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MATSCIENCE REPORT 4

LECTURES ON

GROUP THEORY

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Lectures on

GROUP THEORY

'The Theory of Representations of Finite Groups'

By

S. Swaminathan

and

'Elements of Lie Groups and Lie Algebras'

By

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THE THEORY OF REPRESENTATIONS OF FINITE GROUPS

Introduction

The scope of these lectures delivered at The Institute of Mathematical Sciences, Madras during the summer of 1962 is to develop the theory of representations of finite groups and apply it to the problem of classification of eigenvalues of a quantum-mechanical system.

In developing the theory, stress is laid more on the vector space methods than on matrix-methods. This is done on the belief that such an approach gives better insight into the results of the representation theory, for which the treatment of Schur's Lemma and Burnside's Theorem will bear testimony.

In preparing these lectures I have not hesitated to draw freely from the inspiring standard works of Hammermesh, Lyubarskii, Heine and Van der Waerden.

It is a pleasure to record here my grateful thanks to Professor Alladi Ramakrishnan, Director of the Institute of Mathematical Sciences, for his kind invitation to me to deliver these lectures during summer 1962 and to the authorities of the University of Madras for kindly permitting me to accept the invitation.

Madras

Dated December 1st 1962.

S.Swaminathan.

Lecture I

Elements of Group Theory

The concept of a group is an abstraction of the algebraic properties of numbers (integers, rational numbers, real numbers etc.). When we analyse the properties of the operations of addition and multiplication in these number systems. We find that they uniformly satisfy certain laws which are independent of the nature of the numbers and the method in which the operation is performed. A little reflection helps us to discern that the system of all polynomials (of one variable with a real or complex coefficients) also behave inthe same way with respect to the usual notion of operations of addition and multiplication. These observations lead us to consider a set G of elements whose nature need not be specifically preassigned and introduce a binary operation called 'multiplication' in it which satisfies the following laws :

G 1 : The product of any two elements x , y of G (which is the result of 'multiplying' x and y) is also an element of G . In general the order in which x and y are multiplied matters, i.e. the products xy and yx may be different.

G 2 : Associative law : $(xy/z = x(yz)$ for $x, y, z, \in G$.

G 3 : Existence of an identity : There is an element ~~of~~ e of G called the identity, which is such that

$$ex = xe = x \quad \text{for} \quad x \in G$$

G 4 : Existence of an inverse: For each element $x \in G$ there is an element x^{-1}, called the inverse of x such that

$$xx^{-1} = x^{-1}x = e$$

(Note: x^{-1} is a notation for such an element).

The set G is called a <u>group</u> with respect to the operation 'multiplication' when G1 to G4 are satisfied.

When the number of elements of G is finite, the group is said to be finite, otherwise infinite. The number of elements in a finite group G is called its order.

In G1 we noted that the order in which the elements are multiplied does matter in general. In the particular case where it does not matter, i.e., $xy = yx$, both representing the same element of G, the group is called <u>commutative</u> or <u>abelian</u> (after the Norwegian mathematician Abel).

The group operation can be called 'addition' instead of 'multiplication'. If the additive terminology is used, in G3 we demand the existence of 'zero' and in G4, the 'negative'. Hereagain, it is only a matter of nomenclature.

<u>Examples</u>.

1. The set of all integers (-ve, o , +ve) under the usual operation of addition.

2. The set of all rational numbers (-ve, o , +ve) under the usual operation of addition.

3. The set of all positive non-zero rational numbers under the usual operation of multiplication.

4. The set of all real numbers under the usual operation of addition.

5. The set of all complex numbers under the usual operation of addition.

6. The set of all polynomials in one variable with real coefficients (or complex coefficients) under the usual operation of addition.

7. <u>Transformation groups</u>. Consider twosets S and S^1. By a transformation T (or a mapping) of S into S^1, we mean a rule which assigns to every element x of S, a unique element x^1 of S^1. x^1 is the image of x under T. This is denoted by $x^1 = T(x)$ or $x \to x^1 = T(x)$.

If every element x^1 of S^1 is the image of atleast one x of S, the transformation T is said to be 'onto' i.e., from S onto S^1.

If every element x^1 of S^1 is the image of exactly one element of S, the transformation is said to be <u>one-to-one</u>.

We may take S^1 to be S itself. Then we have transformations of a set S into (or onto) itself. In such a case we can combine ~~fxm~~ two~~x~~ transformations to yield a third transformation T_3 of S into itself by prescribing

$$T_3(x) = T_2\left[T_1(x)\right] \quad \text{for } x \in S.$$

T_2 can be thought of as the 'product' of T_1 and T_2.

Now consider the set T of <u>all</u> transformations of a set into itself under the operation of multiplication as defined above. The axioms $G1$ to $G3$ are satisfied. $G4$ is not satisfied in general. For that we need one-to-one transformations.

The set of all one-to-one transformations of a set S onto itself is a group under multiplication as defined above.

A one-to-one mapping of a set onto itself is called a <u>permutation</u>. When the set consists of a finite number of elements, a permutation can be written by putting the elements of the set, which can be denoted by numbers 1, 2, 3 etc. in a row and their images below them. Thus

$$P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

are two permutations of a set consisting of three elements. Their product is defined to be the permutation

$$P_2 \; P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

The product ~~of~~ $P_1 \; P_2$ is $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$P_1 \; P_2 \neq P_2 \; P_1$. The set of all permutations forms a group which is non-commutative.

This permutation group is also known as the symmetric group S_3. The justification for this is as follows:

Consider an equilateral triangle A_1 A_2 A_3 with centre 0 (the centroid). Consider the set of six geometrical transformations which turn the triangle into a coincident position (without necessarily leaving any vertex fixed). A positive (anti clockwise) rotation through 120^0 about 0 carries A_1, A_2, A_3 into A_2, A_3, A respectively. Denote this rotation by $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$.

A positive rotation through 240^0 about 0 carries A_1, A_2, A_3 into A_3, A_2, A_1. Denote this rotation by $\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}$.

A positive rotation through 0^0 (or 360^0) about 0 leaves every vertex fixed. This rotation can be regarded as an Identity rotation $I = \begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}$

Next consider a rotation through 180^0 about 0 A_1, bringing the triangle through positions out of the plane of the paper. This carries A_1, A_2, A_3 into A_1, A_3, A_2. Note this notation by $\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}$.

we have similar rotations $\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}$ about 0 A_2 and $\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}$ about 0 A_3.

The six rotations permute the vertices ~~vertex~~ of the triangle in six possible ways. Let us write $I = \begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}$.

$$P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \qquad P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

The product of any two of these as defined above by taking successive performance gives us a third rotation. We can form a multiplication table for all the different rotation thus:

Right Factor.

		I	P_1	P_2	P_3	P_4	P_5
	I	I	P_1	P_2	P_3	P_4	P_5
	P_1	P_1	P_2	I	P_5	P_3	P_4
	P_2	P_2	I	P_1	P_4	P_5	P_3
Left factor.	P_3	P_3	P_4	P_5	I	P_1	P_2
	P_4	P_4	P_5	P_3	P_2	I	P_1
	P_5	P_5	P_3	P_4	P_1	P_2	I

To read from this table the product $P_2 . P_5$ (for example) look along the row marked P_2 and down the column marked P_5. Where these intersect we find the product, namely P_3.

Notice that the group axioms can be verified directly from this table.

Similar considerations on the permutations of a set consisting of n elements give us the symmetric group S_n.

The symmetry groups are important in physics and chemistry. The symmetry of a body is described by giving the set of all transformations which preserve the distances between all pairs of points of the body and bring the body into coincidence with itself.For finite bodies such transformations are made up of either rotation through a definite angle about some axis or a mirror ~~minor~~ reflection in a plane. The following situations are some of those which are related to the symmetry of the system in question: The classification of the spectralterms of a polyatomic molecule; the problem of finding the vibration spectrum of the molecule; microscopic structure of crystals; classification of electron energy levels in a crystal. However, we are not going into the details of these here. An excellent treatment of the symmetry groups is given in Chapter 2 of Morton Hamermesh: Group Theory and its applications to physical problems.

2.1. Subgroups: A subset H of a group G can be a group on its own right under the same operation. It is then called a subgroup. Accordingly we have the following criterion for a subgroup.

Theorem: A subset H of a group G is a subgroup if the following conditions hold:

(i) if $h_1 \in H$, $h_2 \in H$, then $h_1 h_2 \in H$ and (2) if $h_1^{+1} \in H$ then $h_1^{-1} \in H$.

(For finite groups, the second condition is redundant.

Examples: (i) Integers $\subset$ rational numbers $\subset$ real nos. $\subset$ complex numbers, all considered as groups under the usual operation of addition.

(ii) The positive rational numbers form a group under multiplication, but is not a subgroup of the additive group of rationals.

(iii) Every group G has two trivial subgroups: the group consisting of the identity element alone and the whole group G itself.

2. 2. Cyclic groups: In any group G, the integral powers a^m of an element a can be defined separately for positive, zero and negative exponents. If $m > 0$, we define

$$a^m = a.a.a \ldots a \text{ (m times) } ; \quad a^0 = e, \quad a^{-m} = (a^{-1})^m$$

It can be easily seen that

$$a^r . a^s = a^{r+s} \quad \text{and} \quad (a^r)^s = a^{rs}$$

But $(ab)^r \neq a^r . b^r$ in general. The equality holds if and only if G is commutative.

Definition . The order of an element a in a group is the least positive integer m such that $a^m = e$; if no positive power of a equals e , then a is said to be of infinite order. A group G is cyclic if every element of G is the power of some element a of G . a is said to generate the group.

For any group G , the powers of an element a form a group which is the cyclic subgroup generated by a . Thus the subgroup of integers within the group of real numbers is a ~~xing~~ cyclic subgroup of the reals.

2. 3. Homomorphism and Isomorphism. Suppose that there is associated with each element a of a group G an element $b = f(a)$ of another group H. The totality of elements $b = f(a)$ of the group H so obtained is denoted by $f(G)$. We say we are considering a mapping of the group G into the group H . If every element of H is an element in $f(G)$ then the mapping is said to be 'on-to'. In general f need not preserve the operation of the group. Consider, for example, the mapping $f : x \rightarrow x^2$. Since $(x + y)^2 \neq x^2 + y^2$ the addition is not preserved by f .

When the group operation is preserved by a mapping f , we call the mapping a homomorphism of G into H , i.e. $f : G \rightarrow H$ is a homomorphism means

$$f(x.y) = f(x).f(y) \text{ for every } x, y \in G$$

From the definition, we immediately have the result.

Theorem. If f is a homomorphic mapping of a group G into a group H, then the set $f(G)$ is a subgroup of H. It should be noted that a homomorphism of G into H maps the identity of G into the identity of H and the inverse of an element of G into the inverse in H of the corresponding element. By the kernel of a homomorphism f of G into H, we mean the set of all elements of G which are mapped into the identity of H. The mappings which define a homomorphism of a group into another need not be one-to-one, (Recall that a mapping of a set G into a set H is one-to-one means no two elements of G have the same image and every element of H is the image of one (and only one) element of G).

When a homomorphism is also a one-to-one mapping of G into H, then we call it an isomorphism of G into H. The groups G and $f(G)$ are said to be isomorphic. They are not different algebraically, for every algebraic theorem, established for G is automatically applicable to $f(G)$ and also to all groups isomorphic to G. From the point of view of group theory, isomorphic groups are identical.

Examples. (1) The group S_3 of permutations of three objects (see ~~Lecture~~ page 6 I) is homomorphic to the group H consisting of the two numbers 1 and -1. The operation in H is multiplication. The homomorphism is given by $I \rightarrow 1$, $P_1 \rightarrow 1$, $P_2 \rightarrow 1$, $P_3 \rightarrow -1$, $P_4 \rightarrow -1$, $P_5 \rightarrow -1$.

(2) In our discussion of the symmetric group S_3 in Lecture I, we have actually shown that S_2 is isomorphic to the group of rotation of an equilatorial triangle.

Lecture III.

Vector Spaces

3.1. The concept of a vector must already be familiar to students from mechanics. Force, displacement, velocity acceleration, electric intensity are familiar examples to students of theoretical physics. Let us consider 'force' as a vector in detail. Two forces f_1 and f_2 acting on a particle are represented by vectors emanating from the origin and terminating at the points whose rectangular coordinates are (x_1 , y_2 , z_2). The resultant force is denoted by $f_1 + f_2$ and is known to be represented by the vector, emanating from the origin which is the diagonal of the parallelogram having the two given vectors as sides. The coordinates of the end point of $f_1 + f_2$ are the sum of the corresponding coordinates of f_1 and f_2 ;

This is the familiar parallelogram law of addition of vectors. Another familiar operation on the vectors is that of multiplying by a real number (or scalar). Thus $k f_1$ is a vector having the same direction as f_1 if k is positive or the opposite direction if k is negative and having a length equal to $| k |$ times that of f_1. We may write $k f_1 = (k x_1 , k y_1 , k z_1)$. To make this valid for $k = 0$ we define $k f_1$ to be the zero vector, namely, the single point the origin.

It is clear from these operations that the nature of the quantity represented by a vector has nothing to do with the operations themselves. Sets of numbers, subject to the operations ~~xxxxx~~ as described above will completely define the force or any one of other quantities mentioned above. Consequently we may simply take a set of numbers as a <u>vector</u>.

