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MATSCIENCE REPORT 6

LECTURES ON

THE MANDELSTAM REPRESENTATION

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(6)

LECTURES ON THE MANDELSTAM REPRESENTATION.

I. "Lectures on Dispersion Theory"

By

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II. "Lectures on the Strip Approximation"

By

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PART I

LECTURES ON DISPERSION THEORY.

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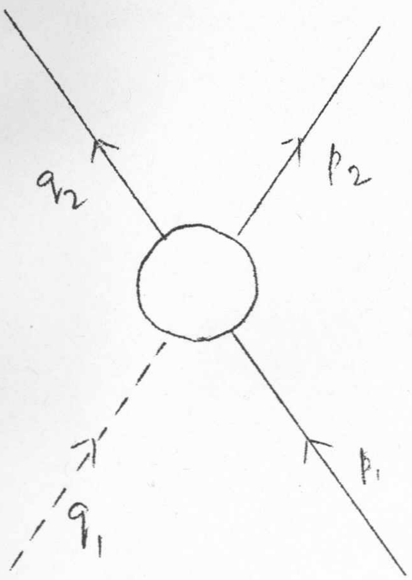
No consistent theory exists for strong interactions based on a field theoretic approach using the concept of a Lagrargian or Hamiltonian. Therefore an abstract approach based on general principles like Lorentz invariance, causality, and unitarity deduced from field theories ~~are~~ to utilized and the mathematical implications are studied. All fields introduced are already renormalized. All divergences arising from splitting the Hamiltonian into unperturbed part plus a perturbation are avoided. No infinities should arise in this theory. (Abstract approach). Whether these can be applied to any physical process completely or not is not yet known. However these principles help us to prove dispersion relations by establishing analyticity properties of matrix elements. This approach leads to relations between experimentally measurable quantities and the matrix elements. Perturbation theory has been extremely helpful in determining the location of poles using the Feynman diagrams of the scattering process. A simple pole occurs when the conservation laws permit a single particle intermediate state and a branch point where production amplitudes compete with scattering amplitudes and real particles can occur in the intermediate states. Chew et al have studied scattering amplitudes for fixed momentum transfer. However a representation for arbitrary energy and momentum transfer has been given by Mandelstam using the theory of two complex variables. The calculation based on the Mandelstam representation done using the unitarity condition enables one to write a complete dynamical description of the strong interaction scattering. processes

First we shall take up the fixed momentum transfer dispersion relations for the case of pion nucleon scattering. A partial wave analysis is made using the dominance of the P_{33} resonance. The S, D, and small P wave phase shifts are determined using the P33 phase shifts.

KINAMATICS



$$p_1 + q_1 = p_2 + q_2$$
$$p = \frac{1}{2}(p_1 + p_2)$$
$$Q = \frac{1}{2}(q_1 + q_2)$$
$$K^2 = \frac{1}{2}(q_1 - q_2)^2 = \frac{1}{2}|\vec{q}|^2(1-\cos\theta)$$
(c.m s)

(1)

The two independent scalars are chosen to be

$$\nu = \frac{P \cdot Q}{M} \qquad \text{and} \qquad K^2$$

and $\nu = \nu_L - \dfrac{K^2}{M}$, $\quad s = M^2 + 1 + 2M\nu_L$

ν_L = meson energy. in lab. system.

Further invariants can be formed with Dirac matrices.

But $(i\gamma \cdot p_1 + M)u_1 = (i\gamma \cdot p_2 + M)u_2 = 0$

and $i\gamma.K = i\gamma.\frac{(p_2 - p_1)}{2}$ so that $i\gamma.Q$ is the only other invariant which can be formed.

Thus

$$T = -A + i\gamma.Q.B. \tag{2}$$

where A and B are functions of ν and R^2. But they are matrices in isospin space.

$$\langle p'|T|p\rangle = u_\alpha^{(\gamma)}(p')\left[-A + i\gamma.Q.B\right]u_\beta^{\delta}(p) \tag{3}$$

Now we will express A and B in terms of the scattering amplitudes of definite angular momentum and parity.

We use the explicit representation for the spinors $u_\beta^{(\delta)}(p)$

$$u_\beta^{(\delta)}(p) = \frac{(M - i\gamma p)_{\beta\delta}}{\sqrt{2M(M+E)}} = \frac{(M - i\rho_2\vec{\sigma}.\vec{p} + \rho_3 E)_{\beta\delta}}{\sqrt{2M(M+E)}}$$

$$u_\alpha^{r}(p') = \frac{M - i\rho_2\vec{\sigma}.\vec{p_1} + \rho_3 E'}{\sqrt{2M(M+E)}}$$

$$\rho_2 = 0$$
$$\rho_2^2 = 1 \tag{4}$$
$$\rho_3 = 1, \quad \rho_3^2 = 1$$

In the centre of mass frame

$$p = (-\vec{q}, E); \quad p' = (-\vec{q}', E) \tag{5}$$

$$\bar{u}(p')u(p) = \frac{1}{2M(M+E)}(M + i\rho_2\vec{\sigma}.\vec{q} + \rho_3 E)(M + i\rho_2\sigma.q + \rho_3 E) \tag{\cancel{5}}$$

$$= \frac{M+E}{2M} - \frac{\sigma \cdot q' \sigma \cdot q}{2M(M+E)} \quad \text{. . . . . (6)}$$

and $\bar{u}(p') i \gamma \cdot Q u(p)$

$$= -\frac{(E+M)(W-M)}{2M} + \frac{-M+W}{2M(M+E)} \sigma q' \cdot \sigma \cdot q \qquad (7)$$

so that

$$T = -\left(\frac{E+M}{2M}\right)(A + \overline{W-M}\, B) + \frac{E-M}{2M}(A - \overline{W+M}\, B)$$

$$(\sigma \cdot q')(\sigma \cdot q) \qquad (8)$$

T has to be computed between $\bar{u}(p')$ and $u(p)$

If $F(W, x) = f_1(W, x) + \sigma \cdot q' \sigma \cdot q\, f_2(W, x) \qquad (9)$

where $x = \cos\theta = q \cdot q'$

the partial wave expansion can be written as

$$F(W,x) = \sum_{l=0}^{\infty} (2l+1)\left[f_{l-}(W) R_{j=l-\frac{1}{2}} + f_{l+}(W) R_{j=l+\frac{1}{2}} \right] P_l(x)$$

(10)

where

$$R_{l-} = \frac{l - \sigma \cdot L}{2l+1} \quad \text{and} \quad R_{l+} = \frac{l+1+\sigma \cdot L}{(2l+1)}$$

$$j(j+1) = l(l+1) + 3/4 + \sigma \cdot L$$

$$F(W, x) = \sum_0^\infty \left[l f_{l-} + (l+1) f_{l+} \right] P_l(x) + (f_{l+} - f_{l-})\, \sigma \cdot L \cdot P_l(x) \qquad (11)$$

With the help of the relations

$$\sigma \cdot L\, P_l(x) = (x - \sigma \cdot q' \sigma \cdot q) P_l'(x); \quad x P_l'(x) = l P_l(x) + P_{l-1}'(x) \qquad (12)$$

$$(l+1) P_l(x) = -x P_l'(x) + P_{l+1}'(x)$$

we get

or
$$F(W, x) = \sum_{l=0}^\infty f_{l+} P_{l+1}'(x) - f_{l-} P_{l-1}'(x)$$

$$+ \sigma \cdot q' \sigma \cdot q \sum_{l=0}^\infty (f_{l-} - f_{l+}) P_l'(x) \qquad (13)$$

where $f_{l\pm}$ are the scattering amplitudes in the state of parity $-(-1)^l$ and total angular momentum $j = l \pm 1/2$ and are of the form

$$f_{l\pm} = \frac{e^{i\delta_{l\pm}(W)} \sin \delta_{l\pm}}{q} \qquad (14)$$

δ is real below the two meson threshold.

The $f_{\ell\pm}$ are normalized so that

$$(j+\tfrac{1}{2}) I_m f_{\ell\pm} = \frac{q}{4\pi}\,\sigma_{\ell\pm} \tag{15}$$

is the total cross-section of the partial wave involved.

If $$A^+ = \frac{1}{3}\left(A^{1/2} + 2A^{3/2}\right)$$

$$A^- = \frac{1}{2}\left(A^{1/2} - A^{3/2}\right)$$

(16)

$$\frac{1}{4\pi} A^{\pm} = \frac{W+M}{E+M} f_1^{\pm} - \frac{W-M}{E-M} f_2^{\pm}$$

$$\frac{1}{4\pi} B^{\pm} = \frac{1}{E+M} f_1^{\pm} + \frac{1}{E-M} f_2^{\pm}$$

(17)

(+) (-) refer to the isotopic symmetric and antisymmetric parts respectively.

DISPERSION RELATIONS

In what follows a drastic assumption of no subtraction is made about the high frequency zero as $\nu \longrightarrow \infty$ so that all the integrals will converge.

Then
$$R_e A^{\pm}(\nu, k^2) = \frac{P}{\pi}\int_{1-\frac{k^2}{M}}^{\infty} d\nu' I_m A^{\pm}(\nu', k^2)\left[\frac{1}{\nu'-\nu} \pm \frac{1}{\nu'+\nu}\right]$$

(18)

$$\mathrm{Re}\, B^{\pm}(\nu, K^2) = \frac{g_r^2}{2M}\left(\frac{1}{\nu_B - \nu} \mp \frac{1}{\nu_B + \nu}\right)$$

$$+ \frac{P}{\pi}\int_{1-\frac{K^2}{M}}^{\infty} d\nu' \,\mathrm{Im}\, B^{\pm}(\nu', K^2)\left[\frac{1}{\nu' - \nu} \mp \frac{1}{\nu' + \nu}\right] \tag{19}$$

where
$$\nu_B = -\frac{1}{2M} - \frac{K^2}{M}$$

Within the integral we assume that only the P_{33} amplitude contributes because of the resonance. Also the nuclear velocity is assumed to be very small.

Retaining $l = 2$ but no higher values we have

$$f_1 = f_S - f_{D3/2} + 3 f_{P3/2}\left(1 - \frac{2K^2}{q^2}\right) + \frac{1}{2} f_{5/2}\left[15\left(1 - \frac{2K^2}{q^2}\right)^2 - 3\right] + \cdots \tag{20}$$

and

$$f_2 = f_{P1/2} - f_{P3/2} + 3\left(1 - \frac{2K^2}{q^2}\right)\left(f_{D3/2} - f_{D5/2}\right) \tag{21}$$

If we set $K^2 = 0$

$$f_1(0) = f_S + 3 f_{P3/2} \qquad \text{since } f_0 \ll f_S \tag{22}$$

Differentiating with respect to K^2 and equating $K^2 = 0$

$$f_1'(0) = -6 P_{3/2}/q^2 \tag{23}$$

$$\therefore f_S = f_1(0) + \tfrac{1}{2} q^2 f_1'(0) + \qquad + \text{D waves} \tag{24}$$

Similarly for $f_{3/2}$, $f_{P/2}$ etc.

On the right hand side of the dispersion relations, we write

$$\frac{1}{4\pi}\,\mathrm{Im}\,A^{\pm} = \left[\frac{3(W'+M)\left(1-\frac{2K^2}{q^2}\right)}{E'+M} + \frac{W'-M}{E'-M}\right]\mathrm{Im}\,f_3^{\pm} \tag{25}$$

So the equation reduces to (neglecting terms $O\left(\frac{1}{M^2}\right)$).

$$\frac{1}{4\pi}\,\mathrm{Im}\,B^{\pm} = \left[\frac{3\left(1-2K^2/q^2\right)}{E'+M} - \frac{1}{E'-M}\right]\mathrm{Im}\,f_3^{\pm} \tag{26}$$

$$\mathrm{Re}\,f_S = f_1^{\pm}(0) + \tfrac{1}{2} q^2 f_1^{(\pm)'}(0)$$

$$= \frac{M}{W}\left\{\frac{-q^2}{2M}\left[\left(1-\frac{\omega}{2M}\right) \pm \left(1+\frac{\omega}{2M}\right)\right] + \frac{2M}{\pi}\int_1^{\infty}\frac{d\omega'}{q'^2}\left[\left(1+\frac{2\omega'}{M}+\frac{\omega}{M}\right) \pm \left(1+\frac{2\omega'}{M}-\frac{\omega}{M}\right)\right]\mathrm{Im}\,f^{(\pm)}(\omega')\right\}$$

The strong energy dependence of f_3 produces no reflection in the S wave energy dependence which to a high degree of approximation. is given by constant K for f_+ and $K\omega$ for f_-

The S-wave contribution can be calculated by writing the equation as

$$\left[f_S = \frac{A}{4\pi}\left(K^2 = 0\right)\right]$$

$$\mathrm{Re}f_s = \left\{ \left[\lambda^+ - \frac{\omega}{2M} \lambda^- \right] \pm \left[\lambda^+ + \frac{\omega}{2M} \lambda^- \right] \right\}$$

$$+ \frac{P}{\pi} \int d\omega' \, \mathrm{Im} f_3^{\pm}(\omega') \left[\frac{1}{\omega' - \omega} \pm \frac{1}{\omega' + \omega} \right] \tag{28}$$

$$\omega = W - M$$

$$\lambda^+ = \frac{g^2}{2M} - \frac{2M}{\pi} \int \frac{d\omega'}{q'^2} \left(1 + \frac{2\omega'}{M}\right) \mathrm{Im} f_3^+(\omega') \tag{29}$$

$$\lambda^- = \frac{g^2}{2M} + \frac{4M}{\pi} \int \frac{d\omega'}{q'^2} \mathrm{Im} f_3^-(\omega') \tag{30}$$

<u>Oehme Equations</u>

From the threshold values for $f^{3/2}$ when $f^{1/2}$ waves do not contribute $\lambda^+ = .01$, $\lambda^- = .6$

Similarly for P waves we have

$$\mathrm{Re}f_{11} = -\frac{8}{3} \frac{f^2 q^2}{\omega^2} + \frac{3}{M} f^2 q^2 + \frac{16}{9} \frac{q^2}{\pi} \int \frac{d\omega'}{q'^2} \frac{\mathrm{Im} f_{33}(\omega')}{\omega' + \omega} \tag{31}$$

$$\mathrm{Re}f_{13} = \mathrm{Re}f_{31} = \frac{1}{4} \mathrm{Re}f_{11} - \frac{3}{4M} f^2 q^2 \tag{32}$$

$$Re f_{33} = \frac{4}{3}\frac{f^2 q^2}{\omega} + \frac{q^2}{\pi} P \int \frac{d\omega'}{q'^2} \, Im f_{33}(\omega') \left[\frac{1}{\omega' - \omega} + \frac{1}{M} + \frac{1}{q}\frac{1}{\omega' + \omega} \right] \tag{33}$$

which is a 'derivation' of the static theory.

If we consider $\frac{1}{q} \ll 1$ and $\frac{1}{M} \ll 1$ we have the solution for equation

$$f_{33} = \frac{\Gamma}{\omega} + \frac{1}{\pi} \int_{m+1}^{\infty} dW' \, \frac{Im f_{33}(W')}{W' - W} \tag{34}$$

by setting $f_{33} = N/D$ and requiring N(W) to contain the pole and $D(W)$ the cut (physical) and with one subtraction.

The solution is

$$f_{33} = \frac{\Gamma/\omega}{1 - \frac{\omega}{\omega_r}} \tag{35}$$

(Chew-Low effective range formula)

The equation does not determine the resonance.

Summarizing we get

1) The P wave phase shifts satisfy the effective range formula and $f_{13} = f_{31}$ as in static theory.

2) The S wave amplitude is given by $\lambda_{\pm}$ and the strong energy dependence of the (3,3) state has no reflection in the S wave energy dependence.

3) It is possible to calculate the D wave phase shifts.

Lecture II

A relativistic analogy of the Chew-Low method is conjectured and the dispersion relations are written down for arbitrary energy-momentum transfers using the theory of two complex variables. The two invariant scalars are

$$t = -(q_1 - q_2)^2$$

$$s = -(p_1 + q_1)^2$$

and $$\bar{s} = (2m^2 + 2\mu^2) - s - t$$

where $$\bar{s} = -(p_1 - q_2)^2$$

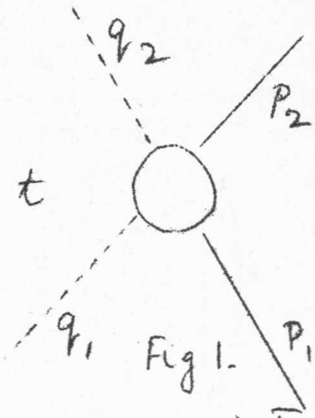


Fig 1.

The Green's function relevant to this process $\pi_1 + N_1 \rightarrow \pi_2 + N_2$ also gives the processes

$$\pi_2 + N_1 \rightarrow \pi_1 + N_2 \qquad (2)$$

$$N_1 + \overline{N_2} \rightarrow \pi_1 + \pi_2$$

The (energy variables)^2 for the three processes are s, s_c and t respectively. The other two variables will be (momentum transfer)^2 for each process.

$$\nu = \nu_L - t/4M \qquad (3)$$

The kinematics for the three reactions are represented by the diagram (Fig.2)

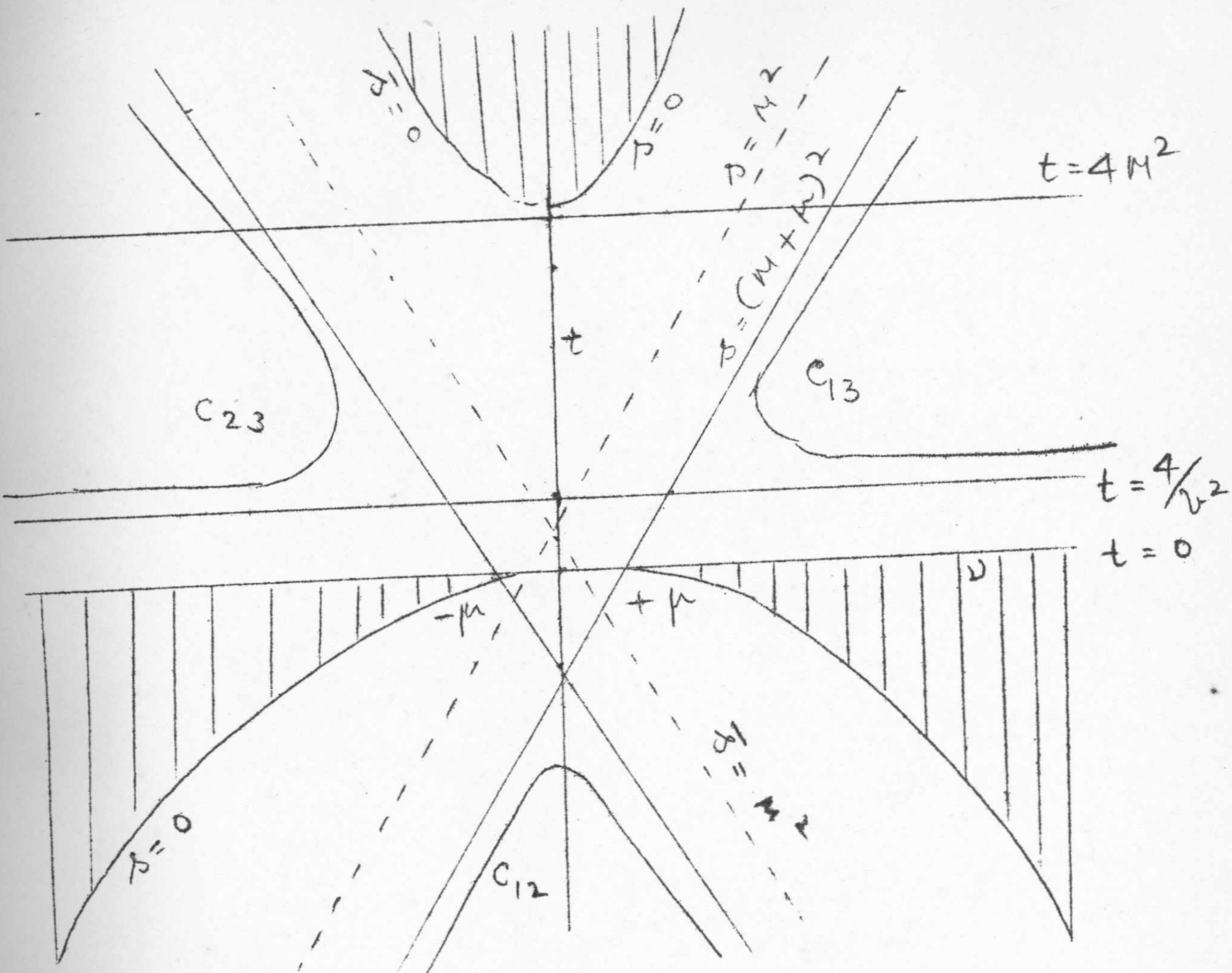


Fig.2

Let us consider the centre of mass for process I.

$$\pi_1 + N_1 \rightarrow \pi_2 + N_2 \qquad (4)$$

Then $$s = (E + \omega)^2 = W^2$$

$$t = -(\vec{q} - q')^2 = -2q^2(1 - \cos\theta_x) \qquad (5)$$

In the physical region for this process

$$s \geqslant (M+\mu)^2 \;, \quad |\cos\theta_I| \leq 1 \tag{6}$$

Now using

$$q^2 = \frac{[s-(M+\mu)^2][s-(M-\mu)^2]}{4s} \tag{7}$$

We get

$$0 \geqslant t \geqslant -s + 2(M^2+\mu^2) - \frac{(M^2-\mu^2)^2}{s}$$

(i.e.)

$$t \leq 0 \;; \quad \bar{s} \leq \frac{(M^2-\mu^2)^2}{s} \leq (M-\mu)^2 \,, \{\nu+\mu\} \tag{8}$$

For process II similarly we have

$$\bar{s} > (M+\mu)^2 \;; \quad t = 0 \text{ for } s = 0 \text{ and } s \leq (M-\mu)^2$$

$$0 \geqslant t \geqslant 2(M^2+\mu^2) - s - \frac{(M^2-\mu^2)}{s} \tag{9}$$

For process III

$$q = (\vec{q}, \omega) \qquad p' = (-\vec{p}, E)$$

$$-q' = (-\vec{q}_1\, \omega) \qquad -p = (\vec{p}, E)$$

$$\omega\sqrt{q^2+\mu^2} = E = \sqrt{p^2+m^2}$$

$$s = -p^2 - q^2 + 2pq\cos\theta$$

$$\bar{s} = -p^2 - q^2 - 2pq\cos\theta \;, \quad t = (2\omega)^2$$

(10)

The first possible intermediate state is a 2π state $t = 4\mu^2$ but it is not physical because $\vec{p}$ will not be real. The physical region starts at

$$t \geq 4M^2, \quad s \leq 0, \quad \bar{s} \leq 0, \quad t \leq 2(M^2 + m^2) - s - \frac{(M^2 - m^2)^2}{s} \tag{11}$$

The Analytic properties of the amplitude $A(\nu, t)$

A as a function of ν for fixed t satisfies the dispersion relation

$$A(\nu, t) = \frac{g^2}{4M}\left(\frac{1}{\nu_B - \nu} + \frac{1}{\nu_B + \nu}\right) + \frac{1}{\pi}\int_{(\mu + t/4M}^{} d\nu' \, \frac{A_1(\nu, t)}{\nu' - \nu} - \frac{1}{\pi}\int_{-\infty}^{-\mu - t/4M} d\nu' \, \frac{A_2(\nu', t)}{(\nu' - \nu)} \tag{12}$$

where $\nu_B = \frac{-\mu^2}{2M} + \frac{t}{4M}$

A_1 and A_2 are the "absorptive parts" associated with the reactions I and II respectively and are given by

$$(2\pi)^4 A_1(\nu, t)\, \delta(p_1 + q_1 - p_2 - q_2) = +(2\pi)^6 \, 4 (P_{01} P_{02} q_{01} q_{02}) \sum_n \langle N(p_1)\pi(q_1)|n\rangle\langle n|N(p_2)\pi(q_2)\rangle \tag{13}$$

$$(2\pi)^4 A_2(\nu,t)\,\delta(p_1+q_1-p_2-q_2) = (2\pi)^6\left(\frac{4P_{01}P_{02}q_{01}q_{02}}{M^2}\right)^{1/2} \sum_n \langle N(P_1)\pi(-q_2)|n\rangle\langle n|N(P_2)\pi(-q_1)\rangle \qquad (14)$$

A_1 and A_2 are nonzero to the right of AB and left of CD respectively. Eqn.(12) states that A is an analytic function with poles at $\pm\nu_B$ and cuts along the real axis from $(\mu + t/4M)$ to ∞ and $-\infty$ to $-\mu - t/4M$

The equation is represented (in this diagram) by an integration along a horizontal line below the ν axis. The poles occur at $s = \mu^2$ and $\bar{s} = M^2$ Apart from then the integrand will be zero between AB and CD (up to $t = -4\mu M$) The equation is valid only if the functions A, A_1 and A_2 tend to zero sufficiently rapidly as $\nu \to \infty$. Otherwise we will have to make one or more subtractions.

