1 p_2 \\
k_1 (p_1 + p_2) &= -k_1 k_1 - k_1 k_2 = -\mu^2 - k_1 k_2 \\
k_2 (p_1 + p_2) &= -k_2 k_2 - k_1 k_2 = -\mu^2 - k_1 k_2
\end{aligned} \tag{6.2}$$

From the first and second pair of equations follows

$$\begin{aligned}
(p_1 - p_2)(k_1 + k_2) &= 0 \\
(k_1 - k_2)(p_1 + p_2) &= 0
\end{aligned} \tag{6.3}$$

Adding and subtracting these two equations

$$\begin{aligned}
2 p_1 k_1 - 2 p_2 k_2 &= 0 \quad , \text{ie} \quad p_1 k_1 = p_2 k_2 \\
2 p_1 k_2 - 2 p_2 k_1 &= 0 \quad , \text{ie} \quad p_1 k_2 = p_2 k_1
\end{aligned} \tag{6.4}$$

Thus if we take p_1k_2 as one variable p_2k_1 cannot be taken and if we take p_1k_1 as onevariable we cannot take p_2k_2 as the other.

From the second pair in eq.(6.2) one gets

$$(k_1 + k_2)(p_1 + p_2) = -2\mu^2 - 2k_1k_2$$

Adding the first two of eq.(6.2) one gets,

$$(p_1 + p_2)(k_1 + k_2) = -2m^2 - 2p_1p_2$$

So,
$$(-1/2)(k_1 + k_2)(p_1 + p_2) = \mu^2 + k_1k_2$$
$$= m^2 + p_1p_2$$

and
$$\begin{aligned}(1/2)(k_1 + k_2)^2 &= (1/2)(p_1 + p_2)^2 \\ &= \mu^2 + k_1k_2 \\ &= m^2 + p_1p_2\end{aligned} \tag{6.5}$$

This shows that p_1p_2 and k_1k_2 cannot serve as variables at the same time.

Also
$$p_1k_1 + p_1k_2 = -m^2 - p_1p_2 \tag{6.6}$$

So if we take p_1k_1 and p_1k_2, then p_1p_2 cannot be taken. So also k_1k_2, since p_1p_2 differs from k_1k_2 by a constant.

Thus we have to choose p_1k_1 and p_1k_2 for the two variables.

The transition amplitude $T(p_1, p_2, k_1, k_2)$ should be written as an invariant function, as the number of particles is an invariant quantity. So T should be a function of the two Lorentz scalars p_1k_1 and p_1k_2.

Choice of Lorentz frames

The most simple Lorentz

system is one in which one of particles is at rest.

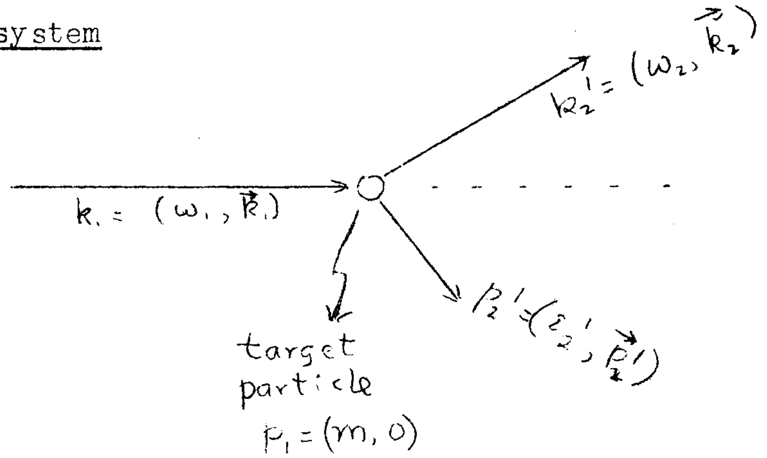
(A) Laboratory system. Here one of the ingoing particles is at rest. Either $\vec{p}_1 = 0$ or $\vec{k}_1 = 0$

(B) Centre of mass system $\vec{p}_1 + \vec{k}_1 = \vec{p}_2 + \vec{k}_2$

(C) Breit system $\vec{k}_1 + \vec{k}_2' = 0$ or $\vec{k}_1 = \vec{k}_2$

We shall calculate the scattering angle in the three different Lorentz frames

a) Lab. system



$$p_1 = (m, 0) \qquad p_2' = (\varepsilon_2', \vec{p}_2') \qquad k_1 = (\omega_1, \vec{k}_1) \qquad k_2' = (\omega_2', k_2')$$

$\omega_1 = \dfrac{p k_1}{m}$ = Lab energy of the incoming particles

$\varepsilon_2 = \dfrac{p_1 p_2'}{m}$ = Lab energy of the target particle after scattering

$\omega_2' = \dfrac{p_1 k_2'}{m}$ = Lab energy of the scattered particle

(6.7)

$$k_1 k_2' = \omega_1 \omega_2' - |k_1||k_2'| \cos \theta_L$$

$$|k_1| = \sqrt{\omega_1^2 - \mu^2}$$

$$|k_2'| = \sqrt{\omega_2'^2 - \mu^2}$$

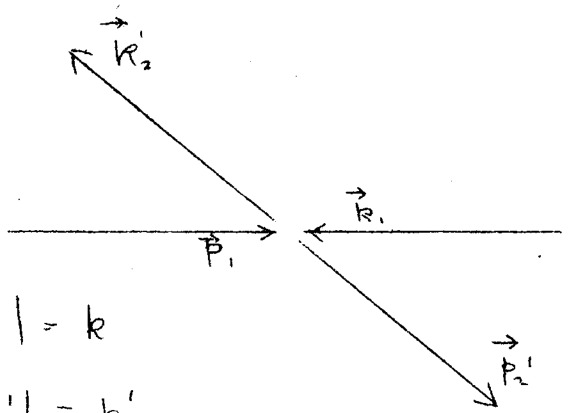
$$\cos \theta_L = \frac{\omega_1 \omega_2' - k_1' k_2'}{|k_1||k_2'|}$$

$$= \frac{\frac{p_1 k_1}{m} \; \frac{p_1 k_2'}{m} - k_1 k_2'}{\sqrt{(\omega_1^2 - \mu^2)(\omega_2'^2 - \mu^2)}}$$

$$= \frac{(p_1 k_1)(p_1 k_2') - (k_1 k_2')\, m^2}{((p_1 k_1)^2 - m^2\mu^2)^{1/2} \, ((p_1 k_2')^2 - m^2\mu^2)^{1/2}} \tag{6.8}$$

This can be expressed in terms of the invariant scalars $p_1 k_1$ and $p_1 k_2$.

(b) <u>Centre of momentum system</u>



$$|\vec{k}_1| = |\vec{p}_1| = k$$

$$|k_2'| = |p_2'| = k'$$

$$\vec{k}_1 + \vec{p}_1 = 0 \quad , \quad \vec{k}_2 + \vec{p}_2{}' = 0$$

$$(k_1 + p_1)^2 = (k_2' + p_2')^2 = E_{cm}^2$$

$$(\omega_1 + \varepsilon_1)^2 = (\omega_2' + \varepsilon_2')^2$$ due to conservation of energy

i.e. $$\left(\sqrt{k^2 + \mu^2} + \sqrt{k^2 + m^2}\right)^2 = \left(\sqrt{k'^2 + \mu^2} + \sqrt{k'^2 + m^2}\right)^2$$

i.e. $$k = k'$$

$$k^2 = |\vec{p}_1| = \frac{[M^2 - (m+\mu)^2][M^2 - (m-\mu)^2]}{4M^2}$$

where $M^2 = (p_1 + k_1)^2$

as we have found earlier.

We take $(p_1 + k_1)^2 = (p_2' + k_2')^2 = s$

as one of the two invariant scalars to describe the scattering.

$$(k_1 - k_2')^2 = 2\mu^2 - 2\omega_1\omega_1' + 2|\vec{k}||\vec{k}_2'| \cos\theta_{cm}$$

$$= 2(\mu^2 - \omega^2 + k^2 \cos\theta_{cm})$$

$$\omega^2 = k^2 + \mu^2$$

So

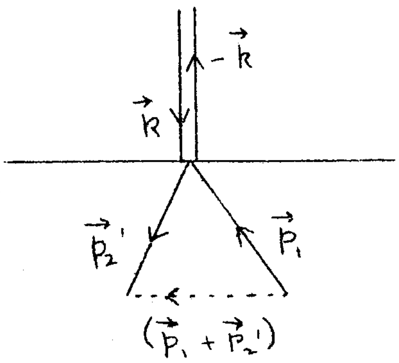
$$(k_1 - k_2')^2 = 2k^2(\cos\theta_{cm} - 1) = t \text{, say.}$$

We take t for the other invariant scalar to describe the scattering.

We can express the momentum k in the Centre of momentum system in terms of s and t.

$$k = \frac{\left[s-(m+\mu)^2\right]\left[s-(m-\mu)^2\right]}{4s} \tag{6.10}$$

C. Breit system (Brick wall system).



$$\vec{k}_1+\vec{k}_2{}' = 0$$
$$\vec{k}_2 = -\vec{k}_1 = -\vec{k}$$

$$\begin{aligned} k_1 &= (\omega, \vec{k}) \\ k_2{}' &= (\omega, -\vec{k}) \\ p_1 &= (\varepsilon_1, \vec{p}_1) \\ p_2{}' &= (\varepsilon_2, \vec{p}_2{}') \end{aligned} \tag{6.11}$$

Energy conservation gives

$$\omega_1 + \varepsilon_1 = \omega_2 + \varepsilon_2$$

But since $\omega_1 = \omega_2 = \omega$,

$$\varepsilon_1 + \varepsilon_2 = \varepsilon$$

so that $|\vec{p}_1| = |\vec{p}_2| = p$

.omentum conservation:

$$\vec{k}_1 + \vec{p}_1 = \vec{p}_2 + \vec{k}_2$$

$$\text{or} \quad \vec{p}_1 - \vec{p}_2{}' = \vec{k}_2{}' - \vec{k}_1 = -2\vec{k}$$

$$\begin{aligned} (\vec{p}_1+\vec{p}_2{}')\cdot 2\vec{k} &= -(\vec{p}_1+\vec{p}_2{}')(\vec{p}_1-\vec{p}_2{}') \\ &= -\vec{p}_1{}^2 + \vec{p}_2{}'^2 \quad = -p^2+p^2 \\ &= 0 \end{aligned}$$

.lso

$$(p_1+p_2{}')(k_1+k_2{}') = (2\varepsilon, \vec{p}_1+\vec{p}_2)\cdot(2\omega, 0) = 4\varepsilon\omega$$

$$t = (k_1-k_2{}')^2 = (0, 2\vec{k})\cdot(0, 2\vec{k})$$

$$= -4|\vec{k}|^2 = \begin{cases}\text{square of the three}\\ \text{momentum transfer}\end{cases}.$$

we can now take ω and $\mathcal{E}$ as the two independent variables to describe the scattering.

$$\omega = \frac{1}{4}(k_1 + k_2')^2$$

$$\mathcal{E} = \frac{1}{2}\,\frac{(p_1 + p_2')(k_1 + k_2')}{\sqrt{(k_1 + k')^2}} \tag{6.12}$$

The scattering angle for 'k' particle is 180° by definition

The scattering for 'p' particle is found as follows.

$$(p_1 - p_2')^2 = 2p^2(\cos\theta_B - 1)$$

But $\quad (p_1 - p_2')^2 = (k_1 - k_2')^2 = t\,,\quad$ hence

$$\cos\theta_B = 1 + \frac{t}{2p^2} \tag{6.13}$$

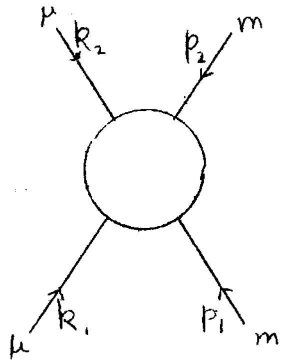
$$t = 2p^2(\cos\theta_B - 1)$$

$$\omega^2 = |\vec{k}|^2 + \mu^2$$

$$= \mu^2 + \frac{t}{4}$$

The Mandelstam Variables s, t, u.

As before, if we do not consider the complications in the scattering amplitude by the spin and isospin of the interacting particles, then the momenta k_1, p_1 and k_2, p_2 completely determine the incoming and out-going states respectively



We introduce the following three Lorentz invariant variables.

$$s = (k_1 + p_1)^2 = (k_2 + p_2)^2$$

$$t = (k_1 + k_2)^2 = (p_1 + p_2)^2$$

$$u = (k_1 + p_2)^2 = (k_2 + p_1)^2$$

and it is easy to check that

$$s + t + u = 2m^2 + 2\mu^2$$

where m and μ are the masses of the two particles, so that out of the three variables s, t and u, only two of them are independent. Now we ask what is the physical significance of these variables. We will now see in the following that these variables are the squares of the energies inthe centre of momentum system in different channels of interaction.

s is the (c.m. energy)^2 if k_1, p_1 or k_2, p_2 are incoming; this we call as the s-channel

$$t = (\text{c.m.energy})^2 \text{ if } (k_1, k_2) \text{ or } (p_1, p_2)$$

are incoming. this we call as the t-channel.

and $u =$ (c.m.energy)^2 if (k_1, p_2) or (k_2, p_1) are incoming.

This we call as the u-channel.

To be more clear, let us take a specific example of the pion-nucleon system. If we denote the pion-momenta by k_1 and k_2 and the nucleon momenta by p_1 and p_2 then the three different channels mentioned above are given by the following processes:

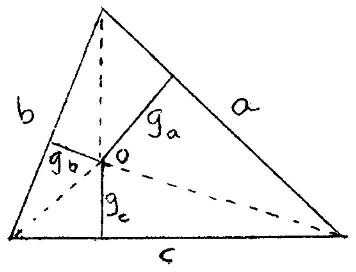
$$\pi + N \longleftrightarrow \pi + N \text{ or } \pi + \bar{N} \leftrightarrow \pi + \bar{N} \quad \text{(s-channel)}$$

$$\pi + \pi \longleftrightarrow N + \bar{N} \quad \text{(t-channel)}$$

and $$\pi + N \longrightarrow \pi + N \quad \text{(u-channel)}$$

The scattering amplitude can thus be expressed as a function of these three variables (though only two are independent). The scattering amplitude $T(s,t,u)$ can be proved to be an analytic function of the three variables s,t, andu. for all the values of these three variables complex, negative,.etc. This, if we know the scattering amplitude over a small region of energy and momentum transfer, then the amplitude may in principle, be determined at all energies and angles since $T(s,t,u)$ is analytic. That $T(s,t,u)$ is an analytic function of s,t, andu means that the physical scattering amplitude is the limitting value of the general function when these variables become real.

Mandelstam Triangle



We now recall an elementary theorem on triangles, If g_a, g_b, g_c are the distances of the sides a, b, c of a triangle from any point O, then

$$a g_a + b g_b + c g_c = 2F = a h_a + b h_b + c h_c$$

where F is the area of the triangle and h_a, h_b, and h_c are the three altitudes of the triangle (to the sides a, b and c respectively). Dividing by c we get

$$\frac{a}{c} g_a + \frac{b}{c} g_b + g_c = h_c$$

Comparing this equation with eqn.

$$s + t + u = 2(m^2 + \mu^2) \tag{6.14}$$

We find that we need only to identify

$$\frac{a}{c} g_a = u \quad ; \quad \frac{b}{c} g_b = s \quad ; \quad g_c = t$$

$$h_c = 2(m^2 + \mu^2)$$

so that any point in the plane(s, t, u) satisfies eq. 6.14
It is perhaps most natural and convenient to take an equilateral triangle.

It is easy to find out the physical region in a given channel (say s-channel) by studying the range of values which the other variables can take. For example, in the s-channel,

$$\begin{aligned} s &= \text{sq. of c.m. energy in the s-channel} \\ &= (k_1+p_1)^2 = k_1^2+p_1^2+2k_1p_1 \\ &= m^2+\mu^2+2(\omega\mathcal{E}-\vec{k}_1\cdot\vec{p}_1) \\ &= m^2+\mu^2+2\omega\mathcal{E}+2k^2 \\ &\geq (m+\mu)^2 \end{aligned}$$

$$k_1=(\omega,\vec{k}_1)\ ;\ k_2=(\mathcal{E},\vec{k}_2)\ ;\ \vec{k}_{1\,cm}=-\vec{p}_{1\,cm}$$

$$t = (k_1-k_2)^2 = -2k^2(1-\cos\theta)$$

(where $|\vec{k}_1|=|\vec{k}_2|=k$; $\theta =$ angle between $\vec{k}_1$ and $\vec{k}_2$

It is assumed here that the outgoing particles have the same masses as the incoming ones).

and we know that

$$k^2 = \frac{[s-(m+\mu)^2][s-(m-\mu)^2]}{4s}$$

so that

$$t = \frac{[s-(m+\mu)^2][s-(m-\mu)^2]}{2s}(\cos\theta-1)$$

and since $\cos\theta$ is real only when

$$0 \leqslant \theta \leqslant \pi$$

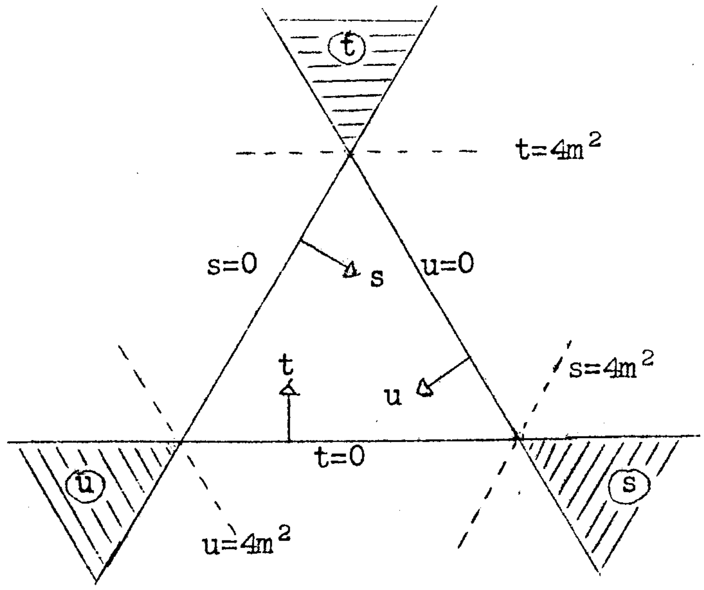
the two limits of t are given by

$$t_{max} = 0 \quad \text{when} \quad \theta=0$$

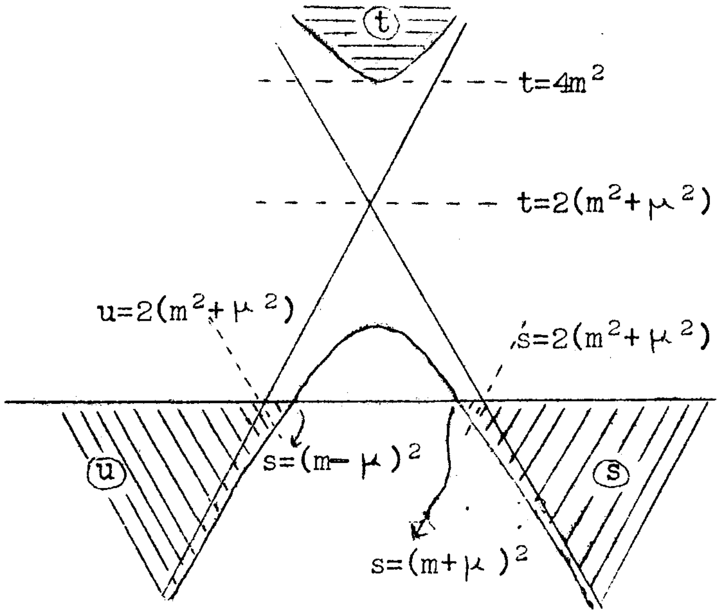
$$t_{min} = -\frac{[s-(m+\mu)^2][s-(m-\mu)^2]}{s} \quad \text{when} \quad \theta=\pi$$

Thus for given s, the eqn. for t is a hyperbola with two branches.

