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LECTURES ON

DIFFERENTIAL EQUATIONS

BY

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LECTURES ON DIFFERENTIAL EQUATIONS *

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ON INFINITE SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

At the outset, we review our knowledge concerning the solutions of ordinary differential equations of the form $y' = f(t,y)$ as the nature of f is increasingly specialized. Here $\underline{t}$ is a real variable and y is an n-dimensional complex vector, each component of which is a function of t; if $y = (y_1, y_2, \ldots, y_n)$, then we define the norm of y according to: $||y|| = \sum_{i=1}^{n} |y_i|$. (For convenience, we shall refer to these systems of differential equations as ''finite systems'', in the sequel.)

Next, we shall consider certain special cases of differential equations of the Kolmogorov type which arise in the study of some stochastic processes, by way of illustrating how many of the properties of the solutions in the above case (where y has a finite number of components) fail to hold in the case where the components of y are denumerable.

I. Let $y' = f(t,y)$ where f is given to be continuous in t and y in a region D containing the point (t_0, y_0) where $y_0 = y(t_0)$ according to the prescribed ''initial condition''. The following features characterize this case:

(1) existence of solutions: there exists at least one solution in a region of the form $|t - t_0| < \rho$, $|| y - y_0|| < \delta$;

(2) holding of the parity principle: the solutions are defined both for $t < t_0$ and for $t > t_0$; and

(3) the solutions involve arbitrary constants (as opposed to arbitrary functions, as in the case of partial differential equations, or, even, infinite systems of linear differential equations, as we shall see below).

II. If, in (I), Lipschitz's condition be also satisfied, namely, that for all $(t, y_1) \in D$, $(t, y_2) \in D$, we have

$$|| f(t, y_1) - f(t, y_2) || < C \, || y_1 - y_2 ||$$

for some constant $C > 0$, then

(4) the solution is unique.

III. If the differential equation is a linear homogeneous equation: $y' = A(t) \, y$, where t belongs to some open interval I, and $A(t) = (a_{jk}(t))$ is a $n \times n$ matrix, each element of which is a continuous function of t over I, so that we have n equations:

$$y_j'(t) = \sum_{k=1}^{n} a_{jk}(t) \, y_k(t) \quad \text{for } 1 \leq j \leq n,$$

then Lipschitz' condition is satisfied (over every bounded subinterval of I) and we have:

(5) there is no null solution other than the trivial solution i.e., if a solution vanishes for some $\tau \in I$, then it is identically zero; and

(6) the set of all possible solutions on I (corresponding to the various possible initial conditions) form an n-dimensional vector-space over the complex field. (Ofcourse, there is a <u>unique</u> solution corresponding to every given initial condition, by property (4) above).

(IV) If, in (III), A (t) is analytic, then

(7) the solutions are analytic;

(8) we have permanence of the functional equation: analytic continuations of the solutions continue to satisfy the same differential equation, and

(9) the singularities of the solutions are fixed: namely, those of A (t) <u>plus</u> the point at infinity.

(V) If, further, A(t) is entire, then

(10) the solutions are entire functions also.

(VI) If, in particular, $A(t) \equiv A = (a_{jk})$, a constant matrix, then the solution is an exponential function, viz.,

(11) $$\overrightarrow{y(t)} = \exp\left[(t-t_0)A\right] \overrightarrow{y(t_0)} .$$

Here $\exp B = E + \sum_{k=1}^{\infty} (B^k/k!)$ denotes the <u>exponential matrix</u> of the matrix B (This is well-defined for any square-matrix B), E being the identity matrix.

Having recapitulated the salient features of the solutions of ''finite systems'' of differential equations, we now turn to the cases where y has, not a finite, but a denumerable, set of components. We consider the simplest case, of linear differential equations of the form:

$$y_j'(t) = \sum_{k=1}^{\infty} a_{jk} y_k(t) \; ; \; j = 1, 2, 3, \ldots\ldots$$

(1) Note first that problems of convergence of the RHS can arise. Even if such problems are absent, others arise, even in comparatively trivial cases; for example, let

$$y_k'(t) = k\, y_k(t) \; ; \; k = 1, 2, 3, \ldots\ldots$$

where the $y_k(0)$ are given to be real. Then $y_k(t) = y_k(0).e^{kt}$

A natural vector-space to consider here is the space $\{m\}$ of all convergent sequences, with the supremum norm: $|| \{y_k\} || = \sup_k | y_k |$. If we take $\{y_k(0)\}$ to be in $\{m\}$, then the solution-vector $\{y_k(t)\}$ belongs to $\{m\}$ for $t < 0$, but, in general, does not belong to $\{m\}$ for $t > 0$ (the only exceptional case being the case where $y_k(0) = 0$ for all k after a certain stage). The equation thus fails to obey the parity principle, concerning itself only with the ''past'' ($t < 0$) and refusing to have anything to do with the ''future'' ($t > 0$).

(2) An example where we have better-behaved solutions is the infinite-dimensional analog of the ''finite system'' given by $y' = Ay$. Instead of A being an $n \times n$ matrix, it now takes the form of a doubly-infinite array, $A = (a_{jk})$; $j, k = 1, 2, \ldots\ldots$ We consider the set of all such ''matrices'' A for which $||A|| = \sup_j \left(\sum_{k=1}^{\infty} |a_{jk}| \right)$ is finite; if AB be defined as $(\sum_p a_{jp} b_{pk})$, then $||AB|| \leq ||A||\,||B||$ and $||\;\;||$ may be taken as a norm in the set of such matrices. For such a matrix A, exp (At) makes sense when defined analogously to the case where

A is a (n x n) matrix, and then the infinite system of differential equations given by $y' = Ay$ where A is a constant matrix has the analog of the ''exponential solution'' as its solution, i.e., the solution is $y(t) = \exp\{A(t-t_0)\} \cdot y(t_0)$.

NOTE: Infinite ''Stochastic matrices'' A have the property $||A|| < \infty$, (in fact, $||A|| = 1$ in such cases), and so our analysis above applies to them.

(3) We finally consider Kolmogorov's differential equations $Y'(t) = A\,Y(t)$, where A and Y are matrices and Y' is the matrix of derivatives. A is not an arbitrary matrix but should be such that $a_{jj} < 0$ for every j, $a_{jk} \geq 0$ for $j \neq k$, and $\sum_k a_{jk} \leq 0$ (These imply that every $\sum_k |a_{jk}|$ is convergent, but not necessarily that $||A||$ is finite). A solution Y(t) will be acceptable (from the probabilist's point of view) if it satisfies the following conditions: if $Y(t) = (p_{jk}(t))$, then we must have $0 \leq p_{jk}(t) \leq 1$; $\sum_k p_{jk}(t) \leq 1$, and $p_{jk}(t) \to \delta_{jk}$ (the Kronecker delta function) as $t \downarrow 0$.

(Such solutions have been discussed by W. Feller and J. L. Doob (around 1940) and by W. Ledermann and G.E. Reuter). In our discussion of the following special cases, we do not confine ourselves to acceptable solutions.

