MATSCIENCE REPORT 13

Notes on

BANACH SPACES

Basic Definitions and Theorems, and Related Topics.

BY

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1. INTRODUCTION

A knowledge of the elements of functional analysis is now essential for students and research workers in applied mathematics and certain branches of mathematical physics. The purpose of these notes is to present a collection, index, or list, of certain facts concerning Banach Spaces and related topics. In particular, we have prepared these notes as the first chapter of a forthcoming book on semigroups of operators and their applications; the applications being in the areas of differential equations, Markov processes, and mathematical physics.

These notes are not intended to serve as a text book, for many topics are omitted, and we give no proofs. We only hope that the notes will (1) tell the student about to study semigroup theory exactly which concepts and results he should be familiar with, and (2) to serve as a reference for definitions and results that may wish to be recalled, and which are sometimes used in the literature without comment.

At the end of these notes we give a Bibliography which lists certain books and monographs in functional analysis that the reader may wish to consult for proofs of theorems given in the notes, and for additional results and comments,

2. Banach Spaces

A. Normed linear spaces, Banach spaces, Banach algebras and Banach lattices.

Throughout this section we denote by F the field of real or complex numbers.

<u>Definition 1</u>. A <u>linear space</u> U over F is a set of elements $x, y, \ldots,$ satisfying

<u>Axiom A</u>: For the elements of U there exists two uniquely defined operations: an addition and a scalar multiplication such that if x and y are arbitrary elements of U and α is an arbitrary complex number, $x + y \in U$ and $\alpha x \in U$. For these operations the following rules hold for all $x, y, z \in U$ and $\alpha, \beta \in F$:

(i) $x + y = y + x$

(ii) $x + (y + z) = (x + y) + z$

(iii) there is a U, dependent on x and y_1 such that

$$x + U = y$$

(iv) $\alpha(x + y) = \alpha x + \alpha y$

(v) $(\alpha + \beta)x = \alpha x + \beta x$

(vi) $\alpha(\beta x) = (\alpha\beta)x$

(vii) $1x = x$ (where 1 is the unit element of F)

If U is a linear space, then (a) there is a unique element θ (called the <u>null element</u>) in L such that $x + \theta = \theta + x = x$, and $\alpha \cdot \theta = \theta x = \theta$ for all $\alpha \in F$ and $x \in U$, (b) $\alpha x = \theta$ if and only if $\alpha = 0$ or $x = \theta$, for each $x \in U$ there is a unique $y \in L$ such that $x + y = y + x = \theta$ and $(-1)x = y$; then for $z, x \in U$ define $z - x = z + (-1)x$ and $-x = \theta - x$.

<u>Definition 2</u>. A <u>normed linear space</u> is a linear space X, over F, on which there is defined a nonnegative real-valued function called the norm (we denote the norm of an element x by $||x||$)

satisfying, for all $x, y \in X$, $\alpha \in F$,

<u>Axiom B</u>: (i) $\| x \| \geqslant 0$, $\| x \| = 0$ if and only if $x = 0$

(ii) $\| \alpha x \| = | \alpha | \; \| x \|$

(iii) $\| x + y \| \leqslant \| x \| + \| y \|$

<u>Definition 3</u>: A normed linear space X is said to be complete, if to every sequence $\{x_n\}$ of elements of the space which satisfy the Cauchy condition $\lim_{m, n \to \infty} \|x_m - x_n\| = 0$, there exists an element x of the space with the property $\lim_{n \to \infty} \|x_n - x\| = 0$. We will call the above axiom of completeness Axiom C.

<u>Definition 4</u>. A normed linear space X is said to be <u>separable</u> if it satisfies the following axiom:

<u>Axiom D</u>: There exists a denumerable sequence of elements $x_n \in X$ such that if ϵ is an arbitrary positive number, and x an arbitrary element of x, at least one of the elements satisfies the inequality $\|x - x_n\| < \epsilon$.

<u>Definition 5</u>. A space X with elements $x, y \ldots$ is called a <u>metric space</u> if for every pair of elements $x, y \in X$ there is defined a real-valued nonnegative function $d(x, y)$, called the distance between x and y, satisfying the postulates of Lindenbaum:

(i) $d(x, y) = 0$, if and only if $x = y$

(ii) $d(x, z) \leqslant d(x, y) + d(y, z)$, if $x, y, z \in X$.

The above properties imply that

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, y) \geqslant 0$.

A normed linear space becomes a metric space if the distance $d(x, y)$ between two elements x and y is defined by $d(x, y) = \| x - y \|$.

Definition 6. In a metric space X, the set of points $\{x: d(x_0, x) < \epsilon\}$ is called a sphere of radius ϵ about x_0.

Definition 7. A set Y is said to be bounded if it is contained in a sphere. The diameter of a bounded set Y is given by $\sup \{d(x, y): x, y \in Y\}$. A set Y is said to be totally bounded if for each $\delta > 0$ Y may be covered by a finite number of spheres of diameter less than δ.

Theorem 1. If X is a metric space, then the following conditions are equivalent to one another:

(1). X has the Bolzano-Weierstrass property (i.e., every infinite subset of X has a limit point);

(2). X is sequentially compact (i.e., every sequence in X has a convergent subsequence);

(3). X is compact (i.e., as a topological space, every open cover has a finite subcover).

Definition 8. A normed linear space $\mathcal{H}$ is said to be a Banach space if it is complete in the metric defined by the norm; that is, if it satisfies Axioms A, B and C given above. If, in addition, $\mathcal{H}$ satisfies Axiom D it is called a separable Banach space. $\mathcal{H}$ is said to be real or complex Banach space according as the field F is real or complex.

<u>Definition 9</u>. A <u>linear subspace</u> $[L]$ of a Banach space is a subset of $\mathfrak{X}$, with the property that, if $x, y \in [L]$ and α and β are complex numbers, then $\alpha x + \beta y \in [L]$. In addition $[L]$ is closed, i.e., if $x_n \in [L]$ and $\lim_{n \to \infty} ||x-x_n|| = 0$ then $x \in [L]$.

<u>Definition 10</u>. If E is a subset of a Banach space $\mathfrak{X}$, the set of all linear combinations $\Sigma \alpha_i x_i$, where the α_i are complex numbers and $x_i \in E$, is called the <u>linear manifold</u> determined by E. We denote it by $L(E)$. A closed linear manifold is a linear subspace.

<u>Definition 11</u>. A system of n elements $x_1, x_2, \ldots, x_n$ of a Banach space $\mathfrak{X}$ is said to be <u>linearly independent</u> in the equation $\sum_{i=1}^{n} \alpha_i x_i = 0$, $\alpha_i \in F$, implies $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$. The elements are <u>linearly dependent</u> if such an equation holds in which at least one $\alpha_i \neq 0$. If $\mathfrak{X}$ contains n linearly independent elements, but every system of $(n + 1)$ elements is linearly dependent, then $\mathfrak{X}$ is said to be of <u>dimension n</u>. If the number of linearly independent systems is not finite, then $\mathfrak{X}$ is said to be of <u>infinite dimension</u>. Any set of n linearly independent elements $\{y_i\}$ is an n dimensional Banach space $\mathfrak{X}$ constitutes a (Hamel) <u>basis</u> for $\mathfrak{X}$, and each element $x \in \mathfrak{X}$ admits a unique representation of the form $x = \sum_{i=1}^{n} \alpha_i y_i$, $\alpha_i \in F$.

<u>Definition 12</u>. The <u>unit sphere</u> or <u>ball</u> of a Banach space $\mathfrak{X}$ is the set $\{x;\ x \in \mathfrak{X}, ||x|| \leq 1\}$.

Definition 13. The positive cone $\mathcal{X}^+$ of a Banach space $\mathcal{X}$ is a set of elements of $\mathcal{X}$ with the following properties: (i) If $x, y \in \mathcal{X}^+$ and α, β are nonnegative constants, then $\alpha x + \beta y \in \mathcal{X}^+$ (ii) If $x \in \mathcal{X}^+$ and $-x \in \mathcal{X}^+$, then $x = \theta$.

Some examples of Banach Spaces.

1. The real line. The real line R with the usual arithmetic operations is the simplest example of a Banach space. In this case the norm $\| x \|$ of an element $x \in R$ is simply $|x|$.

2. The space E_n. The space of all n-tuplets of real numbers $x = (x_1, x_2, \ldots, x_n)$ is a Banach space. The norm of an element x is $\|x\| = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$. If the scalar field F is the real number system E_n is called n-dimensional Euclidean space; and if F is the field of complex numbers E_n is called n-dimensional unitary space, or n-dimensional Hilbert space.

3. The space of continuous functions $C(S)$. The space of all bounded continuous scalar-valued functions $f(s)$ on a topological space S is a Banach space. Here the operations of addition and scalar multiplication are as usually defined for functions. The norm of an element f (1) is given by $\|f\| = \sup_{s \in S} | f(s) |$.

A special case in the space $C[a, b]$, where $S \in [a, b]$, with $-\infty \leq a < b \leq \infty$.

4. The sequence spaces ℓ_p, $1 \leq p \leq \infty$. The sequence $x = \{x_1, x_2, \ldots\}$, $x_i \in F$, is said to belong to the sequence space $\ell_p (1 \leq p < \infty)$ if the infinite series $\sum_{i=1}^{\infty} | x_i |^p$, $\leq$, is convergent. For $x, y \in \ell_p$, $\alpha \in F$, we define

$$\alpha x = \{ \alpha x_1, \alpha x_2, \ldots \}, \quad x + y = \{ x_1 + y_1, x_2 + y_2, + \ldots \}$$

the norm of an element in ℓ_p is given by $||x||_p = \left\{ \sum_{i=1}^{\infty} |x_i|^p \right\}^{1/p}$.