Generalisation of this concept and its study is the scope of the theory of vector spaces. We briefly sum up the main results needed for our purposes. The generalisation can be made in two directions. Firstly we may consider n-coordinates instead of 3. Thus we take an ordered set of n numbers, like $\xi = (x_1, x_2, \ldots, x_n)$ to be a vector. ξ is called an n-triple. The n-triples of real numbers ~~is said to form an~~ underlies the space of n-dimensional geometry. The operation of addition and scalar multiplication follow the same pattern as in the case of 3-triples. Secondly, the numbers used in defining a vector need not always be real numbers. The vector operations depend only on the corresponding operations of the numbers used. So we may take complex numbers instead of reals, or more generally elements from a field F. (A set F is said to be a field, when these two operations + and x defined in it such that (i) under + , it is an abelian group, (ii) the non-zero (zero of +) elements form an abelian group under x , and (iii) the two operations distribute each other, i.e. $a \times (b + c) = a \times b + a \times c$ for $a, b, c \in F$). Thus we may consider the set of all n-triples

$$\xi = (x_1 , x_2 , x_3 , \ldots x_n)$$

where each x_i is chosen from a field F. Denoting this set by $V_n (F)$, we define the vector operations of addition and scalar multiplication by elements of F , which satisfy the usual laws. We give now the abstract definition of a vector space.

Definition. A vector space V over a field F is a set of elements called vectors, such that any two vectors α and β determine a unique vector $\alpha + \beta$ as sum and that any vector α in V and a scalar k from F define a scalar product, denoted by $k \alpha$ with the following properties.:

1. V is an abelian group under addition.
2. $k(\alpha + \beta) = k.\alpha + k\beta$ and $(k + k')\alpha = k\alpha + k'\alpha$
3. $(k.k')\alpha = k(k'\alpha)$ and $1.\alpha = \alpha$

for any k , k' in F and α, β in V .

Examples: (1) The totality of $1 \times n$ matrices over F , the addition and scalar multiplication being the corresponding matrix operations,

(2) The set of all functions of a real variable over the interval $(0, 1)$, under pointwise addition and scalar muultiplication. The sum h of functions f and g is given by $h(x) = f(x) + g(x)$ and $k.f$ is given by $k f(x)$ for every real x .

2.2. Sub-spaces:

A subset A of a vector space $V(F)$ is called a subspace (more accurately a sub-vector space or a vector subspace) when it is itself a vector space over F with respect to the addition and scalar multiplication of $V(F)$.

Any subset need not be a subspace. It is easy to see from the definition that a necessary and dufficient condition for a subset A to a subspace is that the sum of any two vectors of A has in A and the scalar multiple of any vector of A is also a vector of A.

Note the similiarity of the definition to that of a subgroup. In fact, the subspace A is a subgroup of V under addition. As am example, the vectors of the form (0, x_2, 0 and x_4) constitute a subspace of V_4 (F) for any field F.

If $\xi_1, \xi_2, \ldots, \xi_t$ be vectors belonging to V(F) then the vector ξ given by

$$\xi = k_1\xi_1 + k_2\xi_2 + \ldots + k_t\xi_t \qquad --- (1)$$

~~(1)~~

where $k_1, k_2, \ldots, k_t$ are scalars from F , must also belong tto V (F). ξ is said to be a linear combination of ξ_i's, (i = 1, 2, .. t). Consider the set L of all linear combinations of ξ_i 's .This set is obtained by giving all possible values to the k's from F in (1) above. The set L is obviously a subspace of V(F). It is said to be spanned by or generated by the vectors .

2.3. Linear independence and basis.

Vectors $\xi_1, \xi_2, \ldots, \xi_t$ belonging to V(F) are called linearly dependent if there exist scalars k_1, k_2, ... k_t not all zero, such that

$$k_1\xi_1 + k_2\xi_2 + \cdots + k_t\xi_t = 0 \qquad ---- (2)$$

where the '0' on the right denotes the zero vector.

If no such scalars exist, the set $\xi_1, \xi_2, \ldots, \xi_t$ is called linearly independent, i.e. when (2) implies every k_i is zero. Consider the vectors $\xi_1 = (1, 2, 1)$, $\xi_2 = (0, 1, 0)$, $\xi_3 = (2, 0, 2)$ in 3-dimensional space. T~~x~~hey are linearly dependent because

$$2\xi_1 - 4\xi_2 - \xi_3 = 0$$

This merely means that they are coplanar. Linear dependance is a generalisation of 'coplanarity'. On the other hand, the vectors ξ_2 and ξ_3 of this example are linearly independent.

Basis. A basis of a vector space is a linearly independant subset which spans the whole space.

A vector space is called finite-dimensional if it has a basis consisting of a finite number of elements.

The following is an important theorem of the theory of vector spaces. We mention it without proof.

Theorem: Any two bases of a finite-dimensional vector space have the same number of elements.

Hence we define the invariant number of vectors in a basis of a finite-dimensional vector space to be its dimension.

Let $(\alpha_1, \alpha_2, \ldots \alpha_n)$ be a basis of a vector space over complex numbers. Then every vector α of the space can be expressed in the form $\alpha = x_1 \alpha_1 + x_2 \alpha_2 + \ldots + x_n \alpha_n$. Since this expression is unique (because of linear independence) we say that the numbers $(x_1, x_2, \ldots x_m)$ are the coordinates of the vector α with reference to the basis $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Example : The vectors (1, 0, 0), (0, 1, 0) and (0, 0, 1) form a basis for the 3-dimensional euclidean space. Take any vector, say (1, 2, 3). It can be written

$$(1, 2, 3) = (1, 0, 0) + (0, 1, 0).2 + (0, 0, 1).3$$

Operators.

It is usual to refer to a mapping of a vector space into itself as an operator on the space.

An operator T on a vector space is said to be linear when $T(\alpha + \beta) = T(\alpha) + T(\beta)$ and $T(k\alpha) = k\,T(\alpha)$ where α, β denote vectors of the space.

Corresponding to a basis of the vector space, a linear operator can be expressed as a matrix. The matrix is obtained by expressing the transforms of every coordinate. It should be noted that the matrix depends on the choice of the basis.

Scalar products: With each pair of vectors (x , y) of a vector space over complex numbers, we associate a complex number which we denote by (x, y) and call the scalar product of x and y when it satisfies the following conditions:

$(x, y) = (y, x)^*$, where * denotes complex conjugate

$(x, ky) = k(x, y)$.

$(x_1 + x_2, y) = (x_1, y) + (x_2, y)$

$(x, x) \geq 0$ and $(x, x) = 0$ only if $x = 0$.

The length of a vector is the square root of (x, x).

The scalar product of two vectors is independent of the choice of a basis.

Two vectors are said to be orthogonal when their scalar product is zero.

Lecture 4.

Representations of finite groups

4.1. Group Representations.

Among the applications of group theory to physics the most important is the application of theory of representations, which is concerned with the homomorphic mappings of a group on all possible groups of linear operators. Such mappings arise naturally when various types of symmetry are considered.

Consider the set of operators A, B, ... in a vector space L. This set is a group when we take the product of operators A and B to be the single operator

$$C_x = A\,(B_x) \text{ for all } x \in L.$$

The identity of the group is the unit operator which leaves all vectors in L unchanged.

Now let G be any group. Let it be mapped homomorphically on a group of operators $D(G)$ in a vector space L. The operator group $D(G)$ is called a <u>representation</u> of G in the <u>representation space</u> L. The dimension n of the vector space L is called the dimension of the representation. To each element g of G there is an operator $D(g)$ of L into itself which corresponds to g in the group $D(G)$. If g and h are elements of G, we have

$$D(gh) = D(g).\,D(h). \quad D(g^{-1}) = [D(g)]^{-1},$$

$$D(e) = E \quad \text{(the identity operator)}.$$

A <u>linear representation</u> is a representation in terms of linear operators. We shall study only linear representations. So we shall mean always linear representations when we talk of representations in this course. We have ~~seen~~ noted that corresponding to a basis in the space L, the linear operators on L can be described by matrices. The dimension being n, to each linear operator, there is an n x x matrix, so that there is a homomorphic mapping of D (G) on a group of (n x n) matrices. If we consider the direct homomorphism of G into the matrix group thus obtained, we have a matrix representation of this group G . Obviously the matrices should be non-singular and such that

$$D_{ij}(e) = \delta_{ij} = \begin{cases} 1 & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases} \qquad i,j = 1,2,\ldots n.$$

$$D_{ij}(gh) = \sum_k D_{ik}(g) D_{kj}(h)$$

when the homomorphism on D(G) reduces to an isomorphism, the representation is said to be <u>faithful</u>.

<u>Examples:</u> 1. Let G be an arbitrary group. To each element g of G assign the unit operator E of a certain space L . We obtain a representation called the identity representation.

2. Let G be the group of spatial translations along z-axis and L a one-dimensional space. Corresponding to a translation of magnitude α, associate the operator of multiplication by the number $e^{ik\alpha}$ for some fixed k. We obtain a one-dimensional representation of the group of translations.

4.2. Equivalent representations.

We are interested in enumerating all the possible representations of a given group. In studying this two concepts play a fundamental role: the concept of equivalence of representations and the concept of reducibility of a representation.

Given a representation $D(G)$ of a element G, we can construct any number of new representations of G. Let A be any non-singular linear operator of L into itself. Assign to $g \in G$, the operator $D_A(g) = AD(g)A^{-1}$. The correspondence of $g \to D_A(g)$ is a representation of G. For

$$\begin{aligned} D_A(g_1 g_2) &= A D(g_1 \cdot g_2) A^{-1} = A D(g_1) \cdot D(g_2) A^{-1} \\ &= A D(g_1) A^{-1} \cdot A D(g_2) A^{-1} \\ &= D_A(g_1) \cdot D_A(g_2). \end{aligned}$$

Any two representations of the group which are connected by a relation of the type $D_A(G) = AD(G)A^{-1}$ are called equivalent. All representations which are equivalent to a given one are equivalent to one another. Hence all the representations of a given group G can be split into classes of mutually equivalent representations. It is enough to specify any one representation of each class. Accordingly the problem of finding all representations is reduced to the more limited one of finding all mutually non-equivalent representations.

If we consider the corresponding matrices of the equivalent representations, we have an $(n \times n)$ matrix $\{D_{ij}(g)\}$ corresponding to the operator $D(g)$ with reference to a given basis of L, while the matrix corresponding to the operator $D(g)$ with reference to a given basis of L, while the matrix corresponding to $D_A(g)$ is

$$A \; D_{ij}(g) \; A^{-1}$$

which is obtained from $D_{ij}(g)$ by a similarity transformation by A. The equivalent representation can be regarded as obtained by changing the basis in L in accordance with the relation

$$y = A \; x$$

in which the vectors x, y, of L are related by the matrix A.

$$\begin{tikzcd} x \arrow[r, "A"] \arrow[d, "D_{ij}(g)"'] & y \arrow[d, "A D_{ij}(g) A^{-1}"] \\ x' \arrow[r, "A"'] & y' \end{tikzcd}$$

y' is given by $y^1 = (A \, D_{ij}(g) A^{-1}) \, y$. In other words the transition to an equivalent representation corresponds to replacing the components of vectors in the representation space of the matrices $D_{ij}(g)$ by new components in accordance with $y = A\,x$, where A is a non=singular matrix.

Now that it is enough that we know one representation in each class of equivalent representations, we look for one which has simple and useful form for manipulations. A unitary representation, by which we mean a representation whose matrix is unitary, is one such representation. The question arises whether each class of equivalent representations of a group G will contain a unitary representations. For finite groups the answer is yes as is proved in the following theorem.

4.3. Theorem. Every representation of a finite group is equivalent to a unitary representation.

Proof: For a pair of vectors x, y of the representation space L (C) , we construct the expression

$$\{x, y\} = \frac{1}{m}\sum_{g} (D(g)x, D(g)y)$$

where m is the order of the group and the summation extends to all elements of G . The number $\{x, y\}$ satisfies all the requirements for a scalar product viz., we easily verify

1) $\{x,y\} = \{y,x\}^{*}$ where * denotes complex conjugate.

2) $\{x,\alpha y\} = \alpha\{x,y\}$ for all $\alpha \in C$

3) $\{x_1 + x_2, y\} = \{x_1, y\} + \{x_2, y\}$

4) $\{x,x\} \geq 0$ and $\{x,x\} = 0$ only if $x = 0$.

For any element (fixed) a of G , we have

$$\{D(a)x, D(a)y\} = \frac{1}{m}\sum_{g} (D(g)D(a)x, D(g)D(a)y)$$

$$= \frac{1}{m}\sum_{g} (D(ga)x, D(ga)y)$$

a being fixed, as g runs through all elements of G , so does g a , hence the expression on the right is simply $\frac{1}{m}\sum_{g}(D(g)x, D(g)y)$. Hence $\{D(a)x, D(a)y\} = \{x,y\}$ --- I which means the operators of the representation D(G) are unitary with reference to the scalar product { } but not with reference to ( ).

