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MATSCIENCE REPORT

LECTURES ON
AN INTRODUCTION TO
COMPLEX ANGULAR MOMENTUM, REGGE POLES, AND HIGH ENERGY SCATTERING.

K. RAMAN

THE INSTITUTE OF MATHEMATICA' CES, MADRAS-4, INDIA.

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THE INSTITUTE OF MATHEMATICAL SCIENCES

MADRAS - 4 (India)

An Introduction to

Complex Angular Momentum, Regge Poles,

and High Energy Scattering.

Lectures by

K. RAMAN*

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* Senior Research Fellow of the Department of Atomic Energy, Government of India.

PREFACE

This is a series of 12 introductory lectures on Regge poles and ~~their role in high-energy diffraction scattering~~ given during the academic year 1962-63. In these lectures we give a critical survey of the basic ideas of the subject and of the experimental observations from which these ideas stem.

Lecture I is a general introduction to the idea of complex angular momentum. In lectures II - V we give an account of Regge's original work on potential scattering, following closely the paper by Bottino, Longoni, and Regge. [Nuovo Cim. 23, 954 (1962).] Lectures VI - IX deal with high-energy diffraction scattering; starting with the work by A. Salam and P.T.Matthews on dispersion relations and the optical model of diffraction scattering we consider in detail the modification of the semi-classical picture proposed by Lovelace, and finally see how information about the various Regge trajectories could be obtained from an analysis of high-energy data, if there were no branch cuts in the angular momentum plane.

Lecture X gives an account of the extension (to Re $\ell < -\frac{1}{2}$) of the Regge representation proposed by Mandelstam; we follow closely Mandelastm's paper (Annals of Physics 19, 254 (1961)).

In lecture XI, we discuss in detail the conjectures by Frautschi, Gell-Mann and Zacharisen on carrying over the results of potential scattering to a relativistic theory.

Finally, in lecture XII, we begin considering how Regge poles could be obtained from a relativistic theory. After briefly indicating why one should introduce the concept of Regge poles into a relativistic theory, we consider how Regge poles arise naturally in a perturbation theory. In conclusion we discuss the possible role of Regge poles in electrodynamics and weak interactions, presenting some speculation on these.

We shall deal with the various important questions arising in relation to Regge poles and a relativistic theory in future lectures. The recent Brookhaven experiments showing that the πp and Kp diffraction peaks do not shrink make it very likely that branch cuts in the angular momentum plane are important. We shall consider these in detail in future lectures.

No claim to originality is made with regard to the material presented in these lectures. Our purpose was to build a background that enables one to understand the theoretical work on Regge poles, and we have made free use of all the available sources. (A list of references is given on pages (iv) to (ix).)

K. Raman.

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GENERAL INTRODUCTION.

In S-matrix theory, we are familiar with the idea of considering the scattering amplitude as a function of one or two complex variables with well-defined singularities. In potential theory, the variables chosen are k and $\cos\theta$; in the relativistic theory, they are $s = E^2$ and t = the (4-momentum transfer)^2. The S-matrix is physically defined only for real E and real $\cos\theta$; thus to consider it as a function of complex E and complex $\cos\theta$, a prescription is needed to make the necessary analytic continuation, i.e. the various singularities in the product of the E plane and the $\cos\theta$ plane must be known. We enquire whether it is meaningful to consider a continuation in some pair of variables other than E and $\cos\theta$. Regge showed that in potential scattering it is useful to consider the scattering amplitude as a function of a complex energy and a complex angular momentum.

The utility of such considerations will be determined by their success in explaining the different aspects of relativistic interactions. In the first place a useful description of the S-matrix is one that predicts the behaviour of the scattering amplitude and gives the observed bound states and resonances.

The Concept of Complex Angular Momentum.

The consideration of the S-matrix $S(\ell, E)$ as a function of complex ℓ and complex E may be looked upon as a purely formal extension designed to give a more complete understanding of the properties of the function $S(\ell, E)$. However, when complex values of the variables are associated with a physical state, it is desirable to have some interpretation of the real and imaginary parts of the variable

A complex energy is a familiar concept. Consider a physical system whose time dependence is essential (i.e. not just the stationary time-dependence $\exp iEt$). It may be described (non-relativistically) by a time-dependent wave equation:

$$\mathcal{H}(\vec{q}, t)\, \Psi(\vec{q}, t) = i\hbar \frac{\partial}{\partial t} \Psi(\vec{q}, t) \tag{1}$$

Suppose the wave function represents the simplest type of decaying state — an exponentially decaying state; then the time dependence will be contained in a factor,

$$e^{iEt}\, e^{-\frac{1}{2}\Gamma t},$$

i.e.

$$\Psi(\vec{q}, t) = \psi(\vec{q})\, e^{iEt}\, e^{-\frac{1}{2}\Gamma t} \tag{2}$$

$$= \psi(\vec{q})\, e^{i\mathcal{E}t}, \quad \mathcal{E} = E + i\frac{\Gamma}{2}. \tag{2a}$$

Thus if one introduces the notion of a complex energy, the time-dependent (decaying) state may be obtained as a solution of a stationary wave equation with complex eigenvalues:

$$H(\vec{q})\, \psi(\vec{q}) = \mathcal{E}\, \psi(\vec{q}) \tag{3}$$

Thus the time-dpendent Hamiltonian $\mathcal{H}(\vec{q}, t)$ has been replaced by an equivalent time-independent (i.e. 'stationary') but <u>non-Hermitian</u> Hamiltonian. The real part of the eigenvalues $\mathcal{E}$ gives the energy to be associated with the system, while the imaginary part is related the uncertainly in the time coordinate (t) of the system,

$$\mathcal{Im}\,\varepsilon = \frac{1}{2}\Gamma = \frac{1}{2}\cdot\frac{1}{\Delta t} = \frac{1}{2}\cdot\frac{1}{\tau}, \qquad (4)$$

τ being the mean life of the system.

At first sight there seems to be no direct interpretation of a complex angular momentum. This is partly because the angular momentum (J) takes on only discrete values in quantum mechanics.

First we note that angular momentum in classical mechanics i a continuous variable, taking positive real values. It is only in Q.M. that it is quantized. This is somewhat analogous to the energy values of bound states (or more generally, those of a particle in a finite box) being allowed a continuum of values in classical mechanics, but only a discrete set of values in Q.M. For a real particle occupying one of the discrete energy levels E_i in a box, the allowed momentum $\vec{p}_i$ is related to E_i by

$$|\vec{p}_i| = \sqrt{E_i^2 - m_0^2},$$

m_0 being the mass of the particle. We can still conceive of particles with unphysical energies (not in the discrete set) when they appear as exchanged particles.

By a somewhat superficial analogy we may also conceive of particles having physical values of J when they appear as real particles but unphysical J values when they appear as exchanged particles.

Alternatively, if one resolves the amplitude corresponding to the exchange of an unphysical J value into its projections on physical J values, one may consider the exchange of a particle of unphysical J as equivalant to the exchange of a family of particles of physical J .

Just as the real and imaginary values of a complex energy are related to the expectation values of a pair of canonically conjugate variables, viz. E and t , those of a complex angular momentum can be related to the expectation values of J_z and φ_z, which are canonically conjugate.

Consider a wave function corresponding to a state with a definite angular momentum; i.e. an eigenfunction of J^2 and J_z. This may be associated with a particle moving in a closed orbit. The angle coordiate of this particle is completely uncertain; the wave function is of the form (assuming it to be axi-symmetric)

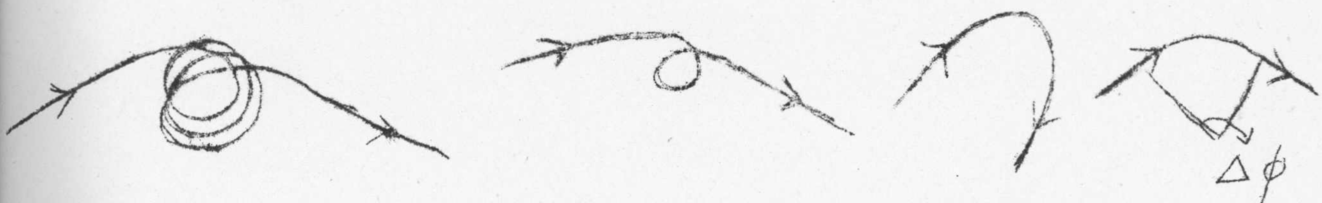
$$\Psi(\rho, z, \phi) = \psi(\rho, z)\, e^{i m_z \varphi}, \tag{5}$$

where ρ, z, ϕ are cylindrical coordinates, and m_z is the eigen-value of J_z .

For a system whose J^2, J_z values ~~are uncertain~~, one may expect the representation

$$\Psi(\rho, z, \phi) = \psi(\rho, z)\, e^{i m_z \phi}\, e^{-\gamma\phi} = \psi(\rho, z)\, e^{i(m_z + i\gamma)\phi}, \tag{5a}$$

where the factor $e^{-\gamma\phi}$ expresses the fact that the particle does not indefinitely go around in the same orbit; as ϕ increases, the probability of the particle's being in the orbit decreases; thus one may visualize an orbit of one of the forms shown (asymptotic orbits)



If we write

$$\Psi(\rho, z, \phi) = \psi(\rho, z)\, e^{i[m_z + i\Delta m_z]\phi} \tag{5b}$$

then we obtain

$$\gamma = \Delta m_z = \frac{\hbar}{\Delta\phi}. \tag{5c}$$

The width of the state is thus inversely related to the angle of orbiting, $\Delta\phi$.

From the partial-wave Schrodinger equation

$$\left[\frac{d^2}{dx^2} + E + \frac{(\beta + \frac{1}{4})}{x^2} - V(x)\right]\psi(x) = 0,$$

$$\beta \equiv -(\ell + \tfrac{1}{2})^2,$$

We can obtain

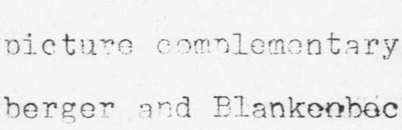
$$\frac{dE}{d\beta} = -\frac{\int_0^\infty \frac{\psi^2}{x^2}\, dx}{\int_0^\infty \psi^2\, dx}.$$

Start with a real value of ℓ, and thus of β. A small increase $-i\Delta\beta$ in β will cause a change $\Delta E = \frac{i\Delta\beta}{R^2}$ in E, where

$$\frac{1}{R^2} = \frac{\int_0^\infty \frac{\psi^2}{x^2}\,dx}{\int_0^\infty \psi^2\,dx}$$

E thus becomes complex. Thus a complex value of β may be interpreted as describing a decaying state.

The diagrams shown above are reminiscent of the orbits made by satellites (or by comets, etc.). Thus complex angular momentum is just an unusual representation of something that is quite familiar. Sommerfeld applied it to the propagation of radio waves around the Earth; It has also been applied to optical diffraction; one thinks of a light ray following the surface of a smooth object for some distance before emerging as the diffracted ray.



The example of optical diffraction takes us to another representation of scattering---that using the idea of the impact parameter. This is a semi-classical picture complementary to the angular momentum representation. Goldberger and Blankenbecler have developed a treatment of scattering phenomena that is a generalization of the impact-parameter representation.

A resonance may be thought of as a state in which two particles orbit around each other for some time and then separate; thus we may expect that a resonance may be described by a complex angular momentum.

Complex Angular Momentum and Composite Particles.

Consider the scattering of 2 particles through the exchange of a composite system like a bound state of two particles. Somewhat fanciful examples are $e^- e^-$ scattering through exchange of a positronium state, $\bar{n} p$ exchange scattering via exchange of a deuteron, etd. The composite particle exchanged may be in any one of its excited states (besides the ground state~~s~~); in general, one must then consider the exchange of a superposition of states with different angular momenta. If the resultant scattering could be approximated by that due to the exchange of a single particle of angular momentum J, then we may expect that, in general, J will not have a physical value; moreover, it~~s~~ may vary with the momentum transfer carried by the exchanged state. Regge poles are a generalization of these composite particles; they have been sometimes called generalized bound states and resonances.

Also, it is not clear how particles of spin greater than $\frac{1}{2}$ could be treated in a relativistic field theory; perturbation theories involving them seem to be unremormalizable . It turns out that the introduction of a complex angular momentum may resolve this difficulty.

Consider elastic $2 \to 2$ scattering, or the scattering of a particle by a potential. The scattering amplitude may be represented by a partial-wave expansion:

$$A(q, z) = \frac{1}{q} \sum_{l=0}^{\infty} (2l+1) \left[\frac{e^{2i\delta_l} - 1}{2i} \right] P_l(z),$$

$$z \equiv \cos\theta. \tag{6}$$

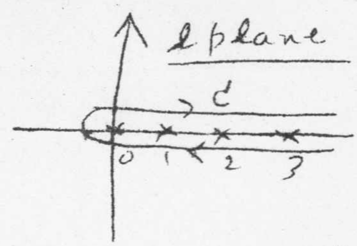
Regge showed that this amplitude could be continued into the complex ℓ plane (keeping E or q fixed) for $\mathrm{Re}\,\ell > -\frac{1}{2}$, for a certain class of potentials. In the ℓ plane, $A(q, \gamma)$ has only poles; for $E < 0$, the poles lie on the real ℓ axis, whereas for $E > 0$, they are at complex ℓ values. The amplitude may be written in the form

$$A(q,\gamma) = \frac{1}{q}\sum_{\ell=0}^{\infty}(2\ell+1)\left[\frac{e^{2i\delta_\ell}-1}{2i}\right]P_\ell(\gamma)$$

$$= \frac{1}{q}\int_C d\ell\,(2\ell+1)\left[\frac{e^{2i\delta_\ell}-1}{4}\right]\frac{P_\ell(-\gamma)}{\sin\pi\ell} \quad \text{--- (7a)}$$

$C \rightarrow$ (contour in the fig.)

$$= \frac{1}{q}\int_{-\frac{1}{2}-i\infty+\epsilon}^{-\frac{1}{2}+i\infty+\epsilon} d\ell\,(2\ell+1)\left[\frac{e^{2i\delta_\ell}-1}{4}\right]\frac{P_\ell(-\gamma)}{\sin(\pi\ell)}$$



$$+ \sum_i \beta_i(E)\,P_{\alpha_i(E)}(-\gamma) \quad \text{--- (7b)}$$

[Note. The contour C encircles the poles clockwise.]

The integral $\rightarrow 0$ as $|\gamma| \equiv |\cos\Theta| \rightarrow \infty$. The pole contributions survive as $|\gamma| \rightarrow \infty$. $\alpha_i(E)$ is the position of the i^{th} pole in the complex angular momentum plane. These poles are the "Regge poles"; for $E < 0$, they correspond to bound states; for $E > 0$, they correspond to resonances.

Near a pole $\ell \approx n$,

$$\frac{1}{\sin(\pi\ell)} \sim \frac{1}{\pi(\ell-n)(-1)^n} \quad (8)$$

One can write the pole terms in (7) as

$$\sum_{\ell=0}^{\infty}\left[\sum_i \frac{\beta_i}{(\alpha_i-\ell)(\alpha_i+\ell+1)}\right]P_\ell(\cos\Theta). \quad (9)$$

$\beta_i / [(\alpha_i - l)(\alpha_i + l + 1)]$ is the contribution of the i^{th} Regge pole to the l^{th} partial wave.

Suppose $Re\, \alpha_i \approx$ an integer $n\ (\geq 0)$ at some energy E_n. Then, for $E \simeq E_n$, the contribution of the i^{th} Regge pole to the l^{th} partial wave is

$$\frac{\beta_i}{(\alpha_i - l)(\alpha_i + l + 1)} = \frac{\beta_i / (\alpha_i + l + 1)}{Re\, \alpha_i(E) + i\, Im\, \alpha_i(E) - l}$$

$$\approx \frac{\beta_i / (\alpha_i + l + 1)}{\left[l + \frac{d}{dE}\left[Re\, \alpha_i(E)\right]_{E_n} (E - E_n) + \cdots \right] + i\, Im\, \alpha_i(E) - l}$$

$$\approx \frac{\beta_i / (\alpha_i + l + 1)}{\eta_i \left(E - E_n + \frac{i}{2} \Gamma_i \right)} \qquad (10)$$

where $\eta_i \equiv \frac{d}{dE}\left[Re\, \alpha_i(E) \right]$, $\qquad (10a)$

and $\Gamma_i \equiv 2 \left[Im\, \alpha_i(E) \right] \left\{ \frac{d}{dE} \left(Re\, \alpha_i(E) \right) \right\}^{-1}$. $\qquad (10b)$

This is of the Breit-Wigner form with a width Γ. Thus when $E_n > 0$, the pole in the l^{th} partial wave induced by the Regge pole may be interpreted as a resonance with mean energy E_n and width Γ. For this to be valid, Γ must be positive. Regge proved that for a bound states and narrow resonances, $\frac{d}{dE}(\mathrm{Re}\,\alpha) > 0$. Thus for $\mathrm{Im}\,\alpha > 0$, one obtains a resonance, for $E_n > 0$. For $E_n < 0$, we must have $\mathrm{Im}\,\alpha = 0$ in order to get a bound state.

In the above, we must restrict ourselves to l values such that $\mathrm{Re}\, l \geq -\frac{1}{2}$; Mandelstam has shown how the Reggee formula (7) may be extended so that we may consider the properties of the scattering amplitude in the whole l plane. (The Reggee formula cannot be extended just by moving the contour to the left of $\mathrm{Re}\, l = -\frac{1}{2}$, as $P_l(-z) \sim z^{-l-1}$ for $l < -\frac{1}{2}$, and the integral diverges.)

LECTURE II (23-8-62)

Complex Angular Momentum in Potential Scattering

It was in potential scattering that the utility of considering the angular momentum as a complex variable was first found out:

Consider the partial wave Schrodinger equation for a central potential $V(r)$;

$$D(k, l)\,\psi(r) = 0, \tag{1}$$

where
$$D(k,l) \equiv \left[\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} - V(r)\right]. \tag{1a}$$

Define $\lambda = (l + \frac{1}{2})$; since we shall be interested in the analytic properties of $V(r)$, we replace r by a complex variable ν; we write

$$D(k,\lambda)\,\psi(\lambda,k,z) = 0, \tag{1b}$$

where $$D(k,\lambda) \equiv \left[\frac{d^2}{dz^2} + k^2 - \frac{(\lambda^2-\frac{1}{4})}{z^2} - V(z) \right]. \tag{1c}$$

(We shall choose the units $\hbar = 1 = c = 2\mu$, μ being the reduced mass of the system.)

In the usual theory, l ranges over the values 0,1,2,........ Here we allow l , and hence λ , to range over the continuum of complex numbers.

We have noted earlier that one of the chief defects of partial wave representation (with integral l) is that the partial wave expansion can be proved to be convergent only in a very limited domain and therefore cannot be directly made to incorporate the properties of the Mandelstam representation. We shall see that this difficulty is overcome when l is allowed to become complex; we shall prove that for particular classes of potentials, the scattering amplitude may be proved to be analytically continuable for arbitrary momentum transfer; the analytic properties found for the scattering amplitude will be seen to be equivalent to those of the Mandelstam representation.

The important physical quantity that is a function of l is the phase shift $\delta(l)$. The extension to complex angular momentum may be viewed as an interpolation of $\delta(l)$ between the physical values at real, integral l . Of the different conceivable ways of performing this interpolation, we shall be interested in those functions $\delta(\lambda)$ that can be generated by a potential (and later, in the relativistic case, in functions $\delta(\lambda)$ with certain conjectured properties.)

In general, we are interested, physically, in scattering amplitudes with certain analyticity properties. A limited number of poles and branch cuts are allowed; these may be interpreted as bound states (or resonances or virtual states) and thresholds (in the relativistic theory) respectively. To obtain an analytic scattering amplitude, we expect that the interaction put in must have certain analyticity properties. Than the Fourier transform of the interaction must have corresponding analyticity properties; in the non-relativistic case, this Fourier transform is just the potential. Alternatively, if the scattering amplitude in the Born approximation has certain analyticity properties, its Fourier transform, the potential, must also have corresponding analyticity properties.

We consider the following class of potentials $V(z)$ ~~, (i) all~~ all $V(z)$ which may be expanded in the form

$$V(z) = \int_{m>0}^{\infty} \sigma(\mu) \frac{e^{-\mu z}}{z} d\mu \tag{2}$$

i.e. a continuous superposition of Yukawa potentials. This immediately tells us that $V(z)$ can be continued into the half-plane $\mathrm{Re}\, z \geqslant 0$. It has an essential singularity at $\mathrm{Re}\, z = -\infty$; so we shall avoid the left-half z-plane.

(ii) We also require that on any ray $\arg z = \phi$, $\phi \leq \frac{\pi}{2} - \epsilon$, we have

$$\int_0^\infty |V(z) z| \, |dz| < M < \infty \tag{3}$$

The value $\phi = \pi/2$ is excluded, because for the Yukawa potential $V(z) = e^{-\mu_1 z}/z$, (i.e. $\sigma(\mu) = \delta(\mu-\mu_1)$), we have

$$\int_0^\infty |V(z)\cdot z|\,|dz| = \int_0^\infty \exp\left[-\mu|z|\cos\phi\right]|dz| \longrightarrow \infty,$$

$$\text{as } \phi \to \pi/2$$

An immediate question is : What are the properties of the Fourier transform of the potential (2) ? Define

$$V(q) = \int d^3\vec{r}\; e^{i\vec{q}\cdot\vec{r}}\; V(r) \tag{4a}$$

$$= \int d^3\vec{r}\; e^{i\vec{q}\cdot\vec{r}} \int_m^\infty d\mu\,\sigma(\mu)\,\frac{e^{-\mu r}}{r}$$

Then

$$V(q) = \int d\mu\,\sigma(\mu) \int d^3\vec{r}\,\frac{e^{-\mu r}\, e^{i\vec{q}\cdot\vec{r}}}{r}$$

$$= \int_m^\infty d\mu\,.\,\sigma(\mu)\,.\;2\pi\int_0^\infty dr\,.\,r\int_{-1}^{+1} d(\cos\theta)\, e^{-\mu r}\, e^{iqr\cos\theta}$$

$$= \int d\mu\,.\,\sigma(\mu)\,.\;2\pi\int_0^\infty dr\;r\,.\,e^{-\mu r}\,.\;2\,\frac{\sin qr}{qr}$$

$$= \int d\mu\,.\,\sigma(\mu)\,.\,\frac{4\pi}{q}\int_0^\infty dr\,.\,e^{-\mu r}\sin qr\,.$$

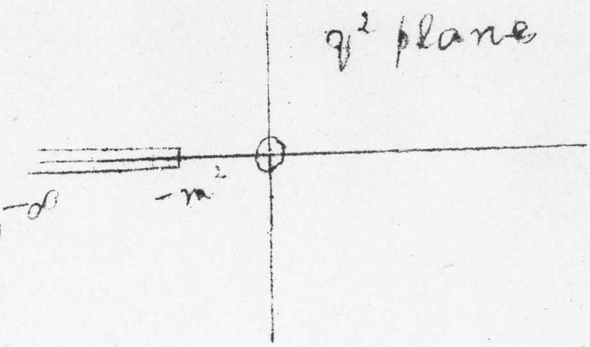
Let $I = \int_0^\infty dr\; e^{-\mu r}\,\sin qr$

$$= \left[-\frac{1}{q}\right] - \frac{\mu}{q}\int \cos qr\,.\,e^{-\mu r}\,dr$$

$$= -\frac{1}{q} - \frac{\mu^2}{q^2} I . \quad \text{Thus} \quad I = \frac{-q}{\mu^2 + q^2}$$

$$\therefore \quad \boxed{V(q) = 4\pi \int_{m>0}^{\infty} d\mu \; \sigma(\mu) \cdot \frac{1}{\mu^2 + q^2}} \qquad (4b)$$

Thus $V(q) = V(q^2)$ has a branch cut in the q^2- plane from $-m^2$ to $-\infty$; it is analytic in the cut q^2- plane. Thus the scattering amplitude in the Born approximation is analytic in this cut q^2- plane. We may note, in passing, that in Feynman perturbation theory, $\frac{1}{\mu^2 + q^2}$ is the amplitude arising from exchange of a particle of mass μ; thus the branch-point at $q^2 = -m^2$ corresponds to the exchange of the lightest particle. [Note that $\vec{q}$ here is the 3-momentum transfer. $\vec{q}^{\,2}$ also = the (4-mom. transfer)^2, for an elastic potential.]



Consider the equation

$$\left[\frac{d^2}{dz^2} + k^2 - \frac{(\lambda^2 - \frac{1}{4})}{z^2} - V(z)\right] \psi(\lambda, k, z) = 0 . \qquad (1c)$$

Solutions may be defined by imposing boundary condition at either the origin or at infinity.

Consider the limits of (1c) for $z \to 0$ and $z \to \infty$.

(i) $\underline{z \to 0}$. $\left[\frac{d^2}{dz^2} - \frac{(\lambda^2 - \frac{1}{4})}{z^2} - V(z)\right]\psi = 0 .$ - (5a)

(ii) $z \to \infty$:

$$\left[\frac{d^2}{dz^2} + k^2 - V(z)\right]\psi = 0 \tag{5b}$$

Thus λ dominates the bahaviour at the origin while k dominates the behaviour at infinity:

Consider the equation

$$D(\lambda, k, z)\,\varphi(\lambda, k, z) \equiv \left[\frac{d^2}{dz^2} + q(\lambda, k, z)\right]\varphi(\lambda, k, z) = h(\lambda, k, z)\,\varphi(\lambda, k, z) \tag{6}$$

Impose the boundary condition

$$\varphi(\lambda, k, z) \to \varphi_1(\lambda, k, z) \text{ at } z \to z_1 \tag{6a}$$

on the equation (1c).

(6) and (6a) together may be expressed by the integral equation

$$\varphi(\lambda, k, z) = \varphi_1(\lambda, k, z) + \frac{1}{W[\varphi_A, \varphi_B]}\int_{z_1}^{z} dz' \cdot \left[\varphi_A(z')\,\varphi_B(z) - \varphi_B(z')\,\varphi_A(z)\right] \cdot h(\lambda, k, z')\,\varphi(\lambda, k, z') \tag{6b}$$

where φ_1 , φ_A , φ_B are solutions of the "homogeneous" equation

$$D(\lambda, k, z)\,\varphi = 0 \tag{6c}$$

and $W[\varphi_A, \varphi_B] = [\varphi_A \varphi_B' - \varphi_A' \varphi_B]$ is the Wronskian of φ_A and φ_B .

Proof: If $z \to z_1$ in the R.H.S. of (6b) ; then $\varphi \to \varphi_1$, so that (6a) is satisfied.

Apply the operator $D(\lambda, k, z)$ to the equation (6b).

We have
$$D(\lambda, k, z)\varphi = h(\lambda, k, z)\,\varphi$$
$$D(\lambda, k, z)\,\varphi_1 = 0$$

To prove (6b), we must prove that

$$D(\lambda, k, z)\,\frac{1}{W[\varphi_A, \varphi_B)}\int_{z_1}^{z} dz' \cdot G(z, z')\, h(\lambda, k, z')\,\varphi(\lambda, k, z')$$
$$= h(\lambda, k, z)\,\varphi(\lambda, k, z),$$

(6d)

where $G(z, z') = [\varphi_A(z')\,\varphi_B(z) - \varphi_B(z')\,\varphi_A(z)]$

For this, consider the two equations

$$D(\lambda, k, z')\,\varphi(z') = h(z')\,\varphi(z') \qquad \text{(i)}$$

and
$$D(\lambda, k, z')\,G(z, z') = 0. \qquad \text{(ii)}$$

Multiply (i) by $G(z, z')$ and (ii) by $\varphi(z')$; substract.

Result:

$$\frac{d}{dz'}\left[G(z, z')\,\varphi'(z') - \varphi(z')\,G'(z, z')\right]$$
$$= G(z, z')\,h(z')\,\varphi(z').$$

$$\therefore \left[G(z, z')\,\varphi'(z') - \varphi(z')\,G'(z, z')\right]_{z_1}^{z}$$
$$= \int_{z_1}^{z} G(z, z')\,h(z')\,\varphi(z')\,dz' . \qquad \text{(iii)}$$

We note that $G(z, z) = 0$, ;

and $$\left[\frac{d\, G(z, z')}{dz'}\right]_{z'=z} = -W(\varphi_A ; \varphi_B).$$

Therefore, the L.H.S. of (iii) is equal to

$$W(\varphi_A, \varphi_B)\, \varphi(z) - \left[G\varphi' - \varphi G'\right]_{z'=z},$$

Operating with $D(\lambda, k, z)$ on this we obtain

$$W(\varphi_A, \varphi_B) \cdot h(\lambda, k, z)\, \varphi(z) + 0$$

Therefore

$$h(\lambda, k, z)\, \varphi(z) = \frac{1}{W(\varphi_A, \varphi_B)}\, D(\lambda, k, z) \int_{z_1}^{z} G(z, z')\, h(z')\, \varphi(z')\, dz'$$

Q.E.D.

and (6d) is proved.

<u>Boundary conditions at the Origin.</u>

Impose the following boundary condition at the origin:

$$\varphi(\lambda, k, z) \sim z^{\lambda + \frac{1}{2}}, \quad \text{as } z \to 0$$

We must choose $D(\lambda, k, z)$ such that $z^{\lambda+\frac{1}{2}}$ is a solution of the equation $D(\lambda, k, z)\, \varphi(\lambda, k, z) = 0$.

The simplest choice is

$$D(\lambda, k, z) = D(\lambda, z) = \left[\frac{d^2}{dz^2} - \frac{(\lambda^2 - \frac{1}{4})}{z^2}\right].$$

i.e. (11) is written

$$\left[\frac{d^2}{dz^2} - \frac{(\lambda^2-\frac{1}{4})}{z^2}\right]\varphi(\lambda,k,z) = \left[V(z)-k^2\right]\varphi(\lambda,k,z)$$

Take

$$\varphi_A = z^{\lambda+\frac{1}{2}}, \qquad \varphi_B = z^{-\lambda+\frac{1}{2}},$$

so that $\quad W[\varphi_A, \varphi_B] = -2\lambda$.

Therefore, the integral equation is (for z in the neighbourhood of the origin)

$$\varphi(\lambda,k,z) = z^{\lambda+\frac{1}{2}} - \frac{1}{2\lambda}\int_0^z \left[\frac{u^{\lambda+\frac{1}{2}}}{z^{\lambda-\frac{1}{2}}} - \frac{z^{\lambda+\frac{1}{2}}}{u^{\lambda-\frac{1}{2}}}\right].$$

$$.\left[V(u)-k^2\right]\varphi(\lambda,k,u)\,du. \tag{7}$$

Let φ_0 be the free solution (i.e. when $V=0$) of (i),
i.e. φ_0 is the solution of

$$\left[\frac{d^2}{dz^2} + k^2 - \frac{(\lambda^2-\frac{1}{4})}{z^2}\right]\psi(z) = 0 \tag{8}$$

Comparing with the Bessel's equation

$$\left[\frac{d^2}{d\zeta^2} + \frac{1}{\zeta}\frac{d}{d\zeta} + 1 - \frac{\lambda^2}{\zeta^2}\right]\chi(\zeta) = 0, \tag{8a}$$

We find that the substitution $\psi(z) = z^{\frac{1}{2}} \chi(z)$, (8b)

$z \to \zeta = kz$, reduces (7) to (7a).

The solution of (7) with the property $\psi \xrightarrow[z \to 0]{} z^{\lambda + \frac{1}{2}}$

is (8c)

$$\psi_0 = C\, z^{\frac{1}{2}}\, J_\lambda(kz),$$

since

$$J_\lambda(z) = \frac{z^\lambda e^{-iz}}{2^\lambda\, \Gamma(\lambda+1)}\, F(\lambda + \tfrac{1}{2} | 2\lambda + 1 | 2iz)$$

$$\to \frac{1}{\Gamma(\lambda+1)} \cdot \left(\frac{z}{2}\right)^\lambda, \text{ as } z \to 0, \quad (9)$$

$$\therefore J_\lambda(kz) \to \frac{1}{\Gamma(\lambda+1)}\, \frac{k^\lambda}{2^\lambda}\, z^\lambda, \quad \text{as } z \to 0. \quad (9a)$$

To satisfy the integral equation (7) with $V = 0$, we must take the constant C in (8 c) to be

$$C = 2^\lambda k^{-\lambda}\, \Gamma(\lambda+1).$$

(8d)

Thus $\psi_0 = 2^\lambda k^{-\lambda}\, \Gamma(\lambda+1)\, z^{1/2} J_\lambda(kz) \to z^{\lambda+\frac{1}{2}}$ as $z \to 0$.

Suppose we treat $V(z)$ as a perturbation; we can find solutions of

$$\left[\frac{d^2}{dz^2} + k^2 - \frac{(\lambda^2 - \frac{1}{4})}{z^2}\right]\phi = V(z)\,\phi$$

(10)

with the boundary condition $\phi \to z^{\lambda + \frac{1}{2}}$ as $z \to 0$ (10a)

for we know that (10) with the R.H.S. $= 0$ is satisfied by $z^{\lambda + 1/2}$ as $z \to 0$.