The sequence space ℓ_p is the space of all bounded sequences.* In this case we have $||x||_\infty = \sup_i |x_i|$.

5. The space c. The space c is the linear subspace of

$\ell_\infty / \ell_\infty$ consisting of all convergent sequences.

6. The space c_0. The space c_0 is the linear subspace of c consisting of all sequences converging to zero.

7. The Labesgue spaces L_p, $1 \leq p \leq \infty$. The function space $L_p(S, \mathcal{a}, \mu)$, defined for any real number p, $1 \leq p < \infty$ and any positive measure spaces $(S, \mathcal{a}, \mu)$, consists of those μ - measurable functions $x(s)$ on S for which the norm

$$|| x ||_p = \left\{ \int_S |x(s)|^p \mu(ds) \right\}^{1/p}$$

is finite. We remark that the elements of the Lebesgue spaces are not actually functions but equivalance classes of functions, two functions belonging to the same equivalance class if they differ only on a set of μ measure zero. The space $L_\infty(S, \mathcal{a}, \mu)$ is defined for a positive measure space $(S, \mathcal{a}, \mu)$, and consists of all essentially bounded, μ -measurable scalar functions $x(s)$. The norm is

$$||x||_\infty = \text{ess.} \sup_{s \in S} | x(s) | .$$

8. The Hardy spaces H_p, $1 \leq p \leq \infty$. The Hardy spaces H_p, $1 \leq p \leq \infty$, consists of all functions $x(s)$ analytic in the unit circle Δ and such that

$$||x||_p = \sup_{0 \leq r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |x(re^{i\theta})|^p d\theta \right\}^{1/p}$$

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* The space of bounded sequences is often denoted by m.

is finite. H_∞ is the class of all functions $x(s)$ analytic in the unit circle such that $|x(s)|$ is bounded when $s \in \triangle$. In this case $\| x \|_\infty = \sup_{s \in \triangle} |x(s)|$.

9. Hilbert space H. A linear space H over F is called an inner-product space (also pre-Hilbert space) if there is defined for pairs of elements $x, y \in H$ a function on $H \times H$ to F, denoted by (x, y) and called the inner product of x and y, such that (i) $(x, y) = \overline{(y, x)}$, (ii) $(\alpha x + \beta y, z) = \alpha (x, z) + \beta (y, z)$, (iii) $(x, x) \geq 0$, and $(x, x) = 0$ if and only if $x = \theta$. The norm of an element $x \in H$ is given by $\|x\| = (x,x)^{1/2}$. If H is complete in the resulting norm topology, then H is called a Hilbert space. Of the spaces listed above, E_n, ℓ_2, L_2, and H_2 are examples of Hilbert spaces.

10. The space BV(I). The space BV(I) is defined for a given interval I, and consists of all scalar functions x on I which are of bounded variation. If a is the left-end point of I, then $\|x\| = \|x(a+)\| + v(x, I)$, where $v(x, I)$ denotes the total variation of x in I.

The space $BV_0(I)$ is defined for a given interval I, and consists of those functions $x \in BV(I)$ which are normalized by the requirement that (i) x is continuous on the right at each interior point of I, and (ii) $x(a+) = 0$, where a is the left-end point of I. In this case $\|x\| = v(x, I)$.

11. The space of measures rca (S). The space of all regular countably additive scalar-valued measures (set functions) μ defined on the σ - field B of all Bosel sets in a

topological space S is a Banach space. The norm of an element μ is $||\mu|| = v(\mu, S)$, the total variation; that is, for every $B \in \mathcal{B}$, $v(\mu, B) = \operatorname{Sup} \sum_{i=1}^{n} |\mu(B_i)|$, where the supremum is taken over all finite sequences of disjoint sets in $\mathcal{B}$ with $B_i \subseteq B$.

Definition 14. A set A is called a real (complex) Banach algebra if (i) A is a real (complex) linear space, (ii) For any two elements $x, y \in A$ there is defined the operation of multiplication, written xy, satisfying the conditions: $\alpha(xy) = (\alpha x)y$; $(xy)z = x(yz)$; $x(y+z) = xy + xz$; $(y + z)x = yx + zx$, for all $x, y, z, \in A$, and all $\alpha \in F$, (iii) A is a Banach space with norm $||.||$, and (iv) For every $x, y \in A$, $||xy|| \leq ||x|| \; ||y||$.

A is said to be a commutative Banach algebra if for any pair of elements $x, y \in A$ the relation $xy = yz$ is satisfied. The unit or identity element of A, within e, is that element, if it exists, such that $ex = xe = e$ for all $x \in A$.

Definition 15. A partially ordered set L is said to be a lattice if every pair of elements $x, y \in L$ has a least upper bound and a greatest lower bound, denoted by $x \vee y$ and $x \wedge y$, respectively. A lattice which is a Banach space and satisfies the condition $|x| \leq |y|$ implies $||x|| \leq ||y||$ is called a Banach lattice. In the above condition $|x| = x^+ - x^-$, where $x^+ = x \vee 0$ and $x^- = x \wedge 0$.

B. Linear functionals.

Definition 16. A functional x^* on a Banach space $\mathfrak{X}$ is a function from $\mathfrak{X}$ to the scalars. The value of x^* for the element $x \in \mathfrak{X}$ is denoted by $x^*(x)$. A functional $x^*(x)$ is linear if (i) $x^*(x + y) = x^*(x) + x^*(y)$, and (ii) $x^*(\alpha x) = \alpha x^*(x)$, for $x, y \in \mathfrak{X}$, $\alpha \in F$.

Definition 17. A functional x^* is said to be bounded if there exists a real constant $M \geq 0$ such that for all $x \in \mathfrak{X}$ $|x^*(x)| \leq M \|x\|$. The smallest M for which the above holds is called the norm or bound of the functional x^*, and we denote it by $\|x^*\|$.

Theorem 2. A linear functional x^* is bounded if and only if it is continuous.

Definition 18. If x_1^* and x_2^* are two functionals, then $x_1^* + x_2^*$ represents the functional whose value at x is $(x_1^* + x_2^*)(x) = x_1^*(x) + x_2^*(x)$. Similarly, αx^*, $\alpha \in F$, represents the functional whose value at x is $(\alpha x^*)(x) = \bar{\alpha} x^*(x)$ where $\bar{\alpha}$ is the complex conjugate of α.

Theorem 3. The set $\mathfrak{X}^*$ of bounded linear functionals on a Banach space $\mathfrak{X}$ is a Banach space.

Definition 19. $\mathfrak{X}^*$ is called the adjoint (conjugate, or dual) space of the Banach space $\mathfrak{X}$.

Theorem 4. If $\mathfrak{X}^*$ is separable, then $\mathfrak{X}$ is separable.

Theorem 5. (Hahn-Banach Extension Theorem). Let $\mathfrak{X}$ be a real Banach space, and let L be a linear manifold (not necessarily closed) in $\mathfrak{X}$. Let x^* be a bounded linear functional defined over L and let $\|x^*\|_L$ denote the norm of x^*. Then there exists a bounded linear functional y^* defined over $\mathfrak{X}$ such that (i) $x^*(x) = y^*(x)$ for $x \in L$, and (ii) $\|y^*\| = \|x^*\|_L$.

The complex analogue of Theorem 5 is the Bohnenblust-Sobczyk Extension Theorem.

Definition 20. The set of all bounded linear functionals on $\mathfrak{X}^*$ is called the second adjoint space to the Banach space $\mathfrak{X}$, and is denoted by $\mathfrak{X}^{**}$.

Definition 21. The Banach space $\mathfrak{X}$ is said to be reflexive if $\mathfrak{X} = \mathfrak{X}^{**}$. If $\mathfrak{X} \neq \mathfrak{X}^{**}$, then $\mathfrak{X}$ is said to be irreflexive.

C. Topologies and convergence concepts.

Definition 22. The topology induced in a Banach space $\mathfrak{X}$ by the metrix $d(x, y) = ||x - y||$ is called the metric, norm or strong topology of $\mathfrak{X}$.

Definition 23. A sequence $\{x_n\}$ of elements in a Banach space $\mathfrak{X}$ converges strongly, or converges in the strong topology, to an element x if $\lim_{n \to \infty} || x_n - x || = 0$.

Definition 24. Let x_o be a fixed element of a Banach space $\mathfrak{X}$, and let the set $S(x_o)$ be defined as follows:

$$S(x_o) = S(x_o ; x_1^*, \ldots\ldots, x_n^* ; \epsilon)$$
$$= \left\{ x: |x_i^*(x) - x_i^*(x_o)| < \epsilon, i = 1, 2, \ldots . n \right\},$$

$x_1^*, \ldots, x_n^* \in \mathfrak{X}^*$; $\epsilon > 0$. The topology defined by the sets $S(x_o)$, $x_o \in \mathfrak{X}$, is called the weak topology of $\mathfrak{X}$.

Definition 25. A sequence $\{x_n\}$ of elements in a Banach space $\mathfrak{X}$ converges weakly, or converges in the weak topology to the element x if (i) the norms of the elements are uniformly bounded, i.e. $||x_n|| \leq M$, and (ii) $\lim_{n \to \infty} x^*(x_n) = x^*(x)$ for every $x^* \in \mathfrak{X}^*$.