Next we consider a set of vectors u_i which are orthonormal with reference to the original scalar product ( ) , and a set v_i which are orthonormal with reference to the new scalar product { } . This means

Let T be the linear operator which takes the u's to the v's

$$v_i = Tu_i$$

Since $Tx = T\sum x_i u_i = x_i \sum Tu_i = \sum x_i v_i$

$$\begin{aligned}
\{Tx, Ty\} &= \{\sum x_i v_i, \sum y_j v_j\} \\
&= \sum_i \{x_i v_i, \sum y_j v_j\} \\
&= \sum_i \{\sum y_j v_j, x_i v_i\}^* \\
&= \sum_j \sum_i \{y_j v_j, x_i v_i\}^* \\
&= \sum_j \sum_i x_i^* \{y_j v_j, v_i\}^* \\
&= \sum_j \sum_i x_i^* \{v_i, y_j v_j\} \\
&= \sum_j \sum_i x_i^* y_j \{v_i, v_j\} = \sum_j \sum_i x_i^* y_j (u_i, u_j) = (X, Y) \qquad \text{---(II)}
\end{aligned}$$

We now consider the equivalent representation defined by

$$D'(g) = T^{-1} D(g) T$$

and find that

$$\begin{aligned}
(T^{-1}D(g)Tx, T^{-1}D(g)Ty) &= \{D(g)Tx, D(g)Ty\} \quad \text{since II can be written } (x,y) = (T^{-1}x, T^{-1}y). \\
&= \{Tx, Ty\} \quad \text{from I} \\
&= (x, y) \quad \text{from II}
\end{aligned}$$

Hence D' (g) is unitary with reference to the original scalar product ( ) and this is equivalent to D (g) . Hence the theorem. (Using the matrices of the representation, we can prove the theorem by forming a Hermitian matrix H which is the sum over the group of the product of each matrix by its adjoint and by diagonalising it and finding the inverse square root, $H = \sum_{k=1}^{m} D^k_{ij} (D^k_{ij})^\dagger$. It is shown that successive similarity transformation of the D^k_{ij} by the matrix U which diagnolises H and by $d^{1/2}$ the sq.root of diagonal from of H , produces a representation which is unitarity] . If certain additional restrictions are impored this theorem remains true for infinite groups whole elements depend on a parameter. An instance in which it is not true is the 4 x 4 Lorentz group .

LECTURE 5.

Reducibility and Irreducibility of Representations

5.1. Reducibility of Representations.

The concept of reducibility of a representation arises when we attempt to express a given representation of a group G in terms of 'simpler' representations. The natural idea of simplicity here is that such a representation should have as low a dimension as possible consider, for example, a three dimensional representation D of a group G, whose matrix is of the form

$$\left(\begin{array}{cc|c} a_i & b_i & e_i \\ c_i & d_i & f_i \\ \hline 0 & 0 & g_i \end{array}\right) \quad \text{---(I)}$$

The product of two such matrices is again a matrix of the same form, e.g. the product for $i = 1$ and 2 will be

$$\left(\begin{array}{cc|c} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 & a_1e_2 + b_1f_2 + e_1g_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1b_2 & c_1e_2 + d_1f_2 + f_1g_2 \\ \hline 0 & 0 & g_1g_2 \end{array}\right)$$

Now the matrices $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ in the upper left-hand corner provide us with a two dimensional representation, while these in the lower right-hand corner as (g_i) give a one-dimensional representation.

We say that the 3-dimensional representation is <u>reducible</u> to two lower dimensional (viz., 2 and 1-dim) representations.

Now in general a representation may not be of the form (I) while an equivalent representation obtained from it by a similarity transformation may be of this form. Then it will not be immediately apparent that the representation is made up of lower dimensional ones.

In general, if we can find a basis in which <u>all</u> the matrices $D(g)$ of an n-dimensional representation can be brought simultaneously to the form

$$D(g) = \left(\begin{array}{c|c} D^1(g) & A(g) \\ \hline O & D^2(g) \end{array}\right) \quad --- (\mathrm{II})$$

where $D^1(g)$ are $m \times m$ matrices, $D^2(g)$ are $(n-m)\times(n-m)$ matrices, $A(g)$ is a $m \times (n-m)$ - rectangular matrix, and O denotes the $(n-m) \times m$ matrix whose elements are all zero, then we say that the representation $D(g)$ is reducible.

It is clear that $D^1(g)$ and $D^2(g)$ are m-dimensional and $(n-m)$-dimensional representations, for the matrix product

$$D(gh) = D(g)D(h) = \left(\begin{array}{c|c} D^1(g)\,D^1(h) & D^1(g)A(h) + A(g)D^2(h) \\ \hline O & D^2(g)\,D^2(h) \end{array}\right)$$

has the same from as (II).

It may happen that we can further split $D^1(g)$ and $D^2(g)$ such that

$$D^1(g) = \left(\begin{array}{c|c} D^3(g) & A'(g) \\ \hline 0 & D^4(g) \end{array}\right) \qquad D^2(g) = \left(\begin{array}{c|c} D^5(g) & A''(g) \\ \hline 0 & D^6(g) \end{array}\right)$$

and continue the process, which clearly comes to an end, since the representations are finite dimensional.

The original representation can be expressed interms of lower ones as shown in the matrix aside, where we have blocks of matrices along the diagonal, zero-matrix blocks in the left-hand lower corner and non-zero rectumgular matrices in the upper right hand corner. The representations $D^1(g), D^2(g), \ldots, D^k(g)$ are irreducible representation of dimensions $m_1, m_2, \ldots, m_k$ where $\sum_{i=1}^{k} m_i = n$

$$\begin{pmatrix} \boxed{D^1} & & & & \\ 0 & \boxed{D^2} & & & \\ 0 & 0 & \boxed{D^3} & & \\ & & & \boxed{} & \\ & 0 & & & \ddots \quad \boxed{} \end{pmatrix}$$

We have introduced the concept of reducibility interms of matrix representations. Let us see what it means in the representation space L. Consider the 3-dimensional representations given in (I) above, again. The effect of the matrix on a vector in the subspace of L spanned by the first two components alone is, as in

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & g \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \\ 0 \end{pmatrix}$$

to yield a vector in the same subspace. In other words the two-dimensional subspace is invariant under all the transformations (linear operators) of the group. But, since

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & g \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} = \begin{pmatrix} e x_3 \\ f x_3 \\ g x_3 \end{pmatrix}$$

the subspace spanned by the third component alone (which is complementary to the other subspace) is not invariant under the grou

Now taking the general case, suppose the representation can be written

$$\left(\begin{array}{c|c} D^1(g) & A(g) \\ \hline 0 & D^2(g) \end{array}\right)$$

the subspace of the first m-components, where m is the dimension of $D^1(g)$ is left invariant, while the complementary subspace of $n-m$ dimensions is not, for

$$\left(\begin{array}{c|c} D^1(g) & A(g) \\ \hline 0 & D^2(g) \end{array}\right) \left(\begin{array}{c} X \\ \hline 0 \end{array}\right) = \left(\begin{array}{c} D^1(g) X \\ \hline 0 \end{array}\right)$$

while

$$\left(\begin{array}{c|c} D^1(g) & A(g) \\ \hline 0 & D^2(g) \end{array}\right) \left(\begin{array}{c} 0 \\ \hline X \end{array}\right) = \left(\begin{array}{c} A(g) X \\ \hline D^2(g) X \end{array}\right)$$

So in the general case, we can say that the representation is <u>reducible</u>, if there exists some subspace of dimension $m < n$ which is invariant under all the transformations of the group. If there is no proper subspace which is invariant the representation is <u>irreducible</u>.

It might happen that, in the above, all $A(g)$, that is, all the rectangular matrices the right of the diagonal blocks, are also zero matrices, so that

$$D(g) = \begin{pmatrix} D^1(g) & 0 \\ 0 & D^2(g) \end{pmatrix} = \begin{pmatrix} D^1(g) & 0 & 0 & & 0 \\ 0 & \boxed{D^2(g)} & 0 & & \\ 0 & 0 & \boxed{} & & 0 \\ & & & \ddots & \\ 0 & 0 & & & \boxed{} \end{pmatrix}$$

The representation is then said to be <u>fully-reducible</u>, (or completely reducible).

In this case the complementary subspace should also be invariant. When the representation is unitary this is always the case. This is proved in the following theorem.

<u>5.2. Theorem: Let D be a unitary reducible representation of a group G in the space L and let L_1, be an invariant subspace. Then the subspace L_2, the orthogonal complement of L_1, is also invariant.</u>

<u>Proof:</u> Let $x \in L_1$, $y \in L_2$. Then $D^{-1}(g)x \in L_1$ and so $(D'(g)x, y) = 0$. But since D is unitary,

$$(D^1(g)x, y) = (D(g)D^{-1}(g)x, D(g)y) = (x, D(g)y).$$

Therefore $(x, D(g)y) = 0$
which means x is orthogonal to $D(g)y$, or $D(g)y$ ~~i.e.~~ $\in L_2$

Hence L_2 is invariant under $D(g)$. Theorem is proved.

In the case of fully reducible representations, the representation space is decomposed in to mutually orthogonal invariant sub spaces. $L = L_1 + L_2$ in the case of Theorem I. In the general case,

$$L = L_1 + L_2 + \dots + L_k$$

each L_i transforming according to irreducible representation of G. Also the representation $D(g)$ itself can be written

$$D(g) = D^1(g) + D^2(g) + \dots + D^k(g)$$

Where each $D^i(g)$ is irreucible. Among the $D^i(g)$'s there may be several which are equivalent to one another (for this, of course, they must have the same dimensions). Since equivalent irredicible representations are not counted as distinct, we may write

$$D = a_1 D^1 + a_2 D^2 + \dots + a_k D^k.$$

where the a_i's are positive integers.

Combining the Theorem (of last lecture) on finite groups and the above theorem we see that:

<u>Every (finite dimensional) representation of a finite group is either irreducible or can be decomposed into a sum of irreducible representations.</u> Also in the case of finite groups reducibility implies fully reducibility and since we are studying finite groups only we will not make a distinction.

<u>LECTURE 6.</u>

<u>Schur's Lemma.</u>

<u>6.1. Schur's Lemma:</u> Irreducible representations have a series of properties which render them very valuable in applications and simplify their determination.

We shall derive a necessary and sufficient condition for a representation to be reducible.

We agree to call the diagonal matrix $\begin{pmatrix} k & 0 & \cdots & 0 \\ 0 & k & & \\ & & \ddots & \\ 0 & & & k \end{pmatrix}$ with identical elements k on the principal diagonal, a multiple of the unit matrix, written kI. In matrix operations, kI behaves like the number k.

Suppose now that we have a reducible representation of a finite group G, for example, a representation whose matrices can be written in the form

$$D(g) = X\,[A(g), B(g), C(g)]\,X^{-1}$$

where by $[A(g), B(g), C(g)]$ we denote the quasi-diagonal matrix $\begin{pmatrix} \boxed{A} & & 0 \\ & \boxed{B} & \\ 0 & & \boxed{C} \end{pmatrix}$, and X is some matrix. Form the matrix

$$Y = X\,[kI, lI, mI]\,X^{-1}$$

where the inner matrix is quasi-diagonal with the same structure as $[A(g), B(g), C(g)]$. It is not hard to see that Y commutes with all the matrices $D(g)$, since

$$D(g)Y = X[A(g), B(g), C(g)]X^{-1}X[kI, lI, mI]X^{-1}$$

$$= X[A(g)k, B(g)k, C(g)k]X^{-1}$$

and $YD(g) = X[kA(g), kB(g), kC(g)]X^{-1}$

and $kA(g) = A(g)k$, etc.

We can choose k, l, m distinct so that the matrix is not a multiple of the unit matrix. So we have proved the result that

If a representation consisting of matrices $D(g)$ is reducible then there exists a matrix Y which is not a multiple of the unit matrix and which commutes with all the matrices $D(g)$.

The converse of this is also true and we shall prove it in the following form which is known as a lemma of Schur.

<u>6.2. Schur's Lemma I:</u> <u>If a representation $D(g)$, $g \in G$ of a group G is irreducible, then every linear operator A commuting with all the operators $D(g)$ is a multiple of the unit operator. In other words from the relation</u>

$$D(g)\, A = A\, D(g)$$

<u>we have the equality $A = \lambda E$ where λ is a number and E is the unit operator.</u>

<u>Proof:</u> A being a non-singular linear operator, it has at le one eigen-value. Let us denote it by λ . Let L_1, denote the subspace of eigenvectors of the operators A which correspond to the number λ . Thus, we have

$$Ax = \lambda x \quad \text{if } x \in L_1 \qquad ---(\text{III})$$

Evidently the subspace L_1 is not empty, other wise λ would not be an eigenvalue. We shall show that the subspace L_1 is invariant. Take $x \in L_1$, and let $D(g)$ be one of the operators of the irreducible representations. Consider the vector $A D(g) x$. We have

$$A D(g) x = D(g) A x = D(g) \lambda x = \lambda D(g) x .$$

Thus $D(g) x$ is an eigen vector of the operator A and corresponds to the eigenvalue λ . In other words, the vector $D(g)x$ is contained in L_1 which proves that L_1 is an invariant subspace. Now since $D(g)$ is irreducible, L_1 must be trivial, eit[h]er empty or the whole space. Since L_1 is not empty, we obtain $L_1 = L$. So the equation III, $Ax = \lambda x$ is valid for all $x \in L$ which means that $A = \lambda E$, Proving the lemma.

Thus we see that a necessary and sufficient condition for a representation to be irreducible is that there should exist no matrix other than a matrix of the form kI which commutes with all the matrices of the representation.

It should be observed that Schur's Lemma I is true for both finite and infinite groups.

6.3. Abelian groups: Let G be an abelian group. Suppose Dg_1, Dg_2 denote the matrices corresponding to elements g_1, g_2 of G in some representation. Then since the products $g_1 g_2$ and $g_2 g_1$ have images $Dg_1 Dg_2$ and $Dg_2 Dg_1$ respectively it follows that $Dg_1 Dg_2 = Dg_2 Dg_1$; so that the matrices forming a representation of an Abelian group commute. Suppose the representation Dg is unitary, (we take G to be not necessarily finite). Then from matrix theory we know that there exists a unitary transformation U such that all matrices $U Dg U^{-1}$ have a purely diagonal form so that there exists an equivalent representation

$$U D_g U^{-1} = [k_g^1, k_g^2, \ldots, k_g^n]$$

consisting of diagonal matrices. Thus, in this case, the representation reduces to n one-dimensional representations, k_g^i $(i = 1, 2, \ldots n)$. In other words, by using a unitary matrix, every unitary representation of an abelian group can be transformed into an equivalent set of one-dimensional representations.