Take

$$\varphi_A = 2^{\lambda} k^{-\lambda} \Gamma(\lambda+1) z^{1/2} J_{\lambda}(kz) \longrightarrow z^{\lambda+\frac{1}{2}}, \text{ as } z \to 0.$$

and --(10b)

$$\varphi_B = 2^{-\lambda} k^{\lambda} \Gamma(-\lambda+1) z^{1/2} J_{-\lambda}(kz) \longrightarrow z^{-\lambda+\frac{1}{2}}, \text{ as } z \to 0.$$

(10b)

Then $W(\varphi_A, \varphi_B) = -2\lambda$. . (Compute it from the values at $z = 0$, since it is a constant.).

Noting that $\Gamma(1+x)\,\Gamma(1-x) = \pi x / (\sin \pi x)$, we obtain the integral equation:

$$\varphi(\lambda, k, z) = \psi_0 - \frac{\pi\sqrt{z}}{2 \sin \pi\lambda} \int_0^z du \cdot u^{1/2} \cdot \left[J_{\lambda}(ku)\, J_{-\lambda}(kz) - J_{\lambda}(kz)\, J_{-\lambda}(ku) \right] \cdot V(u)\, \varphi(\lambda, k, u),$$

ψ_0 being given by eqn (9d) above. (10c)

<u>Note:-</u> Equation (i) is invariant under

(a) $\lambda \to -\lambda$

(b) $k \to -k$

Does this mean that the operations (a) and (b) result in a solution which is the same as the solution $\varphi(\lambda, k, z)$? This will depend on whether or not a singularity in the λ plane or k plane respectively is crossed while making the passage from λ to $-\lambda$

or k to $-k$. This, in turn, depends on the boundary conditions imposed on the differential equations (1).

In general, we my note that the possible singularities in z of the solutions of the equation

$$\frac{d^2}{dz^2}\psi(z) + \left[k^2 - \frac{(\lambda^2-\frac{1}{4})}{z^2} - V(z)\right]\psi(z) = 0$$

will be given directly by considering the singularities of the coefficients of $d\psi/dz$ and ψ in the differential equation; imposing a boundary condition will select a particular solution that may or may not have all the possible singularities.

On the other hand, the analyticity in λ and k is not obvious from the differential equation; it will be more appropriate to consider the integral equation which includes both the differential equation and the boundary conditions.

For equation (10a), (10b) above, since the boundary condition does not involve k and since the coefficients of ψ and $d\psi/dz$ are analytic in k, we may expect that $k \to -k$ will not involve crossing any singularities in the k plane, and hence that $\varphi(\lambda, k, z) = \varphi(\lambda, -k, z)$ gives the same solution. On the other hand, $\lambda \to -\lambda$ will give a different solution, as is evident from the limiting form as $z \to 0$.

$\varphi(\lambda, k, z)$ will be regular at the origin, if $Re\,\lambda > 0$ (i.e. $Re\,\ell > -\frac{1}{2}$), while $\varphi(-\lambda, k, z)$ is irregular at $z = 0$ (since it $\sim z^{-\lambda+\frac{1}{2}}$) for $Re\,\lambda > 0$ (or at least for $Re\,\lambda > \frac{1}{2}$).

On the line $Re\ \lambda = 0$, the solutions $\varphi(\lambda, k, z)$ and $\varphi(-\lambda, k, z)$ cross and exchange their roles. On this line,

$$\varphi \sim z^{(\pm i\, Im\, \lambda + \frac{1}{2})} \qquad \text{as } z \to 0 \ ;$$

thus both solutions have an oscillatory character ([illegible]

We note that $W\left[\varphi(\lambda, k, z), \varphi(-\lambda, k, z)\right] = -2\lambda$.

<u>Boundary Conditions at Infinity</u>.

Define $f(\lambda, k, z)$ as that solution which behaves like $exp(-ikz)$ for large z.

Write the differential equation (1c) as

$$\left[\frac{d^2}{dz^2} + k^2\right] f(\lambda, k, z) = \left[V(z) + \frac{(\lambda^2 - \frac{1}{4})}{z^2}\right] f(\lambda, k, z) \tag{11}$$

Impose the boundary condition $f(\lambda, k, z) \to e^{-ikz}$,

$$\text{as } z \to \infty \tag{11a}$$

Taking the two independent solutions $exp(+ikz)$ and $exp(-ikz)$ of $\left[\frac{d^2}{dz^2} + k^2\right]\phi = 0$, and noting that

$$W\left[e^{ikz}, e^{-ikz}\right] = 2ik,$$

the integral equation incorporating (11) and **(11a) is** (for large z).

$$f(\lambda, k, z) = e^{-ikz} + \frac{1}{2ik}\int_{\infty}^{z} d\xi \left[e^{ik(\xi - z)} - e^{ik(z-\xi)}\right]$$

$$\cdot \left[V(\xi) + \frac{(\lambda^2 - \frac{1}{4})}{\xi^2}\right] \cdot f(\lambda, k, \xi) \qquad \text{(11b)}$$

Let $f_0(\lambda, k, z)$ be the solution with the potential switched off, $V = 0$. As in (8) -- (8b), we may again obtain a Bessel's equation; the solutions with the asymplotic behaviour $e^{\pm ikz}$ are

$$f_0(\lambda, k, z) = \sqrt{\frac{\pi}{2}}\, e^{-i\frac{\pi}{2}(\lambda + \frac{1}{2})} \sqrt{kz}\; H_\lambda^{(2)}(kz)$$

$$\longrightarrow e^{-ikz}, \quad as\ z \to \infty$$

and $\qquad$ (11c)

$$f_0(\lambda, -k, z) = \sqrt{\frac{\pi}{2}}\, e^{+i\frac{\pi}{2}(\lambda + \frac{1}{2})} \sqrt{kz}\cdot H_\lambda^{(1)}(kz)$$

and

$$\longrightarrow e^{ikz}, \quad z \to \infty$$

where $H_\lambda^{(1)}(kz)$ and $H_\lambda^{(2)}(kz)$ are the Hankel functions of the first and second kinds respectively. We note that

$$H_\lambda^{(1)}(kz) = \frac{2^{\lambda+1}(kz)^\lambda e^{+ikz}}{\sqrt{\pi}} U_1(\lambda + \tfrac{1}{2} | 2\lambda + 1 | 2ikz)$$

$$\xrightarrow[z \to \infty]{} \sqrt{\frac{2}{\pi kz}} \cdot e^{ikz - \frac{1}{2}i\pi(\lambda + \frac{1}{2})};$$

(11d)

$$H_\lambda^{(2)}(kz) = 2^{\lambda+1}(kz)^\lambda e^{-ikz}\, U_2(\lambda + \tfrac{1}{2} | 2\lambda + 1 | 2ikz)$$

$$\xrightarrow[z \to \infty]{} \sqrt{\frac{2}{\pi kz}} \cdot e^{-ikz + \frac{1}{2}i\pi(\lambda + \frac{1}{2})},$$

(11e)

U_1 and U_2 being the confluent hypergeometric functions of the 3rd kind.

(Ref. Morse and Feshbach, Vol.1, p.610-625).

The Wronskian of $H^{(1)}_\lambda(kz)$ and $H^{(2)}_\lambda(kz)$ will be $\frac{4}{\pi i k z}$,

thus the Wronskian of the two solutions f is $\frac{\pi}{2} \cdot kz \frac{4}{\pi i k z} = \frac{2}{i}$.

Thus in terms of the free solutions, we may write the integral equation.

$$f(\lambda, k, z) = f_0(\lambda, k, z) +$$

$$+ \frac{i\pi}{4} z^{\frac{1}{2}} \int_z^\infty du \; u^{1/2} \left[H^{(1)}_\lambda(ku) H^{(2)}_\lambda(kz) - H^{(2)}_\lambda(ku) H^{(1)}_\lambda(kz) \right] \cdot V(u) f(\lambda, k, u)$$

(11f)

<u>Analytic properties of $f(\lambda, k, z)$</u>.

In the k plane, $f(\lambda, k, z)$ has a branch-point at $k = 0$ cf. eq.(11c). Thus, although the differential equation is symmetric under $k \to -k$, the solution

$f(\lambda, k e^{-i\pi}, z) \neq$ the solution $f(\lambda, k, z)$.

This is the evident from the asymplotic form of the free solution at $[z \to \infty$:

$$f(\lambda, k, z) \longrightarrow e^{-ikz},$$

$$f(\lambda, ke^{-i\pi}, z) \longrightarrow e^{+ikz}.$$

The Wronskian of the two solutions is $2ik$.

For a hermitian Hamiltonian, i.e. for a $V(r)$ that is real for real r (>0), we can prove that

$$\varphi(\lambda, k, z) = \varphi^*(\lambda^*, k^*, z),$$

and

$$f(\lambda, k, z) = f^*(\lambda^*, -k^*, z).$$

But in order to extend the analytic properties, z will be allowed to be complex in the following.

The Jost functions and analyticity in λ, k.

The Jost function may be defined as the Wronskian of two solutions of the scattering equation, where one is defined by the boundary condition at the origin and the other by the boundary condition at infinity. In contrast to the Wronskian of two solutions defined by boundary conditions either at the origin or at infinity (which are 2λ and $2ik$ respectively) which do not contain any information about the potential, the Jost function is characteristic of the potential. Given the Jost function, the S-matrix is completely determined.

We have proved in an earlier lecture that the Jost function is also the Fredholm determinant of the scattering integral equation (written as a Fredholm equation).

In the next lecture, we shall examine in detail the properties of the Jost function and its relation to the S-matrix.

LECTURE III. (4-?-1962.)

Complex Angular Momentum in Potential Scattering

We define the Jost function $F(\lambda, k)$ as the Wronskian

$$F(\lambda, k) = \frac{1}{2\lambda} W\left[f(\lambda, k, z), \varphi(\lambda, k, z) \right] \tag{1}$$

Any pair of independent solutions of the scattering equation form a linearly independent set in terms of which every other solution may be expressed as a linear combination. Thus we may write:

$$\begin{aligned} \varphi(\lambda, k, z) &= A\, f(\lambda, k, z) + B\, f(\lambda, -k, z), \\ \varphi(-\lambda, k, z) &= C\, f(\lambda, k, z) + D\, f(\lambda, -k, z); \end{aligned} \tag{2}$$

A,B,C,D are independent of z, but in general will depend on λ and k.

Substitute (2) into (1).

$$\therefore \quad F(\lambda, k) = \frac{2ikB \cdot 2\lambda}{2\lambda} = 2ikB$$

Similarly, $F(\lambda, -k) = 2ikA$; $F(-\lambda, k) = 2ikD$; (3)

and $F(-\lambda, -k) = 2ikC$.

Thus

$$\varphi(\lambda, k, z) = \frac{1}{2ik}\left[F(\lambda, k)\, f(\lambda, -k, z) - F(\lambda, -k)\, f(\lambda, k, z)\right] \quad \ldots (4)$$

We have $2\lambda = W\left[\varphi(\lambda, k, z), \varphi(-\lambda, k, z)\right]$

$$= \frac{1}{2ik}\left[F(\lambda, -k)F(-\lambda, k) - F(\lambda, k)\, F(-\lambda, -k)\right],$$

OR

$$F(\lambda, -k)\, F(-\lambda, k) - F(\lambda, k)\, F(-\lambda, -k) = 4ik\lambda. \quad (5)$$

(2) may be inverted to give

$$f(\lambda, k, z) = \frac{1}{2\lambda}\left[F(-\lambda, k)\, \varphi(\lambda, k, z) - F(\lambda, k)\, \varphi(-\lambda, k, z)\right]$$

$$f(\lambda, -k, z) = \frac{1}{2\lambda}\left[F(-\lambda, -k)\, \varphi(\lambda, k, z) - F(\lambda, -k)\, \varphi(-\lambda, k, z)\right]. \quad (6)$$

The free Jost Functions.

When $V = 0$,

$$F(\lambda, k) = \frac{1}{2\lambda} W\left[2^{\lambda}\Gamma(\lambda+1)\, k^{-\lambda} z^{1/2} J_{\lambda}(kz), \sqrt{\tfrac{\pi}{2}}\, e^{i\frac{\pi}{2}(\lambda+\frac{1}{2})} \sqrt{kz} \cdot H_{\lambda}^{(1)}(kz)\right] \quad (7)$$

As this is independent of z, it may be evaluated at $z \to \infty$, noting that $J_{\lambda}(kz) \to \sqrt{\frac{2}{\pi k z}} \cos\left[kz + \frac{\pi}{2}(\lambda + \frac{1}{2})\right]$, as $z \to \infty$.

We then obtain

$$F(\lambda,k) = \frac{1}{2\lambda}\, 2^{\lambda}\,\Gamma(\lambda+1)\, k^{-\lambda}\cdot W\left[\sqrt{\frac{2}{\pi k}}\cos\left[kz+\frac{\pi}{2}(\lambda+\tfrac{1}{2}\right), e^{ikz}\right]$$

$$= \frac{1}{2\lambda}\cdot 2^{\lambda}\,\Gamma(\lambda+1)\sqrt{\frac{2}{\pi}}\; k^{-\lambda+\frac{1}{2}}\; \exp\left[-\frac{i\pi}{2}(\lambda-\tfrac{1}{2})\right] \tag{8}$$

Considering $F(\lambda,k)$ as a function of the 2 complex variables λ, k, we see that it has a branch point at $k=0$; i.e. the Jost functions are multi-valued in k .

<u>Connection between the Jost function and the S-matrix.</u>

The asymptotic behaviour $(at\ z=\infty)$ of the regular free solution is

$$\varphi^{\circ}(\lambda,k,z) = 2^{\lambda}\,\Gamma(\lambda+1)\, k^{-\lambda} z^{\frac{1}{2}} J_{\lambda}(kz)$$

$$\underset{z\to\infty}{\sim} 2^{\lambda}\,\Gamma(\lambda+1)\, k^{-\lambda} z^{1/2}\sqrt{\frac{2}{\pi kz}}\cdot \sin\left[kz-\frac{\pi}{2}(\lambda-\tfrac{1}{2})\right].$$

$$= \exp\left[i\frac{\pi}{2}(\lambda-\tfrac{1}{2})\right]\cdot\frac{1}{k}\, F_0(\lambda,k)\,\sin\left[kz-\frac{\pi}{2}(\lambda-\tfrac{1}{2})\right]. \tag{9}$$

The regular solution for $V\neq 0$ has the asymptotic behaviour

$$\varphi(\lambda,k,z) = \frac{1}{2ik}\left[F(\lambda,k)\, f(\lambda,-k,z) - F(\lambda,-k)\, f(\lambda,k,z)\right]$$

$$\underset{z\to\infty}{\sim} \frac{1}{2ik}\left[F(\lambda,k)\,e^{ikz} - F(\lambda,-k)\,e^{-ikz}\right]$$

$$= e^{i\frac{\pi}{2}(\lambda-\frac{1}{2})}\,e^{-i\delta}\,\frac{1}{k}\,F(\lambda,k)\sin\left[kz-\frac{\pi}{2}(\lambda-\tfrac{1}{2})+\delta(\lambda,k)\right] \tag{10}$$

where

$$e^{2i\delta(\lambda,k)} \equiv \frac{F(\lambda,k)}{F(\lambda,-k)}\,e^{i\pi(\lambda-\frac{1}{2})} \tag{11}$$

OR

$$\boxed{S(\lambda,k) = \frac{F(\lambda,k)}{F(\lambda,-k)}\exp\left[i\pi(\lambda-\tfrac{1}{2})\right]} \tag{12}$$

<u>Analytic Properties of the Partial Wave Functions.</u>

We must find out when the partial waves $\varphi(\lambda,k,z)$ exist and what are their analytic properties. The method adopted is the following:

1) Obtain a formal iterated solution of the integral equation and examine the analyticity properties of each term; deduce the domain of analyticity of the solution (it will be at least as large as the intersection of the analyticity domains of the individual terms.)

2) Place bounds on the solution in such a way that the series converges uniformly inside the domain of analyticity.

The results obtained are as follows:

I) $\varphi(\lambda,k,z)$ and $\frac{d}{dz}\varphi(\lambda,k,z)$ are (i) <u>Integral functions of k</u> (i.e. regular for all k except for an essential singularity at $k=\infty$)

ii) <u>analytic in λ for $Re\,\lambda > 0$</u>; the expansion also converges for $Re\,\lambda = 0$ (for special potentials, the region of analyticity may be extended to the left of $Re\,\lambda = 0$), or,

iii) $\varphi(\lambda, k, z)$ is analytic in λ and k in the topological product of the k plane (excl. $k = \infty$) and the right half λ - plane ($Re\ \lambda > 0$) ; it is continuous for $Re\ \lambda = 0$,

(II) $f(\lambda, k, z)$ is analytic in λ and k in the topological product of the λ plane (excl. $\lambda = \infty$) and the lower half k-plane ($Im\ k < 0$); it is continous for $Im\ k = 0$.

Proof:

1) Consider the integral equation

$$g(\lambda, k, z) = g_0(\lambda, k, z) + \int_{z_1}^{z} L(\lambda, k, z')\, g(\lambda, k, z')\, dz' \tag{13}$$

Then $|g(\lambda, k, z)| \leq |g_0(\lambda, k, z)| + \int_{z_1}^{z} |L(\lambda, k, z')\, g(\lambda, k, z')\, dz'|$

Let $|g_0(\lambda, k, z)| \leq M(\lambda, k, z)$; let $G(\lambda, k, z) \equiv \dfrac{g(\lambda, k, z)}{M(\lambda, k, z)}$.

Then

$$|G(\lambda, k, z)| \leq 1 + \int_{z_1}^{z} |K(z, z')\, G(z')\, dz'|,$$

where

$$K(z, z') = \frac{M(\lambda, k, z')}{M(\lambda, k, z)}\, L(\lambda, k, z')$$

By iteration, we may obtain, formally,

$$|G(\lambda, k, z)| \leq 1 + \int_{z_1}^{z} |K(z, z')\, dz'| + \int_{z_1}^{z} |dz'\, K(z, z')| \int_{z_1}^{z'} |dz''\, (K(z', z'')$$

On changing the ordered integrals into products of integrals, we obtain i.e. $$|G(\lambda,k,z)| \leq 1 + \frac{1}{1!}\int_{z_1}^{z} |K(z,z')\,dz'| + \frac{1}{2!}\left[\int_{z_1}^{z} |K(z,z')\,dz'|\right]^2 + \cdots$$

OR

$$|G(\lambda,k,z)| \leq \exp\left[\int_{z_1}^{z} |K(z')\,dz'|\right] \qquad (14)$$

This may not be a regorous proof; however it enables us to understand that the iterated series would converge if the integral in (14) is bounded.

Consider the integral equation (7) of Lecture II.

$$\varphi(\lambda,k,z) = z^{\lambda+\frac{1}{2}} - \frac{1}{2\lambda}\int_0^z \left[\frac{u^{\lambda+\frac{1}{2}}}{z^{\lambda-\frac{1}{2}}} - \frac{z^{\lambda+\frac{1}{2}}}{u^{\lambda-\frac{1}{2}}}\right] \cdot \left[V(u) - k^2\right] \cdot \varphi(\lambda,k,u)\,du \qquad (15)$$

Here, $z_0(\lambda,k,z) = z^{\lambda+\frac{1}{2}} \equiv M(\lambda,k,z)$

We note that since $0 < u < z$,

$$|K(z,u)| = \left|\frac{u^{\lambda+\frac{1}{2}}}{z^{\lambda+\frac{1}{2}}}\right| \cdot \left|\frac{u^{\lambda+\frac{1}{2}}}{z^{\lambda-\frac{1}{2}}} - \frac{z^{\lambda+\frac{1}{2}}}{u^{\lambda-\frac{1}{2}}}\right|$$

$$= \left|\frac{u^{2\lambda+1}}{z^{2\lambda}} - u\right|$$

$\leq 2u$, when $Re\,\lambda \geq 0$, (as the maximum value of u is z). Considering potentials that fall off at least as fast as $1/r^2$, we have

$$|V(u) - k^2| \leqslant \frac{C}{u^{2-\epsilon}} + D \leqslant \frac{F}{u^{2-\epsilon}},$$

where k is finite, D is the upper limit of k^2 and C, F are constants.

$$\int_0^z |K(\lambda,k,u)\,du| = \frac{1}{2|\lambda|}\int_0^z \left|\frac{u^{\lambda+\frac{1}{2}}}{z^{\lambda+\frac{1}{2}}}\right| \cdot \left|\frac{u^{\lambda+\frac{1}{2}}}{z^{\lambda-\frac{1}{2}}} - \frac{z^{\lambda+\frac{1}{2}}}{u^{\lambda-\frac{1}{2}}}\right| \cdot |V(u) - k^2|\,|du|$$

$$\leqslant \frac{F}{|\lambda|} \cdot \int_0^z \left|\frac{u}{u^{2-\epsilon}}\right| |du| = \frac{F}{|\lambda|} \int_0^z \left|\frac{du}{u^{1-\epsilon}}\right|$$

$$= \frac{F}{|\lambda|}\left[\frac{u^{+\epsilon}}{+\epsilon}\right]_0^z = \frac{F}{\epsilon|\lambda|},$$

which is bounded when $\epsilon \neq 0$, i.e. when the potential falls off more rapidly than $1/r^2$.

Thus $\varphi(\lambda, k, z)$ is bounded for all finite k when $Re\,\lambda > 0$. Therefore, $\varphi(\lambda, k, z)$ is an integral function of k, analytic in the half-plane $Re\,\lambda > 0$.

A similar method may be used to derive the same analytic propertie of $\varphi'(\lambda, k, z)$.

To prove (II).

Consider the integral equation (11b) of Lecture II:

$$f(\lambda,k,z) = e^{-ikz} - \frac{1}{2ik}\int_z^\infty \left[e^{ik(\xi-z)} - e^{-ik(\xi-z)}\right] \cdot \left[V(\xi) + \frac{(\lambda^2-\frac{1}{4})}{\xi^2}\right] \cdot f(\lambda,k,\xi)\, d\xi$$

[This is valid for large z.]

Here, $g_0(\lambda,k,z) = e^{-ikz} \equiv M(\lambda,k,z)$.

$$\int_z^\infty |K(\xi)\, d\xi| =$$

$$= \frac{1}{2|k|}\cdot\int_z^\infty \left|e^{-ik(\xi-z)}\right| \cdot \left|e^{ik(\xi-z)} - e^{-ik(\xi-z)}\right| \cdot \left|V(\xi) + \frac{(\lambda^2-\frac{1}{4})}{\xi^2}\right| d\xi$$

But

$$\left|e^{-ik(\xi-z)}\right| \cdot \left|e^{ik(\xi-z)} - e^{-ik(\xi-z)}\right|$$

$$= \left|1 - e^{-2ik(\xi-z)}\right|$$

$$= \left|1 - e^{2(\operatorname{Im} k)(\xi-z)}\, e^{-2i(\operatorname{Re} k)(\xi-z)}\right|,$$

which is bounded i.e. it is $\leq C \equiv$ a constant, if $\operatorname{Im} k \leq 0$. (Note: $(\xi - z) > 0$).

For potentials decreasing at least as fast as $1/r^2$,

$$\int_z^\infty |K(z')\, dz'| \leq \frac{c}{2|k|}\int_z^\infty \left|\frac{D}{\xi^{2-\epsilon}} + \frac{(\lambda^2-\frac{1}{4})}{\xi^2}\right| d\xi,$$

which is bounded for finite λ , $|k| \neq 0$, and $z \neq 0$.

Thus $f(\lambda, k, z)$ is an integral function of λ , analytic in the half-plane $\mathrm{Im}\, k < 0$. It can be proved to be continuous for $\mathrm{Im}\, k = 0$.

A similar method may be used for the other integral equations.

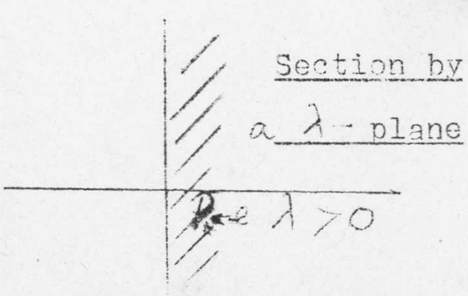
We note that in the above, we have considered z real only. In the next lecture we shall show how to continue $f(\lambda, k, z)$ into the half-plane $\mathrm{Re}\, z \geqslant 0$; we shall deduce the analyticity domain of the Jost function $F(\lambda, k)$ in the k plane.

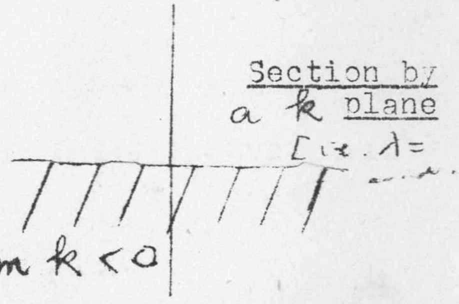
LECTURE IV. (13-9-'62.)

Complex Angular Momentum and Potential Scattering.

From the analyticity domains (I) and (II) of $\varphi(\lambda, k, z)$ and $f(\lambda, k, z)$ given in the last lecture, we can deduce the domain of analyticity of the Jost function .

$F(\lambda, k)$ is analytic in λ , k in the intersection of the analyticity domains of $\varphi(\lambda, k, z)$, $f'(\lambda, k, z)$ $f(\lambda, k, z)$, $\frac{1}{\lambda}$, and $\varphi'(\lambda, k, z)$, i.e. in the product of the half-planes $\mathrm{Re}\, \lambda > 0$, $\mathrm{Im}\, k < 0$; it is continuous on the boundaries $\mathrm{Re}\, \lambda = 0$, $\mathrm{Im}\, k = 0$. It has a branch point at $k = 0$.





To enlarge the analyticity domain of $f(\lambda, k, z)$, consider the Schrodinger euqation (1c) of Lecture II along a ray $z = \rho e^{i\sigma}$ in the z plane (i.e. keep σ fixed) such that $|\sigma| \leq \pi/2$

The equation (1c) then becomes

$$\frac{d^2}{d\rho^2} \psi(\rho) - \frac{(\lambda^2 - \frac{1}{4})}{\rho^2} \psi + k_1^2 \psi - V_1 \psi = 0, \tag{1}$$

where $k_1 = k e^{i\sigma}$, & $V_1 = V(\rho e^{i\sigma}) e^{-2i\sigma}$. (2a)

By proceeding as for the original Schrodinger equation, we shall obtain functions $\varphi_1(\lambda, k_1, \rho)$, $f_1(\lambda, k_1, \rho)$, and a new Jost function $F_1(\lambda, k_1)$.

We wish to find out how the new functions are related to the analytic continunations of the old functions $f(\lambda, k, z)$, $\varphi(\lambda, k, z)$ and $F(\lambda, k)$ with z real.

The general solution of (1) will have the asymptotic behaviour

$$\phi(\lambda, k_1, \rho) \sim \alpha(\sigma) e^{ik_1\rho} + \beta(\sigma) e^{ik_1\rho}, \quad \rho \to \infty \tag{2a}$$

Define $f_1(\lambda, k_1, \rho)$ as the solution of (1) with the behaviour

$$f_1(\lambda, k_1, \rho) \sim e^{-ik_1\rho}, \quad \rho \to \infty, \tag{2b}$$

for all σ

The solution $f(\lambda, k, z)$, which may be continued into the half-plane $Re\, z \geq 0$ may be expressed in terms of the linearly independent solutions e^{-ikz}, e^{+ikz} as

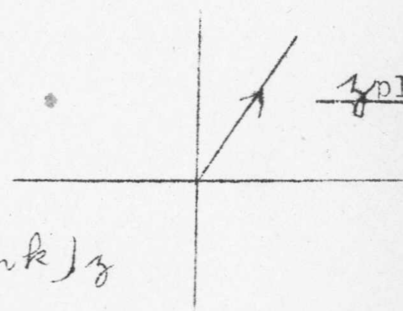
$$f(\lambda, k, z) \sim \alpha(\sigma) e^{-ikz} + \beta(\sigma) e^{+ikz}, \quad |z| \to \infty. \tag{3}$$

We must find $\alpha(\sigma)$ and $\beta(\sigma)$.
For their, consider

$$f(\lambda, k, z) e^{-ikz} \sim \alpha(\sigma) e^{-2ikz} + \beta(\sigma) \tag{3a}$$

We know that if a function $\psi(z)$ is an analytic function in a ~~xx~~ sector $\alpha < \arg z < \beta$ and its continuous and bounded on the boundary rays $\arg z = \alpha$, $\arg z = \beta$, then its limit is the same as $|z| \to \infty$ along any ray in this sector (This may be proved by a conformal transformation of Montel's Theorem, Ref. Titchmarsh : Theory Functions, p. 170).

The function $f(\lambda, k, z) e^{-ikz}$ is analytic in the domain $Re\, z > 0$, $Im\, z > 0$, and is continuous and bounded if $Re\, z = 0$. Its limit as $z \to \infty$ along the ray $Im\, z = 0$ is just $\beta(\sigma)$, when $Im\, k < 0$, for along this ray,



$$e^{-2ikz} = e^{-2i(Re\, k)z} e^{+2(Im\, k)z}$$

$$\to 0, \quad \text{as } z = Re\, z \to \infty, \text{ if } Im\, k < 0.$$

Therefore $f(\lambda, k, z) e^{-ikz}$ must $\to \beta(\sigma)$ along any ray in the sector $0 < \arg z < \pi/2$ i.e. $\beta(\sigma)$ must be independent of σ.

But $\beta(0) = 0$, as for real z ,

$$f(\lambda, k, z) \sim e^{-ikz} , \quad z \to \infty.$$

$$\therefore \quad \underline{\beta(\sigma) = 0.} \qquad (4)$$

$$\therefore\ f(\lambda, k, z) \sim \alpha(\sigma)\, e^{-ikz} \tag{5}$$

$$f(\lambda, k, z)\, e^{ikz} \sim \alpha(\sigma) = \alpha(0), \quad \rho \to \infty,$$

as the limit along any ray must be the same. But $\alpha(0) = 1$

as $f(\lambda, k, z) \sim \exp(-ikz)$ for real z

$$\therefore\ \underline{\alpha(\sigma) = 1} \tag{6}$$

$\therefore\ f_1(\lambda, k_1, \rho)$ is the same as the analytic continuation of $f(\lambda, k, z)$.

Defining solutions of (2) with the boundary condition at $\rho = 0$, we have

$$\varphi_1(\lambda, k, \rho) \sim \rho^{\lambda+\frac{1}{2}}, \quad \rho \to 0.$$

$$\sim z^{\lambda+\frac{1}{2}}\, e^{-i\sigma(\lambda+\frac{1}{2})}, \quad \rho \to 0$$

$$\therefore\ \varphi_1(\lambda, k_1, \rho) = e^{-i\sigma(\lambda+\frac{1}{2})}\, \varphi(\lambda, k, z). \quad \text{---}(7)$$

Then, from the definiton of the Jost function,

$$F_1(\lambda, k_1) = e^{-i\sigma(\lambda+\frac{1}{2})}\, F(\lambda, k) \tag{8}$$

We earlier proved that the old Jost function (defined in terms of λ, k from the Schrodinger equation in z) is analytic in

$\operatorname{Im} k < 0$ and $\operatorname{Re} \lambda > 0$. Applying the same reasoning but working in terms of k_1 and ρ instead of k and z, we obtain that the new Jost function $F(\lambda, k)$ is analytic in $\operatorname{Im} k_1 < 0$, $\operatorname{Re} \lambda > 0$, $k_1 = k e^{i\sigma}$, $|\sigma| < \pi/2$. Thus $F(\lambda, k_1)$ must thus be analytic in all domains of the kind $\operatorname{Im} (k e^{i\sigma}) < 0$. As $|\sigma| < \pi/2$, the union of all those domains is the k plane cut along the positive imaginary axis.

Thus $F(\lambda, k)$ is analytic in the product of the half-plane $\operatorname{Re} \lambda > 0$ with the k-plane cut along the positive $\operatorname{Im} k$ axis. Detailed study shows (as we have shown in previous lectures on the potential) that the cut starts from $\operatorname{Im} k = \frac{m}{2}$, where m is defined from

$$V(z) = \int_{m>0}^{\infty} \sigma(\mu) \frac{e^{-\mu z}}{z} d\mu .$$

$F(\lambda, -k)$ is analytic in the product of the half-plane $\operatorname{Re} \lambda > 0$ with the k plane cut along the negative $\operatorname{Im} k$ axis from $\operatorname{Im} k = -\frac{m}{2}$ to $\operatorname{Im} k = -\infty$.

.

<u>The branch point of $F(\lambda, k)$ at $k = 0$</u>.