Definition 26. Let x_o^* be a fixed element of $\mathfrak{X}^*$, and let the set $S(x_o^*)$ be defined as follows:

$$S(x_o^*) = S(x_o^* ; x_1, \ldots\ldots, x_n, \epsilon)$$
$$= \left\{ x^*: |x^*(x_i) - x_o^*(x_i)| < \epsilon, i = 1, 2, \ldots, n \right\},$$

$x_1, \ldots, x_n \in \mathfrak{X}$, $\epsilon > 0$. The topology defined by the sets $S(x_o^*)$, $x_o^* \in \mathfrak{X}^*$, is called the weak * topology.

Definition 27. A sequence $\{x_n^*\}$ of linear functionals *converges weakly*, or converges in the *weak * topology*, to the linear functional x^* if (i) the norms $||x_n^*||$ are uniformly bounded, i.e., $||x_n^*|| \leq M$, and (ii) $\lim_{x \to \infty} x_n^*(x) = x^*(x)$ for every $x \in \mathfrak{X}$.

Theorem 6. If a sequence $\{x_n\}$ of elements in a Banach space $\mathfrak{X}$ converges strongly to an element $x \in \mathfrak{X}$, then $\{x_n\}$ also converges weakly to x.

D. Banach space-valued measurable functions.

Let $(S, \mathcal{A}, \mu)$ be a measure space (i.e., S is an
$\mathcal{A}$ is a σ-ring of subsets of S
abstract set, and μ is a σ-finite measure defined on $\mathcal{A}$), $\mathfrak{X}$ an arbitrary complex Banach space, and let $x(s)$ be a function on S to $\mathfrak{X}$.

Definition 28. The function $x(s)$ is said to be: (1) a finitely-valued function if it is constant on each of a finite number of disjoint measurable sets E_i and equal to θ on $S - (\bigcup_i E_i)$, (2) a simple function if it is finitely valued and if $\mu\{s: \|x(s)\| > 0\} < \infty$, (3) a countably-valued function if it assumes at most a countable set of values in $\mathfrak{X}$, assuming each value different from θ on a measurable subset.

Definition 29. A function $x(s)$ is said to be (1) separably-valued if its range is separable, (2) almost separably-valued if there exists a nul set $E_0 \in \mathcal{A}$ such that $x(S - E_0)$ is separable.

Definition 30. A function $x(s)$ is said to be (1) weakly measurable if the scalar-valued functions $x^*(x(s))$ are Lebesgue measurable for each $x^* \in \mathfrak{X}^*$; (2) strongly (or Bochner) measurable if there exists a sequence of countably-valued functions converging almost everywhere to $x(s)$.

Theorem 7. A Banach space-valued function $x(s)$ is strongly measurable if and only if it is weakly measurable and almost separably-valued.

Corollary 1. A function $x(s)$ is strongly measurable if and only if it is the uniform limit almost everywhere of a sequence of countably-valued functions.

Corollary 2. If the Banach space $\mathfrak{X}$ is separable, then strong and weak measurability are equivalent.

Definition 31. A countably-valued function $x(s)$ on S to $\mathfrak{X}$ is Bochner integrable if and only if $||x(s)||$ is Lebesgue integrable.

Theorem 8. A necessary and sufficient condition that a function $x(s)$ on S to $\mathfrak{X}$ be Bochner integrable is that $x(s)$ be strongly measurable and that $\int_S || x(s) || \, d\mu < \infty$.

We will denote by $B(S; \mathfrak{X}; \mu)$ the class of function on S to $\mathfrak{X}$ which are Bochner integrable with respect to μ, and by $B(E_n; \mathfrak{X})$ to be the class of Bochner integrable functions on E_n to $\mathfrak{X}$, where is Lebesgue measure.

Theorem 9. If $x(t) \in B(E_1; \mathfrak{X})$, then

$$\lim_{\xi \to 0} \int_{E_1} || x(t + \xi) - x(t) || \, dt = 0.$$

Theorem 10. If $x(t) \in B(E_1; \mathfrak{X})$ and $f(t)$ is a bounded numerically-valued measurable function, then

$$Y(\xi) = \int_{E_1} f(t) \, x(t + \xi) \, dt$$

is a continuous function of ξ

E. The Laplace transform of Banach space-valued functions.

Let $\mathfrak{X}$ be an arbitrary complex Banach space, x(t) a function on $(0, \infty)$ to $\mathfrak{X}$, and let x(t) be of strong bounded variation on finite interval (0, a). Since a function of strong bounded variation has right-and left-hand limits everywhere and is continuous except for a countable set of discontinuities of the first kind, we can normalize ** x (t) in $(0, \infty)$ by assuming $x(0) = \theta$, and $x(t) = \frac{1}{2}(x(t+) - x(t-))$, $t > 0$.

Definition 32. The integral

$$f(t); \lambda) = \int_0^t e^{-\lambda\xi} \, dx(\xi)$$

exists for finite values of λ (λ complex) and finite positive values of t. If, for a particular λ, $\lim_{t\to\infty} f(t;\lambda)$ exists, we write

$$f(\lambda) = \lim_{t\to\infty} f(t;\lambda) = \int_0^\infty e^{-\lambda\xi} \, dx(\xi)$$

and say that the integral converges for this value of λ. The function $f(\lambda)$ is called the abstract Laplace-Stieltjes transform or x(t), and we write $f(\lambda) = \mathcal{L}\{x(t)\}$. If x(t) is absolutely continuous, and $x(t) = \int_0^t y(\xi)\, d\xi$, where $y(\xi) \in$, then

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* A Banach spaced-valued function x(t) is said to be of strong bounded variation in an interval (a, b) if $\sup \sum_i \| x(a_i) - x(a_{i-1}) \| < \infty$, where all finite partitions of (a, b) are allowed.

** In the classical theory, a function f(x) of bounded variation in an interval (a, b) is said to be normalized in (a, b) if $f(a) = 0$ and $f(x) = \frac{1}{2}(f(x+)) - f(x-))$, $x \in (a, b)$.

$$f(\lambda) = \int_0^\infty e^{-\lambda \xi} \quad y(\xi)\, d\xi \quad ,$$

and $f(\lambda)$ is called the <u>abstract Laplace transform</u> of $y(t)$.

In the classical theory it is known that a function $x(t)$, if it is normalized, is uniquely determined by its Laplace-Stieltjes transform $f(\lambda)$. This uniqueness result is valid for the Banach space-valued functions, and we have

<u>Theorem 11</u>. There cannot exist two different normalized representations of $f(\lambda)$, in terms of abstract Laplace-Stieltjes transforms.

Given the abstract Laplace-Stieltjes or Laplace transform $f(\lambda)$ of a function $x(t)$, it is of great interest to consider the inverse problem; namely, how to determine $x(t)$ given $f(\lambda)$. This leads to various inversion theorems and formulas which give $x(t)$ in terms of $f(\lambda)$.

<u>Theorem 12*</u>. Let $f(\lambda)$, as defined earlier, be convergent for $R(\lambda) > \sigma_0$, and let $\gamma > \max(0, \sigma_0)$, and put

$$x(t;\, w) = \frac{1}{2\pi i} \int_{\gamma - iw}^{\gamma + iw} e^{\lambda t} f(\lambda) \frac{d\lambda}{\lambda} .$$

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* In this theorem σ_0 denotes the abscissa of ordinary convergence; that is $\sigma_0 = \lim_{t \to \infty} \sup \frac{1}{t} \log \|x(\infty) - x(t)\|$, where $x(\infty) = \lim_{t \to \infty} x(t)$ or θ according to whether the limit exists or not.

Then

$$\lim_{w \to \infty} x\,(t;\,w) = x(t)\,, \quad \text{for} \quad t > 0$$

$$= 1/2\; x\,(0+), \quad \text{for } t = 0$$

$$= \theta \qquad , \quad \text{for } t < 0$$

the limit existing uniformly with respect to t in any finite interval of continuity of $x(t)$.

Theorem 13*. Let $x(t)$ be a function on $(0, u)$ to $\mathfrak{X}$ for every finite u, i.e., $x(t) \in B\,((0, u);\, \mathfrak{X})$, and let the integral

$$f\,(\lambda) = \int_0^\infty e^{-\lambda t}\; x\,(t)\,dt$$

be absolutely convergent for $\mathcal{R}\,(\lambda) > \sigma_a$. Let $\gamma >$ max $(0, \sigma_a)$, and put

$$\bar{x}\,(t;\,u) = \frac{1}{2\pi} \int_{-u}^{u} (1 - \frac{\tau}{u})\; e^{(\gamma + i\tau)t}\; f(\gamma + i\tau)\,d\tau$$

Then

$$\lim_{u \to \infty} \bar{x}\;(t;\,u) = x(t)\,, \text{ for almost all } t > 0$$

$= 1/2\; (x\,(t+) + x(t-))$ whenever this expression has a meaning

$$= \theta\,, \quad \text{for} \quad t < 0\,,$$

the limit existing uniformly with respect to t in any finite interval of continuity of $x(t)$.

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* In this theorem σ_a denotes the abscissa of absolute convergence; that is $\sigma_a = \lim_{t \to \infty} \sup \frac{1}{t} \log |\, x_*\,(\infty) - x_*\,(t)\,|$, where $x_*\,(t)$ denotes the strong variation of $x(t)$ in $(0, u)$.