Similarly A in terms of A_2 and A_3 or A_1 and A_3 can be written down

$$A(\bar{s},t) = \frac{g^2}{\bar{s}+t-M^2-2\mu^2} - \frac{1}{\pi}\int_{\infty}^{(M-\mu)^2-\bar{s}} dt'\,\frac{A_1(s_c,t')}{t'-t} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt'\,\frac{A_3(s_c,t)}{t'-t} \qquad (15)$$

$$A(s,t) = \frac{g^2}{s - t + M^2 - 2\mu^2} - \frac{1}{\pi}\int_{-\infty}^{(M+6\mu)^2 - s} dt' \, \frac{A_2(s,t)}{t' - t} + \int_{4\mu^2}^{\infty} dt' \, \frac{A_3(s,t)}{t' - t} \tag{16}$$

A_3 is non-zero above the line

Now if A be considered as a function of two complex variables, we can make the simplest assumption that A is analytic in the entire space of the two variables except for the cuts along certain hyper planes. A if considered as a function of the three variables s, $\bar{s}$, t independently will have to be represented in a hyper space. Since there is a relation connecting these three (i.e.) $s + \bar{s} + t = 2(M^2 + \mu^2)$ A has to be described in a hyper plane. The cuts of A are a cut when s is real and $> (M+\mu)^2$ a cut when $\bar{s}$ is real and $> (M+\mu)^2$ and when t is real and $> 4\mu^2$. The discontinuities across these cut will be $2A_1$, $2A_2$ and $2A_3$ respectively. In addition A will have poles when $s = M^2$ and when $\bar{s} = M^2$. By Cauch theorem it can be shown that A can be written as

$$A = \frac{g^2}{M^2 - s} + \frac{g^2}{M^2 - s_c} + \frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} ds' \int_{4\mu^2}^{\infty} dt' \, \frac{A_{13}(s', t')}{(s' - s)(t' - t)}$$

$$+ \frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} d\bar{s}' \int_{4\mu^2}^{\infty} dt' \, \frac{A_{23}(\bar{s}', t')}{(\bar{s} - \bar{s}')(t' - t)} .$$

$$+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} ds' \int_{(M+\mu)^2}^{\infty} d\bar{s}' \, \frac{A_{12}(s', \bar{s}')}{(\bar{s}'-\bar{s})(s'-s)}$$

(17)

A_{13} and A_{23} and A_{12} are nonzero in the regions indicated in the figure. They are called the double spectral functions. The precise boundaries of the double spectral functions will be determined by unitarity. These spectral functions are always zero in the physical region.

The single variable dispersion relations can be immediately derived from (17) writing

$$\frac{1}{(\bar{s}_1-\bar{s})(s'-s)} = \left[\frac{1}{(s'-s)} + \frac{1}{(\bar{s}'-\bar{s})}\right] \frac{1}{s'+\bar{s}'-s-\bar{s}}$$

$$= \frac{1}{(s'-s)(t'-t)} + \frac{1}{(\bar{s}'-\bar{s})(t'-t)} \qquad (18)$$

Hence the third integral becomes

$$-\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} ds' \int_{\infty}^{t_2(s)} dt' \, \frac{A_{12}(s', t')}{(s'-s)(t'-t)}$$

$$-\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} d\bar{s} \int_{-\infty}^{t_2(\bar{s})} dt' \, \frac{A_{12}(\bar{s}, t')}{(\bar{s}'-\bar{s})(t'-t)} \, .$$

When we make change of variables, we should note that A has the same value at the same point and it does not mean that we must take the same function of the new variables.

Writing

$$A_1(s,t) = \frac{1}{\pi}\int_{t_1(s)}^{\infty} dt' \frac{A_{13}(s,t')}{t'-t} - \frac{1}{\pi}\int_{-\infty}^{t_2(s)} dt' \frac{A_{12}(s,t')}{t'-t}$$

$$A_2(\bar{s},t) = \frac{1}{\pi}\int_{t_1(\bar{s})}^{\infty} dt' \frac{A_{23}(\bar{s},t')}{t'-t} - \frac{1}{\pi}\int_{-\infty}^{t_2(\bar{s})-\infty} dt' - \frac{A_{12}(\bar{s},t')}{t'-t} \tag{20}$$

it follows that

$$A = \frac{g^2}{M^2-s} + \frac{g^2}{M^2-\bar{s}} + \frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} ds' \frac{A_1(s',t)}{s'-s} + \frac{1}{\pi^2}\int_{(M+\mu)^-}^{\infty} ds' \frac{A_1(s',t)}{s'-s} \tag{21}$$

t being kept constant.

Thus we have dispersion relations for A_1 and A_2 also in t with s or $\bar{s}$ constant. The equation for A_1 can be represented by an integration along a line parallel to AB and t_1 and

and t_1 and t_2 are the points where this line intersects C_{12} and C_{13}. They satisfy

$$t_1 > 4\mu^2 \quad , \quad t_2 < (M-\mu)^2 - s$$

The absorptive part A_1 has the same analytic properties as a function of momentum transfer t for fixed s & $s > (M+\mu)^2$ as the scattering amplitude except that there is no pole now and the cuts extend only from $-\infty$ to t_2 and t_1 to ∞ and according to the inequalities these cuts do no extend to the interior as much as for A. Thus the region of analyticity of A_1 as a function of t is greater than that of A which has been shown by Lehmann.

$A^{\pm}$ and $B^{\pm}$ satisfy these representations. Crossing symmetry gives

$$A^{\pm}(\nu,t) = \pm A(-\nu,t)$$
$$B^{\pm}(\nu,t) = \mp B(-\nu,t) \tag{22}$$

or

$$A_{13}^{\pm}(s,t) = \pm A_{23}^{\pm}(\bar{s},t)$$
$$A_{12}^{\pm}(s,\bar{s}) = \pm A_{12}(\bar{s},s)$$
$$B_{13}^{\mp}(s,t) = \mp B_{23}(\bar{s},t)$$
$$B_{12}^{\pm}(s,\bar{s}) = \mp B_{12}(\bar{s},s) \tag{23}$$

LECTURE III

The unitarity condition in the one-meson approximation gives

$$A_1(s, \cos\theta_1) = \frac{P}{32\pi^2}\frac{q}{W}\int \sin\theta_2\, d\theta_2\, d\phi_2\, A^*(s, \cos\theta_1)\, A(s, \cos(\vec{\theta_1}, \vec{\theta_2}))$$

or

$$A_1(s, z_1) = \frac{1}{32\pi^2}\frac{q}{W}\int_{-1}^{+1} dz_2 \int_0^{2\pi} d\phi\, A^*(s, z_2)\, A\left\{s, z_1 z_2 + (1-z_1^2)^{1/2}(1-z_2^2)^{1/2}\cos\phi\right\} \quad (1)$$

where $z = \cos\theta$. The nucleon and meson are produced in the intermediate state at an angle θ_2 which is therefore integrated.

$$q^2 = \left\{s - (M+\mu)^2\right\}\left\{s - (M-\mu)^2\right\}/4s$$

$$z = 1 + t/2q^2$$

(2)

The unitarity condition gives A_1 only in the physical region. However A_1 must be obtained in the unphysical region also and hence must be analytically continued, by expressing A in terms or A_2 and A_3 (for fixed) which exist in the unphysical region of process I.

$$\text{(i.e.)}\ A(s, z_2) = \frac{1}{\pi}\int dz_2'\, \frac{A_2(s, z_2') + A_3^*(s, z_2')}{z_2' - z_2} \quad (3)$$

$$A(s, z_1 z_2 + (1-z_1^2)^{1/2}(1-z_2^2)^{1/2}\cos\theta$$

$$= \frac{1}{\pi}\int dz_3'\, \frac{A_2(s, z_3') + A_3(s, z_3')}{z_3' - z_1 z_2 - (1-z_1^2)^{1/2}(1-z_2^2)^{1/2}\cos\theta}.$$

(4)

$A_2(s,z)$ and $(A_3(s,z)$ are non-zero in the regions

$$z < 1 + \frac{(M+\mu)^2 - s}{2q^2}$$

and $z > 1 + 2\mu^2/q^2$ respectively.

Performing the integrations over z_2 and ϕ, A_1 reduces to

$$A_1(s,z_1) = \frac{1}{16\pi^2}\frac{q}{W}\int dz_2' \int dz_3' \frac{1}{\sqrt{K}} \log \frac{z_1 - z_2' z_3' + \sqrt{K}}{z_1 - z_2' z_3' - \sqrt{K}}$$

$$\left\{A_2^*(s,z_2') + A_3^*(s,z_2')\right\}\left\{A_2(s,z_3') + A_3(s,z_3')\right\} \tag{5}$$

where $K = z_1^2 + z_2'^2 + z_3'^2 - 1 - 2z_1 z_2' z_3'$ (6)

we must take that branch of the log which is real in the physical region $-1 < z_1 < 1$

From equation (19) of last lecture the discontinuities of A_1 across the positive and negative real t axis or (z_1) are given by A_{13} and A_{12} respectively.

Thus we have $A_{13}(s,z_1) = \frac{1}{8\pi^2}\frac{q}{W}\int dz_2 \int dz_3 K_1(z_1, z_2, z_3)$

$$\left\{A_2^*(s,z_2)A_2(s,z_3) + A_3^*(s,z_2)A_3(s,z_3)\right\} \tag{7}$$

$$A_{12}(s,z_1) = \frac{1}{8\pi^2}\frac{q}{W}\int dz_2 \int dz_3 K_2(z_1, z_2, z_3)$$

$$\left\{A_2^*(s,z_2)A_3(s,z_3) + A_3^*(s,z_2)A_2(s,z_3)\right\}. \tag{8}$$

The primes on Z_2 and Z_3 are dropped for convenience.

K_1 and K_2 satisfy the conditions

$$K_1 = -1/\sqrt{K} \qquad \text{for } Z_1 > Z_2 Z_3 + (Z_2^2-1)^{1/2}(Z_3^2-1)^{1/2}, \; K>0$$

$$= 0 \qquad \text{for } Z_1 < Z_2 Z_3 + (Z_2^2-1)^{1/2}(Z_3^2-1)^{1/2}, \; K<0$$

$$K_2 = 1/\sqrt{K} \qquad \text{for } Z_1 < Z_2 Z_3 - (Z_2^2-1)^{1/2}(Z_3^2-1)^{1/2}, \; K<0$$

$$= 0 \qquad \text{for } \quad K>0 \tag{9}$$

Going back to the energy momentum variables $s, \bar{s}$ and t

we have

$$A_{13}(s, t_1) = \frac{1}{32\pi^2 q^3 W}\Big[\int dt_2 \int dt_3 K_1(s, t_1, t_2, t_3) \overset{*}{A}(s, t_2) A(s, t_3)$$

$$+ \int d\bar{s}_2 \int d\bar{s}_3 K_1(s, t_1, \bar{s}_2 \bar{s}_3) \overset{*}{A}_2(s, \bar{s}_2) A_2(s, \bar{s}_3) \tag{10}$$

and

$$A_{12}(s, \bar{s}_1) = \frac{1}{32\pi^2 q^3 W} \int dt^2 \int d\bar{s}_2 K_2(s; \bar{s}_1, t_2, \bar{s}_2)$$

$$\Big[A_3^*(s, t_2) A_2(s, \bar{s}_1) + \overset{*}{A}_2(s, \bar{s}_3)(A_3(s, t_2)\Big\} \tag{11}$$

The regions where the $K's$ vanish in terms of these variables are given by (for $s \to \infty$)

$$K_1(s;t_1,t_2,t_3) = 0 \text{ unless } t_1^{1/2} > t_2^{1/2} + t_3^{1/2}$$
$$K_1(s;t_1,\bar{s}_2,\bar{s}_3) = 0 \text{ unless } t_1^{1/2} > \bar{s}_2^{1/2} + \bar{s}_3^{1/2}$$
$$K_2(s;\bar{s},t_2,\bar{s}_3) = 0 \text{ unless } s_1^{1/2} > t_2^{1/2} + \bar{s}_3^{1/2}$$

(12)

These relations are the only ones in the regions $\bar{s} > M^2$ and $t > 4\mu^2$ outside which A_2 and A_3 vanish. Therefore it follows that $A_{13}(s,t)$ for a given t can be calculated in terms of $A_3(s,t')$ and $A_2(s,\bar{s}')$ provided $t(\bar{s})$ is greater than $t'(s_2')$

Again we have from the dispersion relations

$$A_2(s,\bar{s}) = \frac{1}{\pi}\int_{s_2(s)}^{s} ds' \frac{A_{12}(s',\bar{s})}{s'-s} + \frac{1}{\pi}\int_{t_1(s)}^{\infty} ds' \frac{A_{23}(\bar{s}'t)}{t'-t}$$

(13)

and $$A_3(s,t) = \frac{1}{\pi}\int_{s_3(t)}^{\infty} ds' \frac{A_{13}(s',t)}{s'-s} + \frac{1}{\pi}\int_{\bar{s}(t)}^{\infty} d\bar{s}' \frac{A_{23}(\bar{s}',t)}{\bar{s}'-s}$$

(14)

Hence A_2 and A_3 can be found in terms of A_{12} and A_{23} respectively if we neglect the second terms in these equations. Knowing that at $\bar{s} = M^2$ A_2 is a δ function, $g^2\delta(\bar{s}-M^2)$, we can therefore find A_{13}, hence A_3 and then A_{12} for all values of s and successively larger values of $\bar{s}$ and t. It follows A_{13} is zero if $t_1 < 4M^2$. The lowest value of $\bar{s}$ at which there is contribution is $\bar{s} = M^2)$. More precisely the conditions (9) are

$$K_1(s;t_1,t_2,t_3) = 0 \text{ unless } t_1^{1/2} > t_2^{1/2}\left(1+\frac{t_3}{4q^2}\right)^{1/2}$$

$$K_1(s;t_1,\bar{s}_2\ \bar{s}_3) = 0 \text{ unless } + t_3^{1/2}\left(1+t_2/4q^2\right)^{1/2}$$

$$t_1^{1/2} > (\bar{s}_2-u)^{1/2}\left[1+(\bar{s}_3-u)/4q^2\right]^{1/2}$$
$$+(\bar{s}_3-u)^{1/2}\left\{1+(s_2-u)/4q^2\right\}^{1/2}$$

= 0 unless

$$(s_1-u)^{1/2} > t_2^{1/2}\left\{1+(\bar{s}_3-u)/4q^2\right\}^{1/2}$$
$$+(s_3-u)^{1/2}\left(1+t_2/4q^2\right)^{1/2} \tag{15}$$

where

$$u = (M^2-\mu^2)/s$$

The smallest value for which A_{13} is non-zero is given by

$$t^{1/2} = 2\,(M^2-u)^{1/2}\left\{1+(M^2-u)/4q^2\right\}^{1/2} \tag{16}$$

For large s, t → $4M^2$ but as s decreases t becomes larger and larger until at $s = (M + \mu)^2$, t becomes?

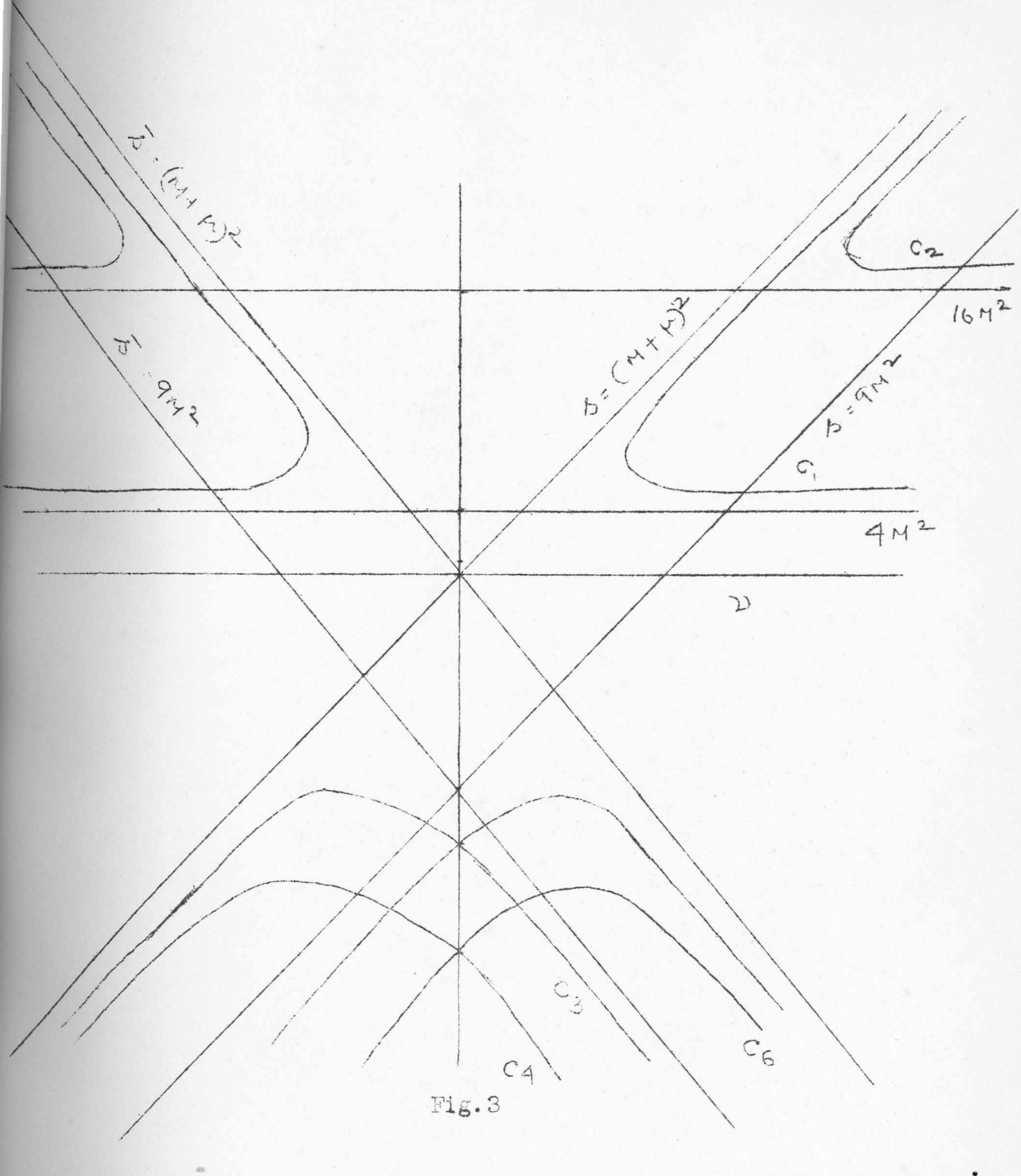


Fig.3

A_{13} will be non-zero within C_1 and near it will behave like $(t-t_0)^{-1/2}$ where t_0 is the value of t given by (16). Thus $\sim$ from (15) it follows that $A_3(s,t)$ is non-zero if $t > 4M^2$ and behaves like $(t - 4M^2)^{1/2}$ just above this limit. This value $t = 4M^2$ is the threshold for the process III. Till now we have neglected the subtractions. However it can be shown when substractions are made that the curve C_1 becomes C_{13} of Fig.2. Above $t = 4M^2$ for a range of values only the δ-function of the second term of A_{13} contributes. At a certain state the other terms in A_2 and A_3 also contribute and if for the moment we neglect the 2nd term it can be shown that the contribution from the first term starts at $\underset{s \to \infty}{t = 16M^2}$ (since A_3 exists only above $t = 4M^2$ or $t_2^{1/2} = t_3^{1/2} = 2M$ and hence $t_1^{1/2} > t_2^{1/2} + t_3^{1/2}$ i.e. $> 4M$. The result is the curve C_2 above which the new contribution starts. This corresponds to the threshold for production of an additional pair of nucleons in process.

Such discont nuities in the higher derivatives of A_{13} occur at $t = 4n^2M^2$.

Now we will study A_{12} in a similar fashion. The lowest value for which K_2 exists is $t_2 = 4M^2$ and $\bar{s}_3 = M^2$ or $\bar{s}_1 > 9M^2$ as $s \to \infty (C_3)$. Automatically from (15) A_2 will be nonvanishing for $\bar{s} > 9M^2$ which is the threshold for pair production in reaction II. We get similar curves C_4 etc. approaching the line $\bar{s} = (2n+1)^2 M^2$ The contribution to A_{13} from second term starts at $t = 4n^2M^2$.