Now let us find out the asymptotes of this hyperbola. We



Physical regions of s, t, u — channels in symmetrical representation when $m = \mu$. Every point in the plane satisfies $s + t + u = 4m^2$.



The physical regions when $m > \mu$.

Chapter 7. Phase space considerations

The phase space plays an important role when we consider a system with infinite numberrof freedom. The behaviour of a given system is made up of two factors, one arising from kinematical phase space, which has its upper hand when the number of particles with which the system is composed is very large, the other arises from the dynamics of the interaction between particles, which is essentially important when only few particles determine the behaviour of the system.

Thus we see that when the degrees of freedom are very large, any process is more governed by kinematical aspects, the dynamical aspects only alter some constants or few parameters characteristic of the system. This is no longer true when the number of degrees of freedom is very small. This is incomplete analogy with classical statistical mechanics. In statistical mechanics, the only dynamical part is the weak Van der Waals forces. All the rest is just kinematics, i.e. the consideration of the phase space. The dynamics enter into picture only in the determination of certain constants characteristic of the system (e.g. specific heat, etc). Only when we study the system when the degrees of freedom are small (say when the system is at a low temperature), do these dynamical aspects take their lead over kinematical aspects.

In elementary particle physics, where the number of degrees we cannot expect the phase-space to decide everything about the system allowing the dynamics to creep through only in determining few constants. Nevertheless, the considerations of phase space give us a statistical description of the system supposed on which is the dynamics of the system, so that any deviation from the statistical energy and momentum distribution of the system gives us some ideas about the dynamics of the system.

Phase Space in Quantum Mechanics

It is now familiear that most of the physical information of a system is contained in the S-Matrix. Let us, ... have a pure initial state $|i\rangle$ formed out a superposition of states forming a complete orthonormal basis in Hilbert space. Let us say that this state is prepared at a time T when the particles are far apart and free and do not interact with each other. After some time, say at time $T \approx 0$, they all come into the sphere of influence of each other. They remain interacting in this space-time region and then move apart with a different complexion (as a result of the interaction that has taken place) after some time, say at time T. Let the final state, which is again a state (containing particles which are free and, far away from the space-time region of interaction) of free particles at time T be $|i'\rangle$. Of course, the complexion $|i'\rangle$ has resulted from $|i\rangle$ due to the specific form of interaction Our understanding about the dynamics of the system lies essentially in our finding out what has happened in the interaction region. However, it is very clear, that the state $|i'\rangle$ has resulted from the state $|i\rangle$ and therefore should be related to that by a unitary transformation (in the limit of $T \to \infty$). That is

$$|i'\rangle = S\,|i\rangle \tag{7.1}$$

S does not depend on the state $|i\rangle$. It then represents just the dynamical behaviour. Assuming that S is known, let us calculate the probability of finding a final state $|f\rangle$ at a time T. That is

$$P_{i\to f} = |\langle f|i'\rangle|^2$$

$$= |\langle f|S|i\rangle|^2 \qquad (7.2)$$

The initial state $|i\rangle$ is usually a two particle state in elementary particle physics, while the final state is, in general, a many particle state. The initial and final characterized by a set of complete quantum numbers, say, four momentum, spin, nucleon number etc. Let us now take the simplest case of scalar particles. Then

$$|i\rangle = |p_1, p_2\rangle$$

$$|i'\rangle = S|p_1, p_2\rangle \qquad (7.3)$$

$$|f\rangle = |p_1', p_2' \cdots p_n'\rangle$$

corresponding to a two-particle initial state and many particle final state.

Eq.(7.2) gives the probability for the two particle state $|i\rangle$ to lead, through scattering, to an n-particle final state $|f\rangle$:

$$P(i\to f) = |\langle p_1' \cdots p_n'|S|p_1 p_2\rangle|^2 \, dp_1' \cdots dp_n' \qquad (7.4)$$

If the number of particles is large, then even the knowledge of S becomes useless, since the calculation becomes unwieldly. Even when n is not very large, it is usual to average out the unwanted degrees of freedom.

Suppose we are interested in the momentum distribution of one set of particles for all orientations of the rest of the particles. We usually average over the things which we are not explicitly interested.

Let F be a well-defined set of final states

$$F = \{ | f \rangle \} \tag{7.5}$$

Disregarding any normalization factor, the probability that the initial state $|i\rangle$ goes over into any one state $|f\rangle$ of the set F is given by

$$P_f \propto \sum_{f \subset F} | \langle p_1' \cdots p_n' | S | p_1 p_2 \rangle |^2$$

$$= \int_F d^4 p_1' \cdots d^4 p_n' | \langle p_1' \cdots p_n' | S | p_1 p_2 \rangle |^2 \tag{7.6}$$

where F denotes the region of phase space in which we are interested. We find that

$$| \langle p_1' \cdots p_n' | S | p_1 p_2 \rangle |^2$$

$$= \delta^4 (p_i - p_f) S (p_1' \cdots p_n' | p_1 p_2) \prod_{i=1}^{n} \delta (p'^2 - m_i^2) \tag{7.7}$$

where the function S is no more restricted to energy and momentum conservations. Thus,

$$P_f \propto \int d^4 p_1' \cdots d^4 p_n' \quad \prod \delta (p_i'^2 - m_i^2)$$

$$\delta^4 \left(\sum_{i=1}^{n} p_i' - (p_1 + p_2) \right) S (p_1' \cdots p_n' | p_1 p_2) \tag{7.8}$$

The question is what the function S is? The factor $\prod_i \delta(p_i'^2 - m_i^2)$ has been introduced for convenience. Since P_f is relativistic invariant and the function S is invariant, the other factors have to be invariant too. Averaging the function S over $p_1' \cdots p_n'$, we can write (7.8) as

$$P_f \propto \bar{S}_n \int_F \prod_{i=1}^n d^4 p_i' \prod_{i=1}^n \delta(p_i'^2 - m_i^2) \, \delta^4\left(\sum_{i=1}^n p_i' - p_1 - p_2\right) \tag{7.9}$$

where $\bar{S}_n$ depends only on p_1 and p_2 and no more on the final momenta p_i'. This averaging is justified by the assumption that when n is very large, S does not depend strongly on the dynamic details of the interaction. The set of final states considered F, may be any domain. For instance, if F is restricted to

$$F = \{ \vec{p}_1 , d^3\vec{p}_1 \} \tag{7.10}$$

and leaving it unrestricted in the other momenta, this will lead to the momentum distribution of particle 1, specifying two three-momenta gives angular correlations etc.

From eq.(7.8) we find that the dynamics is contained in the factor $S(p_1' \cdots p_n' \mid p_1 p_2)$ while the rest is just kinematics. This rest is called the "phase-space factor".

$$R_F = \int_F \cdots \int \prod_{i=1}^{n} \delta(p_i'^2 - m^2)\, \delta^4\left(\sum_{i=1}^{n} p_i' - p_1 - p_2\right) \prod_{i=1}^{n} d^4 p_i'$$

(7.11)

and F is called the "region of phase space". If F is defined in an invariant way then R_F is a relativistic invariant. If there is no restriction on the range F of integration, then R is called the "phase-space integral".

Sec. 7.2 Statistical Theory

In the last lecture, we saw, how when the number of particles is large, the dynamics gets averaged except for one or two important parameters. The whole dynamics is contained in the function. $S(p_1' \cdots p_n' | p_1 p_2)$. It is really difficult to know what this function is. However, in most cases, even an approximate idea of this function may give some insight to the physical problem. Or at least one can understand the peculier behaviour of the S-function by observing its deviation from a smooth statistical behaviour.

In most cases, what we want to know is the total probability for an n-particle final state, and the momentum spectrum of the particles. Let us assume that all the n-particles are of the same type. The probability for n particles to be present in the final state is

$$P_n \propto \int \cdots \int \prod_{i=1}^{n} \delta(p_i'^2 - m_i^2)\, d^4 p_i \quad \delta^4\Big(\sum_{j=1}^{n} p_j' - p_1 - p_2\Big)\, S(p_1' \cdots p_n' \,|\, p_1 p_2)$$

$$= \bar{S}_n \int \cdots \int \prod_{i=1}^{n} \delta(p_i'^2 - m_i^2)\, d^4 p_i' \quad \delta^4\Big(\sum_{j=1}^{n} p_j' - p_1 - p_2\Big)$$

$$= \bar{S}_n \; R_n \tag{7.12}$$

where we denote by $\bar{S}_n$, the weighted average of the S-function $S(p_1' \cdots p_n' \,|\, p_1 p_2)$ over the whole phase space. It is clear that $\bar{S}_n$ is a function of n (the number of particles) so that for sufficiently large n, the interaction will not have a big influence on this function (except when n is very small).[+]

The momentum spectrum of any particle (say particle 1 in the final state) is given by

$$P(\vec{p}_1')\, d^3\vec{p}_1' \propto \bar{S}_n(\vec{p}_1' \cdots) \frac{d^3\vec{p}_1'}{2E_1} \int \prod_{i=2}^{n} \delta(p_i'^2 - m_i^2)\, d^4 p_i' \quad \delta\Big(\sum_{j=2}^{n} p_j - (P - p_1)\Big) \tag{7.12a}$$

+ This is the basic idea of Fermi's statistical model.

Ref. E. Fermi Progr. Theoret. Physics, 5, 570, (1950).

By noticing that

$$d^4 p = \frac{d^3 p}{2E}$$

we can write (7.12a) as

$$P(\vec{p}_1{}') \, d^3\vec{p}_1{}' \propto \bar{S}_n(\vec{p}_1{}') \, \frac{d^3\vec{p}_1{}'}{2E} \int \prod_{i=2}^{n} \delta(p_i'^2 - m^2) \, d^4 p_i \, \delta\left(\sum_{j=2}^{n} p_j' - (P - p_1)\right)$$

$$= d^3 p_1' \, \bar{S}_n(\vec{p}_1{}') \, R_n$$

(7.12b)

The average $\bar{S}_n(\vec{p}_1{}')$ now not only depends on n but depends also on $\vec{p}_1$. The phase-space integral does not depend on the vector $\vec{p}_1$ but depends only on its magnitude if we choose the over-all centre of momentum frame. The dependence on direction $\vec{p}_1$, is essentially given by $\bar{S}_n$. The momentum distribution of this singled-out particle

$$P(\vec{p}_1) \, d^3\vec{p}_1 \propto \bar{\bar{S}}_n(p_1) \, \frac{p_1'^2 \, dp_1'}{2E} \, R_{n-1} \qquad (7.13)$$

where an average has been performed over the directions other than that of particle 1. The kinematical factor $|p_1'|^2 R_{n-1}$ essentially determines the spectrum. The range of p_1' is given by

$$|p_1'|^2 \geqslant \{E-(m_2+m_2+\cdots m_n)\}^2 - m_1^2 \qquad (7.14)$$

The equality in (7.14) gives the maximum value of $\vec{p}_1'$

We can only say that the phase space factor already contains a certain amount of information. The influence on the dynamical aspects becomes less important the more we average. Thus, the study of phase-space factor is worthwhile, particularly when not too many particles are involved. The phase space factor yields a back ground (the behaviour of the system when the matrix element is a constant) from which every significant dynamic property should exhibit itself clearly.

Invariant and non-invariant phase-space

In the last section we saw that the probability for particles to be present on the final state is

$$P_n \propto \bar{S}_n \underbrace{\int \prod_{i=1}^{n} \frac{d^3\vec{p}_i'}{2E_i} \delta^4\left(\sum p_i' - (p_1+p_2)\right)}_{\text{Invariant Phase space}}$$

$$= \frac{\bar{S}_n}{2^n(\overline{E_1 - E_n})} \underbrace{\int \prod_{i=1}^{n} d^3\vec{p}_i' \, \delta^3\left(\sum \vec{p}_i - \vec{p}\right) \delta\left(\sum_j E_j - E\right)}_{\text{Non-invariant phase space}}$$

(7.15)

It should be noticed that in the first step of Eq.(7.15), the function $S(p_1' \cdots p_n' | p_1 p_2)$ is averaged out and is taken out of the integral sign, while in the second step of Eq.(7.15), we have also taken the mean value of

$$\left[\prod_{i=1}^{n} 2\epsilon_i\right]^{-1} = \left(\bar{E}\right)^{-1} \quad \text{(say)}$$

Denoting the product

$$\bar{S}_n \left(\bar{E}\right)^{-1} = \overline{S_n'} \tag{7.16}$$

we see that in the centre of momentum frame two equivalent description emerge.

$$P_n \propto \bar{S}_n R_n = \bar{S}_n \left(\prod_{i=1}^{n} 2\epsilon_i\right)^{-1} \rho_n$$

$$= \overline{S_n'} \rho_n \tag{7.17}$$

We see that while both $\bar{S}_n$ and R_n are independently Lorentz invariant, only the product $\overline{S_n'} \rho_n$ is Lorentz invariant.

In addition, the dimensions of $\bar{S}_n$ and $\bar{S}_n'$, and those of R_n and ρ_n differ by a factor $(\text{energy})^n$ as is obvious from the very definition of S_n' and ρ_n. A rough estimation of this factor may be obtained as follows:

$$\overline{\left(\prod_{i=1}^{n} 2\epsilon_i\right)^{-1}} \approx \left(\overline{\prod_{i=1}^{n} 2\epsilon_i}\right)^{-1}$$

$$\left\{ \overline{\prod_{i=1}^{n} 2E_i} \right\} \approx 2^n \prod_{i=1}^{n} \bar{E}_i$$

Here $\bar{E}_i$ may be calculated from the assumption of equipartition of kinetic energies.

$$\bar{E}_i \approx m_i + \frac{(E - \sum_{j=1}^{n} m_j)}{n}$$

Thus the invariant phase space R_n is defined as

$$R_n(P, m_1, \cdots m_n) = \int \prod_{i=1}^{n} d^4 p_i \, \delta(p_i^2 - m_i^2) \, \delta^4\left(\sum_{j=1}^{n} p_j - P_{total}\right) \quad (7.15a)$$

where P_{total} is the total four momentum of the system.

The non-invariant phase space ρ_n is defined (in the C.M. system) as

$$\rho_n(E, m_1, \cdots m_n) = \int \prod_{i=1}^{n} d^3\vec{p}_i \, \delta\left(E - \sum_{j=1}^{n} E_j\right) \delta^3\left(\sum \vec{p}_i\right) \quad (7.15b)$$

where E is the total c.m. energy of the whole system and

$$E_i = \sqrt{|\vec{p}_i|^2 + m_i^2}$$

is the energy of the particle i.

Fermi has formulated his statistical theory in terms of the non-invariant phase space ρ_n. It is, however, more customary nowadays to use the invariant phase space both in statistical theory calculations and in finding the back-ground from which the dynamic properties should exihibit themselves. As soon as one sticks to c.m. frame, both descriptions become completely equivalent (except for dimensions of R_n and ρ_n).

It should be mentioned that as long as we wish to use the relativistic formula $E_i = \sqrt{|\vec{p}_i|^2 + m_i^2}$ for the energy of the particles, both the fact that these integrals become very much complicated when more than three particles are involved Analytical formulas can be obtained if either of the limiting cases,

$$\sqrt{|\vec{p}_i|^2 + m_i^2} \begin{cases} \nearrow \; m_i + \dfrac{|\vec{p}_i|^2}{2m} & \text{non-relativistic} \\ \searrow \; |\vec{p}_i| & \text{ultra-relativistic} \end{cases}$$

is reached for each particle in the integral. However, numerical integrations are simpler in the invariant phase space, because the Lorentz invariance leads to a recurrence formula. We recall the definition of R_n, the invariant phase space (7.15a)

$$R_n = \int \cdots \int \prod_{i=1}^{n} d^4 p_i \; \delta(p_i^2 - m_i^2) \; \delta^4\Big(\sum_{j=1}^{n} p_j - P_{total}\Big)$$

Let us now separate out the n^{th} integration from this; we get,

$$\begin{aligned} R_n&(P, m_1, \cdots m_n) \\ &= \int\Big\{\int \prod_{j=1}^{n-1} d^4 p_j \; \delta(|\vec{p}_j|^2 - m_j^2) \; \delta^4\Big(\sum_{j=1}^{n-1} p_j - (P - p_n)\Big)\Big\} \\ &\qquad\qquad \delta(p_n^2 - m_n^2)\, d^4 p_n \\ &\equiv R_{n-1} \; \delta(p_n^2 - m_n^2)\, d^4 p_n \end{aligned}$$

when the two factors inside the paranthesis are by definition

$R_{n-1}\left((P-P_n)\,;\,m_1,\cdots m_{n-1}\right)$. Hence

$$R_n(P;m_1\cdots m_n) = \int \frac{d^3p_n}{2E_n} R_{n-1}(P-P_n\,;m_1\cdots m_{n-1}) \tag{7.16a}$$

where we have used the relation

$$\delta(p_n^2-m_n^2)\,d^4p_n \rightarrow \frac{d^3p_n}{2E_n}$$

We now use the relativistic invariance to calculate R_n in the c.m.system and R_{n-1} in the system where

$$(P-P_n) = (E,0)$$

$$E_{cM}^2 = (P-p_n)^2 = (E-E_n)^2-(\vec{P}-\vec{P}_n)^2$$

$$= (E-E_n)^2 - |\vec{P}_n|^2$$

Using

$$E_n^2 = |\vec{P}_n|^2 + m_n^2$$

we obtain the recurrence formula

$$R_n(E,m_1\cdots m_n)$$

$$= \int\frac{d^3p_n}{2\sqrt{|\vec{P}_n|^2+m_n^2}} R_{n-1}\left(\sqrt{E^2+m_n^2-2E\sqrt{|\vec{P}_n|^2+m_n^2}}\,;\; m_1,m_2,\cdots m_{n-1}\right) \tag{7.16b}$$

To start with, one may define a single-particle phase-space in its rest frame as

$$R_1 = \int \frac{d^3p}{2\sqrt{|\vec{p}|^2+m^2}} \delta\left(E-\sqrt{|\vec{p}|^2+m^2}\right)\delta^3(\vec{p})$$

$$= \frac{\delta(E-m)}{2m} \qquad (7.16c)$$

Quite similarly one can calculate R_2 etc.

Sec.7.4

Invariant Mass Distributions

We saw in the preceding sections that the deviation (experimentally found) from the statistical uniform behaviour of the function $S(p_1' \ldots p_n' | p_1 p_2)$ could give us some information about the dynamics of the system. In other words= the anamalous behaviour of the function S_n may give us an idea about the striking and dominant dynamics of the system. We shall now discuss (the striking example of the discovery of the ω-meson) how a dynamical property of a system exhibits itself from the background of pure phase space.

Suppose, in a process, n particles are present in the final state. Let us divide these n particles into two groups, one containing ℓ particles $(\ell < n)$ and the other contains the rest. Each group has a total four momentum and rest mass:

$$P_\ell = \sum_{i=1}^{\ell} p_i \quad ; \quad M_\ell^2 = P_\ell^2$$

$$P_{n-\ell} = \sum_{i=\ell+1}^{n} p_i \quad ; \quad M_{n-\ell}^2 = P_{n-\ell}^2 \tag{7.17}$$

If everything is governed by pure phase space what is the probability that the square of the rest mass M_ℓ^2 lies between M^2 and $M^2 + dM^2$? . For this we will have to calculate the mass distribution. If this mass distribution (which we calculate from the phase space) is compared with the experimental one, any significant deviation has to be interpreted as an indication for a dynamical irregularity.