The free Jost solution is

$$f^{\circ}(\lambda, k, z) = \sqrt{\frac{\pi}{2}}\, e^{-i\frac{\pi}{2}(\lambda+\frac{1}{2})} \sqrt{kz}\, H_{\lambda}^{(2)}(kz) \sim e^{-ikz}, \quad |z| \to \infty .$$

Also,

$$f_0(\lambda, ke^{-i\pi}, z) = \sqrt{\frac{\pi}{2}}\, e^{i\frac{\pi}{2}(\lambda+\frac{1}{2})} \sqrt{kz}\, H_\lambda^{(1)}(kz) \sim e^{ikz}, \quad |z| \to \infty .$$

We have--(from eq. (11c), (11d), p. 23)

$$f_0(\lambda, ke^{-2i\pi}, z) - f_0(\lambda, k, z) = -2i\cos(\pi\lambda)\, f_0(\lambda, ke^{-i\pi}, z)$$

(9a)

and therefore

$$f_0(\lambda, ke^{-3i\pi}, z) = [1 - 4\cos^2\pi\lambda]\, f_0(\lambda, ke^{-i\pi}, z) - 2i\cos(\pi\lambda)\cdot f_0(\lambda, k, z) \quad \text{---(9b)}$$

[These may be proved by using the relation

$$H_\lambda^{(2)}(e^{im\pi} z) = e^{i\lambda\pi}\, \frac{\sin(m\lambda\pi)}{\sin\lambda\pi}\, H_\lambda^{(1)}(z) + \frac{\sin(m+1)\lambda\pi}{\sin\lambda\pi}\, H_\lambda^{(2)}(z$$

, we have

As $\varphi(\lambda, k, z)$ is single-valued in k, we have from the definition of $F(\lambda, k)$ eq. (8), p. 28,

$$F_0(\lambda, ke^{-2i\pi}) - F_0(\lambda, k) = -2i\cos(\pi\lambda)\; F_0(\lambda, ke^{-i\pi}) \tag{10a}$$

and

$$F_0(\lambda, ke^{-3i\pi}) = \left[1 - 4\cos^2 \pi\lambda\right] F_0(\lambda, ke^{-i\pi}) - 2i\cos\pi\lambda \cdot F_0(\lambda, k) \tag{10b}$$

Consider the integral equation for $f(\lambda, k, z)$:

$$f(\lambda, k, z) = f_0(\lambda, k, z) + \frac{i\pi}{4}\sqrt{z}\int_z^{\infty} \xi^{1/2}\left[H_\lambda^{(1)}(k\xi)\, H_\lambda^{(2)}(kz) - H_\lambda^{(2)}(k\xi)\, H_\lambda^{(1)}(kz)\right] \cdot V(\xi)\, f(\lambda, k, \xi)\, d\xi . \tag{11}$$

~~(11)~~

The same equation could be written for $f(\lambda, ke^{-i\pi}, z)$.

Putting $\alpha \equiv -2i\cos\pi\lambda$, we can write

$$\left[f(\lambda, k, z) + \alpha f(\lambda, k e^{-i\pi}, z) \right] =$$

$$= \left[f_0(\lambda, k, z) + \alpha f_0(\lambda, k e^{-i\pi}, z) \right] +$$

$$+ \frac{i\pi}{4} \sqrt{z} \int_z^\infty \xi^{\frac{1}{2}} \cdot \left[H_\lambda^{(1)}(k\xi) H_\lambda^{(2)}(kz) - H_\lambda^{(2)}(k\xi) H_\lambda^{(1)}(kz) \right]$$

$$V(\xi) \left[f(\lambda, k, \xi) + \alpha f(\lambda, k e^{-i\pi}, \xi) \right] d\xi \tag{12}$$

Thus the properties (9a), (9b) hold for the $f(\lambda, k, z)$ also and hence also for the $F(\lambda, k)$.

The branch point at $k = 0$ is thus seen to be purely kinematical, i.e. independent of the interaction, since it plays the same role in $F_0(\lambda, k)$ and $F(\lambda, k)$. On the other hand, the branch line in $F(\lambda, k)$ from $\mathrm{Im}\, k = \frac{m}{2}$ to $\mathrm{Im}\, k = \infty$ is dynamical, i.e. it depends on the interaction (or the potential).

We have $S(\lambda, k) \equiv e^{2i\delta(\lambda, k)} = \dfrac{F(\lambda, k)}{F(\lambda, -k)} e^{i\pi(\lambda - \frac{1}{2})}$

$$S(\lambda, k e^{-2i\pi}) = \frac{F(\lambda, k e^{-2i\pi})}{F(\lambda, k e^{-3i\pi})} e^{i\pi(\lambda - \frac{1}{2})}$$

$$= \frac{F(\lambda, k) - 2i(\cos\pi\lambda) \cdot F(\lambda, k e^{-i\pi})}{(1 - 4\cos^2\pi\lambda) F(\lambda, k e^{-i\pi}) - 2i(\cos\pi\lambda) F(\lambda, k)} e^{i\pi(\lambda - \frac{1}{2})}$$

$$\frac{S(\lambda,k) \;-\; 2i\cos\pi\lambda \; e^{i\pi(\lambda-\frac{1}{2})}}{(1-4\cos^2\pi\lambda) \;-\; 2i(\cos\pi\lambda)\, S(\lambda,k)\, e^{-i\pi(\lambda-\frac{1}{2})}} .$$

$$S(\lambda, k e^{-2i\pi}) = \frac{S(\lambda,k) \;-\; 2\cos(\pi\lambda)\, e^{i\pi\lambda}}{[1-4\cos^2\pi\lambda] \;+\; 2\cos\pi\lambda \cdot S(\lambda,k)\, e^{-i\pi\lambda}} \qquad (13a)$$

nd

$$(\lambda, k e^{-i\pi}) = \frac{F(\lambda, k e^{-i\pi})}{F(\lambda, k e^{-2i\pi})}\, e^{i\pi(\lambda-\frac{1}{2})}$$

$$= \frac{F(\lambda, k e^{-i\pi})\; e^{i\pi(\lambda-\frac{1}{2})}}{F(\lambda,k) \;-\; \cos\pi\lambda \cdot F(\lambda, k e^{-i\pi})}$$

$$S(\lambda, k e^{-i\pi}) = \frac{e^{i\pi(\lambda-\frac{1}{2})}}{S(\lambda,k)\, e^{-i\pi(\lambda-\frac{1}{2})} \;-\; 2i\cos\pi\lambda} .$$

$$= \frac{e^{i\pi(\lambda-\frac{1}{2})}}{S(\lambda,k) \;-\; 2\cos(\pi\lambda)\, e^{i\pi\lambda}} \qquad (13b)$$

We may define the function

$$(\lambda,k) = +i k^{2\lambda}\, \frac{S(\lambda,k) - e^{2i\pi\lambda}}{S(\lambda,k) - 1} \quad , \lambda \text{ being real.} \qquad (14)$$

When λ takes on a physical values $(\ell + \frac{1}{2})$, then

$$Z(\lambda,k) = k^{2\ell+1} \cot \delta(\lambda,k) .$$

Thus $Z(\lambda, k)$ may be considered to be a generalization of the function $k^{2\ell+1} \cot \delta(\ell,k)$ (which expanded about $k^2 = 0$ gives the effective range expansion).

$Z(\lambda, k)$ is regular at $k = 0$, and it is even in k.

We also have

$$S(\lambda,k) = \frac{-Z(\lambda,k) + i k^{2\lambda} e^{2i\pi\lambda}}{-Z(\lambda,k) + i k^{2\lambda}} . \tag{15}$$

Also, from the relation

$$F(\lambda,-k)\, F(-\lambda,k) - F(\lambda,k)\, F(-\lambda,-k) = 4i\lambda k ,$$

we obtain

$$\ldots^{\pi\lambda} S(\lambda,k) - e^{i\pi\lambda} S(-\lambda,k) = -\frac{4k\lambda}{F(\lambda,ke^{-i\pi})\,F(-\lambda,ke^{-i\pi})} \qquad \text{---} (16)$$

~~(16)~~

when λ is pure imaginary (so that $F(\lambda, ke^{-i\pi})$) and $F(-\lambda, ke^{-i\pi})$ are both defined; we must thus have $Re\,\lambda \geqslant 0$, $Re(-\lambda) \geqslant 0$).

LECTURE V.

Complex Angular Momentum and Potential Scattering. (Cont'd).

We first mention a few results about the asymptotic behaviour of the phase shift, obtained by an extension of the W.K.B. method to complex λ and complex k.

(i) $\delta \to 0$ for large λ if $Re\left(\frac{\lambda}{k}\right) > 0$ (1)

Thus for large λ, $S(\lambda, k) \to 1$. (1a)

(ii) A more precise result is that for large λ, the Born approximation is very reliable:

$$\delta \sim \delta_B = \frac{\int_0^\infty \varphi_0^2(\lambda, k, r)\, V(r)\, dr}{k\, F_0(\lambda, k)\, F_0(\lambda, ke^{-i\pi})},$$
as $|\lambda| \to \infty$

(12)

(For large λ, the wave function lies almost entirely outside the potential and is therefore affected very little by it. $\therefore \varphi \to \varphi_0$

for large λ .)

For a superposition of Yukawa potentials, the Born formula gives

$$\tan \delta_B \approx \delta_B = -\frac{1}{2k}\int_m^\infty Q_\ell\left(1+\frac{\mu^2}{2k^2}\right)\sigma(\mu)\,d\mu \tag{3}$$

For large λ ,

$$\delta \sim -\sqrt{\frac{\pi}{2}}\;\frac{\sigma(m)}{m}\;k\;\frac{e^{-\alpha\lambda}}{\lambda^{3/2}}\cdot\sqrt{\sinh\alpha} \tag{4}$$

The Poles of $S(\lambda, k)$.

(A) Consider physical values of λ . Then in the complex k-plane, there may be

(i) poles at $k = i\eta$ (η real and positive) corresponding to bound states,

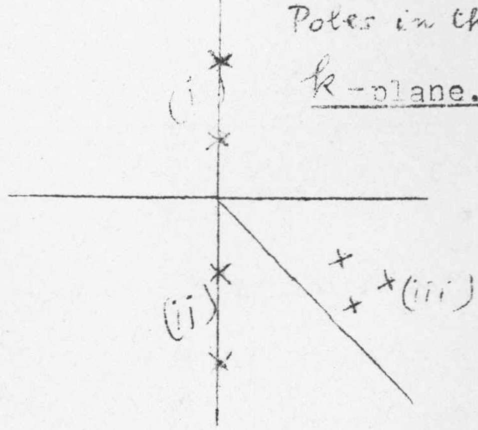
(ii) poles at $k = -i\eta$, called "anti-bound states" or " virtual states ",

and (iii) poles for $\mathrm{Im}\, k < 0$, $\mathrm{Re}\, k \neq 0$,

corresponding to resonances.

In the upper half-k plane, there can be no poles off the imaginary axis.

Poles in the k-plane.



(3) Consider <u>physical values of k</u>. Then in the complex λ -plane, there can be poles only when $Im\ \lambda > 0$, called " <u>the shadow states."</u>

The fact that poles may occur only in the upper-half λ - plane for potential scattering will be proved later. In relativistic scattering, there is also the possibility of poles on the $Re\ \lambda$ axis --- C.D.D. poles, at which $\delta = 0$ and $S(\lambda, k) = 1$. (In potential scattering, there are no C.D.D. poles.)

Considering the S-matrix, $S(\lambda, k)$ as a function of two complex variables λ, k, the existence of poles in 'sections' of the topological product of the λ and k spaces by particular planes implies that $S(\lambda, k)$ has certain singular surfaces (by analytic completion.).

We now ask what are the domains of holomarphy of $S(\lambda, k)$, or alternatively, what are the restrictions on the position of the singularities. We shall derive the restrictions imposed by the continuity equation. There may also be further restrictions imposed by the special properties of a particular potential.

Suppose $S(\lambda, k)$ has a simple pole at $\lambda = \lambda_0$, $k = k_0$. Then $F(\lambda, k e^{-i\pi}) = 0$.

By equation (4), Lecture III,

$$f(\lambda_0, k_0, z) = \frac{F(\lambda_0, k_0) f(\lambda, -k, z) - F(\lambda_0, -k_0) f(\lambda, k, z)}{2ik_0}$$

$$= \frac{F(\lambda_0, k_0)}{2ik_0} f(\lambda_0, k_0 e^{-i\pi}, z).$$

(5)

$$f(\lambda_0, -k_0, z) \sim e^{ik_0 z} \quad \text{at } z = \infty ,$$

$$\varphi(\lambda_0, k_0, z) \to 0 , \quad \text{as } z \to \infty, \quad \text{if } \mathrm{Im}\, k_0 > 0 . \tag{6}$$

Also,

$$\varphi(\lambda_0, k_0, z) \sim z^{\lambda_0 + \frac{1}{2}} , \quad \text{as } z \to 0$$

$$\to 0 , \quad \text{if } \mathrm{Re}\, \lambda_0 > 0 . \tag{7}$$

Thus φ and φ^* will both vanish at $z = 0$ and $z = \infty$ if $\mathrm{Re}\, \lambda_0 > 0$, $\mathrm{Im}\, k_0 > 0$.

From the Schrodinger equations for φ and φ^*, we can easily obtain

$$(\varphi'\varphi^* - \varphi^{*\prime}\varphi)\Big|_0^\infty = (k_0^2 - k_0^{*2}) \int_0^\infty |\varphi|^2 dz - (\lambda_0^2 - \lambda_0^{*2}) \int_0^\infty \frac{|\varphi|^2}{z^2} dz . \tag{8}$$

The L.H.S. $= 0$.

We obtain, noting that $k_0^2 - k_0^{*2} = 4i(\mathrm{Re}\, k_0)(\mathrm{Im}\, k_0)$, ,

$$\text{}k_0)(\mathrm{Re}\,k_0)\int_0^\infty |\phi|^2\,dz - (\mathrm{Im}\,\lambda_0)(\mathrm{Re}\,\lambda_0)\int_0^\infty \frac{|\phi|^2}{z^2}\,dz = 0 \qquad (9)$$

~~(9)~~

each integrand is positive definite, this equation will be consistent only if $(\mathrm{Im}\,k_0)(\mathrm{Re}\,k_0)$ ( ) and $(\mathrm{Im}\,\lambda_0)(\mathrm{Re}\,\lambda_0)$) have the same sign, i.e. if $\mathrm{Re}\,k_0$ and $\mathrm{Im}\,\lambda_0$ have the same sign. Thus a pole cannot occur at λ_0, k_0 if $\mathrm{Re}\,k_0$ and $\mathrm{Im}\,\lambda_0$ have opposite signs, i.e. in the domains

$$\equiv \left\{ \begin{array}{l} \mathrm{Re}\,k_0 > 0 \\ \\ \mathrm{Im}\,\lambda_0 < 0 \end{array} \right\} \quad \text{and} \quad L \equiv \left\{ \begin{array}{l} \mathrm{Re}\,k_0 < 0 \\ \\ \mathrm{Im}\,\lambda_0 > 0 \end{array} \right\} \qquad (10)$$

are two domains of holomorphy of $S(\lambda, k)$. They have a common boundary at $\mathrm{Re}\,k_0 = 0$, $\mathrm{Im}\,\lambda_0 = 0$.

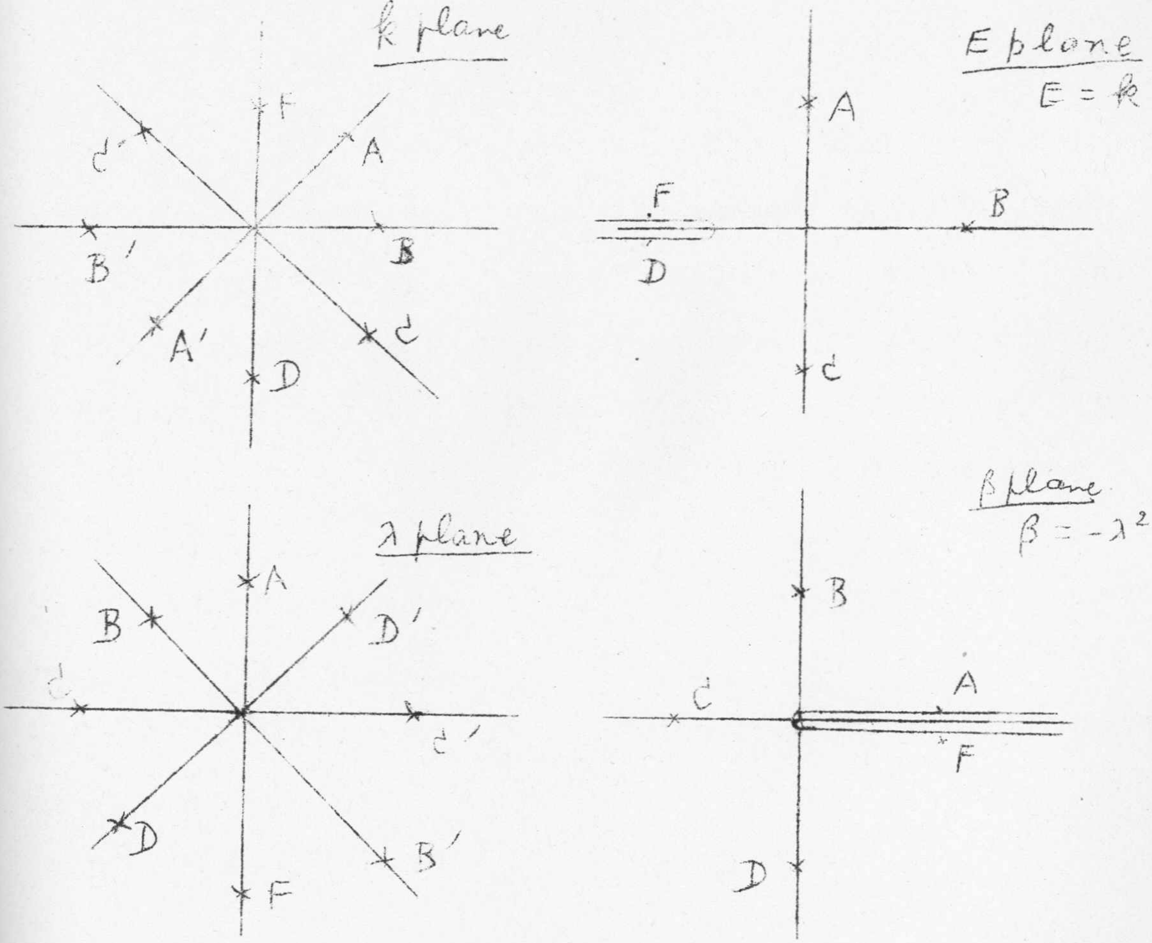
Consider the variables $E = +k^2, \; \beta = -\lambda^2 .$ (11)

cuts on the Imaginary k- axis map on to the part

$0 < E < -\frac{m^2}{4}$ of the negative real axis. Regge also showed

at $S(E,\beta)$ is meromorphic in the β plane cut along the

itive real axis



he mapping $k \to E$, $\lambda \to \beta$ are illustrated in the

iagrams. The domains of interest are $\mathrm{Im}\, k_0 > 0$, $\mathrm{Re}\, \lambda_0 > 0$.

hus we see that a domains of holomorphy (10) map into

$$\left\{ \begin{matrix} \mathrm{Im}\, E > 0 \\ \mathrm{Im}\, \beta > 0 \end{matrix} \right\} \quad \text{and} \quad L \equiv \left\{ \begin{matrix} \mathrm{Im}\, E < 0 \\ \mathrm{Im}\, \beta < 0 \end{matrix} \right\} \qquad (12)$$

Since
$$\frac{dE}{d\beta} = - \frac{\int_0^\infty \frac{\psi^2}{x^2}\, dx}{\int_0^\infty \psi^2\, dx} , \tag{13}$$

r $\beta < \beta_0$ on the real axis there will be no poles if β_0 is a large enough (real) negative number, as poles could then cur only for negative E (and the number n of bound states is mited for the class of potentials considered; so β_0 can be chosen be more negative than the value corresponding to the n^{th} bound state). he real domain $R(E,\beta)$, consisting of the product of $-\infty < \beta < \beta_0$ and $-\frac{m^2}{4} < E < 0$, s a domain of holomorphy of $S(E,k)$ and is a common oundary of the domains U and L. The domain $U+L+R$ can e enlarged by analytic completion to give the complete holomorphy domain f $S(\lambda,k)$.

For this change variables from E,β to

$$W = \frac{4}{m^2} - \frac{1}{E + \frac{m^2}{4}} , \qquad b = \beta - \beta_0 . \tag{14}$$

This mapping takes $\mathrm{Im}\, E > 0$ on to $\mathrm{Im}\, W > 0$, $\mathrm{Im}\, \beta > 0$ onto $\mathrm{Im}\, b > 0$, and the domain $(E,\beta) : -\frac{m^2}{4} \le E \le 0 ,\ \beta < \beta_0$ $R(W,b): -\infty \le W \le 0,\ b < 0$ on to the domain

Consider the function

$$F(\xi W_0, \xi b_0) = \frac{1}{2\pi i} \int_{-\infty}^{0} \frac{S^-(\xi' W_0, \xi' b_0)}{\xi' - \xi} d\xi' +$$

$$+ \frac{1}{2\pi i} \int_{0}^{\infty} \frac{S^+(\xi' W_0, \xi' b_0)}{\xi' - \xi} d\xi'$$

(15)

where S^+ denotes S in U and S^- denotes S in L.

$S^+(W, b) = S^-(W, b)$ if W, b lie in R(W,b).

Assume that the integrals converge.

When $\mathrm{Im}\,\xi \neq 0$, $F(\xi W_0, \xi b_0)$ is holomorphic in ξ. When $W_0, b_0 \in U$, $F(\xi W_0, \xi b_0)$) is holomorphic in W_0, b_0. Therefore $F(W, b)$) is holomorphic for $W = \xi W_0$, $b = \xi b_0$ when $\mathrm{Im}\,\xi \neq 0$ and $W_0, b_0 \in U$.

To show that $F(W, b)$ is an analytic continuation of $S(W, b)$ we prove that

a) for $W_0, b_0 \in R(W_0, b_0)$, $F(\xi W, \xi b) = S^-(\xi W, \xi b)$

(16)

b) for $\mathrm{Im}\,W_0 = 0 = \mathrm{Im}\,b_0$; $\mathrm{Re}\,W_0 > 0, \mathrm{Re}\,b_0 > 0$,

$F(\xi W, \xi b) = S^+(\xi W, \xi b)$.

a) for $W_0, b_0 \in R(W_0, b_0)$,

we have

$\mathrm{Im}\, W_0 = 0 = \mathrm{Im}\, b_0$; $\mathrm{Re}\, W_0 < 0$, $\mathrm{Re}\, b_0 < 0$.

Then the 2nd integral in (15) is

$$= \int_0^{\infty} \frac{S^-(\xi' W_0, \xi' b_0)}{\xi' - \xi}\, d\xi',$$

as $S^+(W, b) = S^-(W, b)$ if $W, b \in R(W, b)$

Thus (15) becomes

$$F(\xi W_0, \xi b_0) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{S^-(\xi' W_0, \xi' b_0)}{\xi' - \xi}\, d\xi'$$

$$= \frac{1}{2\pi i} \oint_C \frac{S^-(\xi' W_0, \xi' b_0)}{\xi' - \xi}\, d\xi',$$

, on completing the contour C

by a semi circle at infinity in the ξ' plane, and assuming that $S^-(\xi' W, \xi' b_0) \to 0$ as $|\xi'| \to \infty$. Thus we obtain $F(\xi W_0, \xi b_0) = S^-(\xi W_0, \xi b_0)$, when $W_0, b_0 \in R(W_0, b_0)$ and $\mathrm{Im}\, \xi \neq 0$; and a) is proved.

b) is proved similarly.

Thus $S(\beta, E)$ is holomorphic in the domain

$$\beta = b + \beta_0 ,$$

$$E = \frac{1}{\frac{4}{m^2} - W} - \frac{m^2}{4} ,$$

where $W = \xi W_0$, $b = \xi b_0$, with , $\operatorname{Im} \xi \neq 0$. (17)

This result is a special case of Glaser's theorem:

Suppose a function $F(z_1, z_2)$ is holomorphic in the two open domains

$$D_1 : \left\{ \begin{array}{l} \operatorname{Im} z_1 > 0 \\ \operatorname{Im} z_2 > 0 \end{array} \right\} \qquad D_2 : \left\{ \begin{array}{l} \operatorname{Im} z_1 < 0 \\ \operatorname{Im} z_2 < 0 \end{array} \right\} ,$$

separated by the common boundary

$$S : \left\{ \begin{array}{l} \operatorname{Im} z_1 = 0 \\ \operatorname{Im} z_2 = 0 \end{array} \right\} ,$$

where $F(z_1, z_2)$ is continuous and bounded. If on the subset T : $\operatorname{Re} z_1 < 0$, $\operatorname{Re} z_2 < 0$ of S , no poles of $F(z_1, z_2)$ occur, then the holomorphy domain of $F(z_1, z_2)$ is obtained by adding to D_1 and D_2 the points $P(z_1, z_2)$ such that

$$\begin{array}{l} \alpha \leq \arg z_1 \leq \alpha + \pi \\ \alpha \leq \arg z_2 \leq \alpha + \pi \end{array} , \qquad 0 < \alpha < \pi .$$

The Analyticity Domain of the Scattering Amplitude.

Notation:

$s = E^2 = (\vec{k})^2 \equiv k^2$ is the square of the energy.

$\Delta^2 \equiv |\vec{\Delta}|^2 = |(\vec{k} - \vec{k}')|^2 = 4k^2 \sin^2 \frac{\theta}{2}$ is the square of the 3-momentum-transfer. (18)

$t \equiv -\Delta^2 = -4k^2 \sin^2 \frac{\theta}{2}$.

Properties of the Scattering amplitude $f(s,t)$:

(i) $f(s,t)$ is unitary:

$$\operatorname{Im} f(k,\cos\theta) = -\frac{1}{4\pi} \int f^*(k,\cos\theta') \cdot f(k, \cos\theta\cos\theta' - \sin\theta\sin\theta'\cos\varphi)\, d\Omega' \tag{19}$$

This may be proved from the partial wave expansion when all the δ_ℓ 's are real, i.e. when $V(z)$ is a real function of z.

(ii) $f(s,t)$ is analytic at least in the Lehmann ellipse in the $\cos\theta$ plane.

Let $z = \cos\theta$, with θ complex.

Then $z = \cos(\operatorname{Re}\theta) \cdot \cosh(\operatorname{Im}\theta) - i\sin(\operatorname{Re}\theta)\sinh(\operatorname{Im}\theta)$

$= x + iy$, say (20)

The family of lines $\operatorname{Im}\theta =$ constant give the family of ellipses

$$\frac{x^2}{\cosh^2(\operatorname{Im}\theta)} + \frac{y^2}{\sinh^2(\operatorname{Im}\theta)} = 1, \tag{21}$$

while the family of lines $Re\ \Theta$ = constant give the hyperbola

$$\frac{x^2}{\cos^2(Re\,\Theta)} - \frac{y^2}{\sin^2(Re\,\Theta)} = 1 .$$

(21b)

Analyticity in the Lahmann ellipse ~~is~~ means analyticity inside the ellipse defined by

$$Im\ \Theta = 1 + \frac{m^2}{2k^2},$$

(22)

where m ~~s~~is defined by the lower bound in the potential

$$V(z) = \int_m^\infty \sigma(\mu)\, \frac{e^{-\mu z}}{z}\, d\mu .$$

Writing $\cosh \alpha = 1 + \frac{m^2}{2k^2}$,

(22a)

the Lehmann ellipse may be defined by

$$Im\ \Theta < \alpha .$$

(22b)

Consider the partial wave expansion

$$f(E, z) = \sum_{l=0}^{\infty} a_l(E)\, P_l(z)$$

(23)

If this Legendre expansion converges ~~for~~ at $z \equiv \cos\theta = z_0$, then it converges inside an ellipse passing through z_0 and with foci at -1 , $+1$, and it represents an analytic function there. (Neumann's Expansion Theorem.)

More precisely,

$$\text{if} \quad a_\ell < c\, e^{-\alpha \ell} \qquad \text{for large } \ell ,$$

then the expansion converges inside the ellipse $\mathrm{Im}\,\theta < \alpha$.

For large ℓ, the phase shift may be approximated by the Born approximation

$$\delta \sim \delta_B = -\frac{1}{2k}\int_m^\infty Q_\ell\left(1+\frac{\mu^2}{2k^2}\right)\sigma(\mu)\,d\mu , \tag{24}$$

so that

$$a_\lambda(k) = \frac{e^{2i\delta(\lambda,k)}-1}{2ik}$$

$$\sim -\sqrt{\frac{\pi}{2}}\cdot\frac{\sigma(m)}{2m}\cdot\frac{e^{-\alpha\lambda}}{\lambda^{3/2}}\sqrt{\sinh\alpha} . \tag{25}$$

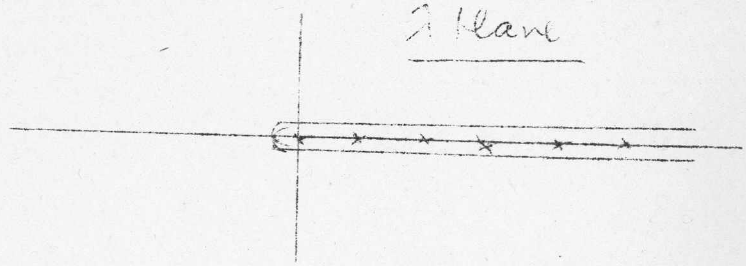
<u>Extension of the analyticity domain in the $\cos\theta$ plane.</u>

By replacing the partial wave expansion by the Regge integral representation in terms of complex λ , the analyticity domain of $f(E,\cos\theta)$ may be enlarged beyond the Lahmann ellipse

$$f(E, \cos\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1) \left[e^{2i\delta(\lambda,k)} - 1 \right] P_l(\cos\theta)$$

$$= \frac{1}{2k} \int_{C'} \frac{e^{2i\delta(\lambda,k)} - 1}{\cos(\pi\lambda)} \, P_{\lambda-\frac{1}{2}}(-\cos\theta) \, \lambda \, d\lambda \qquad (26)$$

λ Plane



where C' is the contour shown in the λ plane.

The integrand has poles at

$$\lambda - \tfrac{1}{2} = 0, 1, 2, \ldots$$

(26) may be proved by noting that when $(\lambda - \frac{1}{2})$ is integral, $\lambda - \frac{1}{2} = n$, say, ~~say,~~

$$P_n(-\cos\theta) = (-1)^n P_n(\cos\theta) \qquad (27)$$

and

$$\frac{1}{\cos\pi\lambda} = \frac{1}{\sin\pi(\lambda-\frac{1}{2})} \approx \frac{(-1)^n}{(\lambda - \frac{1}{2} - n)} \frac{1}{\pi}$$

To prove the covergence of the integral, distort the contour C into C'' composed of $\arg\lambda = \frac{\pi}{2} - \epsilon$ and $\arg\lambda = -\frac{\pi}{2} + \epsilon$. Then if the poles crossed in the course of this deformation are $\lambda - \frac{1}{2} = l_i$ and

with residue β_i, then

$$f(E, \cos\theta) = \frac{1}{2k} \int_{-i\infty}^{+i\infty} d\alpha \, (2\alpha+1) . \frac{e^{2i\delta(\alpha,k)} - 1}{\sin \pi\alpha} . P_\alpha(-\cos\theta)$$

$$+ \, i\pi \sum_i (2\alpha_i + 1) . \frac{\beta_i(E) \; P_{\alpha_i}(-\cos\theta)}{\sin \pi\alpha_i(E)} \qquad (28)$$

where $\alpha \equiv (\lambda - \frac{1}{2})$

The α_i and β_i will be functions of E. For large λ,

$$\left[e^{2i\delta(\alpha,k)} - 1\right] \rightarrow 0 .$$

(which may be proved from a W.K.B. expansion, cf. (1a).

Also, $\left| P_{\lambda-\frac{1}{2}}(-\cos\theta) \right| \approx \left| e^{\pm i(\pi-\theta)\lambda} \right|$, according as $\lambda \rightarrow \pm i\infty$ (along the Im λ axis) — (29)

To see this, we note that for large ν,

$$P_\nu(\cos\theta) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+3/2)}\left(\frac{\pi}{2}\sin\theta\right)^{-\frac{1}{2}}\left\{\cos\left[\left(\nu+\tfrac{1}{2}\right)\theta-\frac{\pi}{4}\right] + O(\nu^{-1})\right\},$$

$$\epsilon \leq \theta \leq \pi-\epsilon, \quad \epsilon > 0$$

When θ is replaced by $(\pi-\theta)$, the result follows.

Thus $\left|P_{\lambda-\frac{1}{2}}(-\cos\theta)\right| \leq e^{|\operatorname{Im}\lambda|\cdot(\pi-\operatorname{Re}\theta)} \cdot e^{(\operatorname{Re}\lambda)|\operatorname{Im}\theta|}$,

as $|\lambda| \to \infty$

(30)

and

$$\frac{P_{\lambda-\frac{1}{2}}(-\cos\theta)}{\cos(\pi\lambda)} \approx e^{-(\operatorname{Re}\theta)|\operatorname{Im}\lambda|}\, e^{(\operatorname{Re}\lambda)|\operatorname{Im}\theta|}$$

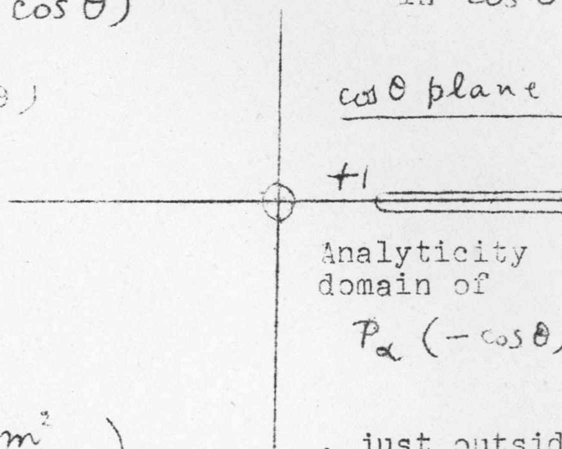
(31)

Along the contour C' , $\operatorname{Re}\lambda = |\lambda| \sin\epsilon$.
Thus (31) is bounded for any $\cos\theta$ such that $\operatorname{Re}\theta > 0$.