So far we have neglected the second terms of eqns. ( ). We have also not made use of crossing symmetry so far. Now in

$$A = \text{pole terms} + \frac{1}{\pi}\int \frac{A_1(\nu',t)}{(\nu'-\nu)}\,d\nu' - \frac{1}{\pi}\int \frac{A_2(\nu',t)}{\nu'-\nu}\,d\nu' \tag{17}$$

A_{23} will contribute only to A_2.

First neglect A_{23} and determine A_{13} and then find A_{23} from A_{13} by crossing symmetry relations and feed that into the expression for A_2 in terms of A_{12} and A_{23} and do the iteration again. While this can be continued for A_{23} the crossing symmetry of the total scattering amplitude is still not satisfied since

$$A_{12}^{\pm}(s,\bar{s}) = A_{12}^{\pm}(\bar{s},s) \tag{18}$$

is not satisfied by A_{12} determined by and unitarity (which is obvious from the curve C_3). Thus it seems we cannot apply unitarity, analyticity and crossing symmetry simultaneously

However we can modify our iteration procedure as follows. We can add a term $A_{12}(s,\bar{s})$ along with A_{23} in the eqns. to satisfy complete crossing symmetry. A_{12} will be non-zero above curve C_5 in Fig.3, i.e. below $s = 9M^2$. The addition of A_{12} violates the unitarity condition in the one meson approximation for values of $s > 9M^2$ (However in this region the one meson approximation is far from correct).

The reason for this violation of the rules is easily understood in perturbation theory

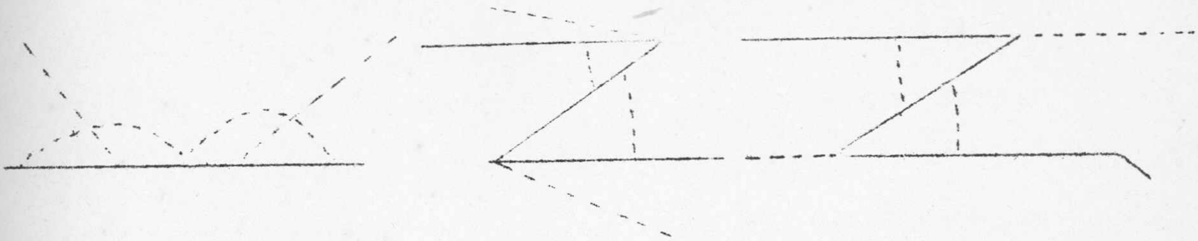


Fig.4

(b) is topologically similar to (a) and has to be added to (a) to avoid square roots in the energy denominators (and hence may not have the required analyticity properties). If (b) is added for crossing symmetry (c) has to be added. However in this graph there is an intermediate state of a nucleon and a pair so that unitarity is violated in the one meson approximation.

We still do not have discontinuities at all possible thresholds.

Lecture IV

Subtraction terms in the Dispersion Relations

So far we have discussed the analytic properties of the scattering amplitudes ignoring all 'subtraction terms' in the dispersion relations. The details of this sort of solution are naturally different from those of the static problem, the reason being that the part of the amplitude corresponding to the lowest angular momentum state~~s~~ appears as a subtraction in the dispersion relation and has not been taken into account.

In general, if the integrals in the scattering amplitude do not converge, we make the 'subtraction' as follows:

We write

$$\frac{1}{\bar{s}'-s} = \frac{(\bar{s}-s_0)}{(\bar{s}_1-\bar{s})(\bar{s}'-\bar{s}_0)} + \frac{1}{\bar{s}_1-\bar{s}_0} \tag{1}$$

and thus

$$\frac{1}{\pi^2}\iint ds'\,d\bar{s}'\,\frac{A_{12}^{\pm}(s',\bar{s}')}{(s'-s)(\bar{s}'-\bar{s})}$$

is rewritten as

$$\frac{\bar{s}-s_0}{\pi^2}\int ds'\int d\bar{s}'\,\frac{A_{12}(s',\bar{s}')}{(s'-s)(\bar{s}'-\bar{s})(\bar{s}'-\bar{s}_0)} + \frac{1}{\pi}\int ds'\,\frac{f_1(s')}{s'-s} \tag{2}$$

where

$$f_1(s') = \frac{1}{\pi}\int d\bar{s}_1 \frac{A_{12}(s', \bar{s}')}{(\bar{s}_1 - \bar{s}_0)} \tag{3}$$

If an s subtraction were necessary we get

$$(s-s_0)(\bar{s}-\bar{s}_0)\int ds'\int d\bar{s}' \frac{A_{12}(s', \bar{s}')}{(s'-s)(s'-s_0)(\bar{s}'-s_0)(\bar{s}'-s)}$$

$$+ (s_l - s_0)\int ds' \frac{f_1(s')}{(s'-s)(s'-s_0)}$$

$$+ (\bar{s} - s_0)\int d\bar{s}' \frac{f_2(\bar{s}')}{(\bar{s}'-\bar{s})(\bar{s}'-\bar{s}_0)} \tag{4}$$

where

$$f_2(\bar{s}') = \frac{1}{\pi}\int ds' \frac{A_{12}(s', \bar{s}')}{s'-s_0} \tag{5}$$

and

$$C = \frac{1}{\pi^2}\int ds'\int d\bar{s}' \frac{A_{12}(s', \bar{s}')}{(s'-s_0)(\bar{s}'-\bar{s}_0)} \tag{6}$$

and the total amplitude with one subtraction in each variable becomes

$$A = \frac{(s-s_0)(t-t_0)}{\pi^2}\int ds'\int dt' \frac{A_{13}(s', t')}{(s'-s_0)(s'-s)(t'-t_0)(t'-t)}$$

$$+ \frac{(t-t_0)(\bar{s}-\bar{s}_0)}{\pi^2}\int dt'\int d\bar{s}' \frac{A_{23}(t', \bar{s}_1)}{(t'-t_0)(t'-t)(\bar{s}_1-\bar{s}_0)(\bar{s}_1-s)}$$

$$+ \frac{(s-s_0)(\bar{s}-\bar{s}_0)}{\pi^2}\int ds'\int d\bar{s}' \frac{A_{12}(s', \bar{s}_1)}{(s'-s)(s'-\bar{s}_0)(\bar{s}'-\bar{s}_0)(\bar{s}'-\bar{s})} + \lambda$$

$$+\frac{s-s_0}{\pi}\int ds'\frac{f_1(s')}{(s'-s_0)(s'-s)}+\frac{t-t_0}{\pi}\int dt'\frac{f_3(t')}{(t'-t_0)(t'-t)}$$

$$+\frac{\bar{s}-s_0}{\pi}\int d\bar{s}_1\frac{f_2(\bar{s}')}{(\bar{s}_1-\bar{s}_0)(\bar{s}'-\bar{s})} \qquad (7)$$

The first three terms are referred to as "double dispersion integrals", the 4th, 5th and the 6th as single dispersion integrals and lastly is the overall subtraction term. If any of the three processes can occur through a discrete intermediate state corresponding to poles they will be represented by δ functions in f_1, f_2 or f_3 below their thresholds. The 4th term in the centre of mass of the 2π system is independent of t and $\bar{s}$ (i.e.) independent of $\cos\theta$. Thus it will contribute to the S wave absorptive part only. Similarly if we make two subtractions for $\bar{s}$ we get terms which contribute~~s~~ to S and P waves only. In view of the strict limitations imposed by unitarity on the partial waves the functions cannot be too singular at high energies. By a dimensional argument it can be shown that all partial wave amplitudes for $\pi - N$ scattering must go to zero at high energies. Thus the single dispersion integrals have no subtraction for this scattering. For boson-boson scattering the single dispersion integrals can have at most one subtraction and the general form of the Mandelstam representation can thus have at most one unknown subtraction constant. The single dispersion integrals can be determined by unitarity. Thus the symmetric treatment of all channels in a given process allows us to avoid the difficulty inherent in the one-dimensional dispersion relations, that the subtractions were unknown functions ~~of one~~ of the invariants. Here the analyticity properties of these subtraction terms are known and their relation for another channel is explicitly seen.

The effects of the subtraction terms in the absorptive parts ~~of~~ for processes I, II and III are given as follows.

$$A_1 = f_1(s) + \frac{t-t_0}{\pi}\int dt' \frac{A_{13}(s,t')}{(t'-t_0)(t'-t)} + \frac{(\bar{s}-\bar{s}_0)}{\pi}\int \frac{d\bar{s}' A_{12}(s,\bar{s}')}{(\bar{s}'-\bar{s}_0)(\bar{s}'-\bar{s})} \tag{9}$$

$$A_2 = f_2(\bar{s}) + \frac{t-t_0}{\pi}\int dt' \frac{A_{23}(t',\bar{s})}{(t'-t_0)(t'-t)} + \frac{s-s_0}{\pi}\int \frac{A_{12}(s',\bar{s})}{(s'-s_0)(\bar{s}'-\bar{s})} ds' \tag{10}$$

$$A_3 = f_3(t) + \frac{s-s_0}{\pi}\int ds' \frac{A_{13}(s',\bar{s})}{(s'-s_0)(s'-s)} + \frac{\bar{s}-\bar{s}_0}{\pi}\int \frac{A_{23}(t,\bar{s}')}{(\bar{s}-\bar{s}_0)(\bar{s}'-\bar{s})} \tag{11}$$

For each process the first term represents the contribution from the S-wave only and the absorptive part corresponding to all the other waves is determined completely by the double dispersion integrals in (7) $f_1 . f_2 , f_3$ naturally depend on the values of s_0, $\bar{s}_0$ and t_0 chosen. The subtractions can be performed in such a way that those functions are just the S wave absorptive parts for the three reactions. To do this we replace

$$\frac{1}{t'-t} \text{ by } \frac{1}{t'-t} - \frac{1}{2}\int_{-1}^{1}\frac{d(\cos\theta)(s,t))}{t'-t} + \frac{1}{2}\int_{-1}^{1}\frac{d(\cos\theta)(s,t))}{t'-t} \tag{12}$$

We have

$$\cos\theta(s,t) = 1 + t/2q^2$$

with

$$4q^2 = s - 2(M^2+m^2) + \frac{(M^2-m^2)^2}{s} \tag{13}$$

Hence

$$-\frac{1}{2}\int_{-1}^{1}\frac{d\cos\theta(s,t)}{t'-t} = \frac{1}{4q^2}\int_{-4q^2}^{0}\frac{dt}{t'-t} = \frac{1}{t_u(s)-t_\ell(s)}\int_{t_\ell}^{t_u}\frac{dt'}{t'-t} \tag{14}$$

Thus

$$\frac{1}{t'-t} \rightarrow \frac{1}{t'-t} - \ell_I^{(0)}(t',s) + \frac{1}{2}\int_{-1}^{+1}\frac{d\cos\theta(s,t)}{t'-t} \text{ where}$$

$$\ell_I^{(0)}(t',s) = \frac{1}{t_u-t_\ell}\log\frac{t'-t_\ell}{t-t_u} \tag{15}$$

As $t' \to \infty$

$$\frac{1}{t'-t} - l_I(0) \to \frac{t - \frac{1}{2}(t_u + t_0)}{t'^2} + O\left(\frac{1}{t'^3}\right) \qquad (16)$$

A second subtraction would be of the form

$$\frac{1}{t'-t} \longrightarrow \left\{ \frac{1}{t'-t} - l_I^{(0)}(t', s) - 3\cos\theta(s,t)\, l_I^{(1)}(t', s) \right\}$$

$$+ \frac{1}{2}\int_{-1}^{1} \frac{d\cos\theta(s,t)}{t'-t} + \frac{3}{2}\cos\theta(s,t)\int_{-1}^{1} \frac{\cos\theta(s,t)}{t'-t}\, d\cos\theta \qquad (17)$$

Thus a subtracted form of a term like

$$\frac{1}{\pi^2}\iint ds'\,dt' \,\frac{A_{13}(s',t')}{(s'-s)(t'-t)}$$

will be

$$\frac{1}{\pi^2}\iint ds'dt' \,\frac{A_{13}(s',t')}{s'-s}\left\{ \frac{1}{t'-t} - l_I^{(0)}(t', s') - 3\cos\theta(s',t')\, l_I^{(1)} \right\}$$

$$+ \frac{1}{\pi^2}\int ds' \frac{1}{s'-s}\left\{ \frac{1}{2}\int_{-1}^{+1} d\cos\theta(s',t) \int \frac{A_{13}(s',t')}{t'-t}\, dt' \right.$$

$$+ \frac{3}{2}\cos\theta(s',t)\int_{-1}^{+1} d\cos\theta(s',t)\, d\cos\theta(s',t) \int \frac{A_{13}(s',t')}{t'-t'}\, dt' \qquad (18)$$

The second term has the form

$$\frac{1}{\pi}\int ds' \frac{1}{s'-s}\left\{ A_I^{(0)}(s') + 3\cos(s',t)\, A_I^{(1)}(s') \right. \qquad (19)$$

The separation of the S and P wave absorptive parts ~~are~~ is thus made clear. The unitarity condition for the states of angular momentum 1/2 to $n - 1/2$ when there are subtractions are to be applied separately. The precise number of subtractions required cannot be determined without calculating the result but it is almost certainly not less than two. We were not able to produce the $P_{3/2}$ resonant behaviour by the above discussions, whereas it follows quite naturally in the Chen-Low formalism. If the coupling for binding the $\pi - N$ in the 3-3 state is large enough we should make at least two subtractions. The direct pole contribution is brought in as a subtractior which contributes to S waves only. We find a ghost state in the first iteration of the calculation of the $j = 1/2$ scattering amplitude. However there are no more ghost states becuase of the crossing symmetry. Let us enumerate the other possible subtractions. It is seen that $A_{13}(s,t) \to \mathrm{Const.}$ as $s \to \infty$ and $B_{13} \longrightarrow$ as $s \to \infty$, when we take the lowest contribution from A_2 (i.e.) the δ-function. Therefore A_3 has a subtraction but B_3 has none. A_{12} and $B_{12} \to 0$ as $s \to \infty$ But we have seen that A_1 and B_1 behave like constants for $s_c \to \infty$ for fixed s and hence by crossing symmetry there should be subtractions in A_2 and B_2 also. We have to use the unitarity condition for A_3 (of reaction III) to calculate the subtraction term for A_3. Only wave is involved. For the first approximation we restrict the intermediate state for process III as 2 pions (or at most two mesons + pair). Thus we require the meson-meson scattering amplitude and this is how the meson-meson coupling is introduced in the $\pi - N$ scattering problem. The value of A_3 will be non-vanishing for $t > 4\mu^2$. Thus the boundary region where the spectral functions do not vanish get changed. In view of the fact that we are applying unitarity separately for $\pi - N$ $J = 1/2$ state and S wave $\pi - \pi$ system, there will be the C-D-D ambiguity due to unstable baryon of spin 1/2 or spin 0 heavy meson. Had there been no interaction with the electromagnetic field, we would have to apply unitarity to the P wave also separately, thus leading to a C.D.D. ambiguity corresponding to unstable particles of spin which corresponds to the Bethe-Beard mixture of vector and scalar mesons which can be renormalized in perturbation theory. As the coupling constant is increased the inadequacy of the one-meson approximation starts when A_{12} gives comparable contribution to A. Hence a cut off is introduced to the contribution of A_2 from the crossing term above the boundary of C_{13}. We have found that $A_3 \neq 0$ for $t > 4\mu^2$ for one meson approximation. Feeding this in the inequalities for $K's$ not to vanish we find that A_{13} does not vanish for ~~vanish for~~ $t > 16\mu^2$ However this is not the precise boundary. The reason is the absence of the pole corresponding to the single pion for process III. However this violates the one meson approximation for process I for

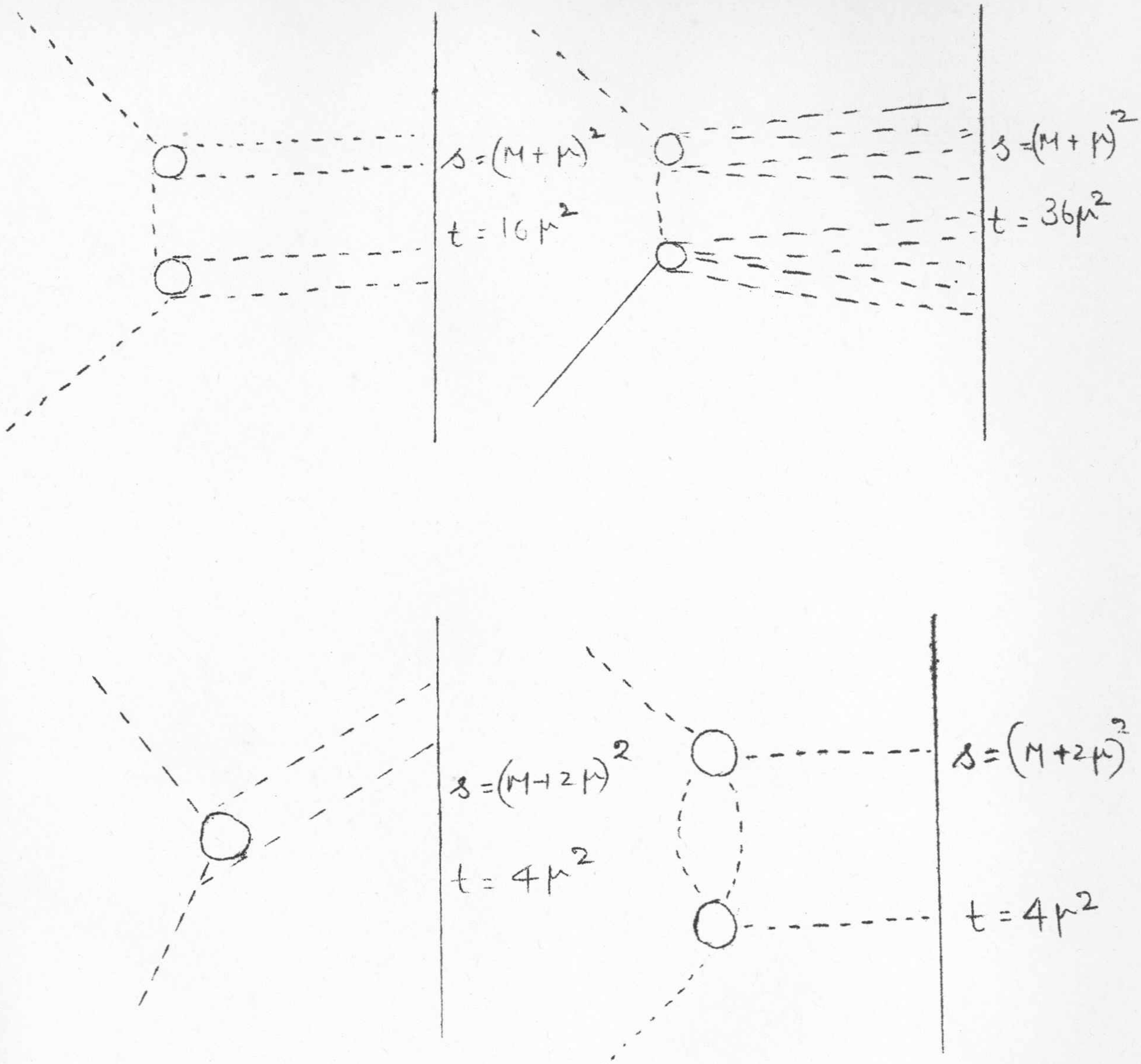
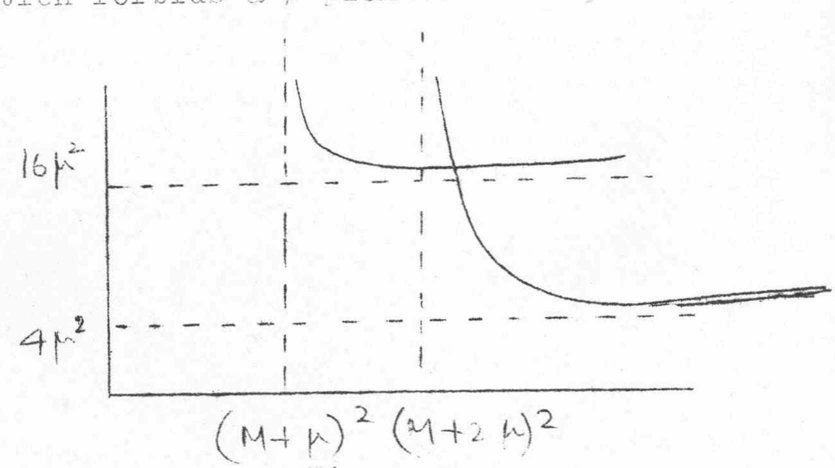


Fig.5

G operation forbids a 3 pion vertex. Thus we have



Lecture V

PARTIAL WAVE DISPERSION RELATIONS FOR πN SCATTERING

The Mandelstam representation for the B amplitude is

$$B^{\pm}(s,\bar{s},t) = \frac{g_r^2}{m^2-s} \mp \frac{g_r^2}{m^2-\bar{s}} + \frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} ds' \int_{(m+\mu)^2}^{\infty} d\bar{s}' \frac{b_{12}^{\pm}(s',\bar{s}')}{(s'-s)(\bar{s}'-\bar{s})}$$

$$+\frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} ds' \int_{4\mu^2}^{\infty} dt' \frac{b_{13}^{\pm}(s',t')}{(s'-s)(t'-t)}$$

$$+\frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty} d\bar{s}' \int_{4\mu^2}^{\infty} dt' \frac{b_{23}^{\pm}(\bar{s}',t')}{(\bar{s}'-\bar{s})(t'-t)} \tag{1}$$

At fixed s

$$B^{\pm}(s,\bar{s},t) = \frac{g_r^2}{m^2-s} \mp \frac{g_r^2}{m^2-\bar{s}} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{b_3^{\pm}(s,t')}{t'-t}$$

$$+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} d\bar{s}' \, b_2(\bar{s}',t)\big/\bar{s}'-\bar{s}$$

$$b_2^{\pm}(\bar{s}',t) = \mp b_1^{\pm}(\bar{s}',t) \tag{2}$$

But

(3)

The last term then becomes

$$\mp \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} d\bar{s}' \; \frac{b_1^{\pm}(\bar{s}', \Sigma - s - \bar{s}')}{\bar{s}' - s}$$

$$= \mp \frac{1}{\pi} \int \frac{d\bar{s}' \, ds' \; b_{12}(s', \bar{s}')}{(\bar{s}' - \bar{s}) \quad (s' - s)}$$

$$+ \int \frac{d\bar{s}'}{\bar{s}' - \bar{s}} \int dt' \, \frac{b_{13}(\bar{s}', t)}{t' + s + \bar{s}' - \Sigma} \qquad (4)$$

and

$$f_{\ell\pm}(W) = \frac{1}{16\pi W} \Big\{ (E+m) \left[A_\ell + (W - m) B_\ell \right] + (E - m) \left[-A_{\ell\pm 1} + (W + m) B_{\ell\pm 1} \right] \Big\} \qquad (5).$$

In the s-plane there is a branch point at s = 0. To avoid this all the singularities of the partial wave amplitudes w be studied in the W plane. In the elastic scattering part the physical region

$$f_{\ell\pm} = \frac{e^{i\delta_{\ell\pm}} \sin \delta_{\ell\pm}}{k}$$

where δ is real. The simple relation

$$\operatorname{Im}\left(1/f_{\ell\pm}\right) = -k$$

immediately follows. The singularities arise due to the vanishing of the denominators in (2) and (4). The denominator in (4) gives rise to the physical cuts i.e. branch cuts in the region $W > m+1$ and $W < -(m+1)$. The right hand cut is the true physical cut on which unitarity canbe applied below the inelastic threshold.