We now calculate the mass distribution. Since

$$\delta(x-y) = \int \delta(x-z)\,\delta(z-y)\,dz$$

we can write $\delta^4\left(\sum p_j - P\right)$ as

$$\delta^4\left(\sum_{j=1}^{n} p_j - P\right) = \int d^4 P_\ell \; \delta^4\left(\sum_{j=\ell+1}^{n} p_j - (P - P_\ell)\right) \delta^4\left(P_\ell - \sum_{j=1}^{\ell} p_j\right)$$

and obtain

$$R_n(P; m_1 \cdots m_n)$$
$$= \int \delta^4\left(P - P_\ell - \sum_{j=\ell+1}^{n} p_j\right) \prod_{j=\ell+1}^{n} \delta(p_j^2 - m_j^2)\, d^4 p_j$$
$$\times \int \delta^4\left(P_\ell - \sum_{j=1}^{\ell} p_j\right) \prod_{i=1}^{\ell} \delta(p_i^2 - m_i^2)\, d^4 P_\ell \; d^4 p_i$$

Introducing a factor 1 of the form

$$1 = \int_0^\infty \delta(M^2 - P_\ell^2)\, dM^2$$

in (7.18a), we get

$$R_n(P; m_1 \cdots m_n) = \int_0^\infty dM^2 \left[\int \delta^4(P - P_\ell - \sum_{j=\ell+1}^{n} p_j) \prod_{j=\ell+1}^{n} \delta(p_j^2 - m_j^2)\, \delta(P_\ell^2 - M_\ell^2)\, d^4p_j\, d^4P_\ell \right]$$

$$\times \left[\int \delta^4(P_\ell - \sum_{j=1}^{\ell} p_j) \prod_{i=1}^{\ell} \delta(p_i^2 - m_i^2)\, d^4p_i \right]$$

(7.18b)

This is a convolution integral over two invariant phase space integrals and therefore we have a general recurrence formula.

$$R_n(P; m_1 \cdots m_n) = \int_0^\infty dM^2 R_{n-\ell+1}(P; M, m_{\ell+1}, \cdots m_n)\, R_\ell(P_\ell; m_1 \cdots m_\ell) \qquad (7.19)$$

$R_{n-\ell+1}(P; M, m_{\ell+1}, \cdots m_n)$ describes a situation in which there are $n-\ell$ particles $m_{\ell+1}, \cdots m_n$ with one 'lumped' particle of mass $M = P_\ell^2$ (representing the system of particles $m_1 \cdots m_\ell$ as one single kinematical object).

$R_\ell(P_\ell; m_1 \cdots m_\ell)$ describes the internal situation of the 'particle' with total momentum P_ℓ which is made up by particles $m_1 \cdots m_\ell$. Eq.(7.19) already gives the mass distribution. Because $R_n(P; m_1 \cdots m_n)$ is to be considered as a simple constant, we may divide by it and obtain

$$\int_0^\infty dM^2 \, \frac{R_{n-\ell+1}(P; M, m_{\ell+1}, \cdots m_n) \; R_\ell(P_\ell; m_1 \cdots m_\ell)}{R_n(P; m_1 \cdots m_n)}$$

$$= 1 . \tag{7.20}$$

from which it follows that the probability distribution $P(M)\,d\,M^2$ is given by

$$P(M)\,dM^2 = \frac{R_{n-\ell+1}(P; M, m_{\ell+1} \cdots m_n)\, R_\ell(P_\ell; m_1 \cdots m_\ell)}{R_n(P; m_1 \cdots m_n)} \tag{7.21}$$

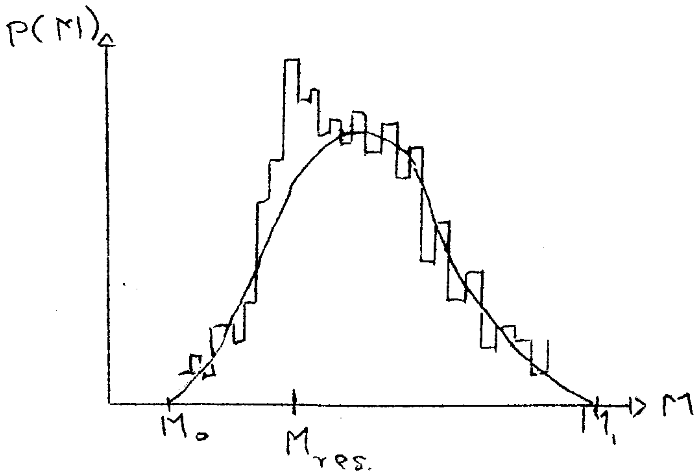
The general form of the mass distribution (7.21) is clear. It becomes zero at the two limits where

$$M = M_0 = \sum_{i=1}^{\ell} m_i$$

and where

$$M = M_1 = E_{cM} - \sum_{j=\ell+1}^{n} m_j$$

and is positive in between. We expect the distribution like Fig.(1)



The ω-meson has been discovered in this way. An experimental histogram of the mass distribution of 3 pions with total charge zero was compared with the mass distribution (7.21) and a sharp peak like the one in Fig.(1) was found. This is then interpreted as an unstable bound state of three mesons and called the ω meson From the width of the peak one can conclude that the life time is $\sim 10^{-22}$ sec. It would be hard to observe such a particle directly, the deviations from pure phase space have lead to its discovery.

Sec.7.5

Production of bound-states in high-energy collisons+

In the secondary beams emerging from targets hit by high energy ($\approx$ 20-30 Gev) protons one observes deuterons, ^3H and ^3He of surprisingly high moments. Until recently, it was not clear whether the elementary deuteron production of the kind

$$p + N \rightarrow d + \text{pions} + \text{others}$$

would be of any importance at all. Now its presence has been established experimentally in a hydrogen bubble chamber by B.Sechi-Zorn at Brookhaven(1). The experiment was done with 2.0 Gev primary protons and gives a deuteron production cross-section in agreement with the one predicted by using statistical theory for 2.3 Gev primary energy while the ratio $\frac{\sigma(d+\pi)}{\sigma(d+2\pi)}$ is experimentally about four times smaller than the one predicted.

In this discussion, let us limit ourselves only to processes of the kind

$$p + p \rightarrow d + (n-1)\,\pi \qquad (7.22)$$

We shall make the hypothesis that the statistical theory is a suitable tool to calculate the production of compound particles and derive the consequences of this supposition. Secondly, we shall critically discuss the validity of the statistical theory in this particular case.

(1) B.Sechi-Zorn: Abstract Washington APS Meeting (April 1962) Bull.Am.Phys.Soc.7, 349 (1962)

+ For a full discussion of this theory and further references see R.Hagedorn, Nuovo Cim.25 (1962) 1017.

If such a bound state of mass M is to be created in a very small region Ω of interaction, then it must be created in a state of extreme contraction. Let ψ_d be the wave function of the deuteron. Then, the probability of the deuteron to be found in the volume Ω is

$$\int_{\Omega} |\psi_d(\vec{x})|^2 \, d^3x$$

Thus, if Ω is the "natural interaction volume".

$$\Omega \sim \frac{4\pi}{3} \left(\frac{1}{\mu}\right)^3$$

$$\mu = \text{pion mass}$$

then for the deuteron production one should use

$$\Omega_d = \Omega \int_{\Omega} |\psi_d|^2 \, d^3x \approx \Omega \cdot \frac{\Omega}{V_d} \qquad (7.23)$$

where V_d is the deuteron volume.

Intuitively the idea is that in a nucleon-nucleon collision, in which several particles are produced, two of the final nucleons can emerge in a bound state only if extremely favourable kinematical conditions are fulfilled. Namely, the binding energy of the deuteron is so small in comparison with the usual kinematic energies of the final particles that one feels it will not be strong enough to bind the two nucleons together, except if they accidentally are created in states of almost equal momenta-such that they fly nearly parallel and with practically equal velocities and would-(should there be a binding force or not)-stay near together for some time in any case. (See fig.(2).

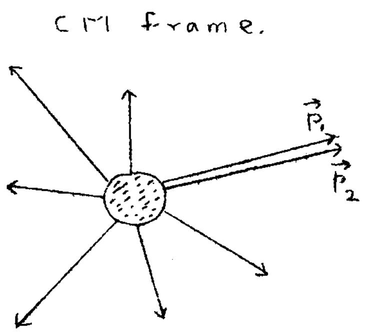


Fig (2)

The condition for this is that the relative momenta are small, they must be contained in that region where the momentum distribution of the nucleons in the deuteron is essentially different from zero. From this point of view the phase space governs essentially the situation and only in those few cases, where, already for pure statistical reasons, two nucleons happen to stay together, can the small force between them come into action. Either it leads them not to a bound state—this case is contained in the statistical weight for free nucleons (which occasionally might have small relative momenta) or it leads to a bound state. As this latter case arises, from the same kinematical situation as the case of small relative moment in the free nucleon case, its probability can be calculated by determining that part of the total phase space, in which this kinematical condition is fulfilled—but, since it describes a bound state, it is entirely something new and has its own statistical weight as a new channel.

We now calculate the probability for two nucleons to emerge with a small relative momentum. Let us formulate this here using the invariant phase space description (discussed in earlier sections of this chapter) which is more elegant.

The invariant phase-space density can be written

$$R_n(P; m_1 \cdots m_n) = \int \delta^4\left(P - \sum_{j=1}^{n} p_j\right) \prod_{i=1}^{n} \delta(p_i^2 - m_i^2)\, d^4 p_i \qquad (7.15)$$

where P is the total four-momentum of all n particles. Let us divide the particles into two groups one with particles $i = 1, \cdots \ell$, the other containing the rest. In the previous section, we saw that in the overall C.M. system we can write the recurrence formula (7.19)

$$P_n(E; m_1 \cdots m_n) = \int_0^\infty dM^2\, R_{n-\ell+1}(M_\ell; m_{\ell+1} \cdots m_n)\, R_\ell(m_1 \cdots m_\ell)$$

We apply this formula to the case $\ell = 2$; $m_1 = m_2 = m$ $M_\ell = M$. Then with $E = \sqrt{|\vec{p}|^2 + m^2}$ we obtain

$$R_2(M, m_1 m_2) = \int \delta(M - (E_1 + E_2))\, \delta(\vec{p}_1 + \vec{p}_2) \frac{d^3\vec{p}_1\, d^3\vec{p}_2}{4E_1E_2}$$

$$= 4\pi \int \frac{\delta(M - E_1 - E_2) \quad p_1^2\, dp_1}{4E_1E_2}\Bigg|_{\text{at } \vec{p}_1 = -\vec{p}_2 = \vec{p}}$$

$$= \frac{\pi}{E_1E_2} \int E_1\, dE_1\, p_1\, \delta(M - E_1 - E_2)$$

$$= \frac{\pi}{E_2}\, \vec{p}_{cm} \int \delta(M - E_1 - E_2) \frac{dE_1}{dM}\, dM$$

$$R_2(M, m_1 m_2) = \frac{\pi}{\epsilon} \vec{P}_{cM} \frac{\epsilon}{2M} .$$

$$= \frac{\pi}{2M} \sqrt{M^2 - 4m^2}$$

where we have used the relations

$$\epsilon_1 \, d\epsilon_1 = p_1 \, dp_1$$

and

$$\vec{p}_1 = -\vec{p}_2 = \vec{p}_{cm}$$

$$\epsilon_1 = \epsilon_2 = \epsilon_{cm}$$

Hence

$$R_n(\epsilon; m_1 \ldots m_n) = \pi \int_{2m}^{\infty} \left(dM \sqrt{M^2 - 4m^2}\right) R_{n-1}(M, m_3, \ldots m_{n-1}; \epsilon)$$

(7.24)

Now the invariant mass of the two considered particles m_1 and m_2 is

$$M^2 = (\epsilon_1 + \epsilon_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$$

$$= 4\epsilon^2 = 4(m^2 + |\vec{p}_{cm}|^2)$$

(7.25)

if ϵ and $\vec{p}$ are taken in the two body rest frame. Therefore we obtain the formula

$$R_n(m_1 \ldots m_n; \epsilon) = 4\pi \int_0^{\infty} \frac{p^2 \, dp}{\sqrt{|\vec{p}|^2 + m^2}} R_{n-1}(M(p), m_3 \ldots m_n; \epsilon) \quad (7.25a)$$

where $M(p)$ is given by (7.24). We now picture the production of a deuteron in the following way.

We introduce in (7.25a) a weight factor $f(p)$ for the p integration, which limits p to those values which are in accordance with the deuteron structure (p is the magnitude of momentum of either nucleon in the rest system of the deuterons. As the two nucleons considered stay near to each other because of their limited relative momentum, the nuclear forces between them can act, while between all the other particles they do not since these particles rapidly reach large mutual distances. Therefore, in the energy balance, the potential energy of these two nucleons must be taken into account. This can be done roughly by replacing

$$M(p) \rightarrow M(p) + V(p) = M^*(p)$$

where $V(p)$ is essentially the Fourier transform of the potential $V(r)$ and $V(p) \leq 0$ for all relevant momenta .

With these two changes we obtain the "partial phase space" for deuteron production

$$\begin{aligned} &\delta R_n (m_1 \cdots m_n , E) \\ &\quad = R_{n-1} \left(E, M_d , m_3 \cdots m_n \right) 4\pi \int_0^\infty \frac{f(p)\, p^2}{\sqrt{p^2+m^2}}\, dp \end{aligned} \tag{7.26}$$

Here the function $R_{n-1} \left(M^*(p) \ldots \right)$ has been taken out of the integral at some average value

$$M^*(\bar{p}) = M(\bar{p}) + V(\bar{p})$$

This average value should clearly be the mass of the deuteron M_d . It remains to evaluate the p-integral. Let the structure of the deuteron be roughly described by

$$|\psi_d|^2 \sim \exp\left(-\frac{r^2}{2r_0^2}\right)$$

It does not matter in this consideration that this is a bad wave function. We define the volume of the deuteron by

$$V_d = 4\pi \int r^2 \exp\left(-\frac{r^2}{2r_0^2}\right) dr$$

$$= \left(2\pi r_0^2\right)^{3/2}$$

The corresponding momentum distribution will be governed by the Fourier transform of $|\psi|^2$ namely

$$f(p) = \exp\left(-\frac{p^2}{2/r_0^2}\right)$$

With this $f(p)$ we obtain for the integral (7.26)

$$4\pi \int_0^\infty \frac{p^2 \exp\left(-\frac{p^2}{2/r_0^2}\right) dp}{\sqrt{p^2+m^2}}$$

$$\approx \left(\frac{2\pi}{r_0^2}\right)^{3/2} \frac{1}{\bar{E}}$$

$$= \frac{(2\pi)^3}{\bar{E}\, V_d}$$

Here $\bar{E}$ is some mean value of $\sqrt{p^2+m^2}$. [As extends from zero to about $2/r_0$ only, we have $\bar{E} \approx m$]

Therefore,

$$\delta R_n(E; m_1 \cdots m_n) = \frac{(2\pi)^3}{\bar{E}\, V_d} R_{n-1}(M_d; E; m_3 \cdots m_n) \tag{7.26}$$

We now consider the probability for finding n particles, two of which are in a bound state, this is given by

$$P_{n,d}(E) \equiv K^n \, \delta \, R_n(m_1 \cdots m_n, E) \qquad (7.27)$$

where K is a certain parameter which plays in the formalism with the invariant phase space, the same role as the interaction volume Ω does in the non-invariant formulation. From(7.26a) and (7.27) we get

$$\begin{aligned} P_{n,d}(E) &= K^{n-2}\left[\frac{K^2(2\pi)^3}{\bar{E}\, V_d}\right] R_{n-1} \\ &= K^{n-2}\, K_d \, R_{n-1} \end{aligned} \qquad (7.27a)$$

$$K_d = \frac{K^2(2\pi)^3}{\bar{E}\, V_d}$$

K_d is defined by this equation.

We now translate this to the non-invariant formulation. As the calculated probabilities must be the same in both discriptions, we have quite generally

$$P_n = K^n R_n = \left(\frac{\Omega}{(2\pi)^3}\right)^n \rho_n$$

Incidentally K^n and $\left(\frac{\Omega}{(2\pi)^3}\right)^n$ are analogues of $\bar{S}_n$ and $\bar{S}_n'$ respectively used in a previous section of this chapter. The invariant phase-space R_n and the non-invariant ρ_n are related to each other by

$$R_n = \int \delta^4 (P - \sum_{j=1}^{n} p_j) \frac{d^3\vec{p}_1 \cdots d^3\vec{p}_n}{2E_1 \cdots 2E_n}$$

$$= \frac{1}{\prod_{i=1}^{n} 2E_i} \int \delta^4 (P - \Sigma p_i) \, d^3\vec{p}_1 \cdots d^3\vec{p}_n$$

$$= \frac{1}{\prod_{i=1}^{n} 2\bar{E}_i} \rho_n$$

The $\bar{E}_i$ are defined by this equation and physically they are mean energies of the particles. This, if different sorts of particles are present, leads then to the correspondence

$$\frac{K_i}{2\bar{E}_i} = \frac{\Omega_i}{(2\pi)^3}$$

Rewriting (7.27a) in the non-invariant formulation, we obtain with this correspondence

$$P_{n,d}(E) = \left(\frac{\Omega}{(2\pi)^3}\right)^{n-2} \cdot \frac{\Omega_d}{(2\pi)^3} \rho_{n-1} \tag{7.27b}$$

$$\Omega_d = \frac{(2\pi)^3 K_d}{2\bar{E}_d}$$

$$= \frac{(2\pi)^3 (2\pi)^3 K^2}{2\bar{E}_d \; \bar{E} \; V_d}$$

Since $\bar{E} \approx m$ was the average energy of a nucleon in the deuteron while $\bar{E}_d$ is the average energy of the deuteron in the c.m. system of the n-particles, we have roughly

$$\bar{E}_d \approx 2\bar{E} \approx 2m$$

The deuterons are non-relativistic in the c.m.) Then, using again the correspondence between K and Ω we obtain

$$\Omega_d \approx \left[\frac{(2\pi)^3 K}{2E} \right] \frac{1}{V_d} \tag{7.23a}$$

$$= \Omega \frac{\Omega}{V_d}$$

With Ω being the natural interactions volume. The consideration just carried through aims rather to show, if the statistical theory (with the above deuteron interaction volume) gives unexpectedly large numbers of deuterons in high energy $p-p$ collisions, then this is not in contradiction with the intuitive argument that the deuterons will appear only if the kinematical conditions favour it. In fact, we have just shown that the quantitative formulation of this intuitive argument leads to exactly the formula used. Then the unexpectedly large number of deuterons (in experiment and in calculation) only teaches us that our intuition is right, concerning the mechanism, but it is wrong in guessing how often in the average the special kinematical conditions are met.

For a detailed account of the computation made and fit with experiments the reader is advised to refer to R.Hagedorn. Nuovo Cim.25, (1962) 1017.

Sec. 7.6

The Dalitz Plot

In the technique of the Dalitz plot, one looks for the deviation of the function $S(p_1' \cdots p_n' | p_1 p_2)$ from its statistical behaviour. If there is an anamalous behaviour (by which, we mean deviation from the statistical behaviour) of the function S, we may be able to say something about the structure of the dynamics. This method has been applied with success to the case of ω -meson, for example.