(I) $$y_k'(t) + k\,y_k(t) = \sum_{n=k+1}^{\infty} y_n(t);$$

$k = 1, 2, 3, \ldots\ldots\ldots$

The corresponding matrix A has the form

$$\begin{pmatrix} -1 & 1 & 1 & 1 & - & - & - \\ 0 & -2 & 1 & 1 & - & - & - \\ 0 & 0 & -3 & 1 & - & - & - \\ 0 & 0 & 0 & -4 & - & - & - \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \end{pmatrix}$$

If $\{l\}$ denotes the set of all sequences $\{a_n\}$ such that $\Sigma\, |a_n| < \infty$, then we require that the solution $\{y_n(t)\} \in \{l\}$. Let it be given that $\{y_n(0)\} \in \{l\}$. The following elegant solution is due to Reuter. Set $z_{k+1}(t) = \sum_{(k+1)}^{\infty} y_n(t)$. (Then, the sequence $\{z_k(t)\}$ is not only in the space of $\{m_0\}$ of convergent null sequences, i.e., the space of all sequences converging to zero, but is even of bounded variation since $\sum_k |z_k(t) - z_{k+1}(t)| = \sum_k |y_k(t)| < \infty$). Let us assume further that the $z_k(t)$ are differentiable with respect to t. (We know only that their sucessive differences, the $y_k(t)$ are differentiable). Then we have the relations:

$$z_k'(t) - z_{k+1}'(t) + k\left[z_k(t) - z_{k+1}(t)\right] = z_{k+1}(t)$$

which gives

$$z_k'(t) + k\, z_k(t) = z_{k+1}'(t) + (k+1)\, z_{k+1}(t) \quad \text{for all } k,$$

i.e., $z_k'(t) + k\, z_k(t)$ is independent of k and so $\equiv f(t)$, say. Let us take $f \in L[0, \omega]$, the class of all Lebesgue integrable functions over $[0, \omega]$. Then for $0 \leq t \leq \omega$, we have on integration

$$z_k(t) = z_k(0)\, e^{-(k+1)t} + \int_0^t e^{(k+1)(u-t)} f(u)\, du$$

i.e.,

$$y_k(t) = z_{k-1}(0)\, e^{-kt} - z_k(0)\, e^{-(k+1)t} + \int_0^t e^{k(u-t)} \left[1 - e^{u-t}\right] f(u)\, d$$

The obtained set of values $\{y_k(t)\}$ satisfy the condition $\{y_k(t)\} \in \{\ell\}$ for $t \geq 0$, and further $y_k(t) \rightarrow y_k(0)$ as $t \rightarrow 0+$. The presence of the kernel $\left[1 - e^{(u-t)}\right]$ vanishing at $u = t$ enables us to assert the existence <u>everywhere</u> in $(0, \omega)$ of $y_k'(t)$, even in the case where $f(u)$ merely belongs to $L\,[0, \omega]$ and is not necessarily continuous there. On substitution, we find that the obtained set of values $\{y_k(t)\}$ satisfy the given differential equation, so that in deed they constitute a solution thereof. In general $y_k''(t)$ exists only almost everywhere on $(0, \omega)$ and $y_k'''(t)$ does not exist (unless we impose further restrictions on f).

Then we have a solution-system depending on an arbitrary <u>function</u> (and not merely on arbitrary constants: cf. property (3) of finite systems). For $t < 0$, in general, $\{y_k(t)\}$ does not belong to $\{\ell\}$, depending on the constants $\{z_k(0)\}$. For such t, again, not only absolute convergence of the series

$\sum_{k} y_k(t)$, but even ordinary convergence fails to hold. Thus, property (2) of finite systems fails to hold. Properties (4) and (5) also fail to hold, since there are infinitely many null solutions in view of the arbitrariness of f. If f is taken to be an analytic function, then (7) holds, but not in general. (9) cannot hold, since the arbitrary f again affects the singularities. (10) certainly does not hold, and so neither does (11). Thus, here we have an almost perfect example of violation of the conditions satisfied by solutions of finite systems.

(II) We can generalize the situation in (I) by weighting the $y_k(t)$'s with suitable coefficients other than unity on the RHS's of the equations.

(III) Birth-and-death processes: These are represented by ''Kolmogorov matrices'' which have all entries zero except for those on the main diagonal and on the diagonals immediately above and below it. A typical Kolmogorov matrix is the following: the entries on the main diagonal are the negatives of the natural numbers in successin; on the diagonal just below it are the natural numbers in succession, and on the diagonal just above it are all 1's. The corresponding system of differential equations is:

$$y_1'(t) = -y_1(t) + y_2(t)$$

$$y_2'(t) = y_1(t) - 2y_2(t) + y_3(t)$$

$$\text{-------------------}$$

$$y_n'(t) = (n-1)\, y_{n-1}(t) - n\, y_n(t) + y_{n+1}(t).$$

$$\text{-------------------}$$

We are not concerned here with solutions acceptable to probabilists (i.e., corresponding to a birth-and-death stochastic process), but with all solutions. If we take $y_1(t) \equiv f(t) \in C^{\infty}$ (a real-valued function having derivatives of all orders, for all $t \in I$), then the system can be solved completely in terms of f(t), by successively computing $y_2(t), y_3(t), \ldots$.

If the polynomials $P_0(\lambda) \equiv 1,\ P_1(\lambda) = \lambda+1, \ldots$ denote the polynomials satisfying the relation

$$B \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \end{bmatrix} = \lambda \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \end{bmatrix}$$

where B is the Kolmogorov-matrix we are considering, then we can write the solution-system symbolically as :

$$y_n(t) = \left[P_{n-1}\left(\frac{d}{dt}\right)\right] f(t) .$$

The solution-system is thus not unique since $f \in C^{\infty}$ is at our choice. In particular, we may specifically require that f and all its derivatives vanish at the origin.

Lecture 2 24th February 1963.

ON THE COMMUTATOR EQUATION IN BANACH ALGEBRAS

1. The Commutator Operator:

The object of this lecture is to discuss some properties of the commutator equation and apply them to some linear differential equations in Banach Algebras (see (3)).

Let $\mathcal{B}$ be a non-commutative Banach algebra with identity element e. Let a be an element of B not belonging to the centre of B, i.e., a does not commute with every element of B.

We define two operations

$$L_a[x] = ax \text{ (left-multiplication by a)}$$

and

$$R_a[x] = xa \text{ (right-multiplication by a)}$$

The operators L_a and R_a are obviously bounded operators and belong to the operator-algebra $\mathcal{E}(\mathcal{B})$ of bounded operators over B.

Let C_a be defined by: $C_a[x] = ax - xa = L_a[x] - R_a[x]$.

Then C_a is the commutator-operator. Being obviously bounded $(\|C_a\| \leq 2\|a\|)$, C_a is alson an element of $\mathcal{E}(\mathcal{B})$.

2. Spectral Relations:

With a is associated its 'Spectrum' $\sigma(a)$, namely, the set of all complex numbers λ for which $(\lambda e - a)$ is singular (has no inverse). In other words, if S denote the algebra of singular elements of B, $\sigma(a) = \{\lambda \mid \lambda e - a \in S\}$. The operators L_a and R_a also have spectra associated with them as members of $\mathcal{E}(\mathcal{B})$, denoted by $\sigma(L_a)$ and $\sigma(R_a)$ respectively.