$P_{\lambda-\frac{1}{2}}(z)$ is an integral function in λ ; the integrand in (28) is a continuous function of λ (or of α), and the integral itself converges.

$\therefore$ The singularities of $f(E, \cos\theta)$ in $\cos\theta$ will be just the singularities of $P_{\lambda-\frac{1}{2}}(\cos\theta)$ in $\cos\theta$.

i.e. the scattering amplitude $f(E, \cos\theta)$ will be analytic in the $\cos\theta$ - plane cut from $+1$ to $+\infty$ along the real axis.



The cut actually begins at $\left(1 + \frac{m^2}{2k^2}\right)$, just outside the Lehmann ellipse.

Since $t = -2k^2(1 - \cos\theta)$,

$f(E, t)$ will be, analytic in the t - plane cut from m^2 to $+\infty$ along the real axis.

This is the same as the analyticity in t conjectured by Mandelstam .

<u>Analyticity in k for fixed t :</u>

(26) may be also written as

$$f(E, \cos\theta) = \frac{1}{2k}\int_C \frac{\lambda P_{\lambda-\frac{1}{2}}(\cos\theta)}{\cos\pi\lambda} e^{-i\pi(\lambda+\frac{1}{2})}[S(\lambda, k) - 1]\, d\lambda \tag{32}$$

noting that $e^{-i\pi(\lambda+\frac{1}{2})} = (-1)^{\lambda+\frac{1}{2}}$.

The integral may be taken along the imaginary axis. $\frac{\lambda P_{\lambda-\frac{1}{2}}(\cos\theta)}{\cos\pi\lambda}$ is an odd function of λ.

$$\left[\ P_{\ell}(z) = -P_{-\ell-1}(z).\right.$$

$$\therefore\ P_{-\lambda-\frac{1}{2}}(z) = P_{-(\lambda-\frac{1}{2})-1}(z) = P_{\lambda-\frac{1}{2}}(z).\Big]$$

Only the odd part of the factor $e^{-i\pi(\lambda+\frac{1}{2})}\left[S(\lambda,k)-1\right]$ will contribute to the integral.

This odd part is

$$e^{-i\pi(\lambda+\frac{1}{2})}\left[S(\lambda,k)-1\right] - \frac{1}{2}e^{-i\pi(-\lambda+\frac{1}{2})}\left[S(-\lambda,k)-1\right]$$

$$\frac{1}{2}e^{-i\pi/2}\left\{e^{-i\pi\lambda}S(\lambda,k) - e^{i\pi\lambda}S(-\lambda,k)\right\} - \frac{1}{2}e^{-i\pi/2}\left(e^{-i\pi\lambda} - e^{i\pi\lambda}\right)$$

$$\frac{i}{2}\left\{e^{-i\pi\lambda}S(\lambda,k) - e^{i\pi\lambda}S(-\lambda,k)\right\} + \sin\pi\lambda$$

$$\frac{2ik\lambda}{F(\lambda, ke^{-i\pi})\,F(-\lambda, ke^{-i\pi})} + \sin(\pi\lambda). \qquad -(33)$$

~~(33)~~

Note:-

$$F(\lambda,-k)\,F(-\lambda,k) - F(\lambda,k)\,F(-\lambda,-k) = 4i\lambda k$$

$$\frac{F(-\lambda,k)}{F(-\lambda,-k)} - \frac{F(\lambda,k)}{F(\lambda,-k)} = \frac{4i\lambda k}{F(\lambda,-k)\,F(-\lambda,-k)}$$

$$e^{-i\pi\lambda}\,S(-\lambda,k) - e^{i\pi\lambda}\,S(\lambda,k)$$

$$= \frac{4i\lambda k\,e^{-i\pi/2}}{F(\lambda,-k)\,F(-\lambda,-k)}$$

$$= \frac{4\lambda k}{F(\lambda,-k)\,F(-\lambda,-k)} \quad \text{-- (34)}$$

(34)

From the expressions for the free Jost functions, we obtain,

$$F_0(\lambda,-k)\quad F_0(-\lambda,-k) =$$

$$= \frac{2}{\pi}\cdot\Gamma(\lambda+1)\,\Gamma(-\lambda+1)\,(-k)\,e^{i\pi/2} =$$

$$= \frac{2}{\pi} \cdot \frac{\pi\lambda}{\sin\pi\lambda} (-k) \cdot i = \frac{-2ik\lambda}{\sin\pi\lambda},$$

$$\left[\text{since} \quad \Gamma(1+\lambda)\,\Gamma(1-\lambda) = \frac{\pi\lambda}{\sin\pi\lambda}.\right] \tag{34a}$$

The R.H.S. of (33) gives

$$\sin\pi\lambda \cdot \left[1 - \frac{F_0(\lambda,-k)\,F_0(-\lambda,-k)}{F(\lambda,-k)\,F(-\lambda,-k)}\right]. \tag{33a}$$

and (32) becomes

$$f(E,\cos\theta) = \frac{1}{2k}\int_{-i\infty}^{i\infty} d\lambda \cdot \lambda\, P_{\lambda-\frac{1}{2}}(\cos\theta)\cdot\tan(\pi\lambda)\cdot\left[1 - \frac{F_0(\lambda,-k)\,F_0(-\lambda,-k)}{F(\lambda,-k)\,F(-\lambda,-k)}\right] \tag{35}$$

The function inside the square brackets in (33a) and (35) does not have any cut in $\Im m\, k > 0$.

Also, for real t , $P_{\lambda-\frac{1}{2}}(\cos\theta) = P_{\lambda-\frac{1}{2}}\left(1+\frac{t}{2k^2}\right)$ does not introduce any singularity in $\mathcal{I}m\, k > 0$, since $P_{\lambda-\frac{1}{2}}(\cos\theta)$ has only a cut from $-\infty$ to -1 .

Thus $f(k,\cos\theta)$ has only poles in $\mathcal{I}m\, k > 0$

For unphysical values of λ , $f(k,\cos\theta)$ is regular in $\mathcal{I}m\, k > 0$; for physical (i.e. half-integral) values of λ , there are simple poles (with $\mathcal{R}e\, k = 0$ in the upper half k-plane), corresponding to the bound states. For the class of Yukawa potentials, the number of these poles is finite.

The analyticity in t proved above, combined with the single-variable dispersion relation (in s) is equivalent to the Mandelstam representation (for scattering by the class of potentials

$$V(r) = \int_m^\infty d\mu\, \sigma(\mu)\, \frac{e^{-\mu r}}{r} .)$$

LECTURE VI.

High-Energy Diffraction Scattering and Regge Poles:

The salient experimental features of high energy scattering are the following

1) Total cross-sections tend to reach energy-independent behaviour, i.e. they approach constant values, with increasing energy. They roughly satisfy Pomeranchuk's Theorems, viz.

(i) $\sigma_{AB} \rightarrow$ const., $s \rightarrow \infty$. The const. $\sim \lambda_\pi^2$

(ii) $(\sigma_{AB} - \sigma_{A\bar{B}}) \rightarrow 0, \; s \rightarrow \infty$

(iii) $(\sigma^I_{AB} - \sigma^{I'}_{AB}) \rightarrow 0, \; s \rightarrow \infty$

$s =$ the square of the total c.m. energy of A and B. I, I' refer to different isospin channels.

λ_π is the pion compton wavelength

2) Scattering at high energies is characterised by smooth variations with respect to energy, in contrast to the numerous resonances found at low energy in strongly interacting systems. At present there is no evidence of any very-high-energy resonances. The S-matrix as a whole seems to have a simple form at very high energies.

3) All elastic cross-sections show the characteristic diffraction pattern with a forward peak. The earlier data suggested that the width of the peak when plotted as a function of the momentum transfer was nearly independent of the energy and was of the order of m_π. Also, it was noticed that apart from the diffraction peak, the elastic scattering was quite small.

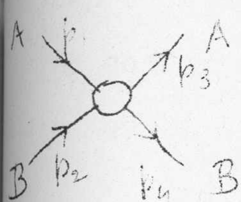
Recent data in pp scattering indicate that there is a shrinking of the diffraction peak with increasing energy. For πp scattering this shrinking has not yet been observed. Probably experiments at much higher energies are needed.

4) At high energies, the elastic scattering amplitude is largely imaginary. In πN scattering, this is true from about 1.5 Gev upwards.

5) For not too large energies, peripheral formulae seem to give the total cross-sections well. They also give the cross-sections for inelastic processes well, mainly for those processes for which there are two well-separated, forward and backward, cones in the c.m.s., i.e. when the two-centres model is valid.

Can these facts be correlated by a simple theory?

Consider 2 $\rightarrow$ 2 scattering of equal-mass particles; for the present, these are taken to be spinless particles.



$$s = (p_1 + p_2)^2 = E^2 \tag{1}$$

$$t = (p_1 - p_3)^2 = -4k^2 \sin^2\frac{\theta}{2} = -\Delta^2 \tag{2}$$

We shall first see how the diffraction scattering of elementary particles can be formulated theoretically, following the treatment of P.T. Mathews and A. Salam. Consider the S-matrix

$$S_{if} = \delta_{if} + (2\pi)^4 i\, \delta(P_i - P_f)\, T_{if} \tag{3}$$

The Unitarity Condition may be written

$$2\, \mathrm{Im}\, T = T\, (2\pi)^4 \delta(P_i - P_f)\, T^{\dagger} \tag{4}$$

OR, in terms of 2-particle channels α (in the 2-particle approximation), as

$$\mathrm{Im}\langle \alpha, \theta | T(E) | \alpha, 0\rangle = \sum_l \langle \alpha, \theta | \alpha, l\rangle \langle \alpha, l | T (2\pi)^4 \delta(P_i - P_f) T^{\dagger} \cdot | l, \alpha\rangle \langle \alpha, l | \alpha, 0\rangle \tag{5}$$

where

$|l,\alpha\rangle$ is the state in a spherical wave representation;

$$|l,\alpha\rangle \simeq 2\sqrt{\pi}\; i^{l}\; \frac{\sin(kr-\frac{1}{2}l\pi)}{kr}$$

and $|\alpha,\theta\rangle$ is the state in a plane wave representation

$$\therefore \langle\alpha,\theta|\beta,l\rangle = \delta_{\alpha\beta}\sqrt{2l+1}\; P_l(\cos\theta) \tag{6}$$

Consider the partial wave expansion of T(E):

$$T(E) = \frac{1}{2q}\sum_{l=0}^{\infty}(2l+1)\frac{[e^{2i\delta_l}-1]}{i}P_l(\cos\theta) \tag{7}$$

At high energies, there is a large number of open inelastic channels, as the energy will be above a number of multi-particle thresholds. These may be taken into account by giving δ_l an imaginary part:

$$\delta_l = a_l + i\beta_l \tag{8}$$

For large inelasticity, β_l will be large, thus

$$\delta_l \simeq i\beta_l$$

The amplitude is ~~propo~~rtional to i, that is, it is purely imaginary

i.e. $\quad T(E) \simeq i\,\mathrm{Im}\,T(E)$, for large E. (9)

We have

$$\langle l,\alpha|\,T.(2\pi)^4\,\delta(P_i-P_f)\,T^{\dagger}|\alpha,l\rangle = \sigma_l^{tot}(E)\,\frac{4pE}{(2l+1)} \tag{10}$$

$p \equiv$ the relative momentum in the c.m.s.

Now,

$$d\sigma_{el}/d\Omega = \left(\frac{1}{8\pi E}\right)^2 \cdot |\langle\alpha,\theta|T|\alpha,0\rangle|^2 \tag{11}$$

Thus,

$$\frac{d\sigma_{el}}{d\Omega} = \left(\frac{1}{4\pi}\right)^2 \cdot \left| \sum_l p\,\sigma_l^{tot}(E)\; P_l(\cos\theta) \right|^2 \tag{11a}$$

We have, from the partial-wave expansion,

$$\frac{d\sigma(E)}{d\Omega} = \frac{1}{p^2} \left| \sum_l \left(l+\tfrac{1}{2}\right) a_l\; P_l(\cos\theta) \right|^2 \tag{11b}$$

In this equation, replace $\sum_l$ by an integral over the impact parameter b, defined by $l = bp$,

or rather, $bp \simeq \sqrt{l(l+1)} \simeq \left(l+\frac{1}{2}\right)$ (12)

Using the fact that

$$P_l(\cos\theta) \approx J_0\left[\sqrt{l(l+1)}\cdot\theta\right] \quad \text{, for small } \theta \tag{13}$$

we obtain

$$\frac{d\sigma_{el}}{d\Omega} = p^2 \left| \int a_b\, J_0(pb\sin\theta)\, b\, db \right|^2 \tag{14}$$

For a 'black' target, taken to be a 'black' sphere,

$$\begin{aligned} a_b &= 1, \quad b \leq R \\ &= 0, \quad b > R \end{aligned} \tag{15}$$

and

$$\frac{d\sigma}{d\Omega} = \left| \frac{R\, J_1(pR\sin\theta)}{\sin\theta} \right|^2 \tag{16}$$

hich is the usual diffraction formula. The first minimum is at

$$\theta_1 = \frac{\pi}{4pR} \tag{17a}$$

r, in terms of t, at $t_1 = p\,\theta_1 = \frac{\pi}{4R}$ (17b)

he "peak width" may be defined by θ_1. R corresponds to the radius f the interaction. Thus the width of the diffraction peak is inversely roportional to the radius of the interaction, for scattering by a black ptical sphere.

he range R of the interaction may be connected with the (4-momentum ransfer)^2 Δ^2 in the interaction roughly by a relation of the type

$$\Delta^2 \propto \frac{1}{R^2}$$

hus the width of the diffraction peak may be given in terms of Δ^2 by saying that the forward diffraction peak corresponds to values of Δ^2 less tha ome value Δ_d^2 : $\Delta^2 \leqslant \Delta_d^2$,

Δ_d^2 being related to the mass of the highest particle that may be exchanged in the interaction. Thus for NN scattering, we expect

$$\Delta_d^2 \approx -m_\pi^2$$

Experimentally, there is no reason why elementary particles should be ave like black spheres; however, we may expect that

$$\mathrm{Im}\,T(E) >> \mathrm{Re}\,T(E).$$

aking an average value of the relevant $a_\ell\ (\ell < \ell_{max})$ to be a, one obtains

$$\frac{d\sigma}{d\Omega} = a^2 \left| \frac{R J_1(pR\sin\theta)}{\sin\theta} \right|^2 ; \quad \sigma_{el} = R^2 a^2 ; \quad \sigma_{tot} = (2\pi R^2)\, a, \tag{18}$$

a is called the opacity parameter.

Besides the main diffraction peak, there are also a series of subsidiary diffraction peaks.

Semi-Elastic or Charge-Exchange diffraction Scattering.

Consider the scattering of systems like (pn) which are not in pure isospin states. Then there will be elastic and charge-exchange scattering. In terms of isospin eigenstates,

$$\langle f|T|i\rangle = \sum_I \langle f|I\rangle T_I \langle I|i\rangle \equiv \sum_I \alpha_I^{fi} T_I, \text{ say} \tag{19}$$

$$\sum_I \alpha_I^{fi} = \delta_{fi} \tag{19a}$$

Then

$$\frac{d\sigma_{fi}}{d\Omega} = \frac{1}{p^2}\left|\sum_l (l+\tfrac{1}{2}) \sum_I \alpha_I^{fi} a_I^l P_l(\cos\theta)\right|^2 . \tag{20}$$

If the scattering amplitude is the same for all I,

then $\langle f|T|i\rangle = T_I \sum_i \alpha_I^{fi} = T_I \delta_{fi}$

For charge-exchange scattering, $f \neq i$. $\therefore \langle f|T|i\rangle = 0$.

But in, general, appreciable charge-exchange scattering is expected.

For pn scattering,

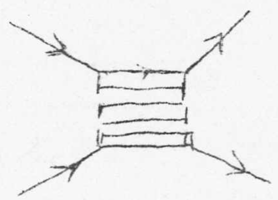
$$\frac{\sigma_{ch,ex.}}{\sigma_{elas.}} = \frac{|a_0 - a_1|^2}{|a_0 + a_1|^2} \tag{21}$$

For a black sphere, this $\simeq 0$ ∅.

Experimental data in this connection are not very clear. For πN Scattering, Von Dardel's experiments seem to show that $\sigma_{3/2}^{tot} \approx \sigma_{1/2}^{tot}$ is reached at about 3 Gev (c.m.) energy.

The energy-dependence of charge-exchange scattering is important; as it will enable us to find out whether high-energy diffraction scattering is dominated by the exchange of a particle with the quantum numbers of the vacuum - the "Pomeranchuk pole."

We now try to see if the observed features of elastic diffraction scattering at high energies can be obtained from a relativistic theory. (We may recollect that the observed fact that the diffracted elastic amplitude is largely imaginary implies that the diffraction scattering is accompanied by scattering into a large number of coupled channels. Thus we expect that Feynman diagrams of the type shown below:



are the ones that determine elastic diffraction scattering.

We now show, following Matthews and Salam, that from dispersion relations, it is possible to derive an approximate expression for high-energy elastic scattering that is of the diffraction form.

Normalize T so that

$$\mathrm{Im}\, T = (2kE)\,\sigma_{tot} \tag{22}$$

Thus, in the high-energy limit $(E \gg M)$,

$$\mathrm{Im}\, T = s\,\sigma_{tot}. \tag{22a}$$

$$\Big[E = \sqrt{k^2+M^2} + \sqrt{k^2+\mu^2}$$

$$\therefore k^2 = \frac{[E^2-(M+\mu)^2][E^2-(M-\mu)^2]}{4E^2} \approx \frac{E^2}{4}, \text{ as } E \to \infty$$

$$\therefore k \approx \frac{E}{2}, \quad \text{and} \quad s = E^2 \approx 2kE.$$

$$\frac{d\sigma}{d\Delta^2} = -\frac{d\sigma}{dt} = \frac{1}{64\pi E^2 k^2} |T|^2,$$

$$(\text{for } t = -\Delta^2).$$

(23)

To obtain $\mathrm{Im}\, T$, write a dispersion relation for fixed, small $t\ (= -\Delta^2)$ for the function

$$\frac{T^t(s,u)}{(s_c - s)^{1/2} (u_c - u)^{1/2}},$$

(the superscript t denotes that t is kept fixed) (24)

where s_c and u_c are the thresholdenergies in the s and u channels respectively.

For large positive s (and hence large negative u, since $s + t + u = 2(M^2 + \mu^2)$),

we have the dispersion relation

$$\frac{\mathrm{Im}\, T^t(s,u)}{(s - s_c)^{1/2} (u_c - u)^{1/2}} = \frac{1}{\pi} \int_{s_c}^{\infty} \frac{\mathrm{Re}\, T^t(s')\, ds'}{(s' - s_c)^{1/2} (u_c - u)^{1/2} (s' - s)} +$$

$$+ \frac{1}{\pi} \int_{u_c}^{\infty} \frac{\mathrm{Re}\, T^t(u')}{(s_c^+ - s)^{1/2} (u' - u_c)^{1/2} (u' - u)}\, du'.$$

(25)

We may expect that the main contribution comes from the neighbourhood of the singularity in the first integral. We have assumed that at such high energies there are no resonances, so that pole terms (occurring at low energies) will be unimportant. Also neglect the 2nd integral over u. To determine $\mathrm{Re}\, T$ for large s, use the single-variable dispersion relation for fixed s,

$$\mathrm{Re}\, T^s(t,u) = \frac{g^2}{\mu^2 - t} + \frac{g^2}{\mu^2 - u} + \frac{1}{\pi}\int_{t_c}^{\infty} \frac{\mathrm{Im}\, T^s(t')\, dt'}{t' - t} + \frac{1}{\pi}\int_{u_c}^{\infty} \frac{\mathrm{Im}\, T^s(u')\, du'}{u' - u}. \tag{26}$$

For $s \gg M^2$, the physically important value of t are small and of the order of $-\mu^2$.

Thus

$$u = 2(M^2 + \mu^2) - t - s$$
$$\approx 2(M^2 + \mu^2) + \mu^2 - s,$$

which is large compared to μ^2 and negative.

The first pole term may be expected to dominate; thus we have

$$\mathrm{Re}\, T^s(t,u) = \frac{g^2}{\mu^2 - t} = \frac{g^2}{\mu^2 + \Delta^2}. \tag{27}$$

$$\frac{T^t(s,u)}{(\ldots)^{1/2}(u_c - u)^{1/2}} = \frac{g^2}{\pi}\int_{s_c}^{\infty} \frac{1/(\mu^2 - t)}{(s' - s_c)^{1/2}(u_c - u)^{1/2}} \frac{ds'}{(s' - s)} \tag{28}$$

OR

$$\mathrm{Im}\, T^{t}(s,u) = \frac{g^2}{(\mu^2-t)} \frac{1}{\pi} f(s), \tag{29}$$

where

$$f(s) = (s-s_c)^{1/2}(u_c-u)^{1/2}\int_{s_c}^{\infty}\frac{ds'}{(s'-s_c)^{1/2}(u_c-u)^{1/2}(s'-s)} \tag{30}$$

$$T(s,t) = T(s,-\Delta^2) = \frac{g^2}{(\mu^2-t)}\left[1+\frac{i}{\pi}f(s)\right]. \tag{31}$$

For large s , $\quad f(s) \simeq \log\left(\frac{4s}{s_c}\right)$

Thus for $s \to \infty$,

$$\left.\begin{aligned} \mathrm{Im}\, T &\to \log\left(\frac{4s}{s_c}\right), \\ \mathrm{Re}\, T &\to \text{const.}, \\ \sigma^{el} &\to \frac{1}{s^2}\left[\log\frac{4s}{s_c}\right]^2 \\ \sigma_{tot} &\to \frac{1}{s}\log\left(\frac{4s}{s_c}\right) \end{aligned}\right\} \tag{32}$$

and

$$\frac{d\sigma}{dt} \longrightarrow \left(\frac{1}{\Delta^2+\mu^2}\right)^2 = \left(\frac{1}{t-\mu^2}\right)^2 \tag{33}$$

This gives an angular distribution which is sharply peaked near $\theta \simeq 0^\circ$, with $\Delta^2 \leq \Delta^2_{max} \simeq \mu^2$, i.e. $t \simeq -\mu^2$,

corresponding to a forward peak of angular width

$$\theta_{max} \simeq \frac{\mu}{k}. \tag{34}$$

This may be written in the diffraction form

$\theta_{max} \approx \frac{1}{k R}$, with $R \approx 1/\mu$ being the range of the interaction.

Thus the above crude approximation to the dispersion relations gives a scattering that is of the classical diffraction form.

We must now ask how the optical theory of diffraction scattering agrees with experiment.

The 2-parameter optical theory above can be made to fit the facts fairly well for πp and pp scattering, if we take

$$R \approx 1/\mu \,, \quad \text{and} \quad a = \frac{2\sigma_{el}}{\sigma_{tot}} \approx \frac{1}{3}.$$

The value of t corresponding to the first minimum outside the forward peak is

$$t \approx -25\mu^2,$$

The objections to this theory are:

1) In this approximation (i.e. without introducing any additional parameters), it is not clear how one could modify the theory so as to get rid of the secondary peaks which result from the sharp boundary of the diffraction region and are unphysical.

2) The theory gives too large a value for the wide-angle scattering, even if only an average value is taken for the latter. Introductionof ideas like peripheral collisions does not reduce it sufficiently.

3) The assumption that the amplitude is pure imaginary means that it has certain analytic properties (as a consequence of the Mandelstam representation). The optical approximation amplitude does not have these analytic properties.

4) Recent experiments show a behaviour of the diffraction peaks with increasing energy which contradicts what is to be expected from a simple optical picture.

(?) Experiments on pp scattering show that the total cross-section $\simeq$ a constant, about 40 mb, from 10-28 Gev (lab), but the width of the diffraction peak decreases logarithmically with increasing s.

$$\mathrm{Im}\, A(s,t) = \beta(t)\, s^{\alpha(t)} = \beta(t)\, e^{\alpha(t)\log s}$$

At large s, $\log s$ varies little with s, as

$$\frac{d}{ds}[\log s] = \frac{1}{s} \to 0 \qquad \text{as} \quad s \to \infty$$

$\alpha(0) = 1$ leads to $\sigma_{tot} \simeq$ constant at high energy.

The width of the diffraction peak may be defined as the change in the t or $\cos\theta$ required forthe amplitude or cross-section to come down to some specified fraction of its peak value, say $\frac{1}{n}$, for fixed s.

Take $\frac{1}{i} A(s,t) = \mathrm{Im}\, A(s,t) = \beta(t)\, e^{\alpha(t)\log s}$.

$$A(s,0) = \beta(0)\, e^{\log s}, \quad \text{if} \quad \alpha(0) = 1.$$

Let $A(s,t_1) = \frac{1}{n} A(s,0)$.

Then

$$\log A(s,0) - \log A(s,t_1) = \log n.$$

or

$$\log\left[\frac{\beta(0)}{\beta(t_1)}\right] - [\alpha(t_1) - 1]\log s = \log n.$$

OR

$$\alpha(t_1) - 1 = \frac{1}{\log s} \cdot \log\left[\frac{\beta(0)}{n\beta(t_1)}\right]$$

In the physical region, $t < 0$. $\therefore t_1 < 0$

If $\frac{d\alpha}{dt} > 0$ in this region, then $\alpha(t_1) < \alpha(0)$.

As t_1 decreases to more negative values, $\alpha(t_1)$ also decreases, and conversely.

From the expression above, $\alpha(t_1)$ decreases logarithmically with s ; thus for small t_1 ,

putting

$$\alpha(t_1) = \alpha(0) + t_1\left(\frac{d\alpha}{dt}\right)_{t=0}$$

$$\approx \alpha(0) + c t_1 ,$$

find that where $c \approx$ a constant,

t_1 also decreases logarithmically with increasing s . i.e. the width of the diffraction pattern decreases as $\frac{1}{\log s}$.

σ_{el} also decreases logarithmically.

This shrinking of the diffraction peak, i.e. the decrease in θ_{max}, does not cause any immediate difficulty, as one might say it corresponds to an increase in R, the size of the diffracting object, or the range of the interaction responsible for the diffraction. One might say that at higher energies, the long-range peripheral interactions become more important.

But the total cross-section also remains constant. Since

$\sigma_{tot} = (2\pi R^2)\, a$, it means that with increasing energy, the opacity parameter <u>a</u> must decrease, i.e. the diffracting object must become more and more transparent. Such an observing sphere whose radius and transparency increase with energy cannot be obtained from any simple classical picture. It is, however, naturally explained by a picture in terms of Regge poles, first put forward by Lovelace.

LECTURE VII.

High-Energy Diffraction Scattering and Regge Poles.

(cont'd)

Last time we compared the observed features of high-energy diffraction scattering and noted in what respects the optical model was defective.

As we showed in an earlier lecture, for potential scattering, the scattering amplitude has the asymptotic behaviour

$$A(s,t) \approx \beta(s)\, t^{\alpha(s)}, \quad t \to \infty \tag{1}$$

The domain of interest in diffraction scattering is that of high energy (i.e. large s) and low momentum transfer (i.e. small t). Mandelstam pointed out that in relativistic 2 -body scattering with crossing symmetry, the domain $t \to \infty$, which is the domain of interest in (1), corresponds to high energy in the third crossed reaction. Thus one may conjecture that, in the relativistic case, the asymptotic behaviour at high energy would be

$$A(s,t) \approx \beta(t)\, s^{\alpha(t)}, \quad s \to \infty \tag{2}$$

$\alpha(t)$ must be complex in the double spectral region $t \geqslant 4\mu^2$

Writing

$$s^{\alpha(t)} = s^{\mathrm{Re}\,\alpha(t)}\, e^{i[\,\mathrm{Im}\,\alpha(t)\,\log s\,]}, \tag{2a}$$

we notice that the double spectral function oscillates. Mandelstam also showed that these oscillations of the double spectral function were important in resolving the problem of divergence due to bad asymptotic behaviour.

$$T(s,t) = \frac{1}{\pi} \int ds' \, \frac{\rho(s',t)}{s'-s} , \qquad \text{for fixed } t . \quad (3)$$

uppose $\rho(s',t)$ behaved badly as $t \to \infty$, for some value f s'. (This would mean divergence at high <u>energies</u> in the crossed eaction.) Then if $\rho(s',t)$ did not oscillate, the integral would till diverge at large t, but of $\rho(s',t)$ does oscillate then there s a possibility that a cancellation removes the divergence, and we obtain n amplitude with an asymptotic behaviour that depends on . This is hat we obtain with a Regge type of behaviour.

According to (2a), the amplitude and frequency of the oscillations ncrease with energy; thus the positive and negative peaks in the double pectral function in (3) will tend to cancell as ($s' - s$) becomes larger, i.e. as s is further and further removed from the double pectral region. Thus wide-angle scattering would be strongly suppressed.

<u>egge Poles</u>:

We have seen that for elastic scattering of a particle by a potential,

$$, \cos\theta) = \frac{1}{2i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{A(\alpha, E) \cdot P_{\alpha}(-\cos\theta)}{\sin \pi\alpha} +$$

$$+ \sum_i \beta_i(E) \, \frac{P_{\alpha_i(E)}(-\cos\theta)}{\sin \pi\alpha_i(E)} , \qquad (4)$$

ound states occur when $E < 0$, and $\alpha_i(E_l) = l$, $l = 0, 1, 2, \ldots$ (5)

ar the bound state, the amplitude is expected to be dominated by the rresponding pole term, and is of the form

$$\frac{\beta_i(E_l)}{\pi \alpha_i'(E_l)} \cdot \frac{P_l(\cos\theta)}{(E - E_l)} \qquad (5a)$$

Resonances occur when for $E > 0$,

$$\mathrm{Re}\,\alpha_i(E_\ell) = \ell\,, \quad \ell = 0, 1, 2, \ldots \tag{6}$$

and the amplitude is of the Breit-Wigner form

$$\frac{\beta_i(E_\ell)}{\pi} \cdot \frac{P_\ell(\cos\theta)\;/\eta}{(E - E_\ell) + i\,\dfrac{\mathrm{Im}\,\alpha_i(E_\ell)}{\eta}}, \tag{6a}$$

where

$$\eta = \frac{d}{dE}\left(\mathrm{Re}\,\alpha_i\right)\Big|_{E = E_\ell}. \tag{6b}$$

As $|z| \to \infty$, $P_\alpha(z) \sim z^\alpha$. ; the integral $\to 0$ as $z^{-1/2}$ or faster, and the pole terms dominate.

If, for some energy, the contribution of the pole with the max. value of α dominates, the amplitude will be

$$A(z, E) \underset{z \to \infty}{\approx} \frac{\beta_i(E)\; z^{\alpha_i(E)}}{\sin \pi\alpha_i(E)}. \tag{7}$$

Suppose we consider the region where one pole is dominant; say $\alpha_i(E) = \ell$

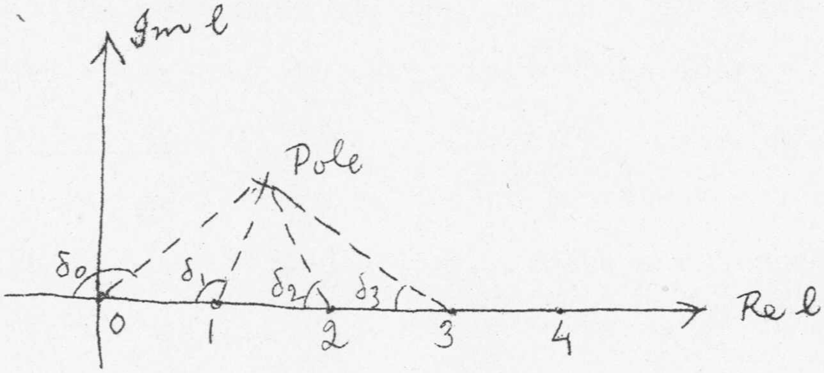
From unitarity, the S-matrix S satisfies

$$S(\ell)\, S^*(\ell^*) = 1. \tag{8}$$

$$\therefore \quad S = \frac{l^* - \alpha^*(s)}{l - \alpha(s)} = \exp(2i\delta_l) \tag{9}$$

$$\underline{\text{OR}} \qquad \delta_l = -\arg\,[\,l - \alpha(s)\,]. \tag{9a}$$

Thus the various values of δ_l, where $l = 0, 1, 2, \ldots$ would be given by the angles shown in the figure (e.g. see Lovelace: Nuovo Cim. 25, 730 (1962).)



As E varies, the position of the pole, $\alpha(E)$, varies. When $\alpha(E) = l_i$, $\delta_{l_i} = 90^\circ$, and we have a resonance for $E > 0$.

The width $\operatorname{Im}\alpha_i(E_l) / \left[\frac{d}{dE}(\operatorname{Re}\alpha)\right]_{E_l}$ may be pictured as the "angular velocity" of the pole at resonance. It can be zero only for a point on the real axis, for which $\delta_l = 0$.

The result obtained by generalising (9a) to the case of several poles is

$$= -\sum_n \left\{ \arg\left[1 - \frac{l}{\alpha_n(s)}\right] + l\left[\operatorname{Im}\left(\frac{1}{\alpha_n(s)}\right) - C(s)\right\}. \tag{10}$$

where $c(s)$ is determined by the condition that as $\ell \to +\infty$, $\delta_\ell \to 0$.