In the latter region,, we have by symmetry relation

$$f_{\ell+}(-W) = -f_{(\ell+1)-}(+W)$$

This allows us to apply unitarity on the left hand cut also. When W is negative one finds

$$-f_{\ell+}(W+i\epsilon) = \frac{e^{-i\delta_{(\ell+1)-}(-W)}\sin\delta_{(\ell+1)-}(W)}{k} \qquad (7)$$

It is seen that the two partial wave amplitudes corresponding to the same total angular momentum are boundary values,, in different regions of the complex W plnae, of the single analytic function $f_{\ell+}(W)$. We can forget $f_{\ell-}$ hereafter. It is found convenient to work with the amplitude.

$$h_{\ell}(W) = \frac{W}{E+M}\,\frac{f_{\ell+}(W)}{k^{2\ell}}$$

$$= \frac{1}{16\pi}\left\{\frac{A_\ell}{k^{2\ell}} + (W - m)\frac{B_\ell}{k^{2\ell}} + (E-m)^2 \times\right.$$

$$\left.\left[\frac{-A_{\ell+1}}{k^{2\ell+2}} + (W+m)\frac{B_{\ell+1}}{k^{2\ell+2}}\right]\right\} \tag{8}$$

By dividing $f_{\ell+}$ by $k^{2\ell}$ we are not introducing any new singularity since A_1 and B_1 behave like $k^{2\ell}$ for $k^2 \approx 0$ This can be seen by multiplying (1) by $P_\ell(\cos\theta)$ and integrating over $d\cos\theta$. This is true except when $\ell = 0$ in which case $k^{2\ell} = 1$ and no singularity is introduced and hence $f_\ell \sim k^{2\ell}$ for $W \approx m \pm 1$. $k^2 \equiv 0$ also when $W = -(m \pm 1)$ and from (5) it follows that $f_{\ell+} \sim (E+m)\, k^{2\ell}$ thereby justifying the analyticity of $h_\ell(W)$. Thus $h_{\ell+}$ approaches a constant at both the left and right hand physical thresholds.

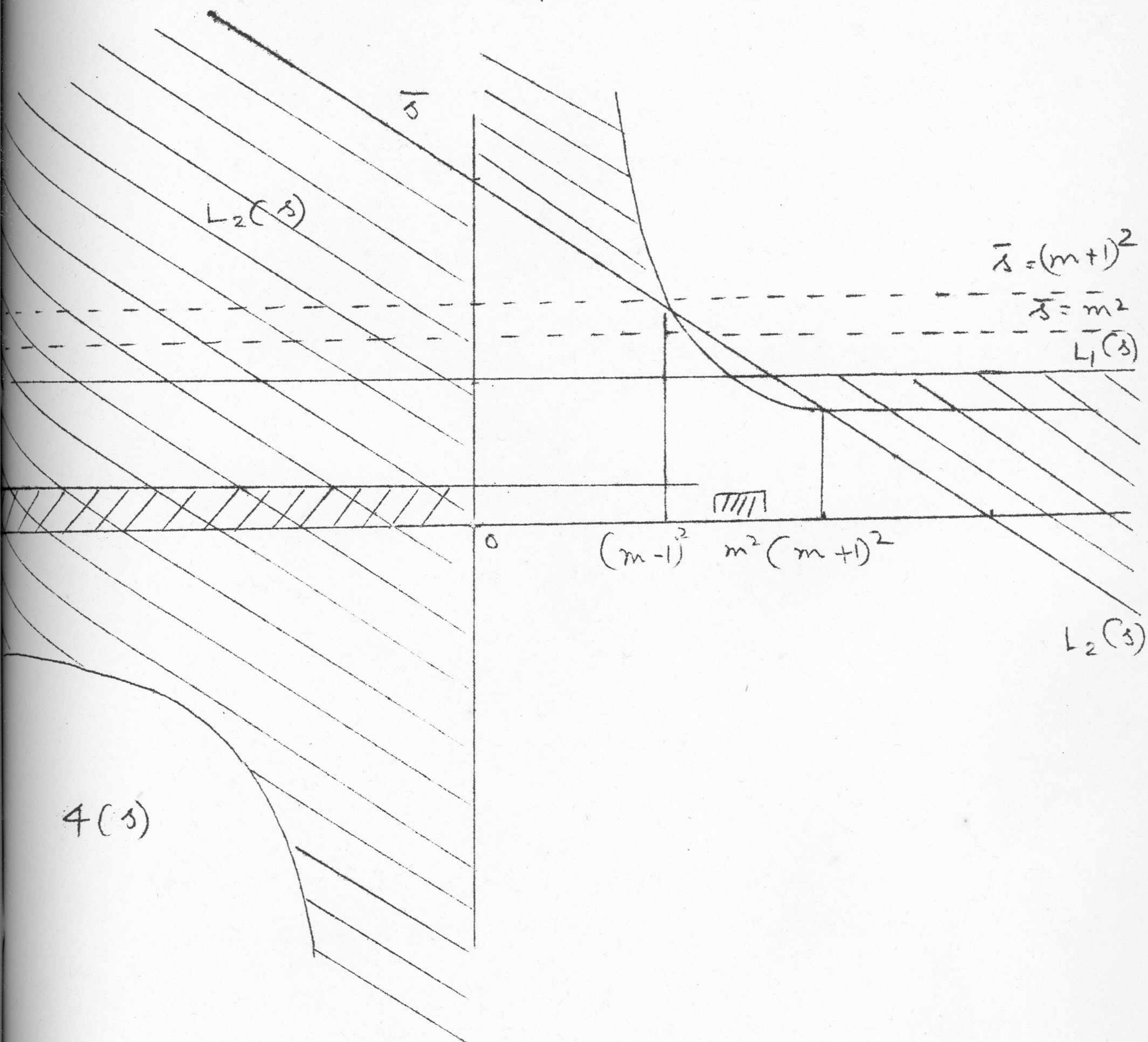
Finally W is introduced to remove the singularity at W = 0.

The second term in (4) does not introduce any additional singularity since it was introduced artificially through the separation into partial fraction of one of the terms. The direct pole contributes only to B_0 (and hence only to h_0). The contribution to B_ℓ from the 2nd and 3rd terms are

$$B_\ell^{\pm}(s) = \pm\frac{1}{2k^2}\int_{L_1(s)}^{L_1(s)} d\bar{s}\; P_\ell\left(1 + \frac{\Sigma - s - \bar{s}}{2k^2}\right)\left[\frac{g^2}{m^2 - \bar{s}}\right.$$

$$+ \frac{1}{\pi}\int_{(m+1)^2}^{\infty} d\bar{s}' \, \frac{b_1^{\pm}(\bar{s}', \Sigma - s - \bar{s}')}{\bar{s}' - s} \tag{9}$$

where $L_1(s)$ and $L_2(s)$ are the limiting curves corresponding to $\cos\theta = -1$ and $\cos\theta = +1$ respectively.



$$L_1(s) = (m^2-1)^2/s$$

$$L_2(s) = 2m^2+2-s$$

(10)

The pole at $\bar{s} = m^2$ gives a branch cut in the regions ~~and~~ $s < 0$ and $(m^2 - 2 + 1/m^2) \leq m^2 + 2$

The continuum beginning at $\bar{s}' = (m+1)^2$ gives a branch cut in the region $s \leq (m-1)^2$. In the W plane, these cuts are given by the entire imaginary axis and

$$-(m^2+2)^{1/2} \leq W \leq -m + 1/m$$

$$-(m+1) \leq W \leq m-1$$

$$m - 1/m \leq W \leq (m^2+2)^{1/2}$$

(11)

Now let us study the singularities coming from the vanishing of of the denominator of the 3rd term in equation (2). In the k^2 complex plane, the branch cut is given by

$$k^2 \leq -1 \quad \begin{cases} \cos\theta = 1 + t/2k^2 \\ t = 4r^2 \end{cases}$$

(12)

In the s-plane this maps into a cut along the negative real axis plus a cut along a circle centred at the origin with radius

$$r^2 = m^2 - 1$$

$$s = (\omega_N + \omega_\pi)^2 = 2k^2 + m^2 + 1 = \sqrt{(k^2+m^2)(k^2+1)}$$.

For $K^2 = -m^2$, $\quad s = -m^2+1$

$k^2 = -1 \qquad s = m^2 - 1$

For

$-m^2 \leq K^2 \leq -1$ let $K^2 = -1-\Lambda$

and

$-1-\Lambda = -m^2 \quad (\Lambda > 0)$

Then

$$s = -2-2\Lambda + m^2+1+2\sqrt{(-1-\Lambda+m^2)(-\Lambda)}$$

$$= (m^2-2\Lambda-1) + 2i\sqrt{\Lambda}\sqrt{(-1-\Lambda+m^2)}$$

$$|s|^2 = s_x^2 + s_y^2 = \left[(m^2-2\Lambda-1)^2 + 4\Lambda(-1-\Lambda+m^2)\right)$$

$$= (m^2-1)^2 = R^2$$

The radius $= (m^2-1)$

and in the W plane $\quad r = \sqrt{m^2-1}$

Thus

$$h_e(w) = \frac{1}{\pi}\int_{m+1}^{\infty} dw' \frac{\mathrm{Im}\, h_e(w')}{w'-w} + \int_{-\infty}^{-m-1} dw' \frac{\mathrm{Im}\, h_e(w')}{(w'-w)}$$

$$-\frac{g^2\delta_{l_3}}{2(m+w')} + \frac{1}{\pi}\int_{-(m^2+2)^{1/2}}^{-m+1/m} dw'^2 \frac{\alpha_1^{\,l}(w')}{w'-w} + \frac{1}{\pi}\int_{m-1/m}^{(m^2+2)^{1/2}} dw' \frac{\alpha_1^{\,l}(w')}{w'-w}$$

$$+\frac{1}{\pi}\int_{-m+1}^{m-1} dw' \frac{\alpha_2^{\,l}(w')}{w'-w} + \frac{r}{\pi}\int_0^{2\pi} i\,d\phi \frac{\alpha_3^{\,l}(\phi)}{w\,e^{-i\phi}-r}$$

$$+\frac{1}{\pi}\int_{-\infty}^{+\infty} dy\, \frac{\alpha_4^{\ell}(y)}{y+iW}$$

(13)

The pole of $g^2/2(m+|W|)$ lies onthe branch line

$$-(m^2+2)^{1/2} \leqslant W < -m + 1/m \tag{14}$$

Now we will calculate the discontinuities across these cuts. Consider the discontinuity $\alpha_1(W)$ across the cuts arising from $\bar{s} = m^2$

We have

$$\mathrm{Im}\, B_\ell = \pm \frac{1}{2K^2}\, \mathrm{Im} \int_{L_1(s)}^{L_2(s)} d\bar{s}\; P_\ell\left[1+\frac{\Sigma - s - \bar{s}}{2K^2}\right] \frac{g_r^2}{m^2-\bar{s}}$$

(15)

If $W > 0$ and $W + i\epsilon$

$$L_1(s) = L_1(s) - i\Lambda$$

$$L_2(s) = L_2(s) + i\Lambda$$

(16)

Then $\bar{s} = \bar{s} - i\Lambda$ and

$$\mathrm{Im}\, B_\ell = \pm \frac{1}{2K^2}\, \mathrm{Im} \int_{L_1(s)}^{L_2(s)} d\bar{s}\; P_\ell \left(1 + \frac{\Sigma - s - \bar{s}}{2K^2}\right) \frac{g_r^2}{m^2 - \bar{s} + 3\Lambda}$$

$$= \mp \frac{1}{2K^2}\, \pi P_\ell \left(1 + \frac{\Sigma - s - m^2}{2K^2}\right) g_r^2 \qquad W > 0$$

and $$= \pm \frac{1}{2K^2}\, \pi P_\ell \left(1 + \frac{\Sigma - s - m^2}{2K^2}\right) g_r^2 \qquad W < 0$$

(17)

$\mathrm{Im}\, A_\ell = 0$ Let $\varepsilon(W) = 1$ for $W > 0$, -1 for $W < 0$

(18)

$$\alpha_1^{\ell\pm}(W) = \mp \left[\varepsilon(W)\pi g^2/8K^2\right]\Big[(W - m) k^{-2\ell} P_\ell(x_0) + (E - m)^2 (W + m) k^{-2\ell-2} P_{\ell+1}(x_0)\Big] \tag{19}$$

where $$X_0 = 1 + \frac{\Sigma - s - m^2}{2K^2} \tag{(20}$$

and (21)

$$g^2 = g_r^2/4\pi$$

The discontinuity $\alpha_2^{\ell}(W)$ in the region $-(m-1) \leq W \leq (m-1)$ can be expressed in terms of $\pi - N$ scattering cross-sections by crossing symmetry.

For this region

$$\mathrm{Im}\, B_\ell^{\pm}(W) = \frac{\mp\, \mathcal{E}(W)}{2K^2} \int_{L_1(s)}^{L_2(s)} d\bar{s}\, P_\ell[x(W, \bar{s})]\, b_1^{(\pm 1)}(\bar{s}, \Sigma - W^2 - \bar{s}) \tag{22}$$

and

$$\mathrm{Im}\, A_\ell^{\pm}(W) = \pm \frac{\mathcal{E}(W)}{2K^2} \int_{L_1(s)}^{L_2(s)} d\bar{s}\, P_\ell\, a_1(\bar{s}, \Sigma - W^2 - \bar{s}) \tag{23}$$

and

$$\alpha_2^{\ell(\pm)}(W) = \frac{\mathcal{E}(W)}{32\pi K^2} \int_{L_1(s)}^{L_2(s)} d\bar{s} \Big\{ K^{-2\ell} P_\ell(x) \left[-a_1^{\pm} + (W - m) b_1 \right.$$

$$\left. + (E - m)^2 K^{-2\ell-2} P_{\ell+1}(x) \left[a_1^{\pm} + (W+m) b_1^{\pm} \right] \right\} \tag{24}$$

a_1 and b_1 have a direct physical meaning. The regions referred to are in the physical regions of process II. The conditions for this are $s > (m+1)^2$, $t \leq 0$, $s\bar{s} \leq (m^2-1)^2$

But we have $\bar{s} \geq L_1(s) = (m^2-1)^2/s$ and since $s < (m-1)^2$ in this region $\bar{s} \geq (m+1)^2$.

In this region also $s > 0$ and it can be shown that $-1 \leq \cos\bar{\theta} \leq 1$ for $s > 0$ and if $\bar{s}$ lies between $L_1(s)$ and $L_2(s)$. If the energy and momentum transfer variables are in the physical region it follows that $a_1 = \mathrm{Im}\, A$ and we can write

$$[a_1(\bar{s}, \Sigma - s - \bar{s})] = 4\pi \Big\{ \frac{\bar{W} + m}{\bar{E} + m} \sum_{l=0}^{\infty} \Big[\mathrm{Im}\, f_{l+}(w') P'_{l+1}(\cos\theta) - \mathrm{Im}\, f_{l-}(\bar{w}) P'_{l-1}(\cos\bar{\theta}) \Big]$$

$$- \frac{\bar{W} - M}{E - M} \sum_{l=1}^{\infty} P'_l(\cos\bar{\theta}) \Big[\mathrm{Im}\, f^{\pm}_{l-}(w') - \mathrm{Im}\, f^{\pm}_{l+}(\bar{w}) \Big] \Big\} \tag{25}$$

Feeding this into expression (24) we can have a formula for $\alpha_2^l(w)$ but this is not an explicit one. This involves $f_{l\pm}$ and hence a complete treatment of $\pi - N$ problem will involve coupled integral equations.

The discontinuity across the circle $W = r e^{i\phi}$ where $r^2 = m^2 - 1$ involves a_3 and b_3 which in turn can be found in the physical region of process III.

$$A_l^{\pm} = \frac{1}{\pi} \int_{-1}^{1} dx \int_{L_1}^{\infty} dt' \, \frac{P_l(x)\, a_3^{\pm}(t', w^2)}{t' + 2K^2(1-x)} \tag{26}$$

and K^2 for any point on the circle is given by

$$K^2 = \frac{\left[r^2 e^{2i\phi} - (m+1)^2\right]\left[r^2 e^{2i\phi} - (m-1)^2\right]}{4r^2 e^{2i\phi}}$$

$$= \frac{1}{4r^2} e^{-2i\phi}\left[r^4 e^{2i\phi} - (m+1)^2 r^2 e^{2i\phi} - (m-1)^2 r^2 e^{2i\phi} + r^4\right]$$

$$= \frac{1}{4}\left[2r^2 \cos 2\phi - \Sigma\right] \tag{27}$$

and at points outside or inside the circle by

$$K^2\left[(r \pm \epsilon) e^{i\phi}\right] = K^2(re^{i\phi}) \pm i\epsilon \sin 2\phi \tag{28}$$

Then for $K^2 \leq -1$

$$\alpha_3^{\ell\pm}(\phi) = \frac{1}{2i}\left\{h_\ell^{\pm}(\overline{r+\epsilon\, e^{i\phi}}) - h_\ell^{\pm}(\overline{r-\epsilon\, e^{i\phi}})\right\} \tag{29}$$

It has been shown that

$$a_3^{\pm}(t,s) = \sum_{J=0}^{\infty}(J+\tfrac{1}{2})\left(\frac{8\pi}{p_-^2}\right)(ip-q)^J\left[P_J(\cos\theta_3)\mathrm{Im}\, f_+^J(t)\right.$$

$$\left. - \left[J(J+1)\right]^{-1/2} m \cos\theta_3 P_J'(\cos\theta_3)\mathrm{Im}\, f_-^J(t)\right] \tag{30}$$

Similarly for

$$b_3, \quad \cos\theta_3 = \frac{q^2 - p_-^2 + s}{2\, i\, p - q}. \tag{31}$$

For W on the circle, $\cos\theta_3$ is complex and the convergence of the series (30) has to be investigated. Finally we have for the discontinuity across the imaginary axis $W = iy$

$$\alpha_4^{\ell}(y) = \frac{1}{2i}\left[h_\ell(iy+\epsilon) - h_\ell(iy-\epsilon)\right] \tag{31}$$

and

$$k^2 = \frac{-\left[y^2+(m+1)^2\right]\left[y^2+(m-1)^2\right]}{4y^2} \tag{32}$$

CHEW - LOW EFFECTIVE RANGE FORMULAR

The (3,3) amplitude is given by

$$h_{33}(w) = h_1^{+}(w) - h_1^{-}(w)\,..$$

We neglect all but the singularities closest to the physical region (i.e.)

$$h_{33}(w) \approx \frac{1}{\pi}\int_{m-1/m}^{(m^2+2)^{1/2}} dw' \frac{\alpha_1^{1(+)}(w') - \alpha_1^{1(-1)}(w')}{w'-w}$$

$$+\frac{1}{\pi}\int_{m+1}^{\infty} dw' \frac{\operatorname{Im} h_{33}(w')}{w'-w}$$

$$= \frac{\Gamma}{m-w} + \frac{1}{\pi}\int_{m+1}^{\infty} dw' \frac{\operatorname{Im} h_{33}(w')}{w'-w}$$

(33).

where $\Gamma = -\frac{2}{3} f^2$ where $f^2 = g^2/4m^2$

Following Chew and Mandelstam (see next lecture) and writing $h_{33}(W) = N/D$ where N contains the pole and D the cut and using the unitarity condition, we have

$$\frac{2}{3} f^2 \frac{k^3 \cot \delta_{33}}{\omega} \frac{E+m}{W} = 1 - \frac{\omega}{\omega_r} + P\omega^2 + \frac{\Gamma\omega^2}{\pi} P\int_{m+1}^{\infty} \tag{34}$$

where 3 subtractions have been made. Taking the first two terms on the right hand side gives the Chew-Low effective range formula

$$\frac{4}{3} f^2 \frac{k^3 \cot \delta_{33}}{\omega} = 1 - \frac{\omega}{\omega_r} \tag{35}$$

$$h_{33} = N/D, \quad N = \frac{\Gamma}{m-W} \text{ if } D(\infty) \neq 0 \tag{36}$$

Say α Then

$$\mathrm{Re}[D(W) - \alpha] = \frac{1}{\pi} P\int_{m+1}^{\infty} \frac{\mathrm{Im}\, D(W')}{W' - W} dW'$$

$$\text{(or) } \mathrm{Re}\, D(W) = \alpha + \frac{1}{\pi} P\int dW' \frac{\mathrm{Im}\, D(W')}{W' - W} \tag{37}$$

$$\text{Also } h_{33} = \frac{W}{E+m} \frac{1}{k^3} e^{i\delta_{33}} \sin \delta_{33} \tag{38}$$

Hence $\mathrm{Im}\, D = N\, \mathrm{Im} \frac{1}{h_{33}} = \frac{-\Gamma}{m - W} \; \frac{E + m}{W} \; k^3$ (39)

$$R_e D = \frac{\Gamma}{m - W} \; \frac{E + M}{W} \; k^3 \cot \delta_{33} \tag{40}$$

Hence $\frac{2}{3} f^2 k^3 \cot \delta_{33} / W \times \frac{E + M}{W} = \alpha + \frac{1}{\pi} P \int \frac{k'^3 (E' + M)}{W'(W' - m)(W' - W)}$ (41)

$$\frac{1}{W' - W} = \frac{1}{W' - m} + (W - m) \frac{1}{(W' - m)^2} + (W - M)^2 \frac{1}{(W' - m)^3} \; \frac{1}{(W' - W)(\;\;)} 4 \tag{42}$$

Now

( )

Substituting within the integral we have

$$R.H.S = \alpha + \frac{\Gamma}{\pi} P \int \frac{dW'\, k'^3 (E' + m)}{W'(W' - m)^2} + \frac{\Gamma}{\pi} (W - m) P \int \frac{k'^3 (E' + m)}{W'(W' - m)^3}.$$

$$+ \frac{\Gamma}{\pi} (W - M)^2 P \int \frac{k'^3 (E' + m)}{W'(W' - m)^4} \, dW' + \frac{\Gamma}{\pi} (W - m)^3 \int \frac{dW'\, k'^3 (E' + m)}{W'(W' - m)^4 (W' - W)}$$

$$= 1 - \frac{\omega}{\omega_r} + P\omega^2 + \frac{\Gamma \omega^3}{\pi} + P \int_{m+1}^{\infty} dW' \frac{k'^3 (E' + m)}{W'(W' - m)^4 (W' - W)}$$

(43)

with α_j suitably chosen

Lecture VI

Theory of Low Energy Pion-Pion Interactions

We found that the $\pi - N$ scattering problem can be completely solved only if the $\pi - \pi$ scattering was solved. Chew and Mandelstam have applied the double dispersion representation to this problem and have derived integral equations under certain approximations - (i.e.) which satisfy analyticity properties suggested by the Mandelstam representation and crossing symmetries exactly and the unitarity condition approximately. There are two approaches to this problem:

(1) Effective range theory of Chew & Mandelstam.