Let $p_i = (E_i, \vec{p}_i)$ be the four momenta of the three pions in the decay

$$\omega \to \pi^+ + \pi^- + \pi^0$$

The differential invariant phase space giving the momentum distribution in the ω -rest frame is

$$d^9 R(M; m_1 m_2 m_3)$$

$$= \delta(M - (E_1 + E_2 + E_3))\, \delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \frac{d^3\vec{p}_1 \; d^3\vec{p}_2 \; d^3\vec{p}_3}{8 E_1 E_2 E_3} \tag{7.28}$$

where M is the mass of the ω -meson and E_1, E_2, E_3 are the energies of the three pions coming out. The energy momentum conservation tells us that

$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$$
$$E_1 + E_2 + E_3 = M \tag{7.29}$$

The differential invariant phase space given by (7.28) depends on three momenta, that is, nine variables; while (7.29) puts four constraints on them. Even then, we are left with five variables, namely E_1, E_2, E_3 and the two angles between the directions of three pions in the final state, for example. However, one may reduce the number of variables by integrating over the unwanted informations.

In the Dalitz plot, what one does, is to integrate over the angles and considers E_1, E_2, E_3 as the three variables. By equation (7.29) $M = E_1 + E_2 + E_3$, so that of the three E_1, E_2 and E_3, only two of them are independent. We want now to calculate the distribution function of any two of the three energies (E_1, E_2), (say). From (7.28) by integration over $\vec{p}_3$ we obtain

$$d^6 R = \delta\left(M - (E_1 + E_2 + E_3)\right) \frac{d^3\vec{p}_1 \, d^3\vec{p}_2}{8 E_1 E_2 E_3}$$

$$\begin{aligned} E_3 &= \sqrt{|\vec{p}_3|^2 + m^2} \\ &= \sqrt{[\vec{p}_1 + \vec{p}_2]^2 + m^2} \\ &= \sqrt{|\vec{p}_1|^2 + |\vec{p}_2|^2 + 2|\vec{p}_1||\vec{p}_2|\cos\phi + m^2} \end{aligned} \qquad (7.30)$$

m = mass of the pion

ϕ = angle between the directions of $\vec{p}_1$ and $\vec{p}_2$

Now $\vec{p}_1$ and $\vec{p}_2$ are arbitrary. Replace now $d^3\vec{p}_2$ by $2\pi p_2^2 \, dp_2 \, d(\cos\phi)$ and perform the first angular integration. Since the second angular integration is not going to depend on the direction any more we can replace $d^3 \vec{p}_1$ by $4\pi \, p_1^2 \, dp_1$. Hence we obtain

$$d^2 R = \int \frac{\delta(M-(E_1+E_2+E_3))}{8 E_1 E_2 E_3} 8\pi^2 p_1^2 \, dp_1 \, p_2^2 \, dp_2 \, d(\cos\phi)$$

$$= \int \frac{8\pi^2}{8 E_1 E_2 E_3} \delta(M - E_1 - E_2 - E_3) \, p_1 p_2 E_1 E_2 \, dE_1 \, dE_2 \, d(\cos\phi)$$

since

$$\begin{aligned} p_1 \, dp_1 &= E_1 \, dE_1 \\ p_2 \, dp_2 &= E_2 \, dE_2 \\ E_1 &= \sqrt{|\vec{p}_1|^2 + m^2} \\ E_2 &= \sqrt{|\vec{p}_2|^2 + m^2} \end{aligned} \tag{7.31}$$

($\vec{p}_1$ and $\vec{p}_2$ arbitrary).

Then

$$d^2 R = \int \frac{\pi^2}{E_3} p_1 p_2 \, \delta(M - E_1 - E_2 - E_3) \, dE_1 \, dE_2 \, d(\cos\phi)$$

$$= \int \frac{\pi^2}{E_3} p_1 p_2 \, \delta(M - E_1 - E_2 - E_3) \, dE_1 \, dE_2 \, \frac{d(\cos\phi)}{dM} \, dM \tag{7.32}$$

Now

$$\frac{dM}{d\cos\phi} = \frac{d(E_1 + E_2 + E_3)}{d\cos\phi}$$

$$= \frac{dE_3}{d\cos\phi} = \frac{d\left(\sqrt{|\vec{p}_1|^2 + |\vec{p}_2|^2 + 2 p_1 p_2 \cos\phi + m^2}\right)}{d\cos\phi}$$

$$= \frac{1}{2E_3} 2 p_1 p_2 = \frac{p_1 p_2}{E_3} \tag{7.33}$$

Thus, Equation (7.32) becomes

$$d^2 R = \frac{\pi^2 p_1 p_2}{E_3} \, dE_1 \, dE_2 \, \frac{E_3}{p_1 p_2}$$

$$= \pi^2 \, dE_1 \, dE_2 \qquad (7.34)$$

Therefore, the probability distribution we wish is

$$P(E_1 E_2) \, dE_1 \, dE_2 = (\text{constant}) \, dE_1 \, dE_2 \qquad (7.35)$$

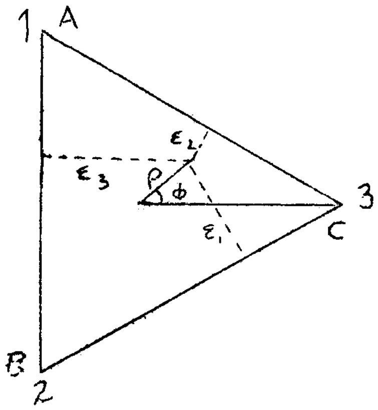
That is

$$P(E_1 E_2) = \text{constant independent of } E_1 \text{ and } E_2 \qquad (7.35a)$$

If we now plot the experimental distribution against a constant distribution, every deviation would be due to the matrix element, if we could be sure that all events we use to plot the experimental distribution are coming from genuine $\omega \rightarrow 3\pi$ decay. However, there is always a considerable back ground of (3π) states which accidentally (from pure phase space considerations) have just the mass M_ω; these are those lying below the statistical curve. Any significant deviation from the constant $(E_1 \, E_2)$ distribution can come only from those lying well above the statistical curve. As far as the ω-meson is concerned, this was in fact sufficient to determine its spin and parity with a rather good reliability.

The triangular plot.

Instead of plotting only E_1 and E_2 , one can as well plot all the three E_1, E_2 and E_3 with a constraint $E_1 + E_2 + E_3 = M$. This is just similar to what we did in the case of Mandelstam variables s, t and u with $s + t + u = 4m^2$. In this Dalitz plot, we choose an equilateral triangle of hight $M - 3m = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = Q$ (say) where $\varepsilon_i = (E - m)$ are the kinetic energies and Q is the total kinetic energy



Any point in the triangle can now be characterized by the three distances ε_1 , ε_2 and ε_3 of the point from the sides. Alternatively, one can introduce polar coordinates as well. At the centre of the triangle $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = Q/3$ Any other point can be specified by its distance ρ from the origin and the angle ϕ given by figure above. Now

$$
\begin{aligned}
\varepsilon_1 &= \frac{Q}{3}\left(1 + \rho \cos \phi_1\right) \\
\varepsilon_2 &= \frac{Q}{3}\left(1 + \rho \cos \phi_2\right) \qquad (7.36) \\
\varepsilon_3 &= \frac{Q}{3}\left(1 + \rho \cos \phi\right)
\end{aligned}
$$

where ϕ_1 , ϕ_2 and ϕ_3 are the angles which the vector makes with the three medians of the triangle and

$$\phi_1 = \phi - \frac{2\pi}{3} \quad ; \quad \phi_2 = \phi + \frac{2\pi}{3}$$

Then where do the physical events lie ? There should be some boundary within which all physical events lie. Let us now find some restrictions on the distribution of events inside this triangle.

No physical event can lie in a corner of our triangle, because this would mean that $\varepsilon_1 = Q$, $\varepsilon_2 = \varepsilon_3 = 0$. That is, particles 2 and 3 are at rest while particle 1 carries the whole energy. However, this can not happen, since the momentum conservation tells that $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$.

We shall now derive the boundary curve outside of which no event can lie. From momentum conservation we have,

$$(\vec{p}_1 + \vec{p}_2)^2 = (\vec{p}_3)^2$$

$$\vec{p}_1^{\,2} + \vec{p}_2^{\,2} + 2\vec{p}_1 \cdot \vec{p}_2 = \vec{p}_3^{\,2}$$

$$p_1^2 + p_2^2 - p_3^2 = -2 p_1 p_2 \cos\theta$$

$$\therefore \quad 4 p_1^2 p_2^2 \cos^2\theta = (p_1^2 + p_2^2 - p_3^2)^2$$

Since $\cos^2\theta \le 1$ for physical events, we have the inequality

$$(p_1^2 + p_2^2 - p_3^2)^2 \leqslant 4 p_1^2 p_2^2 \qquad (7.37)$$

Also we know that

$$p_i^2 = E_i^2 - m^2 = (\mathcal{E}_i + m)^2 - m^2 = \mathcal{E}_i^2 + 2m\mathcal{E}_i$$

$\therefore$ Equation (7.37) becomes

$$4(\mathcal{E}_1^2 + 2m\mathcal{E}_1)(\mathcal{E}_2^2 + 2m\mathcal{E}_2)$$
$$\geq \left[(\mathcal{E}_1^2 + 2m\mathcal{E}_1) + (\mathcal{E}_2^2 + 2m\mathcal{E}_2) + (\mathcal{E}_3^2 + 2m\mathcal{E}_3)\right]^2$$

That is

$$4\left[(\mathcal{E}_1\mathcal{E}_2)^2 + 4m^2\mathcal{E}_1\mathcal{E}_2 + 2m\mathcal{E}_1\mathcal{E}_2(\mathcal{E}_1 + \mathcal{E}_2)\right]$$
$$\geq \left[\mathcal{E}_1^2 + \mathcal{E}_2^2 + \mathcal{E}_3^2 + 2m(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)\right]^2 \qquad (7.38)$$

Now

$$\mathcal{E}_1^2 + \mathcal{E}_2^2 + \mathcal{E}_3^2 = (\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)^2 - 2(\mathcal{E}_1\mathcal{E}_2 + \mathcal{E}_2\mathcal{E}_3 + \mathcal{E}_3\mathcal{E}_1)$$
$$= Q^2 - 2(\mathcal{E}_1\mathcal{E}_2 + \mathcal{E}_2\mathcal{E}_3 + \mathcal{E}_3\mathcal{E}_1)$$

$$\mathcal{E}_1\mathcal{E}_2 = \frac{Q^2}{9}\left[1 + \rho(\cos\phi_1 + \cos\phi_2) + \rho^2\cos\phi_1\cos\phi_2\right]$$

$$\mathcal{E}_2\mathcal{E}_3 = \frac{Q^2}{9}\left[1 + \rho(\cos\phi_2 + \cos\phi) + \rho^2\cos\phi_2\cos\phi\right]$$

$$\mathcal{E}_3\mathcal{E}_1 = \frac{Q^2}{9}\left[1 + \rho(\cos\phi + \cos\phi_1) + \rho^2\cos\phi\cos\phi_1\right]$$

$$\therefore \mathcal{E}_1\mathcal{E}_2 + \mathcal{E}_2\mathcal{E}_3 + \mathcal{E}_3\mathcal{E}_1 = \frac{Q^2}{9}\Big[3 + 2\rho(\cos\phi_1 + \cos\phi_2 + \cos\phi)$$
$$+ \rho^2(\cos\phi_1\cos\phi_2 + \cos\phi_2\cos\phi + \cos\phi\cos\phi_1)\Big]$$
$$= \frac{Q^2}{9}\left[3 + 2\rho(0) + \rho^2\left(\tfrac{3}{2}\right)\right]$$

$$\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1 = \frac{Q^2}{9}\left[3+\frac{3\rho^2}{2}\right] = \frac{Q^2}{3}\left[1+\frac{\rho^2}{2}\right] \qquad (7.39)$$

since

$$\cos\phi_1+\cos\phi_2+\cos\phi=0$$

$$\cos\phi_1\cos\phi_2+\cos\phi_2\cos\phi+\cos\phi\cos\phi_1 = 3/2$$

Finally we get for the right hand side of equation (7.38)

$$\left[\frac{Q^2}{3}\left(1+\frac{\rho^2}{2}\right)+2mQ\right]^2$$

The left hand side of Equation (7.38) becomes

$$4\left(\varepsilon_1^2+2m\varepsilon_1\right)\left(\varepsilon_2^2+2m\varepsilon_2\right) = \left[\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2+2m\left(\varepsilon_1+\varepsilon_2-\varepsilon_3\right)\right] \qquad (7.40)$$

Substituting for $\left(\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2\right)$ and $\left(\varepsilon_1+\varepsilon_2-\varepsilon_3\right)$ from equation (7.36) finally we get for equation (7.38) (for the boundary curve given by the equality)

$$\rho^2\left[(4-2\sigma+\sigma^2)^2+2\sigma\rho\cos 3\phi\right]=(2-\sigma)^2 \qquad (7.41)$$

$$\sigma = Q/M$$

which can be brought to the form

$$\rho^2 = \frac{1}{1+\frac{2\sigma}{(2-\sigma)^2}\left(1+\rho\cos 3\phi\right)} \qquad (7.42)$$

Points corresponding to physical events must be always inside the boundary curve defined by (7.42). Let us discuss this curve in the two limiting cases.

$$\sigma = Q/M = 0 \quad \text{(non-relativistic case)}$$

$$\sigma = Q/M = 1 \quad \text{(ultra relativistic case).}$$

Nonrelativistic case:- $\sigma = 0$: $\rho^2 = 1$

This defines an inscribed circle since for $\phi = \pi$ (for the point lying along the CO meridian, one gets $\varepsilon_3 = 0$; hence the circle touches the sides of the triangle.

Ultra relativistic case: $\sigma = 1$: $2\rho^3 \cos 3\phi + 3\rho^2 - 1 = 0$

Since $\cos 3\phi = 4\cos^3\phi - 3\cos\phi$, this gives

$$8(\rho\cos\phi)^3 + 3\rho^2(1 - 2\rho\cos\phi) - 1 = 0$$

The solution of this equation is

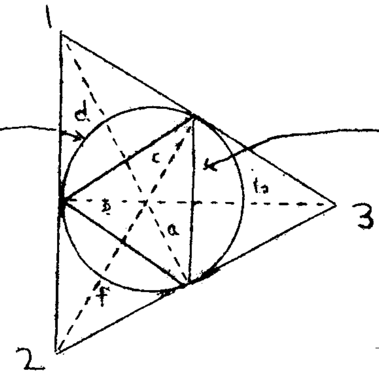
$$\rho\cos\phi = 1/2$$

This is a straight line inscribed. Since there is symmetry about a rotation of 270°; the boundary curve in the ultrarelativistic case is an inscribed triangle.

The curves for the other cases $0 < \sigma < 1$ lie in between the inscribed circle and the inscribed triangle.

Let us now see the kinematic situation in different regions of the Dalitz triangle.

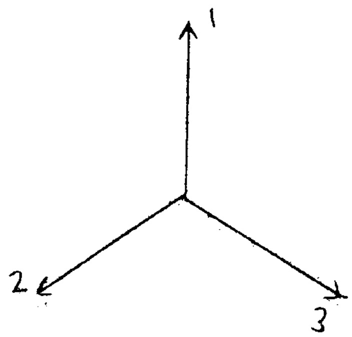
non-relati-
vistic
$\varepsilon_i \ll m$



ultrarelativistic
$\varepsilon_i \gg m$

It is obvious from the figure that there is a sextant symmetry so that it is sufficient to consider events in one sixth of the Dalitz triangle (say in the region between symmetry lines a and b).

The centre of the triangle corresponds to three equal kinetic energies and hence through equal momenta. Thus one may picture this situation by saying that the three particles coulc have flown off from the centre of the triangle toward the three corners.



(At the centre)

<u>The line centres</u>

a is a symmetry line along which

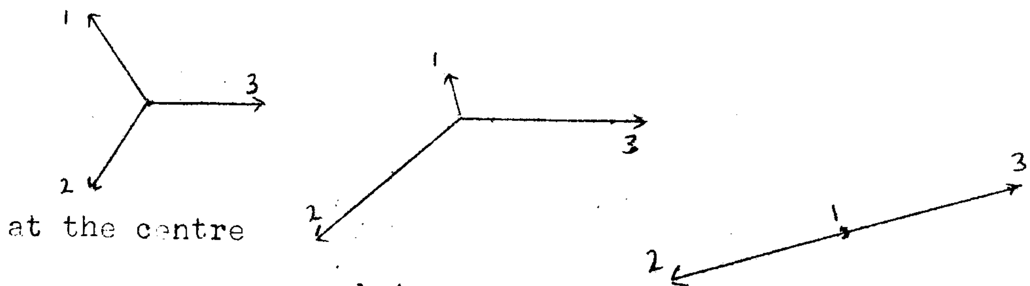
$$\varepsilon_2 = \varepsilon_3 = \frac{1}{2} (Q - \varepsilon_1)$$

For $\varepsilon_1 = Q/3$ the centre and for $\varepsilon_1 = 0$ the boundary curve is reached. Hence this symmetry line corresponds to

$$0 \leqslant \varepsilon_1 \leqslant Q/3$$

b is a symmetry line along which $\varepsilon_1 = \varepsilon_2$; whereas ε_3 varies between $Q / 3$ at the centre and max value which is given by the intersection of b line with the boundary.

At the boundary curve, the three momenta are collinear

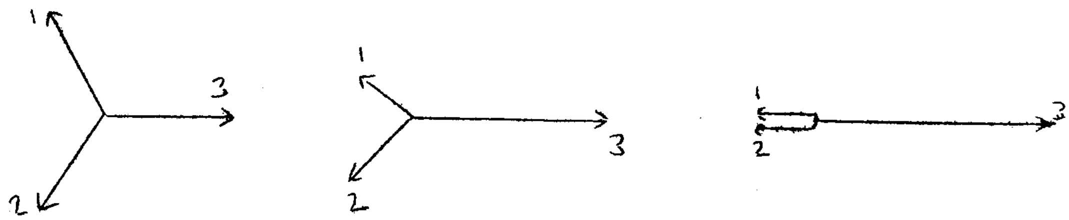


at the centre

between centre and boundary

On the intersection of line a with the boundary.

(a)

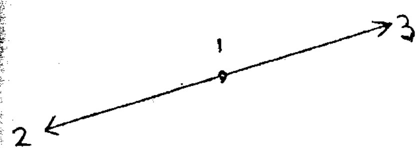


at the centre

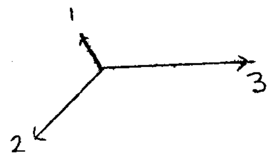
between centre and boundary

at the intersection of b with the boundary.

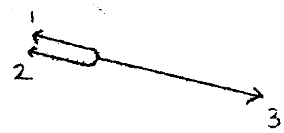
(b)



at the intersec-
tion with a



between the intersec-
tions with a and b



at the intersec-
tion with b.

On boundary line between a and b .

If the situation would be governed by pure kinematics, the distribution of events inside the boundary curve of the Dalitz triangle should be a constant corresponding to where an event lies, we may conclude what situations are favoured or forbidden by the matrix element when the experimental distribution is not a constant. For the case of ω -meson, this technique has worked very well.