1.3 Linear Operators on Banach spaces

A. Introduction. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces of the same scalar type, that is, the scalar fields over which $\mathcal{X}$ and $\mathcal{Y}$ are defined are both real or both complex. We consider mappings, or transformations, T from $\mathcal{X}$ into $\mathcal{Y}$. We can denote such a mapping in several ways; for example

(1) $T: \mathcal{X} \to \mathcal{Y}$

(2) $\mathcal{X} \overset{T}{\to} \mathcal{Y}$

(3) $T: x \to y, \quad x \in \mathcal{X}, \quad y \in \mathcal{Y}$

(4) $Tx = y \quad , \quad x \in \mathcal{X}, \quad y \in \mathcal{Y}$

We assume that the domain of T, written $\infty(T)$, is a subset of $\mathcal{X}$, and that the range of T, written $\mathcal{R}(T)$, is a subset of $\mathcal{Y}$. We also assume that $\infty(T)$ is a linear subspace of $\mathcal{X}$.

B. Bounded linear transformations: General properties.

Definition 33. A transformation T is said to be linear if it is (1) additive, i.e., $T(x_1 + x_2) = Tx_1 + Tx_2$. $x_1, x_2 \in \mathfrak{X}$ and (2) homogeneous, i.e., $T(ox) = oTx$, $x \in \mathfrak{X}$, $\quad \in F$.

Definition 34. A transformation T on one Banach space to another is said to be bounded if it takes bounded sets into bounded sets. If T is additive and bounded

$$||T|| = \sup_{||x|| \leqq 1} ||Tx||$$

is called the bound or norm of T.

Definition 35. A transformation T is continuous at $x = x_0$ if for every set E the condition $x_0 \in \overline{E}$ (the closure of E) implies that $Tx_0 \in \overline{T(E)}$. T is said to be continuous if it is continuous at all elements.

Theorem 14. An additive transformation is continuous everywhere if it is continuous at a single element.

Theorem 15. An additive transformation T on one Banach space to another is continuous if and only if it is bounded. In this case

$$||Tx|| \leqq ||T|| \; ||x||,$$

for all $x \in \mathfrak{X}$.

Theorem 16. (Uniform Boundedness Theorem). Let $\{T_i\}$ be a family of bounded additive transformations on one normed space $\mathfrak{X}$ to another. If $\sup_i ||T_i x|| < \infty$ for each x belonging to a given subset $X \subset \mathfrak{X}$ of second category, then $\sup_i ||T_i|| < \infty$.

Definition 36. A one-to-one continuous linear transformation of one Banach space onto another is called a homeomorphism.

Definition 37. An isomorphism between two Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$ is a one-to-one continuous linear transformation T on $\mathfrak{X}$ to $\mathfrak{Y}$ with $T(\mathfrak{X}) = \mathfrak{Y}$. When such an isomorphism exists, the Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$ are called equivalent or isomorphic. An isomorphism T such that $\| Tx \| = \| x \|$ is called an isometric isomorphism; and the spaces are said to be isometrically equivalent.

Definition 38. Let $\mathfrak{X} \times \mathfrak{Y}$ denote the product of the Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$. The graph of a linear transformation T is the set of pairs of elements $\{ (x, Tx); \ x \in \infty(T) \} \subset \mathfrak{X} \times \mathfrak{Y}$.

Definition 39. A linear transformation T is said to be closed if its graph is a closed subset of the product space $\mathfrak{X} \times \mathfrak{Y}$. That is to say, T is closed if whenever $x_n \to x$, $\{x_n\} \subset \infty(T)$, and $\lim_{n \to \infty} y_n = \lim_{n \to \infty} Tx_n = y$, then $x \in \infty(T)$ and $y = Tx$.

Definition 40. A linear transformation on $\infty(T) \subset \mathfrak{X}$ to $\mathfrak{Y}$ is said to be bounded if $\|Tx\| \leqq M \| x \|$, where $M \geqq 0$, for all $x \in \infty(T)$. The norm of T is $\|T\| = \sup_{\|x\| \leqq 1} \| Tx \|$, where $x \in \infty(T)$.

Definition 41. Let T_1 and T_2 be linear transformations on $\mathfrak{X}$ to $\mathfrak{Y}$ with domains ∞_1 and ∞_2, respectively. If (i) $\infty_1 \subset \infty_2$, and (ii) $T_1 x = T_2 x$ for $x \in \infty_1$, then T_2 is called an extension of T_1; and we write $T_1 \subset T_2$.

Theorem 17. If T is a bounded linear transformation, with $\mathcal{D}(T) \subset \mathcal{X}$, range $\mathcal{Y}$, and norm $\|T\|$, then T has a unique bounded linear extension $\widetilde{T}$ on $\overline{\mathcal{D}(T)}$ with $\|\widetilde{T}\| = \|T\|$.

Definition 42. If T, T_1, and T_2 are bounded linear transformations, then $T_1 + T_2$ and αT, α F, are defined for all $x \in \mathcal{X}$ by

$$(T_1 + T_2)\,x = T_1 x + T_2 x\ , \ \mathcal{D}(T_1 + T_2) = \mathcal{D}(T_1) \cap \mathcal{D}(T_2)$$
$$(\alpha T)\,x = \alpha Tx.$$

The null transformation $\mathcal{O}$ on $\mathcal{X}$ to $\mathcal{Y}$ is defined by

$$\mathcal{O}\,x = \theta\ .$$

for all $x \in \mathcal{X}$.

Theorem 18. The following relation holds for the norms of the bounded linear transformations T_1, T_2 and $T = T_1 + T_2$:

$$\|T\| \leq \|T_1\| + \|T_2\|.$$

We denote by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ the class of all bounded linear transformations with domain $\mathcal{X}$ and range in $\mathcal{Y}$.

Theorem 19. $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is a Banach space, the norm of an element T being given by $\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$, $x \in \mathcal{D}(T)$.

Theorem 20. (Banach-Steinhaus). Let $\{T_n\}$ be a sequence of bounded linear transformations in $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that (i) the T_n are uniformly bounded, i.e. $\|T_n\| \leqq M$ for all n, and (ii) $\lim_{n \to \infty} T_n x$ exists for every x in a set E which is dense in a sphere S. Then $\lim_{n \to \infty} T_n x = Tx$ dxists for all $x \in \mathcal{X}$, where T is a bounded linear transformation, and $\|T\| \leqq \lim_{n \to \infty} \inf \|T_n\|$.

C. The inverse transformation.

Definition 43. A bounded linear transformation T on $\mathcal{X}$ to $\mathcal{Y}$ is said to have an inverse, denoted by T^{-1}, if for every $y \in \mathcal{Y}$ the equation $Tx = y$ has a unique solution.

Theorem 21. The transformation T^{-1} which is the inverse of a bounded linear transformation T is also linear.

Theorem 22. If T is a bounded linear transformation whose inverse T^{-1} exists, then T^{-1} is bounded.

Theorem 23. If a closed transformation T has an inverse T^{-1}, then T^{-1} is also closed.

Theorem 24. If T and T^{-1} are bounded linear transformations, and if $\infty(T)$ is closed, then $\mathcal{R}(T)$ is also closed.

D. <u>The adjoint transformation.</u>

<u>Definition 44</u>. Let T be a linear transformation with $\infty(T)$ dense in $\mathcal{X}$ and with range $\mathcal{Y}$, and let $\mathcal{X}^*$ and $\mathcal{Y}^*$ denote the adjoint spaces of $\mathcal{X}$ and $\mathcal{Y}$, respectively. The adjoint transformation T^* of T is defined as follows: $\infty(T^*)$ is the set of all functionals $Y^* \in \mathcal{Y}^*$ for which there exists an $x^* \in \mathcal{X}^*$ such that $y^*(Tx) = x^*(x)$ for all x $\infty(T)$. In this case we define $T^* y^* = x^*$.

<u>Theorem 25</u>. Let T be a linear transformation with $\infty(T)$ dense in $\mathcal{X}$ and range $\mathcal{Y}$. Then T^* is a closed linear transformation with $\infty(T^*) \subset \mathcal{Y}^*$ and range $\mathcal{X}^*$. If, in addition, T is bounded, then $T^* \in \mathcal{B}(\mathcal{Y}^*, \mathcal{X}^*)$ and $||T^*|| = ||T||$.

<u>Theorem 26</u>. If T be a linear transformation with $\overline{\infty(T)} = \mathcal{X}$, then $(T^*)^{-1}$ exists if and only if $\overline{\mathcal{R}(T)} = \mathcal{Y}$.

<u>Theorem 27</u>. Let T be a linear transformation with an inverse T^{-1} and such that $\overline{\infty(T)} = \mathcal{X}$ and $\overline{\mathcal{R}(T)} = \mathcal{Y}$. Then $(T^*)^{-1} = (T^{-1})^*$; and T^{-1} is bounded if and only if $(T^*)^{-1}$ is bounded on $\mathcal{X}^*$.

E. Closed linear transformations.

Theorem 28. If t is a closed linear transformation whose range $\mathcal{R}(T)$ is of second category in $\mathcal{Y}$, then (i) $\mathcal{R}(T) = \mathcal{Y}$ (ii) there is a positive constant M such that to every $y \in \mathcal{Y}$ there is an x $\infty(T)$ with $y = Tx$ and $||x|| \leq M\, ||y||$; and (iii) if T^{-1} exists then it is bounded.

Theorem 29. (Closed Graph Theorem). If T is a linear transformation of $\mathcal{X}$ into $\mathcal{Y}$, then T is continuous if and only if its graph is closed.

Another form of the closed Graph Theorem is as follows: If T is a closed linear transformation with $\infty(T)$ of second category in $\mathcal{X}$, then $\infty(T) = \mathcal{X}$ and T is bounded.

Throughout this Notes we will denote by $C(\mathcal{X}, \mathcal{Y})$ the class of all closed linear transformations of $\mathcal{X}$ into $\mathcal{Y}$.