When only a finite number of poles is included, their positions $\alpha_i(s)$ can be deduced from the physical δ_ℓ and conversely. However, a finite number of poles cannot reproduce the correct centrifugal barrier; thus the above will not be valid when the contrifugal barrier is important, i.e. for small δ_ℓ.

Also above the inelastic threshold, the above representation $\delta_\ell = - \arg\left[\ell - \alpha(s)\right]$ is not valid because the pole and zer are then not at conjugate points. For a resonance, now, the pole, physical point and zero must all the lie on a straight line (in the ℓ-plane But the pole determines the asymptotic behaviour.

<u>C.D.D. Poles:</u> If for $E > 0$, a Regge pole crosses the real ℓ - axis near a physical point (integral ℓ), then $\delta_\ell = 0$ at that point; $\therefore S = 1$. These correspond to C.D.D. zeros; in the Jost representation of S-matrix in terms of the ratio $\dfrac{F(\lambda, k)}{F(\lambda, -k)}$, both the numerator and denominator have a zero. C.D.D zeros correspond to poles that are independent of the dynamics of the seattering system; they may be interpreted as unstable particles independent of the forces responsible for the scattering but having the same quantum numbers as the scattering system.

Since a potential, by definition, includes only the interaction, there can be no C.D.D. zero in potgntial scattering; thus here, all Regge poles must have $\operatorname{Im} \ell > 0$.

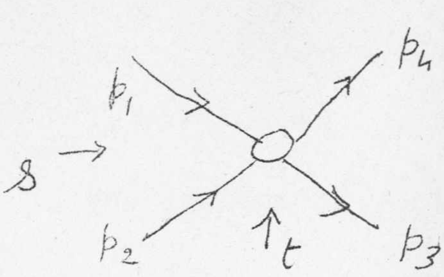
For $E < 0$, all poles lie on the real axis; they correspond to bound states, if $\alpha(s)$ is at a physical value.

<u>Generalisation to Relativistic Scattering</u>

Consider elastic scattering; take $\pi\pi$ scattering as an example.

$\pi\pi$ Scattering:

$$p_1 + p_2 = p_3 + p_4$$



Consider the t channel. Because the pions are bosons, the amplitude will be even or odd under $p_1 \leftrightarrow p_3$ (or $s \leftrightarrow u$) according as the isospin I is even or odd. (Here, $s = (p_1 + p_2)^2$, $u = (p_1 - p_4)^2$; $t = (p_1 - p_3)^2$.]

$$T_I(s,t,u) = A(t,s) \pm A(t,u)$$

$$\sum_l \frac{1}{2} A_l(t) \left\{ P_l\left(1 + \frac{2s}{t-4m^2}\right) \pm P_l\left(1 + \frac{2u}{t-4m^2}\right) \right\}$$

$$\sum_l \frac{1}{2} A_l(t) \left\{ P_l\left(1 + \frac{2s}{t-4m^2}\right) \pm e^{i\pi l} P_l\left(-1 - \frac{2u}{t-4m^2}\right) \right\}.$$

i.e. only even or odd l waves appear according as $(-1)^I$ is even or odd.

Suppose we assume that as for a potential, the single particle states and resonant states with the same quantum numbers occurring in a particular scattering system are dynamically related (assuming there are no C.D.D. poles), then we may conjecture that the scattering amplitude is given by an expression analogous to that for potential scattering. At high energies, pole terms will again dominate.

$$A(s,t,u) = \sum_i \tfrac{1}{2}\beta_i(t)\left[P_{\alpha_i(t)}\left(1+\frac{2s}{t-4m^2}\right) \pm e^{i\pi\alpha_i(t)} P_{\alpha_i(t)}\left(-1-\frac{2u}{t-4m^2}\right)\right]$$

$$= \sum_i \tfrac{1}{2}\beta_i(t)\left[1 \pm e^{i\pi\alpha_i(t)}\right] P_{\alpha_i(t)}(\cos\theta_t),$$

where m is the pion mass and θ_t is the scattering angle in the t channel.

The sign in $\left[1 \pm e^{i\pi\alpha_i(t)}\right]$ is known as the "signature" of the Regge trajectory $\alpha_i(t)$. Such a factor is expected for all (2-particle) scattering amplitudes, as the presence of an exchange potential will give rise to a term that alternates in sign at successive physical J values.

Bound states occur when

$$Re\, \alpha_i(t_s) = l,$$

where l must be even or odd according to the signature of the trajectory $\alpha_i(t)$.

At very high energies, again

$$A(s,t,u) \approx \sum_i \frac{\beta_i(t)\left[1 \pm e^{i\alpha_i(t)}\right] s^{\alpha_i(t)}}{\sin\pi\alpha_i(t)},$$

as the signature is positive for all Regge trajectories with even I and negative for odd I.

Froissart has proved that for .

Suppose elastic scattering at very high energies is dominated by a single pole term. This pole must have the maximum value of $\alpha(t)$, as $t \to 0^-$, of all the possible poles, in order that $s^{\alpha_i(t)}$ is maximum. $[\alpha_i(t)]_{max} = 1$. We notice that if we take this ominant pole to have $\alpha(0) = 1$, then the Pomeranchuk theorem of constancy of total cross-sections is fulfilled.

For $\operatorname{Im} T(s,0) = s\sigma_{tot}$,

So that

$$\sigma_{tot} = \frac{1}{s} \operatorname{Im} T(s,0)$$

$$= \frac{1}{s} \frac{\beta_p(t) \sin \pi\alpha_p(t)}{\sin \pi\alpha_p(t)} s^{\alpha_p(t)} = \beta_p(t) \cdot s^{\alpha_p(t)-1}.$$

$$\xrightarrow[t\to 0^-]{} \text{const.}, \quad \text{if } \alpha_p(0) = 1.$$

It is thus attractive to assume that such a pole called the Pomeranchuk (P) -- dominates very high energy scattering. Chew and Frautschi interpret this by saying that strong interactions saturate unitarity -- i.e. they have the maximum strength consistent with unitarity (since $\alpha_p(t)$ has this maximum value) -- The Principle of Maximum strength for Strong Interactions.

We also obtain

$$T_p \approx i\beta_p(0)\, s^{\alpha_p(t)}$$, which is pure imaginary,

$$\beta_p(0) = \sigma_{tot}(\infty),$$

and

$$\left(\frac{d\sigma}{dt}\right) \Big/ \left(\frac{d\sigma}{dt}\right)_{t=0} = s^{2[\alpha_p(t)-1]},$$

or

$$\frac{d\sigma}{dt} = \frac{4\pi}{s}\frac{d\sigma}{d\Omega} = \frac{1}{\pi}\left|\frac{T}{4s}\right|^2 \approx \frac{1}{\pi}\left|\frac{\operatorname{Im} T}{4s}\right|^2.$$

The Pomeranchuk pole must have the quantum numbers of the vacuum, $I=0$, $S=0$, $B=0$, $G=+1$, in order that it dominate high-energy scattering in <u>all</u> systems. It is taken to have an even signature, as it has $I=0$ and is assumed to occur in $\pi\pi$ scattering.

Immediate consequences of the above hypotheses are that

(i) The 1st Pomeranchuk theorem holds,

(ii) Different scattering systems will show the same high-energy behaviour (for only the <u>coupling</u> to the Pomeranchuk trajectory can be different; $\alpha_P(t)$ is always the same).

(iii) The 2nd Pomeranchuk theorem, that

$$(\sigma_{AB} - \sigma_{A\bar{B}}) \rightarrow 0, \qquad s \rightarrow \infty,$$

or particle and anti-particle scattering cross-sections are the same, holds.

The last can be seen the follows:

$AB \rightarrow A\bar{B}$ is just the 2nd crossed reaction, or the u reaction, if $AB \rightarrow AB$ is the s reaction,

$$T(s,t,u) = \frac{\beta_P(t)}{2\sin\pi\alpha_P(t)}\left[P_{\alpha_P(t)}(\cos\theta_t) + P_{\alpha_P(t)}(-\cos\theta_t)\right]$$

$P_\alpha(\cos\theta)$ has a branch cut in the α-plane from $\alpha=-\infty$ to $\alpha=-1$ along the real axis, with discontinuity $2i(\sin\pi\alpha)\,P_\alpha(-\cos\theta_t)$

Thus <u>in the s channel,</u> where $\cos\theta_t = 1 + \frac{s}{2q_t^2} \geq 1, \because s>0,$

only the 2nd term in ( ) contributes to $\mathrm{Im}\,T$;

$$\mathrm{Im}\,T = -i\beta_P(t)\,P_{\alpha_P(t)}(\cos\theta_t).$$

In the u channel, where $\cos\theta_t = -1 - \frac{u}{2q_t^2} \leq -1$,

only the 1st term in ( ) contributes, and

$$\mathrm{Im}\, T = i\beta_p(t)\; P_{\alpha_p}(-\cos\theta_t)$$

$$\xrightarrow[t\to 0]{} i\beta_p(t)\; (-1)^{\alpha_p(0)}\; P_{\alpha_p(0)}(\cos\theta_t)$$

$$= -i\beta_p(t)\; P_{\alpha_p(0)}(\cos\theta_t)\;, \quad \text{as } t \to 0$$

$$(\text{for } \alpha_p(0) = 1).$$

As $t \to 0$, the total cross-sections (related to $\mathrm{Im}\, T$ by the optical theorem) will be the same in the s and u channels, i.e. for $(A + B)$ and $(A + \bar{B})$ elastic Scattering.

Wetherell has analysed the pp scattering data and deduced the form of the dependence of $\alpha_p(t)$ on t.

We have already noted that

$$\frac{d\sigma}{d\Omega} = \left|\frac{1}{8\pi W} T\right|^2 \;; \quad \frac{d\sigma}{dt} = \frac{4\pi}{s}\frac{d\sigma}{d\Omega} = \frac{1}{\pi}\left|\frac{T}{4s}\right|^2$$

$$\approx \frac{1}{\pi}\left|\frac{\mathrm{Im}\, T}{4s}\right|^2 = \frac{1}{16\pi}\left|\sigma_{tot}\right|^2, \text{ for } t = 0, \text{ as } s \to \infty.$$

For $t \neq 0$, $\mathrm{Im}\, T = s\,\sigma_{tot}\, g(s,t)$,

where $g(s,t)$ is the "form factor" for diffraction scattering.

To find the nature of $g(s,t)$, plot $\frac{d\sigma}{dt}$ as a function of t for fixed s. A rapid increase in the slope of the curves is obtained on a logarithmic plot. It is found that at large s, $g(s,t)$ depends mainly on t and varies little with s.

We may write

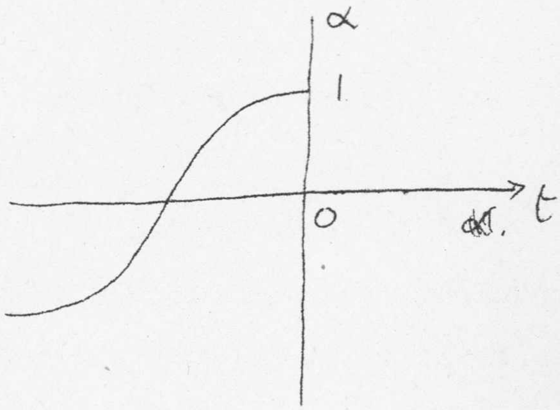
$$\frac{d\sigma}{dt} = F(t)\, s^{2[\alpha_P(t)-1]} .$$

As $t \to 0$, $\alpha(t) \to 1$; thus the dependence of $\frac{d\sigma}{dt}$ on becomes very weak as $t \to 0$.

$$\log\left(\frac{d\sigma}{dt}\right) = \log F(t) + 2[\alpha_P(t)-1] \log s .$$

If $\log\left(\frac{d\sigma}{dt}\right)$ is plotted as a function of s for fixed t , with values taken from experiment, we expect a family of straight lines for different t , each with a slope $2[\alpha_P(t)-1]$. The data can be fitted only very roughly by straight lines; the result is as shown in the figure.

$\alpha_P(t)$ seems to $\to -1$ as $t \to -\infty$.



The Problem of the Ghost

For the Pomeranchuk trajectory, $\alpha(t=0)=1$, and $\frac{d\alpha}{dt} > 0$ for $t < 0$. For some $t = t_0 < 0$, we will have $\alpha(t=t_0)=0$ As $J=0$ is physical for positive signature, this will give a pole in the physical region of the scattering amplitude. (Since this pole would correspond to a particle with $-$ve $(\text{mass})^2$ this 'ghost' state causes an apparent difficulty. Gell-Mann pointed out that this difficulty could be got rid of by postulating that the coupling of all particles to this ghost vanished as

$t \to t_0$, i.e. $\beta(t) \to 0$ as $t \to t_0$. Then the ghost state would be decoupled from all physical systems and there woul be no problem. It has also been pointed out that this is similar to man nuclei lacking a lowest zero-spin state and is not an unnatural occurren

LECTURE VIII.

High-Energy Diffraction Scattering and Regge Poles.

Last time we saw how the hypothesis of Regge poles could be generalized to a relativistic system and how the hypothesis of the Pomeranchuk pole with the quantum numbers of the vacuum would explain the broad features of very high-energy scattering. We also saw how the trajectory $\alpha_p(t)$ could be inferred from a study of pp scattering data at very high energies.

Today we shall look at πN diffraction scattering above a few Gev.

The main observed features of πN scattering at about 5 Gev are a very sharp forward peak containing about 95% of the events between $\cos\theta = 1$ and $\cos\theta = 0.09$, and thus very little wide-angle scattering.

A forward peak may be obtained by the exchange of a 2π state, but this would still give too large a scatterings at angles off the forward direction. Lovelace showed how the Regge pole hypothesis gives a good fit to high energy πN scattering.

If the amplitude is pure imaginary, it will satisfy a dispersion relation for large s .

We can write

$$T(s,t,u) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{\rho_1(s,t')\,dt'}{t'-t} + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} \frac{\rho_2(s,u')\,du'}{u'-u} \qquad (1)$$

Forward scattering corresponding to $|t|$ small, $u \to -\infty$, so that (1) $\approx$ the 1st term

Backward scattering corresponding to $|u|$ small, $t \to -\infty$, so that (1) $\approx$ the 2nd term

Experimentally, backward scattering is negligible. Thus the R.H.S. of (1) may be approximated by the first integral.

Consider the πN scattering amplitude:

$$T = -A + (i\gamma\cdot Q)B \qquad (2)$$

We have

$$A(\pi^- p) = A^{(+)} + A^{(-)}$$

$$A(\pi^+ p) = A^{(+)} - A^{(-)} \qquad (3)$$

Similarly for the B's .

For the reaction $\pi + \pi \to N + \bar{N}$, $A^{(+)}$, $A^{(-)}$ are the amplitudes for the $I = 0$ and $I = 1$ states respectively.

Thus in πN scattering, $A^{(+)}$ will get a pole contribution from scattering via exchange of a $I = 0$ particle, which may be the A B C pole or the Pomeranchuk, while $A^{(-)}$ has a pole in t correspond to the exchange of a $I = 1$ particle, which may be the ρ meson

For πN scattering,

$$A^{(+)} = \frac{1}{3} A^{1/2} + \frac{2}{3} A^{3/2} ,$$

$$A^{(-)} = \frac{1}{3} A^{1/2} - \frac{1}{3} A^{3/2} .$$

(4)

Suppose we assume an asymptotic Regge behaviour:

$$A^{(+)} \sim \beta^{(\pm)}(t)\, s^{\alpha_{\pm}(t)} .$$

Experimentally, one observes that the $\pi^+ p$ and $\pi^- p$ differential cross-sections behave similarly at momenta $>$ 2Gev/c (Lab).

This may be either because one of the two terms $A^{(+)}$ and $A^{(-)}$ is small compared to the other or because both the amplitudes $A^{(+)}$ and $A^{(-)}$ are similar in their variation with energy.

If one assumes that the vacuum trajectory makes the dominant contribution to $A^{(+)}$, and the ρ to $A^{(-)}$, then one might expect that <u>either</u> the ρ is coupled much more weakly than the Pomaranchuk (P), or $\alpha_\rho(t)$ is very similar to $\alpha_P(t)$, <u>or</u> both. There seems to be evidence for the 1st hypothesis; little is known about $\alpha_\rho(t)$, except that $\alpha_\rho(0) \simeq$ 0.03 to 0.4 and $\alpha_\rho(m_\rho^2) = 1$. The slopes of all the known trajectories seem to be similar.

The exchange of a 2π resonance of spin J, or 'spin' $\alpha(t)$, will give rise to the following asymptotic behaviour of A and B:

$$A \sim \frac{P_\alpha(\cos\theta_t)}{t - m^2} \sim s^{\alpha(t)} , \quad \text{as } s \to \infty$$

$$B \sim \frac{P_\alpha'(\cos\theta_t)}{t - m^2} \sim s^{\alpha(t)-1} , \quad \text{as } s \to \infty ,$$

(5)

We may write

$$A(s,t) \approx \beta_1(t)\, s^{\alpha(t)},$$

$$B(s,t) \approx \beta_2(t)\, s^{\alpha(t)-1}. \tag{6}$$

We note that in terms of the amplitudes f_1 and f_2 defined by CGLN,

$$\left.\begin{aligned} A &\to \frac{1}{8\pi}(f_1+f_2); \\ B &\to \frac{1}{8\pi\sqrt{s}}(f_1+f_2) \end{aligned}\right\} \text{as } s \to \infty. \tag{7}$$

Thus $\beta_1(t)$, $\beta_2(t)$ in (6) will, in general, be different.

Consider the coherent and spin-flip amplitudes f, g defined by

$$T = f + g\,(\vec{\sigma}\cdot\vec{N}) \tag{8}$$

where $\vec{N}$ is a unit vector parallel to $(\vec{k}_i \times \vec{k}_f)$, $\vec{k}_i$ and $\vec{k}_f$ being the initial and final c.m. momenta.

One also defines

$$T = f + g'\,\vec{\sigma}\cdot(\hat{\vec{k}}_i \times \hat{\vec{k}}_f), \tag{8a}$$

where $\hat{k}_i$ and $\hat{k}_f$ are <u>unit</u> vectors.

$$\therefore\ g = g' \sin\theta_{if}, \tag{8b}$$

θ_{if} being the angle between $\vec{k}_i$ and $\vec{k}_f$.

It is useful to note the different notations used.

Mandelstam uses

$$T = a + b(\vec{\sigma}\cdot\vec{k}_i)(\vec{\sigma}\cdot\vec{k}_f) \tag{8c}$$

$$= \left[a + b(\vec{k}_i\cdot\vec{k}_f)\right] + b\, i\vec{\sigma}\cdot(\vec{k}_i\times\vec{k}_f) \tag{8d}$$

le have

$$= a = \frac{(E+m)}{8\pi W}\left\{A + (W-m)B\right\} \rightarrow \frac{1}{8\pi}\left[A + \sqrt{s}\cdot B\right]$$

$$= b = \frac{(E-m)}{8\pi W}\left\{-A + (W+m)B\right\} \rightarrow \frac{1}{8\pi}\left[-A + \sqrt{s}\cdot B\right] \tag{9}$$

Thus

$$(s,t) = \frac{1}{8\pi W}\Big\{\left[(E+m) - (E-m)\,\vec{k}_i\cdot\vec{k}_f\right]A + \left[(E+m)(W+m) + \vec{k}_i\cdot\vec{k}_f\,(E-m)(W+m)\right]B\Big\}$$

$$\rightarrow \frac{1}{8\pi}\left[(1-\cos\theta)A + \sqrt{s}\,(1+\cos\theta)B\right], \text{ as } s\rightarrow\infty$$

$$\rightarrow \frac{1}{4\pi}\left[-\frac{t}{s}A + \sqrt{s}\cdot B\right], \quad \text{as } s\rightarrow\infty \tag{10a}$$

and

$$s,t) = b = \frac{(E-m)}{8\pi W}\left[-A + (W+m)B\right] \rightarrow \frac{1}{8\pi}\left[-A+\sqrt{s}\cdot B\right], \text{ as } s\rightarrow\infty$$

$$g = g'\sin\theta \tag{10b}$$

We also have $t = -4k^2 \sin^2\frac{\theta}{2} \rightarrow -k^2\theta^2$, as $\theta \rightarrow 0$

OR

$$\sin\theta \approx \theta \approx \frac{\sqrt{-t}}{k} \rightarrow \frac{2\sqrt{-t}}{\sqrt{s}}, \quad \text{as } s \rightarrow \infty \tag{11}$$

Thus

$$\left.\begin{aligned} f(s,t) &\rightarrow \frac{1}{8\pi}\, 2s^{[\alpha(t)-\frac{1}{2}]}\, \beta_2(t), \\ g(s,t) &\rightarrow \frac{1}{8\pi}\, 2\sqrt{-t}\, s^{[\alpha(t)-\frac{1}{2}]}\, \beta_1(t) \end{aligned}\right\} \tag{12}$$

For initially unpolarised nucleons,

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi}\right)^2 \cdot \left[|f|^2 + |g|^2\right]$$

$$= \left(\frac{1}{4\pi}\right)^2 \left[\cancel{4}|\beta_2(t)|^2 - t\,|\beta_1(t)|^2\right] s^{2\alpha(t)-1} \tag{13}$$

$t < 0$ in the physical region; thus, in this region, $\alpha(t)$ is real.

Lehmann showed that from the fixed t dispersion relations, one obtains

$$\begin{aligned} \mathrm{Re}\,\beta_1(t) &= -\cot\left[\frac{\pi}{2}\alpha(t)\right] \mathrm{Im}\,\beta_1(t), \\ \mathrm{Re}\,\beta_2(t) &= -\cot\left[\frac{\pi}{2}\alpha(t)\right] \mathrm{Im}\,\beta_2(t) \end{aligned} \tag{14}$$

$$\therefore \frac{d\sigma}{d\Omega} = \mathcal{F}(t)\, s^{2\alpha(t)-1}, \tag{15}$$

where

$$(t) = \left\{ \left[\operatorname{Im} \beta_2(t)\right]^2 \operatorname{cosec}^2\left[\frac{\pi}{2}\,\alpha(t)\right] - t\left[\operatorname{Im} \beta_1(t)\right]^2 \cdot \operatorname{cosec}^2\left[\frac{\pi}{2}\,\alpha(t)\right] \right\} \left(\frac{1}{4\pi}\right)^2 \tag{16}$$

According to the Mandelstam representation, $\operatorname{Im} A$ and $\operatorname{Im} B$ satisfy dispersion relations with cuts on both sides. However, neglect of backward scattering leaves us with only 1 branch-cut, from

t_0 to ∞ ,

where

$$t_0 = 4 + \frac{16\left[s + 3M^2 - 3\right]}{\left[s - (M+2)^2\right]\left[s - (M-2)^2\right]} \tag{17}$$

where we have taken $m_\pi \simeq 1$.

We have

$$\operatorname{Im} T(s, t, u) = \frac{1}{\pi}\int_{t_0}^{\infty} \frac{\rho_1(s, t')\, dt'}{t' - t} .$$

But

$$\operatorname{Im} A = \operatorname{Re} \beta_1(t) \cdot s^{\operatorname{Re}\alpha(t)} \sin\left[(\operatorname{Im}\alpha)(\log s)\right] + $$
$$+ \operatorname{Im} \beta_1(t)\, s^{\operatorname{Re}\alpha(t)} \cos\left[(\operatorname{Im}\alpha)(\log s)\right].$$

Similarly for $\operatorname{Im} B$, with $\alpha(t)$ replaced by $\alpha(t) - 1$.
Thus we may expect that $\operatorname{Im} \beta_1(t)$ and $\alpha(t)$ satisfy dispersi relations in t with the same cut.

nalysis of the data:

We have $\dfrac{d\sigma_{el}}{d\Omega} = |\,\mathrm{Im}\,T|^2$

We may write the dispersion relation

$$\frac{d\sigma_{el}}{d\Omega}(s,t) = \frac{1}{\pi}\int_{t_0(s)}^{\infty} \frac{\rho(s,t')\,dt'}{t'-t} \qquad (18)$$

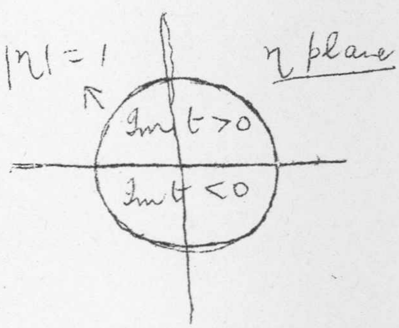
ince the square of $\mathrm{Im}\,T$ will have the same cuts as $\mathrm{Im}\,T$,

Normally, there would be the difficulty of finding out $\rho(s,t)$ n the unphysical region before being able to calculate $\dfrac{d\sigma}{d\Omega}$ in the hysical region. Use of a conformal mapping technique enables us to alculate $\rho(s,t)$ in the unphysical region from the measured alues of $d\sigma_{el}/d\Omega$, by an extrapolation to $t > 0$.

For this, plot the observed values of $\dfrac{d\sigma_{el}}{d\Omega}$ as a function of the ew variable

$$\eta = \frac{-t}{\left[\sqrt{t_0} + \sqrt{t_0 - t}\,\right]^2}, \qquad (19)$$

here to is defined by (17).



his conformal mapping maps (i) the t plane nto the interior of the circle $|\eta| = 1$ s shown, (ii) the physical region in t into art of the segment $0 \leq \eta < 1$.

nd

iii) the cut in $t \geq t_0$ into the circle $|\eta| = 1$.

The double spectral function $\rho(s,t)$, i.e. the discontinuity cross the cut in t, is just the imaginary part on the unit circle n the η-plane.

Thus $\frac{d\sigma_{el}}{d\Omega}$ must be analytic in η inside the unit circle.

The procedure for extrapolation:

(i) Plot the experimental points as a function of η ;

(ii) Fit these points by a function of η analytic in $|\eta| < 1$,

(iii) Calculate the imaginary part on the circle $|\eta| = 1$, and

(iv) transform this imaginary part back as a function of t to get the double spectral function, using the inverse mapping

$$t = \frac{-4t_0\eta}{(1-\eta)^2} \tag{20}$$

The extrapolation is not unique, but bounds can be placed on the errors.

Results of the extrapolation:

1) The data can be fitted as indicated above by a Gaussian form

$$\frac{d\sigma}{d\Omega} = a \exp[-b\eta^2], \tag{21}$$

where $a = 31.49$ mb/sr and $b = (16.9 \pm 0.8)$. There is a sharp forward peak for $\cos\theta > 0.84$.

2) The final result obtained by approximating $\alpha(t)$ and $\beta(t)$ by polynomials in η , and (noting that $\beta(t)$ varies much more slowly than $s^{\alpha(t)}$) approximating $\beta(t)$ by a constant, is

$$\frac{d\sigma}{d\Omega}(\pi^- p) = 1.814\, q^2\, s^{-0.298 - 2.7\eta^2} \tag{22}$$

3) The formula above has been obtained from $\frac{d\sigma}{d\Omega}$ at 5 Gev. and σ_{tot} at different energies. The formula fits $d\sigma/d\Omega$ at other energies very well. However, at lower energies ($\sim$ 2.5 Gev/c), wide - angle

cattering sets in, presumably owing to the contributions of Regge poles ther than the Pomeranchuk pole.

) The width of the forward peak decreases as s increases.

Thus this 2-parameter theory gives a much better fit to the data han the optical model; it thus supports the general hypothesis of the ole of Regge poles (and in particular of the dominance of a single pole t high energies) in diffraction scattering.

efects of Lovelace's Theory:- (i) The approximation $\beta(t) \simeq$ a constant has no justification. (ii) It was later pointed out that the extrapolation violated the Regge condition

$$\frac{d\alpha}{dt} > 0, \qquad 0 < t < 4\mu^2,$$

hich can be proved in field theory.

One of the main results of Lovelace's* extrapolation was the prediction of a possible $I = 0$ D-wave $\pi\pi$ resonance below an energy of 4μ..

A D-wave $\pi\pi$ resonance is expected on other grounds: a) The Pomeranchuk trajectory, if taken to be approximately a straight time, would ass through $J = 2$ at about 1.2 Bev.

) Analysis of the small phase shifts in pion-nucleon scattering leads s to expect a D-wave $I = 0$ resonance.

An $I = 0$ resonance at about 1.2 Bev has been reported at the ERN conference, 1962. If it has $J = 2$, it could be the D-wave ole on the vacuum trajectory. (Lovelace has also speculated that if his $J = 2$ resonance exists, it may be a strongly coupled heavy graition, which would give rise to a strong, short-range "gravitational" orce.)

The hypothesis of the dominance of one pole at very high energies oes not help us to understand the differences between the behaviour of the amplitudes of different reactions. We shall see in the next lecture how the total cross-sections for different reactions may be compared to give us information about the Regge trajectories of the ABC pole, and the ω and ρ mesons; they indicate that an additional 'Vacuum' trajectory P' may also exist.

LECTURE IX

High-Energy Diffraction Scattering and Regge Poles (Cont'd)

First we shall briefly summarise the details of the high-energy data (Ref: S.Drell: Proc. Aix-en-Provence Conference, 1961;
and A.M.Wetherell: Proc.Phys.Soc. 80,)

I) pp scattering:

(i) σ_{pp}^{tot} is a constant $\approx$ 40 mb from 10Gev to 20 Gev, and appears to be so beyond 10^4 Gev from cosmic ray experiments.

(ii) σ_{pp}^{el} $\approx$ 9 mb at 25 Gev. The diffraction peak in elastic scattering becomes narrower with increasing energy.

e.g. σ^{el} at $\sqrt{t} = -1$ Gev/c is less than 2×10^{-4} of its forward value for $E_p(lab) = 24$ Gev.

For the same value of t. it is $\simeq 2 \times 10^{-3}$ of its forward value at $E_p \simeq 6$ Gev.

II) $p\bar{p}$ Scattering:

$\sigma_{tot}(p\bar{p})$ is still slowly decreasing with increasing energy from 52 mb at 13 Gev to 46 mb at 20 Gev. The asymptotic Pomeranchuk limit does not seem to have been reached yet.

σ^{tot}, σ^{inel} and σ^{el} all decrease with increasing energy; this is difficult to understand.

(i) One may ask whether the decrease itself may be caused by the ...ge behaviour. Udgaonkar suggested that the decrease may be due the ... ADC pole, but this would require too large a coupling strength for ... ABC pole.

(ii) The decrease may be a result of the contribution of the higher ...rips.

(iii) The pion itself may be a Regge pole; this would result in a ...garithmic decrease in the one-pion exchange contribution to σ^{inel}, similar ... that for σ^{el}.

...) πN Scattering:

We have already dealt with Lovelace's analysis in detail in the last ...cture.

σ^{tot} is still falling slowly at the highest energies observed.

...$_\pi$ (in Gev)	$\sigma^{tot}_{\pi^- p}$ (in mb)	$\sigma^{tot}_{\pi^+ p}$ (in mb)	$(\sigma_{\pi^+ p} - \sigma_{\pi^- p})$ (in mb)
6	29	27	2
10	27	25	2
16	25		

The table gives the $\pi^+ p$ and $\pi^- p$ total cross-sections; their ...fference is not accurately known, but seems to be slowly decreasing ...cording to recent data (cf. S.Drell: CERN conf., 1962)

$\sigma^{el}(\pi^- p)$ decreases from 5 mb in the 5 Gev region to 4 mb at ... Gev. The angular distribution resembles that of diffraction scatter...g, with the forward peak narrowing as the energy is increased.

...) Strange-particle production cross-sections are less than 3 mb; de...iled data are not available. The $K^+ p$ total cross-section seems to ...proach a constant, while the $K^- p$ total cross-section is still ...lling.

In the various reactions in which the total cross-sections have not become constant at the highest energies observed, one generally says that the energy is not high enough for the Pomeranchuk theorems to be applicable. Can anything can be inferred from the way the cross-sections vary? Udgaonkar showed that information about Regge trajectory other than the Pomeranchuk could be deduced from the variation of the cross-sections.

We have already seen how, once we find a reaction in which the Pomeranchuk limit is attained, we can use it to find $\alpha_P(t)$. Other reactions in which the limit is observed to be attained could then be used to verify the hypothesis of the dominance of the Pomeranchuk pole and check the shape of $\alpha_P(t)$.

When the energy is not sufficiently high, the effect of poles other than the Pomeranchuk will be appreciable; as $\alpha_P(t)$ is known, information about these other poles may be obtained by subtraction.

In a Regge theory, the objective is to find all the $\alpha_i(t)$ and $\beta_i(t)$. Theoretically not much work has been done on these. However Chew and Frautschi conjectured that the $\alpha_i(t)$ may all have the simple form of straight lines. They have made a plot of J vs. M^2 for all the known strongly interacting particles and resonances shown on the next page. Assuming that all these particles are Regge poles that move with energy, particles with the same internal quantum numbers (i.e. parity π baryon no. B, strangeness S, isospin I and G-parity G) are joined by a trajectory. e.g. The nucleon and the third pion-nucleon resonance are assumed to lie on a straight-line trajectory with $I = \frac{1}{2}$, $S = 0$, $B = 1$. Similarly the Λ and the Y_0^{***} (at 1815 Mev) ~~and the~~ ~~to 1520~~ now lie on a straight line trajectory with $I = 0$, $B = 1$, $S = -1$.

The K and the K* may also be joined by a straight line. Although the Chew-Frautschi plot is only a conjecure, it is useful in that it provides a theoretical background against which all data on Regge poles may be viewed. It gives a rough idea of the slope of the trajecories and suggests where we may look for new particles. For instance, as pointed out in an earlier lecture, the Pomeranchuk trajectory passes through $J = 2$ at about 1.2 Bev. An $I = 0$ resonance at about 1.25 Bev has been found; it would be some support for the Regge pole hypothesis if this turns out to have $J = 2$.