The following proof shows how all matrices can be simultaneously reduced to diagonal form so as to yield one-dimensional representations

Theorem: <u>All unitary irreducible representations of an Abelian group are one-dimensional.</u>

<u>Proof:</u> Consider any representation $D_{ij}(g)$, $D_{ij}(h)$, ... of the abelian group G. $(g, h, \ldots \in G)$. All matrices are unitary. Now we know that any unitary matrix can be reduced to diagonal form by another unitary matrix.

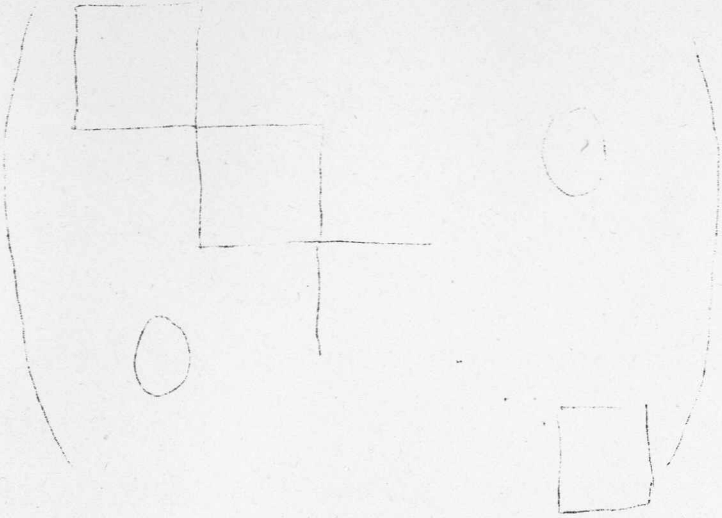
Let $U^{-1} D(g) U = D'(g)$ be the diagonal matrix

$$\begin{pmatrix} \lambda_1 & & & & & & & & & \\ & \lambda_1 & & & & & & & 0 & \\ & & \ddots & & & & & & & \\ & & & \lambda_1 & & & & & & \\ & & & & \lambda_2 & & & & & \\ & & & & & \lambda_2 & & & & \\ & 0 & & & & & \ddots & & & \\ & & & & & & & \lambda_2 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \ddots \end{pmatrix}$$

Put $U^{-1} D(h) U = D'(h)$. Since G is abelian

$$D'(g) D'(h) = D'(h) D'(g) .$$

Now the left side of this equation is just the matrix $D'(h)$ with each row multiplied by the corresponding λ_m of $D'(g)$ while the right side is $D'(h)$ with each column multiplied by the corresponding λ_m of $D'(g)$. Thus $D_{ij}(h) = 0$ if the λ_m's in the i-th and j-th rows of $D'(g)$ are unequal. Thus $D(h)$ has the form



where blocks correspond to equal diagonal elements of $D'(g)$

Now $D'(h)$ can be reduced to diagonal form by a transfor-tion $U'^{-1} D'(h) U'$ with U' also having the form above. This will not alter $D'(g)$ because we are making transformations between rows having the same λ_m. .. Thus both $D(g)$ and $D(h)$ can be reduced similtaneously to diagonal form. Similarly all other matrices can be reduced. Hence proof:

LECTURE 7.

Orthoganality Theorem I

7.1. Group functions: Definition: If to each element of a group G a certain real or complex number $\phi(g)$ is assigned it is said that a function is defined on the group.

Let D be any (reducible or irreducible) n-dimensional repr-sentation of a group G in a certain space L. Consider the matri $D_{ij}(g)$ corresponding to some basis in L. The elements

$$D_{ij}(g) \quad i = 1, 2, \ldots, n; \; j = 1, 2, \ldots, n.$$

form a set of n^2 group functions on G. They satisfy the important algebraic relations

$$D_{ij}(gh) = \sum_{l=1}^{8} D_{il}(g) D_{lj}(h)$$

which is only a restatement of the homomorphism character of the representation.

7.2. Scalar product: We define the scalar product of two group functions $\phi(g)$ and $\psi(g)$ by the expression

$$(\phi, \psi) = \frac{1}{m} \sum_{g \in G} \phi(g) \overline{\psi(g)}$$

The functions $\phi(g)$ and $\psi(g)$ are orthogonal when their scalar product vanishes.

We can now state and prove the first orthogonality theorem.

7.3. Orthogonality Theorem I. The irreducible representation D of n dimensions generates n^2 mutually orthogonal functions $D_{ij}(g)$ $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, where

$$(D_{ij}, D_{i'j'}) = \frac{1}{n} \delta_{ii'} \delta_{jj'}$$

the δ's denoting the Kronecker delta function.

Proof: We use Schur's Lemma I in the proof. Let B be any linear operator acting in the same space L as the operators $D(g)$.

Let us define an operator by the formula

$$A = \frac{1}{m} \sum_{g \in G} D(g) \, B \, D(g^{-1}).$$

With such a definition we verify easily that A commutes with all the operators $D(g)$. For, let $D(h)$ be any operator of the representation.

$$D(h)A = \frac{1}{m}\sum_{g\in G} D(h)\,D(g)\,B\,D(g^{-1})$$

$$= \frac{1}{m}\sum_{g\in G} D(hg)\,B\,D(g^{-1})\,D(h^{-1})\,D(h)$$

$$= \frac{1}{m}\sum_{g\in G} D(hg)\,B\,D[(hg)^{-1}]\cdot D(h)$$

$$= \frac{1}{m}\sum_{g\in G} D(g)\,B\,D(g^{-1})\cdot D(h)$$

$$= A\cdot D(h)$$

since for a fixed h, gh runs through all the elements of G. Hence by Schur's Lemma I, A is a multiple of the identify operator $A = \lambda E$,

$$\text{or} \qquad \frac{1}{m}\sum_{g} D(g)\,B\,D(g^{-1}) = \lambda(B)\,E$$

where obviously λ depends on B.

Let us pass from this operator equation to the corresponding matrix elements

$$\frac{1}{m}\sum_{g} D_{i\alpha}(g)\,B_{\alpha\beta}\,D_{\beta j}(g^{-1}) = \lambda(B)\,\delta_{ij} \qquad \text{-- (I)}$$

where we have used the summation convention over twice repeated indices. Since $D(g)$ is unitary,

$$D_{\beta j}(g^{-1}) = [D_{\beta j}(g)]^{-1} = [D_{j\beta}(g)]^{+}$$

$$\text{i.e. } D_{\beta j}(g^{-1}) = \overline{D_{j\beta}(g)}$$

So (I) becomes

$$\frac{1}{m}\sum_{g\in G} D_{i\alpha}(g)\overline{D_{j\beta}(g)}\cdot B_{\alpha\beta} = \lambda(B)\delta_{ij}$$

i.e.,

$$\left(D_{i\alpha}(g),\ D_{j\beta}(g)\right)B_{\alpha\beta} = \lambda(B)\delta_{ij} \quad — \quad (\mathrm{II})$$

We put $i=j$ in (I), to compute $\lambda(B)$ and sum over i both sides.

Then
$$\lambda(B)\cdot n = \frac{1}{m}\sum_{g\in G} D_{i\alpha}(g)D_{\beta i}(g^{-1})\cdot B_{\alpha\beta}$$
$$= \frac{1}{m}\sum_{g\in G} D_{\beta i}(g^{-1})D_{i\alpha}(g^{-1})\, B_{\alpha\beta}$$
so that
$$= \frac{1}{m}\sum_{g\in G} D_{\beta\alpha}(e)\, B_{\alpha\beta} = \frac{1}{m}\cdot m\,\delta_{\beta\alpha}\, B_{\alpha\beta} = B_{\alpha\alpha}.$$

so $\lambda(B) = \frac{1}{n} B_{\alpha\alpha}$

Remembering that B was arbitrarily chosen, we can take B so that

$$B_{\alpha\beta} = \begin{cases} 1 & \text{when } \alpha = i',\ \beta = j' \\ 0 & \text{otherwise} \end{cases}$$

then $B_{\alpha\alpha} = \delta_{i'j'}$ so that $\lambda(B) = \frac{1}{n}\delta_{i'j'}$.

Hence, from (II) we have

$$\left(D_{ii'},\ D_{jj'}\right) = \frac{1}{n}\,\delta_{ij}\,\delta_{i'j'}.$$

which completes the proof.

Remarks: (1) The theorem can be expressed in a form without the use of scalar products thus: If $(D_{ij}(g))$ is the matrix which is an n-dimensional, irreucible, unitary representation of G. then

$$\sum_{g \in G} D_{ij}(g)\overline{D_{i'j'}(g)} = \frac{m}{n}\delta_{ij}\delta_{i'j'} .$$

(2) The maximum number of linearly independent group functions on a group G of order m is equal to m. In fact we can take m linearly independent functions defined by

$$\phi_i(g_j) = \delta_{ij}$$

$$i = 1, 2, \ldots m, \quad j = 1, 2, \ldots m$$

and express any group function in terms of these.

Since mutually orthogonal functions are linearly independent we must have

$$n^2 \leq m$$

LECTURE 8.

Schur's Lemma II and Second Orthogonality Theorem.

8;1. Schur's Lemma II. Let $D^1(g)$ and $D^2(g)$ be two non-equivalent irreducible representations of a group G. The operators $D^1(g)$ act in the space L_1, those of $D^2(g)$ in the space L_2. If the linear operator A which transforms vectors of L_2 into L_1 satisfies the commutability relation:

$$D^1(g) \cdot A = A \cdot D^2(g) \quad (g \in G)$$

then it vanishes identically, i.e. $A = 0$.

Proof: Let n_1 and n_2 denote the dimensions of L_1, and L_2. Case (i): $n_1 > n_2$. Let M denote the subspace of L_1, which contains all vectors of L_1, which are images of vectors of L_2 by A, i.e.,

$$M = \{ x \mid x_1 \in L_1,\ A x_2 = x_1 \text{ for } x_2 \in L_2 \}.$$

We show that M is invariant for all the operators $D^1(g)$. If $x_1 \in M$ then

$$D^1(g)\, x_1 = D^1(g) A x_2 = A D^2(g) x_2$$
$$= A y_2 \ (\text{for } y_2 \in L_2) = y_1 \in M.$$

Since $D^1(g)$ is irreducible, this means that M coincides with L_1 or is null.

Now $\dim M \leq n_2 < n_1$ and so M cannot be equal to L_1. Therefore M should be null, i.e., M consists to of null vectors, i.e, A must be the zero operator. $A = 0$.

Case(ii): $n_1 = n_2$. L_1 and L_2 have the same number of dimensions. (Now the Operator A should be singular. If it were non-singular A^{-1} exists, then the commuting relation gives

$$D^1(g) = A D^2(g) A^{-1}$$

which means that $D^1(g)$ is equivalent to $D^2(g)$ contrary to hypothesis). Now A can be expressed as a $(n_1 \times n_1)$ - matrix which is singular and so will transform vectors of L_2 into those of L_1, such that the image space is of lesser dimension. Hence $\dim M < n_1$ and the argument of case (i) applies.

Case (iii) $n_1 < n_2$. Again we remember that A is singula Let N denote the subspace of L_2 consisting of these vectors whic are transformed into the zero vector of L_1, by A.

$$N = \{ x_2 \mid x_2 \in L_2 ,\ A x_2 = 0 ,\ 0 \in L_1 \}$$

Since A is singular, N is not null. i.e., $\dim N \neq 0$. N is invariant relative to the operator $D^2(g)$. For this we must show that when $x_2 \in N$, $D^2(g) x_2$ is also in N, i.e., $A D^2(g) x_2 = 0$. But $A D^2(g) x_2 = D^1(g) A x_2 = 0$, since $A x_2 = 0$.

. Since $D^1(g)$ is irreducible, N should coincide with L_2 this means every operator of L_2 is taken to zero by A, for which A ought to be zero itself.

Schur's Lemma II is proved.

We can prove the Second orthogonality Theorem now using the second Lemma of Schur

8.2 Second Orthogonality theorem.

The functions $D^1_{ij}(g)$ generated by the irreducible representati $D^1(g)$ are orthogonal to functions $D^2_{ij}(g)$ generated by any other non-equivalent irreducible representation $D^2(g)$:

$$(D^1_{ij}(g) , D^2_{ij}(g)) = 0$$

Proof: The proof is analogous to that of the first orthogonality theorem. We take an arbitrary linear operator B which transforms vectors of L_2 into L_1 . As before we define the operator

$$A = \frac{1}{m} \sum_{g \in G} D^1(g)\, B\, D^2(g^{-1})$$

A satisfies the commuting relation

$$D^1(g)\, A = A\, D^2(g)$$

for

$$D^1(h)\, A = \frac{1}{m} \sum_{g \in G} D^1(h)\, D^1(g)\, B\, D^2(g^{-1})\, D^2(h^{-1})\, D^2(h)$$

$$= \frac{1}{m} \sum_{g \in G} D^1(hg)\, B\, D^1((hg)^{-1})\, D^2(h)$$

$$= A\, D^2(h)$$

Consequently by Schur's Lemma II, A vanishes. It follows that

$$\frac{1}{m} \sum_{g \in G} D^1(g)\, B\, D^2(g^{-1}) = 0$$

$$\text{or} \quad \frac{1}{m} \sum_{g \in G} D^1_{i\alpha}(g)\, B_{\alpha\beta}\, D^2_{\beta j}(g^{-1}) = 0$$

$$\text{or} \quad \left(\frac{1}{m} \sum_{g \in G} D^1_{i\alpha}(g)\, \overline{D^2_{j\beta}(g)} \right) B_{\alpha\beta} = 0$$

$$\text{or} \quad \left(D^1_{i\alpha}, D^2_{j\beta} \right) B_{\alpha\beta} = 0.$$

As before we choose B such that $B_{\alpha\beta} = 1$ if $\alpha = i'$, $\beta = j'$, and $= 0$ otherwise, so that $(D^1_{ii'}, D^2_{jj'}) = 0$. Hence proof.