F. Compact linear transformations.

Definition 45. A linear transformation T is said to be compact if every bounded infinite set of elements $x \in \mathcal{H}$ contains a sequence x_n such that $y = \lim_{n \to \infty} Tx_n$ exists, and $y \in \mathcal{Y}$

Theorem 30. Every compact linear transformation is bounded.

Theorem 31. Every linear transformation whose domain is a Banach space of finite dimension is compact.

Theorem 32. If T is a compact linear transformation then the adjoint transformation T^* is also compact.

Theorem 33. If T_1 and T_2 are compact linear transformations, and α, β are arbitrary complex numbers, then $\alpha T_1 + \beta T_2$ is compact. In particular, the null transformation (4) is compact.

G. The Banach algebra of endormorphisms.

1. Operators and endomorphisms.

Definition 46. A linear transformation on a Banach space $\mathfrak{X}$ to itself is called an operator on $\mathfrak{X}$. A bounded operator on $\mathfrak{X}$ is called an endomorphism of $\mathfrak{X}$.

For a given Banach space $\mathfrak{X}$ we denote the set of all endomorphisms on $\mathfrak{X}$ by $\mathcal{E}(\mathfrak{X})$, rather than by $\mathcal{B}(\mathfrak{X}, \mathfrak{X})$. By Theorem 19, $\mathcal{E}(\mathfrak{X})$ is a Banach space, hence we have the operations of addition and scalar multiplication defined for elements of $\mathcal{E}(\mathfrak{X})$. For endomorphisms the operation of multiplication is also defined; that is, if $T_1, T_2 \in \mathcal{E}(\mathfrak{X})$, and $x \in \mathfrak{X}$, then $(T_1T_2)x = T_1(T_2 x)$, i.e., multiplication is defined by composition. We also have $||T_1T_2|| \leqq ||T_1|| \, ||T_2||$. If I denotes the identity operator, we have $||I|| = 1$.

Theorem 34. The set of all endomorphisms $\mathcal{E}(\mathfrak{X})$ of a Banach space $\mathfrak{X}$ is a Banach algebra with unit element the identity operator. $\mathcal{E}(\mathfrak{X})$ is noncommutative if the dimension of $\mathfrak{X}$ is greater than one.

Theorem 33 holds for compact endomorphisms of $\mathfrak{X}$; but since multiplication is defined for elements of $\mathcal{E}(\mathfrak{X})$ we have the following result.

Theorem 35. If $T_1, T_2 \in \mathcal{E}(\mathfrak{X})$, with T_2 compact, then T_1, T_2 and $T_2 T_1$ are compact. If $\mathfrak{X}$ is of infinite dimension, then the identity operator I is not compact.

Definition 47. Let M denote a closed linear manifold in $\mathcal{H}$. We assume there exists a closed linear manifold $N \subset \mathcal{H}$ such that $\mathcal{H}$ is the direct sum of M and N. The manifold N is said to be complementary to M, and we denote by (M, N) the complementary pair of manifolds M and N.

Definition 48. An endomorphism P of $\mathcal{H}$ satisfying $P^2 = P$ is called a projection operator. We say P projects $\mathcal{H}$ onto $\mathcal{R}(P)$.

Theorem 36. If (M, N) is a complementary pair of manifolds in $\mathcal{H}$, then there exists a projection operator which projects $\mathcal{H}$ onto M (or N).

Definition 49. If T is an endomorphism of $\mathcal{H}$, the null-space of T is the set $(x: Tx = \theta)$. The null-space is a closed linear manifold in $\mathcal{H}$.

Definition 50. The range of an endomorphism T is the set $(y: y = Tx, x \in \mathcal{H})$. The range is a linear manifold in $\mathcal{H}$; however it may not be closed.

Definition 51. The set $(x: Tx = x, x \in \mathcal{H})$ is called the set of fixed points of the operator T.

Definition 52. A linear bounded operator T is said to be a contraction operator if $\|T\| \leqq 1$, and a proper contraction operator if $\|T\| < 1$.

Definition 53. An endomorphism $T \in \mathcal{E}(\mathcal{H})$ is said to be positive if it maps $\mathcal{H}^+$ into itself.

Definition 54. A positive contraction endomorphism $T \in \mathcal{E}(\mathfrak{X})$ with the property that $||Tx|| = ||x||$ for all $x \in \mathfrak{X}^+$ is called a transition operator.

2. Operator topologies and convergence concepts.

In Section 1.2 C we considered the weak and strong topologies in Banach space and the convergence of sequences of elements in these topologies. We now consider analogous notions for the Banach algebra of endomorphisms $\mathcal{E}(\mathfrak{X})$

Definition 55. The uniform operator topology in $\mathcal{E}(\mathfrak{X})$ is the metric topology of $\mathcal{E}(\mathfrak{X})$ induced by its norm $||T|| = \sup_{||x|| \leq 1} ||Tx||$, $T \in \mathcal{E}(\mathfrak{X})$, $x \in \mathfrak{X}$.

Definition 56. The strong operator topology in $\mathcal{E}(\mathfrak{X})$ is the topology defined by the set of neighborhoods

$$N(T) = N(T;\ E,\ \epsilon) = $$
$$= (S : S \in \mathcal{E}(\mathfrak{X}),\ ||(T - S)x|| < \epsilon,\ x \in E),$$

where E is an arbitrary finite subset of $\mathfrak{X}$, and $\epsilon > 0$ is arbitrary.

Definition 57. The weak operator topology in $\mathcal{E}(\mathfrak{X})$ is the topology defined by the set of neighborhoods

$$N(T) = N(T;\ E,\ F,\ \epsilon) =$$
$$= \{S : S \in \mathcal{E}(\mathfrak{X}),\ ||x^*((T - S)x)|| < \epsilon,$$
$$x \in A,\ x^* \in B\},$$

where A and B are arbitrary finite subsets of $\mathcal{X}$ and $\mathcal{X}^*$, respectively, and $\epsilon > 0$ is arbitrary.

<u>Definition 58</u>. Let (T_n) be a sequence of endomorphisms in $\mathcal{E}(\mathcal{X})$, and let $T \in \mathcal{E}(\mathcal{X})$. If $\lim_{n \to \infty} ||(T_n - T)x|| = 0$ for every $x \in \mathcal{X}$, then T_n is said to <u>converge strongly</u> to T, or <u>converge in the strong operator topology</u> of $\mathcal{E}(\mathcal{X})$

<u>Definition 59</u>. Let (T_n) be a sequence of endomorphisms in $\mathcal{E}(\mathcal{X})$, and let $T \in \mathcal{E}(\mathcal{X})$. If $\lim_{n \to \infty} ||x^*((T_n - T)x)||=0$ for every $x \in \mathcal{X}$, $x^* \in \mathcal{X}^*$, then T_n is said to <u>converge weakly</u> to T, or <u>converge in the weak operator topology</u> of $\mathcal{E}(\mathcal{X})$.

The following relations obtain for convergence in the above operator topologies; uniform convergence implies strong convergence; and strong convergence implies weak convergence.

3. <u>Operator-valued measurable functions</u>.

As in Section 1.2 D, let $(S, \mathcal{a}, \mu)$ denote a measure space, and let $T(s)$ be a function on S to $\mathcal{E}(\mathcal{X})$

Definition 60. The operator-valued function $T(s)$ is said to be (1) uniformly measurable if there exists a sequence of countably-valued operator functions in $\mathfrak{E}(\mathfrak{X})$ converging almost everywhere to $T(s)$ in the uniform operator topology of $\mathfrak{E}(\mathfrak{X})$; (2) strongly measurable if the Banach space-valued function $T(s)x$ is strongly measurable in the sense of Definition 30 for all $x \in \mathfrak{X}$; (3) weakly measurable if $x^* (T(s)x)$ is Lebesgue measurable for all $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$.

Theorem 37. A necessary and sufficient condition that $T(s)$ be (1) strongly measurable is that $T(s)$ be weakly measurable and that $T(s)x$ be almost separably-valued in $\mathfrak{X}$ for each $x \in \mathfrak{X}$; (2) uniformly measurable is that $T(s)$ be weakly measurable and almost separably-valued in $\mathfrak{E}(\mathfrak{X})$.

Theorem 38. If $T(s)x \in B(S, \mathfrak{X}, \mu)$ for each $x \in \mathfrak{X}$, then

$$Ux = \int_S T(s)\, x\, d\mu$$

defines an endomorphism of $\mathfrak{X}$.

Theorem 39. Let $T(s)$ be an everywhere defined uniformly measurable function on $(0, \infty)$ to $\mathfrak{E}(\mathfrak{X})$ such that for S_1, $S_2 \in (0, \infty)$

$$T(S_1 + S_2) = T(S_1)\, T(S_2).$$

Then $T(s)$ is continuous in the uniform operator topology for all positive values of S.

1.4 Hilbert Space

A. Definitions and general properties.

Definition 61. A complex linear space X is called an inner-product space (or pre-Hilbert space) if there is defined on $X \times X$ a complex-valued function (x_1, x_2), called the inner-product of x_1 and x_2, with the following properties:

(i) $(x_1 + x_2, x_3) = (x_1, x_3) + (x_2, x_3)$

(ii) $(x_1, x_2) = \overline{(x_2, x_1)}$

(iii) $(\alpha x_1, x_2) = \alpha (x_1, x_2)$

(iv) $(x, x) \geqq 0$, and $(x, x) \neq 0$ if $x \neq \theta$.