(2) The Cini-Fubini method which is equivalent to (1) and which results in converting the double-variable dispersion relations to single variable ones.

The effective range theory leads to approximate formulas for partial amplitudes valid in a small range of energies that include nearby poles and branch points but ignore distant singularities. These distant singularities are approximated by an arbitrary constant which is determined by experiment. The approximation made is most valid for the $\pi - \pi$ problem in view of the absence of the 3 pion intermediate states due to G parity conservation and thus the first difficult branch point occurs at such a high energy that we believe the omitted effects can be approximated by one or two real parameters. In the conventional Lagrangian formalism also we have an independent coupling constant appearing in the $\pi - \pi$ interaction.

Kinematics and Symmetries

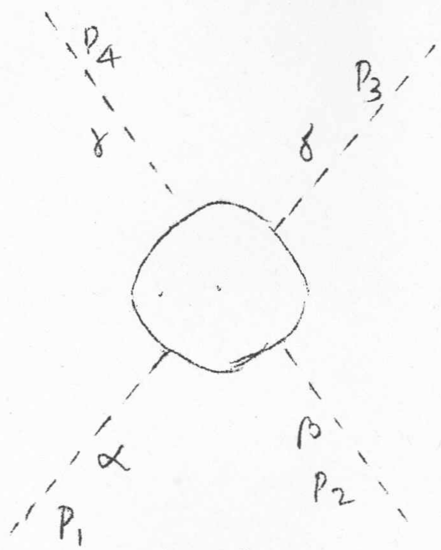


Fig 8.

We define the variables

$$s = (p_1 + p_2)^2 = 4(q^2 + \mu^2)$$

$$u = (p_1 + p_4)^2 = -2q^2(1 + \cos\theta)$$

$$t = (p_1 + p_3)^2 = -2q^2(1 - \cos\theta) \quad (1)$$

$$s + t + u = 4\mu^2 \quad (2)$$

Since I = 0, 1, 2 can occur, we must have three independent invariant function of s, t and u. We write

$$T = A(s,t,u)\,\delta_{\alpha\beta}\,\delta_{\gamma\delta} + B(s,t,u)\,\delta_{\alpha\gamma}\,\delta_{\beta\delta} + C(s,t,u)\,\delta_{\alpha\delta}\,\delta_{\gamma\beta} \quad (3)$$

and

$$A^{(0)} = 3A + B + C$$

$$A^{(1)} = B - C$$

$$A^{(2)} = B + C \quad (4)$$

where $A^{(0)}$, $A^{(1)}$ and $A^{(2)}$ are the amplitudes for the isotopic spin states 0, 1 and 2 respectively.

Crossing symmetry gives the following relations.

$$t \to u\,;\; s \to s \begin{bmatrix} A \to A \\ B \to C \end{bmatrix} \quad s \to t\,;\; u \to u \begin{bmatrix} A \to B \\ C \to C \end{bmatrix} \tag{5}$$

and

$$s \to u\,, \quad t \to t \quad \begin{Bmatrix} A \to C \\ B \to B \end{Bmatrix} \tag{6}$$

The first expresses Pauli principle but the last two impose powerful conditions on the energy and angular dependence of the amplitude. $t \to u$ means $\cos\theta \to -\cos\theta$ and since under this $A^0 \to A^0$ and $A^2 \to A^{(2)}$ but $A^1 \to -A^1$

We find that only even powers of $\cos\theta$ appear in A^0 and A^2 and only odd powers in A^1.

We have

$$A^{(i)}(q, \cos\theta) = \sum_{l} (2l+1) A_l^i(q)\, P_l(\cos\theta) \tag{7}$$

and

$$A_l^i(q) = \frac{\sqrt{q^2+\mu^2}}{q}\, e^{i\delta_l^{(i)}} \sin\delta_l^{(i)}(q) \tag{8}$$

from unitarity, where δ_l are real for $q^2 < 3\mu^2$

$$\operatorname{Im} \frac{1}{A_l^{(i)}(q)} = -\frac{q}{\sqrt{q^2+m^2}} \qquad 0 \leq q^2 \leq 3\mu^2 \tag{9}$$

The Double Dispersion Representation

$$A = \frac{1}{\pi^2}\left[\iint \frac{ds'\,dt'\,A_{13}}{(s'-s)(t'-t)} + \iint \frac{A_{12}\,ds'\,du'}{(s'-s)(u'-u)} + \iint \frac{A_{23}\,dt'\,ds'}{(t'-t)(u'-u)}\right] \quad (10)$$

We have by crossing symmetry

$$A_{13}(x,y) = A_{12}(y,x) = B_{13}(y,x) = B_{32}(x,y)$$

$$= C_{12}(x,y) = C_{32}(y,x)$$

$$A_{32}(x,y) = B_{12}(x,y) = C_{13}(x,y) \quad (11)$$

The region of the $x-y$ plane in which the spectral functions
are non-vanishing is given by the curves

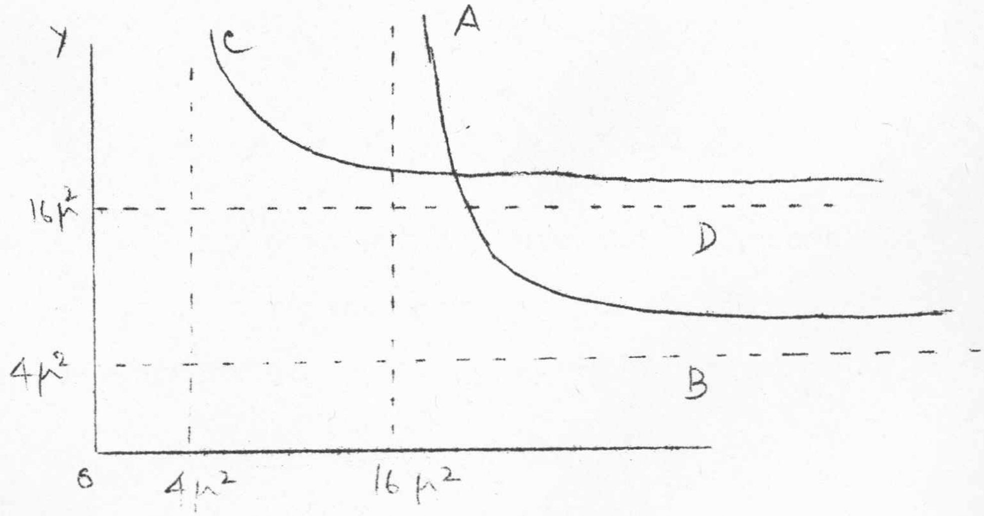


Fig.9

$$x = 16\mu^2 y / (y - 4\mu^2) \qquad \text{for } x > y$$

and

$$y = 16\mu^2 x / (x - 4\mu^2) \qquad \text{for } y > x \tag{12}$$

The large distance to the boundary from $x = y = 4\mu^2$ is due to the absence of the 3π state and any pole. A point of maximum symmetry in the s, t, u variables is the point $s = t = u = 4\mu^2/3$ (non-physical point) where A, B, C are all real and equal to each other. Then it is appropriate to include the $\pi - \pi$ coupling constant λ through the definition

$$\lambda = -A\left(\tfrac{3}{4}\mu^2, \tfrac{4}{3}\mu^2, \tfrac{4}{3}\mu^2\right) = -B = -C \tag{13}$$

At this point

$$A^0 = -5\lambda, \qquad A^1 = 0, \qquad A^2 = -2\lambda \tag{14}$$

Normally a coupling constant is introduced through the residue of a pole,, but here there are no poles. Hence λ is introduced as a subtraction at the symmetry point.

Analyticity properties of the Partial wave Amplitude.

We concentrate on the angular momentum states. We have

$$A_\ell^I(q^2) = \frac{1}{2}\int_{-1}^{1} d\cos\theta \, A^I(q^2, \cos\theta)\, P_\ell(\cos\theta) \qquad (15)$$

At fixed s this corresponds to integration over dt or du in the range 0 and $-4q^2$ moving in opposite directions. It is obvious that all the singularities lie on the real axis in the s-plane. The circular cut has radius M^2-1 for $\pi - N$ scattering but 0 for $\pi - \pi$

The three denominators give rise to the following branch points. Denominators containing s have a branch point at $q^2 = 0$ (threshold of physical region). The next branch point is at $q^2 = 3\mu^2$ Therefore there is a cut along 0 to ∞, — 'right hand cut'.

Denominators containing t or u give the "left hand" or "unphysical cut" from $-\mu^2$ to $-\infty$. The scattering amplitude is real in the gap between $-\mu^2$ and 0 on the real axis. The imaginary part -- discontinuity across right hand cut is given by (9). The calculation of the discontinuity across left hand cut is much more involved. We have to use crossing symmetry and express the imaginary parts of A. B & C in terms of absorptive parts in the physical regions of processes II & III.

We have

$t \to u$ $\qquad A_s \to A_s$ $\qquad s \to t$ $\quad A_t \to B_s$ $\quad s \to u$ $\quad A_u \to C_s$

$B_s \to B_s$; $\qquad B_t \to A_s$ $\qquad B_u \to B_s$

$C_s \to C_s$ $\qquad C_t \to C_s$ $\qquad C_u \to A_s$

(16)

We have

$$Im\, A(q^2, \cos\theta) = -A_u(q^2, \cos\theta) - A_t(q^2, \cos\theta) \quad \text{for } q^2 < 0 \tag{17}$$

Define

$$q'^2 = t/4 - \mu^2 ; \quad \cos\theta' = 1 + \frac{s}{2q'^2} = 1 + \frac{2(q^2+\mu^2)}{q'^2}$$

$$\bar{q}'^2 = u/4 - \mu^2 ; \quad \cos\bar{\theta}' = -1 - s/2\bar{q}'^2 = \frac{-1-2(q^2+\mu^2)}{\bar{q}'^2}$$

$$q^2 = s/4 - \mu^2 ; \quad \cos\theta = 1 + \frac{2(q'^2+\mu^2)}{q^2} = \frac{-1-2(\bar{q}'^2+\mu^2)}{q^2} \tag{18}$$

Then we have

$$Im\, A(q^2, \cos\theta) = -C_s(\bar{q}'^2, \cos\bar{\theta}') - B_s(q'^2, \cos\theta') \tag{19}$$

where q'^2 ranges from $-q^2 - \mu^2$ to $-\mu^2$

as $\cos\theta$ goes from -1 to +1. C_s and B = 0 for $0 > q^2 > -\mu^2$

We have

$$Im\, A^l(q^2) = 1/2 \int_{-1}^{+1} d\cos\theta\, P_l(\cos\theta)\, Im\, A(q^2, \cos\theta)$$

$$= \int_0^{-q^2-\mu^2} \frac{dq'^2}{q^2} \, C_s\left(q'^2, -1 - \frac{2(q^2+\mu^2)}{q'^2}\right) P_\ell\left(1 - \frac{q'^2+\mu^2}{q^2}\right)$$

$$+ \int_0^{-q^2-\mu^2} \frac{dq'^2}{q^2} \, B_s\left(q'^2, 1 + \frac{2(q^2+\mu^2)}{q'^2}\right) P_\ell\left(1 + \frac{2q'^2+\mu^2}{q^2}\right) \qquad (20)$$

Thus we have expressed the imaginary part of the amplitude on the left hand cut in terms of amplitudes on the right hand cut.

The corresponding expressions for well-defined isotopic spins are

$$\operatorname{Im} A^{(t)I}(q^2) = \int_0^{-q^2-\mu^2} \frac{dq'^2}{q^2} P_\ell\left(1 + \frac{2q'^2+\mu^2}{q^2}\right) \times \qquad (21)$$

$$\sum_{I'=0} \alpha_{II'} A^{I'}_{(s)}(q'^2, \cos\theta')$$

where

$$\alpha_{II'} = \begin{bmatrix} 2/3 & 2 & 10/3 \\ 2/3 & -1 & -5/3 \\ 2/3 & -1 & 1/3 \end{bmatrix} \qquad (22)$$

Under the integrals in (20) appear the absorptive parts of scattering amplitudes at values of q'^2 less than. From the boundary curves and their equations it is possible to say that the Legendre polynomial expansion of $A^I_s(q'^2, \cos\theta)$ converges for values of $\cos\theta$ required so long as $q^2 > -9\mu^2$. For the "effective range" approach this limit may be considered as $-\infty$.

Then we have

$$\operatorname{Im} A^{\ell}(q^2) = \int_0^{-q^2-\mu^2} \frac{dq'^2}{q^2} P_\ell\left(1+\frac{2(q'^2+\mu^2)}{q^2}\right)\times$$

$$\sum \alpha_{II'} A^{(\ell')(I')}(q'^2)\, P_{\ell'}\left(1+\frac{2(q^2+\mu^2)}{q'^2}\right)$$

$$\ell', I' = 0, 1, 2$$

$$\text{for } q^2 < -\mu^2 \qquad (23)$$

If the Legendre expansion is rapidly convergent the sum over ℓ' in (23) can be terminated at an early state. The absence of a single - pion exchange reduces the range of the force to $\sim 1/2\mu$ and hence improves the convergence of partial wave amplitude.

With no subtractions we can write

$$A(q^2, \cos\theta) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{A_t(q^2, 1+t'/2q^2)}{(t'-t)} + \int_{4\mu^2}^{\infty} du' \frac{A_u(q^2, -1-\frac{u'}{2q^2})}{(u'-u)}$$

$$= \frac{1}{\pi}\int_0^{\infty} dq'^2 B_s\left(q'^2, 1+\frac{2q^2+\mu^2}{q'^2}\right)\left[\frac{1}{q'^2+\mu^2+\frac{1}{2}q^2(1-\cos\theta)}\right.$$

$$\left. + \frac{1}{q'^2+\mu^2+\frac{1}{2}q^2(1-\cos\theta)}\right] \qquad (24)$$

B_s is generally complex but the imaginary part of B_s vanishes in the lower range of the integral. Since we have

q^2 and $q'^2 > 0$

$$\operatorname{Im} B_s\left[q'^2, 1 + \frac{2(q^2+\mu^2)}{q'^2}\right] = B_{st}\left(4(q'^2+\mu^2), 4(q^2+\mu^2)\right) \qquad (25)$$

which is zero outside the shaded region.

So if we make a subtraction to suppress high energy effects, the remainder will be almost entirely real for small q^2 (i.e.) we are ignoring singularities of B_3. However when q^2 becomes large, the imaginary part cannot be suppressed. The subtraction can be made by removing the S-wave part of the amplitude.

For I = 0, 2

$$A^{\pm}(q^2, \cos\theta) = A^{(0)I}(q^2) + \frac{1}{\pi}\int_0^\infty dq'^2 \sum_{I'} \alpha_{II'} A_s^{I'}\left[q'^2, 1 + \frac{2(q^2+\mu^2)}{q'^2}\right]$$

$$\left\{\frac{1}{2}\left[\frac{1}{q'^2+\mu^2+\frac{1}{2}(1-\cos\theta)} + \frac{1}{q'^2+\mu^2+\frac{1}{2}q^2(1+\cos\theta)}\right]\right.$$

$$- \frac{1}{q^2}\log\left(1 + \frac{q^2}{q'^2+\mu^2}\right) \qquad (26)$$

For I = 1 we subtract the P wave

$$A^I(q^2, \cos\theta) = 3\cos\theta\, A^{(1)}(q^2) + \frac{1}{\pi}\int_0^\infty dq'^2 \sum_{I'} \alpha_{II'} A_s^{I'}$$

$$\left\{ \tfrac{1}{2}[\ldots] - \frac{3\cos\theta}{q^2}\left[\left(1 + \frac{2q'^2+\mu^2}{q^2}\right]\log\left(1 + \frac{q^2}{q'^2+\mu^2}\right) - 2\right]\right\} \qquad (27)$$

These are exact expressions but we shall hereafter replace $A_s^{I'=0,2}$ within the integral by

$$\operatorname{Im} A^{0,2}(q'^2) \text{ for } q'^2 > 0$$

and

$$A_s^{I=1}(q'^2, \cos\theta') \approx 3\cos\theta' \operatorname{Im} A^{(1)}(q'^2) \text{ for } q'^2 > 0$$

Formulation of Integral Equations

With $\nu = q^2/\mu^2$ we have

$$A^{(l)I}(\nu^2) = \frac{1}{\pi}\int_{-\infty}^{-1} d\nu' \frac{\operatorname{Im} A^{lI}(\nu')}{\nu' - \nu} + \frac{1}{\pi}\int_0^\infty d\nu' \frac{\operatorname{Im} A^{(l)I}(\nu')}{\nu' - \nu}$$

(28)

Using unitarity we have the partial wave amplitudes behaving like

$$\operatorname{Im} A^{lI}(\nu) \longrightarrow \tfrac{1}{2}$$

(29)

and

$$R_e A^{(\ell)I}(\nu) \to 0$$

We also know that partial wave amplitude of order ℓ behaves at the origin as ν^{ℓ} so we can consider

$$A'^{(\ell)I}(\nu) = \frac{1}{\nu^{\ell}} A^{(\ell)I}(\nu) \qquad (30)$$

which satisfies (28) and except for $\ell = 0$ behaves like $1/\nu^{\ell}$ at ∞ The higher the angular momentum the smaller is the relative contribution from highe energy (ν') in the dispersion integral. It is only for the S waves that the distant contributions are expected to be important (i.e.) short range effects are not felt by the centrifugally excluded higher ℓ waves

Let us consider the S-wave amplitude first. We have

$$A^{(\ell)I}(\nu) = N_0^I(\nu) / D_0^I(\nu) \qquad (31)$$

where N_0 and D_0 are both real analytic function: the numerator contains the left hand cut and denominator contains the right hand cut and D_0 should have no zeros.

Then we have

$$\left.\begin{aligned} Im\, N_0^I(\nu) &= D_0^I(\nu)\, Im\, A^{(0)I}(\nu) \\ Im\, D_0 &= 0 \end{aligned}\right\} \text{for } \nu < -1$$

$$Im\, N_0 = Im\, D_0 = 0 \qquad \text{for } -1 < \nu < 0$$

$$\operatorname{Im} N_0^I(\nu) = 0$$

$$\left.\begin{aligned}\operatorname{Im} D_0^I(\nu) = N_0^I(\nu) \operatorname{Im} \frac{1}{A^{(0)I}(\nu)}\end{aligned}\right\} \text{for } \nu > 0$$

(33)

(33) normalizes the S-wave amplitude to a_I at the point $\nu = 0$. Therefore we set $N_0^I(\nu_0) = a_I$ and $D_0^I(\nu_0) = 1$. Further $A^{(0)I}(\nu) \to$ constant at ∞ and so we assign constant asymptotic behaviour to the numerator and require the denominator not to vanish.

Then we have

$$N_0^I(\nu) = a_I + \frac{\nu - \nu_0}{\pi} \int_{-\infty}^{-1} d\nu' \frac{\operatorname{Im} A^{(0)}(\nu') D_0^I(\nu')}{(\nu' - \nu)(\nu' - \nu_0)}$$

(34)

and so we make a subtraction for the S wave at the symmetry point $\nu_0 \to -2/3$ to obtain

$$A^{(0)I}(\nu) = a_I + \frac{\nu - \nu_0}{\pi} \int_{-\infty}^{-1} d\nu' \frac{\operatorname{Im} A^{(0)I}(\nu')}{(\nu' - \nu)(\nu' - \nu_0)}$$

$$+ \frac{\nu - \nu_0}{\pi} \int_0^{\infty} d\nu' \frac{\operatorname{Im} A^{(0)I}(\nu')}{(\nu' - \nu)(\nu' - \nu_0)}$$

(35)

and for higher ℓ values

$$\frac{A^{\ell}(\nu)}{\nu^{\ell}} = \frac{1}{\pi}\int_0^{\infty} d\nu' \frac{Im\, A^{(\ell)}(\nu')}{(\nu'-\nu)\nu'^{\ell}} + \int_{-\infty}^{-1} d\nu' \frac{Im\, A^{\ell}(\nu')}{\nu'^{\ell}(\nu'-\nu)} \tag{36}$$

The important contribution in the integral comes from the elastic scattering region where we have

$$Im\left[A^{\ell}(\nu)\right]^{-1} = -\left(\frac{\nu}{\nu+1}\right)^{1/2} \tag{37}$$

and for the left hand cut we take only $\ell = 0$ and 1 terms .