8.3 From the orthogonality theorems we can immediately deduce that the number of non-equivalent irreducible representations of a finite group should be <u>finite</u>. For, suppose D^{μ} is an arbitrary irreducible representation and by varying μ we obtain all such representations which are not equivalent to one another. Let n_μ denote the dimension of D^μ . Then we have n_μ^2 group functions $D^\mu_{ij}(g)$, for $i,j = 1,2,\ldots n_\mu$, which are orthogonal to one another and the group functions $D^\nu_{ij}(g)$ for $i,j = 1,2,\ldots n_\nu$, of another such representation. We have already remarked above that the number of linearly independent orthogonal group functions for a finite group of order m should be less than m . Thus

$$\sum_\mu n_\mu^2 \leq m$$

which means that, the n_μ's, being integral, must be finite in number.

Actually the inequality above can be shown to reduce an equality. The proof of this requires the introduction of two more concepts, that of character of a group and of regular representations to which we shall now turn our attention.

8.4 <u>Characters</u> Let $D(g)$ and $D'(g)$ be two inequivalent irreducible representations of orders p and q respectively of a group G. We denote by $\chi(g)$ and $\chi'(g)$ the traces of the matrices $D_{ij}(g)$ and $D'_{ij}(g)$, i.e, the sum of their diagonal elements:

$$\chi(g) = \sum_{i=1}^{p} D_{ii}(g), \qquad \chi'(g) = \sum_{k=1}^{q} D'_{kk}(g);$$

the numbers $\chi(g)$ and $\chi'(g)$ are called the characters of the representations $D(g)$ and $D'(g)$, .

Equivalent representations have identical characters. For, if C be any matrix (C_{ij}) ,

$$\sum_i (C D(g) C^{-1})_{ii} = \sum_{ikl} C_{ik} D_{kl}(g) C^{-1}_{li}$$

$$= \sum_{kl} \delta_{kl} D_{kl}(g)$$

$$= \sum_k D_{kk}(g)$$

From remark 1 following the first orthogonality theorem we have

$$\sum_g D_{ij}(g) \overline{D_{i'j'}(g)} = \frac{m}{n} \delta_{ij} \delta_{i'j'},$$

Whence

$$\sum_g D_{ii}(g) \overline{D_{kk}(g)} = \frac{m}{n} \delta_{ik}$$

Summing this formula over i and k we obtain

$$\sum_g \chi(g) \overline{\chi(g)} = m \qquad --- (I)$$

Similarly from the second orthonality theorem, we have

$$\sum_g D_{ii}(g) \overline{D_{kk}(g)} = 0$$

Summing tjos formula over i and k, we obtain

$$\sum_g \chi(g) \overline{\chi'(g)} = 0 \qquad (II)$$

(I) and (II) are the orthogonality relations for characters. They can be used to prove some theorems about characters.

Theorem : . . A necessary and sufficient condition for two irreducible representations to be equivalent is that their characters be identical.

Proof: We have already seen that equivalent representations whether reducible or not have the same characters. Hence the necessity of the condition. For sufficiency, suppose for irreducible representations have the same characters, i.e, $\chi(g) = \chi'(g)$. for all $g \in G$.

Then by (I) we have $\sum_g \chi(g) \overline{\chi'(g)} = m$. - - - - - - (III)

If the representations were not equivalent, then (II) gives

$$\sum_g \chi(g) \overline{\chi'(g)} = 0$$

contradicting (III). Hence they should be equivalent.

LECTURE 9.

Regular representations and Burnside's Theorem.

9.1. Burnside's Theorem

As a preliminary to this Theorem we first prove the following theorem:

Theorem: The set of functions

$$D^{\alpha}_{ij}(g) \quad (\alpha = 1,2,\ldots\ldots,q, \quad i,j = 1,2,\ldots.n_{\alpha})$$

generated by all non-equivalent irreducible representations, of a finite group G, where q denotes the number of such representations and n_α the order of the representation D^α, is a basis of the vector space of all group functions.

Proof: To prove this we consider representations of G over the vector space of all group functions on G, and introduce the notion of a regular representation.

To each of $g \in G$, consider the operator $R(g)$ acting in the vector space L of group functions, given by for $\psi \in L$,

$$R(g)\ \psi(h) = \psi(gh) = \emptyset(h)\ (\text{say})$$

$\psi(gh)$ is considered as a function of h and, in general, differs from the function $\psi(h)$. To see that the correspondence $g \to R(g)$ is a homoruorphism of G on the group of operators over L, we have

$$[R(g')\cdot R(g)]\ \psi(h) = R(g')\cdot[R(g)\ \psi(h)] = R(g')\ \psi(gh) = \psi(g'gh)$$
$$= R(g'g)\ \psi(h).$$

Thus $R(g)$ is a representation of G over L. This representation is called the regular representation.

Since every representation of G can be expressed as a sum of irreducible representations, $R(g)$ can be written

$$R(g) = D^1(g) + \ldots + D^\alpha(g) + \ldots + D^p(g)$$

where each $D^\alpha(g)$ is reducible. The space L is correspondingly split into subspaces thus:

$$L = L_1 + L_2 + \ldots + L_\alpha + \ldots + L_p$$

where L_α is the subspace of L left invariant by D^α.

Now let $\phi_1^\alpha, \phi_2^\alpha, \ldots, \phi_{n_\alpha}^\alpha$ denote a basis in L_α. We shall

show that each ϕ_j^α $(j=1,2,\dots,n_\alpha)$ can be expressed interms of the group functions D^α generated by irreducible representation D.
For this consider $R(g)$ as operating on ϕ_j^α.

$$R(g)\cdot\phi_j^\alpha = \chi \text{ same function in } L.$$

But $[D^1 + \dots + D^\alpha + \dots + D^b](g)\phi_j^\alpha = D^\alpha(g)\phi_j^\alpha$ which means χ should be a function in L_α. Let $\chi = \sum_{k=1}^{n_\alpha} a_k \phi_k^\alpha$. a_k is the K-th coordinate of χ w.r.t. the basis $\{\phi_k^\alpha\}$. To evaluate it, consider the matrix form $D^\alpha(g)\,\phi_j^\alpha = \chi$.

$$\begin{pmatrix} D^\alpha_{11} & \cdots & D^\alpha_{1j} & \cdots & D^\alpha_{1n_\alpha} \\ D^\alpha_{21} & \cdots & D^\alpha_{2j} & \cdots & D^\alpha_{2n_\alpha} \\ - & - & - & - & - \\ - & - & - & - & - \\ D^\alpha_{n_\alpha 1} & \cdots & D^\alpha_{n_\alpha j} & \cdots & D^\alpha_{n_\alpha n_\alpha} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ - \\ 1\,(j\text{-th place}) \\ - \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_k \\ \vdots \\ a_{n_\alpha} \end{pmatrix}$$

$$a_1 = D^\alpha_{1j},\quad a_2 = D^\alpha_{2j},\ \dots,\ a_k = D_{kj},\ \dots,\ a_{n_\alpha} = D^\alpha_{n_\alpha j}$$

Therefore $\chi = \sum_{k=1}^{n_\alpha} D^\alpha_{kj}(g)\phi_k^\alpha\ (h)$

But $R(g)\ \phi_j^\alpha\ (h)$ is, by definition of $R(g)$, $\phi_j^\alpha\ (gh)$. Hence

$$\phi_j^\alpha\ (gh) = \sum_{k=1}^{n_\alpha} D^\alpha_{kj}(g)\ \phi_k^\alpha(h)$$

Putting $h = e$, we obtain

$$\phi_j^\alpha\ (g) = \sum_{k=1}^{n_\alpha} D^\alpha_{kj}\ (g)\ \ \phi_k^\alpha(e)$$

Thus ϕ_j^α can be expressed in terms of the group functions generated by the irreducible representation D^α.

Now any function on the group G can be expressed as a linear combination of functions belonging to a basis of L, and each basis function of L can be expressed in terms of group function $D^{\alpha}_{ij}(g)$ generated by some irreducible representation, it follows that any group function can certaintely be expressed in terms of group functions generated by irreducible representations of G.

Since any two group functioned generated by ineducible representations are orthogonal, the set of such group functions is linearly independent and we have just now seen how this set spans the vector space of group functions on G. Hence the set is a basis of this vector space, and the proof of the theorem is complete.

It follows from this theorem that the number of the functions belonging to the set of all group functions generated by all ineducible representations of G is the number of dimension of L, which is $\underline{m}$, the order of the group G, since we have already seen that the maximum number of lin. independent functions on G is m. On the other hand the total number of functions generated by the ineducible representations is equal to the sum of the square of the dimensionalities of these representations. We thus obtain the result that

$$n_1^2 + n_2^2 \ldots\ldots + n_q^2 = m.$$

i.e., the sum of the squares of the number of dimension of all ineducible non-equivalent representations is equal to the order of the group. This result is known us Burnside's Theorem.

LECTURE 10.

The classification of states of a quantum mechanical system.

In this lecture, we discuss the application of the representation theory to classification of states of quantum mechanical syst[em]. We follow closely the treatment in Heines Group Theory in quantum mechanics.

10.1 Symmetry transformations.

We begin with a consideration of how symmetry properties of a physical system are expressed mathematically. Suppose a body with a point P on it is rotated about a point O. If P has coordinates (x, y, z) the point P' to which it goes after a clockwise rotation by an angle α about the z-axis is given by

$$\begin{aligned} x &= X\cos\alpha - Y\sin\alpha \\ y &= X\sin\alpha + Y\cos\alpha \qquad (1) \\ z &= Z \end{aligned}$$

Instead of a rotation like this, we can keep the body (and so P) fixed and change the axes of reference suitably so that P has coordinates (X, Y, Z) with reference to the new axes, the coordinates being given by the above equations. These two points of view are important in mathematics, especially in geometry. The single transformation (1) can be regarded either as a change in coordinates of a point when we rotate a body by an angle $-\alpha$, or change in the coordinates of a fixed point when the axes are rotated by an angle $+\alpha$.

If we consider a perfectly round plate without any markings on it, we say it is symmetrical about a vertical axis through its centre, say Z-axis, since we cannot recognise a rotation after it is performed. On the other hand for a fixed position of the plate, we could also say that the various physical properties such as moments of inertia about x—, y— and z— axes must be the same, no matter how these axes are chosen. This latter can be regarded as a mathematical expression of the symmetry of the plate. Generally the symmetry of a physical system is manifested mathematically in the equations of the system by the invariance of symmetry transformations. When a transformation such as (1) is applied to a differential equation, its effect on the differential operators and on the functions in the equations are different.

Consider the effect of such a transformation on the time-independent Schrodinger equation

$$H\psi = E\psi$$

where H is the Hamiltonian operator, E the energy value belonging to the eigenfunction ψ .

The Hamiltonian for an atom with n electrons, considering the nucleons as fixed and spin ignored, is

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{n}\nabla_i^2 - \sum_{i=1}^{n}\frac{ne^2}{r_i} + \sum\sum_{i<j=1}^{n}\frac{e^2}{r_{ij}}$$

where m is the mass of the electron, e the charge on the photon, and

$$\nabla_i^2 = \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + \frac{\partial^2}{\partial z_i^2}$$

$$r_i^2 = x^2 + y^2 + z^2$$

$$r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2.$$

When the transformation (1) is applied, we find that

$$\nabla^2 = \frac{\partial^2}{\partial X_i^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z_i^2}$$

$$r_i^2 = X^2 + Y^2 + Z^2$$

$$r_{ij}^2 = (X_i - X_j)^2 + (Y_i - Y_j)^2 + (Z_i - Z_j)^2$$

Hence, substituting in the expression for the Hamiltonian, we see that it has precisely the same form as before.

$$H(x_i, y_i, z_i) = H(X_i, Y_i, Z_i)$$

This is expressed by saying that the transformation (1) leaves H invariant, i.e., (1) is a symmetry transformation of H.

But when (1) is applied to ψ, the eigen function of H, in general, (1) does not leave it invariant. Consider for instance the $2p$ wave functions for a hydrogen atom,

$$\psi(2p_x) = x f(r), \quad \psi(2p_y) = y f(r), \quad \psi(2p_z) = z f(r)$$

where $f(r)$ is a function of r, the modulus of the position vector $\vec{r}$.

$x f(r)$ changes into $(x \cos\alpha - Y \sin\alpha)$ f(R) which is obviously a different functional form.

Thus the Schrödinger equation

$$H(x_i, y_i, z_i)\ \psi_1(x_i, y_i, z_i) = E\psi_1(x_i, y_i, z_i)$$

changes into

$$H(X_i, Y_i, Z_i)\ \psi_2(X_i, Y_i, Z_i) = E\psi_2(X_i, Y_i, Z_i)$$

where ψ_2 has a different functional form, in general, from that of ψ_1. If follows that the symmetry transformation of a Hamiltonian can be used to relate the different eigen functions of one energy level to one another and hence label them and discuss the degeneracy of the levels.

The transformation (1) is not the only one for which H is invariant. There are others. The transformation

$$(x_1, y_1, z_1) = (X_2, Y_2, Z_2)$$

$$(x_2, y_2, z_2) = (X_1, Y_1, Z_1)$$

$$(x_i, y_i, z_i) = (X_i, Y_i, Z_i) \quad i = 3, 4, \ldots, n.$$

which is a permutation of the coordinates of 1 and 2, is also a symmetry transformation of H. Similarly any permutation of the coordinates of (x_i, y_i, z_i) is a symmetry transformation.