Then we have

Theorem 1: (i) $\sigma(a) = \sigma(R_a) = \sigma(L_a)$

(ii) $\sigma(C_a) \subseteq \sigma(a) - \sigma(a) = \{\alpha - \beta \mid \alpha, \beta \in \sigma($

Proof: (i) is almost trivial; if $\lambda e - a \in S$, $(\lambda e - a)x = y$ cannot be solved for general values of y, and so $y \in S$, or L_a has the same spectral value.

(ii) is not so simple. This has been proved by Frobenius for the spectrum of matrices. For the general case, we have to use Gelfand's, theory of commutative rings. The operators R_a and L_a commute and we can find a commutative sub-algebra of $\mathcal{E}(\mathcal{B})$ which contains C_a and has the same spectral relations as in $\mathcal{E}(\mathcal{B})$. From Gelfand's theory for this commutative sub-algebra, (ii) will follow for the general case. We omit the details here.

It follows that 0 is a possible eigen-value of C_a. Let $P(a)$ be any polynomial in a. Then, clearly, $C_a[P(a)] = 0$, i.e., 0 is an eigen-value and $P(a)$ is an eigen-function belonging to it. Therefore, 0 belongs to the point-spectrum $P_\sigma[C_a]$. It can be shown that $\sigma[C_a] \neq \{0\}$.

Our next result considers the case when B is a prime ring in the sense of N.Jacobson, i.e., if, for given $x, y \in \mathcal{B}$, $x z y = 0$ for all $z \in \mathcal{B}$, then either x = 0 or y = 0 (The ring of bounded infinite matrices we considered in Lecture 1 of this series is an example of a prime ring).

Theorem 2: Let B be a prime ring. If α and β belong to $P_\sigma(a)$, then $(\alpha - \beta)$ belongs to $P_\sigma[C(a)]$.

Proof: If $\alpha = \beta$, the result is obvious. If $\alpha \neq \beta$, there exist non-null elements x,y such that $ax = \alpha x$ and $ya = \beta y$. B being a prime ring, there exists $z \in B$ such that $xzy \neq 0$. Now

$$a(xzy) - (xzy)a = \alpha(xzy) - (xzy)\beta = (\alpha - \beta)xzy$$

since $xzy \neq 0$, $(\alpha - \beta) \in P_\sigma[C(a)]$.

In the case of matrices, every element of the spectrum of a is an eigen-value so that every value $\alpha - \beta$ is an eigen-value of C_a, and the spectrum of C_a is contained in $\{(\alpha - \beta)\}$. So, it follows from the two theorems above, that every value $(\alpha - \beta)$ is an eigen-value and so, in Theorem 1 (ii), equality holds.

Theorem 3:

The eigen-vectors of C_a are nil-potent elements of B when they do not belong to the eigen-value 0.

We recall that x is 'nil-potent' if there exists a positive integer n such that $x^n = 0$.

Suppose $x \neq 0$ and $\lambda \in P_\sigma[C_a]$, i.e., $C_a[x] = \lambda x$, or $ax - xa = \lambda$

Then, $C_a[x^2] = ax^2 - x^2a = ax^2 - xax + xax - x^2a$

$$= (ax - xa)x + x(ax - xa) = 2\lambda x^2$$

So, $2\lambda \in P_\sigma[C_a]$ and x^2 is an eigen-vector, or $x^2 = 0$

In the latter case, we are through. Otherwise, $x^2 \neq 0$ and we form $ax^3 - x^3a$ and proceed in the same way, getting successively in case x is not nil-potent

$$ax^3 - x^3a = 3\lambda x^3, \text{------}, ax^n - x^na = n\lambda x^n.$$

But, C_a is a bounded operator, and $\|C_a\| \leqq 2\|a\|$, so that all spectral values μ of C_a must satisfy $|\mu| \leqq 2\|a\|$. But, in the above process, we must have $|n\lambda| = n|\lambda| > 2\|a\|$ for all sufficiently large n since $\lambda \neq 0$, and so x must be nil-potent.

In a more or less similar manner, we prove that if $\lambda \in P_\sigma[C_a]$ with x as an eigen-vector and $\mu \in P_\sigma[C_a]$ with y as an eigen-vector, then either $\lambda + \mu \in P_\sigma[C_a]$ or $xy = 0$.

3. Integral Representation of Resolvents:

Next suppose $|\lambda| > 2\|a\|$. We solve the equation $\lambda y - C_a[y] = x$. The solution is called the Resolvent of C_a and is denoted by $R(\lambda; C_a)x$. Yu.L.Daletski (1) has given a formula for the solution from which the resolvent of C_a can be read off.

Let Γ_ε be a contour consisting of one or more closed rectifiable curves such that σ(a) lies in the interior of Γ_ϵ at a distance ε from Γ_ε. Γ_ε bounds a closed region Δ_ε, namely the interior of Γ_ϵ plus its boundary. This region is closed and contains a finite number of maximal components. It is bounded and we may assume that it is contained in the disk

$$|\lambda| < \|a\| + \varepsilon$$

which contains σ (a). Let

$$\Sigma_\epsilon = \{\alpha - \beta \mid \alpha, \beta \in \Delta_\varepsilon\}$$

This is also a closed set with a finite number of components and it is located in the disk

$$|\lambda| < 2(\|a\| + \varepsilon)$$

Finally, let $\Lambda_\varepsilon = C(\Sigma_\varepsilon)$, the complement of Σ_ε. This is an open set, in general not connected, which contains the set

$$|\lambda| > 2(\|a\| + \varepsilon)$$

We now form the double integral

$$\frac{1}{(2\pi i)^2} \int_{\Gamma_\varepsilon} \int_{\Gamma_\varepsilon} \frac{R(\alpha, a)\, x\, R(\beta, a)}{\lambda - \alpha + \beta}\, d\alpha\, d\beta$$

where $\lambda \in \Lambda_\varepsilon$. The integral exists since the denominator is not zero for $\alpha \in \Gamma_\varepsilon$, $\beta \in \Gamma_\varepsilon$, $\lambda \in \Lambda_\varepsilon$.

Theorem 4: For $\lambda \in \Lambda_\varepsilon$,

$$x = \left[1/(2\pi i)^2\right] \int_{\Gamma_\epsilon} \int_{\Gamma_\epsilon} \frac{R(\alpha, a)\, y\, R(\beta, a)}{\lambda - \alpha + \beta}\, d\alpha\, d\beta$$

is the only solution of $\lambda x - C_a[x] = y$.

Proof: The operator $\lambda I - (L_a - R_a)$ can be applied under the integral sign; using

$$(\lambda e - a)\, R(\alpha, a) = e = R(\beta, a)\, (\beta e - a),$$

we have

$$(\lambda I - C_a)\, \frac{1}{(2\pi i)^2} \int_{\Gamma_\epsilon} \int_{\Gamma_\epsilon} \frac{R(\alpha, a)\, y\, R(\beta, a)}{\lambda - \alpha + \beta}\, d\alpha\, d\beta$$

$$= \frac{1}{(2\pi i)^2} \int_{\Gamma_\epsilon} \int_{\Gamma_\epsilon} R(\alpha, a)\, y\, R(\beta, a)\, d\alpha\, d\beta$$

$$+ \int_{\Gamma_\epsilon} R(\beta, a) \left\{ \frac{1}{2\pi i} \int_{\Gamma_\epsilon} \frac{d\alpha}{\lambda - \alpha + \beta} \right\} d\beta$$

$$- \int_{\Gamma_\epsilon} R(\alpha, a) \left\{ \frac{1}{2\pi i} \int_{\Gamma_\epsilon} \frac{d\beta}{\lambda - \alpha + \beta} \right\} d\alpha.$$

The first term on the right reduces to y since

$$(2\pi i)^{-1} \int_{\Gamma_\epsilon} R(\alpha, a)\, d\alpha = e$$

while the last two are zero, since the resolvent is analytic and the Cauchy-theory applies to Banach Algebra-valued functions. The first follows from the fact that, in the neighbourhood of the point at infinity, $R(\alpha, a) = \frac{e}{\alpha} + \frac{a}{\alpha^2} + \frac{a^2}{\alpha^3} + \cdots$ - while the second follows from the fact that the number of times Γ_ε is described in relation to $\lambda + \beta$ and $\lambda - \alpha$ is zero.