A real linear space X is called an inner-product space if there is defined on $X \times X$ a real-valued function (x_1, x_2) with properties (i)-(iv), as given above, except that (ii) becomes $(x_1, x_2) = (x_2, x_1)$.

Theorem 40. If X is an inner-product space, $(x, x)^{1/2}$ has the properties of a norm. Hence, for $x \in X$ we write $||x|| = (x, x)^{1/2}$.

Definition 62. A complete inner-product space (or pre-Hilbert space) is called a Hilbert space.

Theorem 41. Every finite dimensional pre-Hilbert space is complete, hence is a Hilbert space.

Theorem 42. In any Hilbert space:

(i) If $x_n \to x$ and $y_n \to y$, then $(x_n, y_n) \to (x, y)$.

(ii) If (x_n) and (y_n) are Cauchy sequences, then $\{(x_n, y_n)\}$ is a Cauchy sequence of scalars.

Definition 63. If x and y are elements of a Hilbert space, x and y are said to be orthogonal elements, written $x \perp y$ in case $(x, y) = 0$.

Definition 64. A set S of elements in a Hilbert space is said to be an orthogonal set in case $x \perp y$ whenever x and y are distinct elements of S. A sequence (x_n) (finite or infinite) is called an orthogonal sequence if $x_j \perp x_k$ whenever $j \neq k$.

Definition 65. A set S of elements in a Hilbert space is said to be orthonormal in case (i) S is orthogonal, and (ii) $||x|| = 1$ for every $x \in S$. A sequence (x_n) (finite or infinite) is called an orthonormal sequence if (i) $x_j \perp x_k$ whenever $j \neq k$, and (ii) $||x_k|| = 1$ for all k.

Definition 66. A set S of elements in a Hilbert space H is said to be total in case the only element $y \in H$ which is orthogonal to every $x \in S$ in the element $y = \theta$. A sequence (x_n) (finite or infinite) is called a total sequence if $y \perp x_k$ for all k, then $y = \theta$.

Definition 67. A Hilbert space is said to be separable if it possesses a total sequence (finite or infinite).

Theorem 43. The following conditions on a Hilbert space H are equivalent:

(1) H is separable

(2) H has an orthonormal basis.

Definition 68. A Hilbert space H_1 is said to be isomorphic to a Hilbert space H_2 if there exists a mapping T (called a Hilbert space isomorphism) which assigns to each element $x \in H_1$ one and only one element $Tx \in H_2$ in such a way that the following conditions hold:

(i) If $x_1, x_2 \in H_1$, with $x_1 = x_2$, then $Tx_1 = Tx_2$.

(ii) If y is any element in H_2, then there is an element $x \in H_1$ such that $y = Tx$.

(iii) $T(x_1 + x_2) = Tx_1 + Tx_2$ for all $x_1, x_2 \in H_1$.

(iv) $T(\lambda x) = \lambda Tx$, for all $x \in H_1$, and all scalars λ

(v) $(Tx_1, Tx_2) = (x_1, x_2)$, for all $x_1, x_2 \in H_1$.

Theorem 44. Let H be a separable Hilbert space. (1) If H is finite-dimensional, it is isomorphic to E_n. (2) If H is infinite-dimensional it is isomorphic to ℓ_2.

Theorem 45. Let H be a Hilbert space, and let $x^*(x)$ be a bounded linear functional on H. Then there exists a uniquely determined element $y \in H$ such that $x^*(x) = (x, y)$ for every $x \in H$. The norm of $x^*(x)$ is $||y||$.

Theorem 46. H^*, the set of all bounded linear functionals on a Hilbert space H, is also a Hilbert space.

Theorem 47. A Hilbert space is reflexive.

Theorem 48. If in a Hilbert space a sequence (x_n) converges weakly to x and $\lim_{n \to \infty} \| x_n \| = \| x \|$, then (x_n) converges strongly to x.

B. Operators in Hilbert space.

1. Types of operators in Hilbert space and their properties.

Let H be a separable Hilbert space, and let (φ_n) be a complete orthonormal sequence in H. If T is an endomorphism of H, we define a finite or infinite matrix $U = (u_{ij})$ as follows:

$$u_{ij} = (T\varphi_j, \varphi_i).$$

The elements u_{ij} of U are complex numbers, and U is finite if and only if H is finite-dimensional. U is the matrix representation of the endormorphism T.

Theorem 49 If (φ_n) is a complete orthonormal sequence in a separable Hilbert space H, then $u_{ij} = (T\varphi_j, \varphi_i)$ defines a one-to-one correspondence between the algebra $\mathcal{E}$(H) of all endomrphisms T of H and the set of all matrices U, the elements u_{ij} of which satisfy the condition $\sum_j \left| \sum_i u_{ji}\alpha_i \right|^2 \leqq M \sum_i |\alpha_i|^2$ for a positive constant M and (depending on U) and all sequences $[\alpha_i] \in \ell_2$.

Let $\mathcal{D}$(T) denote the domain of an operator T in a Hilbert space H. We assume $\mathcal{D}$ (T) is dense in H. Given such a T, consider the pairs (y, z) elements in H such that

$$(Tx, y) = (x, z)$$

for all $x \in \mathcal{D}$ (T). Now, let Y denote the set of all first elements y in the pairs (y, z) satisfying the above relation and let T^* be the operator defined on $\mathcal{D}(T^*)$ by $T^*y = z$

<u>Definition 69.</u> The operator T^* defined by

$$(Tx, y) = (x, T^* y)$$

for $x \in \mathcal{D}(T)$, $y \in \mathcal{D}(T^*)$ is called the <u>Hilbert space adjoint</u> of T.

<u>Definition 70.</u> Let T be a linear operator with domain $\mathcal{D}(T)$ dense in a Hilbert space H. Let T^* denote the adjoint of T, and let $\mathcal{D}(T^*)$ denote its domain. T is said to be <u>self-adjoint if</u>

(i) $\mathcal{D}(T) = \mathcal{D}(T^*)$

(ii) $Tx = T^* x$ for all $x \in \mathcal{D}(T)$.

<u>Theorem 50</u> An endomorphism T is self-adjoint if and only if (Tx, x) is real for all $x \in H$.

<u>Definition 71.</u> An operator T is called <u>symmetric</u> if $(Tx, y) = (x, Ty)$ for all $x, y \in \mathcal{D}(T)$.

<u>Theorem 51.</u> If T is a self-adjoint linear operator with $\mathcal{D}(T) = H$, then T is bounded.

<u>Definition 72.</u> An endomorphism T is said to be <u>normal</u> if $T^*T = TT^*$.

<u>Definition 73.</u> An endomorphism T is said to be <u>unitary</u> if $T^*T = TT^* = I$, that is, whenever T^{-1} exists with $\mathcal{D}(T^{-1}) = H$ and satisfies $T^{-1} = T^*$.

<u>Theorem 52.</u> In Hilbert space, the following conditions on an operator are equivalent:

(i) T is isometric.

(ii) $T^* T = I$.

(iii) $(Tx_1, Tx_2) = (x_1, x_2)$ for all x_1, x_2 H.

<u>Theorem 53.</u> In Hilbert space, the following conditions on an operator T are equivalent:

(i) T is unitary.

(ii) T^* is unitary.

(iii) T and T^* are isometric.

(iv) T is one-to-one, and $T^{-1} = T^*$.

<u>Definition 74.</u> If a bounded self-adjoint operator T on a Hilbert space H is such that $(Tx, x) \geqq 0$ for all $x \in H$, then T is called a <u>positive</u> operator.

<u>Definition 75.</u> An endomrphism T in Hilbert space is called a <u>projection</u> if $T^2 = T$ and $T^* = T$.

<u>Definition 76.</u> A family of projection operators $[E(\lambda), -\infty < \lambda < \infty]$ is called a <u>resolution of the identity if</u> and only if

(i) $\lambda_1 < \lambda_2$ implies $E(\lambda_1) < E(\lambda_2)$;

(ii) $\lambda_n \to -\infty$ implies $E(\lambda_n)$ H strongly;

(iii) whenever $\{\lambda_n\}$ is an increasing sequence such that $\lambda_n \to \lambda$ then $E(\lambda_n) \to E(\lambda)$ strongly.

The largest number m such that $E(m) = H$ and $E(m + \epsilon) \neq H$ for $\xi > 0$ is called the lower bound of the resolution; and the number M such that $E(m + \varepsilon) = I$ and $E(M - \varepsilon) \neq I$ for all $\varepsilon > 0$ is called the upper bound of the resolution.

Theorem 54 Let $E(\lambda)$ be a bounded resolution of the identity with lower bound m and upper bound M. The integral

$$T = \int_m^M \lambda \, dE(\lambda)$$

exists in the uniform operator topology and represents a self-adjoint operator. T commutes with all operators S which commute with $E(\lambda)$ for every value of λ. The norm of T is given by $\| T \| \leqq \max(|m|, |M|)$, and $mI \leqq T \leqq MI$.

Theorem 55. Let $E(\lambda)$ be a bounded resolution of the identity for which $E(0) = H$ and $E(2\pi + 0) = I$. Then the integral

$$T = \int_0^{2\pi} e^{i\lambda} dE(\lambda)$$

exists in the uniform operator topology, and represents a unitary operator.

C. Operator algebras in Hilbert space.

Let $\mathcal{E}(H)$ denote the algebra of endomorphisms in a Hilbert space H. A subalgebra $\mathcal{E}_0(H)$ of $\mathcal{E}(H)$ is said to be self-adjoint if it is closed with respect to the adjoint operation, that is for every $T \in \mathcal{E}_0(H)$, $T^* \in \mathcal{E}_0(H)$.