The <u>inverse</u> transformation defined by

$$x_i = -X_i, \quad y_i = -Y_i, \quad z_i = -Z_i \text{ for all } i$$

is also a symmetry transformation for H.

As an example of a transformation which is not symmetric for H we may cite the transformation to cylindrical polar, coordinates given by $x_i = R_i \cos\Theta_i \quad y_i = R_i \sin\Theta_i \quad z_i = Z_i$

which transforms ∇_i^2 into

$$\frac{1}{R_i}\frac{\partial}{\partial R_i}\left(R\frac{\partial}{\partial R_i}\right)+\frac{1}{R_i^2}\frac{\partial^2}{\partial \Theta_i^2}+\frac{\partial^2}{\partial Z_i^2}.$$

10.2 Theorem: All the symmetry transformations of a Hamiltonian form a group.

Taking the product of 2 symmetric transformations to be their successive applications, it is clear that (i) such a product is also a symmetry transformation, (ii) the identity transformation is ~~xxxx~~ a symmetry transformation and (iii) the inverse of a symmetry transformation is also a symmetry transformation is also a symmetry transformation.

There is no certain method of finding out all the symmetry transformations of a given hamiltonian. One generally arrives at a subgroup of the group of all such transformations. Some of the symmetry transformations may not be obvious. For certain systems, some of the more subtle ones have been discovered recently (See for example Jauch and Rohrlich, Theory of Photons and Electrons, 1955, Add.Wesley, p. 143; Baker, Phy. Rev. 103, 1119, 1956).

10.3 Example. Consider three protons fixed at the points

$$\vec{r}_1=(0,2\sqrt{3}a,0),\quad \vec{r}_2=(-3a,-\sqrt{3}a,0),\quad \vec{r}_3=(3a,-\sqrt{3}a,0)$$

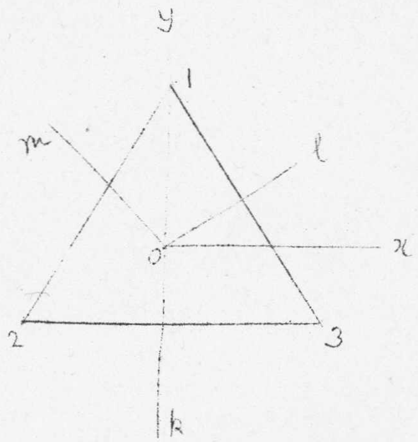
forming an equilatoral triangle about the origin. The Hamiltonian for one electron moving the field of the three protons is

$$H=-\frac{\hbar^2}{2m}\nabla^2+\frac{e^2}{|\vec{r}-\vec{r}_1|}+\frac{e^2}{|\vec{r}-\vec{r}_2|}+\frac{e^2}{|\vec{r}-\vec{r}_3|}.$$

The symmetry of this system is closely related to that of an ozone molecule or that of an ion situated between three water molecules in a hydrated crystal of a salt . The physical system has the same rotational symmetry as for an equilaterial triangle. We have discussed the group of rotations of an equilaterial triangle which leave the triangle invariant in lecturel.

The group G of rotations with reference to the adjoining figure has elements

A, B, K, L, M, E, where



A: rotation of 120° about x-axis

B; .. 240° (or -120°) about y-axis

K: .. 180° about OR axis

L: .. 180° .. Ol ..

M: .. 180° .. Om ..

E: .. 0° (or 360°) about any axis (no rotation).

when the transformations for these rotations are written down, we have

E: $(x, y, z) = (X, Y, Z)$

A: $x = -\frac{1}{2}X + \frac{1}{2}\sqrt{3}\,Y$ B: $x = -\frac{1}{2}X - \frac{1}{2}\sqrt{3}\,Y$

$y = -\frac{1}{2}\sqrt{3}\,X + \frac{1}{2}Y$ $y = \frac{1}{2}\sqrt{3}\,X - \frac{1}{2}Y$

$z = Z$ $z = Z$

$$K: \quad (x, y, z) = (X, Y, Z_1)$$

$$L: \quad x = \tfrac{1}{2}X + \tfrac{1}{2}\sqrt{3}\,Y \qquad\qquad M: \quad x = \tfrac{1}{2}X - \tfrac{1}{2}\sqrt{3}\,Y$$

$$y = \tfrac{1}{2}\sqrt{3}\,X - \tfrac{1}{2}Y \qquad\qquad y = -\tfrac{1}{2}\sqrt{3}\,X - \tfrac{1}{2}Y$$

$$z = -Z_1 \qquad\qquad z = -Z_1$$

It can be easily verified that these transformations are symmetry transformations.

We shall find the representations of this group G.

Consider the transformations above operating in the functions

$$xe^{-r}, \quad ye^{-r}$$

where $r^2 = x^2 + y^2 + z^2$.

$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} xe^{r} \\ ye^{r} \end{pmatrix}$ is the effect of A on the vector $\begin{pmatrix} xe^{r} \\ ye^{r} \end{pmatrix}$

We can write this as

$$A\,xe^{-r} = -\tfrac{1}{2}\,xe^{-r} + \frac{\sqrt{3}}{2}\,ye^{-r}$$

$$A\,ye^{-r} = -\frac{\sqrt{3}}{2}\,xe^{-r} - \tfrac{1}{2}\,ye^{-r}$$

Corresponding to A we can ~~t~~take the representation matrix as

$$D_{ij}(A) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}$$

Where we have transfered the matrix of A. Actually the functions xe^{-r}, ye^{-r} are only auxilary for the finding of the representation. The representation~~s~~ is obtained by considering linear transformation in the vector space generated by xe^{-r} and ye^{-r}. These functions are called a basis of the representation. Obviously there can be other bases for the representation.

We can operate the transformation A, B, etc. on the function ze^{-r} also and generate a representation $\mathcal{A}$ of 1 x 1 matrices. The matrices are not all different. Then there is the identity representation $\mathcal{I}$ which associates the 1 x 1 matrix (1) with every element. The irreducible representations of the group G are, thus,

Representation	E	A	B	K	L	M
Γ	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
$\mathcal{A}$	1	1	1	−1	−1	−1
$\mathcal{I}$	1	1	1	1	1	1

We can prove that Γ is ineducible as follows:

If Γ were reducible, it would have to be reducible into two one-dimensional representations, so that it would be possible to find two vectors in a space (ϕ_1, ϕ_2), where ϕ_1 and ϕ_2 a form basis for Γ such that each of the vectors is invariant under the group G. It is, therefore, sufficient to prove that there is no vector $\phi = C_1\phi_1 + C_2\phi_2$ which is invariant under all transformations of the group.

Applying the transformation K to $\phi_1\phi_2$, we have

$$K\phi_1 = -\phi_1 \quad \text{and} \quad K\phi_2 = +\phi_2$$

Suppose there exists a vector $\phi = c_1\phi_1 + c_2\phi_2$ which is invariant under all transformations of G. Then

$$K\phi = \gamma\phi$$

γ a constant. Then

$$K(c_1\phi_1 + c_2\phi_2) = \gamma(c_1\phi_1 + c_2\phi_2)$$
$$= -c_1\phi_1 + c_2\phi_2$$

Therefore $-c_1 = \gamma c_1$ and $c_2 = \gamma c_2$.

Either
$$\gamma = -1 \quad c_1 \neq 0 \quad c_2 = 0$$
or
$$\gamma = 1 \quad c_1 = 0 \quad c_2 \neq 0$$

The invariant vector must be ϕ_1 or ϕ_2. But any other representation, say A, shows that this cannot be so. Hence no invariant vector exists, and so Γ' is irreducible.

10.4. We can now state the applications to quantum mechanics of the representation theory for the classification of eigen-values.

<u>Theorem</u>: If a Hamiltonian is invariant under a group G of symmetry transformations, then the eigenfunctions belonging to one energy level form a basis for a representation of G.

If T is a symmetry transformation belonging to G, applying it to the equation $H\psi(q_i) = E\psi(q_i)$, we get

$H\psi'(q_i) = E\psi'(q_i)$ where $T\psi = \psi'$.

ψ' is also an eigen function of H belonging to E.
Any eigenfunction belonging to E is transformed by T to another eigen function belong to E. Also if ψ_1, ψ_2 are two such functions, then $c_1\psi_1 + c_2\psi_2$ is also such a function.

Therefore all eigen functions belonging to one energy level form a vector space which is invariant under G, and is therefore a basis for a representation of G.

Thus we can label and describe an energy level (and its eigen-functions) simply by naming the representation associated with it. This, of course, does not tell us everything about eigen-functions, expecially numerical calculations, but it does indicate the symmetry properties.

For example consider the lithium atom with 3 electrons. Its Hamiltonian is invariant under S_3, the group of permutations of the electron coordinates. $(x_1, y_1, z_1) \to (x_i, y_i, z_i)$, $(x_2, y_2, z_2) \to (x_j, y_j, z_j)$, $(x_3, y_3, z_3) \to (x_k, y_k, z_k)$ where i, j, k is some permutation of 1,2,3, S_3 is isomorphic to the group G we had for the equilateral triangle and therefore any representation of G is also a representation of S_3. Thus S_3 has 3 different (non-equivalent) irreducible representations Γ, ~~it~~, $\mathcal{A}$, and ϑ. By the theorem the eigen-functions associated with one energy level form an invariant vector space under the group S_3 and this vector space can be reduced into subspaces each of which transforms according to one of the representations Γ, $\mathcal{A}$ and ϑ.
A wave function transforming according to the representation $\mathcal{A}$ is anti-symmetric in the usual quantum mechanical sense, i.e., it

It is known experimentally that the anti-symmetric states corresponding to A are the only ones ever found in nature. So that this sorting out of the wave functions according to irreducible representations of S_3 is very important.

Corollary: If a Hamiltonian is invariant under a group G of transformations, then eigenfunctions of the Hamiltonian transforming according to one irreducible representation of G belong to the same energy level.

For, from the theorem, the vector space of eigenfunctions belonging to one level is either irreducible or reducible to subspaces each of which is invariant under an irreducible representation. It thus never happens that eigen-functions belonging to the same irreducible vector space belong to different levels.

The question now arises whether eigenfunctions belonging to different irreducible vector spaces always belong to different levels. In general the answer is negative. However, if we find several irreducible vector spaces to be associated with each energy level in a systematic way, there must be some symmetry property which accounts for this degeneracy. Hence if we include all possible symmetry transformation in the group G, we would expect different irreducible vector spaces to have different energy levels, simply because no symmetry property is left. This conclusion is borne out by experience. Nevertheless a few accidental degeneracies remain.

10.5 The effect of perturbations:

We consider next the classification of energy levels and eigenfunctions of a complicated Hamiltonian H, which can be written

$$H = H_o + H_p$$

where H_o is a simple part, and H_p, that due to a perturbation. H_o is taken to be simpler in the sense that it has a higher degree of symmetry than H. For instance it is easier to calculate the energy levels of an electron in a spherically symmetric potential than in a potential varying in some directions. If H_o has a higher symmetry than H then the energy levels of H_o are more degenerate, because there are more symmetry transformations to make more eigenfunctions to have the same energy. The effect of H_p can be taken to be to split these levels. The degree of such splitting is limited by the number of irreducible components of the representation corresponding to an energy level of H. This is made precise in the following result.

If H, H_o and H_p are all invariant under a group G of symmetry transformations, and if the eigenfunctions of an energy level of H_o transform according to the representation

$D = D^{(1)} + D^{(2)} + \ldots + D^{(n)}$ where $D^{(i)}$'s are irreducible components of D, then the greatest splitting that H_p can cause is into n levels.

To see the truth of this statement, consider the eigenfunctions and energy levels of the Hamiltonian

$$H_\epsilon = H_o + \epsilon H_p$$

where ϵ varies continuously from 0 to 1. This can be thought of as a mathematical device or even sometimes physically achieved by reducing the perturbing magnetic field to zero steadily.

For an $\epsilon \neq 0$, let E_x be an energy level. The eigenfunctions of this level transform acc^{ord} to some irreducible representation $D^{(\alpha)}$ or according to a reducible representation $D^{\alpha} + D^{\alpha'}$ etc., if G does not contain all the symmetry elements of H.

As ϵ is varied continuously, the energy levels and eigenfunctions also vary continuously and the representation D^{α}, etc. cannot change discontinuously to some different (say non-equivalent) representation. So when $\epsilon \to 0$, several energies coalesce into one so that the representation $D^{(1)} + D^{(2)} + \ldots\ldots + D^{(n)}$ of the composite level corresponds exactly to the components $D^{(\alpha)}, D^{(\beta)}, \ldots\ldots D^{(\nu)}$ which have coalesced. Looking at this the other way round, we can say that the degenerate level is split by H_p into a maximum of n levels associated with the n-irreducible components $D^{(1)}$ to $D^{(n)}$.

Example: Consider a free hydrogen atom with the electron in $2p$ level The Hamiltonain is

$$H_0 = -\frac{\hbar^2}{2m}\nabla^2 - e\,V(r)$$

where $V(r)$ is the spherically symmetric potential due to the proton. The p-eigenfunctions have the form

$$\psi_x = x f(r), \quad \psi_y = y f(r), \quad \psi_z = z f(r)$$

where $f(r)$ is some function of r. If the atom is placed in an electric potential $V_{32}(r)$ with trigonal symmetry, i.e., given rise to by three atoms in an equilateral triangle

$$H_p = -e\,V_{32}(\vec{r})$$

H_0, H_ϕ, H_p are all invariant under the group of rotations R_{32}. In the free state, i.e., in the absence of H_p, the three eigenfunctions all have the same energy (This is because the hamiltonian H_0 is invariant under all rotation and it can be shown that the functions transform according to an irreducible representation of the rotation group, so that they belong to the same level). However they transform according to the reducible representation $\Delta = \Gamma + \vartheta$ of the group G. . So we may expect H_p to split the triply degenerate level of H_c to a non-degenerate level ϑ and a doubly degenerate level Γ.