We may note the following features of the Chew-Frautschi plot.

(i) All trajectories have roughly the same slope, of $\sim 1 \text{ Gev}^{-2}$.

From the relation $d(\alpha + \frac{1}{2})^2 / dp^2 = R^2$, where R is the 'average radius' of the bound state corr to (the Regge pole) λ, one may say that all particles have the same 'size'.

(ii) Successive particles on the same trajectory obey the interval rule $J = 2$. The spacing in m^2 is about 2 $(\text{Gev})^2$

(iii) Only one particle on each trajectory is stable, as expected.

(iv) The vacuum trajectory lies the highest; as one goes lower, one gets further away from the vacuum quantum numbers. One may associate maximum "coherence" (leading to max. scattering in the forward direc tion) with the exchange of the quantum numbers of the vacuum; this is true of $\pi\pi$ scattering (as may be seen from the crossing matrix and may be assumed to be a general property. (iv) is consistent with this, as the vacuum trajectory lies the highest, it is the only one that reaches the Froissart limit.

It is clear that one can obtain only discrete points on the trajectories from the mass spectrum of particles and resonances. By studying the forward and backward peaks in scattering, we can trace out parts of the curve $\alpha(t)$, as we saw in connection with Wetherell's work on pp scattering.

We shall now consider how information about the various trajectories may be inferred. The sources are the variation with energy (at high energies) of (i) differential cross-section in the forward and backward directions, and (ii) total cross-sections .

Consider the amplitudes A^-, A^+ for pion-nucleon scattering.

$$A^- = \frac{1}{3}\left[A^{1/2} - A^{3/2}\right],$$
$$A^+ = \frac{1}{3}\left[A^{1/2} + 2A^{3/2}\right].$$

A^- will have contributions from the exchange of I = 1, G = 1 trajectories, presumably mainly the ρ trajectory. A^+ will receive contributions of I = 0, G = 1 trajectories, like the vacuum, and ABC trajectories.

The optical theorem gives

$$\frac{1}{2}\left[\sigma(\pi^-p) + \sigma(\pi^+p)\right] = \frac{1}{q_L}\,\mathrm{Im}\,A^+(\theta=0),$$
$$\frac{1}{2}\left[\sigma(\pi^-p) - \sigma(\pi^+p)\right] = \frac{1}{q_L}\,\mathrm{Im}\,A^-(\theta=0),$$

where $q_L \equiv$ the pion lab. momentum.

(Note: $E_\pi(\mathrm{lab}) = \dfrac{s - (m_\pi^2 + m_N^2)}{2m_N}$

Thus $q_L \to \dfrac{s}{2m_N}$ at high energies.)

We obtain

$$\sigma(\pi^-p) - \sigma(\pi^+p) \propto s^{-[1-\alpha_\rho(0)]}$$
$$\sigma(\pi^-p) + \sigma(\pi^+p) \propto a + b s^{-[1-\alpha_{ABC}(0)]},$$

since $\alpha_P(0) = 1$. 1.

The ABC virtual state occurs near $s \approx 4\mu^2$, and ≈ 0 here. We may expect that $\alpha_{ABC}(0) \simeq 0$.

Then $\sigma(\pi^- p) + \sigma(\pi^+ p) = a + \dfrac{b}{E}$.

Such a fit was obtained empirically by Lindarbaum et al.

(Note: There is a close relation between $I = 0$ $\pi\pi$ scattering and the sum of $\pi^\pm p$ total cross-sections at high energies, as both depend on the ABC and vacuum poles.)

Regarding the ρ trajectory, we knew that $\alpha_\rho(30\mu^2) = 1$; $\alpha_\rho(s=0) < 1$.

Recent data on the decrease of $\left[\sigma(\pi^- p) - \sigma(\pi^+ p)\right]$ with increasing energy indicates that $\alpha_\rho(0) \simeq 0.3$ to 0.4

Considering the various $S = 0$ mesons P, ABC, ρ, ω, η, π, ...

One may represent any high-energy total cross-section as

$$\sigma^{tot}(s) = P_n + \rho_n\, s^{-[1-\alpha_\rho(0)]} + \omega_n\, s^{-[1-\alpha_\omega(0)]} + \cdots\cdots$$

The larger the value of s, the more rapidly does the series converges.

For the various cross-sections, we obtain

$$= \sigma(\infty) - \pi\epsilon_\omega g^2_{pp\omega}\left(\frac{s}{s_0}\right)^{\alpha_\omega(0)-1} - \pi\epsilon_\rho g^2_{pp\rho}\left(\frac{s}{s_0}\right)^{\alpha_\rho(0)-1} + \pi\epsilon_{ABC}\, g^2\left(\frac{s}{s_0}\right)^{\alpha_{ABC}(0)-1} + \cdots$$

$$= \sigma(\infty) + \pi\epsilon_\omega g^2_{\bar{p}p\omega}\left(\frac{s}{s_0}\right)^{\alpha_\omega(0)-1} + \pi\epsilon_\rho g^2_{\bar{p}p\rho}\left(\frac{s}{s_0}\right)^{\alpha_\rho(0)-1} + \pi\epsilon_{ABC}\, g^2\left(\frac{s}{s_0}\right)^{\alpha_{ABC}(0)-1} + \cdots$$

$$= \sigma(\infty) - \pi\epsilon_\omega g^2_{pn\omega}\left(\frac{s}{s_0}\right)^{\alpha_\omega(0)-1} + \pi\epsilon_\rho g^2_{pn\rho}\left(\frac{s}{s_0}\right)^{\alpha_\rho(0)-1} + \pi\epsilon_{ABC}\, g^2\left(\frac{s}{s_0}\right)^{\alpha_{ABC}(0)-1} + \cdots$$

$$= \sigma(\infty) + \pi\epsilon_\omega g^2_{pp\omega}\left(\frac{s}{s_0}\right)^{\alpha_\omega(0)-1} - \pi\epsilon_\rho g^2_{pn\rho}\left(\frac{s}{s_0}\right)^{\alpha_\rho(0)-1} + \pi\epsilon_{ABC}\, g^2\left(\frac{s}{s_0}\right)^{\alpha_{ABC}(0)-1} + \cdots$$

[Ref. S. Drell : CERN Conf., 1962.]

To find the different parameters, one starts with scattering data above about 2 Gev and tries to fit the data to the above expressions. The individual contributions of different Regge trajectories may be isolated by taking suitable co~~ntr~~^{mb}ibutions of the cross-sections.

<u>Analysis of the data</u>:

First we note that the difference $\sigma(pp) - \sigma(\bar{p}p)$, which is due to the ω and ρ, is about 20 mb at 10 Gev, whereas the difference $\sigma(np) - \sigma(pp)$, which is due to the ρ above, is only about 0 mb at 10 Gev. Thus we may expect that we can ignore the ρ trajectory in interpreting the pp, $p\bar{p}$ difference. (Better still, we could take the difference between $\sigma(pd)$ and $\sigma(\bar{p}d)$ in which the ρ contribution is completely eliminated).

Comparison with experiment gives $\alpha_{\omega}(0) \approx 0.4$, which agrees with the Chew-Frautschi plot.

However, if we interpret the pp and $\bar{p}p$ cross-sections as a result of the Pomeranchuk and ω trajectories, the question arises: Why is the observed pp total cross-section $\approx$ an energy-independent 40 mb from 10 Gev to 28 Gev ?

From the expressions for the total cross-section, we would expect that the pp and $\bar{p}p$ cross-sections would approach the Pomeranchuk limits symmetrically from below and above respectively. Why do the observed cross-sections behave differently ?

This difficulty would be resolved if we postulate a third trajectory P', with the quantum numbers of the vacuum, the same coupling strength as the ω, and with $\alpha_{P'}(0) = \alpha_{\omega}(0) \approx 0.4$, but with a signature opposite to that of the ω. At $t = 0$, this would contribute an imaginary part to the amplitude that would just cance~~l~~l the contribution of the ω in the pp amplitude but add to it in the $\bar{p}p$ amplitude

This would explain the constancy of the pp total cross-section. The real parts of the amplitude coming from the ω and P' largely add in pp scattering, resulting in 10% increase in $\sigma_{el}(pp)$ above the optical theory value at 20 Gev.

We now ask: Is a new P' trajectory needed or is it one of the old particles ? It presumably cannot be identified with the ABCE meson, as $\alpha_{P'}(0) \approx 0.4$, and we expect $\alpha_{ABC}(0) \lesssim 0$, unless the ABC trajectory turns around and comes back into the physical region at $\alpha(0) \approx 0.4$.

It may be that additional trajectories are not really necessary. Inclusion of correction terms to the asymptotic behaviour of $P_\alpha(\cos\theta_t)$ at not too large values of s may account for the discrepancies.

Now consider the K^+p and K^-p total cross-sections. Their difference must be due to the ω ; it is consistent with $\alpha_\omega(0) \approx 0.4$. The K^+p cross-section is a constant ≈ 18 mb at energies above 5 Gev; thus again there must be a cancellation between the ω and P' contributions. This requires that $\frac{g_{P'N}}{g_{\omega N}} = \frac{g_{P'K}}{g_{\omega K}}$, where $g_{P'N}$ is the strength of the coupling of the P' trajectory to the nucleon, etc.

The ω will not contribute to $\pi^\pm p$ scattering owing to its negative G parity. Thus we expect that the π^+p and π^-p cross-sections should both decrease towards the Pomeranchuk limit, due to the P' contribution; this is what is observed. Thus the $\pi^\pm p$ cross-sections support the postulate of a P' trajectory. In fact, the first suggestion for another vacuum trajectory with $\alpha(0) \approx \frac{1}{2}$ was put forward by K.Igi, who observed that dispersion relations for πN forward scattering (without charge-exchange) with one subtraction corr. to the Pomeranchuk contribution led to an appreciable discrepancy between theory and experiment.

If the P' exists, it is important to see whether $\alpha_{P'}$ passes through zero, giving rise to a "Ghost" which must be got rid of as before by requiring the coupling to vanish as $\alpha_{P'}$ passes through zero.

About the ρ trajectory, little is known besides that its coupling strength is much smaller than that of the ω. The $\pi^+ p$, $\pi^- p$ difference is due to the ρ above; it indicates $\alpha_\rho(0) \approx 0.3$ to 0.4. The difference $\sigma(pn) - \sigma(pp)$ gets contributions from the ρ and the π; the pion has a lower trajectory and presumably gives a much smaller contribution. Thus we expect that the $(pn - pp)$ difference has the same sign as the $[(\pi^- p) - (\pi^+ p)]$ difference.

Faissner's experiments on $\pi^- p$ charge-exchange scattering at 4 Gev/c show the presence of a sharp peak for events with $\sqrt{|t|} \lesssim 150$ Mev/c. The total elastic cross-section of 110^{-45}_{+80} mb agrees with the theoretical estimate from the observed $(\pi^- p)$, $(\pi^+ p)$ difference if $\alpha_\rho(0) \approx 0.4$, and ~~0.4,~~ $Re\, \alpha_\rho' \approx \frac{1}{m^2}$

Mahn's experiments on $pn \to np$ (charge-exchange) elastic scattering give an (elastic) cross-section of about $100\,\mu b$. The angular distribution is roughly isotropic up to $|t| \approx 2\text{Gev}^2$ at a lab. energy of 25 Gev. The cross-section is an order of magnitude greater than that expected from the ρ if $\sigma(np) - \sigma(pp) \lesssim 1\,mb$.
It is similar to that expected from the exchange of an unreggeized pion.

Looking briefly at KN cross-sections, we note that the following combinations get contributions from the trajectories indicated.

$[\sigma(K^+p) - \sigma(K^+n)]$	ρ
$[\sigma(K^+n) - \sigma(K^-p)]$	ω
$[\sigma(K^-p) - \sigma(K^+p)]$	ρ, ω
$[\sigma(K^-p) + \sigma(K^+p)]$	P, ABC
$[\sigma(K^+p) + \sigma(K^+n)]$	P, ABC, ω

<u>The Nucleon as a Regge pole:</u> Blankenbecler and Goldberger speculated that the nucleon may lie on a Regge trajectory. This may be tested by determining whether the differential cross-section near the backward direction, for a fixed value of u, is independent of s (as it would be for an unreggeized nucleon) or whether it decreases with increasing s (as it would if the nucleon were a Regge pole).

Gribov pointed out that the factor $s^{2[\alpha_N(u)-1]}$ in the backward peak would give rise to rapid osillations in the angular pattern of the peak. The oscillations increase as u increases (i.e. as one gets closer to the backward direction).

$\pi^+ p$ backward scattering gets contributions from both N exchange and N* exchange.

Finally we consider the reaction $p+N \to p+N^*$. The Pomeranchuk can give rise to production of only the 2nd and 3rd πN resonances (with $I = \frac{1}{2}$); thus production of the (3,3) resonance should decrease with increasing energy, as compared to the higher resonance, as seems to be observed. (In pp scattering at high energies, a no. of bumps are observed in the cross-section, corr. to production of the resonances. The (3,3) bump disappears as the energy increases.)

Production of the (3,3) resonance will take place at lower energies via the pion trajectory.

The above has been studied by Frautschi, ~~et al.~~ Contogouris and Wong. Production of ρ mesons in πp collisions is being studied by Caldwell et al; the preliminary results give a lower limit for this process

A general criticism to be made of all the above is that if branch cuts in ℓ exist, as suggested by Amati, Fubini et al, their effect would mask the contributions of all poles other than the Pomeranchuk and would prevent the determination of all but the vacuum trajectory.

LECTURE X.

AN EXTENSION OF THE REGGE REPRESENTATION†

†[This lecture closely follows S. Mandelstam: ANN. PHYS. 19, 254 (1962)

The original Regge representation of the partial wave expansion as an integral plus pole terms, viz.

$$A(k,z) = \sum_{l} (2l+1)\, A(k,l)\, P_l(z) \qquad \text{(1a)}$$

$$= \frac{1}{2i} \int_C dl\, (2l+1)\, A(k,l)\, \frac{P_l(-z)}{\sin \pi l} \qquad \text{(1b)}$$

$$\frac{1}{2i} \int_{-i\infty - \frac{1}{2}}^{+i\infty + \frac{1}{2}} dl\, (2l+1)\, A(k,l)\, \frac{P_l(-z)}{\sin \pi l} + \\ + \sum_{Re\,\alpha_i > -\frac{1}{2}} (2\alpha_i+1)\, \beta_i(k)\, \frac{P_{\alpha_i}(-z)}{\sin \pi \alpha_i}. \qquad \text{(1c)}$$

was restricted to $Re\, l > -\frac{1}{2}$, for only then did the "background integral" in equation (1c) converge.

For $P_l(-z) \sim (-z)^l$, as $|z| \to \infty$, for $l > -\frac{1}{2}$ (2)

and the background integral $\sim z^{-1/2}$, as $|z| \to \infty$.

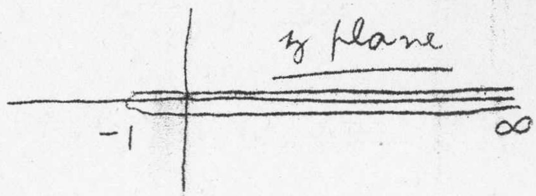
But for $l < -\frac{1}{2}$, $P_l(-z) \sim z^{-l-1}$, (3)

and the integral diverges as $|z| \to \infty$.

Thus the representation (1c) must be modified before the contour can be shifted to the left of the line $Re\, l = -\frac{1}{2}$.

This modification is achieved by introducing the Legendre functions of the 2nd Kind, $Q_\ell(-z)$, which have the following properties:

1) $Q_\ell(-z)$ is analytic in the z-plane cut from -1 to ∞ along the real axis.



2) The discontinuity across the cut is given by ~~Disc~~

$$\text{Disc}\ Q_\ell(-x) = -i\pi P_\ell(-x), \quad -1 < x < 1 \tag{4a}$$

$$= 2i(\sin\pi\ell)\, Q_\ell(x), \quad x > 1, \tag{4b}$$

where Disc $Q_\ell(-x) \equiv Q_\ell(-x+i\epsilon) - Q_\ell(-x-i\epsilon)$. (4c)

3) $Q_\ell(z)$ is ~~a meromorphic~~ function of ℓ in the entire ℓ-plane. The poles are at $\ell = -1, -2, -3, \ldots$

4) $$Q_\ell(z) \sim z^{-\ell-1}, \quad |z| \to \infty, \text{ for } \ell > 0. \tag{5}$$
$$\sim z^{\ell} \quad \text{for } \ell < 0.$$

5) $$\frac{P_\ell(z)}{\sin\pi\ell} - \frac{Q_\ell(z)}{\pi\cos\pi\ell} = -\frac{Q_{-\ell-1}(z)}{\pi\cos\pi\ell}.$$

(6)

(7)

The function $\left[\dfrac{P_\ell(z)}{\sin\pi\ell} - \dfrac{Q_\ell(z)}{\pi\cos\pi\ell}\right] \sim z^\ell, \ |z| \to \infty.$

(13)

for all ℓ , positive or negative

Write

$$A(k,z) = \Sigma\,(2\ell+1)\,A(k,\ell)\,P_\ell(z) +$$

$$+ \frac{1}{\pi} \sum_{\ell=1,2,3,\dots}^{\infty} (-1)^{\ell-1}\, 2\ell \cdot A(k,\ell-\tfrac{1}{2})\, Q_{\ell-\frac{1}{2}}(z) -$$

$$- \frac{1}{\pi} \sum_{\ell=0,1,2,}^{\infty} (-1)^{\ell-1}\, 2\ell\; A(k,\ell-\tfrac{1}{2})\, Q_{\ell-\frac{1}{2}}(z).$$

(9)

The mofified formula obtained is

$$(k,z) = \frac{1}{2i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\ell\,(2\ell+1)\,A(k,\ell)\left[\frac{P_\ell(-z)}{\sin\pi\ell} - \frac{Q_\ell(-z)}{\pi\cos\pi\ell}\right] -$$

$$- \frac{1}{\pi} \sum_{\ell=0}^{\infty} (-1)^{\ell-1}\, 2\ell\; A(k,\ell-\tfrac{1}{2})\, Q_{\ell-\frac{1}{2}}(z) +$$

$$\sum_{\alpha_i > -\frac{1}{2}} (2\alpha_i+1)\,\beta_i(k)\left[\frac{P_{\alpha_i}(-z)}{\sin\pi\alpha_i} - \frac{Q_{\alpha_i}(-z)}{\pi\cos\pi\alpha_i}\right].$$

(10)

The integration contour may now be moved to the left, to $\mathrm{Re}\,\ell = -L$, say. The extra terms to be added are the poles of $A(k,\ell)$ crossed by the contour (which results in the summation in the last term of (10) being over $\mathrm{Re}\,\alpha_i > -L$) and the poles of

$$\left[\frac{P_\ell(-z)}{\sin\pi\ell} - \frac{Q_\ell(-z)}{\pi\cos\pi\ell}\right].$$

The latter poles occur at $\ell = -n-\frac{1}{2}$, n integral; the residues at these poles are $\frac{1}{\pi^2}(-1)^{n-1} Q_{n-\frac{1}{2}}(z)$. They thus give rise to the additional terms

$$\sum_{-N+1}^{0} \frac{1}{\pi}(-1)^{n-1}\, 2n.\ A(k,-n-\tfrac{1}{2})\, Q_{n-\frac{1}{2}}(z),$$

$$= -\sum_{0}^{N-1} \frac{1}{\pi}(-1)^{n-1}\, 2n.\ A(k, n-\tfrac{1}{2})\, Q_{-n-\frac{1}{2}}(z).$$

N being the integer nearest to L $\left[-N-\frac{1}{2} < L < -N+\frac{1}{2}\right]$. This just cancels the first $(N-1)$ terms of the first summation in equation (10).

The result is

$$(k,z) = \frac{1}{2i}\int_{-L-i\infty}^{-L+i\infty} d\ell\,(2\ell+1)\, A(k,\ell)\left[\frac{P_\ell(-z)}{\sin\pi\ell} - \frac{Q_\ell(-z)}{\pi\cos\pi\ell}\right] -$$

$$- \frac{1}{\pi}\sum_{n=N}^{\infty}(-1)^{n-1}\, 2n\ A(k, n-\tfrac{1}{2})\, Q_{n-\frac{1}{2}}(z) +$$

$$+ \sum_{Re\,\alpha_i > -L}(2\alpha_i+1)\,\beta_i(k)\left[\frac{P_{\alpha_i}(-z)}{\sin\pi\alpha_i} - \frac{Q_{\alpha_i}(-z)}{\pi\cos\pi\alpha_i}\right].$$

(11)

In this expansion, the "background" terms, i.e. the integral and the first sum, decrease at least as fast as z^{-L}, as $|z| \to \infty$, while the Regge pole terms (in the last sum) go as z^{α_i} as $|z| \to \infty$.

We enquire what happens to the background terms as $(-L) \to -\infty$. The background term does <u>not</u> $\to 0$ as $(-L) \to \infty$. If it did $\to 0$, it could be proved that Disc $A(k, z)$ across the cut for $z > 1$ was analytic, which is not true if $A(k, z)$ has a Mandelstam representation.

As the background term does not vanish as $(-L) \to -\infty$, the last sum in equation (11) is an asymptotic series.

Also, because of the properties of $Q_{\alpha_i}(-z)$, every term of the series

$$\sum_{(\alpha_i > -L)} (2\alpha_i + 1)\, \beta_i(k) \left[\frac{P_{\alpha_i}(-z)}{\sin \pi\alpha_i} - \frac{Q_{\alpha_i}(-z)}{\pi \cos \pi\alpha_i} \right]$$

has a cut in z from -1 to ∞ along the real axis. However, the scattering amplitude $A(k, z)$ has a cut starting from $z = z_0 > 1$, as seen earlier.

Consider the partial-wave Schrondinger equation for scattering by a potential $V(x)$:

$$\psi''(x) - \left[\frac{\ell(\ell+1)}{x^2} + V(x) \right] \psi(x) = 0. \tag{12}$$

OR

Take the class of potentials which can be expanded into a power series:

$$V(x) = \sum_{n=0}^{\infty} v_n x^{n-1}. \tag{13}$$

Regge showed that the solution of (12) was an analytic function of ℓ for $Re\,\ell > -\frac{1}{2}$, and that the scattering amplitude was a meromorplic function of ℓ in this half-plane. We wish to extend these results into the whole ℓ-plane. For this, first note that equation (12) is invariant under $\ell \to -\ell - 1$. But the boundary condition at the origin will be different; the two solutions behave like $x^{-\ell}$ and $x^{\ell+1}$ at the origin $(x = 0)$. Thus the two solutions coincide at $\ell = -\frac{1}{2}$. If the first solution can be continued into the half-plane $Re\,\ell < -\frac{1}{2}$, we shall obtain the 2nd solution.

To find whether the first solution can be analytically continued into the left half-plane, we solve the equation by a power series.

$$\text{Put} \quad \psi(x) = \varphi(x)\, e^{ikx}, \tag{14}$$

which is permissible,

as $\psi(x) \sim e^{ikx}$ as $x \to \infty$, for k^2 positive. The sign of the exponent is chosen so that it becomes a decreasing exponential when $k = i\kappa$, $\kappa > 0$.

$\phi(x)$ obeys the equation

$$(x) + 2ik\,\varphi(x) - \frac{\ell(\ell+1)}{x^2}\varphi(x) - \left(\sum_{n=0}^{\infty} v_n x^{n-1}\right)\varphi(x) = 0 \tag{15}$$

Put

$$\varphi(x) = \sum_{r=0}^{\infty} a_r x^{s+r} \tag{16}$$

Substituting (16) into (15), we obtain the recursion relation

$$[(s+r)(s+r-1) - l(l+1)]\,a_r + 2ik(s+r-1)\,a_{r-1} - \sum_{n=0}^{\infty} v_n a_{r-1-n} = 0. \tag{17}$$

For $r = 0$, only the 1st term contributes.

$$\therefore \quad s(s-1) = l(l+1)$$

$$\underline{\text{OR}} \qquad s = l+1 \ \underline{\text{or}} - l \tag{18}$$

Whichever solution we start with, all the higher terms can be calculated successively. The only singularities these terms can have in the finite part of the l- plane are poles at the negative integers or half-integers where the coefficient of a_r vanishes.

If r is sufficiently large, we can find an M (for given l) such that

$$\left| \frac{a_{r+1}}{a_r} \right| < \frac{M}{r} \tag{19}$$

First, suppose the last term (which is a sum of a series) in (15) has only a finite number of terms.

Then

$$\frac{a_r}{a_{r-1}} = -\frac{2ik(s+r-1)}{[(s+r)(s+r-1) - l(l+1)]} + \frac{\sum_{n=0}^{N} v_n a_{r-1-n}}{[(s+r)(s+r-1) - l(l+1)]}$$

For n sufficiently large,

$$\frac{a_n}{a_{n-1}} = -\frac{2ik}{n},$$

and thus the above assertion is proved.

When $V(x) = \sum_{n=0}^{\infty} v_n x^{n-1}$ contains an infinite number of terms, for a superposition of Yukawa potentials, these terms decrease with n like $\frac{1}{n!}$ and thus the same result holds.

Thus the power series solution for $\varphi(x)$ converges for all ℓ and x and converges uniformly w.r.t. ℓ in any region of the ℓ-plane. Therefore, since the individual terms are meromorphic in the whole ℓ-plane, so is the whole series (because of the uniform convergence). Then we can prove that the scattering amplitude is meromorphic in ℓ in the whole ℓ-plane.

<u>Behaviour of the Scattering Amplitude for $Re\, \ell < -\frac{1}{2}$:-</u>

Although, in general, the scattering amplitude for ℓ and $-\ell-1$ are unrelated, as they correspond to solutions with different boundary conditions, we can require that

$$\left.\begin{aligned} A(k,\ell) &= A(k,-\ell-1) && \text{for } \ell \text{ half-integral} \\ &= -A(k,-\ell-1) && \text{for } \ell \text{ integral.} \end{aligned}\right\} \text{for } \ell < -\tfrac{1}{2} \qquad (20)$$

For half-integral or integral ℓ less than $-\frac{1}{2}$, the solution behaving at the origin like $x^{\ell+1}$ does not exist, for in the power series for $\varphi(x)$, the coefficients $a_{2|\ell|-1}$ and higher order coefficients become infinite. (For the solution behaving like $x^{\ell+1}$ at the origin, $s = \ell + 1$. The coefficient of a_r in the recursion relation (17) is then

$$(l+1+r)(l+r) - l(l+1) = (l+1)r + r(l+r)$$

$$= (2l+1+r)r ,$$

which is zero for $r = -2l-1 = 2|l|-1$.

$\therefore a_{2|l|-1} = \infty$, as $s+r-1 = l+r \neq 0$.

All the higher order coefficients a_r are also infinite, as $(s+r-1) = l+r \neq 0$ for $r > 2|l|-1$. Thus to define a power-series solution, we must multiply the whole series by a factor which is equal to zero at negative half-integral and integral l (less than $-\frac{1}{2}$), e.g. $\frac{1}{\Gamma(l)\,\Gamma(l+\frac{1}{2})}$. This leaves unaltered the analytic properties of φ and the value of the scattering amplitude (as the wave function is always undetermined to a constant factor)

For negative integral or half integral l, all the coefficients up to $a_{2|l|-2}$ would be zero and higher order coefficients would be finite.

In the corresponding positive solution (obtained by the change $l \to -l-1$) the same holds true for the corresponding coefficients i.e. all the coefficients up to a_{2l} are zero and higher order coefficients are finite.

The scattering amplitude is related to the wave function by

$$A(k,l) = \frac{F(l,-k)}{F(l,k)}\, e^{i\pi l} \qquad (21)$$

When $l \to -l-1$, $e^{i\pi l} \to e^{-i\pi l} e^{-i\pi} = e^{i\pi l}$, for half-integr

$= -e^{i\pi l}$, for integral.

Thus it is generally possible to require the relation (20) between the scattering amplitudes for l and $-l-1$. However, if k passes through such a value that the first $2|l|-2$ terms add to zero.

the recursion relation (17) will leave $a_{2|l|-1}$ determined.

To study this situation, consider values of l in the neighbourhood of an integral or half-integral value l_0: $l = (l_0 + \Delta l)$, say. The coefficient of $a_{2|l|-1}$ in (17) will also be small, & so will be the sum of the first $2|l_0|-2$ terms in (17). Now vary k around k_0. This would give rise to rapid changes in $a_{2|l|-1}$ and higher order terms.

Thus for Δl small, we can get any desired change in the coefficients by moving k around k_0. If ψ behaves asymptotically like $\alpha e^{ikx} + \beta e^{-ikx}$, the ratio β/α can be varied as desired by moving k around k_0. In particular, β/α can be made zero and the amplitude A has a pole at that point (since there is only an outgoing wave). (A different way of seeing this is that the solution with the required boundary condition at the origin is now also the solution that behaves like e^{ikx} at infinity and since the Jost function $F(l,k)$ is the Wronskian of just these two solutions, we have $F(l,k) = 0$, and A has a pole.)

The residue at the above pole tends to zero as l tends towards a negative integer or half-integer. At these points the symmetry relation (20) does not hold.

Consider the behaviour of the Regge poles when V is small or $|k|$ large. Then the recursion relation (17) becomes

$$[(s+r)(s+r-1) - l(l+1)]\,a_r + 2ik(s+r-1)\,a_{r-1} = 0$$

If l is a negative integer, the solution behaving as z^{-l} at the origin (i.e. with $s = l$) will terminate at z^0, as $a_{l+1} = 0$, since $(s+r-1)$ is then $= 0$, and the coefficient of a_{l+1} is non-zero. Thus the solution $V(x) = \phi(x)\,e^{ikx}$ will behave like e^{ikx} at infinity. Again the residue at this 'pole' vanishes, since l is a negative integer. But as V is increased or $|k|$ reduced, Regge poles appear at values removed from the negative integer values of l.

Since the pole farthest to the right dominates the asymptotic behaviour, we have the result that as V is increased from zero or as $|k|$ is decreased from infinity, the scattering amplitude behaves as $1/r$. This is what one expects from the Born approximation. For an attractive potential, the poles move to the right as V is increased or $|k|$ decreased and the asymptotic behaviour is stronger than in the limiting case. For a repulsive potential it is weaker.

We now go back to equation (11).

The Regge pole terms $$\Sigma\,(2\alpha_i+1)\,\beta_i(E)\left[-\frac{Q_{-\alpha_i-1}(-z)}{\cos\pi\alpha_i}\right]$$

can at first sight become infinite at negative half-integral values of α_i also. However, at these values, either

(i) the residue at the pole vanishes, and the amplitude remains finite

OR

(ii) two Regge poles with the same residue pass simultaneously through ℓ_0 and $-\ell_0 - 1$, and they cancel, since $_0+1) + [2(-\ell_0 - 1) + 1] = 0$, and there is no infinity.

Thus the scattering amplitude becomes infinite only at positive integral ℓ, where we have a bound state.

We may expect that at least part of the above may hold in a relativistic case.

LECTURE XI.

Experimental Consequences of the Hypothesis of Regge Poles

We shall begin by summarising various results we have obtained in earlier lectures.

Consider a general two-body reaction

$$a + b \rightarrow c + d \quad . \tag{1}$$

In the relativistic case we conjecture that the behaviour of the invariant scattering amplitude $A(s,t)$ at large s is of the form

$$\sum_i \beta_i(t) \frac{\left[1 \pm e^{i\pi\alpha_i(t)}\right] P_{\alpha_i}(x_t)}{\sin \pi\alpha_i(t)} , \tag{2}$$

where the sign inside the brackets is known as the "signature" of the Regge trajectory.

(2) may be interpreted as expressing that in the Regge formalism, the scattering is expressed as a sum of pole terms, each of which corresponds to the exchange of the set of the possible particle combinations with given internal quantum numbers Π, B, I, S, and G. (Π is the parity). As we have mentioned earlier, each

pole term represents the exchange of a coherent superposition of angular momentum states, which may be expressed as the exchange of an equivalent unphysical energy-dependent spin $\alpha(t)$.

The conjecture in the relativistic case may be expressed in the form of a set of rules for writing the contribution of a given Regge pole term (ref. Frautschi, Gell-Mann, and Zachariasen, Phys. Rev. 126, 2204 (1962).). This replaces the set of rules that one uses in writing the contribution of pole terms in Feynman perturbation theory.

Rule 1. Consider a complete set of linearly independent invariant scattering amplitudes $A(s,t)$ free of kinematical singularities and zeros in s and t .

Rule 2. For the t reaction, take any set of values of the conserved quantum numbers B, I, S, G, C, Π except J . As a function of J , consider the contribution to the amplitudes A_i of a hypothetical exchanged 'particle' with these quantum numbers (and arbitrary 'mass' M). For each A_i , this contribution will be a sum of terms containing Legendre functions $P_j(x_t)$ and $P_j'(x_t)$. At large s , each such function of x_t is asymptotic to a power of s , with the exponent varying with j like ($j+$ const.). Thus the contribution to A_i is of the form

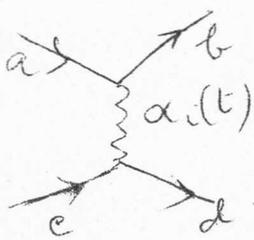
$$c_i \, s^{(j-\nu_i)} \frac{1}{(t-M^2)}, \quad \text{as } s \to \infty$$

where the c_i are constants determined by the reaction in question.