Sec. 7.7

Application of Phase Space Methods to Unstable Particles

In a production process, along with stable particles, there may also be unstable particles which decay so fast that we observe only their decay products. The problem is this, suppose we know the energy spectrum of the decaying particle in over-all centre-of-momentum frame. What will be the energy spectrum of the decay products in the same frame of reference? For this we will have to know the angular distribution of decaying particle and those of the decay products in the rest frame of the decaying particle. This is a really complicated to find. Instead, we consider the decaying particle which isotropically distributed in the C.M. frame and is unpolarized. Then, to a very good degree of approximation, the decaying particle is going to decay isotropically. Suppose we consider two and three-body decays of unstable particles. In the case of two-body decays, the matrix element determines the life-time and if this is small, then the energy spectrum of the decay products is essentially going to be determined by pure kinematics. However in the case of three-body decay, the energy distribution of the decaying particles does depend on the matrix element. Therefore, in this case, we have to make assumptions about the matrix element and check the results against an experimental result (Dalitz plot for example)

The kinematical problem with which we have to work is a) first calculate the momentum spectra of the decay products in the rest frame of the decaying particle. b) For a given C.M. momentum of the initial particle, Lorentz transform these spectra tothe C.M. frame and c) integrate over the initial particle spectrum with the condition that the energy of the decay products, as seen from the C.M. frame lies between η and $\eta + d\eta$

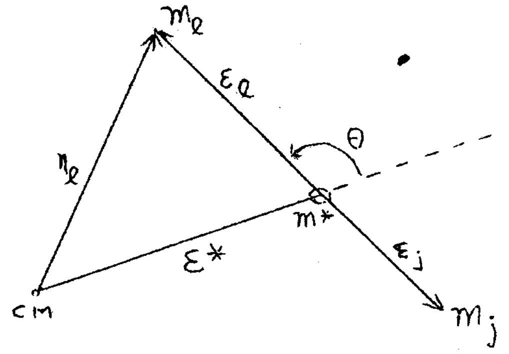
Two-Body decay

$$m^* \rightarrow m_\ell + m_j$$

Let $(\varepsilon^*, \vec{p}^*)$ be the four momentum of m^* in the C.M. frame

Let $(\varepsilon_i, \vec{p}_i)$ be the four momentum of either of the decay product in the rest frame of m^*.

Let η_ℓ be the total energy of either of the decay product in the C.M. frame



We first calculate the energy η_ℓ assuming that m^* has a four momentum $P^* = (\varepsilon^*, \vec{p}^*)$ and m_ℓ has (in the rest frame of m^*) the four momentum. $P_\ell = (\varepsilon_\ell, \vec{p})$ Then the C.M. can be thought of as a fictitious particle, which in the rest of m^* has the four momentum $P_{CM} = (\varepsilon^*, -\vec{p}^*)$

Therefore, in the m^* rest frame we have:

$$\begin{array}{lll} \ell \text{ particle} & P_\ell = (\varepsilon_\ell, \vec{p}) & \\ \text{C.M. particle} & P_{CM} = (\varepsilon^*, -\vec{p}^*) & (7\,43) \end{array}$$

We know that if P_ℓ and P_{CM} are the four-momenta of any two particles in any Lorentz frame, the energy of the particle l, seen from "particle C.M." is

$$\eta_\ell = \frac{P_\ell P_{CM}}{m^*} = \frac{1}{m^*}\left(\varepsilon_\ell \varepsilon^* + \vec{p}\cdot\vec{p}^*\right)$$

$$= \frac{1}{m^*}\left(\varepsilon_\ell \varepsilon^* + |\vec{p}|\,|\vec{p}^*| \cos\theta\right)$$

$$= \frac{1}{m^*}\left(\varepsilon_\ell \varepsilon^* + |\vec{p}|\left(\sqrt{\varepsilon^{*2} - m^{*2}}\right)\cos\theta\right)$$

where θ is the angle between $\vec{p}$ and $\vec{p}^*$. The momentum $|\vec{p}|$ and the energy ε_i of the decay products seen from the m^* rest frame are fixed by three masses m^*, m_ℓ, m_j. We know then

$$p^2(m^*; m_\ell; m_j) = \frac{[m^{*2} - (m_\ell + m_j)^2]^2 [m^{*2} - (m_\ell - m_j)^2]^2}{4m^{*2}}$$

$$\mathcal{E}_\ell (m^*, m_\ell ; m_j) = \frac{m^{*2} + (m_\ell^2 - m_j^2)}{2m^*}$$

$$\mathcal{E}_j (m^*, m_\ell, m_j) = \frac{m^{*2} + (m_j^2 - m_\ell^2)}{2m^*} \qquad (7.45)$$

The situation now is as follows.

$W^*(\mathcal{E}^*)d\mathcal{E}^*$ is the probability that the particle m^* has an energy betwwen $\mathcal{E}^*$ and $\mathcal{E}^* + d\mathcal{E}^*$

The probability that the particle m_ℓ is emitted at an angle Θ to the m^* direction is

$$W_\ell(\Theta)\, d\cos\Theta = \tfrac{1}{2}\, d(\cos\Theta)$$

$W_\ell^{(2)}(\eta, m_\ell, m^*)\, d\eta$ is the probability that the m_ℓ particle has an energy (C.M.energy) between η and $\eta + d\eta$.

$W_\ell^{(2)}(\eta)$ is now calculated by finding the probability for the m^* particle to have n energy $\mathcal{E}^*$ and multiplying it by the probability for the m_ℓ particle to go in a direction making an angle Θ with the m^* direction and integrating over $\mathcal{E}^*$ and Θ under the condition that the m_ℓ particle has an energy η in the C.M. frame.

$$W_\ell^{(2)}(\eta) = \int d\mathcal{E}^*\, d(\cos\Theta)\, W^*(\mathcal{E}^*)\, \delta(\eta - \eta_\ell)$$

$$\eta_\ell = \frac{\mathcal{E}^* \mathcal{E}_\ell + p^* p_\ell \cos\Theta}{m^*}$$

$$p^* = \sqrt{\mathcal{E}^{*2} - m^2} \qquad (7.46)$$

The $\delta(\eta - \eta_\ell)$ assures that the m_ℓ particle has an energy η in the C.M. frame

$$\therefore \int W_\ell^{(2)}(\eta)\, d\eta = \int W^*(\varepsilon^*)\, d\varepsilon^* \qquad (7.47)$$

We carry out the integral (7.48) by using

$$\delta(a - bx) = \frac{1}{b}\,\delta(x - a/b)$$

We then get

$$W_\ell^{(2)}\eta = \frac{m^*}{2p}\int d\varepsilon^* \frac{W^*(\varepsilon^*)}{\sqrt{\varepsilon^{*2} - m^{*2}}}\int_{-1}^{+1} dx\, \delta\left(x - \frac{m^*\eta - \varepsilon_\ell \varepsilon^*}{p\sqrt{\varepsilon^{*2} - m^{*2}}}\right) \qquad (7.49)$$

$$x = \cos\theta$$

The second integral gives one when

$$-1 \leq \frac{m^*\eta - \varepsilon_\ell\, \varepsilon^*}{p\sqrt{\varepsilon^{*2} - m^{*2}}} \leq 1 \qquad (7.50)$$

Solving this for ε^* we get two values ε_0^* and ε_1^* between which ε^* has to lie to satisfy Eq.(7.47).

$$W_\ell^{(2)}(\eta\,; m_\ell\, m_j\, m^*) = \frac{m^*}{2p(m^*, m_\ell, m_j)}\int_{\varepsilon_{\ell_0}}^{\varepsilon_{\ell_1}^*} d\varepsilon^* \frac{W^*(\varepsilon^*)}{\sqrt{\varepsilon^{*2} - m^{*2}}}$$

where

$$\varepsilon^{*}_{\ell_{0}^{\pm}} (m^{*}, m_\ell, m_j)$$

$$= \frac{m^{*}}{m_\ell^2}\left[\varepsilon_\ell (m^{*} m_\ell m_j)\eta \pm p(m^{*} m_\ell m_j)\sqrt{\eta_\ell^2 - m_\ell^2}\right]$$

Three-body-decay

$$m^{*} \longrightarrow m_\ell + m_j + m_k$$

can be reduced to a two-body decay

$$m^{*} \longrightarrow m_\ell + M_{jk}$$

where

$$m_j + m_k \leqslant M_{jk} \leqslant m^{*} - m_\ell \qquad (7.52)$$

We have to calculate the proability distribution $P(M_{jk})\, dM^2_{jk}$ for the three body decay

We must first use the two-body formula with a given M_{jk} and integrate it over M^2_{jk} after multiplication with $P(M_{jk})$. We therefore have,

$$W^{(3)}_\ell (\eta; m_\ell m_j m_k m^{*})$$

$$= \int_{(m_j+m_k)^2}^{(m^{*}-m_\ell)^2} P(M_{jk})\, W^{(2)}_\ell (\eta; m_\ell, M_{jk} m^{*})\, dM^2_{jk} \qquad (7.53)$$

The superscripts (2) and (3) denote the two-body and three-body decay spectra. The only task which remains is to make use of the eq (7.21) to calculate $P(M_{jk})$ explicitly. From (7.21) we have

$$P(M_{jk}) = \frac{R_2(m^* M_{jk}, m_\ell)\, R_2(M_{jk}\, m_j\, m_k)}{R_3(m^*, m_\ell\, m_j\, m_k)} \qquad (7.54)$$

where R_2 and R_3 are invariant phase spaces for 2 and 3 particles respectively. It can be shown that

$$R_2(m^*, M_{jk}\, m_\ell) = \frac{\pi p}{m^*}$$

$$p = p(m^*, M_{jk}\, m_\ell) \qquad (7.55)$$

Therefore,

$$P(M_{jk}) = \frac{\pi^2}{R_3}\left(\frac{1}{m^*}\right)\left(\frac{1}{M_{jk}}\right) p(m^* M_{jk}\, m_\ell)\; p(M_{jk}\, m_j\, m_k) \qquad (7.56)$$

Finally we obtain

$$W_\ell^{(3)}(\eta ; m_\ell\, m_j\, m_k\, m^*)$$

$$= \frac{\pi}{R_3}\int dM_{jk}^2\, \frac{1}{m^*}\, \frac{1}{M_{jk}}\; p(m^*, M_{jk} m_\ell)\, p(M_{jk}, m_j\, m_k)$$

$$\times \frac{m^*}{2p(m^*, M_{jk}\, m_\ell)} \int_{\varepsilon_0^*}^{\varepsilon_1^*} d\varepsilon^*\, \frac{W^*(\varepsilon^*)}{\sqrt{\varepsilon^{*2} - m^{*2}}}\,.$$

Hence

$$W_\ell^{(3)}\,(\eta; m_\ell m_j m_k m^*)$$

$$= \frac{\pi}{R_3(m^*, m_\ell m_j m_k)} \int_{m_j+m_k}^{m^*-m_\ell} dM \;\; p(M, m_j m_k)$$

$$\int_{\varepsilon_{\ell_0}^*}^{\varepsilon_{\ell_1}^*} \frac{d\varepsilon^* W^*(\varepsilon^*)}{\sqrt{\varepsilon^{*2} - m^{*2}}}$$

By interchanging ℓ with j and k one can obtain the other two spectra.

CHAPTER 8

Relativistic Notation

The contravariant four vectors x and p are defined as

$$x = (t, \vec{x}) \qquad p = (E, \vec{p}) \tag{8.1}$$

The metric tensor that we use is

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \tag{8.2}$$

Then the covariant components are obtained as

$$x_\mu = g_{\mu\nu} x^\nu = (t, -\vec{x}) \tag{8.3}$$

Using the metric in any tensor, any index can be raised or lowered. In particular

$$g_\mu^{\ \rho} = g_{\mu\nu} g^{\nu\rho} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$= \delta_{\mu\rho} \tag{8.4}$$

The invariant scalar product of two four vectors (say p and x) is

$$p x = p_\mu x^\mu = p^\nu g_{\nu\mu} x^\mu = p^\nu x_\nu$$

$$= E t - \vec{p} \cdot \vec{x} \qquad (8.5)$$

Also

$$p^2 = p_\mu p^\mu = E^2 - |\vec{p}|^2 = m^2 \qquad (8.6)$$

If $F(x_t, \vec{x})$ is an invariant function, so also is

$$dF = \frac{\partial F}{\partial x'_\mu} d x_\mu$$

$$= \frac{\partial F}{\partial x^\nu} dx^\nu = \text{invariant} \qquad (8.7)$$

Therefore,

$$\frac{\partial}{\partial x_\mu} = \partial^\mu \quad \text{are contravariant components}$$

$$\frac{\partial}{\partial x^\mu} = \partial_\mu \quad \text{are covariant components} \qquad (8.8)$$

In our metric

$$\frac{\partial}{\partial x_\mu} = \partial^\mu = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial \vec{x}} \right)$$

$$\frac{\partial}{\partial x^\mu} = \partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \vec{x}} \right) \qquad (8.9)$$

Defining the Klein-Gordon operator as

$$\begin{aligned}\Box &= \partial_\mu \partial^\mu \\ &= \frac{\partial^2}{\partial t^2} - \nabla^2\end{aligned} \tag{8.10}$$

in our metric.

The electromagnetic field tensor is defined as

$$F = F^{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & H_z & -H_y \\ -E_y & -H_z & 0 & H_x \\ -E_z & H_y & -H_x & 0 \end{bmatrix} \tag{8.11}$$

In our metric, the Maxwell's equation

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = j^\mu$$

reads as

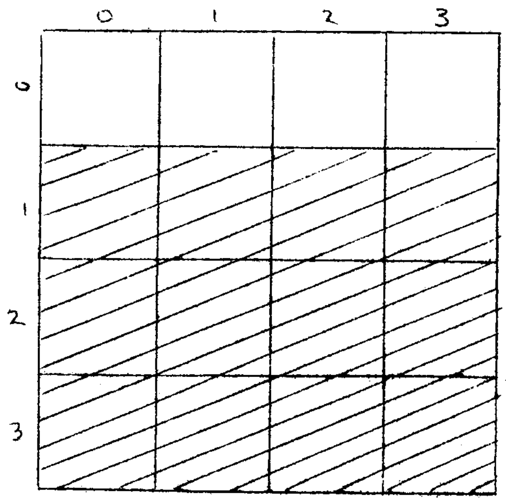
$$\left.\begin{aligned} \partial^\rho F^{\mu\nu} + \partial^\nu F^{\rho\mu} + \partial^\mu F^{\nu\rho} &= 0 \\ j^\mu &= (\rho, \rho\vec{\beta}) \end{aligned}\right\} \tag{8.12}$$

$$
\begin{aligned}
F_{\mu}^{\ \nu} &= g_{\mu\rho} F^{\rho\nu} \\
&= F^{\mu\nu} \qquad \text{for} \quad \mu = 0 \\
&= -F^{\mu\nu} \qquad \text{for} \quad \mu = 1, 2, 3
\end{aligned}
\tag{8.13}
$$

$$
\begin{aligned}
F_{\nu}^{\ \mu} &= g_{\nu\rho} F^{\rho\mu} \\
&= F^{\nu\mu} \qquad \text{for} \quad \mu = 0 \\
&= -F^{\nu\mu} \qquad \text{for} \quad \mu = 1, 2, 3
\end{aligned}
$$

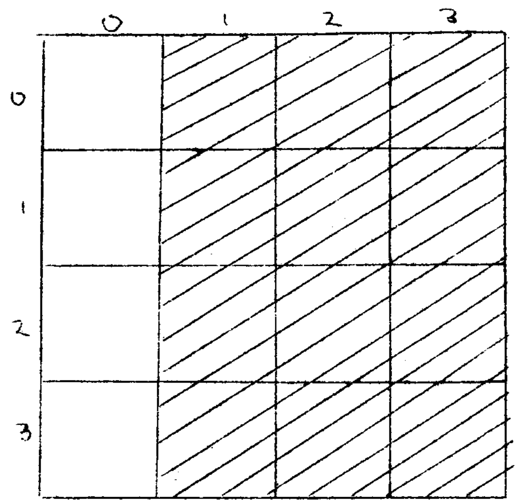
It looks that the shaded portion changes sign

$$
F_{\mu}^{\ \nu} = g_{\mu\rho} F^{\rho\nu} = \begin{cases} F^{\mu\nu} & \text{for } \mu = 0 \\ -F^{\mu\nu} & \text{for } \mu = 1, 2, 3 \end{cases}
$$



$$F^{\mu}_{\ \nu} = g_{\nu\rho} F^{\rho\mu}$$

$$= \begin{cases} F^{\nu\mu} & \text{for } \nu = 0 \\ -F^{\nu\mu} & \text{for } \nu = 1,2,3 \end{cases}$$



CHAPTER 9

Polarization of Particles in electromagnetic field

Let us consider a particle with spin σ. The component m of this spin along some given direction (say $\vec{e}$ is the unit vector in this direction) $\vec{e}$, will have $(2\sigma+1)$ possible eigenvalues ranging from $m=\sigma$ to $m=-\sigma$ Suppose we do the same for a beam of particles (having spin having some frequency distribution $W(\vec{e}, m)$. The average over this distribution is

$$\langle \vec{\sigma}\cdot\vec{e} \rangle = \sum_{m=-\sigma}^{\sigma} W(\vec{e}, m)\, m \qquad (9.1)$$

This is the expectation value of the spin component in the direction $\vec{e}$. From correspondence principle, we know, that the expectation value of any quantum mechanical operator corresponds to a classical observable and thus obeys some classical equation of motion. If the expectation value $\langle \vec{\sigma}\cdot\vec{e} \rangle$ is zero for all choices of direction $\vec{e}$, then the beam is unpolarized. If this is not the same, then there exists a certain direction $\vec{e}_0$ along which this expectation is a maximum (corresponding to the maximum allignment of spins). Thus,

$$\langle \vec{\sigma}\cdot\vec{e} \rangle_{max} = \langle \vec{\sigma}\cdot\vec{e}_0 \rangle = S \qquad (9.2)$$

$$0 \leqslant S \leqslant \sigma$$

We then call this 'S' as the degree of polarization. Now we define the polarization vector $\vec{S}$ as

$$\vec{S} = \vec{e}_0 \cdot S \qquad (9.3)$$

Since $\vec{s}$ is defined in terms of expectation values, it should obey a classical equation of motion. From classical physics we know that in the rest system of the particles considered this equation of motion is

$$\frac{d\vec{s}}{dt} = g\mu_0 \quad \vec{s} \times \vec{H} \qquad (9.4)$$

where $g\mu_0\sigma$ is the magnetic moment of the particle considered, g is the geomagnetic factor, $\mu_0 = e/2m$ for charged particles. In Dirac theory $g = 2$, but in quantum electrodynamics corrections are obtained such that $g \neq 2$ for electrons and muons

The problem now is whether we can generalize this equation of motion (9.4) into a covariant equation of motion by defining a polarization four vector. Viz.,

$$S = (s_0 \, \vec{s})$$

such that in the rest frame it is space-like

That is,

$$S_R = (0, \vec{s}) \qquad (9.5)$$

Since degree of polarization is an invariant quantity (also easily seen from eq.(9.4) by scalarly multiplying both sides with $\vec{s}$) we have

$$S^2 = -|\vec{s}|^2 = s^2 \qquad (9.6)$$

If polarization four vector can at all be defined, it is really not obvious how its time component should be zero in the rest frame. Therefore, our task is to see whether such a definition for the polarization four vector $S = (s_0, \vec{s})$ with $S_R = (0, \vec{s})$ is consistent with equations of motion.