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ELEMENTS OF LIE GROUPS AND LIE ALGEBRAS

By

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ELEMENTS OF LIE GROUPS AND LIE ALGEBRAS.

1. Lie group.

An abstract group is a set G of elements which possesses a binary law of composition, such that

(i) for any two elements a, b of G, $a.b.$ is an element of G,

(ii) if a, b, c are in G, $a.(b.c) = (a.b).c$,

(iii) there exists an element I in G such that whatever be a in G, $a.I = I.a = a$,

(iv) for each element a in G, there exists an element a^{-1} of G such that $a.a^{-1} = a^{-1}.a = I$

(Condition (ii) : is known as the associative law for the binary operation. 'I' is called the identity element of G and a^{-1} the inverse of a .)

If whatever be a, b in a group G, $ab = ba$, then G is said to be abelian.

A topological space is a set certain subjects of which are designated as 'open', the 'open subsets satisfying the following conditions :

(i) The intersection of any finite number of open subsets is open,

(ii) The union of any number of open subsets is open,

(iii) The empty subset and the whole space are open,

(iv) to each pair of distinct points there are open sets containing them which do not intersect.

A set G of elements is a <u>topological group</u> if

(i) G is an abstract group,

(ii) G is a topological space,

(iii) (a) If a and b are elements of G then for each open set W of $a.b.$ there exists open subsets U, V containing a, b respectively such that $U.V \subset W$ i.e. whatever be x in U, y in V, xy is in W,

(b) If a is in G, then for every open set V containing a^{-1} there exist open sets U containing a such that $U^{-1} \subset V$ (i.e. whatever be x in U, x^{-1} is in V).

A topological group G is called a <u>Lie Group</u>, if the following contions (i) and (ii) are satisfied :

i) A <u>coordinate system</u> can be introduced in G. By this we mean that to every r-triple $(s^1 \ldots s^r)$ in an open sets containing $(o, \ldots , o)$ of a r-dimensional euclidean space we can associate an element s in an open set U of G containing I in a one-one bicontinuous manner that the r-triple $(o, \ldots , o)$ corresponds to I. $(s^1 \ldots s^r)$ are called <u>coordinates</u> of s.

Let W be a sufficiently small open set containing I and contained in U so that whatever be s, t in W, $s.t$ is also in W. Then the coordinates of $s.t$ are functions of the coordinates of s and the coordinates of t. Thus, if $s.t = u$ and u^α, the α^{th} coordinate of u,

$$u^{\alpha} = f^{\alpha}(s^1, \ldots, s^r; t^1, \ldots, t^r), \quad \alpha = 1, \ldots, r \qquad (1)$$

(ii) The function f^{α} is differentiable an arbitrary finite number of times with respect to the parameters, s^1, $\ldots$, s^r; t^1, $\ldots$, t^r.

The number r is called the <u>dimension</u> of G.

For sake of brevity, we write 1 as

$$u^{\alpha} = f^{\alpha}(s, t), \quad \alpha = 1, \ldots, r \qquad (1^1)$$

2. <u>Structure constants.</u>

Let G be a Lie Group. As $I = (o, \ldots, o)$ and $uI = Iu = u$, from (1) we get

$$\begin{aligned} f^{\alpha}(s^1, \ldots, s^r; 0, \ldots, 0) &= s^{\alpha} \\ f^{\alpha}(0, \ldots, 0; t^1, \ldots, t^r) &= t^{\alpha} \end{aligned} \qquad (2)$$

In view of (ii) in the definiton of a Lie Group, we can develop f^{α} as a Taylor's series which we give below with summation convention as

$$f^{\alpha}(s^1, \ldots, s^r; t^1, \ldots, t^r)$$
$$= s^{\alpha} + t^{\alpha} + a^{\alpha}_{\beta\gamma} s^{\beta} t^{\gamma} + g^{\alpha}_{\beta\gamma\delta} s^{\beta} s^{\gamma} t^{\delta} + h^{\alpha}_{\beta\gamma\delta} s^{\beta} t^{\gamma} t^{\delta} \longrightarrow \qquad (3)$$

If $s\tilde{s} = I$, then from (3) we can manage to get the α^{th} coordinate $\tilde{s}^{\alpha}$ in terms of the coordinates of s as

$$\tilde{s}^{\alpha} = -s^{\alpha} + a^{\alpha}_{\beta\gamma} s^{\beta} s^{\gamma} + \ldots \qquad (4)$$

If s and t are elements G, consider the element $q(s,t) = s\,t\,s^{-1}\,t^{-1}$ (called the <u>commutator</u> of s and t). Then from (3) and (4) we obtain

$$q^{\alpha}(s,t) = c^{\alpha}_{\beta\gamma}\, s^{\beta} t^{\gamma} + \dots \qquad (5)$$

The constants $c^{\alpha}_{\beta\gamma}$ for α, β, γ varying from $1, \dots, r$ are called the <u>structure constants</u> of G. From (3) and (5) we can deduce that $c^{\alpha}_{\beta\gamma} = a^{\alpha}_{\beta\gamma} - a^{\alpha}_{\gamma\beta}$. Hence

$$c^{\alpha}_{\beta\gamma} = - c^{\alpha}_{\gamma\beta} \qquad (6)$$

From the associate law for G, we can deduce another condition on the r^3 structure constants of G. If s, t, u are in G, as $(s.t).\,u = s.\,(t.\,u)$,

$$f^{\alpha}(f^{\alpha}(s,t),u) = f^{\alpha}(s, f^{\alpha}(t,u)) \quad \text{(cf. (1)}$$

Substituting (3), for the elements of third order in the equation (which is identically fulfilled inthe first and second order), we have for the coefficient of $s^{\beta} t^{\gamma} u^{\delta}$, the relation

$$a^{\alpha}_{\sigma\delta}\, a^{\sigma}_{\beta\gamma} - a^{\alpha}_{\beta\sigma}\, a^{\sigma}_{\gamma\delta} = h^{\alpha}_{\beta\gamma\delta} + h^{\alpha}_{\beta\delta\gamma} - g^{\alpha}_{\beta\gamma\delta} - g^{\alpha}_{\gamma\beta\delta} \qquad (7)$$

As $h^{\alpha}_{\beta\gamma\delta} = - h^{\alpha}_{\gamma\beta\delta}$ etc., the right hand side vanishes when summed with respect to α while the left hand side in view of (6) becomes

$$c^{\alpha}_{\beta\sigma}\, c^{\sigma}_{\gamma\delta} + c^{\alpha}_{\gamma\sigma}\, c^{\sigma}_{\delta\beta} + c^{\alpha}_{\delta\sigma}\, c^{\sigma}_{\beta\gamma} = 0 \qquad (8)$$

<u>N.B.</u> The structure constants depend on the coordinate system chosen for the euclidean space.

3. <u>Lie Algebra</u>.

Let g be the r-dimensional vector space in which the following operation of composition of vectors is defined:

(i) to every pair ξ, η of vectors, there corresponds a vector $\zeta = [\xi, \eta]$ called the Lie bracket of ξ, η.

(ii) $[\xi_1, c_1\eta_1 + c_2\eta_2] = c_1[\xi_1, \eta_1] + c_2[\xi_1, \eta_2]$ when c_1, c_2 are constants and ' + ' is vector addition,

(iii) $[\xi, \eta] + [\eta, \xi] = 0$

(iv) $[[\xi, \eta], \zeta] + [[\eta, \zeta], \xi] + [[\zeta, \xi], \eta] = 0$ for any triple ξ, η, ζ. (This property is known as the <u>Jacobi identity</u>.) Then g is called a <u>Lie Algebra</u>.

Let $\{e_\alpha\}$, $\alpha = 1, \ldots, r$ be a basis for g. Then $[e_\beta, e_\gamma] = c^\alpha_{\beta\gamma} e_\alpha$ (written with summation convention) for a suitable choice of r^3 constants $c^\alpha_{\beta\gamma}$, α, β, γ varying from 1 to r. $\{c^\alpha_{\beta\gamma}\}$ are called the <u>structure constants</u> of g (These obviously depend on the choice of the basis).

From (iii) and (iv) it follows that

(a) $c^{\alpha}_{\beta\gamma} = - c^{\alpha}_{\gamma\beta}$

(b) $c^{\alpha}_{\beta\sigma} c^{\sigma}_{\gamma\delta} + c^{\alpha}_{\gamma\sigma} c^{\sigma}_{\delta\beta} + c^{\alpha}_{\delta\sigma} c^{\sigma}_{\beta\gamma} = 0$

Conversely, if $\{ c^{\alpha}_{\beta\gamma} \}$ are r^3 constants satisfying (a) and (b), then by defining $[e_\beta, e_\gamma] = c^{\alpha}_{\beta\gamma} e_\alpha$ and extending by linearity the bracket operation for any two vectors of a r-diml. vector space for which $\{ e_\alpha \}$ is a basis, we get a Lie algebra structure $\mathfrak{g}$ for which $\{ c^{\alpha}_{\beta\gamma} \}$ are the structure constants. Thus a Lie Algebra is completely specified by its structure constants.

4. <u>The Lie Algebra of a Lie Group</u>.

A collection ($(s(\tau))$) of elements in a Lie group G depending continuously on a real parameter τ varying on a real interval such that s (0) = I is called a <u>curve</u> in G .

We shall say that the curve ($s(\tau)$) has a <u>tangent</u> if the derivatives $\xi^\alpha = \frac{d s^\alpha(\tau)}{d\tau}\Big|_{\tau=0}$ exist. The r-vector whose components are ξ^α , $\alpha = 1, \dots r$ is called the <u>tangent vector</u> of the curve in question.

Thus we <u>associate with a r-dimensional Lie group</u> G , <u>r-dimensional vector space</u> $\mathfrak{g}$ composed of all tangents to the curves in G . This association can be shown to be independent in a natural manner of the choice of a basis for the euclidian space which gives the coordinates of elements of G .

If ξ, η are tangents to the curves $s(\tau)$, $t(\tau)$ respectively and if $u(\tau)$ is the curve such that

$u(\tau) = s(\tau).t(\tau)$, the tangent ζ to $u(\tau)$ is given by $\zeta = \xi + \eta$.

With the same notations as in previous para let $q(\tau)$ be the curve such that

$$q(\tau) = s(\tau)\, t(\tau)\, (s(\tau))^{-1}\, (t(\tau))^{-1}$$

Introducing the parameter $\sqrt{\tau}$ (the positive root of τ) let ζ be the tangent vector of $q(\tau)$. We define a bracket operation on g thus : For ξ , η in g define $[\xi, \eta] = \zeta$, ζ defined as above.

We can check up that the bracket operation defined on g satisfies the conditions of a Lie algebra. Further the structure constants of the Lie algebra are the same as those of the Lie group G in corresponding coordinates. <u>g is called the Lie algebra of the Lie group G</u> .

<u>We</u> quote <u>Lie's Fundamental Theorem</u> :

To every Lie group there corresponds a Lie algebra of same dimension; conversely, every Lie algebra determines unique upto 'local isomorphism' a Lie group (cf. Pontrjagin).

5. <u>Special classes of Lie Groups and Algebras</u>.

In sec. 1 , we defined an abelian group. <u>For an abelian Lie group</u>, inview of (6) in §. 2, <u>all structure constants are zero</u>.

<u>A commutative Lie algebra</u> is one where the bracket of any two elements is 0 . <u>The structure constants of a commutation Lie algebra are zero</u>.

A subgroup of a group G is a non-void subset which with respect to the induced operation constitutes a group. An <u>invariant subgroup</u> S is a subgroup such that whatever be x in G, $x\,S\,x^{-1}$ is contained in S.

A subring of a Lie algebra $\mathfrak{g}$ is a subgroup of $\mathfrak{g}$ with respect to vector addition which is closed for bracket operation. An <u>ideal</u> J of $\mathfrak{g}$ is a subring of such that whatever α in $\mathfrak{g}$ and ξ in J, $[\alpha, \xi]$ is in J.

By a <u>simple group</u>, we mean a group which does not contain any invariant subgroup other than the whole group and the identity element, considered as subgroups. A group is <u>semi-simple</u> if <u>it does not containany abelian invariant subgroup</u>.

The <u>Lie $\mathfrak{g}$ algebra</u> of a simple (respectively semi-simple) Lie group is itself said to be <u>simple (semi-simple)</u>.

6. <u>Some Properties of Semi-simple Lie Algebras</u>.

Let $\mathfrak{g}$ be a Lie algebra with structure constants $\{c^{\alpha}_{\beta\gamma}\}$, α, β, γ ranging from 1 to r. <u>We define the symmetric tensor g_{ik} thus</u> :

$$g_{ik} = g_{ki} = c^{\alpha}_{i\beta}\, c^{\beta}_{k\alpha} \qquad (1)$$

<u>Theorem 1</u>. (Cartan's criterion) : The necessary and sufficient condition that a Lie algebra $\mathfrak{g}$ be semi-simple is that $\det|g_{ik}| \neq 0$.

By a <u>Linear Lie algebra</u> , we mean a Lie algebra the elements of which are linear operators acting on a vector space.