Daletsky has generalized this formula in many directions and has also given some applications.

The solvability of an equation of the type

$$\lambda x - C_a [x] = y$$

plays a decisive role in the Frobenius-Schur theory of linear differential equations. This was pointed out for the matrix case, in 1930, by J. A. Lappo-Danilevski.

Consider the equation

$$z(dw/dz) = f(z)\omega.$$

In the matrix case, we take $f(z) = \sum_{k=0}^{\infty} a_k z^k$, where a_k is an nxn matrix. The solution is of the form $\omega(z) = \left(\sum_0^\infty c_k z^k\right) z^{a_0}$. In the general case, we take a_k to be an element of the Banach algebra B. Then we have

$$\sum_1 c_k (a_0 + ke) z^{ke + a_0} = \sum \left(\sum_{j=0}^{k} a_j c_{k-j} \right) z^{ke + a_0}$$

Identifying co-efficients,

$$c_0 a_0 = a_0 c_0$$

$$ke - a_0 c_k - c_k a_0 = \sum_{j=1}^{k} a_j c_{k-j} \quad \text{for all } k \geq 1.$$

Take $a_0 = e$. Then $ke - a_0 c_k - c_k a_0 = [kI - C_{a_0}] a_k$.

In particular, $e - a_0 c_1 - c_1 a_0 = a_1 c_0$. The RHS is known since $a_0 = e$. If no integer belongs to $\sigma(C_{a_0})$ these equations can be solved, and the resulting series gives a solution.

Hille (3) has considered the case where the values α, β of $\sigma(C_0)$ such that $\alpha - \beta = n$ are poles of $R(\lambda, C_0)$ following the Frobenius method. In this connection, the work of S.R.Foguel (2) is of importance. He showed, among other things, that if α and β are poles of $R(\lambda, a)$ of orders μ_1 and μ_2 respectively, then $(\alpha - \beta)$ is a pole of order $\leq \mu_1 + \mu_2 - 1$. If several choices of α and β give the same $(\alpha - \beta)$, then the order does not exceed $\max(\mu_1 + \mu_2 - 1)$. Hille has shown that if B is a prime ring, then the order <u>equals</u> max $(\mu_1 + \mu_2 - 1)$.

4. <u>Unbounded Operators:</u>

Let X be a Banach space, A a linear unbounded operator whose domain and range lie in X. Further let the spectrum $\sigma(A)$ be limited to a horizontal half-strip, say, Σ_1, with $R(\lambda) \leqq \rho$ and $|I_m(\lambda)| \leqq \sigma$ (R and I_m denote the real and imaginary parts).

Let $\delta \| R(\lambda, a) \| \leq M$, when λ lies outside Σ and δ the distance between λ and Σ_1.

Let $\mathcal{E}(X)$ be the Algebra of linear bounded operators of X onto X. For $S \in \mathcal{E}(X)$, we consider the operator equation

$$\lambda T - AT + TA = S.$$

The solution of Daletzky can be applied so that we have the integral

$$\left(\frac{1}{2\pi i}\right)^2 \int_\Gamma \int_\Gamma \frac{R(\alpha, A)\, S\, R(\beta, A)}{\lambda - \alpha + \beta}\, d\alpha\, d\beta.$$

Γ is a curve from $-\infty$ to $-\infty$ in two straight lines. The integral appears to exist; so this leads to a possible solution. We cannot work further with A under the integral sign, since the resulting integral exists atmost as a Cauchy's principal value. Also the denominator $\lambda - \alpha + \beta$ cannot be got rid of as it is needed for convergence properties.

We can possibly apply this to the Heisenberg's equation

$$PQ - QP = (h/2\pi i)\,E.$$

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25th February 1963.

Lecture 3

GREEN'S TRANSFORMS AND THEIR APPLICATIONS TO PROBLEMS CONNECTED WITH ORDINARY LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

Introduction: In studying the solutions of linear second order differential equations (D.E.'s), one can obtain much information directly from the equation without the use of explicit representations of the solutions. This observation goes back to Sturm and Liouville in the 1830's. They used quadratic identities obtained from the equation, to prove oscillation theorems. In their discussions, all the variables entering into the differential equation as well as the parameters were real. Sporadic attempts were made to use such identities also for complex values of the variables or parameters. Thus, for instance, the proof that the Bessel function $J_\alpha(z)$ has only real zeros for $\alpha > 1$ due to P. Schafheitlin (Über die Gaussche und Besselsche Differentialgleichung, Journal für die reine und angewandte Mathematik, Vol.114, 1895), is based on this idea. Likewise, such an identity plays an important role in Hermann Weyl's discussion of singular boundary-value problems (1).

In papers appearing during the period 1918-1927, E. Hille made a systematic study of such identities for complex values of the variables (see, for instance, (2)). If the differential equation is

$$w'' + G(z)w = 0$$

where $G(z)$ is holomorphic in a domain D, then we call the relation

$$(1)\quad \left[\overline{\omega(t)}\,\omega'(t)\right]_{z_0}^{z} - \int_{z_0}^{z} |\omega'(t)|^2\,dt + \int_{z_0}^{z} G(t)\,|\omega(t)|^2\,dt = 0$$

the <u>Green's transform</u> of the equation. Here z_0, z and the path of integration —all are located in D. This identity obtained from the D.E. by multiplying the D.E. by $\overline{w}$, integrating from z_0 to z, and then integrating by parts to get rid of ω''.

In most of E.Hille's early papers, this identity was used to obtain qualitative information concerning the distribution in the complex plane of zeros of the solutions. This technique was used successfully in the study of many equations well-known in Mathematical Physics, such as those of Bessel, Hermite-Weber, Laguerre, Legendre and Mathieu. In a later paper (3), the same identity was used to study the solutions of the equation

$$\omega'' - \lambda F(x)\,\omega = 0$$

where and F are real-valued but λ is complex. Finally, in an address to the Edinburgh Mathematical Congress in 1958, E.Hille used the transform for the study of certain aspects of Weyl's singular boundary-value problem. This address is to appear as (4).

We shall touch upon all the three problems mentioned above in the course of this lecture.