The natural generalization of Eq.(9.4) for polarization four-vector will be

$$\frac{dS}{d\tau} = Z$$

where τ is the proper time of the particle, S is the polarization four vector to be found. Z will then depend upon (1) the polarization fourvector S generalizing Eq.(9.4) (2) the electromagnetic tensor $F_{\mu\nu}$ which will be antisymmetric, generalizing $\vec{H}$ occuring in eq.(9.4) and (3) the four velocity vector V. $F_{\mu\nu}$ has the well-known form

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & H_3 & -H_2 \\ -E_2 & -H_3 & 0 & H_1 \\ -E_3 & H_2 & -H_1 & 0 \end{bmatrix} \quad ; \quad \begin{array}{l} F^{ij} = H_k \\ F^{0k} = E_k \end{array} \tag{9.7}$$

The four velocity has the form

$$V = (\gamma, \vec{\beta}\gamma) \tag{9.8a}$$

which in the rest frame reads as

$$V_R = (1, 0) \tag{9.8b}$$

We also know that

$$S_R = (0, \vec{s}) \tag{9.5}$$

so that

$$(S V)_R = 0 \tag{9.9}$$

Since $S V$ is an invariant, in any frame

$$S V = 0 \tag{9.9a}$$

Differentiating this we get

$$S \frac{dV}{dt} = - \frac{dS}{dt} V \tag{9.10}$$

Eq.(9.10) in the rest frame becomes

$$\frac{d s_0}{dt} = - S \frac{dV}{dt} \tag{9.11}$$

since $V_R = (1, 0)$

and $\frac{dS}{dt} = \left(\frac{dS_0}{dt} \; \frac{d\vec{s}}{dt} \right)$

$$\therefore \left(\frac{dS}{dt}\right)_R = \left(\frac{dS_0}{dt}, \frac{d\vec{s}}{dt}\right)_R$$

$$= \left(-S\frac{dV}{dt}, g\mu_0 \vec{s} \times \vec{H}\right)$$

(9.12)

using eq.(9.11) and the equation of motion in the rest frame, namely

$$\frac{d\vec{s}}{dt} = g\mu_0 \vec{s} \times \vec{H} \qquad (9.4)$$

Now the covariant generalization consists in

(1) replacing $\frac{d}{dt}$ by $\frac{d}{d\tau}$ (denoted by $\cdot$)

(2) $SF = S_\mu F^{\mu\nu} = -F^{\nu\mu}S_\mu = -FS$

Also

$$(SF)_R = \left(\vec{S}_R \times \vec{H}\right)$$

Therefore, from eq.(9.12) we have

$$\dot{S}_R = \left(-S\dot{V}, g\mu_0 (SF)_R\right) \qquad (9.13)$$

We notice that $\dot{S}_R$ is (a) homogeneous and linear in S (b) linear in F and (3) linear in $\dot{V}$ since $\dot{V}$ is a function of F. The only non-constant <u>four vectors</u> that can be formed out S, F, V and $\dot{V}$ satisfying all the four conditions mentioned above are

$$S F \quad , \quad V(S\dot{V}) \quad , \quad V(SFV) \tag{9.14}$$

A product $SF\dot{V}$ is not permitted since this is not linear in F (noticing that $\dot{V}$ is a function of F)

A product SVF is either a scalar or a tensor and therefore is not allowed.

The most general form for $\dot{S}$ is therefore

$$\dot{S} = a\,SF + b\,V(S\dot{V}) + c\,V(SFV) \tag{9.15}$$

To determine the constants a,b and c we go to the rest frame of Eq.(9.15) and compare it with eq. (9.13). That is

$$\dot{S}_R = \left(a(SF)^{o}_{R} + b(S\dot{V})^{o}_{R} + c(SF)^{o}_{R} \quad , \quad a(SF)_R \right)$$

$$= \left(-S\dot{V} \quad , \quad g\mu_o (SR)_R \right)$$

Thus

$$b = -1 \quad , \quad a = -c = g\mu_o$$

(we have tacitly assumed that $g\mu_o$ is a constant and does not undergo any change under Lorentz transformation).

Finally eq.(9.15) becomes

$$\dot{S} = g\mu_o \left[SF - V(SFV) \right] - V(S\dot{V}) \tag{9.16}$$

This is a unique generalization of Eq.(9.13) in the rest frame. Suppose $\dot{S}$ contains an overall additional constant multiplying it.

Since a constant does not undergo any change under Lorentz transformation, it should be present in $\dot{S}_R$ also. However the form of $\dot{S}_R$ given by (9.13) does not contain any overall constant. Thus, there cannot be any overall multiplicative constant. There can be no multiplicative four vector in (9.16) since $\dot{S}$ will then no longer be a four vector. Let us see whether there can be any four vector added to eq.(9.16). This is not possible since this added four vector should be $(0, 0)$ in the rest frame as dictated by eq.(9.13) and remains as $(0, 0)$ throughout since the Lorentz transformation considered here is homogeneous. Thus, we have proved that $\dot{S}$ given by eq.(9.16) is the most general and unique generalization of eq.(9.13).

In such a generalization, we have not complicated the problem by taking a more general classical equation including electric moment and quadruple moments.

In a homogeneous field, the equation of motion of a charged particle can be verified to be

$$\dot{V} = -\frac{e}{m}(FV) \tag{9.17}$$

In this case, from (9.16) using (9.17) we get

$$\dot{S} = g\mu_0 SF + \left(\frac{e}{m} - g\mu_0\right) V(SFV) \tag{9.18}$$

Putting $g\mu_0 = g\frac{e}{2m}$ for charged particles we have

$$\begin{aligned} \dot{S} &= g\mu_0 SF + (2\mu_0 - g\mu_0) V(SFV) \\ &= \mu_0 \left[g SF - (g-2) V(SFV)\right] \end{aligned} \tag{9.19}$$

We have to check for the consistency of these formulas:

$$S^2 = (-s^2)$$
$$S V = 0 \tag{9.20}$$

with our four vector S.

From (9.16) we get

$$\dot{S} V = g \mu_0 \left[S F V - V^2 (S F V) \right] - V^2 (S \dot{V})$$

$$= -(S \dot{V}) \qquad \text{since } V^2 = 1$$

Therefore
$$\dot{S} V + S \dot{V} = 0$$

$$S V = \text{constant} \tag{9.21}$$

We see thus, since V is a four vector, S defined in eq.(9.5) is a four vector and in particular the constant coming in eq. (9.21) is zero since $S_R = (0, \vec{s})$ and $V_R = (1, 0)$

Thus

$$S V = 0 \tag{9.22}$$

Furthermore, multiplying eq.(9.19) with S, we get

$$S \dot{S} = \mu_0 (2-g) \; S V \; (S F V)$$
$$= 0$$

where we have used the fact

$$S F S = 0 \text{ since } F \text{ is antisymmetric}$$

and $S V = 0$ by eq.(9.22)

$$\dot{S} S = S \dot{S} = \frac{1}{2} \frac{d}{d\tau} (S S)$$

$$= \frac{1}{2} \frac{d}{d\tau} (S^2) = 0$$

Thus eq.(9.20) is consistent with the generalized equation of motion (9.18). Thus S defined in eq.(9.19) is a four vector.

Sec. 9.2

In the last section, we proved that the generalized polarization vector defined earlier is in fact a four vector and we could also generalize the equation of motion in a covariant way. Let us now see how this polarization four vector looks like in the lab. frame, for instance, where the electromagnetic field is simply described, though, of course, only the polarization four vector in the rest frame has a direct significance to the word polarization. Let us start with the polarization four vector in the rest frame S_R

$$S_R = (0, \vec{s})$$

Now make a Lorentz transformation on this to get the polarization four vector in the lab. frame. We get,

$$S_L = (S_{0_L}, \vec{S}_L)$$

$$= \left(\gamma \vec{\beta}\, \vec{S}_R\,,\ \vec{S}_R + \vec{\beta}\, \frac{\gamma^2}{\gamma+1}\, \vec{\beta}\cdot\vec{S}_R\right)$$

where $\vec{\beta}$ is the velocity of the polarized beam in the lab.frame. One more thing which we know is that $S_R^2 = S_L^2 = -\lambda^2 =$ invariant, so that

$$S_L^2 = S_{0_L}^2 - |\vec{s}_L|^2 = -s^2 = \text{invariant}$$

or

$$|s_L|^2 = S_{0_L}^2 - S_L^2 = S_{0_L}^2 + s^2$$

Hence
$$|\vec{S}_L|^2 = s^2 + (\gamma\vec{\beta}\,\vec{S}_R)^2$$
$$= s^2 + \gamma^2|\vec{\beta}|^2|\vec{S}_R|^2\cos^2\theta_R$$
$$= s^2 + \gamma^2\beta^2 s^2 \cos^2\theta_R$$
$$= s^2(1+\gamma^2\beta^2\cos^2\theta_R)$$

where θ_R is the angle between the directions $\vec{\beta}$ and $\vec{S}_R$ and we have used the fact that

$$S_R^2 = -s^2 = 0 - (\vec{S}_R)^2$$
$$\therefore \quad |\vec{S}_R|^2 = s^2$$

Thus

$$|\vec{S}_L|^2 = s^2(1+\gamma^2\beta^2\cos^2\theta_R)$$

Suppose $\cos\theta_R \neq 0$. Then if $\gamma^2 \gg 1$, we see that $|\vec{S}_L|^2$ is directly proportional to γ^2 (Remember that $\gamma^2 = \frac{1}{1-\beta^2}$, $\vec{\beta} = \vec{v}$ in our units). For massless particles, however, $v^2 = \beta^2 = 1$ and hence $\gamma^2 \to \infty$. Therefore in such case, we will have to adopt a different method. From the general expression for $\vec{S}_L$, namely

$$\vec{S}_L = \vec{S}_R + \vec{\beta}\,\frac{\gamma^2}{\gamma+1}\,(\vec{\beta}\cdot\vec{S}_R)$$

it is obvious that $\vec{S}_L$ depends on the direction of $\vec{\beta}$ (from the second term). Suppose now $\gamma >> 1$ so that

$$\vec{S}_L \approx \vec{\beta} \left(\frac{\gamma^2}{\gamma+1} \vec{\beta} \cdot \vec{S}_R \right)$$

Therefore, in this case, $\vec{S}_L$ is parallel to $\vec{\beta}$ if $\vec{\beta} \cdot \vec{S}_R > 0$ and is antiparallel when $\vec{\beta} \cdot \vec{S}_R < 0$. The angle Θ_L between the directions of $\vec{S}_L$ and $\vec{\beta}$ (when γ is not very large) is given by

$$\cos^2 \Theta_L = \cos^2 (\vec{S}_L , \vec{\beta})$$

$$= \frac{(\vec{S}_L \cdot \vec{\beta})^2}{|\vec{S}_L|^2 |\vec{\beta}|^2}$$

From the expressions for $\vec{S}_L$, $\vec{\beta}$ and $|\vec{S}_L|^2$ which we have already written down, we get,

$$\vec{S}_L = \vec{S}_R + \vec{\beta} \frac{\gamma^2}{\gamma+1} \vec{\beta} \cdot \vec{S}_R$$

$$\vec{\beta} \cdot \vec{S}_L = \frac{\beta^2 \gamma^2}{\gamma+1} (\vec{\beta} \cdot \vec{S}_R) + (\vec{\beta} \cdot \vec{S}_R)$$

$$= (\gamma - 1)(\vec{\beta} \cdot \vec{S}_R) + (\vec{\beta} \cdot \vec{S}_R)$$

$$= \gamma \vec{\beta} \cdot \vec{S}_R$$

$$\beta^2 |\vec{S}_L|^2 \cos^2\Theta_L = \gamma^2\beta^2 |\vec{S}_R|^2 \cos^2\Theta_R$$

$$|\vec{S}_L|^2 \cos^2\Theta_L = \gamma^2 s^2 \cos^2\Theta_R$$

But $$|\vec{S}_L|^2 = s^2(1+\gamma^2\beta^2\cos^2\Theta_R)$$

$$\therefore \quad \cos^2\Theta_L = \frac{\gamma^2\cos^2\Theta_R}{1+\gamma^2\beta^2\cos^2\Theta_R}$$

Let us now introduce the concept of helicity here.

Define helicity h as the component of polarization along the direction of flight. Then

$$h = \frac{(\vec{S}_L\cdot\vec{\beta})}{\beta} = \frac{S_0}{\beta} = \gamma\frac{\vec{S}_R\cdot\vec{\beta}}{\beta}$$

$$= \gamma s \cos\Theta_R$$

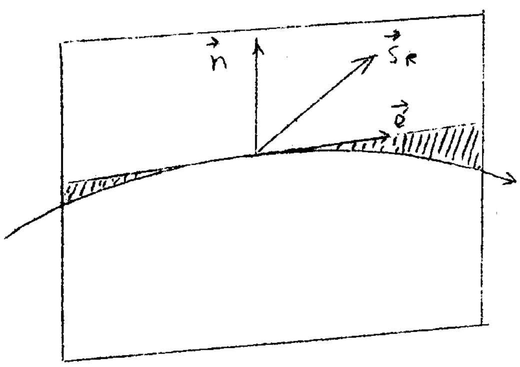
This shows that polarization of a beam has a physical meaning only in the rest system. That is, the polarization vector depends on $\vec{\beta}$ both in direction and magnitude.

Sec. 9.3 The rate of change of polarization

The polarization is determined completely when we know S_R . $S = (s_0, \vec{s})$ is a four vector. So S^2 = Lorentz invariant scalar. Also $S^2 = -s^2$ is constant as $\frac{d}{d\tau} S^2 = 0$ which follows from the equation of motion. The degree of polarization is unaltered when the beam of particles go through electromagnetic fields where the inhomogenity can be neglected. We are concerned with the direction of the polarization at any time.

To fix the polarization vector whose magnitude is constant we want two angles. We can prove that only one angle is relevant, namely the angle which it makes with the direction of motion.

We take unit vector $\vec{e}$ in the direction of motion and another unit vector perpendicular to $\vec{e}$ such that $\vec{S}_L$ is in the plane of $\vec{e}$ and $\vec{n}$.



We have

$$\vec{e} = \vec{\beta}/\beta$$

$$\vec{n}\cdot\vec{e} = 0$$

$$\vec{n}^2 = \vec{e}^2 = 1$$

$$\vec{S}_R = s\cos\theta\,\vec{e} + s\sin\theta\,\vec{n}$$

Θ_R is the angle between $\vec{S}_R$ and $\vec{e}$

$$S_L = (s_{0L}, \vec{s}_L) = \left(\gamma\vec{\beta}\cdot\vec{S}_R \,,\; \vec{S}_R + \vec{\beta}\,\frac{\gamma^2}{\gamma+1}\,\vec{\beta}\cdot\vec{S}_R\right)$$

$$= s\Big(\beta\gamma\cos\Theta_R \,,\; \vec{e}\cos\Theta_R + \vec{n}\sin\Theta_R + \vec{e}\,\frac{\beta^2\gamma^2}{\gamma+1}\cos\Theta_R$$

$$\beta^2\gamma^2 = \gamma^2 - 1$$

so $$\frac{\beta^2\gamma^2}{\gamma+1} + 1 = \frac{\gamma^2-1}{\gamma+1} + 1$$

$$= \gamma$$

so

$$S_L = s\left(\beta\gamma\cos\Theta_R \,,\; \gamma\,\vec{e}\cos\Theta_R + \vec{n}\sin\Theta_R\right)$$

$$= s\,L\cos\Theta_R + s\,N\sin\Theta_R$$

where

$$L = (\beta\gamma \,,\, \gamma\vec{e}) \qquad N = (0, \vec{n})$$

$$L^2 = N^2 = -1$$

$$LN = 0 - \gamma\vec{e}\cdot\vec{n} = 0$$

$$LV = (\beta\gamma, \gamma\vec{e})\cdot(\gamma, \gamma\vec{\beta}) = 0$$

$$NV = (0, \vec{n})\cdot(\gamma, \gamma\vec{\beta}) = 0$$

Differentiating $L^2 = N^2 = -1$, we get

$$L\dot{L} = N\dot{N} = 0$$

Differentiating $LN = 0$, we get

$$L\dot{N} = -N\dot{L}$$

We can introduce the expression for S_L in the equations of motion

$$\dot{S} = g\mu_0 SF + \left(\frac{e}{m} - g\mu_0\right) V(SFV)$$

$$= \Delta\left[\dot{L}\cos\Theta_R + \dot{N}\sin\Theta_R + \dot{\Theta}_R(N\cos\Theta_R - L\sin\Theta_R)\right]$$

$$= g\mu_0 SF + \left(\frac{e}{m} - g\mu_0\right) V(SFV)$$

$$= \Delta\Big[g\mu_0(LF\cos\Theta_R + NF\sin\Theta_R)$$
$$-\left(g\mu_0 - \frac{e}{m}\right) V(LFV\cos\Theta_R + NFV\sin\Theta_R)\Big]$$

So the equation to be co sidered is

$$\dot{L}\cos\Theta_R + \dot{N}\sin\Theta_R + \dot{\Theta}_R(N\cos\Theta_R - L\sin\Theta_R)$$
$$= g\mu_0(LF\cos\Theta_R + NF\sin\Theta_R)$$
$$-(g\mu_0 - e/m)V(LFV\cos\Theta_R + NFV\sin\Theta_R)$$

Multiply by N from the right to get

$$\dot{L}N\cos\Theta_R - \dot{\Theta}_R\cos\Theta_R = g\mu_0 LFN\cos\Theta_R$$

$$\dot{\Theta}_R = \dot{L}N - g\mu_0 LFN$$

We would have got the same equation on multiplying the original equation by L .

$$L = (\beta\gamma , \vec{e}\gamma) \qquad \dot{L} = (\beta\dot{\gamma} , \dot{\vec{e}}\gamma + \dot{\gamma}\vec{e})$$

$$\dot{L}N = 0 - \dot{\vec{e}} \cdot \vec{n}\gamma + \dot{\gamma}\,\vec{e}\cdot\vec{n} = -\dot{\vec{e}}\cdot\vec{n}$$

To get $\dot{\vec{e}}$ we use the expression for $\dot{V}$ where

$$V = (\gamma , \gamma\beta\vec{e})$$

$$\dot{V} = \left(\dot{\gamma} , \gamma\beta\dot{\vec{e}} + (\gamma\beta)^{\cdot}\vec{e}\right)$$

$$\dot{V}N = -\gamma\beta\,\vec{n}\cdot\dot{\vec{e}}$$

So

$$\dot{L}N = \frac{1}{\beta}\dot{V}N = -\frac{e}{m}\frac{1}{\beta}NFV$$

using $\dot{V} = -\frac{e}{m}FV$ which was proved inthe previous section.

Introducing the expression for $\dot{L}N$ in the equation for $\dot{\Theta}_R$ we get

$$\dot{\Theta}_R = -\frac{e}{m}\frac{1}{\beta}NFV - g\mu_0 LFN$$

$$= \frac{e}{m\beta}VFN - g\mu_0 LFN \qquad \text{, since } F \text{ is antisymmetric.}$$

$$= \left(\frac{e}{m\beta}V - g\mu_0 L\right)FN$$

Now $(FN)^{\mu} = F^{\mu\nu} N_{\nu}$

When $\mu = 0$, $(FN)^{0} = F^{0\nu} N_{\nu}$

$$= -F^{0k} N_k \qquad , k = 1,2,3$$

$$= \vec{E} \cdot \vec{n}$$

$\mu = 1$, $(FN)^{1} = F^{1\nu} N_{\nu}$

$$= F^{1k} N_k$$

$$= -H_3 N_2 + H_2 N_3$$

$$= (\vec{H} \times \vec{n})_1$$

Therefore $FN = (-\vec{E} \cdot \vec{n}, -\vec{n} \times \vec{H})$

$$\frac{e}{m\beta} V - g\mu_0 L = \frac{e}{m}\left(\frac{\gamma}{\beta}, \vec{e}\gamma\right) - g\mu_0(\beta\gamma, \vec{e}\gamma)$$

$$= \gamma\left[\frac{e}{m\beta} - g\mu_0\beta \quad , \quad \vec{e}\left(\frac{e}{m} - g\mu_0\right)\right]$$

Inserting this expression the equation for $\dot{\theta}_R$ becomes

$$\frac{d\theta_R}{dt} = (\vec{E} \cdot \vec{n})\left(g\mu_0\beta - \frac{e}{m\beta}\right) + \left(g\mu_0 - \frac{e}{m}\right) \vec{e} \cdot \vec{H} \times \vec{n}$$

This is valid for any particle with magnetic moment $g\mu_0\sigma$ and charge e. If the charge $e \neq 0$

$$g\mu_0\sigma = g\left(\frac{e}{2m}\right)\sigma$$

$$\frac{d\theta_R}{dt} = \frac{e}{2m}\left[(\vec{E} \cdot \vec{n})(g\beta - 2/\beta) + (g-2)\vec{e} \cdot \vec{H} \times \vec{n}\right]$$

$$g\beta - 2/\beta = \frac{g\beta^2 - 2}{\beta}$$

$$= \frac{(g-2) - g/\gamma^2}{\beta}$$

$$\frac{d\theta_R}{dt} = \frac{e}{2m}\left[(\vec{E}\cdot\vec{n})\,\frac{(g-2) - g/\gamma^2}{\beta} + (g-2)\,\vec{e}\cdot\vec{H}\times\vec{n}\right].$$

$\vec{E}$ and $\vec{H}$ are homogeneous fields in the lab system. $\frac{d\theta_R}{dt}$ is independent of the degree of polarization. Also $\dot{\theta}_R$ is an invariant and is expressed in terms of invariant quantities VFN and LFN. We have chosen the suitable reference system—here the lab system—for its expression.