We have then

$$Im\, A^{(\ell)I}(\nu) = \frac{1}{\nu}\int_0^{-\nu-1} d\nu'\, P_\ell\left(1+\frac{2\nu'+1}{\nu}\right)$$

$\nu < -1$

$$\left[\alpha_{I0}\, Im\, A^{(0)(0)}(\nu') + \alpha_{I2}\, Im\, A^{(0)2}(\nu') + 3\left(1+\frac{2(\nu+1)}{\nu'}\right)\alpha_{I1}\, Im\, A^{(1)1}(\nu')\right] \tag{38}$$

$$D_0^I(\nu) = 1 - \frac{\nu-\nu_0}{\pi}\int_0^{\infty} d\nu' \left(\frac{\nu'}{\nu'+1}\right)^{1/2} \frac{N_3^I(\nu')}{(\nu'-\nu)(\nu'-\nu_0)} \tag{39}$$

or we have if $\omega = -\nu$, $E_0^I(\omega) = D_0^I(\nu)$ and

$$\operatorname{Im} A^I(\nu) = f_\ell^I(\nu)$$

$$E_0^I(\omega) = 1 + (\omega + \nu_0) K(\omega, -\nu_0)\, a_I + \frac{\omega + \nu_0}{\pi} \int_1^\infty d\omega' \frac{K(\omega, \omega') f_0^I(\omega') E_0^I(\omega')}{\omega' + \nu_0} \tag{40}$$

with

$$K(\omega, \omega') = \frac{1}{\pi} \int_0^\infty d\nu'' \frac{[\nu''/(1+\nu'')]^{1/2}}{(\nu'' + \omega)(\nu'' + \omega')}$$

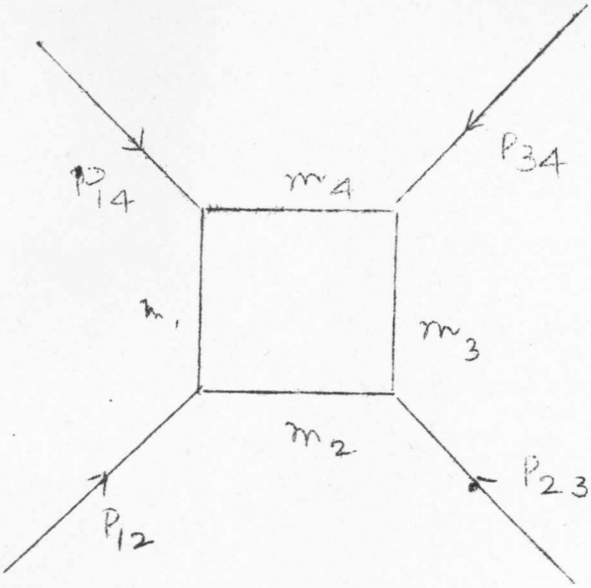
$$= \frac{2}{\pi(\omega - \omega')} \left\{ \left(\frac{\omega}{\omega - 1}\right)^{1/2} \log\left[\sqrt{\omega} + \sqrt{(\omega - 1)}\right] - \left(\frac{\omega'}{\omega' - 1}\right)^{1/2} \log\left[\sqrt{\omega'} + \sqrt{\omega' - 1}\right] \right\} \tag{41}$$

If f_0^I is known this would be a Fredholm equation which is soluble. All that we have achieved now is a non-linear inte - gral equation.

Chew, Mandelstam and Noyes have solved these equations assuming the dominance of the S-waves which corresponds to a small value of λ. But an increase in the value of λ still gave small P-waves so that is was not possible to reproduce the resonance in the (33) state. A possible reason for this may be their complete neglect of the double spectral function on the lef hand cut.

Lecture VII

SPECTRAL REPRESENTATION IN 4TH ORDER PERTURBATION THEORY



In this lecture, we shall study spectral representations for scattering amplitudes in perturbation theory. The fourth order amplitude is given by

$$A^{(4,i)} = \frac{i g^4}{(2\pi)^4} \int \frac{d^4 q}{[(p_{12}-q)^2 - m_1^2][(p_{23}+q)^2 - m_3^2][(p_{12}+p_{14}-q)^2 - m_4^2][q^2 - m_2^2]} \tag{1}$$

We have to find the region where this matrix element is an analytic function of the two variables s and t

We have

$$p_{12} + p_{23} + p_{34} + p_{41} = 0 \tag{2}$$

and

$$p_{13} = p_{12} + p_{23} = -(p_{34} + p_{41})$$
$$p_{24} = p_{23} + p_{34} = -(p_{41} + p_{12})$$

(3)

Define

$$P_{k\ell}^2 = m_k^2 + m_\ell^2 - 2 m_k m_\ell y_{k\ell} \tag{4}$$

The four variables associated with the single particle invariant are subject to the stability conditions

$$y_{12} > -1, \; y_{23} > -1, \; y_{34} > -1, \; y_{41} > -1 \tag{5}$$

which states that at each vertex the external mass is less than the sum of the two masses to which it is coupled. The integral can be parametrized and written as

$$A \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^1 dx_4 \; \frac{\delta(1 - x_1 - x_2 - x_3 - x_4)}{m_1 m_2 m_3 m_4 D^2} \tag{6}$$

where

$$D = \sum_{k=1}^{4} x_k^2 + 2 \sum_{k=1}^{4} \sum_{\ell=k+1}^{4} x_k x_\ell y_{k\ell} \tag{7}$$

The equation shows that this function is analytic if any one variable is complex while the others are real and hence

$$A(y_{13}, y_{24}) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{-\infty}^{\bar{y}_{13}(y_{24})} \frac{\operatorname{Im} A(t + i\epsilon, y_{24})}{t - y_{13}} \, dt \tag{8}$$

$\bar{y}_{13}$ is this boundary on the real axis. D when

$$D = (x_1 + x_3 - x_2 - x_4)^2 - 2 x_1 x_3 (y_{13} - 1) + 2 x_2 x_4 (y_{24} - 1)$$
$$+ 2\left[x_1 x_2 (1 + y_{12}) + x_2 x_3 (1 + y_{23}) + x_3 x_4 (1 + y_{24}) + x_1 x_4 (1 + y_{14})\right] \tag{9}$$

The stability conditions imply that the denominator does not vanish if

$$y_{13} > 1 \ , \quad y_{24} > 1 \tag{10}$$

However if $y_{13} < -1$ and or $y_{24} < -1$ then D may vanish $A(y_{13}, y_{14})$ is an analytic function of both variables in this region (Y_{13}, Y_{24}). This plane is contained in a larger region R in which A is an analytic function. of Y_{13} and Y_{24}. The problem is to find the region R in which the expression D cannot vanish if x is in T,

$$\left\{ x_K \geqslant 0 \qquad x_1 + x_2 + x_3 + x_4 = 1 \right\} \tag{11}$$

T consists of the interior and boundary of an equilateral tetrahedron. x_K's are barycentric coordinates. At the vertices $x_K = 1, K = 1, 2, 3, 4)$, $D = 1$

We find the region R_i such that the expression D is positive on every edge of T, if and only if Y_{13}, Y_{24} are in R_i.

Next we find a smaller region R_{ii} such that the expression of D is on every face of T if and only if Y_{13}, Y_{24} are in R_{ii}. Finally we find R for which D cannot vanish at all in T. The three cases are discussed as follows:

<u>Case (i)</u>: Let D take non-positive values on some edge of T. On an edge $x_K = x_l = 0$. The smallest value that D can take on an edge is

$$\min \left[1 \ ; \ \tfrac{1}{2}(1 + y_{Kl}) \right]$$

However due to the stability conditions we have this value positive for Y_{12}, Y_{23}, Y_{34}, Y_{14}. Therefore we need consider only $x_1 = x_3 = 0$ or $x_2 = x_4 = 0$. The region Ri is therefore.

$$Ri\,(y_{13} > -1,\ y_{24} > -1) \tag{12}$$

Case (ii) We assume $(y_{13}$ & $y_{24})$ is in Ri and consider the possibility that D has a non-positive value on the face of T defined by $x_K = 0$ say $x_4 = 0$. Then we have the triangular conditions $x_i > 0$ and $x_1 + x_2 + x_3 = 1$

The denominator reduces to

$$D_6 = x_1^2 + x_2^2 + x_3^2 + 2(x_1 x_2 y_{12} + x_1 x_3 y_{13} + x_2 x_3 y_{23}) \tag{13}$$

Then condition $y_{12} > -1$, $y_{13} > -1$ and $y_{23} > -1$ implies that D_6 does not vanish on the edge Let us consider

$$y_{12} + y_{13} \geqslant 0 \tag{14}$$

$$y_{12} + y_{13} < 0 \quad \text{(in this case } y_{12} < 1,\ y_{13} < 1) \tag{15}$$

writing $\lambda = \min\,[1, y_{12}]$ and assume $Y_{12} = Y_{13}$

$$y_{12} \leq y_{13}$$

Then we write for the first case

$$D_0 = (x_2 - x_3 + \lambda x_1)^2 + (1-\lambda^2)\, x^2 \left[x_2 x_3 (1 + y_{23}) + x_1 x_2 (y_{12} - \lambda) + x_1 x_3 (y_{13} + \lambda) \right] \qquad (16)$$

Each term in this expression is non-negative and hence D_0 cannot vanish if $y_{23} > -1$. In the second case

$$D_0 = (x_1 + x_2 y_{12} + x_3 y_{13})^2 + \left[x(1 - y_{12}^2)^{1/2} - x_3 (1 - y_{13}^2)^{1/2} \right]^2 + 2 x_2 x_3 \left\{ y_{23} - y_{12} y_{13} + (1 - y_{12}^2)^{1/2} (1 - y_{13}^2)^{1/2} \right\} \qquad (17)$$

Hence D_0 is positive since $y_{12} + y_{13} < 0$ $(y_{12} < 1, y_{13} < 1)$

if

$$y_{23} > y_{12} y_{13} - \left[(1 - y_{12}^2)^{1/2} (1 - y_{13}^2)^{1/2} \right]$$

$$= \cos(\theta_{12} + \theta_{13}) \text{ if } \begin{array}{l} y_{12} = \cos(\theta_{12}) \\ y_{13} = \cos \theta_{13} \end{array} \qquad (18)$$

The condition $y_{12} + y_{13} < 0,\ y_{12} < 1,\ y_{13} < 1$ (19)

implies that

$$\theta_{12} + \theta_{13} > \pi \qquad (20)$$

Thus we have for the different faces of T the Region Rii defined by $Rii \left\{ y_{13} > L_{13},\ y_{24} > L_{24} \right\}$ (21)

where

$$L_{24} = \max[L_1, L_3]$$

$$L_{13} = \max[L_2, L_4]$$

(22)

and

$$\begin{array}{ll} L_1 = -1 & \gamma_{23} + \gamma_{34} \geq 0 \\ L_1 = \cos(\theta_{23} + \theta_{34}) & \gamma_{23} + \gamma_{34} < 0 \\ L_2 = -1 & \gamma_{14} + \gamma_{34} \geq 0 \\ L_2 = \cos(\theta_{14} + \theta_{34}) & \gamma_{14} + \gamma_{34} < 0 \\ L_3 = -1 & \gamma_{12} + \gamma_{14} \geq 0 \\ L_3 = \cos(\theta_{12} + \theta_{14}) & \gamma_{12} + \gamma_{14} < 0 \\ L_4 = -1 & \gamma_{12} + \gamma_{23} \geq 0 \\ L_4 = \cos(\theta_{12} + \theta_{23}) & \gamma_{12} + \gamma_{23} < 0 \end{array}$$

(23)

Case (iii) Similarly if D should not vanish in the interior of T then

a) $\gamma_{12} < 1, \ \gamma_{23} < 1, \ \gamma_{34} < 1, \ \gamma_{14} < 1$ (23)

b) $2\pi < \theta_{12} + \theta_{23} + \theta_{34} + \theta_{41} < 2\pi + 2\min[\theta_{12}, \theta_{23}, \theta_{34}, \theta_{41})$

and Region R is

$$R: \left\{\gamma_{24} > \gamma_{24}^{0}, \ \gamma_{13} > L_{13}\right\} + \left\{\gamma_{13} > \gamma_{13}^{0}, \ \gamma_{24} > L_{24}\right\} + \left\{\Delta(\gamma_{13}, \gamma_{24}) > 0\right\} \qquad (24)$$

where

$$\Delta(y_{13}, y_{24}) = \begin{vmatrix} 1 & y_{12} & y_{13} & y_{14} \\ y_{12} & 1 & y_{23} & y_{24} \\ y_{13} & y_{23} & 1 & y_{34} \\ y_{14} & y_{24} & y_{34} & 1 \end{vmatrix} \qquad (25)$$

and

$$\Delta(L_{13}, y_{24}^{0}) = 0 = \Delta(y_{13}^{0}, L_{24}) \qquad (26)$$

In terms of W^2 we have the physically accessible region given by

$$W^2 = (P_{12}+P_{23})^2 = (P_{14}+P_{34})^2$$
$$> max[(M_{12}+M_{23})^2, (M_{14}+M_{34})^2 \qquad (27)$$

$$\Delta^2 = -(P_{12}+P_{14})^2 = -(P_{23}+P_{34})^2$$
$$> max[-(M_{12}-M_{14})^2, -(M_{23}-M_{34})^2] \qquad (28)$$

In terms of y_{13} we have

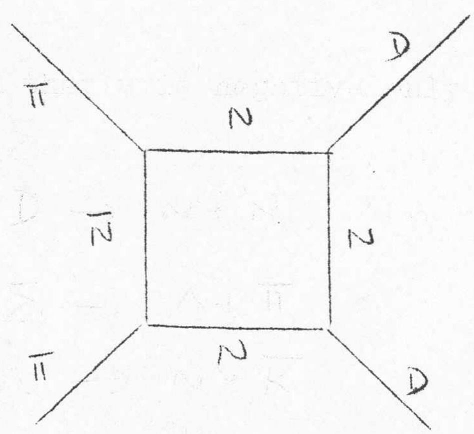
$$y_{13} = \frac{m_1^2 + m_3^2 - W^2}{2 m_1 m_3} \qquad (29)$$

The normal threshold is given by $L_{13} = -1$ Equations ( ) show that $L_{13} > -1$ only if sufficient number of y_{Kl}'s are negative. For each vertex say $M_{13} \to m_1 + m_3$ we calculate $\cos\theta = y = \dfrac{m_1^2 + m_3^2 - M_{13}^2}{2 m_1 m_3}$ and tabulate

It is found that y is negative only for the vartices

$D \rightarrow N + N$	$-\ .995$
$\bar{\Sigma} \rightarrow \Lambda + \bar{\pi}$	$-\ .50$
$\Lambda \rightarrow N + \bar{K}$	$-\ .13$
$\Sigma \rightarrow N + \bar{K}$	$-\ .31$
$\Xi \rightarrow \Lambda + \bar{K}$	$-\ .24$
$\Xi \rightarrow \Sigma + \bar{K}$	$-\ .07$

Thus we conclude that for $\pi + N \rightarrow \pi + N$ <u>no</u> anomalous threshold exists and $L_{13} = -1$ and $W^2 = (M_N + M_\pi)^2$ is the threshold. For πD scattering we have



$$L_2 = L_4 = L_{13} = \cos(174^\circ + 8^\circ) = -.999 > -1$$

and

$$W^2 = (2M_N^2) - .002 M_N^2 \tag{29}$$

which is slightly different from the normal threshold.

For ΣN scattering, We have,

$$L_2 = L_4 = L_3 = \cos(86^\circ + 120^\circ) = -.899 > -1$$

$$W^2 = (M_N + M_\Lambda)^2 - .202\, M_N M_\Lambda \tag{30}$$

The corresponding regions Ri, Rii and R are given by

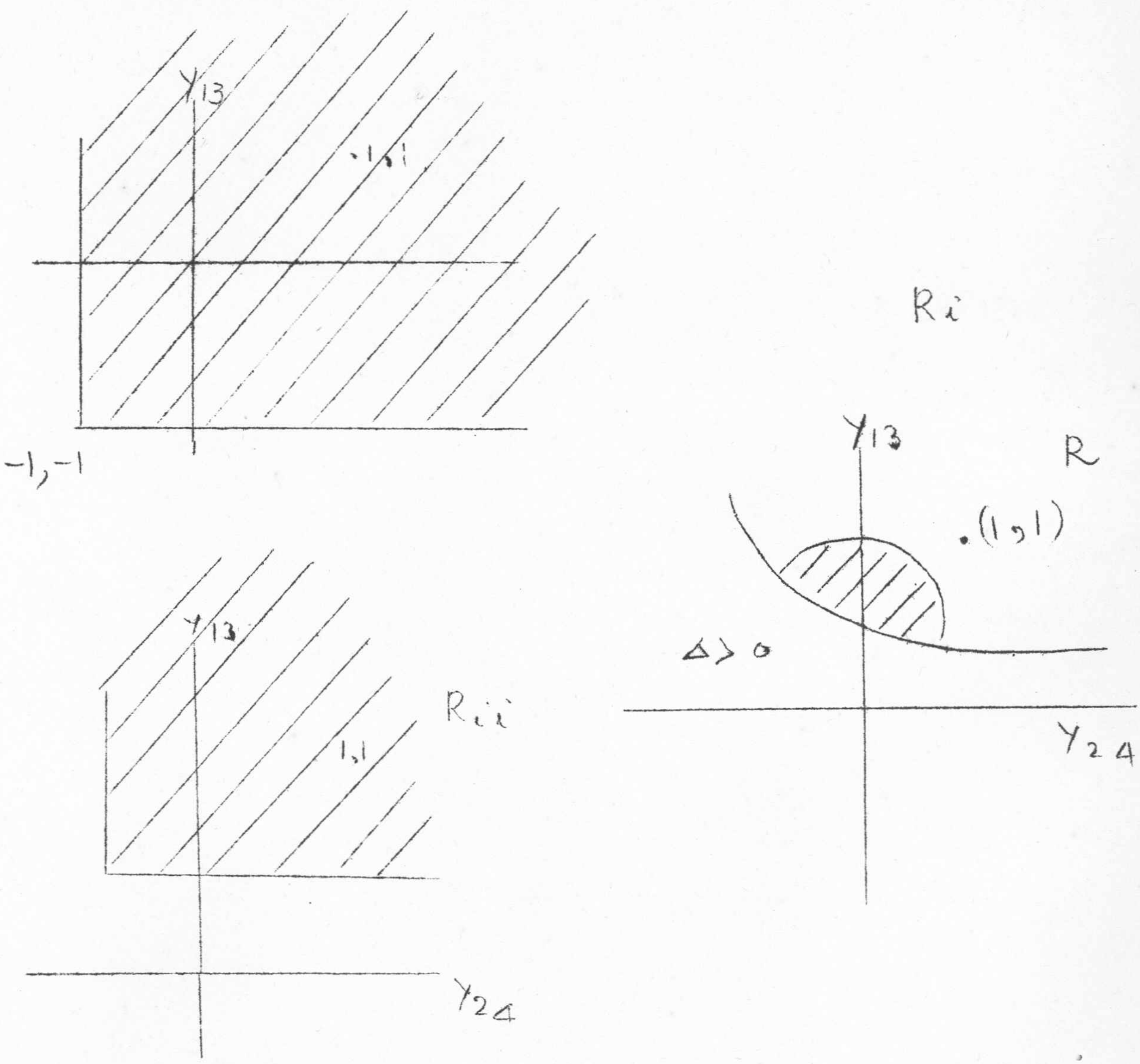
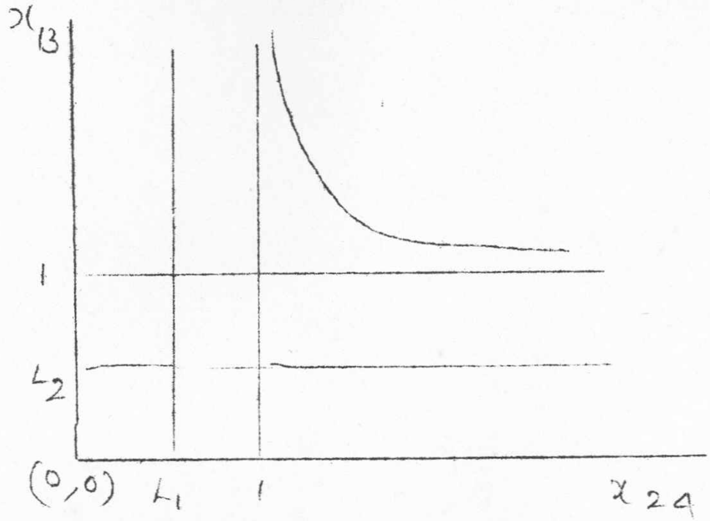


Fig.10

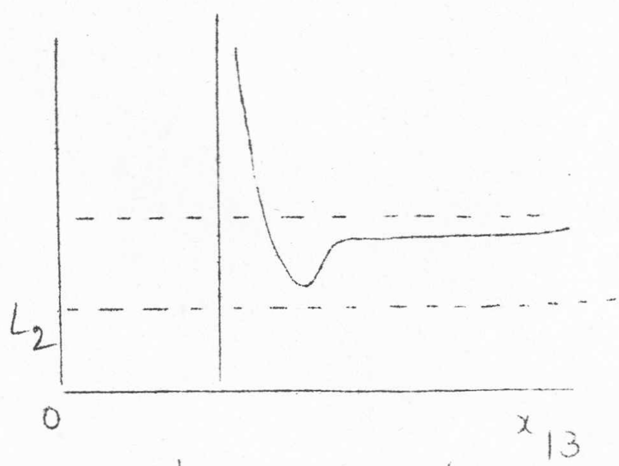
The spectral functions A_{13} then do not vanish between L_{13} and $x_{13} = - Y_{13} = 1$



$K_2 = 0$ or $K_4 = 0$ given by the lines L_4 and L_2. $K_1 = 0$ or $K_3 = 0$ give L_1 and L_3.

$$K = 1 - Y_{12}^2 - Y_{13}^2 - Y_{23}^2 + 2Y_{12} \cdot Y_{23} Y_{13} \tag{31}$$

When there is anomalous threshold in process I we have this curve modified as



Anomalous threshold means that vertical or horizontal tangency for this curve exist besides the usual ones within which A_{13} is nonvanishing

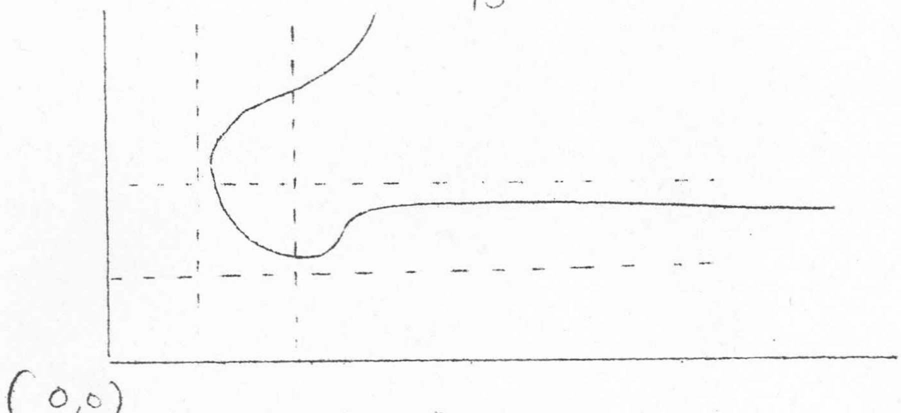


Fig 12

Lecture VIII

TRANSITION AMPLITUDES IN PERTURBATION THEORY

The objective is to express the fourth order Feynman diagram in the form of the Mandelstam representation. In general the reducible graphs contributes only to the single dispersion integrals. The irreducible diagrams represented by Fig.13 contributes to the matrix element,

$$A^{4(1)} = \frac{i g^4}{(2\pi)^4} \int \frac{d^4 q}{\left[(P_1 - q)^2 - M^2\right]\left[(P_1 + P_2 - q)^2 - M^2\right] \times \left[(P_3 + q)^2 - M^2\right] \times \left[q^2 - M^2\right]} \tag{1}$$

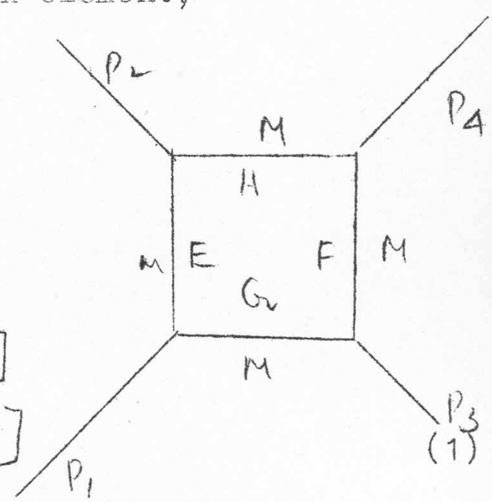


Fig 13.