Rule 3. Write $j=\alpha$ for integral spin (in the t reactions) and $j = \alpha + \frac{1}{2}$ for half-integral spin.

Make an analytical continuation to complex α . Then each Regge term depends on s as stated in Rule 2 (when $s \to \infty$), where α now depends on t , and with $\frac{c_i}{t-M^2}$ replaced by $\frac{1 \pm e^{-i\pi\alpha}}{2\sin\pi\alpha(t)}\, c_i(t)$. Each Regge term is associated with a particular set of conserved quantum numbers in the t reaction and with a definite signature g ; the latter can be $+1$ or -1 for each set of quantum numbers. When $g = +1$, poles occur at even α ; for $g = -1$, poles are at odd α .

For a reaction in which a given set of quantum numbers can be exchanged, the amplitude will contain the corresponding Regge term, e.g. the set of quantum numbers characterising the vacuum can occur in $\pi\pi$, πN , NN , KN , etc. scattering; hence the vacuum Regge term will occur in all these. The residues $\beta(t)$ will be different in all these reactions. An important factorization property of these residues has been conjectured by Gell-Mann and independently by Gribov and Pomeranchuk. This has been proved for potentials. When α takes physical values, the residue $\beta(t)$ factors into the coupling strengths of the trajectory $\alpha_i(t)$ to the initial and final states in the t - channel.



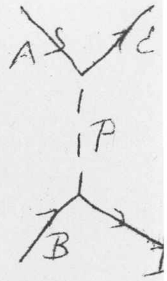
The Factorization Hypothesis: The conjecture is that this factorisation property holds all along the Regge trajectory, i.e. even at unphysical values of α .

We have certain situations in which the coupling strengths of a given Regge trajectory to all possible systems must vanish at same value of t. We have already mentioned the example of the "ghost" in the Pomeranchuk trajectory which arises because $\alpha_P(t)$ becomes zero at a negative value of t, and we have a physical state with negative $(\text{mass})^2$. In nuclear physics also one has systems with ground state spin 2. At $t = E_o^2$, where E_o is the energy of the ground state, $\alpha = 2$. Suppose $\alpha(t)$ continues to decrease as t decreases. Then it attains the physical value $\alpha = 0$ below the ground state; this difficulty would be resolved if the residue vanished at this point. We also note in passing that this property of the residue vanishing at particular values of J is something we have already encountered in a different context, viz. at negative half-integral ℓ for a potential. (See the last lecture.)

Comparison between pology in dispersion theory and Regge pology.

A Regge pole must give the same result as a pole contribution in dispersion theory when the trajectory crosses a physical point. This gives a relation between the residues at the Regge pole and the coupling constants in perturbation theory.

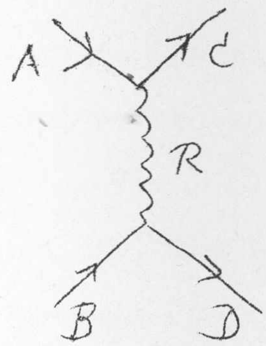
The pole term in dispersion theory is



$$A = \frac{g_{APC}\ g_{BPD}}{(t - m_P^2)} P_\ell(\cos\theta_t),$$

where $m_P^2 = m_0^2 - i\,\Gamma m_0$ for a slightly unstable particle P.

The pole term in the Regge formalism is, at a physical value of α,



$$A = \frac{\beta(t)\ [1 \pm e^{-i\pi\alpha}]}{2 \sin \pi\alpha(t)} P_\alpha(\cos\theta_t) \longrightarrow$$

$$\longrightarrow \frac{\beta_{ACRBD}(t)\ P_{\alpha(t)}(\cos\theta_t)}{\pi\ \mathrm{Re}\,\alpha'(t_R)\cdot[t - m_R^2]}$$

where $\beta_{ACRBD}(t) = G_{ACR}(t)\ G_{BDR}(t)$,

by the factorization hypothesis, and

$$m_R^2 = t_R - i\,\frac{\mathrm{Im}\,\alpha(t_R)}{\mathrm{Re}\,\alpha'(t_R)}.$$

Comparison:

	Disp. Theory	Regge pole, at $t \simeq t_R$
(i) The coupling Strengths:	$g_{ACP}\, g_{BDP}$	$\dfrac{\beta_{ACR}(t)\ \beta_{BDR}(t)}{\pi\ \mathrm{Re}\,\alpha_R'(t_0)}$
(ii) The "propagator":	$\dfrac{1}{t - m_R^2}$	$\dfrac{(-1)^\ell\, \pi\ \mathrm{Re}\,\alpha_R'(t_0)}{\sin \pi\alpha_R(t_0)}$

(iii) Asymptotic behaviour:

Dispersion theory: $A \to f(t)\, s^{\ell}$, as $s \to \infty$

Regge formalism: $A \to F(t)\, s^{\alpha(t)}$, as $s \to \infty$

At a physical pole only do these coincide.

(Ref. S.Drell: Report at CERN Conf., 1962)

Ex.1 $\pi\pi$ Scattering:

According to Rule 1, take the independent amplitudes as the isospin eigenamplitudes $T^{I}(s,t)$, $I = 0, 1, 2$. Consider the set of quantum numbers $I = 1$, $P = -1$, $G = +1$ (with $B = 0$, $S = 0$). As we saw in an earlier lecture (on diffraction scattering), the boson nature of pions requires that the signature $\mathcal{J} = -1$ for $I = 1$.

Rule 2. The exchange of a particle with the above quantum numbers contributes the following to the isospin eigen-amplitudes in the s-channel:

$$\begin{bmatrix} T^0 \\ T^1 \\ T^2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \frac{P_J(x_t)\; c}{(t - M^2)}$$

Rule 3. For large x_t or large s, the Regge ~~turn~~ term has the form

$$\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} s^{\alpha} \frac{\left(1 - e^{-i\pi\alpha_\rho(t)}\right)}{2 \sin \pi\alpha_\rho(t)}\, c(t).$$

The index ρ in $\alpha_\rho(t)$ indicates that we are referring to a particular trajectory (characterised by a particular set of quantum numbers) — the ρ trajectory. From the observed ρ resonance we require $\mathrm{Re}\,\alpha_\rho(t) = 1$ at $t = m_\rho^2$.

Define the dimensionless quantity $b(t)$ by

$$c(t) = \frac{4 m_\pi^2}{(2 m_\pi^2)^{\alpha_\rho(t)}}\, b(t)$$

In the neighbourhood of $t = m_\rho^2$, the contribution of the Regge term is approximately

$$\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} s \, \frac{2 b(m_\rho^2)}{\pi \epsilon_\rho} \, \frac{(-1)}{t - m_\rho^2 + i I_\rho \epsilon_\rho^{-1}},$$

where $\epsilon_\rho = \mathrm{Re}\,\alpha_\rho'(m_\rho^2)$; $I_\rho = \mathrm{Im}\,\alpha_\rho(m_\rho^2)$.

In dispersion theory, the exchange of an unstable ρ would give the contribution

$$\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} P_1(x_t) \cdot \left[-2 \gamma_{\rho\pi\pi}^2 (m_\rho^2 - 4 m_\pi^2)\right] \frac{(-1)}{t - m_\rho^2 + i \Gamma_\rho m_\rho}$$

$$\longrightarrow \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} 8 \gamma_{\rho\pi\pi}^2 \, s \, \frac{(-1)}{t - m_\rho^2 + i \Gamma_\rho m_\rho}$$

Comparison gives

$$\frac{b(m_\rho^2)}{\pi \epsilon_\rho} = 4 \gamma_{\rho\pi\pi}^2 ,$$

$$I_\rho \epsilon_\rho^{-1} = \Gamma_\rho m_\rho .$$

The coupling constant $\gamma_{\rho\pi\pi}^2$ may be related approximately to the observed width Γ_ρ by

$$\Gamma_\rho m_\rho = \frac{1}{3} \frac{\gamma_{\rho\pi\pi}^2}{4\pi} \cdot \frac{(m_\rho^2 - 4 m_\pi^2)^{3/2}}{m_\rho}$$

Thus we obtain

$$\frac{\pi I_\rho}{b(m_\rho^2)} = \frac{(m_\rho^2 - 4 m_\pi^2)^{3/2}}{48\pi} m_\rho^{-1}$$

(The assumptions involved are

(i) The decay mode $\rho \to 2\pi$ is dominant, so that $\Gamma_\rho \approx \Gamma(\rho \to 2\pi)$

(ii) Γ_ρ is small,)

The contribution of the ρ trajectory to the charge-exchange cross-section will be

$$\frac{d\sigma}{dt}^{I=0} - \frac{d\sigma}{dt}^{I=2} \xrightarrow[s \to \infty]{} 3 F_{\rho\pi\pi}(t) \left(\frac{s}{2 m_\pi^2}\right)^{2\alpha_\rho(t) - 2} ,$$

where

$$F_{\rho\pi\pi}(t) = \frac{1}{16\pi} \left| b_{\rho\pi\pi}(t) \left(\frac{1 - e^{-i\pi\alpha_\rho(t)}}{\sin \pi\alpha_\rho(t)} \right) \right|^2 .$$

The factor 3 comes from the coefficient $(-2)^2-(1)^2$. The factorisation hypothesis gives

$$F_{\rho\pi\pi}(t) \equiv F_{\pi\pi\rho\pi\pi}(t) = \left[G_{\pi\pi\rho}(t)\right]^2 .$$

The contribution of a Pomeranchuk pole to $\pi\pi$ scattering may be written down similarly as

$$\left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right]\left(\frac{s}{2m_\pi^2}\right)^{\alpha_P(t)}\left\{\frac{1+e^{-i\pi\alpha_P(t)}}{2\sin\pi\alpha_P(t)}\right\}4m_\pi^2\, b_{P\pi\pi}(t)$$

$\alpha_P(t) \simeq 1$ at $t \approx 0$; thus the Pomeranchuk pole dominates the forward pure elastic scattering if there is no other pole with $\alpha(0) \simeq 1$. As we have seen in an earlier lecture, the dominance of the Pomeranchuk leads to

(i) The pure imaginary nature of the scattering amplitude:

$$T^I(s,0) \xrightarrow[s\to\infty]{} -\left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right] i s\, b_{P\pi\pi}(0),$$

(ii) the constancy of the total $\pi\pi$ cross-section:

$$\sigma_T^{\,I} \xrightarrow[s\to\infty]{} -\frac{1}{s}\, \mathrm{Im}\, T^I(s,0) = b(0),$$ and

(iii) the shrinking of the diffraction peak with increasing energy.

$$\frac{d\sigma^I}{dt} \rightarrow F_{\rho\pi\pi}(t) \left(\frac{s}{2m_\pi^2}\right)^{2\alpha_\rho(t)-2}$$

$$= F_{\rho\pi\pi}(t) \exp\left[-2|t|\,\alpha_\rho'(0) \log\left(\frac{s}{2m_\pi^2}\right)\right],$$

where

$$F_{\rho\pi\pi}(t) = \frac{1}{16\pi}\left| b_{\rho\pi\pi}(t) \left(\frac{1+e^{-i\pi\alpha_\rho(t)}}{\sin\pi\alpha_\rho(t)}\right)\right|^2 .$$

$\pi\pi$ scattering (in the s channel) is symmetric in t and u i.e. in the forward and backward directions.

<u>Ex.2. πN Scattering :</u>

$$\begin{array}{lcccc} & \pi & + & N & \longrightarrow & \pi & + & N \\ & q & & p & & q' & & p' \\ \text{charge index} & \sigma & & & & \sigma' & & \end{array}$$

Write the amplitude as

$$T = A^+ \delta_{\sigma'\sigma} + A^- \frac{[\tau_{\sigma'}, \tau_\sigma]}{2} +$$

$$+ \left\{ B^+ \delta_{\sigma'\sigma} + B^- \frac{[\tau_{\sigma'}, \tau_\sigma]}{2} \frac{(\not{q} + \not{q}')}{2} \right\} .$$

<u>Rule 1.</u> Choose the amplitudes as $A^{\pm}$ and $B^{\pm}$.

<u>Forward Scattering:</u>

<u>Rule 2.</u> As before consider the sets of quantum numbers

(i) $I = 1, \; \Pi = -1, \; G = +1$ (with $B = 0, \; S = 0$)

and (ii) $I = 0, \; \Pi = +1, \; G = +1$ (" " " ")

First consider (i), i.e. the ρ-trajectory.

The amplitudes in the t channel $\pi + \pi \rightarrow N + \bar{N}$ may be written in terms of partial waves as

$$A^{(\pm)} = + \frac{8\pi}{p^2} \sum_J (J+\tfrac{1}{2}) (p_t^N q_t^\pi)^J \left\{ \frac{m \cos\theta_t}{\sqrt{J(J+1)}} P_J'(\cos\theta_t) f_{J-}^{(\pm)} - P_J(\cos\theta_t) f_{J+}^{(\pm)} \right\},$$

$$B^{(\pm)} = 8\pi \sum_J \frac{(J+\frac{1}{2})(p_t^N q_t^\pi)^{J-1}}{\sqrt{J(J+1)}} P_J'(\cos\theta_t) f_{J-}^{(\pm)}$$

where $t = 4(q_t^2 + m_\pi^2) = 4(p_t^2 + m_N^2)$,
[cf. Frautschi & Walecka: Phys. Rev $\underline{120}$, 1486.(1960)]
q_t and p_t being the initial and final c.m. momenta in $\pi + \pi \rightarrow N + \bar{N}$.

$A^{(+)}$ and $B^{(+)}$ correspond to a pure $I = 0$ state in this channel; thus only $A^{(-)}$ and $B^{(-)}$ receive contributions from the $I = 1$ ρ trajectory. The exchange of a single angular momentum j gives the contribution

$$A^{(-)} = \frac{c^{(1)} P_j(x_t) + m_N (M^2 - 4m_\pi^2)^{\frac{1}{2}} (M^2 - 4m_N^2)^{-\frac{1}{2}} c^{(2)} x_t P_j'(x_t)}{t - M^2},$$

$$B^{(-)} = c^{(2)} \frac{P_j'(x_t)}{t - M^2}$$

where $$x_t = \cos\theta_t = -\frac{(s - m_N^2 - m_\pi^2 + \frac{1}{2}t)}{2q_t p_t}$$

We have

$$P_j(x_t) \sim s^j, \quad s \to \infty$$

$$P_j'(x_t) \sim j s^{j-1}, \quad s \to \infty$$

Rule 3. gives

$$-A^{(-)} \to \frac{1 - e^{i\pi\alpha_\rho(t)}}{2\sin\pi\alpha_\rho(t)} \left(\frac{s}{2m_\pi m_N}\right)^{\alpha_\rho(t)} \cdot 2m_\pi \left[b^{(1)}_{\pi\pi\rho NN}(t) - \right.$$

$$\left. - \alpha_\rho(t)\, b^{(2)}_{\pi\pi\rho NN}(t) \right] + \cdots$$

$$-B^{(-)} \to \frac{1 - e^{i\pi\alpha_\rho(t)}}{2\sin\pi\alpha_\rho(t)} \left(\frac{s}{2m_\pi m_N}\right)^{\alpha_\rho(t)-1} \cdot 2\alpha_\rho(t)\, b^{(2)}_{\pi\pi\rho NN}(t) + \cdots$$

There are 2 unknown functions of t corresponding to the two possible states in the $N\bar{N}$ system.

Evaluating these functions $b(t)$ at $t = m_\rho^2$ gives

$$\frac{b_\rho^{(1)}(m_\rho^2)}{\pi\,\epsilon_\rho} = 2\gamma_{\rho NN}\gamma_{\rho\pi\pi},$$

$$\frac{b_\rho^{(2)}(m_\rho^2)}{\pi\,\epsilon_\rho} = 2\gamma_{\rho NN}\gamma_{\rho\pi\pi} - 4m_N \mu_{\rho NN}\gamma_{\rho\pi\pi}$$

The asymptotic no-spin-flip and spin-flip amplitudes are:

a) No Spin-flip:

$$f^{-} = -\frac{m_N}{4\pi\sqrt{s}}\left(A^{-} + \frac{s}{2m_N}B^{-}\right)$$

$$= \frac{m_N}{4\pi\sqrt{s}}\left(\frac{1-e^{-i\pi\alpha_\rho}}{2\sin\pi\alpha_\rho}\right)\left(\frac{s}{2m_\pi m_N}\right)^{\alpha_\rho} 2m_\pi b_\rho^{(1)} + \cdots$$

b) Spin-flip

$$g^{-} = \frac{1}{16\pi}\left(-A^{-} + s^{1/2}B^{-}\right)$$

$$= \frac{1}{16\pi}\left(\frac{1-e^{-i\pi\alpha_\rho}}{2\sin\pi\alpha_\rho}\right)\left(\frac{s}{2m_\pi m_N}\right)^{\alpha_\rho} 2m_\pi\left(b_\rho^{(1)} - \alpha b_\rho^{(2)}\right) + \cdots$$

The charge-exchange cross-section is

$$\frac{d\sigma^{-}}{d\Omega} = \left\{ |f^{-}|^2 + |g^{-}|^2 \sin^2\theta \right\}$$

OR

$$\frac{d\sigma^{-}}{dt} \rightarrow \left(\frac{s}{2m_\pi m_N}\right)^{2\alpha_\rho - 2} F_{\rho\pi N}(t) ,$$

where

$$F_{\rho\pi N}(t) = \frac{1}{16\pi}\left\{ |b^{(1)}|^2 - \frac{t}{4m_N^2}\,|b^{(1)} - \alpha b^{(2)}|^2 \right\} \times$$

$$\times \left| \frac{1-e^{-i\pi\alpha_\rho}}{\sin\pi\alpha_\rho} \right|^2$$

The factorisation hypothesis gives

$$F_{\rho\pi N}(t) = G_{\rho\pi\pi}(t)\, G_{\rho NN}(t)$$

We would obtain a similar expression for nucleon-nucleon charge-exchange scattering (assuming it to be dominated by the ρ), with

$$F_{NN\rho}(t) = \left[G_{\rho NN}(t)\right]^2 .$$

Thus we would obtain the following relations for the charge-exchange cross-sections:

$$\sigma^2_{\pi N} = \sigma_{\pi\pi} \cdot \sigma_{NN}$$

Now consider the set of quantum numbers (ii), corresponding to the Pomeranchuk. This contributes only to $A^{(+)}$ and $B^{(+)}$.

The resultant asymptotic forms for $A^{(+)}$, $B^{(+)}$ and $f^{(+)}, g^{(+)}$ may be obtained by making the replacements $A^{(-)} \to -A^{(+)}$, $B^{(-)} \to -B^{(+)}$, $f^{(-)} \to -f^{(+)}$, $g^{(-)} \to -g^{(+)}$; $\alpha_\rho(t) \to \alpha_P(t)$ etc. in the asymptotic forms given above.

At very high energies,

$$f^{(+)} \to \frac{\sqrt{s}}{8\pi}\, i\, b_P^{(1)}(0) + \ldots$$

$$\sigma_{tot}^{(+)} \to \frac{8\pi}{\sqrt{s}}\, \mathrm{Im}\, f^{(+)} = b_P^{(1)}(0).$$

$\frac{d\sigma^{(+)}}{dt}$, for pure elastic scattering, is given by an expression analogous to $\frac{d\sigma^{-}}{dt}$ above, with the replacements $\alpha_\rho \to \alpha_P$, etc....

Turning to backward πN scattering, the lack of symmetry between the t and u channels leads to a difference between forward and backward scattering. It is a general feature of meson-baryon scattering that the t channel will have poles with $\mathcal{B} = 0$ i.e. meson poles, while the u channel will have poles with $\mathcal{B} = 1$, i.e. baryon poles. As the limit u small, $s \to \infty$ corresponds to backward scattering, the baryon poles will be detected in the 'backward peak.'

Thus the nucleon and pion-nucleon resonances will contribute poles in backward πN scattering, the Λ, Σ, Y_0^*, Y_1^*, etc. to KN scattering & $\pi + N \to K + Y$ in the backward direction, etc.

Consider backward, πN scattering.

Rule 1:

Choose the $A^{(+)}$ and $\mathcal{B}^{(+)}$ amplitudes as before.

Rule 2:

Write down the partial-wave expansion in the u channel.

$$A = 4\pi \left[\frac{W+m}{E+m} f_1 - \frac{W-m}{E-m} f_2 \right] =$$

$$= 4\pi \left[\frac{W+m}{E+m} \left\{ \sum_{\ell=0}^{\infty} f_{\ell+} P'_{\ell+1}(x) - \sum_{\ell=2}^{\infty} f_{\ell-} P'_{\ell-1}(x) \right\} - \right.$$

$$\left. - \frac{W-m}{E-m} \left\{ \sum_{\ell=1}^{\infty} (f_{\ell-} - f_{\ell+}) P'_{\ell}(x) \right\} \right] ,$$

$$\mathcal{B} = 4\pi \left[\frac{1}{E+m} f_1 + \frac{1}{E-m} f_2 \right]$$

$$= 4\pi \left[\frac{1}{E+m} \left\{ \sum_{\ell=0}^{\infty} f_{\ell+} P'_{\ell+1}(x) - \sum_{\ell=2}^{\infty} f_{\ell-} P'_{\ell-1}(x) \right\} + \right.$$

$$\left. + \frac{1}{E-m} \left\{ \sum_{\ell=1}^{\infty} (f_{\ell-} - f_{\ell+}) P'_{\ell}(x) \right\} \right]$$

where $f_{\ell+}$ refers to the states $j = \ell + \frac{1}{2}$ and $f_{\ell-}$ to states with $j = \ell - \frac{1}{2}$. Thus a 'Particle' with spin j and 'Mass' M contributes to the u channel the following pole terms:

(i) To partial waves $\ell = j + \frac{1}{2}$ only $f_{\ell-}$ contributes

Thus

$$A = \frac{1}{u - M^2}\left[+ \frac{W+m}{E+m}\, c\, P'_{j-\frac{1}{2}}(\cos\theta_u) + \frac{W-m}{E-m}\, c\, P'_{j+\frac{1}{2}}(\cos\theta_u)\right]$$

$$B = \frac{1}{u - M^2}\left[- \frac{1}{E+m}\, c\, P'_{j-\frac{1}{2}}(\cos\theta_u) + \frac{1}{E-m}\, c\, P'_{j+\frac{1}{2}}(\cos\theta_u)\right]$$

(ii) To partial waves $\ell = j - \frac{1}{2}$ only $f_{\ell+}$ contributes

Thus

$$A = \frac{\cancel{-}1}{u - M^2}\left[\frac{W+m}{E+m}\, D\, P'_{j+\frac{1}{2}}(\cos\theta_u) + \frac{W-m}{E-m}\, D\, P'_{j-\frac{1}{2}}(\cos\theta_u)\right]$$

$$B = \frac{1}{u - M^2}\left[\frac{1}{E+m}\, D\, P'_{j+\frac{1}{2}}(\cos\theta_u) - \frac{1}{E-m}\, D\, P'_{j-\frac{1}{2}}(\cos\theta_u)\right]$$

W, E, θ_u, etc. all refer to the variables in the u channel.

A particle with $(B = 1,\ S = 0)$ and given spin j and parity Π will contribute to only one of the partial waves $\ell = j + \frac{1}{2}$ or $j - \frac{1}{2}$, such that $\Pi = (-1)^{\ell+1}$, as the πN relative parity is odd.

Thus a particular pole term will be characterised by only one constant (c or D).

Rule 3:

Put $j = \alpha + \frac{1}{2}$

Noting that $P_j'(\cos\theta_u) \sim s^{j-1}$ as $s \to \infty$, we can obtain the asymptotic behaviour.

For Regge trajectories with an even signature, α must be an even integer for the pole terms. For the partial wave $\ell = j + \frac{1}{2} = \alpha + 1$, the parity is $(-1)^{\ell+1} = (-1)^{\alpha}$ which is positive, while for the partial wave $\ell = j - \frac{1}{2} = \alpha$, the parity is negative. The reverse holds good for an odd signature trajectory. Thus for a trajectory with even signature and positive parity, only the $\ell = j + \frac{1}{2}$ partial wave or $f_{\ell-}$ contributes, giving an asymptotic behaviour

$$A \to + \frac{\left(1 + e^{i\pi\alpha(u)}\right)}{2\sin\pi\alpha(u)} \left(\frac{s}{2m_\pi m_N}\right)^{\alpha(u)} (W_u - m_N)\, b(u) + \cdots,$$

$$B \to \frac{\left(1 + e^{i\pi\alpha(u)}\right)}{2\sin\pi\alpha(u)} \left(\frac{s}{2m_\pi m_N}\right)^{\alpha(u)} b(u) + \cdots$$

For a trajectory with odd signature and odd parity also, only $f_{\ell-}$ contributes, and the asymptotic behaviour is obtained by merely altering the signature factor above to $\left[1 - e^{i\pi\alpha(u)}\right]$

On the other hand, for trajectories with an even signature and negative parity, or with an odd signature and positive parity, only the $\ell = j - \frac{1}{2}$ partial wave or $f_{\ell+}$ contributes, and the asymptotic behaviour is given by replacing $(W - m)$ above by $(W + m)$ in A, and changing the sign of A, and putting in the correct signature factor.

Consider the contribution of the N with $J = \frac{1}{2}$, $I = \frac{1}{2}$ and $+ve$ parity, the N^* at 1238 Mev with $J = \frac{3}{2}$, $I = \frac{3}{2}$ and $+ve$ parity $(P_{3/2})$, the N^{**} at 1520 Mev with $J = 3/2$, $I = \frac{1}{2}$ and $-ve$ parity $(D_{3/2})$, and the N^{***} at 1680 Mev with $J = 5/2$, $I = \frac{1}{2}$ and $+ve$ parity $(f_{5/2})$.

The N has $\alpha = 0$, the N^* has $\alpha = 1$, the N^{**} has $\alpha = 1$, and the N^{***} has $\alpha = 2$. We can assume that the N and N^{***} both belong to the same trajectory, which has $I = \frac{1}{2}$, an even signature and positive parity. This will contribute only to the $f_{\ell-}^{(\pm)}$ partial wave, with a factor $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, from isospin. The upper signs $(\pm)$ refer to the isospin decomposition. Note that the crossing matrix between the s and u channels is given by $\begin{bmatrix} A_s^{(+)} \\ A_s^{(-)} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} A_u^{1/2} \\ A_u^{3/2} \end{bmatrix}$

The N^*, belonging to an odd signature and positive parity trajectory, contributes only to the $f_{\ell+}^{(\pm)}$ partial wave, with a factor $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$; and the N^{**} with an odd signature and negative parity, contributes to the $f_{\ell-}^{(\pm)}$ partial wave with a factor $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The residues $b(u)$ at the physical points may be related to the coupling constants of dispersion theory,

e.g. $$\frac{b(m_N^2)}{\pi \frac{d}{du} \alpha(u) \Big|_{m_N^2}} = g_{NN\pi}^2$$

KN, $\bar{K}$N Scattering, and Associated Production

An analysis similar to the above may be carried out for KN and $\bar{K}$N scattering.

In KN scattering, we have the P, ρ, and ω poles in the t channel which determine the forward scattering. The ρ and ω have odd signature and odd parity. The non charge-exchange (n.c.e.) amplitude will be dominated by the P and ω and a P' if one exists, while the charge-exchange (c.e.) amplitude will be determined mainly by the ρ trajectory. The c.e. amplitude will be similar to that in πN scattering, but the n.c.e. amplitude will be more complicated.

The backward scattering would be determined by the Λ and Σ poles and the various Y^* resonances. One may expect an even signature trajectory with positive parity, $S=-1$ and $I=0$ carrying the Λ and the Y_0^{***} at 1815 Mev, an even signature, positive parity $I=1$, $S=-1$ trajectory with the Σ, an odd signature, negative parity $I=0$, $S=-1$ trajectory with the $d_{3/2}$ Y_0^{**} at 1520 Mev, etc., The spins and parities of the Y_1^*, Y_0^* and the Y_2^* at 1550-1580 Mev are not yet established. (The last would give a $I=2$ baryon trajectory).

In $\bar{K}N$ scattering, one has the same poles in the t channel and none in the u channel, unless some K^+N resonance exists.

Associated production $\pi + N \rightarrow K + Y$ would be governed in the forward direction by the K^* trajectory with odd signature, odd parity, $S = \pm 1$ and $I=\frac{1}{2}$ (and by other possible $(K\pi)$ resonances) and in the backward direction by the Λ, Σ, and the various Y^* trajectories.

We may also consider reactions involving photons, e.g.

$$\gamma + N \to \pi + N, \qquad \gamma + N \to K + Y.$$

These are similar to the corresponding reactions with incident pions except that in the forward direction the π and K trajectories respectively would also contribute. (Note that these additional contributions must vanish at the physical points $t = m_\pi^2$ and $t = m_K^2$ respectively, by angular momentum conservation

<u>Baryon - Baryon Scattering.</u>

We might take as examples N-N scattering and ΛN scatteri

The problem is complicated by the presence of the spins.

<u>Rules 1:</u>

Spin alone gives 5 independent amplitudes [for NN scattering and 6 for ΛN] and for NN scattering, this is doubled by the isospin. choice of Amplitudes free of kinematical singularities may be obtained by taking a particular allows only a particular combination of the helicity amplitudes.

The complete analysis has been given by Gell-Mann, and by Itzykson and Jacob; we shall consider this in a later lecture. A rough treatment which assumes that the main features do not depend much on the spin was given by Hadjioannou, Rarita and Phillips (Phys. Rev. Lett 9, 183 (1962)) ;
they fit the data assuming that forward NN scattering is dominated by the P, ω and P' trajectories. We have discussed this in an earlier lecture.

Inclusion of spin effects leads to predictions regarding the polarisation produced (ref. Y. Hara : Phys. Rev. Lett. 9, 246 (1962)
A single Regge trajectory does not give rise to any polarisation (with intially unpolarised nucleons); any polarisation produced

would thus be evidence for the presence of 2 or more Regge Trajectories.

The polarized cross-section at a fixed (small) value of t is predicted to decrease with increasing energy. (cf. Itzykson & Jacob, preprint

NN scattering is symmetrical in the t and u channels and hence in the forward and backward directions. However, it is not so with $N\bar{N}$ scattering. $N\bar{N}$ scattering with Regge poles has been studied by various people [e.g. S. Minami : Prog. Theor. Phys. 2 (1962)...

Forward $N\bar{N}$ scattering will be determined by the same poles as NN scattering, with certain differences in sign that we discussed earlier in connection with high-energy diffraction scattering. Backward $N\bar{N}$ scattering may show the deuteron trajectory, with odd signature, positive parity, $B = 2$, $I = 0$, etc. The virtual singlet $^1S_0$ state with even signature, positive parity, $I = 1$ (?) may also appear.

Regge poles in reactions involving the scattering of α's etc. are being studied.

Inelastic reactions (with particle production)

The above conjectures may be extended to reactions in which 3 or more particles are produced, provided these are characterised by sharply separated forward and backward cones at high energies (as seems to be true). This was first done by Frautschi, Contogouris and Wong for the reaction $N + N \rightarrow N + N^*$

Considering reactions like

$$N + N \rightarrow N + N + n\pi \quad , \quad \pi + N \rightarrow N + m\pi$$
$$\gamma + N \rightarrow N + m\pi$$

one expects that the conjectures will be more nearly valid the better production reactions can be approximated by $2 \rightarrow 2$ reactions. That is, as pointed out by Frautschi *et al*, we can carry over the original conjectures to reactions like

$$N + N \rightarrow N + N + \pi \quad , \quad \pi + N \rightarrow \pi + N + \pi$$
$$\rightarrow N + N + 2\pi \quad , \quad \rightarrow \pi + N + 2\pi$$

provided these occur mainly as isobar production:

$$N + N \rightarrow N + N^{*} \quad , \quad \pi + N \rightarrow \pi + N^{*} ,$$
$$\rightarrow N^{*} + N^{*} \qquad\qquad \rho + N$$
$$\rho + N^{*}$$

Frautschi, Contogouris and Wong have attempted to fit the data for $N + N \rightarrow N + N^{*}$ with the Pomeranchuk trajectory and π trajectory respectively for production of the $I = 1/2$ and $I = 3/2$ resonances.

One might speculate whether the above can be generalised to any reaction in which there are well-separated forward and backward cones (not necessarily correlated as isobars). This would require us to define a coupling of a Regge trajectory to more than 2 particles.

In the next lecture we shall begin a study of Regge poles in a relativistic theory, i.e. S-matrix theory and perturbation theory.

COMPLEX ANGULAR MOMENTUM IN A RELATIVISTIC THEORY

So far we have seen how a Regge behaviour, proved for scattering by a wide class of potentials, could be conjectured to hold for a general two-body relativistic scattering problem, leading to definite predictions about the asymptotic behaviour of cross-sections. We shall now see how much of this conjecture can be proved in the relativistic case.

We shall consider (a) Regge poles in relation to perturbation theory, and (b) Regge poles from the Mandelstam representation.