<u>Definition 73</u>. A self-adjoint subalogebra of $\mathcal{E}(H)$ is called a C^* -<u>algebra</u> if it is closed in the uniform operator topology, and is called W^* -<u>algebra</u> if it is closed in the weak operator topology. W^* -algebras are also called <u>Mings of operators</u> or <u>von Neumann algebras</u>.

5. Spectral Theory

A. Introduction.

In this section we consider some results from the spectral theory of linear operators; in particular, we consider the following topics: (1) resolvent and spectrum of a linear operator, (2) operational calculus; (3) spectral theory of linear operators in Hilbert space, and (4) spectral theory of positive operators. Throughout this section, unless otherwise stated, we will be concerned with a linear operator T, not necessarily bounded, whose domain $\mathcal{D}$ (T) and range $\mathcal{R}$ (T) lie in a complex Banach space $\mathfrak{X}$.

B. Resolvent and spectrum of a linear operator.

1. Generál definitions and results. Associated with every linear operator T is linear operator, also defined on $\mathcal{D}(T)$.

$$T(\lambda) = (\lambda I - T),$$

where λ is a scalar and I is the identity operator. Quite often we shall suppress I and simply write $T(\lambda) = (\lambda - T)$

Definition 78. The values of λ for which $T(\lambda)$ has a bounded inverse $T^{-1}(\lambda) = (\lambda - T)^{-1} = R(\lambda ; T)$ with domain dense in $\mathfrak{X}$ form the resolvent set $\rho(T)$ of T. The set of all these operators $R(\lambda ; T)$ is called the resolvent of T. All values of λ not in $\rho(T)$ comprise the set of called the spectrum $\sigma(T)$ of T.

Theorem 56. The resolvent set $\rho(T)$ is open. The function $R(\lambda ; T)$ is analytic in $\rho(T)$.

Theorem 57 The closed set $\sigma(T)$ is bounded; and $\sigma(T)$ is non-void if T is bounded.

Definition 79. The values of λ for which $T(\lambda)$ has an unbounded inverse with domain dense in $\mathfrak{X}$ form the continuous spectrum $C\sigma(T)$ of T. The values of λ for which $T(\lambda)$ has an inverse with domain not dense in $\mathfrak{X}$ form the residual spectrum $R\sigma(T)$ of T. The values of λ for which $T(\lambda)$ has no inverse form the point spectrum $P\sigma(T)$

of T. The union of the three sets $C\sigma(T)$, $R\sigma(T)$ and $P\sigma(T)$ is the spectrum $\sigma(T)$ of T.

Theorem 58 No two of the four sets $\rho(T)$, $C\sigma(T)$ $R\sigma(T)$, and $P\sigma(T)$ have common points, and their union is the whole complex λ-plane.

Definition 80 The extended spectrum $\sigma_e(T)$ of a linear operator T is a point set in the extended complex plane whose elements are the singular points of $R(\lambda; T)$. If $T \in \mathcal{E}(\mathfrak{X})$, then $\sigma_e(T) = \sigma(T)$. If $T \notin \mathcal{E}(\mathfrak{X})$ then $\sigma_e(T) = \sigma(T) \cup [\infty]$.

Theorem 59 $\sigma_e(T)$ is nonvoid.

Definition 81 A subset of $\sigma_e(T)$ is called a spectral set of the operator T if it is both open and closed in the relative topology of $\sigma_e(T)$ as a subset of the extended complex plane.

Theorem 60 The equation $(T - \lambda_0)x = \Theta$ has a solution $x_0 \neq \Theta$ if and only if $\lambda_0 \in P\sigma(T)$.

Definition 82 If $\lambda_0 \in P\sigma(T)$, then λ is called a characteristic value of T; and the solutions $x \neq \Theta$ of the equation $(T - \lambda_0)x = \Theta$ are called the characteristic functions of T, associated with λ_0. The set of all characteristic functions is a linear manifold $M(\lambda_0, T)$ in $\mathfrak{X}$, and its closure $\bar{M}(\lambda_0, T)$ is called the characteristic subspace of λ_0.

We remark, that, whenever T is bounded $M(\lambda_o, T) = \bar{M}(\lambda_o, T)$.

Definition 33. A complex number λ_o is said to have index ν (ν a positive integer, or zero) with respect to an operator T in case $T^{\nu+1}(\lambda_o)x_o = \theta$ implies $T^{\nu}(\lambda_o)x = \theta$ and there is an x_o such that $T^{\nu}(\lambda_o)x_o = \theta$ and $T^{\nu-1}(\lambda_o)x_o \neq \theta$. If $\nu = 0$, this is taken to mean that $T(\lambda_o)$ has an inverse. λ_o is said to be of infinite index if no such integer ν exists.

In section 3B we defined the extension of an operator (Definition 41). It is of interest to know what happens to the spectrum of an operator T under extensions of T. Firstly, we observe that if T_2 is an extension with domain $\mathcal{D}(T_2)$ of an operator T_1, then $T_2(\lambda) = \lambda I - T_2$ is an extension of $T_1(\lambda)$ with domain $\mathcal{D}(T_2(\lambda)) \supset \mathcal{D}(T_1(\lambda))$ and range $\mathcal{R}(T_2(\lambda)) \supset \mathcal{R}(T_1(\lambda))$. If $T_2(\lambda)$ admits an inverse, so does $T_1(\lambda)$ and $T_2^{-1}(\lambda)$ is an extension of $T_1^{-1}(\lambda)$.

Theorem 61 If T_2 is an extension of T_1, then $P_\sigma(T_1) \subseteq P_\sigma(T_2)$ and $M(\lambda_o, T_1) \subseteq M(\lambda_o, T_2)$ for every $\lambda_o \in P_\sigma(T_1)$. Also, $R_\sigma(T_2) \subseteq \rho(T_1) \cup R_\sigma(T_1)$, and $\rho(T_2) \subseteq \rho(T_1) \cup R_\sigma(T_1)$.

Finally, we state a result for adjoint operators.

<u>Theorem 62</u>. If T is a linear operator with $\overline{\mathcal{D}(T)} = \mathfrak{X}$ and $\mathcal{R}(T) \subset \mathfrak{X}$, then $\rho(T) = \rho(T^*)$ and $R^*(\lambda; T) = R(\lambda; T^*)$.

2. <u>Results for bounded linear operators.</u> If T is a bounded linear operator on $\mathfrak{X}$, (i.e. an endomorphism of $\mathfrak{X}$) then $T \in \mathfrak{E}(\mathfrak{X})$.

<u>Definition 84</u>. For $T \in \mathfrak{E}(\mathfrak{X})$, the quantity

$$r(T) = \sup_{\lambda \in \sigma} \{|\lambda|\}$$

is called the <u>spectral radius</u> of T.

<u>Theorem 63</u>. The spectral radius satisfies the following properties:

(i). $r(T) = \lim_{n \to \infty} \|T^n\|^{1/n}$,

(ii) $r(T^n) = r^n(T)$,

(iii) $r(\alpha T) = |\alpha| r(T)$, α a scalar

(iv) $r(T) \leq \|T\|$.

<u>Theorem 64</u>. If $T \in \mathfrak{E}(\mathfrak{X})$, $\sigma(T)$ is a closed, bounded non empty point set. (This has already been stated as Theorem 57)

<u>Theorem 65</u>. The spectrum $\sigma(T)$ is a compact subset of the complex plane; if $\lambda \in \sigma(T)$, then $|\lambda| \leq \|T\|$.

<u>Theorem 66</u>- If $T \in \mathfrak{E}(\mathfrak{X})$, the resolvent $R(\lambda; T)$ admits the power series representation

$$R(\lambda; T) = \sum_{n=1}^{\infty} \lambda^{-n} T^{n-1}$$

if $|\lambda| > r(T)$. This series also represents $R(\lambda;T)$ if the series converges and $|\lambda| = r(T)$. The series diverges if $|\lambda| < r(T)$.

Theorem 67 If $\lambda \in \rho(T)$, then $\mathcal{D}(R(\lambda;T)) = \mathfrak{X}$.

Theorem 68 If $\lambda, \mu \in \rho(T)$, $T \in \mathfrak{C}(\mathfrak{X})$, then

(*) $\quad R(\lambda;T) - R(\mu;T) = -(\lambda - \mu)\, R(\lambda;T)\, R(\mu;T)$.

Equation (*) is called the first resolvent equation, and gives a relation for resolvent operators as a function of elements of the resolvent set.

Theorem 69 Let $S, T \in \mathfrak{C}(\mathfrak{X})$ and suppose that $\lambda \in \rho(S) \cap \rho(T)$

that $\lambda \in \rho(S) \cap \rho(T)$ so that $R(\lambda; S)$ and $R(\lambda; T)$ exist. Then

(**) $\quad R(\lambda;S) - R(\lambda;T) = R(\lambda;S)\ (S - T)\ (S - T)\, R(\lambda;T)$.

Equation (**) is called the second resolvent equation and gives a relation for resolvent operators as a function of elements of $\mathfrak{C}(\mathfrak{X})$.

Theorem 70 If $T \in \mathfrak{C}(\mathfrak{X})$ and $\lambda \in \rho(T)$ then $\lambda \in \rho(T^*)$.