Consider the equation

$$\frac{d\theta_R}{dt} = \frac{e}{2m}\left[\vec{E}\cdot\vec{n}\,\frac{(g-2) - g/\gamma^2}{\beta} + (g-2)\,\vec{e}\cdot\vec{H}\times\vec{n}\right]$$

which is true when $e \neq 0$

Remarks

1. When $\vec{H}$ is parallel to $\vec{e}$ the second term vanishes.
2. When $\beta \to 0$ the first term goes to infinity. This means if the particle is at rest it gets accelerated instantaneously. The θ_R also changes instantaneously.

Problem I.

We shall now give a full discussion when

$$\vec{E} \times \vec{e} = \vec{H} \times \vec{e} = 0$$

We first verify whether the condition $\vec{E} \times \vec{e} = \vec{H} \times \vec{e} = 0$ is compatible with the equation of motion $VF = \frac{m}{e} \frac{dV}{d\tau}$ always. That is, we have to check whether $\vec{e} \times \vec{H} = 0 = \vec{E} \times \vec{e}$ is conserved for all time.

$$\frac{dV}{d\tau} = \dot{V} = \frac{e}{m} VF = \frac{e}{m} \gamma \left[\beta \vec{E} \cdot \vec{e} \, , \vec{E} + \beta \, \vec{e} \times \vec{H} \right]$$

at $t = t_0$, $\vec{e} \times \vec{H}$ and $\vec{E} \times \vec{H}$. Writing $\vec{E} = E \vec{e}$ at $t = t_0$ we have

$$\left. \frac{dV}{d\tau} \right|_{t = t_0} = \frac{e}{m} \gamma \left[\beta E , \; E \vec{e} \right]$$

At the same time from

$$V = (\gamma, \gamma\beta\vec{e}) \; ; \quad \dot{V} = \left(\dot{\gamma} , (\gamma\beta)^{\cdot} \vec{e} + \dot{\vec{e}} (\gamma\beta) \right)$$

Equating these two expression for $\dot{V}$ we have

$$\frac{e\gamma}{m} E \vec{e} = (\gamma\beta)^{\cdot} \vec{e} + (\gamma\beta) \dot{\vec{e}}$$

Either $\dot{\vec{e}} \parallel \vec{e}$ or $\dot{\vec{e}} = 0$. Since

$\vec{e}^{\,2} = 1$, $\vec{e} \cdot \dot{\vec{e}} = 0$ So $\dot{\vec{e}}$ cannot be parallel to $\vec{e}$ without being zero.

i.e. $\dot{\vec{e}} = 0$ at $t = t_0$. Similarly we can prove all the other time derivatives are zero at $t = t_0$. So $\vec{e}$ is constant. $\vec{E} \times \vec{e} = \vec{H} \times \vec{e} = 0$ is conserved at all times.

From the equation for $\dot{\theta}_R$

$$\frac{d\theta_R}{dt} = (\vec{E}\cdot\vec{n})\left(g\mu_0\beta - \frac{e}{m\beta}\right) + \left(g\mu_0 - \frac{e}{m}\right)\vec{e}\cdot\vec{H}\times\vec{n}$$

We see $\frac{d\theta_R}{dt} = 0$ as $\vec{E} = E\vec{e}$ is $\perp r$ $\vec{n}$

and $\vec{e}\cdot\vec{H}\times\vec{n}$ as $\vec{H} \parallel \vec{e}$

i.e. θ_R is constant.

$$\dot{S} = s\left(\dot{L}\cos\theta_R + \dot{N}\sin\theta_R + \dot{\theta}_R(N\cos\theta_R - L\sin\theta_R)\right)$$

$$= s\left(\gamma\frac{e}{m}E\cos\theta_R\ ,\ \beta\gamma\frac{e}{m}E\vec{e}\cos\theta_R + g\mu_0\vec{n}\times\vec{H}\sin\theta_R\right)$$

when $\dot{\theta}_R = 0$.

The second of these expressions is obtained by substituting

$$NF = -FN$$

$$= \left(E\cdot n\ ,\ \vec{n}\times\vec{H}\right)$$

and similar other expressions for LF , etc. , inthe expansion for $\dot{S}$.

Thus c omparing the two expressions for $\dot{S}$, we have

$$\dot{L} = \frac{e}{m} E \gamma (1, \vec{\beta})$$

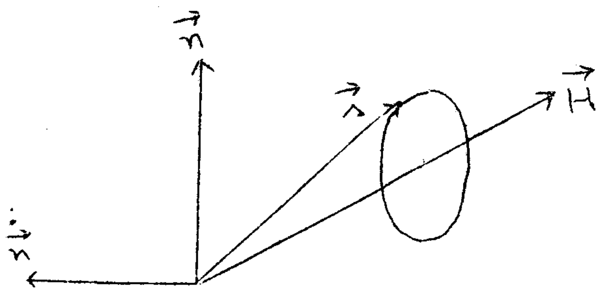
$$\dot{N} = g \mu_0 (\vec{n} \times \vec{H})$$

These equations state that L changes because the particle is accelerated and that N precesses in a left screw and with constant angular frequency around $\vec{H}$.

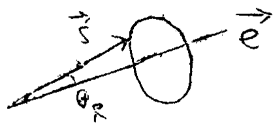
$$\frac{dN}{dt} = \frac{d\vec{n}}{dt} = \frac{1}{\gamma}\dot{N}$$

$$= \frac{g\mu_0}{\gamma} \vec{n} \times \vec{H}$$

$$\left|\frac{d\vec{n}}{dt}\right| = \omega = \frac{g\mu_0 |\vec{H}|}{\gamma}$$



Also since Θ is constant and $\dot{s} \neq 0$, $\vec{\lambda}$ process about $\vec{q}$ which is constant



Problem 2

We shall consider the case $\vec{E} = 0$, $\vec{H} \cdot \vec{n} \times \vec{e} = H$

(a) We check whether the above conditions are conserved

$$\frac{d\theta_R}{dt} = \frac{e}{2m}\left[\vec{E}\cdot\vec{n} + (g-2)\ \vec{e}\cdot\vec{H}\times\vec{n}\right]$$

$$= \frac{e}{2m}\ (g-2)\ H$$

The equation of motion is

Also

$$\dot{V} = \left(\dot{\gamma},\ \dot{\vec{e}}(\beta\gamma) + \vec{e}\,(\beta\gamma)^{\cdot}\right)$$

$$\dot{V} = \frac{e}{m} VF$$

$$= \frac{e}{m}\gamma\left(\beta\vec{E}\cdot\vec{e},\ \vec{E} + \beta\,\vec{e}\times\vec{H}\right)$$

$$= \frac{e}{m}\gamma\left(0,\ \beta H\vec{n}\right)$$

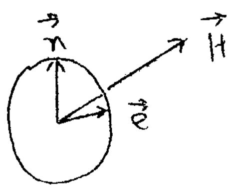
Comparison of these different expression for $\dot{V}$ gives

$$\dot{\vec{e}} = \frac{e}{m} H\vec{n}, \quad \because\ \dot{\vec{e}} \text{ is } \perp r\ \vec{n},\ (\beta\gamma)^{\cdot} = 0$$

at a time t.

$$\frac{d\vec{e}}{dt} = \frac{eH}{m\gamma}\vec{n}$$

$\vec{e}$ turns around with frequency $eH/m\gamma$ in a left screw way.



Therefore the condition $\vec{H} = H(\vec{n}\times\vec{e})$ and $\vec{E} = 0$ is conserved.

The frequency $eH/m\gamma$ is called Lamo r's frequency.

(b) $$\frac{d\theta_R}{dt} = \left(g\mu_0 - \frac{e}{m}\right) H = \frac{eH}{m}\left(\frac{g}{2} - 1\right)$$

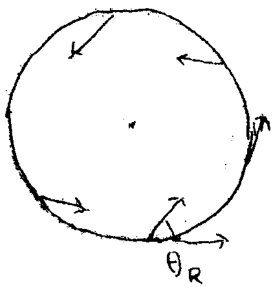
for a charged particle. Right hand side is positive for $g > 2$ and negative for $g < 2$ and zero for $g = 2$. But always the right hand side is a constant. So the angle between the direction of motion and direction of polarization is constantly decreasing for $g < 2$, increases with $g > 2$ and remains the same for $g = 2$. This is independent of the velocity of the particle. The particle move around a circle and turn around once in the time $T = 2\pi/\omega_0 = 2\pi\,(m\gamma/eH)$ as $\vec{e}$ and $\vec{n}$ rotate with angular frequency $\frac{d\vec{e}}{dt} = \frac{eH}{m\gamma}$.

After one turn the $\vec{e}$ returns to its old position the change in the angle, θ_R , say $\Delta\theta_R$ given by

$$\Delta\theta_R = \theta_R(T) - \theta_R(0)$$

$$= T\,\frac{d\theta_R}{dt} = 2\pi\gamma\left(\frac{g}{2} - 1\right)$$

is the change in the directionx of polarization per turn.



In the figure we have assumed $g > 2$.

This fact can be used to find the g factor of the μ .

Problem 3

In the case of a fictitious particle with spin and with $e \neq 0$ but without magnetic moment $(g = 0)$ one sees that

$$\frac{d\theta_R}{dt} = -\frac{e}{m}\left(\frac{\vec{E}\cdot\vec{n}}{\beta} + \vec{e}\cdot\vec{H}\times\vec{n}\right)$$

Physical intuition tells us, however, that the polarization should not change if there is no magnetic moment show that there is no contradiction.

Since Θ_R is the angle between polarization and the direction of motion and since we expect the polarization S_R to remain constant, we have to show that the change in Θ_R is brought about by the change in the direction of motion.

(a) $\dot{V} = \dfrac{dV}{d\tau} = \gamma\,\dfrac{dV}{dt} = \dfrac{\gamma e}{m}\left(\beta\,\vec{E}\cdot\vec{e}\ ,\ \vec{E} + \beta\,\vec{e}\times\vec{H}\right)$

Denoting $\dfrac{d}{dt}$ by prime $(')$

$V = (\gamma, \gamma\vec{\beta})$

$\dot{V} = \gamma V' = \gamma(\gamma',\ \beta\gamma\vec{e}\,' + \vec{e}\,(\beta\gamma)')$

Comparing the two expression for V' we get

$$\gamma' = \frac{e}{m}\,\vec{E}\cdot\vec{e}\,\beta$$

Further

$$(\beta\gamma)' = \frac{\gamma'}{\beta}$$

$$= \frac{e\,\vec{E}\cdot\vec{e}}{m}$$

This we have got by comparing the time components.

By comparing the space components

$$\frac{e}{m}\left(\vec{E} + \beta\,\vec{e} \times \vec{H}\right) = \beta\gamma\,\vec{e}\,' + \vec{e}\,(\beta\gamma)$$

$$= \beta\gamma\,\vec{e}\,' + \vec{e}\,\frac{\gamma'}{\beta} = \beta\gamma\,\vec{e}\,' + \vec{e}\left(\frac{e\,\vec{E}\cdot\vec{e}}{m}\right)$$

$$\beta\gamma\,\vec{e}\,' = \frac{e}{m}\left[\vec{E} + \beta\,\vec{e} \times H - \vec{e}\,(\vec{E}\cdot\vec{e})\right]$$

<u>The Case of Mass Zero</u>

The polarization four vector

$$S^{\mu} = \left(\gamma\vec{\beta}\,\vec{\Delta}_R\ ,\ \vec{\Delta}_R + \vec{\beta}\,\frac{\gamma^2}{\gamma+1}\,\vec{\beta}\cdot\vec{\Delta}_R\right)$$

does not work when $\gamma \to \infty$. But $\gamma \to \infty$ as mass zero particles moves with unit velocity.

$\vec{\Delta}_R$ is now meaningless. Also the polarization is meaningful only in the rest system of the particle which we cannot reach by Lorentz transformation. So the whole concept fails when $m \to 0$. Yet particles with mass zero and spin $\neq 0$ exists.

The problem can be solved by replacing the four vector V_μ by four momentum P_μ in the above analysis. For $P_\mu = mV = (\varepsilon, \vec{p})$ even for $m = 0$. So let us multiply S_μ by m and call it W_μ.

$$\vec{e}\,' = \frac{e}{m\beta\gamma}\left[\vec{E} - \vec{e}\,(\vec{E}\cdot\vec{e}) + \beta\,\vec{e} \times \vec{H}\right]$$

(b) At a given instant t_0 we take in the lab system a constant four vector $A = S(t_0)$. Then $A = L\cos\Theta + N\sin\Theta$ since $\Theta(t_0) = \theta_R$

$$A' = L'\cos\Theta + N'\sin\Theta + \Theta'(N\cos\Theta - L\sin\Theta)$$

$$= 0$$

$\therefore A = \text{constant}$

Multiplying by N the last equation becomes

$$\Theta' = NL' = \gamma\,\vec{n}\cdot\vec{e}$$

for $N\dot{N} = 0$, $N^2 = -1$, $LN = 0$

Using the expression for $\vec{e}\,'$ calculated in (a)

$$\Theta' = -\frac{e}{m}\left[\frac{\vec{E}\cdot\vec{n}}{\beta} + \vec{e}\times\vec{H}\cdot\vec{n}\right]$$

This shows $\Theta' = d\theta_R/dt$.

W_μ exists even when $m \to 0$

$$W_\mu = \left(m\gamma\beta\,\vec{s}\ ,\ m\,\vec{s}_R + \vec{\beta}\,\frac{m\gamma^2}{\gamma+1}\,\vec{\beta}\cdot\vec{s}\right)$$

W_μ transforms like a four vector.

$$W_\mu(m\to 0) = s(\varepsilon, \vec{e}\,\varepsilon)$$

$$= s\,p^\mu \quad (m=0)$$

We call s the degree of polarization and find

$$m=0\ : \quad W^\mu = s\,p^\mu$$

$$\vec{\zeta} = s\,\vec{e}$$

$$W^\mu W_\mu = s\,W^\mu p_\mu = s^2 p^\mu p_\mu = 0$$

Since W_μ and P_μ are four vectors Δ is invariant. The direction of polarization is either $\parallel$ to the momentum (when $\Delta > 0$) or antiparallel to momentum ($s < 0$). This direction is the same for all Lorentz system and we call $\pm\Delta$ the helicity. We do not need any equation for the polarization. We can apply this to light quanta.

Sec.9.4

Relation between polarization four-vector and angular momentum tensor

If we recall the definition of polarization in quantum mechanics (as the maximum value of the expectation value of the spin) which obeys classical equations of motion, it is very natural to expect a relation between the polarization four vector and angular momentum.

Suppose we consider a system of N spinless particles of mass m_i with coordinates and momenta

$$x_i^\mu = (t, \vec{x}_i) \qquad , i = 1, \cdots N$$

$$p_i^\mu = (\varepsilon_i, \vec{p}_i)$$

The angular momentum of the system is then

$$M^{\mu\nu} = \sum_i (x_i^\mu p_i^\nu - x_i^\nu p_i^\mu)$$

$\mu, \nu, \lambda = 0, 1, 2, 3$ $\qquad$ i = particle label

$j, k, l = 1, 2, 3$ $\qquad$ $= 1, 2, \ldots N$

The time component of $M^{\mu\nu}$ is then

$$M^{0k} = t \sum_i (p_i^k - x_i^k \varepsilon_i)$$

In the rest system, namely when $\sum p_i = 0$ we obtain

$$\left(M_i^{ok}\right)_{rest} = -\left(x_i^k \varepsilon_i\right)_{rest}$$

This quantity should remain constant in time. So, we now define in the rest system the four vector

$$(x^\mu)_R = \left(t, \frac{1}{\sum \varepsilon_i} \sum \vec{x}_i \varepsilon_i\right)_R$$

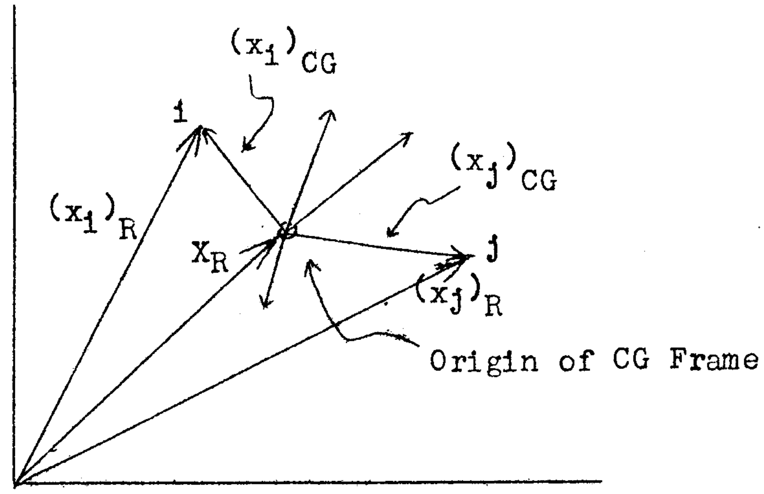
Since this is defined invariantly, we can Lorentz transform this to any frame of reference so that

$$x^\mu = L\,(x^\mu)_R$$

$$= L\left(t, \frac{\sum x_i \varepsilon_i}{\sum \varepsilon_i}\right)_R$$

This x^μ will be different from the one which we may obtain after Lorentz transforming each x_i^μ and p_i^μ and then constructing $\left(t, \frac{\sum x_i \varepsilon_i}{\sum \varepsilon_i}\right)$. However, we define it in the above way to ensure that x^μ is a four vector. Now in rest system (when $\sum \vec{p}_i = 0$), we may introduce new coordinates $x_{i\,cg}^{\mu}$ by measuring all distances from the centre of gravity (We call it as centre of gravity because in the non-relativistic limit

$$\frac{\sum x_i \varepsilon_i}{\sum \varepsilon_i} \longrightarrow \frac{\sum x_i m_i}{\sum m_i}$$



Origin of
rest frame

It is obvious that

$$(x_i)_R = X_R + (x_i)_{CG}$$

It should be noted that $(x_i)_{CG}$ has no time components.
The angular momentum tensor then becomes

$$(M^{\mu\nu})_R = \sum \left(x^\mu p^\nu - x^\nu p^\mu \right)_R$$

$$= \sum \left(x^\mu p^\nu - x^\nu p^\mu \right)_{CG} + \left(x^\mu p^\nu - x^\nu p^\mu \right)_R$$