1. <u>An Application of Green's transform to the problem of distribution of zeros:</u>

Consider the equation

$$\omega'' - z\,\omega = 0$$

and a solution $\omega(z)$ such that $\omega(0)\,\omega'(0) = 0$. Here $G(z) \equiv -z$ is an entire function, and a solution $\omega(z)$ such that $\omega(0)\,\omega'(0) = 0$ is either of the form $K_1(z^3)$ of of the form $z K_2(z^3)$, where $K_1(t)$, $K_2(t)$ are power-series in t, according to whether $\omega'(0)=0$ or $\omega(0)=0$. This assertion, as well as the fact that $\omega(0)$ and $\omega'(0)$ cannot simultaneously be zero, is a direct consequence of the D.E. itself.

Let z be a zero of such a solution $\omega(z)$. Taking the simplest path—the straight line—from 0 to z, we have from (1) that, if $|z|=R$, $\arg z=\theta$ (so that $t=re^{i\theta}$, $0\le r\le R$ on the path), then

$$\int_0^R |\omega'(t)|^2 e^{-i\theta}\,dr+\int_0^R re^{i\theta}\,|\omega(t)|^2 e^{i\theta}\,dr=0$$

i.e.

$$\int_0^R |\omega'(t)|^2\,dr+e^{3i\theta}\int_0^R r\,|\omega(t)|^2\,dr=0$$

Since $\omega(t)\not\equiv 0$ on the path, it follows, on equating first the imaginary parts and then the real part of the L.H.S. to zero, that we must have $e^{3i\theta}=-1$, i.e., $\theta=\frac{\pi}{3},\pi$ or $\frac{5\pi}{3}$ $(0\le\theta<2\pi)$

Thus, if $\omega(0)\,\omega'(0)=0$, then all the zeros of $\omega(z)$ lie on one or other of these three rays.

Let us now consider an unrestricted solution $\omega(z)$ of the D.E. i.e., a solution for which $\omega(0)\,\omega'(0)$ is not necessarily zero. If $z_0=x_0+iy_0$ be a zero of $\omega(z)$ in the open second quadrant, then there can be no zero either on the vertical line or on the horizontal line passing through z_0. Furthermore, there can be at most one zero on the positive imaginary axis; if there is one, then there can be no other zero on the whole of the imaginary axis. These assertions can be proved as below: Let z_0 be a zero of $\omega(z)$ such that $x_0\le 0$, $y_0\geqq 0$ $(z_0=x_0+iy_0)$.

Then, if $z_1 = x_0 + iy_1$ be a zero, (1) gives:

(2) $$\int_{y_0}^{y_1} |w'(t)|^2 dy - \int_{y_0}^{y_1} (x_0 + iy)) \, |w(t)|^2 dy = 0 \quad (t = x_0 + iy).$$

Considering the imaginary part alone, we find that there can be no zero z_1, with $y_1 \geqq 0$, $y_1 \neq y_0$. (The same assertion holds, for the same reason, if $x_0 \geqq 0$ also). If we consider the real part, then $x_0 \leqq 0$ implies that there can be no zero z_1 with $y_1 \neq y_0$. Hence, if z_0 is in the open second quadrant, there is no other zero on the vertical through z_0; and if z_0 is on the positive imaginary axis, there is no other zero on the whole of the imaginary axis.

Let next, $z_1 = x_1 + iy_0$ be a Zero (1) gives:

(3) $$\int_{x_0}^{x_1} |w'(t)|^2 dx + \int_{x_0}^{x_1} (x + iy_0) \, |w(t)|^2 dx = 0 \quad (t = x + iy_0).$$

If $y_0 > 0$, this is impossible, as seen from the imaginary part.

Thus, the above assertions have been completely proved.

Next, suppose that z_0 is a zero of ww', with $x_0 \leq 0$, $y_0 > 0$ and $z_1 = x_1 + iy_1$ is a zero of ww', with $x_1 < x_0$ and $y_1 \leq 0$. Integrate along a broken line from $x_0 + iy_0$ to $x_1 + iy_1$, through $x_0 + iy_1$. It turns out that the imaginary part of the LHS of (1) with $z = z_1$ and $G(t) = -t$ is negative both on the vertical part and on the horizontal part of the line of integration. This being impossible, we conclude that all the zeros of ww' to the left of the vertical line $x = x_0$ lie in the strip $0 < y < y_0$.

It may be shown, for instance, by a continuity argument, that there are infinitely many zeros in this strip. The preceding argument shows that the ordinates and the abscissae form decreasing sequences. Moreover, again by a continuity argument, one may show that the zeros of W alternate with those of W'; and it may also be shown that the zeros of WW' from the vertices of an infinite convex polygon and that their ordinates decrease to zero.

2. <u>Applications of Green's transforms to the investigation of the behaviour of solutions of equations of the form</u> $w'' = \lambda F(x)$

Let $F(x)$ be positive and continuous over $[0, \infty)$, while λ is a complex parameter. Using fairly simple arguments, we can prove

<u>Theorem 2.1</u>:

A necessary and sufficient condition for the system

$$W'' = \lambda F(x) W$$

to have a fundamental solution-system of the form

$$W_1(x) = x\left[1 + o(1)\right], \quad W_2(x) = 1 + o(1) \quad \text{as } x \to \infty \tag{4}$$

for some fixed $\lambda \neq 0$ is that $x F(x) \in L(0, \infty)$, the class of all Lebesgue-integrable functions over $(0, \infty)$. If this condition be satisfied, then (4) holds for every λ, and we further have

$$w_1'(x) = 1 + o(1), \quad W_2'(x) = o(1) \quad \text{as } x \to \infty$$

Having thus disposed of the case $x F(x) \in L(0, \infty)$, we now turn to the case: $x F(x) \notin L(0,\infty)$.

Let $\Lambda = \{ \lambda \mid \lambda$ is complex, $\lambda \not\leq 0 \}$. If λ is negative, the solutions of the given D.E. are normally oscillatory, and their behaviour is totally different from that holding for the case $\lambda \in \Lambda$, and is fairly well-known. We will confine our attention below to the case $\lambda \in \Lambda$.

The case $\lambda > 0$: The behaviour for $\lambda > 0$ is also well-known, and sets the pattern for the rest of Λ. Let $W_0(x, \lambda)$, $W_1(x, \lambda)$ be the fundamental solution-system of the given D.E. determined by the initial conditions

$$W_0(0,\lambda) = W_1'(0, \lambda) = 0 ; \; W_0'(0,\lambda) = W_1(0,\lambda) = 1 .$$

For $x > 0$, these solutions are positive, monotone increasing and convex downwards, and $W_K(x, \lambda)/x \to \infty$ as $x \to \infty$ $(K = 0, 1)$. For fixed x, the solutions $W_K(x, \lambda)$ are entire functions of λ of order $\leq \frac{1}{2}$. The formula

(5)
$$W_+(x, \lambda) = W_1(x, \lambda) \int_x^\infty \frac{ds}{[W_1(s, \lambda)]^2}$$

defines a 'sub-dominant' solution of the D.E.; it is positive, monotono decreasing, convex downwards, and tends to zero as $x \to \infty$

The case λ complex, $\mathrm{Im}(\lambda) \neq 0$: The descriptive properties of $W_K(x, \lambda)$ for such λ are more complicated than for $\lambda > 0$, but we can prove the following results, using the Green's transform method to prove assertions (b) and (d):

<u>Theorem 2.2:</u> If $xF(x) \notin L(0,\infty)$, if $\lambda = \mu + i\nu$, $\nu \neq 0$, then for $k = 0, 1$,

(a) $W_k(x,\lambda)$ describes a spiral $S_k(\lambda)$ from k to ∞ in the complex W-plane as x goes from 0 to $+\infty$; and $\frac{1}{\nu} \arg W_k(x,\lambda)$ increases steadily from 0 to $+\infty$ as $x \to +\infty$.