This is a function of both the invariants s & t. First we show that keeping t fixed, real and negative, this integral is an analytic function of s with no singularities in the complex plane. The imaginary part which appears in the integrand is then calculated from (1) As we have already seen in the previous lecture for fixed t real and positive, the amplitude has no complex singularities.

If the condition for anomalous threshold is also not satisfied then the denominators does not vanish also on the real axis below the normal threshold.

$$A^4(s,t,u_1) = \frac{1}{\pi}\int_{(2M)^2}^{\infty} \frac{Im\, A^{(11)}(s',t,u_1)}{(s'-s)}\, ds'$$

(2)

Calculation of $Im\, A$

The propagators are written as

$$\frac{1}{(p_1-q)^2-M^2+i\epsilon} = \frac{P}{(p_{12}-q_2)-(\vec{p}_1-\vec{q})^2-M^2}$$

$$-\frac{i\pi}{2q_0}\delta\left\{p_{10}-q_0+\sqrt{(\vec{p}_1-\vec{q})^2+M^2-i\epsilon}\right\}-\frac{i\pi}{2q_0}\delta\left\{(p_{10}-q_{10})-\sqrt{(\vec{p}_1-\vec{q})^2+M^2-i\epsilon}\right.$$

(3)

The imaginary part of (1) should therefore contain an even number of δ- functions. However the integral of this product of the four principal parts vanish. The product of the four δ- functions is also zero. Thus we are left with terms which have 2 δ -functions and 2 principal parts.

If in such a term the arguments of the two δ -functions have imaginary parts of opposite sign then this multiplies the term by two. Otherwise the term does not contribute. The arguments of this δ- function vanish when the intermediate particle is on the mass shell. Two adjacent lines cannot be on the mass shell since at a vertex we cannot have three real lines. Further the two opposite lines E and F cannot be simultaneously on the mass shell and have δ functions which contribute to $\Im m\, A$. The two δ functions are

$$\delta\left[-p_{10}+q_0 \pm \sqrt{(\vec{p}_1+\vec{q}\,)^2+m^2-i\epsilon}\right.$$

and

$$\delta\left[p_{30}+q_0 \pm \sqrt{(\vec{p}_3+\vec{q}\,)^2+m^2-i\epsilon}\right. \tag{4}$$

If they are to be non-zero and have opposite signs for $i\epsilon$ then $(p_{10} - q_0)$ and $(p_{30} + q_0)$ must have the same sign. However the two lines are on the mass shell and hence time like, and the fourth components of their moments must be timelike. This is t and by assumption it is space like. It is therefore not possible to have E & F on the mass shell. This leaves the pair G & H. (i.e.)

$$\delta\left(q_0 - (p_1+p_2) \pm \sqrt{q^2+m^2-i\epsilon}\right.$$

$$\delta\left(q_0 \pm \sqrt{q^2+m^2-i\epsilon}\right. \tag{5}$$

If this to exist $(p_1+p_2)_0 - q_0$ and q_0 must have the same sign and must therefore be positive as the incoming particles have positive energy.

Therefore

$$\mathrm{Im}\,A = \frac{g^4}{8\pi^2}\int d^4q \frac{1}{\{(p_1-q)^2-M^2\}\{(p_3+q)^2-M^2\}}$$

$$\theta(q_0)\,\theta((p_1+p_2)_0-q_0)\,\delta(q^2-M^2) \qquad s>(2M)^2 \tag{6}$$

Writing $t_{ie} = (p_1-q)^2$

and $t_{io} = (p_3+q)^2$

and making use of the δ - functions, only two integrations remain and we have

$$\mathrm{Im} = \frac{g^4 k}{32\pi^2\omega}\int d^2\vec{n}_i \frac{1}{(t_{ie}-M^2)(t_{io}-M^2)} \qquad s>(2M)^2 \tag{7}$$

and since

$$t_{io(ie)} = 2k^2(z_{io(ie)}-1) \tag{8}$$

and

$$z_{io} = z z_{ip} + (1-z^2)^{1/2}(1-z_{ie}^2)^{1/2}\cos\phi \tag{9}$$

We have $\mathrm{Im}\,A = \dfrac{g^4 k c m}{82\pi^2\omega}\displaystyle\int_{-1}^{1} dz_{ip}\int_0^{2\pi} d\phi\,.$

$$\times \frac{1}{[2k^2(z_{ip}-1)-M^2]\,[2k^2\{z_{ip}z+(1-z^2)(1-z_{ip}^2)\cos\theta-1\}-M^2]} \tag{10}$$

Replacing Z by t and writing

$$k^2 = \frac{s - 4M^2}{4} \qquad [k = \text{centre of mass momentum}$$

we find that

$$Im\, A = \frac{1}{16\pi \{K(s,t)\}^{1/2}} \log \frac{\alpha(s,t) + \left(\frac{k}{s}\right)^{1/2} [K(s,t)]^{1/2}}{\alpha(s,t) - \left(\frac{k}{s}\right)^{1/2} [K(s,t)]^{1/2}}$$

$$= 0 \text{ for } s < 2M^2$$

(11)

where

$$K(s,t) = 4st[s,t - 4M^2(s+t) + 12M^4]$$

$$\alpha(s,t) = st - 2M^2 s - 4M^2 t + 6M^4$$

(12)

The r.h.s. of equation (2) is an analytic function of t in the complex t plane except for a cut along that portion of the real axis where k and t are both positive. The discontinuity across the cut is $-1/\{4[K(s,t)]^{1/2}\}$

Hence we may write

$$Im\, A^{(4)}(s,t,u_1) = \frac{1}{\pi} \int dt' \frac{A_{13}^{(4)}(s,t')}{(t'-t)}$$

(13)

where

$$A_{13}(s,t) = \frac{1}{[8K]^{1/2}} \qquad K > 0, \; t > 0, \; s > 2M^2$$

(14)

Thus

$$A^4(s,t) = \frac{1}{\pi^2} \int ds' dt' \frac{A_{13}^4(s',t')}{(s'-s)(t'-t)} \tag{15}$$

The boundary of the region where A_{13} is non zero is given by

$$st - 4M^2(s+t) + 12M^4 = 0 \tag{16}$$

This approaches $s = 4M^2$ asymptotically

In the more general case when the mass are not equal, we have

$$A_{13} = -\frac{1}{8 q_e q_0 q_i \, W\{L(z_1, z'_{ip}, z'_{i0})\}^{1/2}}, \quad z \geq z_1 \quad (17)$$

$$= 0 \text{ for } z < z_1$$

where

$$z_1 = z'_{ip} z'_{i0} + (z'^2_{ip} - 1)^{1/2} (z'_{i0} - 1)^{1/2} \tag{18}$$

<u>Kinematics of general scattering processes and the Mandelstam Representation</u>

The masses of the four particles involved are shown in fig. 14. For convenience it is assumed that

$$m_1 \geq m_2 \geq m_3 \geq m_4 \geq 0$$

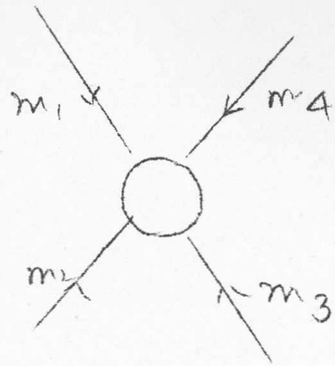


Fig.14

$$\text{If } m_4 > m_1 + m_2 + m_3 \qquad (20)$$

the decay process is also energetically possible.- As usual

$$p_1 + p_2 + p_3 + p_4 = 0$$

$$p_i^2 = m_i^2$$

$$r = (p_1 + p_2)^2 = (p_3 + p_4)^2$$

$$s = (p_1 + p_3)^2 = (p_2 + p_4)^2$$

$$t = (p_1 + p_4)^2 = (p_2 + p_3)^2$$

(23)

and $r + s + t = K = m_1^2 + m_2^2 + m_3^2 + m_4^2$

(24)

In the c.m.s.

$$p_1 = (E_1, \vec{q}_1), \quad p_2 = (E_2, -\vec{q}_1)$$

$$p_3 = (-E_3, -\vec{q}_3), \quad p_4 = (-E_4, \vec{q}_3)$$

$$r = W^2, \quad W = E_1 + E_2 = E_3 + E_4$$

$$4rq_1^2 = [r - (m_1 + m_2)^2][r - (m_1 - m_2)^2]$$

$$4rq_2^2 = [r - (m_3 + m_4)^2][r - (m_3 - m_4)^2]$$

(25)

The invariants are related to the scattering angle by

$$2s = K - r + 4q_1 q_3 z - \frac{(m_1^2 - m_2^2)(m_3^2 - m_4^2)}{r}$$

$$2t = K - r - 4q_1 q_3 + \frac{(m_1^2 - m_2^2)(m_3^2 - m_4^2)}{r} \tag{26}$$

where $z = \cos(\vec{q}_1\ \vec{q}_3)$

The condition for physical scattering is given by

$$r > (m_1 + m_2)^2 \quad \text{or} \quad r < (m_1 - m_2)^2 \tag{27}$$

If the scattering angle is to be real

$$\begin{vmatrix} P_1^2 & P_1 \cdot P_2 & P_1 \cdot P_3 \\ P_2 \cdot P_1 & P_2^2 & P_2 \cdot P_3 \\ P_3 \cdot P_1 & P_2 \cdot P_3 & P_3^2 \end{vmatrix} \tag{28}$$

This may be written as

$$rst > (r + s + t)^2 \; (ar + bs + ct) \tag{29}$$

where

$$k^3 a = (m_2^2 m_1^2 - m_3^2 m_4^2)(m_1^2 + m_2^2 - m_3^2 - m_4^2)$$

$$k^3 b = (m_1^2 m_3^2 - m_2^2 m_4^2)(m_1^2 + m_3^2 - m_2^2 - m_4^2)$$

$$k^3 c = (m_1^2 m_4^2 - m_2^2 m_3^2)(m_1^2 + m_4^2 - m_2^2 - m_3^2) \tag{30}$$

r,s,t may be regarded as homogeneous coordinates in a plane in which the line at infinity is $r + s + t = 0$. Then the region (29) is given by a cubic curve in this plane whose asymptotics are $r = 0$, $s = 0$ and $t = 0$. The curve intersects its asymptotes on the line $ar + bs + ct = 0$ The regions I, II, III and IV are the physical regions for the corresponding processes (Fig. 15) Since

$$m_1 \geqslant m_2 \geqslant m_3 \geqslant m_4, \quad a \geqslant b \geqslant c$$

a and b are necessarily positive [from $K^3 a = \cdots$]

although C may have either sign. If C is negative the line $ar + bs + ct = 0$ passes within the rst triangle. and therefore the region III includes part of this triangle. r,s,t can all be positive for this process.

The lines $r = (m_1 + m_2)^2$ or $(m_1 - m_2)^2$ etc are tangential to the curve so that the entire boundary of each of the physical regions is a part of the curve. If (20) is satisfied

$$(m_1 - m_2)^2 < r < (m_1 + m_2)^2$$
$$(m_3 - m_4)^2 < r < (m_3 + m_4)^2 \tag{31}$$

do not overlap that region IV is allowed kinematically. If (20) is not satisfied the trips overlap o that the region IV is exluded. We assume that ordinary dispersion relations, hold for integration along AB. Poles occur when AB crosses $r = m_a^2$ or $s = m_{a'}^2$ where m_a or $m_{a'}$

is the mass of a single particle intermediate state. Mandelstam's A_{rs} will be non-zero in the region $r > r_a$ and $s > s_a$ (say). Proceeding in the same way as that of Mandelstam using unitarity for process I, we have

$$A_{1r}(r,s) = \tfrac{1}{2}(2\pi)^{-2}\int d^4p_5\, d^4p_6\, \delta(p_5^2 - m_5^2)$$

$$\delta(p_6^2 - m_6^2)\,\theta(p_{50})\,\theta(p_{60})\,\delta(p_5+p_6-p_1-p_2)\,A_3(r,s_3)\,A_2(r,s_2) \qquad (32)$$

A_1, A_2 and A_3 now refer to the processes

$$\begin{array}{lll} I_1 : & 1+2 \to 3+4 & p_1+p_2+p_3+p_4 = 0 \\ I_2 : & 1+2 \to 5+6 & p_1+p_2+p_5+p_6 = 0 \\ I_3 : & 3+4 \to 5+6 & \therefore\ p_3+p_4 = (p_5+p_6) \end{array} \qquad (33)$$

We now convert the p_5 integrations into one over p_5^2, $(p_5-a)^2$ $(p_5-b)^2$ and $(p_5-c)^2$ where a, b, c are any three time like vectors. The Jacobian for the transformation is

$$J = \tfrac{1}{16}\left[-\Delta(a,b,c,p_5)\right]^{1/2} \qquad (34)$$

where

$$\Delta(a,b,c,p) = \begin{bmatrix} a^2 & a.b & a.c & a.p \\ b.a & b^2 & b.c & b.p \\ c.a & c.b & c^2 & c.p \\ p.a & p.b & p.c & p^2 \end{bmatrix} \qquad (35)$$

It is convenient to choose $a = p_1 + p_2,\ b = p_1,\ c = -p_3$ (36)

so that

$$(p_5 - a)^2 = p_6^2,\ (p_5 - b)^2 = s_2,\ (p_5 - c)^2 = s_3 \tag{37}$$

$$\Delta(a,b,c,p) = \frac{1}{16}\begin{vmatrix} 2r & r + m_1^2 - m_2^2 & r + m_3^2 - m_4^2 & r + m_5^2 - m_6^2 \\ r + m_1^2 - m_2^2 & 2m_1^2 & m_1^2 + m_3^2 - s_1 & m_1^2 + m_5^2 - s_2 \\ r + m_3^2 - m_4^2 & m_1^2 + m_3^2 - s_1 & 2m_3^2 & m_3^2 + m_5^2 - s_3 \\ r + m_5^2 - m_6^2 & m_1^2 + m_5^2 - s_2 & m_3^2 + m_5^2 - s_3 & 2m_5^2 \end{vmatrix}$$

$$= \Delta(r, s_1, s_2, s_3) \tag{38}$$

This transformation is not one to one since the products are not changed by changing the sign of component of p_5 perpendicular to a, b and c. Therefore we have an extra factor 2.

$$A_{1r}(r,s) = \frac{1}{64\pi^2}\int_{\Delta<0} ds_2 \int ds_3 \left[-\Delta(r, s_1, s_2, s_3)\right] A_3^*(r, s_3)\, A_2(r, s_2) \tag{39}$$

If $r > (m_1+m_2)^2,\ r > (m_3+m_4)^2,\ r > (m_5+m_6)^2$ (40)

and take s_1 to be in the physical region for process I. The condition $\Delta < 0$ implies that s_2 and s_3 are in the physical regions for processes I_2 and I_3. Thus the denominators of dispersion integrals for A_2 and A_3 will not vanish and we may ignore the $i\epsilon$ term. Then perform s_2 and s_3 integrations using

$$z_1 = \cos(q_1, q_2) \quad z_2 = \cos(q_1, q_5) \quad z_3 = \cos(q_3 q_5)$$

$$\Delta(r, s_1, s_2, s_3) = r\, q_1^2\, q_3^2\, q_5^2\, K(z_1 z_2 z_3) \tag{41}$$

where

$$K = z_1^2 + z_2^2 + z_3^2 - 1 - 2 z_1 z_2 z_3$$

The spectral functions can immediately be found from the discontinuity across the cut (real s_1 axis) Thus A_{ins} reduces to the same form as that of Mandelstam.

Lecture IX

Equality of the π^+ and π^- total interaction Cross-section at High Energies

We define the nucleon (p and p') and pion (q and q') four-momentum by

$$p = p' = (0, M) \quad , \quad q = q' = (\vec{q}\,, \omega) \tag{1}$$

The forward scattering cross-section is given by

$$\sigma_{Tot} = (2\pi)^4 \frac{1}{q} \sum_n (4\pi)^2 \frac{\delta\,(P_n - p - q)}{N_n^2 \; V^n} \left| F_n(n, p, q) \right|^2 \tag{2}$$

where

$$F_{\beta\alpha}(\omega) = \frac{1}{4\pi} \left\{ \delta_{\beta\alpha}\left(A^+ + \omega B^+\right) + \tfrac{1}{2}\left[\tau_\beta , \tau_\alpha\right]\left(A^- + \omega B^-\right) \right\} \tag{3}$$

and $4\pi \; \mathrm{Im} \; F_{\alpha\alpha}(\omega) = \tfrac{1}{2} q \, \sigma$ (4)

The dispersion Relations are

$$A^{\pm}(\omega) = \frac{1}{\pi} \int_m^\infty \frac{d\omega'}{\omega' - \omega} \, \mathrm{Im}\, A^{\pm}(\omega') + \frac{1}{\pi} \int_{-\infty}^{-m} \frac{d\omega'}{\omega' - \omega} \, \mathrm{Im}\, A^{\pm}(\omega')$$

$$B^{\pm}(\omega) = \frac{-g^2}{m^2 + 2M\omega} \pm \frac{g^2}{m^2 - 2M\omega} + \frac{1}{\pi} \int_m^\infty \frac{d\omega'}{\omega' - \omega} \, \mathrm{Im}\, B^{\pm}(\omega')$$

$$+ \frac{1}{\pi} \int_{-\infty}^{-m} \frac{d\omega'}{\omega' - \omega} \, \mathrm{Im}\, B^{\pm}(\omega') \tag{5}$$

Crossing symmetry gives

$$\mathrm{Im}\, A^{\pm}(-\omega) = \mp\, \mathrm{Im}\, A^{\pm}(\omega)$$

$$\mathrm{Im}\, B^{\pm}(-\omega) = \pm\, \mathrm{Im}\, B^{\pm}(\omega)$$

(6)

so that

$$\mathrm{Re}\, A^{+}(\omega) = \frac{2}{\pi} P \int_{m}^{\infty} \omega' d\omega' \frac{\mathrm{Im}\, A^{+}(\omega')}{(\omega'^2 - \omega^2)}$$

(7)

$$\mathrm{Re}\, A^{-}(\omega) = \frac{2\omega}{\pi} P \int_{m}^{\infty} d\omega' \frac{\mathrm{Im}\, A^{-}(\omega')}{(\omega'^2 - \omega^2)}$$

(8)

$$\mathrm{Re}\, B^{+}(\omega) = \frac{-g^2}{M} \frac{\omega}{\omega - \left(\frac{\cancel{m}^2}{2M}\right)^2} + \frac{2\omega}{\pi} P \int_{m}^{\infty} d\omega' \frac{\mathrm{Im}\, B^{+}(\omega')}{\omega'^2 - \omega^2}$$

(9)

$$\mathrm{Re}\, B^{-}(\omega) = \frac{g^2}{2M^2} \frac{m^2}{\omega^2 - \left(\frac{m^2}{2\pi}\right)^2} + \frac{2}{\pi} P \int_{m}^{\infty} \omega' d\omega' \frac{\mathrm{Im}\, B^{(-)}(\omega')}{\omega' - \omega^2}$$

(10)

In terms of isotopic spin states

$$F^{+} = \frac{1}{4\pi}\left(A^{+} + \omega B^{+}\right) = \frac{1}{3}\left[F^{1/2} + 2F^{3/2}\right] \tag{11}$$

$$F^{(-1)} = \frac{1}{4\pi}\left(A^{-} + \omega B^{-}\right) = \frac{1}{3}\left(F^{1/2} - F^{3/2}\right)$$

(12)

However

$$\tfrac{1}{2}(F_{\pi^-} - F_{\pi^+}) = \tfrac{1}{3}(F^{1/2} - F^{3/2})$$

$$\tfrac{1}{2}(F_{\pi^-} + F_{\pi^+}) = \tfrac{1}{3}(F^{1/2} + F^{3/2}) \tag{13}$$

Writing

$$g^2/4\pi \left(\frac{m}{2M}\right)^2 = f^2, \left(\frac{m^2}{2M}\right)^2 = \omega_B^{\,2} \tag{14}$$

and using optical theorem we have

$$\mathrm{Re}\left[F_{\pi^-}(\omega) - F_{\pi^+}(\omega)\right] = \frac{2\omega f^2}{\omega^2 - \omega_B^2} + \frac{\omega}{2\pi^2} P$$

$$\int_m^\infty \frac{q' d\omega'}{\omega'^2 - \omega^2} \; \frac{\sigma_{\pi^-}(\omega') - \sigma'_{\pi^+}(\omega')}{2} \tag{15}$$

This converges if $\sigma_{\pi^-}(\omega) - \sigma_{\pi^+}(\omega) \to 0$ faster than $1/\log \omega'$ as $\omega' \to \infty$. Experimental data indicate that $\sigma_{\pi^+}, \sigma_{\pi^-}$ tend to a constant at high energies and using the above equation one can determine the coupling constant

$$f^2 = .082 \pm .015 \tag{16}$$

We also see that

$$\tfrac{1}{2} \mathrm{Re}\left[F_{\pi^-}(\omega) + F_{\pi^+}(\omega) \right] = \frac{1}{4\pi} A^+(0) - \frac{g^2}{4\pi} \frac{\omega^2}{M} \frac{1}{\omega^2 - \omega_B^2}$$

$$+ \frac{\omega^2}{2\pi^2} P \int_m^\infty \frac{q^2 d\omega'}{\omega'} \frac{1}{\omega'^2 - \omega} \frac{\sigma_{\pi^-}(\omega') + \sigma_{\pi^+}(\omega')}{2} \tag{17}$$

where $A^+(a)$ is an unknown subtraction constant which can be expressed in terms of the amplitude at threshold.

$$\tfrac{1}{2} \mathrm{Re}\left[F_{\pi^-}(\omega) + F_{\pi^+}(\omega) \right] = \tfrac{1}{2} \mathrm{Re}\left[F_{\pi^-}(m) + F_{\pi^+}(m) \right]$$

$$+ \frac{f^2}{M} \frac{q^2}{\omega^2 - \omega_B^2} \frac{1}{1 - \omega_B^2/m^2} + \frac{q^2}{2\pi^2} \int_m^\infty \frac{\omega' d\omega'}{q^2} \frac{1}{\omega'^2 - \omega^2} \tfrac{1}{2}\left(\sigma_{\pi^-}(\omega') + \sigma_{\pi^+}(\omega') \right) \tag{18}$$

This equation seems to be in good agreement with expreiments.