One tries to take over the concept of a Regge behaviour into a relativistic theory by coupling the Regge representation with the idea of crossing symmetry. The behaviour of the scattering amplitude $A(s, t)$ at large t is given as

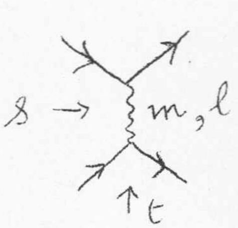
$$A(s,t) \sim \beta(s)\, t^{\alpha(s)}, \quad t \to \infty, \; s \text{ fixed}.$$

Crossing symmetry implies that the behaviour at large s is

$$A(s,t) \sim \beta(t)\, s^{\alpha(t)}, \quad s \to \infty, \; t \text{ fixed}.$$

The question of the asymptotic behaviour of the amplitude is crucial in a dispersion theory. Two types of difficulties arise in this connection:

1) When a particle of spin ≥ 1 is exchanged in two-body scattering, the amplitude diverges at large energy, as it is of the form $A(s,t) \sim \dfrac{P_\ell(\cos\theta_t)}{t - m^2}$,



and

$$P_\ell(\cos\theta_t) \sim s^\ell, \quad s \to \infty$$

2) The Gribov paradox:

(Ref. Froissart's lectures at the International Atomic Energy Agency Seminar, Trieste, 1962)

The observed behaviour of the scattering cross-sections at high energies was explained by Pomeranchuk on the basis of a semi-classical diff-raction picture which implies the assumption

$$A(s,t) \sim i s f(t) \qquad \text{-- (1)}$$

Gribov pointed out that such a picture was not consistent with the requirements of analyticity and unitarity implied by the Mandelstam representation: Using (1) and unitarity, the double spectral function $\rho(s,t)$ can be calculated; the result obtained is

$$\rho(s,t) \sim s \log s \, |f(t)|^2 \qquad \text{-- (2)}$$

However, (1) implies that $\rho(s,t) \sim s \, \mathrm{Im} f(t)$, which is not consistent with (2). This is the Gribov paradox. The paradox shows that the old diffraction model for high-energy scattering is not valid; it would be resolved if the amplitude had an asymptotic behaviour

$$A(s,t) \sim \beta(t) \, s^{\alpha(t)},$$

with $\alpha(t)$ real and < 1 for $t < 0$ and complex for $t > 0$; this is just the Regge behaviour.

One may try to prove that a scattering amplitude satisfying some restrictions of analyticity and unitarity must have a Regge behaviour. Work on this has done by Barut and Zwanziger, Bardacki, Oehme and by Mandelstam.

We shall consider the following topics:

A) Regge poles from Perturbation Theory.

B) Regge poles from the Mandelstam representation.

The following questions immediately arise:

1) How does one define a continuation of the amplitude to complex ℓ in a way that is unique and preserves unitarity?

2) Does the scattering amplitude $A(s, \ell)$ have only poles in complex ℓ, or also other types of singularities, esp. branch cuts? And more specifically, what are the domains of holomorphy and meromorphy of $A(s, \ell)$?

3) What is the relation between analyticity in angular momentum and the analyticity in s and t implied by the Mandelstam representation? Can perturbation theory give us a clue to the analyticity of the amplitude in ℓ?

We shall first ask what is the relation of Regge poles to high-energy behaviour in perturbation theory. The question of obtaining a Regge behaviour from unitarity and analyticity we shall consider later.

One of the questions emphasized by Chew and Frautschi in their early speculations on Regge poles was whether the various particles could be distinguished as elementary or non-elementary.

The exchange of a single particle of spin ℓ gives an amplitude that behaves as s^{ℓ} for large s, ℓ being a fixed number independent of t. The exchange of a single Regge pole would give an amplitude that behaves asymptotically as $s^{\alpha(t)}$. By obtaining the power $\alpha(t)$ from the peak width in forward or backward diffraction scattering and examining whether it is a constant ($= \ell$) or whether it varies with t, one may deduce whether the exchanged particle is an elementary particle or whether it is a Regge pole. [If no distinction between elementary and non-elementary particles is possible, all particles must be Regge poles.]

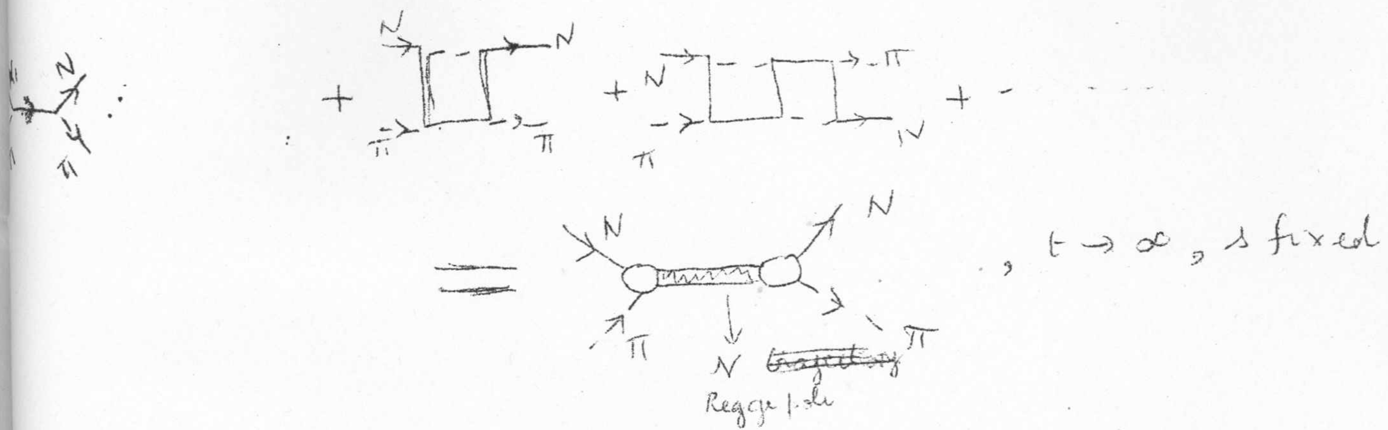
The argument given above has the fallacy that it is only the lowest order contribution — the 'pole term' — that behaves asymptotically as as^{ℓ} , with ℓ fixed. Higher order contribution may add so as to give a behaviour $s^{\alpha(t)}$. In fact, Amati, Fubini et al showed that the ladder diagrams in high-energy scattering sum to give a Regge behaviour. Lévy suggested that the radiative corrections to electron scattering when summed are equivalent to the contribution of an exchanged positronium pole. 1)

This suggests that perturbation theory, when considered to all orders, leads quite generally to a Regge behaviour of the scattering amplitude. We shall prove this, following the method given by Polkinghorne. 2)

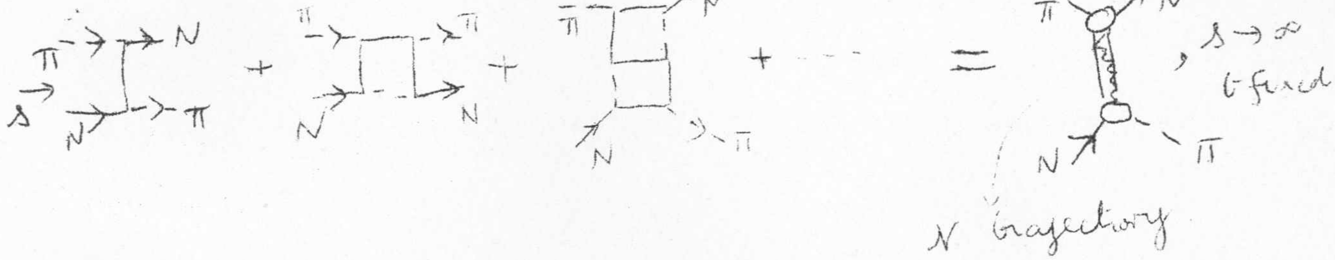
First we note the difference between stating that a scattering amplitude has a Regge behaviour and the statement that this Regge behaviour corresponds to the 'reggeisation' of some particular elementary particle pole. The first statement seems to be true in a large number of cases; however, in most of these, the Regge poles lie (for large Δ) at = -1, -2, -3. The second statement would imply much more and requires that some of the Regge poles obtained by summing perturbation theory would lie at = 0, 1, etc, ... depending on the particle exchanged, and have the properties expected of Regge trajectories corresponding to elementary particles.

1) Phys. Rev. Lett 9, 235 (1962)
2) Cambridge preprint.

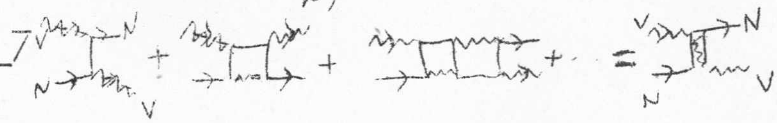
The simplest way to think of an elementary particle as a member of a Regge trajectory is in the framework of a theory in which all particles are composite, which is the basis of the bootstrap philosophy of Chew, Zachariasen and Zemach and others. In such a framework, one can think of elementary particle Regge trajectories as composite states formed of suitable components (e.g. the nucleon trajectory may be a pion-nucleon composite state bound by the exchange of N^*, $N^* + \pi$, etc...... as given by the ladder diagrams in the s channel (in the limit $t \to \infty$, fixed s)



The exchange of such a composite nucleon would then be represented by the crossed diagrams



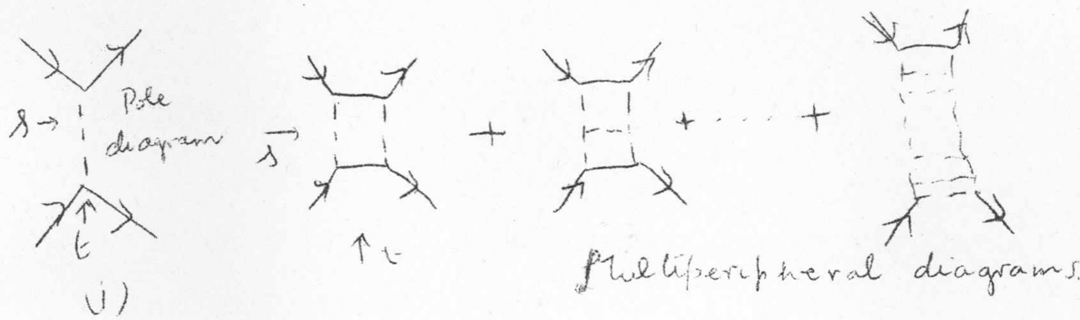
Gell-Mann and Goldberger * I have suggested that the nucleon may be reggeized in an essentially different way that does not correspond to its being a composite state. According to them the iteration of the Born approximation, i.e., the set of ladder diagrams may, in the limit $s \to \infty$, t fixed, be equivalent to the exchange of a nucleon trajectory. [They find that this happens only when the virtual meson involved is a vector meson (V), and not when it is a pseudoscalar meson.]



If this is proved it would mean that even when the nucleon is considered as ~~xxx~~ an elementary particle, the effect of radiative corrections wouldbe to make it behave as a Regge trajectory. If this turns out to be a correct picture of elementary particle Regge trajectories, then it would seem to be in contradiction to the view that all particles are composite.

We now consider the question of high-energy behaviour in perturbation theory. We shall follow the method given by Polkinghorne for finding the asymptotic behaviour of a class of Feynman diagrams.

We consider the class of ~~ladder~~ multiperipheral diagrams shown, taking all the particles to be scalar, and assuming trilinear (Yukawa) non-derivative couplings.



The multiperipheral diagrams may be defined as the class of diagrams with a two-particle intermediate state in the t-channel and any number of particles in the s-channel intermediate state, but being such that no two lines cross.

The pole diagram (i) gives a contribution with the asymptotic behaviour $\sim s^{0} \sim$ constant, for $s \to \infty$ and t fixed, as the exchanged particle is spinless. If the total amplitude (i.e. the sum over all Feynman diagrams) has the asymptotic behaviour $A(s, t) \to \beta(t)\, s^{\bar{\alpha}(t)}$, t fixed, $s \to \infty$, we may write $\beta(t)\, s^{\bar{\alpha}(t)} = \beta(t)\, s^{\alpha_0 + \alpha(t)}$, (where α_0 is the integer nearest to α (o),)

$= \beta(t)\, s^{\alpha_0}\, e^{\alpha(t) \log s}$

$= \beta(t)\, s^{\alpha_0}\left[1 + \alpha(t) \log s + \frac{[\alpha(t)]^2}{2!} \log^2 s + \cdots\right]$

$= \beta(t) s^{\alpha_0} + \alpha(t)\beta(t)\, s^{\alpha_0} \log s + \frac{1}{2!}[\alpha(t)]^2 \beta(t)\, s^{\alpha_0} \log^2 s + \cdots$

The first term would correspond to the pole contribution if $\alpha_0 = 0$.

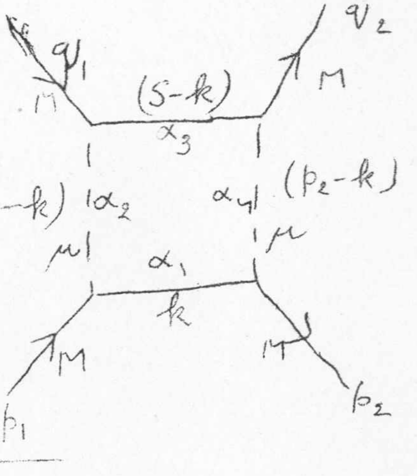
If we find that higher-order Feynman diagrams contribute terms to the amplitude that behave asymptotically like $s^{\alpha_0} \log s$, $s^{\alpha_0} \log^2 s$, - - - - - - - - , then we may conclude that the pole of fig. (i) is 'reggeised' by the higher-order corrections. However, if the amplitude has a Regge behaviour,

$$A(s,\, t) \underset{s \to \infty}{\longrightarrow} \beta(t)\ s^{\bar{\alpha}(t)}, \qquad (i)$$

and α_0 is not zero but some negative number, $\alpha_0 = -1, -2$, etc. , as happens in potential scattering, then this does not correspond to any particle being reggeised.

Take the 4th order contribution of fig. (ii). The Feynman integral for this may be written

$$\mathcal{M} = \frac{g^4}{16\pi^2} \int_0^1 d\alpha_1 \ldots d\alpha_4\, \delta(1 - \Sigma\alpha_i) \int \frac{d^4k}{[F]^4}$$



where

$$F = (\Sigma\alpha_i)\, k^2 - 2k\cdot(\alpha_2 p_1 + \alpha_3 S + \alpha_4 p_2) + c,$$

$$S = p_1 + q_1\ ; \quad S^2 = -s\ ,$$

$$c = (\alpha_1 + \alpha_3)\, M^2 - \alpha_3 s\ .$$

This may be simplified to give

$$M = c \int_0^1 d\alpha_1 \ldots d\alpha_4 \ \delta(1 - \Sigma\alpha_i) \ \frac{1}{[D]^2}$$

where $D = D(s, t, \alpha) = -[\alpha_1\alpha_3 s + \alpha_2\alpha_4 t + d(\alpha, M^2, \mu^2)]$.

what is the asymptotic behaviour of

$$\int_0^1 d\alpha_1 \ldots d\alpha_4 \ \delta(1 - \Sigma\alpha_i) \frac{1}{[\alpha_1\alpha_3 s + \alpha_2\alpha_4 t + d(\alpha, M^2, \mu^2)]^2} ?$$

Polkinghorne has given the following method of finding this.

We want the limit of $\mathcal{M}(s, t)$ for fixed t as $s \to \infty$.

If the coefficient of s in the denominator, viz., $\alpha_1 \alpha_3$, did not vanish on part of the boundary of the domain of integration (eg. if the limits of integration had been 2 and 3) the integral would have had the asymptotic behaviour $\frac{1}{s^2}$. However, when $\alpha_1 = 0$, $\alpha_3 = 0$, the coefficient of s vanishes; this results in the integral decreasing less rapidly as $s \to \infty$. [If the coefficient vanished in some point within the region of integration but not on the boundary, the asymptotic behaviour would still be $\frac{1}{s^2}$.] The asymptotic behaviour is thus determined by the contribution of a small region near $\alpha_1 = 0$. $\alpha_3 = 0$. This is

$$I \underset{s \to \infty}{\longrightarrow} \int_0^\epsilon d\alpha_1 d\alpha_3 \frac{1}{[\alpha_1\alpha_3 s + d(\alpha_2, \alpha_4, t)]^2}$$ putting $d_1 = 0$, $\alpha_3 = 0$ in all but the first term of the denominator. Thus

$$I \longrightarrow \int_0^\epsilon d\alpha_1 \frac{1}{(\alpha_1 s)} \frac{(-1)}{[\alpha_1\alpha_3 s + d]} \Bigg|_{\alpha_3 = 0}^{\alpha_3 = \epsilon} =$$

$$= \int_0^\epsilon d\alpha_1 \frac{\epsilon}{\alpha_1 s \epsilon + d} = \frac{1}{s} \log\left(1 + \frac{s\epsilon^2}{d}\right) \underset{s \to \infty}{\longrightarrow} \frac{1}{s} \log s.$$

Thus the 4th order diagram gives a contribution that behaves as $\frac{1}{s} \log s$ for $s \to \infty$. This will evidently not reggeise the pole (with spin o) in second order. We may say that the radiative corrections considered here will not reggeize the spin zero pole in the second order diagram. If these were the only radiative corrections permissible, the pole would then correspond to a fixed 'elementary particle' pole.

We may ask what is the nature of the radiative corrections required to reggeize the pole considered above.

Before considering that, we briefly give the result obtained by summing the ladder diagrams to all orders. The result is

$$A(s,t) \underset{\substack{s\to\infty \\ t \text{ fixed}}}{\sim} b(t)\, s^{\alpha(t)},$$

where

$$\alpha(t) = -1 + g^2 K(t),$$

$$b(t) = g^2 ,$$

and

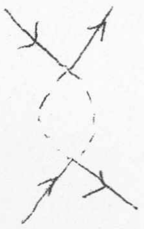
$$K(t) = \frac{1}{16\pi^2} \int_0^1 \frac{d\alpha_3 \, d\alpha_4 \; \delta(1-\alpha_3-\alpha_4)}{\left[\alpha_3\alpha_4 s - (\alpha_3+\alpha_4)^2\right]}$$

We note the following:

1) The result given above is obtained by summing the asymptotic contributions of the individual ladder diagrams. This may not be the same as the asymptotic behaviour of the sum of the diagrams, in general.

2) $\alpha_0 = -1$ for the ladder diagrams considered above. Thus the pole that is reggeized has $\alpha_0 = -1$. Diagrams obtained from the ladder diagrams by crossing some of the internal meson lines (in the s-channel intermediate state) contribute terms that reggeize poles with $\alpha_0 = -2, -3, \cdots$; they also add correction terms to the expressions for $\alpha(t)$ and $\beta(t)$ given above.

3) In the fourth order diagram considered, the asymptotic behaviour is determined by contributions (to the Feynman integral) from $\alpha_1 = 0$, $\alpha_3 = 0$. This corresponds to taking the contracted diagram shown below, in which the particles in the intermediate state in the t channel are on the mass shell. In general, we expect that the Regge pole terms will be obtained by taking the real-particle intermediate states in the crossed channel.



We now return to the question of what type of radiative corrections are required to reggeize poles with $\alpha_0 = 0, 1$, etc.

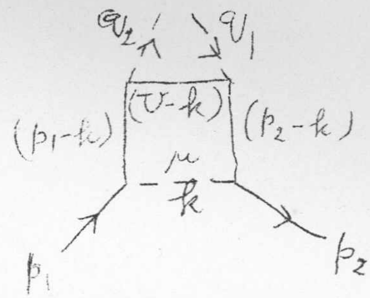
Evidently, we require contributions with a higher power of the momentum in the numerator. These may be obtained by considering derivative couplings and/or particles of higher spin.

For the nucleon pole in πN scattering, the situation is similar to the above, so long as we consider only the radiative corrections arising from virtual pions.

The fourth-order diagram gives terms behaving like $\frac{1}{s}$ logs but none that behave as logs. The matrix element for the fourth order diagram is

$$\bar{u}(p_2) \left[\int d^4k \; \frac{N}{D} \right] u(p_1) \; ,$$

where



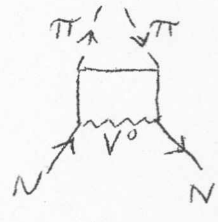
$$N = i\gamma_5 \left[i(\not{p}_2 - \not{k}) - m \right] i\gamma_5 \cdot \left[i(\not{v} - \not{k}) - m \right] i\gamma_5 \left[i(\not{p}_1 - \not{k}) - m \right] i\gamma_5$$

$$= \left[-i(\not{p}_2 - \not{k}) - m \right] \left[i(\not{v} - \not{k}) - m \right] \cdot \left[-i(\not{p}_1 - \not{k}) - m \right], \qquad v = (p_1 - q_2)$$

and $D = (k^2 - 2p_2\cdot k)\left[k^2 - 2v\cdot k + (M^2 - u)\right](k^2 - 2p_1\cdot k)(k^2 + \mu^2)$.

As the γ_5's can be removed by merely changing the sign of some of the momenta, and since $i\not{p}_1$, $i\not{p}_2$ taken between spinors are constants, the diagram above has the same behaviour as the one with scalar particles, i.e., it behaves as $\frac{1}{s}$ logs, $s \to \infty$, with u fixed.

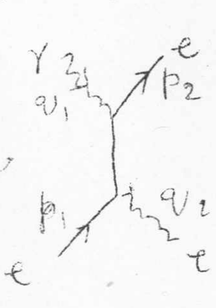
However, if the γ_5's at the vertices are replaced by γ_μ's, as (in Compton) scattering, or when the exchanged pion in the above diagram is replaced by a neutral vector meson, e.g., a ρ^0 or a ω^0 the matrix element will contain additional powers of momentum. The matrix element for the latter case will have

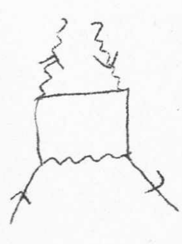


$$N = \gamma_\mu \left[i(\not{p}_2 - \not{k}) - m \right] i\gamma_5 \left[i(\not{v} - \not{k}) - m \right] i\gamma_5 \cdot \left[i(\not{p}_1 - \not{k}) - m \right] \gamma_\mu ,$$

whereas for Compton scattering, it would have

$$N = \gamma_\mu \left[i(\not{p}_2 - \not{k}) - m \right] \not{q}_2 \cdot \left[i(\not{v} - \not{k}) - m \right] \not{q}_1 \left[i(\not{p}_1 - \not{k}) - m \right] \gamma_\mu ,$$





with D as before.

In operating $\not p_1$ and $\not p_2$ on the spinors, one must write

$$\gamma_\mu \not p_2 = - \not p_2 \gamma_\mu + 2 p_{2\mu} \; ; \; \not p_1 \gamma_\mu = - \gamma_\mu \not p_1 + 2 p_{1\mu}$$

This gives a term that has an extra power of momentum

$$(p_1 \cdot p_2) = \frac{1}{2}(t - 2m^2) = \frac{1}{2}(2\mu^2 - s - u)$$

(Note: We take u fixed, $s \to \infty$.)

This extra power of mome n tum gives terms that behave as logs. As the nucleon pole and electron pole have $\alpha_0 = 0$, such a fourth order correction, together with higher-order corrections, may 'reggeize' the nucleon or electron pole.

REGGE POLES IN ELECTRODYNAMICS AND WEAK INTERACTION.

As the concept of Regge poles seems to be a useful one in dealing with strong interactions, it would be natural to ask whether it is a useful one in the domain of electromagnetic and weak interactions also.

Blankenbecler, Cook and Goldberger have speculated on the possible consequences of the photon being a Regge pole. [Phys. Rev. Lett. 8, 463 (1962)]. If all mesons and baryons are Regge poles, one may expect that leptons and the photon should also be so, if strong, electromagnetic, and weak interactions are to be treated in a unified way. However, the photon and the neutrino seem to be zero mass particles, and this may give them a unique position. It may be that they are the only 'elementary' particles.

The argument given by Blankenbecler et al for a reggeised photon was that if the photon were elementary, then the one-photon exchange contribution to pp scattering would dominate over the purely strong contribution from Pomeranchuk exchange, as the latter has a damping factor $s^{\alpha_P(t)-1}$, $\alpha_P(t) < 1$ when $t < 0$, (i.e. in the physical region), which damps its contribution at high energies. [The 1-photon exchange term has no such factor.] It seems unreasonable that the electromagnetic contribution to pp scattering should dominate over the purely strong contribution; Blankenbecler et al reasoned that this would be averted if the photon, too, were assumed to be a Regge pole.

We have, however, seen that this is not necessary; higher order contributions to the scattering may give a Regge-type damping factor even if they do not reggeize the photon pole.

In fact, it is known that the high-energy infra-red radiative corrections multiply the lowest-order scattering amplitude by an exponential factor. Levy has attempted to interpret this exponential factor as arising from the exchange of the positronium Regge pole. If this is correct, one may say that the effect of the infra-red radiative corrections is equivalent to that of the positronium Regge pole, just as the effect of the radiative corrections to backward Compton scattering is equivalent to considering the electron as a Regge pole. If the photon and the electron are Regge poles, this would imply a deviation from the predictions of quantum electrodynamics taken to the first 2 or 3 orders. The consequences of the reggisation of the photon have been noted by Blankenbecler, Cook and Goldberger. All the invariant amplitudes now contribute, eg., $e - \pi$ scattering is determined by 2 amplitudes and $e - \mathcal{N}$ scattering by six. The 'form factors' now depend on both the energy and momentum transfer. There would be deviations from the Rosenbluth formula for $e\text{-}\mathcal{N}$ scattering. [A method of finding $\alpha_\gamma(t)$ would be to use a spin-zero target and find the ratio of the scattering cross-sections for a fixed value of t and two different incident energies E_1, E_2; this ratio can be shown to be

$$\frac{\sigma(E_1, t)}{\sigma(E_2, t)} \approx \left[1 + 2\{\alpha_\gamma(t) - 1\} \log \frac{E_1}{E_2} \right].$$

- Blankenbecler et al: Phys. Rev. Lett **8**, 463 (1962)].

It is known that radiative corrections arising from multi-photon exchange give qualitatively the same deviations from

;pwest-order (one-photon exchange) electron scattering (as the reggeisation of the exchanged photon). To find out whether the exchange of 1, 3, 5,... photons may seem to reggeize the photon pole, one has to sum these contributions to the matrix element. [Note:- As the photon is odd under charge-conjugation, [illegible], only the exchange of an odd number of photons could contribute to the reggeization of the photon pole. The exchange of an even number of photons could give a trajectory with C = + 1.]

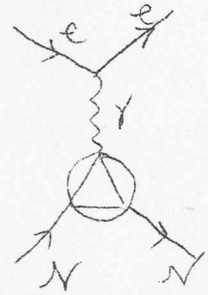
Lévy has summed a large number of Feynman graphs involving the exchange of an arbitrary number of virtual photons; he does not find a behaviour of the form $e^{\alpha(t)}$ and concludes that the photon is probably not a Regge pole,

Whether the electron is a Regge pole, as seems likely, may be tested by isolating the electron pole contribution in Compton scattering and in pair annihilation; Contogouris has suggested how this may be done from electron-electron and electrol-positron colliding beam experiments. [Phys Lett. 3, 103 (1962)]

Regge poles and form factors:

We have already seen how the reggeisation of the photon would affect the electromagnetic form factors. However, even if the photon were not reggeized, the electromagnetic form factors would still be affected by the reggeization of the strongly interacting particles, Mc.Millan and Predazzi have examined this question.

[Neuvo Cim. 25, 838 (1962).]
[Also see G. Domokos + Wolf; Phys Lett. 1, 349 (1962).]

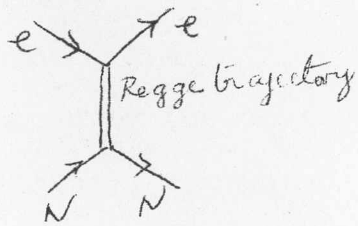


When a J = 1 photon is coupled to the Regge trajectories that dominate the nucleon form factors, it projects out the

J = 1 part of the contribution of these trajectories. This may be regarded as an improvement over the 'pole approximation' in that the exchanged pole now includes a whole family of particles (with the correct quantum numbers), and thus includes a number of higher-ordercorrections to the form factors.

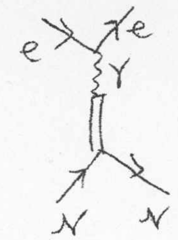
A Regge trajectory α (t) gives the contribution

$$b(t) \, \frac{[2\alpha(t)+1]}{\sin \pi\alpha(t)} \, P_{\alpha(t)}(\cos\theta).$$



to the scattering amplitude.

The photon projects out the J = 1 part of this.



Noting that

$$\frac{1}{2}\int_{-1}^{+1} P_{\alpha(t)}(\cos\theta) \; P_1(\cos\theta)\, d(\cos\theta) =$$

$$= \frac{1}{\pi} \; \frac{\sin \pi\alpha(t)}{[\alpha(t)-1]\,[\alpha(t)+2]},$$

the form factors are obtained as

$$G_{V,S}(t) = \frac{c_{V,S}(t)}{1-\alpha_{V,S}(t)}.$$

Approximating α (t) by a linear function of t gives back the pole approximation of dispersion theory; taking it to be quadratic gives deviations from the pole approximation, etc.

Mc.Millan and Predazzi have fitted the observed data on form factors using a quadratic form for α (t),

$$1-\alpha(t) \approx a + bt + ct^2 .$$

They obtain the two-pole formulae

$$F_V^{1,2}(t) \approx \frac{1}{\left(1-\frac{t}{28}\right)} \frac{1}{\left(1-\frac{t}{60}\right)} ;$$

$$F_S^{1,2}(t) \simeq \left(\frac{1}{1-t/28}\right) \left(\frac{1}{1-t/600}\right)$$

The above results were obtained by taking the electromagnetic interaction to the lowest order. One may next take the electromagnetic interaction acting twice and couple the two-photons to the strong Regge trajectories. Freund or Kemmer 1) suggested that this contribution would be enhanced by the coupling to the spin 2 particle on the Pomeranchuk trajectory, for which there now seems to be some evidence. This would give a correction to the Rosenbluth formula (whereas just coupling a 1-photon exchange to the Regge trajectories would give just the Rosenbluth expression for e-N scattering.)

1) Ref. Nuovo Cim 26 (1962)

<u>Regge poles and weak Interactions:</u>

A natural question to ask would be whether Regge poles play a role in weak interactions also. [cf. K. Raman: Proc. First Anniversary Symposium, 1963, Matscience Report-1]. Regge poles may enter naturally into a description of weak interactions in two ways.

(i) One may try to relate composite-state models of weak interactions to Regge poles.

(ii) Both the universal Fermi Theory and the charged vector boson theory seem to diverge in higher orders. The solution to this apparent divergence may be that the sum to all orders has an exponential Regge-type factor that damps the amplitude at high-energies. For a theory with vector bosons, the radiative corrections may prove to give an exponential factor as seems to happen with neutral vector bosons. The ladder diagrams may also sum to give a Regge behaviour.

An important question is whether there is any relation between Regge behaviour and the renormalizability of a theory.

An immediate consequence of the reggeization of the weak current would be the violation of the local action of lepton currents. The high-energy differential cross-section of a neutrino reaction $\nu + T \to F + \ell$ is no longer a quadratic function of the neutrino momentum p_ν, the lepton energy E_ℓ or $\cos\theta_{\nu F}$ (as it should be for a local lepton current; it is asymptotically of the form $s^{2\alpha(t)-1}$. As for a reggized photon, the weak form factors would become a function of the energy as well as the momentum transfer; their energy dependence would be asymptotically mainly in a factor $s^{\alpha(t) - 1}$.

Some other consequences have been discussed in the reference fiven above. [K. Raman: Proc. First Anniversary Symposium, 1963, Matscience Report 1.]

We merely note here that in a theory in which the vector boson is replaced by a Regge trajectory, the failure to detect the vector boson may be explained if this trajectory has an odd signature, like the Pomeranchuk trajectory.

The role of Regge poles in weak interaction form factors may be investigated as McMillan and Predazzi have done for the electromagnetic form factors. A difference is that here there is an axial vector current also, in addition to the vector current. The effect of Regge poles in low-energy weak interactions, i.e., of the usual pole terms, e.g., the pion pole in β decay, the K, K$'$, and K* poles in K and Y decay, etc., being replaced by Regge poles, is being studied.

The idea of conserved currents has played an important role in the theory of electromagnetic and weak interactions; it implies that the strength of coupling of the current is universal. We may conjecture a possible generalisation of this to an S-Matrix theory with a reggeized photon or a reggeized weak current viz., that the universality of coupling $\beta(t)$ holds all along the Regge trajectory.

This would imply that the electromagnetic form factors or weak vector form factors of all particles are the same. (This would give relations between K decay and Y decay, π decay and n decay, etc.)

We conclude by mentioning some of the questions that are to be answered in relation to Regge poles.

1) Do moving branch-cuts occur in the ℓ plane?

These seem to be connected to multi-particle intermediate states; we shall examine this question in a later lecture.

2) What is the experimental status regarding diffraction scattering?

Recent experiments at Brookhaven seem to show that the πp and Kp diffraction peaks do not seem to shrink with increasing energy; the pp peak also seems to shrink less than was formerly thought. This may be experimental evidence for moving branch-cuts in the ℓ-plane.

3) Can the baryons and mesons be proved to be reggeized (starting from a relativistic theory) ?

Very recently, Gell-Mann, Goldberger, Low and Zachariasen have attempted to prove that the nucleon is reggeised by vector meson exchange. *

We shall deal with these and other topics in future lectures.

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* Private communication from A.P. Balachandran.