(b) $S_k(\lambda)$ has a positive radius of curature everywhere, and is concave towards the origin;

(c) if $\mu \geqq 0$, $|W_k(x,\lambda)|$ is monotone increasing and $|W_k(x,\lambda)|/x \to \infty$ with x when $\mu > 0$; and

(d) for all $\lambda \in \Lambda$ (so that $\lambda > 0$ is allowed), $|W_k(x,\lambda)|^{-1} \in L_2(1,\infty)$.

<u>Note:</u> The Green's transform of the given D.E. for the interval $(0,x)$ is used in the proof and has the form

$$\overline{w_k(x,\lambda)}\, w_k'(x,\lambda) = \int_0^x |w_k'(s,\lambda)|^2 ds + \lambda \int_0^x F(s)\, |w_k(s,\lambda)|^2 ds.$$

The results of theorem 2.2 hold, at least for all sufficiently large x, for all solutions, with one striking exception. This is the <u>sub-dominant</u> or <u>exceptional</u> solution $W_+(x,\lambda)$, given by relation (5) above (which makes sense for all $\lambda \in \Lambda$ and satisfies the given D.E.). The behaviour of this solution is given by

<u>Theorem 2.3</u>.

If $xF(x) \in L(0,\infty)$, and if λ be such that $\nu = \mathrm{Im}(\lambda) \neq 0$, then

(a) $W_+(x,\lambda)$ describes a spiral $S_+(\lambda)$ in the complex w- plane as x goes from 0 to $+\infty$, the sense of description being opposite to that of $S_1(\lambda)$, and $-\frac{1}{\nu}\arg\left[W_+(x,\lambda)\right]$ increases to $+\infty$ with x;

(b) $S_+(\lambda)$ has a positive radius of curvature and is convex towards the origin;

(c) $|W_+(x,\lambda)|$ is monotone decreasing if $\mu \geqq 0$ and tends to zero if $\mu > 0$, and

(d) for each $\lambda \in \Lambda$ (so that $\lambda > 0$ is allowed), we have that $W_+(x,\lambda)\,W_+'(x,\lambda) \to 0$ as $x \to \infty$, and $F(x)\,|W_+(x,\lambda)|^2$ and $|W_+'(x,\lambda)|^2$ belong to $L(0,\infty)$

<u>Note:</u> The Green's transform we use in the proof of (parts of) Theorem 2.3 is that for the given D.E. over the interval (x,∞), which takes the form:

$$\overline{W_+(x,\lambda)}\;W_+'(x,\lambda) = -\int_x^\infty |w_+'(s,\lambda)|^2\,ds - \lambda\int_x^\infty F(s)\,|W_+(s,\lambda)|^2\,ds$$

3. <u>Green's transforms and singular boundary-value problems:</u>

Consider the D.E.

$$u'' - [\lambda + q(x)]\,u = 0 \tag{6}$$

where $q(x)$ is real-valued and belongs to $C[0,\infty)$; the class of all continuous functions over $[0,\infty)$. . If we impose the conditions

$$\left.\begin{aligned} &u(x,\lambda) \in L_2(0,\infty) \\ &\cos\alpha\, u(0,\lambda) - \sin\alpha\, u'(0,\lambda) = 0 \end{aligned}\right\} \tag{7}$$

on the solution $u(x,\lambda)$, then we raise a so-called 'singular boundary-value problem' for the equation (6). The conditions (7) were introduced by H.Weyl in his dissertation; he proved that such a solution for system (6) exists for any λ with $\operatorname{Im}(\lambda) \neq 0$. If $u_\alpha(x,\lambda)$ be the solution corresponding to such a value of λ, then

$$u_\alpha(x,\lambda) = u_1(x;\alpha,\lambda) + m(\alpha,\lambda)\, u_2(x;\alpha,\lambda)$$

where

$$u_1(0;\alpha,\lambda) = u_2'(0;\alpha,\lambda) = \cos\alpha; \quad -u_1'(0,\alpha,\lambda) = u_2(0;\alpha,\lambda) = \sin\alpha$$

and $m(\alpha,\lambda)$ is a multiplier independent of X. For fixed α and x, the functions $u_K(x;\alpha,\lambda)$; $K=1,2$, are entire functions of λ, of order 1/2. As a function of λ, $m(\alpha,\lambda)$ is holomorphic in $\operatorname{Im}\lambda > 0$ as well as in $\operatorname{Im}\lambda < 0$. Further, $m(\alpha,\bar{\lambda}) = \overline{m(\alpha,\lambda)}$ and $\operatorname{sgn}\{\operatorname{Im}[m(\alpha,\lambda)]\} = -\operatorname{sgn}[\operatorname{Im}\lambda]$.

Normally, $m(\alpha,\lambda)$ in $\operatorname{Im}\lambda > 0$ is not an analytic continuation of $m(\alpha,\lambda)$ in $\operatorname{Im}\lambda < 0$. $m(\alpha,\lambda)$ satisfies the interesting functional relation (satisfied by the tangent-function);

$$\frac{m(\alpha,\lambda) - m(\beta,\lambda)}{1 + m(\alpha,\lambda)\, m(\beta,\lambda)} = \tan(\alpha-\beta).$$

Now, on multiplying the relation

$$u''(x,\lambda) - [\lambda + q(x)]\, u(x,\lambda) = 0$$

by $\overline{u(x,\lambda)}$ and integrating from a to b (a,b real and non-negative), we obtain the Green's transform:

(8) $$\left[\overline{u(x,\lambda)}\, u'(x,\lambda)\right]_a^b - \int_a^b |u'(s,\lambda)|^2\, ds + \int_a^b [\lambda + q(s)]\, |u(s,\lambda)|^2\, ds = 0$$

The imaginary part of this relation gives:

(9) $$\left[\mathrm{Im}\left\{\overline{u(x,\lambda)}\, u'(x,\lambda)\right\}\right]_a^b + \nu \int_a^b |u(s,\lambda)|^2\, ds = 0 \quad (\lambda = \mu + i\nu)$$

This identity has played a basic role in the treatment of singular boundary-value problems, since the days of H. Weyl. We shall see below that the real part can be exploited to advantage in such discussions. Thus, we have

<u>Theorem 3.1</u>: Let $q(x)$ be bounded below, say $q(x) \geq q_0$ for all x in $[0, \infty)$ and let $u(x,\lambda)$ be a solution of (6) belonging to $L_2(0, \infty)$, where $\nu = \mathrm{Im}\, \lambda \neq 0$. Then

(a) $|q(x)|^{\frac{1}{2}}\, u(x,\lambda) \in L_2(0,\infty)$

(b) $u'(x,\lambda) \in L_2(0,\infty)$; and

(c) $\overline{u(x,\lambda)}\, u'(x,\lambda) \to 0$ as $x \to \infty$.

Further, $u(x,\lambda)$ is unique except for an arbitrary multiplicative constant, and the D.E. is of the 'limit-point type at ∞'.