<u>Lemma</u> : Let $\mathfrak{g}$ be a Lie algebra and A an element of $\mathfrak{g}$. Define for each X in $\mathfrak{g}$, A (x) as the element $[A, x]$ of $\mathfrak{g}$. Then A (x) is a linear operator on the vector space underlying $\mathfrak{g}$. (The operator A (x) is 0 if and only if A commutes with each element of $\mathfrak{g}$).

With usual addition of operators and defining the bracket of two operators defined in the above lemma as the operator defined by the bracket of the elements determining them, we can prove that such operators constitute a (linear) Lie algebra. As a semi-simple Lie algebra cannot contain a commutative ideal, it follows that such a Lie algebra is identical with the Lie algebra constituted by the operators A (x). Hence we have

<u>Theorem 2</u> . Every semi-simple Lie algebra is a linear Lie algebra.

For linear operators we have the usual product operator which is an associatve operation. Thus in any semi-simple Lie algebra we have an associative product denoted by $\cdot$.

We now define the <u>CasimirForm</u> of a semi-simple Lie algebra $\mathfrak{g}$
Let g^{ik} be normalized cofactors of det $|g_{ik}|$ (cf. (1) above), i.e. $g^{i\alpha} g_{\alpha k} = \delta_i^k$. As $\det |g_{ik}| \neq 0$, $g^{i\alpha}$ exist.

<u>The quadratic form</u>,

$$F = g^{\alpha\beta} e_\alpha \cdot e_\beta \tag{2}$$

(where $e_\alpha . e_\beta$ is the associative product referred to above) is called the Casimir Form of $\mathcal{g}$.

Themrem 3. (Casimir). The casimir form F of a semi-simple Lie algebra commutes with every element (equivalently, $[F , e_r] = 0$).

The following theorem relates semi-simple Lie algebras to simple ones.

Theorem 4. (Cartan) Every semi-simple Lie algebra is a 'direct sum' of all its simple ideals.

7. The Standard Form of a Semi-simple Lie Algebra.

Let $\mathcal{g}$ be a Lie algebra of dimension r . Consider the eigen value problem of the operator A (x) defined in the Lemma in §. 6 . i.e. $A(x) \equiv [A , x] = \rho x$. If the secular equation of the operator has r distinct roots, then we have r linearly independent eigen vectors which can be used as a basis for the vector space underlying $\mathcal{g}$. If, however, the secular equation has degenerate roots, r linearly independent vectors may not exist. Hence, a coordinate system for $\mathcal{g}$ cannot be arrived at by the above method. But for semi-simple Lie algebras we have the following

<u>Theorem</u> (Cartan) : For a semi-simple Lie algebra $\mathfrak{g}$ if we choose A so that the secular equation of A (x) has the maximum number of distinct roots (which we can), the only degenerate root is $\rho = 0$ and if ℓ is the multiplicity of the root, there exist corresponding to this root, [illegible] ℓ linearly independent eigen vectors any two of which commute.

The number ℓ is called <u>rank</u> of $\mathfrak{g}$.

We shall choose as basis the ℓ linearly independent eigen vectors (say) $H_1 , \dots , H_\ell$ corresponding to the degenerate root $\rho = 0$ [illegible] together with the ($r - \ell$) linearly independent eigen vectors E_α , E_β . . . corresponding to the distinct roots α, β, ,

The commutational relations for $H_1 , \dots , H_\ell$; E_α , E_β, ... can be obtained to be

$$[H_i, H_k] = 0 \qquad (1)$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha \qquad (2)$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \text{ if } \alpha+\beta \text{ is not a vanishing root} \qquad (3)$$

$$[E_\alpha, E_{-\alpha}] = \alpha^i H_\alpha \qquad (4)$$

The structure constants are then,

$$c_{ik}^{\tau} = 0 ; \; c_{i\alpha}^{\tau} = \alpha_i \delta_\alpha^\tau ; \; c_{\alpha\beta}^{\alpha+\beta} = N_{\alpha\beta} ; \; c_{\alpha\beta}^{\tau} = 0 \text{ if } \tau \neq \alpha+\beta$$

Further, $[A, H_i] = 0$ (5)

$$[A, E_\alpha] = \alpha E_\alpha \qquad (6)$$

As A is a eigenvector of $[A, x] = \rho x$,

$$A = \lambda^i H_i \qquad (7)$$

From (6), (7) and (2), it follows that

$$\alpha = \lambda^i \alpha_i \qquad (8)$$

8. The concept of Root and Schouten Diagrams.

The form in (8) of sec. 7 is called a root of the semi-simple Lie algebra g. It can be thought of as a vector in a l-dimensional vector space.

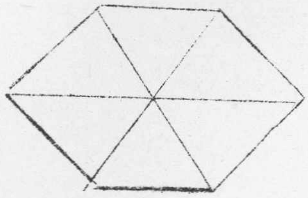
Let α, β be vectors. $(\alpha\beta)$ denotes their scalar product.

Theorem 1. If α and β are two roots of g, then $\frac{2(\alpha\beta)}{(\alpha\alpha)}$ is an integer and $\beta - \frac{2(\alpha\beta)}{(\alpha\alpha)}\alpha$ is a root.

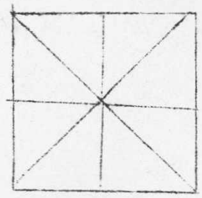
Theorem 2. Let α, β be two roots and φ angle between them. Then φ can have only the values 0°, 30°, 45°, 60° and 90°. Further the ratio of the lengths of the two vectors α, β are $\sqrt{3}$ for 30°, $\sqrt{2}$ for 45°, 1 for 60° and indeterminate for 90°. If $\varphi = 0$, $\alpha = \beta$.

By a Schoten diagram of a vector diagram we mean the graphical representation of the root vectors of g. All simple Lie algebras (and hence simple Lie groups) can be derived from these diagrams.

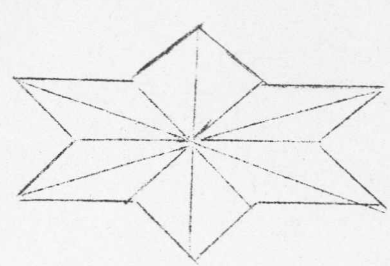
The diagrams, the only possible, in two dimensions are given below:



A_2



B_2



G_2

9. The classification of Simple Lie Algebras.

Simple Lie algebras can be classified by their Schouben diagrams. Let l be an integer and $\{e_i\}$ an orthonormal basis in a l-dimensional vector space. We have the following classes of vector diagrams and hence simple Lie groups:

A_l : The collection of $l(l+1)$ differences $\{e_i - e_k\}$ of $l+1$ unit vectors e_i . The dimension of the algebra is $(l+1)^2-1$. For $l = 2$, we get the diagram A_2.

B_l : Consists of $\{\pm e_i\}$ and $\{\pm e_i \pm e_k\}$, $i,k = 1,\ldots,l$ numbering $2l^2$. The dimension of the algebra is $l(2l+1)$ For $l = 2$, we get B_2 .

C_l : Another generalisation of B_2; consists of $\{\pm 2e_i\}$ and $\{\pm e_i \pm e_k\}$.

D_l : The collection $\{\pm e_i \pm e_k\}$ for $l > 2$. There are $2l(l-1)$ of them, the dimension of the algebra being $l(2l-1)$.

There are however just five exceptions:

1. G_2 (Cf. § 8)

2. F_4 : The diagram of B_4 with 16 more vectors $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$. (Total 48 vectors and dimension is 52).

3. E_6 : The diagram A_5 , the vectors $\pm \sqrt{2}\, e_7$ and $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6) \pm e_7/\sqrt{2}$ where in the former fraction three signs positive and three negative. Number of vectors 72 and dimension 78.

4. E_7 : The diagram A_7 and the vectors $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8)$ when we take four positive and four negative signs Numbers of vectors 126 and dimension 133.

5. E_8 : The diagram D_8 and the vectors $\frac{1}{2}(\pm e_1 + e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8)$ with each sign occurring an even number of times. There are 240 vectors and dimension of algebra is 248.

Realizations of the groups A_ℓ , B_ℓ , C_ℓ and D_ℓ are the so-called classical groups.

10. Representation of Lie Groups and Lie Algebras.:

Let G be a Lie group. If to each element g of G we can associate a linear operator $T(g)$ of a certain n-dimensional vector space V such that to $g_1 \cdot g_2$ in G the operator $T(g_1) \cdot T(g_2)$ corresponds and the association $g \rightarrow T(g)$ is further continuous, then T is a n-dimensional representation of G .

Let $\mathfrak{g}$ be a Lie algebra. If to each element ξ of $\mathfrak{g}$ we can associate an operator $A(\xi)$ acting on V such that

$$A(\xi+\eta) = A(\xi) + A(\eta)$$
$$A(c\cdot\xi) = c\,A(\xi)$$
$$A([\xi,\eta]) = [A(\xi), A(\eta)]$$

then A is said to be a n-dimensional representation of $\mathfrak{g}$.

Theorem 1 . Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then any representation of G gives rise to a representation of $\mathfrak{g}$ and vice versa.

Theorem 2 : Any representation of a Lie Algebra (and hence of a Lie group) is determined by r matrices D_ρ which are such that $[D_\rho, D_\sigma] = c^{\tau}_{\rho\sigma} D_\tau$ ($c^{\tau}_{\rho\sigma}$ being the structure constants).

Two representations $\xi \to A_1(\xi)$, $\xi \to A_2(\xi)$ of a Lie Algebra are equivalent if there is an operator U such that $U\,A_1(\xi)\,U^{-1} = A_2(\xi)$ whatever be ξ .

A representation $\xi \longrightarrow A(\xi)$ is reducible if the operators $A(\xi)$ leave a proper subspace of V invariant. If a representation A is reducible, there is an equivalent representation A^1 such that for each ξ, $A^1(\xi)$ has the matrix form. $\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}$. A not reducible representation is called irreducible.

A representation $\xi \longrightarrow A(\xi)$ is decomposable if the operators $A(\xi)$ leave a two mutually orthogonal sub-spaces which together span the whole space V, invariant.

If a representation A is decomposable then there is an equivalent representation A^1 such that for each ξ, $A^1(\xi)$ has the matrix form $\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$.

11. Decomposability of Representations of a Semi Simple Lie Al~~gb~~ebra.

We give the steps to the result that every finite dimensional representation of a semi-simple Lie Algebra can be decomposed into irreducible ones.

Theorem 1. (Brauer). If all representations of a Lie algebra of the form

$$\begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}$$

where A is an irreducible representation are decomposable, then all finite dimensional representations are decomposable.

Theorem 2. The eigenvalue λ of the Casimir form (cf. s. 6) defined for any semi-simple Lie algebra is different from zero for every non-trivial finite dimensional irreducible representation of the Lie algebra.

By application of Schur's lemma, we get.

Theorem 3. Every finite dimensional representation of a semi-simple Lie algebra can be decomposed into irreducible representations.

12. The concept of weight.

Consider a n-dimensional matrical representation of a semi simple Lie algebra $\mathfrak{g}$. The representation is completely specified by r matrices (r being dimension of $\mathfrak{g}$) D_ρ which satisfy the equation

$$[D_\rho, D_\sigma] = c^{\tau}_{\rho\sigma} D_\tau$$

when $c^{\tau}_{\rho\sigma}$ are the structure constants of $\mathfrak{g}$.

Let us express the representation with respect to the standard form of a semi simple Lie algebra discussed in §.7 . With notation as in that section, let H_1' , ... H_ℓ' ; E_α' , E_β' . . . be the matrices in the representation corresponding to the basis $H_1, \ldots, H_\ell ; E_\alpha, E_\beta, \ldots$ of $\mathfrak{g}$.

Let u be a simultaneous eigen vector of the matrices H_1' , .. , H_ℓ' so that $H_i' u = m_i u$. Let m be the ℓ-vector whose components are (m_1 , ... , m_ℓ) The vector m is called the weight of the eigen vector u .

Theorem 1. Every representation has atleast one weight.

Theorem 2. A vector u of weight m which is a linear combination of vectors u_k of weights m^k , $m^k \neq m$ for each k , must vanish.

Theorem 3. There exist at the most n linearly independent weights corresponding to a representation.

Theorem 4. If u is a vector of weight m , then $E_\alpha' u$ and $H_i' u$ have weights m and m + α respectively where $\alpha = (\alpha_i)$ and $[H_i, E_\alpha] = \alpha_i E_\alpha$ (cf. § 7)

Theorem 5. If a representation is irreducible, then all the H_i' 's can be simultaneously expressed in diagonal form.

Theorem 6. For any weight m and root α, $\frac{2(m\alpha)}{(\alpha\alpha)}$ is an integer and $m - \frac{2(m\alpha)}{(\alpha\alpha)}\alpha$ is a weight.

Theorem 7. The set of all weights is invariant under the group S of transformations generated by the reflections with respect to the hyperplanes passing through the origin and perpendicular to the roots.

13. The classification of Irreducible Representations.

Two weights are said to be equivalent if they can be obtained from each otherby transformations belonging to the group S in Theorem 7 (sec. 12). A weight is positive if the first non-vanishing component is positive. One weight is higher than the other if their difference is positive. A weight higher than its equivalent, is said to be dominant. A weight is said to be simple if it belongs to only one eigenvector.

Theorem 1. If a representation is irreducible, then its highest weight is simple.

Theorem 2. Two irreducible representations are equivalent, if their highest weights are equal.

Theorem 3. For every simple Lie algebra of rank ℓ (cf. S. 8) there are ℓ dominant weights (called fundamental dominant weights) so that every dominant weight is a non-negative integral linear combination of them.

Theorem 4. There exist ℓ fundamental irreducible representations $A_1, \ldots, A_\ell$ which have the fundamental weights as their highest weights.