Theorem 71 If $\mathfrak{X}$ is a separable Banach space, or Hilbert space (separable or not), if $T \in \mathfrak{C}(\mathfrak{X})$ and if $\lambda \in R_\sigma(T)$, then $\lambda \in P_\sigma(T^*)$

3. Some results for unbounded linear operators.

The results of the last subsection do not apply if the operator

$T \notin \mathcal{E}(\mathfrak{X})$ i.e. it is unbounded. We now assume that T is closed, with domain $\mathfrak{D}(T)$ and range $\mathfrak{R}(T)$ both contained in $\mathfrak{X}$, that is $T \in \mathcal{C}(\mathfrak{X})$. Then $T(\lambda) = \lambda - T$ defined on $\mathfrak{D}(T)$ is closed, and when $T^{-1}(\lambda)$ exists it will also be closed. If $\lambda \in \rho(T)$ then $T^{-1}(\lambda) = R(\lambda;T) \in \mathcal{E}(\mathfrak{X})$. Thus for $\lambda \in \rho(T)$ we have the following relations:

$$(\lambda - T)R(\lambda; T)x = x, \quad x \in \mathfrak{X}$$

$$R(\lambda; T)(\lambda - T)x = x, \quad x \in \mathfrak{D}(T).$$

When $T \in \mathcal{C}(\mathfrak{X})$ Theorems 68 and 69 become;

<u>Theorem 68'</u> If $T \in \mathcal{C}(\mathfrak{X})$ and if $\lambda, \mu \in \rho(T)$ then

$$R(\lambda; T) - R(\mu; T) = -(\lambda - \mu)R(\lambda; T)R(\mu; T)$$

<u>Theorem 69'</u> Let $S, T \in \mathcal{C}(\mathfrak{X})$, and suppose that $\lambda \in \rho(S) \cap \rho(T)$ so that $R(\lambda; S)$ and $R(\lambda; T)$ exist. If $\mathfrak{D}(T) \subset \mathfrak{D}(S)$, then

$$R(\lambda; S) - R(\lambda; T) = R(\lambda; S)(S - T)R(\lambda; T)$$

As a final result we have the following.

<u>Theorem 72</u> Let $T \in \mathcal{E}(\mathfrak{X})$ and $T \in \mathcal{C}(\mathfrak{X})$ and suppose $\lambda_0 \in \rho(T)$. Then $\lambda \in \rho(T)$ if and only if $I - (\lambda_0 - \lambda)R(\lambda_0; T)$ is regular* in $\mathcal{E}(\mathfrak{X})$. In

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* An element a of a Banach algebra A with unit e is said to be regular if there is an element a^{-1}, the inverse of a, such that $aa^{-1} = a^{-1}a = e$.

this case

$$R(\lambda; T) = R(\lambda_0; T)\left[I - (\lambda_0 - \lambda) R(\lambda_0; T)\right]^{-1}$$

4. <u>Results for compact linear operators.</u> We now consider the case where T is a compact linear operator belonging to $\mathfrak{E}(\mathfrak{X})$.

<u>Theorem 73</u>. If T is compact, or one of its iterates is compact, the number of different characteristic values λ_n of T is finite, or denumerably infinite, and in the latter case $\lim_{n \to \infty} \lambda_n = 0$. The same statement holds for T^*.

<u>Theorem 74.</u> If $T \in \mathfrak{E}(\mathfrak{X})$ is compact, then $P\sigma(T)$ consists of isolated points, with the possible exception of $\lambda = 0$.

<u>Corollary</u>. If $T \in \mathfrak{E}(\mathfrak{X})$ is compact, then $R\sigma(T)$ consists of isolated points, with the possible exception of $\lambda = 0$.

<u>Theorem 75</u>. if $T \in \mathfrak{E}(\mathfrak{X})$ is compact, then $\sigma(T) = P_\sigma(T) \cup R_\sigma(T) \cup [0]$.

<u>Theorem 76</u>. Let $T \in \mathfrak{E}(\mathfrak{X})$ be compact, and let $\lambda \in \sigma(T)$, $\lambda \neq 0$; then the following statements are equivalent:

(1) $T(\lambda)$ has a bounded inverse on $\mathfrak{X}$.

(2) $T^*(\lambda)$ has a bounded inverse on $\mathfrak{X}^*$

(3) $T(\lambda)x = y$ has a solution $x \in \mathfrak{X}$ for every $y \in \mathfrak{X}$.

(4) $T^*(\lambda)\ x^* = y^*$ has a solution $x^* \in \mathfrak{X}^*$ for every $y^* \in \mathfrak{X}^*$

5. Results for operators with compact resolvents.

<u>Theorem 77</u> Let $R(\lambda)$, with domain G, a subset of the complex plane, and range $\mathcal{E}(\mathfrak{X})$ satisfy the first resolvent equation. If for some $\lambda_0 \in G$, $R(\lambda_0)$ is compact, then $R(\lambda)$ is compact for all $\lambda \in G$.

<u>Theorem 78</u> If T is a closed linear operator, i.e. $T \in \mathcal{C}(\mathfrak{X})$ and possesses a compact resolvent, then T is unbounded and has a pure point spectrum consisting of isolated points.

C. Operational calculus

We will not consider the operational calculus for bounded linear operators belonging to $\mathcal{E}(\mathfrak{X})$ but will restrict our attention to the operational calculus of closed linear operators belonging to $\mathcal{C}(\mathfrak{X})$.

Theorem 79. Let Δ be an open subset of the extended complex plane not containing the point λ_0, $\lambda_0 \neq \infty$. Let $A(\Delta)$ be the complex algebra of all function $f(\lambda)$, locally holomorphic in Δ, with the usual arithmetic operations, and with a sequence topology*. Further, let $\mathcal{B}(\Delta)$ be the complex algebra of functions $f(T)$, locally analytic in $\mathcal{J}(\Delta)$ (where $\mathcal{J}(\Delta)$ is the set of all operators $T \in \mathcal{C}(\mathfrak{X})$ such that $\sigma_e(T) \subset \Delta$) and having values in $\mathcal{E}(\mathfrak{X})$ the arithmetic operations being defined as in $\mathcal{E}(\mathfrak{X})$.

Then, there exists an isomorphic mapping $f(\lambda) \to f(T)$ on a subalgebra of $\mathcal{B}(\Delta)$ such that

(i) $1 \to I$

(ii) $I(\lambda_0 - \lambda)^{-1} \to R(\lambda_0; T)$

(iii) $f_n \to f$ implies $\| f_n(T) - f(T) \| \to 0$ uniformly in $\mathcal{B}(\Delta)$.

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* That is, $f_n \to f$ denoting that $f_n(\lambda)$ converges pointwise to $f(\lambda)$, the convergence being uniform in each compact subset of Δ.

This mapping is unique and is defined by the Cauchy formula

$$f(T) = \alpha f(\infty) + \frac{1}{2\pi i} \int_{\Gamma} f(\zeta)\ R(\zeta\,;\, T)\, d\zeta ,$$

where Γ is an oriented envelope of $\sigma_e(T)$ with respect to $f(\lambda)$, and $\alpha = 1$ or 0 according as Γ contains $\lambda = \infty$ in its interior or not.

The mapping from functions $f(\lambda)$ to operators $f(T)$ is called an <u>operational calculus</u>.

We now state a <u>spectral mapping</u> theorem, this theorem being concerned with the mapping $T \longrightarrow f(T)$, where $T \in C(\mathfrak{X})$ and $f(T) \in \mathcal{E}(\mathfrak{X})$.

<u>Theorem 80</u>. Let Δ be an open proper subset of the extended complex plane. Suppose $f(\lambda) \in A(\Delta)$ and $T \in \mathcal{B}(\Delta)$. Then $\sigma(f(T)) = f[\sigma_e(T)]$.

D. <u>Spectral theory of linear operators in Hilbert space</u>

In this section we restrict our attention to some spectral properties of bounded linear operators in Hilbert space, that is operators $T \in \mathfrak{L}(H)$. We do not give the spectral representations of self-adjoint and unitary operators, as these were given in theorems 53 and 54.

<u>Theorem 81.</u> If T is unitary, $\sigma(T)$ consists only of numbers λ satisfying $|\lambda| = 1$; and $R_\sigma(T)$ is empty.

<u>Theorem 82.</u> If T is normal, then $R\sigma(T)$ is empty.

<u>Theorem 83.</u> If T is self-adjoint, then

$$\|T\| = \sup_{\lambda \in \sigma(T)} \{|\lambda|\} = r(T).$$

<u>Thhorem 84.</u> If T is self-adjoint, $\sigma(T)$ is a subset of the real axis; and $R\sigma(T)$ is empty.

E. Spectral theory of positive operators.

In this section we state two theorems concering the spectral properties of positive operators.*

Theorem 85. If T is a positive operator, then $1 \in \sigma(T)$.

The above theorem asserts that $r(T)$ belongs to the spectrum (not necessarily the point spectrum) of a positive operator.

The next theorem establishes the existence of characteristic functions belonging to $\mathfrak{X}^+$, and proves an ergodic theorem.

Theorem 86. If 1 is a pole of order k of $R(\lambda; T)$ then

$$\frac{\Gamma(k+1)}{n^k} \sum_{i=1}^{n} T^i \to (T-I)^{k-1} P,$$

in the uniform operator topology, where P is a projection operator. The point $1 \in P\sigma(T)$, and there exists, associated with 1, nonzero elements of $\mathfrak{X}^+$ which are characteristic functions. Furthermore, there exists an element

$x^* \in (\mathfrak{X}^*)^+$, $x^* \neq 0$, for which $T^*x^* = x^*$

In theorem 85

$$P = \frac{1}{2\pi i} \int_C R(\xi; T)d\xi$$

where C is a positively oriented circle enclosing $r(T)$ and having no other elements of $\sigma(T)$ in its interior or on its boundary.

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