<u>Note:</u> The D.E. (6) will be of the 'limit-point type at ∞' (' if, for a particular complex number l_0, not every solution of the system $u'' - [\lambda + q(x)]\, u = l_0 x$ belongs to $L_2(0,\infty)$.

<u>Proof:</u> Take a = o, b = x in (8). Then, for any solution $u(x,\lambda)$, we have

$$\overline{u(x,\lambda)}\, u'(x,\lambda) = A + \int_0^x |u'(s,\lambda)|^2\, ds + \int_0^x [\lambda + q(s)]\, |u(s,\lambda)|^2\, ds,$$

where $A = \overline{u(0,\lambda)}\, u'(0,\lambda)$. Taking the imaginary parts, we find that

$$\lim_{x\to\infty} \mathrm{Im}\left[\overline{u(x,\lambda)}\, u'(x,\lambda)\right] = \mathrm{Im}\, A + \nu \int_0^\infty |u(s,\lambda)|^2\, ds$$

exists finitely if $u(x, \lambda)$ is in $L_2(0, \infty)$. This, of course, is well-known. The real part gives

$$\frac{1}{2}\frac{d}{dx}|u(x,\lambda)|^2 = Re(A) + \int_0^x |u'(s,\lambda)|^2 ds + \int_0^x [\mu + q(s)]|u(s,\lambda)|^2 ds$$

(10) $$= Re\, A + \int_0^x |u'(s,\lambda)|^2 ds + \int_0^x [q(s) - q_0]|u(s,\lambda)|^2 ds + (\mu + q_0)\int_0^x |u(s,\lambda)|^2 ds$$

The integrands of the first and second integrals on the right one non-negative functions and so each of these two integrals is a monotone non-decreasing function of x, and thus tends to a non-negative limit, finite or infinite, as $x \to \infty$. Also, since $u(x, \lambda) \in L_2(0, \infty)$, the last number on the right tends to a finite limit as $x \to \infty$. Thus, the R.H.S. tends to to a limit l, either finite or $+\infty$, as $x \to \infty$. Then;

$$\lim_{x\to\infty} \frac{d}{dx}|u(x,\lambda)|^2 = 2\ell \qquad (-\infty < \ell \leq +\infty).$$

The requirement $u(x, \lambda) \in L_2(0, \infty)$ implies that ℓ can only be zero. This shows that

$$\int_0^\infty |u'(s,\lambda)|^2 ds \quad \text{and} \quad \int_0^\infty [q(s) - q_0]|u(s,\lambda)|^2 ds$$

exist finitely, by virtue of (10). Hence $\int_0^\infty |q(s)|\,|u(s,\lambda)|^2 ds$

$$\leqq \int_0^\infty [q(s) - q_0]|u(s,\lambda)|^2 ds + |q_0|\int_0^\infty |u(s,\lambda)|^2 ds < \infty.$$

Thus, assertions (a) and (b) of the theorem are proved. Finally, since $u(x, \lambda)$ and $u'(x, \lambda)$ are in $L_2(0, \infty)$, it follows that $\overline{u(x,\lambda)}\, u'(x, \lambda) \in L_1(0, \infty)$ and so

$\lim_{x\to\infty} \overline{u(x,\lambda)}\, u'(x,\lambda)$ --which exists--can only be zero.

Thus, (c) is also proved.

To verify the last assertions, we need only note that, for any α satisfying $\sin\alpha \cos\alpha = 0$, the solution $u_1(x; \alpha, \lambda)$ of the given D.E., satisfying the conditions:

$u_1(0; \alpha, \lambda) = \cos\alpha, \quad u_1'(0, \alpha, \lambda) = -\sin\alpha$ has the property

$$\mathrm{Im}\left[\overline{u_1(x,\alpha,\lambda)}\, u_1'(x,\alpha,\lambda)\right] = \nu \int_0^x |u_1(s,\alpha,\lambda)|^2 ds$$

which certainly does not tend to zero as $x \to \infty$ $(\nu \neq 0)$. Hence, by (c), $u_1(x,\alpha,\lambda) \notin L_2(0,\infty)$. Thus, the D.E. is of the 'limit-point type at ∞', and so $u(x,\alpha,\lambda)$ is unique up to a multiplic tive constant.

Under the same condition on $q(x)$, --that of being bounded below--we c n obtain more information on $m(\alpha,\lambda)$. We have, in fact,

<u>Theorem 3.2</u>: If $q(x) \geq q_0$ for all x in $[0,\infty)$, then $m(\alpha,\lambda)$ is holomorphic in the hlaf-plane $\mathrm{Re}\,\lambda > -q_0$ except possibly for a single simple pole on the real axis. There are no poles at all if (for example) $0 < \alpha < \pi/2$.

<u>Proof:</u> We substitute $u_\alpha(x,\lambda)$ in (8) with $a = 0$ and let $b \to \infty$. By Theorem 3.1 and the initial conditions, we have, for λ with $\mathrm{Im}\,\lambda \neq 0$,

$$\sin\alpha\cos\alpha\left[1 - |m(\alpha,\lambda)|^2\right] - \cos 2\alpha\, \mathrm{Re}\left[m(\alpha,\lambda)\right] - i\,\mathrm{Im}\left[m(\alpha,\lambda)\right]$$

(11) $$= \int_0^\infty |u_\alpha'(s,\lambda)|^2 ds + \int_0^\infty [\lambda + q(s)]\, |u_\alpha(s,\lambda)|^2 ds.$$

In this, we set $\alpha = \pi/4$ and take the real part, obtaining;

(12) $$1 - \left|m\left(\tfrac{\pi}{4},\lambda\right)\right|^2 = 2\int_0^\infty |u_{\frac{\pi}{4}}'(s,\lambda)|^2 ds + 2\int_0^\infty [\mu + q(s)]\, |u_{\frac{\pi}{4}}(s,\lambda)|^2 ds.$$

This shows that

(13) $$\left|m\left(\tfrac{\pi}{4},\lambda\right)\right| < 1 \quad \text{if } \mathrm{Re}(\lambda) \geq -q_0,\ \mathrm{Im}\,\lambda \neq 0.$$

Further,

(14) $$2(\mu+q_0)\int_0^\infty |u_{\frac{\pi}{4}}(s,\lambda)|^2 ds < 1 \quad (\mathrm{Im}\,\lambda \neq 0)$$

We make repeated use of these inequalities. Firstly, from (11), on equating the absolute values of the imaginary parts

$$|\mathrm{Im}(m(\tfrac{\pi}{4},\lambda))| = |\nu|\int_0^\infty |u_{\frac{\pi}{4}}(s,\lambda)|^2 ds < \frac{|\nu|}{2(\mu+q_0)} \quad \text{from (14)}$$

We conclude that

(15) $$\lim_{\nu\to 0} \mathrm{Im}\left[m\left(\tfrac{\pi}{4},\lambda\right)\right] = 0 \quad \text{for } \lambda=\mu+i\nu,\ \mu>-q_0,$$

this limit being uniform in μ for $\mu \geqq -q_0+\epsilon$, for every $\epsilon > 0$.