We find that when the frame and centre of gravity frame coincide, then

$$x^\mu p^\nu - x^\nu p^\mu = 0$$

From the equation

$$M^{\mu\nu} = S^{\mu\nu} + L^{\mu\nu}$$

by multiplying by P_ν on both sides and using the fact that $S^{\mu\nu} P_\nu = 0$ we get

$$M^{\mu\nu} P_\nu = L^{\mu\nu} P_\nu$$

Thus, P_ν serves to project out the $L^{\mu\nu}$ part from $M^{\mu\nu}$. The projection of $L^{\mu\nu}$ is carried out as follows:

We introduce the completely antisymmetric tensor $\varepsilon_{\mu\nu\rho\sigma}$ of rank four

$$\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } (\mu\nu\rho\sigma) \text{ is even} \\ -1 & \text{if } (\mu\nu\rho\sigma) \text{ is odd} \end{cases}$$

Even or odd refer to the number of permutation operation one has to perform on (0,1,2,3) to get $(\mu\nu\rho\sigma)$. If two of the indices are equal then ;

$\varepsilon_{\mu\nu\rho\sigma} = 0$ if two of the indices are equal.

$$\varepsilon_{\mu\nu\rho\sigma} = - \varepsilon_{\nu\rho\sigma\mu}$$

Furthermore, the metric tensor $g^{\mu\nu}$ changes the sign only when the index is 1,2,3; it does not change sign if the index is 0. With $\varepsilon_{\mu\nu\rho\sigma}$ we define the four component objects

$$W^{\mu} = \frac{1}{2} \varepsilon^{\mu}_{\ \nu\rho\sigma} P^{\nu} M^{\rho\sigma}$$

$$= \frac{1}{2} \varepsilon^{\mu}_{\ \nu\rho\sigma} P^{\nu} S^{\rho\sigma}$$

$$S^{\mu} = \frac{1}{m} W^{\mu} \quad , \quad m = \sqrt{P^2}$$

We have the first equation since ε is actually a projection operator for

$$\varepsilon^{\mu}_{\ \nu\rho\sigma} P^{\nu} (x^{\rho} p^{\sigma} - x^{\sigma} p^{\rho}) = 0$$

by the antisymmetric property of ε. Suppose we go to the rest frame, then,

$$(W^{0})_{R} = \frac{1}{2} \varepsilon^{0}_{\ 0\rho\sigma} (P^{0} S^{\rho\sigma})_{R}$$

$$= 0$$

since $\varepsilon_{\mu\nu\rho\sigma} = 0$ if any two of the indices are the same. Also

$$(W^{1})_{R} = \frac{1}{2} \varepsilon^{0}_{\ 0\rho\sigma} (P^{0} S^{\rho\sigma})_{R}$$

$$= 0$$

$$(W^2)_R = m\,(S^{31})_R$$

$$(W^3)_R = m\,(S^{12})_R$$

Therefore,

$$(W^\mu)_R = m\,(0,\ S^{23},\ S^{31},\ S^{12})_R$$

$$= m\,(S^\mu)_R$$

$$= (0,\ S_R)$$

Thus, $(W^\mu)_R$ is a three vector $(m\,\vec{S}_R)$ whose three components are m times the components of intrinsic angular momentum tensor. Thus, though W^μ transforms as a four vector and $S^{\mu\nu}$ as a tensor, in the rest frame they have such a relation.

Suppose these N spinless particles collapse together (so that we can no longer talk about the constituents!) to form a single-like particle of mass m, momentum P and intrinsic angular momentum or spin $S^{\mu\nu}$. It should be clearly understood that we are only giving an analogy; by this kind of argument, one cannot obtain the half integral spin values even by invoking quantum mechanics. The quantum mechanical spin does indeed behave as our $S^{\mu\nu}$; in particular obeys the master equation

$$S^{\mu\nu} P_\nu = 0$$

Thus, $S^{\mu\nu}$ cannot be understood from our knolwedge on structure of rotation of the particle

If for example, consider an electron as a rotating sphere of mass m , moment of inertial $2/5\, m r^2 = I$ and angular momentum. $S = \frac{1}{2}\hbar = \omega I$, where ω is the angular velocity given by v/r (and v is the velocity at its equator. Then

$$v = r/2 \left(\frac{\hbar}{I}\right) = 5/4 \left(\hbar/mr\right)$$

Putting $r = e^2/mc^2$ one finds

$$\frac{v}{c} = \frac{5}{4}\left(\frac{\hbar c}{e^2}\right) = \frac{5}{4} \times 137 \gg c$$

which contradicts velocity.

We now have got a projection operator to project out the $S^{\mu\nu}$ part of $M^{\mu\nu}$ (the $L^{\mu\nu}$ part vanishing because of the antisymmetric property of ε) as

$$S^{\mu\nu} = \frac{1}{m^2} \varepsilon^{\mu\nu}{}_{\alpha\beta}\, P^\alpha W^\beta$$

Thus, since $S^{\mu\nu}$ satisfies this covariant equation,

$$(S^{12})_R = \frac{1}{m^2} \varepsilon^{12}_{\ \ 03} (p^0 W^3)$$

$$= \frac{1}{m^2} \varepsilon^{12}_{\ \ 03} \cdot m \cdot m \cdot (S^{12})_R = (S^{12})_R$$

$$(S^{0k})_R = \frac{1}{m^2} \varepsilon^{0k}_{\ \ 0i} (p^0 W^i)_R = 0, \text{ since } \varepsilon^{0k}_{\ \ 0i} = 0$$

Introducing the expression

$$W^\beta = \frac{1}{2} \varepsilon^{\beta}_{\ \lambda\rho\sigma} p^\lambda M^{\rho\sigma}$$

for W in the expression for $S^{\mu\nu}$ we get

$$S^{\mu\nu} = \left(\frac{1}{2m^2} \varepsilon^{\mu\nu}_{\ \ \alpha\beta} \varepsilon^{\beta}_{\ \lambda\rho\sigma} p^\alpha p^\lambda\right) M^{\rho\sigma}$$

$$= \Sigma^{\mu\nu}_{\ \ \rho\sigma} M^{\rho\sigma}$$

where

$$\Sigma^{\mu\nu}_{\ \ \rho\sigma} = \left(\frac{1}{2m^2} \varepsilon^{\mu\nu}_{\ \ \alpha\beta} \varepsilon^{\beta}_{\ \lambda\rho\sigma} p^\alpha p^\lambda\right)$$

$$= \varepsilon^{\mu\nu}_{\ \ \alpha\beta} \left(\frac{p^\beta p_\lambda}{2m^2}\right) \varepsilon^{\lambda\alpha}_{\ \ \rho\sigma}$$

Thus $\Sigma^{\mu\nu}_{\ \ \rho\sigma}$ is the required projection operator that projects the $S^{\mu\nu}$ from $M^{\mu\nu}$. The expression for $S^{\mu\nu}$ is indeed consistent with the equation

$$S^{\mu\nu} p_\nu = 0$$

since

$$\varepsilon^{\mu\nu}{}_{\alpha\beta}\ \varepsilon^{\beta}{}_{\lambda\rho\sigma}\ p^{\alpha} p_{\nu} = 0$$

because of the antisymmetry of ε .

Thus, we have achieved the following thing: We have expressed $M^{\mu\nu}$, the angular momentum tensor as

$$M^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$$

$L^{\mu\nu}$: orbital part

$S^{\mu\nu}$: Intrinsic part

Only $S^{\mu\nu}$ survives in the C.G. system and even for $S^{\mu\nu}$ only the components S^{23}, S^{31}, S^{12} are different from zero. The above considerations hold for a single particle with spin.-

Let us now turn to the case of a system of particles with spin. Let each particle in the system have momentum p , mass m , coordinate x^{μ} and total angular momentum $M^{\mu\nu} = (x^{\mu} p^{\nu} - x^{\nu} p^{\mu}) + S^{\mu\nu}$. The total angular momentum of the system is

$$M^{\mu\nu} = \sum_{i=1}^{N} M_i^{\mu\nu} = \sum_{i=1}^{N} (x_i^{\mu} p_i^{\nu} - x_i^{\nu} p_i^{\mu}) + S_i^{\mu\nu}$$

$$= \sum_{i=1}^{N} L_i^{\mu\nu} + \sum_{i=1}^{N} S_i^{\mu\nu}$$

This separation, however, is not useful because $L^{\mu\nu}$ already contains an orbital spin part, the spin due to orbital motion of particles relative to the centre of gravity. Thus $M^{\mu\nu}$ is actually,

$$M^{\mu\nu} = \underbrace{(x^\mu p^\mu - x^\nu p^\mu) + S^{\mu\nu}_{orbital}}_{\sum_{i=1}^{N} L_i^{\mu\nu}} + S^{\mu\nu}_{spin}$$

If we really want a nice separation between orbital angular momentum and spin, we write the above equation

$$S^{\mu\nu} = M^{\mu\nu} - (x^\mu p^\nu - x^\nu p^\mu)$$

$$= M^{\mu\nu} - L^{\mu\nu}$$

where now $L^{\mu\nu}$ part does not include $S^{\mu\nu}_{orbital}$ and

$$S^{\mu\nu} = \sum_{i=1}^{N} S_{i\,orbital}^{\mu\nu} + S_{i\,spin}^{\mu\nu}$$

For the system as a whole, then $M^{\mu\nu}$, $L^{\mu\nu}$ are known, so that $S^{\mu\nu}$ can be defined in a covariant way, but not $S^{\mu\nu}_{spin}$ separately (unless we know $S^{\mu\nu}_{orbital}$!). To know $S^{\mu\nu}_{orbital}$ of course, we need to have an idea of the individual particles. Thus, from the characteristics of the system as a whole, it is possible to separate only $S^{\mu\nu}_{orbital} + S^{\mu\nu}_{spin}$

Let us now see the effect of our projection operator $\sum^{\mu\nu}_{\rho\sigma}$ on $M^{\rho\sigma}$ now. We have,

$$\sum^{\mu\nu}_{\rho\sigma} M^{\rho\sigma} = \sum^{\mu\nu}_{\rho\sigma} \left(L^{\rho\sigma} + S^{\rho\sigma}_{orbital} + S^{\rho\sigma}_{spin} \right)$$

$$= \sum^{\mu\nu}_{\rho\sigma} S^{\rho\sigma}_{orbital} + \sum^{\mu\nu}_{\rho\sigma} S^{\rho\sigma}_{spin}$$

$$= S^{\mu\nu}_{orbital} + \left(\sum^{\mu\nu}_{\rho\sigma} S^{\rho\sigma}_{spin} \right)$$

This reduction is because of the property of $\sum^{\mu\nu}_{\rho\sigma}$ namely

$$\sum^{\mu\nu}_{\rho\sigma} M^{\rho\sigma} = S^{\mu\nu}_{orbital}$$

However, we do not know anything about its effect on $S^{\rho\sigma}_{spin}$. The last term, in general, is not equal to $S^{\mu\nu}_{spin}$. But,

$$\left(\sum^{\mu\nu}_{\rho\sigma} S^{\rho\sigma}_{spin} \right) P_\nu = 0$$

since $$\Sigma^{\mu\nu}_{\rho\sigma} P_\nu = 0$$

Also we have

$$S^{\mu\nu}_{spin} P_\nu = \left(\sum_i S_i^{\mu\nu} \right) P_\nu$$

$$\neq \sum_i S_i^{\mu\nu} P_{i\nu} = 0$$

Therefore, in general

$$\Sigma^{\mu\nu}_{\rho\sigma} S^{\rho\sigma}_{spin} \neq S^{\mu\nu}_{spin}$$

$$\Sigma^{\mu\nu}_{\rho\sigma} M^{\rho\sigma} \neq S^{\mu\nu}$$

$$S^{\mu\nu} P_\nu \neq 0$$

The reason for this is the following. Writing $M^{\mu\nu}$ as

$$M^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}_{orbital} + S^{\mu\nu}_{spin}$$

We see that by going to the C.G. system we can make the time components of <u>only</u> $(L^{\mu\nu} + S^{\mu\nu}_{orbital})$ to vanish so that

$$\left[\left(L^{\mu\nu} + S^{\mu\nu}_{orbital} \right) P_\nu \right]_{CG} = 0$$

However, in the C.G. system $\left(S^{\mu\nu}_{spin}\right)_{CG}$ will have a time component so that $\left(S^{\mu\nu}_{spin} P_\nu\right)_{CG} \neq 0$.

This is again because, $S^{\mu\nu}_{i\,spin}$ will have a zero time component in the rest system of i. Thus $\left(S_i^{\mu\nu}\right)_{R_i}$ will have zero time component. But $\left(S_i^{\mu\nu}\right)_{CG}$ is obtained by Lorentz transforming $\left(S_i^{\mu\nu}\right)_{R_i}$ so that it will, in general, have a time component. Thus, it seems that for a system of spinning particles the separation of $L^{\mu\nu}$ and $S^{\mu\nu}$ is difficult. However, there are some exceptions, the cases in which the individual momenta of all the N particles are known (a) If only very few particles are present, e.g. $M \to m + \mu$ and (b) if we have a beam of like particles with equal sharp momentum the scheme works. In the case (b), if we know the total momentum P, we do know about the individual momenta (since the N particles have equal momenta in this case, so that

$$\left(S^{\mu\nu}_{orbital}\right)_R = 0$$

since in this system all the particles are at rest. and no orbital motion around the origin exists. And since not only all particles together, but as well each individual particle will at rest, each $\left(S^{\mu\nu}_{spin}\right)_i$ will have no time components in their own rest frames, so that $\left(\sum_j S_j^{\mu\nu}\right)_R$ has no time component.

Thus, in the rest frame

$$\left(M^{\mu\nu}\right)_R = \left(L^{\mu\nu}\right)_R + \left(S^{\mu\nu}_{spin}\right)_R$$

$$\doteq \left(L^{\mu\nu}\right)_R + \left(\sum_j S_j^{\mu\nu}\right)_R$$

By a Lorentz transformation, we can get

$$M^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}_{spin}$$

since

$$\left(S^{\mu\nu}_{orbital}\right)_R = 0 \Rightarrow \left(S^{\mu\nu}_{orbital}\right)_{\text{in any frame}} \doteq 0$$

The projection operator works here:

$$S^{\mu\nu}_{spin} P_\nu = N \sum_j \left(S_j^{\mu\nu} p_\nu^{\gamma}\right) = 0$$

because $p_\nu = \frac{P_\nu}{N}$ if the N particles have the same momentum and

$$S_j^{\mu\nu} p_\nu = 0$$

Also

$$\Sigma^{\mu\nu}_{\rho\sigma} M^{\rho\sigma} \doteq S^{\mu\nu}_{spin}$$

This is true since $\sum^{\mu\nu}_{\rho\sigma}$ can be written in the form

$$\sum^{\mu\nu}_{\rho\sigma} = \frac{1}{2\ m^2}\ \varepsilon^{\mu\nu}_{\alpha\beta}\ \varepsilon^{\beta}_{\lambda\rho\sigma}\ P^{\alpha}\ P^{\lambda}$$

$$= \frac{1}{2m^2}\ \varepsilon^{\mu\nu}_{\alpha\beta}\ \varepsilon^{\beta}_{\lambda\rho\sigma}\ \frac{p^{\alpha}}{N}\ \frac{p^{\lambda}}{N}$$

$$= \frac{1}{2N^2m^2}\ \varepsilon^{\mu\nu}_{\alpha\beta}\ \varepsilon^{\beta}_{\lambda\rho\sigma}\ p^{\alpha}\ p^{\lambda}$$

since $P = N\,p$ (for particles with equal momenta).

$\sum^{\mu\nu}_{\rho\sigma}$ thus projects out each $S^{\mu\nu}_{j\,,spin}$ and thus the sum

$$S^{\mu\nu}_{spin} = \sum_j S^{\mu\nu}_j$$

Thus, for a system consisting of like particles with spin, it is possible to separate the true spin part from orbital part $L^{\mu\nu}$ so that $S^{\mu\nu}$ and $W^{\mu} = mS^{\mu}$ defined earlier can be used to describe the state of polarization of the particle. However, it only $M^{\mu\nu}$ (and not $S^{\mu\nu}$ or $L^{\mu\nu}$) is conserved.

Since in quantum mechanics a beamof like particles with sharp momentumis described as a plane wave, we expect a close relation between $S^{\mu\nu}$ and S^{μ} and quantum mechanical operators. We shall explicitly write down this relation for the case of Dirac particles.

We choose a particular representation of γ relation.

$$\left[\gamma^{\mu}, \gamma^{\nu}\right]_{+} = 2g^{\mu\nu}$$

in which the components of spinors naturally split up into two large and two small ones in the non-relativistic limit

$$\gamma^{k} = -\gamma_{k} = \begin{pmatrix} 0 & \sigma^{k} \\ -\sigma^{k} & 0 \end{pmatrix}$$

$$\gamma^{0} = \gamma_{0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^{5} = -\gamma_{5} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^{1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma^{2} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma^{3} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

γ's are (4 x 4) matrices and σ's are (2 x 2) matrices.
For free particles.

$$\psi(x) = e^{-ipx}\, U(p)$$

$$U(p) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

$$\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where $u_1(p)$, $u_2(p)$ are abitrary. Consider two classes of operators

$$\Sigma^{\mu} = i\gamma^5\gamma^{\mu}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]$$

$$\Sigma^{0} = i\gamma^5\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$$\Sigma^{k} = i\gamma^5\gamma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix}$$

$$\sigma^{0k} = i\begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}$$

$$\sigma^{jk} = \begin{pmatrix} \sigma^{\ell} & 0 \\ 0 & \sigma^{\ell} \end{pmatrix}$$

Define $$S^{\mu} = \langle \Sigma^{\mu} \rangle = \bar{\psi}\, i\gamma^5 \gamma^{\mu} \psi$$

$$, \quad \bar{\psi} = \psi^* \gamma^0$$

$$S^{\mu\nu} = \langle \sigma^{\mu\nu} \rangle = \bar{\psi}\, \frac{i}{2} [\gamma^{\mu}, \gamma^{\nu}]_- \psi$$

We will now show that S^{μ} and $S^{\mu\nu}$ are identical with our old S^{μ} and $S^{\mu\nu}$ respectively.

Since these two are covariant, it is sufficient to show the relationin the rest frame where $u_3, u_4 = 0$

From the definition for Σ's and σ's it is obvious that

$$(\Sigma^0) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}$$

Hence

$$\langle \Sigma^0 \rangle_R = \langle \sigma^{0k} \rangle_R = 0$$

Moreover,

$$\langle \Sigma^k \rangle_R = \widetilde{u_1^* \, u_2^*} \; \sigma^k \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$= \langle \sigma^k \rangle_R$$

$$\langle \sigma^{jk} \rangle_R = \widetilde{u_1^* \, u_2^*} \; \sigma^{\ell} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$= \langle \sigma^{\ell} \rangle_R \qquad (j, k, \ell \text{ cyclic})$$

Hence
$$(S^{\mu})_R = (0, \langle\sigma\rangle_R)$$

$$(S^{\mu\nu})_R = \left\{ \begin{array}{l} \langle\sigma^{0k}\rangle = 0 \\ \langle\sigma^{jk}\rangle = \langle\sigma\rangle \end{array} \right\}_R$$

Thus in the rest frame and hence in all frames we have the correspondence

$$S^{\mu\nu} \longleftrightarrow \sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right]_-$$

$$S^{\mu} \longleftrightarrow \Sigma^{\mu} = i\,\gamma^5\gamma^{\mu}.$$