Secondly, we can compute the value of the integral

(16) $$\int_0^\infty u_\alpha(s,\lambda_1)\, u_\alpha(s,\lambda_2)\, ds = \frac{m(\alpha,\lambda_1)-m(\alpha,\lambda_2)}{\lambda_2-\lambda_1}$$

by the classical method of Sturm. In particular,

(17) $$\int_0^\infty [u_\alpha(s,\lambda)]^2 ds = -\frac{\partial}{\partial\lambda} m(\alpha,\lambda).$$

Combining this formula with (14), we get

(18) $$2(\mu+q_0)\left|\frac{\partial}{\partial\lambda} m\left(\tfrac{\pi}{4},\lambda\right)\right| < 1 \quad (\mathrm{Im}\,\lambda\neq 0).$$

It follows from (13) and (18) that the set of functions of μ: $\{m(\frac{\pi}{4},\mu+i\nu),\ \nu>0\}$ are uniformly bounded and equi-continuous in any interval of the form $[-q_0+\epsilon,\infty)$, for every $\epsilon > 0$. Hence, by Ascoli's lemma, for any interval $[-q_0+\epsilon, A]$, there exists a sequence $\{\nu_n\}$ (in general, depending on ϵ and A) with every $\nu_n > 0$, such that $\{m(\frac{\pi}{4},\mu+i\nu_n)\}$ converge uniformly to a function of μ on $[-q_0+\epsilon, A]$, which is therefore also bounded and continuous there. Denoting this function by $m_{\epsilon,A}\left(\frac{\pi}{4},\mu\right)$, we have from (15) that it should be real-valued on $[-q_0+\epsilon, A]$, and from (18) again that

$$\left| m\left(\frac{\pi}{4}, \mu+i\nu\right) - m_{\epsilon,A}\left(\frac{\pi}{4},\mu\right)\right| < \frac{\nu}{2(\mu+q_0)}$$

Hence,

$$\lim_{\nu\to 0+} m\left(\frac{\pi}{4}, \mu+i\nu\right) \equiv m_{\epsilon,A}\left(\frac{\pi}{4},\mu\right), \text{ say},$$

exists as a real-valued continuous function of μ in $[-q_0+\epsilon, A]$. Since A and $\epsilon > 0$ are arbitrary, it follows that

$$\text{(19)} \qquad \lim_{\nu\to 0+} m\left(\frac{\pi}{4}, \mu+i\nu\right) \equiv m\left(\frac{\pi}{4},\mu\right), \text{ say}$$

exists as a real-valued continuous function for all $\mu > -q_0$. It then follows that further $\left|m\left(\frac{\pi}{4},\mu\right)\right| \leqq 1$ for such μ.

So far, we have assumed that $\nu > 0$; but the relation $m(\alpha, \bar{\lambda}) = \overline{m(\alpha, \lambda)}$ shows that the same result is obtained if we let $\nu \to 0$ from below. The relation $\lim_{\nu\to 0} m(\pi/4, \mu+i\nu) = m(\pi/4,\mu)$ is a continuous function of μ for $\mu > -q_0$; the relation $m\left(\frac{\pi}{4}, \bar{\lambda}\right) = \overline{m\left(\frac{\pi}{4}, \lambda\right)}$ and the fact that $m\left(\frac{\pi}{4}, \lambda\right)$ is analytic in $\operatorname{Im}\lambda > 0$ and in $\operatorname{Im}\lambda < 0$, ensure, by Schwarz's symmetry Principle, that $m\left(\frac{\pi}{4}, \lambda\right)$ is holomosphic also on the real axis, for $\mu > -q_0$; and, further, $\left|m\left(\frac{\pi}{4}, \lambda\right)\right| \leqq 1$ for $\operatorname{Re}\lambda > -q_0$

By virtue of (19),

$$\lim_{\nu\to 0+} u_{\pi/4}(x,\mu+i\nu) = u_1(x,\tfrac{\pi}{4},\mu) + m(\tfrac{\pi}{4},\mu)\,u_2(x,\tfrac{\pi}{4},\mu)$$

exists as a uniform limit over any interval $(0,W)$, $W<\infty$, $\mu \geq -q_0+\epsilon$.

Hence

$$\lim_{\nu\to 0+}\int_0^W |u_{\pi/4}(x,\mu+i\nu)|^2\,dx \text{ exists and } = \int_0^W |u_{\pi/4}(x,\mu)|^2\,dx.$$

We then have, by (14), that

$$\int_0^W |u_{\pi/4}(x,\mu)|^2\,dx \leq \frac{1}{2(\mu+q_0)}.$$

This holds for every W and so $u_{\pi/4}(x,\mu) \in L_2(0,\infty)$. Further, (12), (16), (17) and (18) carry over for real values of $\lambda > -q_0$ also. In particular, $m(\frac{\pi}{4},\mu)$ is a monotone decreasing function of μ in $[-q_0,\infty)$; using the relation

$$\frac{m(\alpha,\lambda)-m(\beta,\lambda)}{1+m(\alpha,\lambda)\,m(\beta,\lambda)} = \tan(\alpha-\beta),$$

valid for $\operatorname{Im}\lambda \neq 0$, we can now extend these results to arbitrary α (assumed positive, for definiteness). We have

$$m\left(\frac{3\pi}{4},\lambda\right) = -\left[m\left(\frac{\pi}{4},\lambda\right)\right]^{-1}$$

and, if $\alpha \not\equiv \frac{3\pi}{4} \pmod{\pi}$, then

$$m(\alpha,\lambda) = \frac{m(\frac{\pi}{4},\lambda)-\tan(\alpha-\pi/4)}{1+m(\frac{\pi}{4},\lambda)\tan(\alpha-\pi/4)}$$

These relations show that $m(\alpha,\lambda)$ can be extended to the real axis for $\mu > -q_0$ and is holomorphic there except possibly for poles: the poles can occur only at points λ_0 such that

$m\left(\frac{\pi}{4}, \lambda_0\right) = \cot\left(\frac{\pi}{4} - \alpha\right)$. Since $m\left(\frac{\pi}{4}, \lambda\right)$ is decreasing in $(-q_0, \infty)$, there can be at most one such point λ_0. Further, since $\left|m\left(\frac{\pi}{4}, \lambda\right)\right| \leq 1$, $m(\alpha, \lambda)$ cannot have a pole in $(-q_0, \infty)$ if $0 < \alpha < \pi/2$. This completes the proof.

Equating the imaginary parts of the two sides of (11), and setting $\lambda = i\nu$ (λ is purely imaginary), we can prove (with the help of a few auxiliary results and arguments)

Theorem 3.3: If $q(x) \in C[0, \infty)$ and if $\alpha \not\equiv 0 \pmod{\pi}$, then

$$R(\nu) = \sqrt{\nu}\,\left|m(\alpha, i\nu) + \cot\alpha\right|$$

is bounded away from zero and infinity, as $\nu \to \infty$; for $\alpha = 0$, the same assertion is true of

$$S(\nu) = \nu^{-1/2}\left|m(0, i\nu)\right|$$

Remark:

Analogous results hold even in certain other cases, such as, for instance, the case $\left|\arg\lambda \pm \frac{\pi}{2}\right| \leq \frac{\pi}{2} - \epsilon$ ($\epsilon > 0$ arbitrary).

The proof of Theorem 3.3 consists in showing that $R(\nu)$ (respectively, $S(\nu)$) lies between the roots of a certain quadratic equation, the limiting form of which, as $\nu \to \infty$, has (strictly) positive roots.

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