With the neglect of ω_B^2/m^2 we have

$$\tfrac{1}{2} \mathrm{Re}\left[F_{\pi^-}(\omega) + F_{\pi^+}(\omega) \right] = \frac{\omega}{2\pi} \mathrm{Re}\left[F_{\pi^-}(m) - F_{\pi^+}(m) \right]$$

$$- \frac{2 f^2 q^2 \omega}{m^2(\omega^2 - \omega_B^2)} + \frac{q^2 \omega}{2\pi^2} P \int_m^\infty \frac{d\omega'}{q'} \frac{1}{\omega'^2 - \omega^2} \tfrac{1}{2}\left(\sigma_{\pi^-} - \sigma_{\pi^+} \right) \tag{19}$$

Writing $D_\pm(\omega) = \mathrm{Re}\, F_{\pi^\pm}(\omega)$ (20)

We find

$$D_+(\omega) = \tfrac{1}{2}\left(1+\frac{\omega}{m}\right)D_+(m) + \tfrac{1}{2}\left(1-\frac{\omega}{m}\right)D_-(m)$$

$$+ \frac{2f^2}{m^2}\,\frac{q^2}{\omega^2 - \omega^2/2M} + \frac{q^2}{4\pi^2}\,P\int_m^\infty \frac{d\omega'}{q}$$

$$\left[\frac{\sigma_\pi(\omega')}{\omega'-\omega} + \frac{\sigma_{\pi^-}(\omega')}{\omega'+\omega}\right]$$

$$D_-(\omega) = \tfrac{1}{2}(1+\omega/m)\,D_-(m) + \tfrac{1}{2}(1-\omega/m)\,D_+(m) \qquad (21)$$

$$-\frac{2f^2q^2}{m^2(\omega+m^2/2M)} + \frac{q^2}{4\pi^2}\,P\int_m^\infty \frac{d\omega'}{q'}\left[\frac{\sigma_{\pi^-}(\omega')}{\omega'-\omega} + \frac{\sigma_{\pi^+}(\omega')}{\omega'+\omega}\right]$$

(22)

Now the Pomeranchuk theorem states that if $\sigma_+(\omega)$ and $\sigma_-(\omega)$ approach constant values when $\omega \to \infty$ then $\sigma_+(\infty) = \sigma_-(\infty)$. That they are constants follows from that fact that all strong interaction approach exponentially for large values of the impact parameter ρ. This constancy of the cross-section cannot be due to the screening effect in atomic electrons since this could happen at only $E \sim 10^{14}$ ev. On the other hand the observed constancy is from 10^{10} ev.

Let us call $\sigma_{\pi^+}(\omega) - \sigma_{\pi^-}(\omega) = \delta\sigma(\omega)$

(23)

Then assuming the limits are not equal we will show the contradiction that arises. (i.e.) we assume

$$\lim_{\omega \to \infty} {}^{\delta}\sigma(\omega) \Rightarrow \delta\sigma \neq 0 \tag{28}$$

Let us examine the high energy behaviour of $D_+(\omega)$

For $\omega >> \Omega$ where Ω is the energy beyond which the cross-section are effectively constants we have

$$D_+(\omega) \sim \frac{\omega}{2m}\left(D_+(m) - D_-(m) + \frac{2f^2}{m^2}\omega\right.$$

$$+ \frac{\omega^2}{4\pi^2} P\int_m^{\Omega} \frac{d\omega'}{q'} \frac{2\omega'}{\omega'^2-\omega^2}\sigma_{\pi -}(\omega') + \frac{\omega^2\sigma_-}{4\pi^2} P\int_{\Omega}^{\infty} \frac{d\omega' 2\omega'}{q'\omega'^2-\omega^2}$$

$$+ \frac{\omega^2}{4\pi^2} P\int_m^{\Omega} \frac{d\omega'}{q'} \frac{\delta\sigma(\omega')}{\omega'+\omega} + \frac{\omega^2\delta\sigma}{4\pi^2}\int_{\Omega}^{\infty} \frac{d\omega'}{q'}\frac{1}{\omega'+\omega} \tag{29}$$

The leading term is the last one and we have for $\omega >> \Omega$

$$D_+(\omega) \sim \frac{\delta\sigma}{4\pi^2}\,\omega \log \omega/\Omega \tag{30}$$

Thus we have

$$R_e F_{\pi+}(\omega) \sim d\sigma/4\pi^2\, \omega \log\frac{\omega}{m} + o(\omega)$$

$$I_m F_{\pi+}(\omega) \sim \omega/4\pi\, \sigma_+ \tag{31}$$

This result contradicts the conclusion that the interaction decays exponentially for large distances. Let us write out th general expression for the elastic scattering amplitude A(o) for the angle zero. For simplicity we shall not take account of spin noting that ~~in~~ orbital angular momentum is very large at high energies so that in replacement of ℓ by $\ell \pm 1/2$ whic must be made in order to take into account spin, we change nothing. At high energies

$$A_{(0)} = \frac{1}{2\iota\omega} \Sigma_{\ell} \left(e^{2\iota \eta_{\ell}} - 1 \right) \ell \tag{32}$$

Where η_{ℓ} is the phase of the wave with orbital angular momentum ℓ. For large we may use the semiclassical impact parameter ℓ/E and the effective upper limit on ℓ is $\omega \rho$ Since the modulus of $\left(e^{2\iota k} - 1\right)$ is not greater > 2 the magnitude of the modulus of A(o) is $< C\omega\rho^2$ Thus $D_{\pm}(\omega)$ cannot contain terms $\propto$ to $\left[\sigma_{+}(\infty) - \sigma_{-}(\infty)\right]$ $\log \omega/\eta$ Therefore

$$\sigma_{+}(\infty = \sigma_{-}(\infty)$$

Similarly at high energies the $\left(K^{+}, K^{0}, \overline{K^{0}}, \overline{K}\right), p, n, \overline{p},$ have constant cross-sections.

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PART II.

LECTURES ON THE STRIP APPROXIMATION.

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LECTURES ON THE STRIP APPROXIMATION

Lecture 1.

The dispersion theory of the present day elementary particle physics arose out of the gross inadequacy of the conventional Lagrangian method of field theory in dealing with strong interactions involving pions, nucleons, and the strange particles. Such relations were well known in electrodynamics where the real and imaginary parts of the refractive index could be related by expressing one as an integral over the other. An extension to cases where the scattered particle has finite mass followed. Microscopic causality which is a statement of the relativistic requirement that no signal can propagate with velocity greater than that of light in quantum mechanical terms, namely, that the commutator of two field operators belonging to two space-time points vanish for space-like separation of the two points - was the physical basis of the dispersion relations. The Cauchy integral formula for an analytic function was the mathematical basis. For any point on the real axis this leads to the Hilbert transforms, i.e., the real part of an analytic function is expressed as integral over the imaginary part and vice-versa. These, if the analytic function considered is the scattering amplitude considered as a function of the energy for fixed momentum transfer or of the momentum transfer for fixed energy are precisely the single variable dispersion relations. Causality has to be involved to continue the scattering amplitude which is a function of a real variable into the complex plane. (Alternatively, we can use the Titchmarsh theorem which states that

under certain restrictions, the real and imaginary parts of the Fourier transform of a function which vanishes for negative values of the argument, are Hilbert transforms of each other).

Though rigorous proofs for the dispersion relations were given for some cases. (pion-pion and pion-nucleon scattering), such a proof was not possible in other cases. (In the case of nucleon-nucleon scattering, the relations could not be proved even for forward scattering). The restrictions on the momentum transfers to achieve a proof led the unphysical relations between the masses of the particles involved in the reaction (e.g. for the case of nucleon-nucleon scattering, a proof is possible only if the mass of the pion $\geq$ ($\sqrt{2}$ - 1) times the mass of the cucleon).

Mandelstam made the intuitively appealing conjecture that the scattering amplitude is simultaneously an analytic function of the variables, the energy and momentum transfer, apart from the poles and, branch cubs necessitated by unitarity in each of the three channel representing the three different processes given by the graph for two particles going over to two particles. The pole arises when there is a intermediate state withthe same quantum numbers as the initial (or final) system of particles. The branch cuts arise for two - or higher particle intermediate states. Admitting the Mandelstam conjecture, the Mandelstam representation follows by a double application of the Cauchy formula. For the case of scattering of spinless particles it can be written as

$$A(s,u,t) = \frac{R_1}{s-s_1} + \frac{R_2}{u-u_1} + \frac{R_3}{t-t_1}$$

$$+ \iint \frac{A_{13}(s',t')}{(s'-s)(t'-t)} \, ds' \, dt'$$

$$+ \iint \frac{A_{12}(s',u')}{(s'-s)(u'-u)} \, ds' \, du'$$

$$+ \iint \frac{A_{23}(u',t')}{(u'-u)(t'-t)} \, du' \, dt' \qquad (1)$$

Here

$$s = (p_1 + p_2)^2$$
$$u = (p_1 - p_4)^2$$
$$t = (p_1 - p_3)^2$$

with

$$s + u + t = \sum_i M_i^2 \qquad (2)$$

are respectively the square of the total energy (each in its own centre of mass) of the three reactions with particles (2,2), (1,4) and (1,3) as the initial state. s_1, u_1 and t_1 are the square of the masses of the single particle (if there is any) in the intermediate state of the three reactions the R_i's being the corresponding residues The integrals are always over the positive axis of the variables, the cut in each case starting from square of the sum of mass of the lowest two particle intermediate state concerned. Thus the double-spectral functions A_{13}, A_{12} and A_{23} are non-vanishing only in the unphysical region (in the physical region when one of the variables, say s_1 is positive, the other independent variable t representing a momentum transfer, will be negative)

Though Mandelstam had himself given a method for computing the double spectral function, in the earlier application of the representation (the Cini - Fubini and Chew-Mandelstam approaches) they were completely ignored and the double variable representation was converted into a single variable one. Let us consider for simplicity pion-pion scattering by the Cini-Fubini method. In this case all the integrals range from 4 to ∞ (using the pion mass as the unit). The essential argument used in the method is that no two variables s, t or u can reach their lower limits at the same time. This can be seen easily by expanding the graphs for each of the channels in terms of unitarity graphs. Thus, for channel I, with s as the energy variable we have the situation represented by Fig. 1.

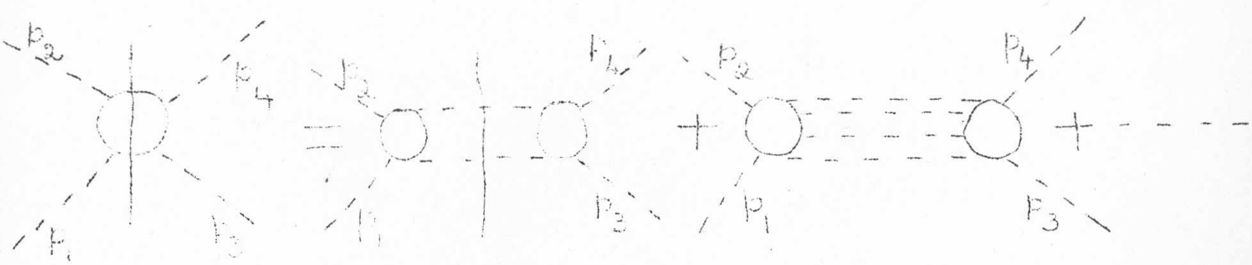

Fig. 1

If we confine ourselves to the elastic region of the reaction only the first graphs on the right hand side has to be taken into account. This graph can be further split horizontally as shown in Fig. 2.

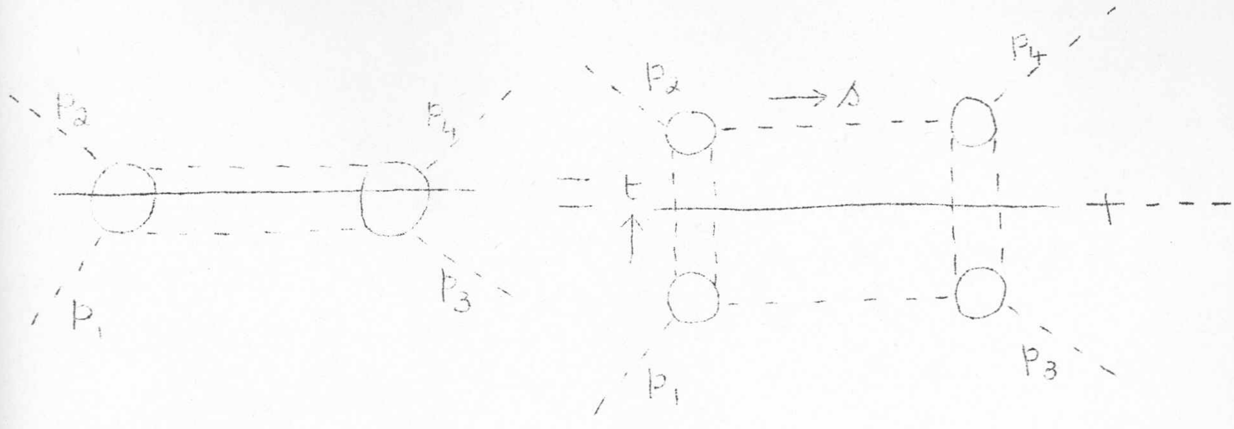

Fig. 2

The requirement of invariance under G-parity which forbids pion vertices with an odd number of pions necessitates four pions being exchanged in the t direction when two particles "exist" in the s channel. Similar arguments follow for the other channels so that if we confine ourselves to the elastic region in each case, the dependence on one of the variables in each of the double integrals will be strong and the other weak. (since the latter has its lower limit exactly at the upper limit of the elastic region).

Now we can write the first of the double integrals as

$$\int_4^\infty \int_4^\infty \frac{A_{13}(s',t')\,ds'dt'}{(s'-s)(t'-t)} = \frac{1}{2}\left[\int_4^\infty \frac{ds'}{s'-s}\int_{16}^\infty dt'\frac{A_{13}(s',t')}{(t'-t)} + \int_4^\infty \frac{dt'}{t'-t}\int_{16}^\infty ds'\,\frac{A_{13}(s',t')}{(s'-s)}\right] \tag{3}$$

Now the denominat~~io~~ns in each of the second integrals in the two terms on the right hand side will never banish if we deal with elastic scattering below the threshold for inelastic processes i.e. when the variables have values < 16. We can therefore club the first term on the right hand side with a similar term:

$$\frac{1}{2}\int_{4}^{\infty} \frac{ds'}{s'-s} \int_{16}^{\infty} du' \; \frac{A_{12}(u',s')}{u'-u} \tag{4}$$

arising from the 5th term of (1) and expand the denominators in the "weak" integrals. (over t' and u'). Using the symmetry property.

$$A_{13}(s,t) = A_{12}(u,s)$$

We can therefore write the sum as

$$\int_{4}^{\infty} \frac{ds'}{s'-s} \int_{16}^{\infty} \frac{dy}{y} \, A_{13}(s',y) \left[1 + \frac{t+u}{y} + - - - \right]$$

the series being terminable at any order we want. Suppose we keep duly the first term. Then we can write the double integrals as

$$\int_{4}^{\infty} \frac{ds'}{s'-s} \, \rho_0(s')$$

where

$$\rho_0(s') = \int \frac{dy}{y} \, A_{13}(s',y)$$

Writing x for the dummy variables s', t', u' and noting that there are no poles in the problem, we can write the scattering amplitude as

$$A(s,t,u) = \int_4^\infty \frac{dx}{x-s}\, \rho_0(x) + \int_4^\infty \frac{dx}{x-u}\, \rho_0(x) + \int_4^\infty \frac{dx}{x-t}\, \rho_0(x) \tag{5}$$

Thus the double variable integrals have been reduced to the single variable form. To determine $\rho_0(x)$ we make a partial wave expansion of the scattering amplitude.

$$h_\ell(\nu) = \frac{1}{2}\int_{-1}^{1} d\cos\theta\, P_\ell(\cos\theta)\, F(\nu,\cos\theta) \tag{6}$$

where $\nu = q^2$ the square of the centre of ~~wave~~ mass momentum. Substituting the representation (5), into (6) we get

$$h_0(\nu) = \frac{1}{\pi}\int_4^\infty \frac{\rho_0(x)}{x-4-4\nu-i\epsilon} \tag{7}$$

$+$ terms with non vanishing denominators for $\nu > 0$. This gives us

$$\operatorname{Im} h_0(\nu) = \rho_0(x)$$
$$\operatorname{Im} h_\ell(\nu) = 0 \text{ for } \ell \neq 0$$

Thus the equation (5) becomes

$$A(\nu,\cos\theta) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\, h_0(\nu')\,d\nu'}{\nu'-\nu-i\epsilon} + \frac{4}{\pi}\int_0^\infty \frac{\mathrm{Im}\, h_0(\nu')\,d\nu'}{4\nu'+4+2\nu(1-\cos\theta)} + \frac{4}{\pi}\int_0^\infty \frac{\mathrm{Im}\, h_0(\nu')\,d\nu'}{4\nu'+4+2\nu(1+\cos\theta)} \tag{8}$$

By projecting out the s^{th} partial wave, we obtain an integral equation for $h_0(\nu)$

$$h_0(\nu) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\, h_0(\nu')}{\nu'-\nu-i\epsilon}\,d\nu' + \frac{2}{\pi}\int_{-1}^{1} d\cos\theta \int_0^\infty \frac{\mathrm{Im}\, h_0(\nu')\,d\nu'}{4\nu'+4+2\nu(1-\cos\theta)} + \frac{2}{\pi}\int_{-1}^{1} d\cos\theta \int_0^\infty \frac{\mathrm{Im}\, h_0(\nu)\,d\nu'}{4\nu'+4+2\nu(1+\cos\theta)} \tag{9}$$

The Chew-Mandelstam approach is also similar in that they keep only the nearest singularities in the scattering amplitude and obtain the same type of integral equation as (9) for the partial wave amplitudes with the difference that the integrals involving the angles are now in the form of a left-hand cut.

In addition to the fact that the above approximations do not do full justice to the double variable natrre of the Mandelstam representation, they may not even work in some cases. Consider the process

$$\gamma + \pi \longrightarrow N + \overline{N}$$

which is the third channel in the problem of the photoproduction of pions on nucleus. There are two low-mass intermediate states possible, -- a two-pion state corresponding to the isoscalar part of the photon interaction and a three-pion state corresponding to isovector part. Now the graphs arising from the isoscalar part and their contribution to the photoproduction problem can be treated in the Cini - Fubini approximation as is clear from cutting the figures 3a and 3b horizontally and vertically. Whereas the approximation fails for the isovector part since both variables may now reach their lower limits at the same time (Fig. 3b)

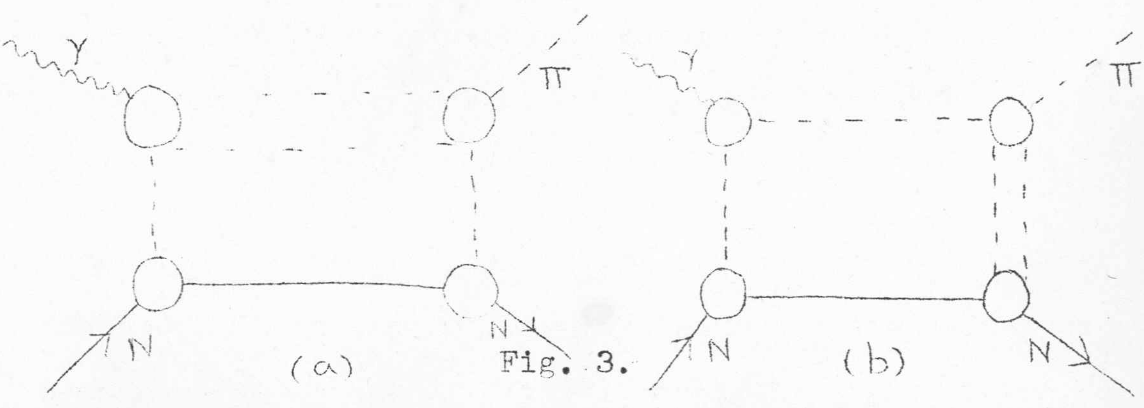

Fig. 3.

Thus a better approximation to the Mandelstam representation which takes into account the double-spectral functions is desirable.

LECTURE II

The "strip" approximation recently suggested by Chew and Frautschi and also independently by Ter-Martirosyan, Gribov, and others indicates a way out of the difficulty encountered in the earlier approximations, viz. the fact that one had to confine oneself to the low-energy region. Chew and Frautschi treat the dynamics of both high and low energy strong interactions in a unified way. The difficulties they met with in trying to incorporate the p -wave resonances into the pion-pion and pion-nucleon systems made them believe that they could be resolved only by explicit consideration of higher energies and inelastic effects.

To start with they define a generalized potential for relativistic scattering which seems to be a natural consequence of the Mandelstam representation, ignoring the pole terms, and fixing the energy variable s, in equation (1) of the last lecture, we can write the scattering amplitude as

$$A(s,t,u) = \frac{1}{\pi}\int dt' \frac{A_3(t',s)}{t'-t} + \frac{1}{\pi}\int \frac{A_2(u',s)\,du'}{u'-u} \tag{1}$$

where

$$A_2(s,t,u) = \frac{1}{\pi}\left[\int \frac{A_{12}(s',u)}{s'-s}ds' + \int dt' \frac{A_{23}(t',u)}{t'-t}\right]$$

$$= V_2(s,u) + \frac{1}{\pi}\int_4^{16} ds' \frac{A_{12}^{(el)}(s',u)}{s'-s} \qquad (2)$$

$$+ \frac{1}{\pi}\int_4^{16} dt' \frac{A_{23}^{(el)}(t',u)}{t'-t}$$

(2)

$$
\begin{aligned}
A_3(s,t,u) &= \frac{1}{\pi}\int \frac{A_{13}(s',t)}{s'-s}\,ds' \\
&\quad + \int du'\,\frac{A_{23}(u',t)}{u'-u} \\
&= V_3(s,t) + \frac{1}{\pi}\int_4^{16} ds'\,\frac{A_{13}^{(el)}(s',t)}{s'-s} \\
&\quad + \frac{1}{\pi}\int_4^{16} du'\,\frac{A_{23}^{(el)}(u',t)}{u'-u}
\end{aligned}
\qquad (3)
$$

V_3 and V_3 which are integrals over the double spectral functions in the inelastic region of the variables are called the direct and exchange potentials, because the equation (1), (2), (3) bear a close resemblance to the set of equations determining nonrelativistic potential scattering (of the type which obeys a Mandelstam representation) if the "elastic" double spectral functions in these equations are represented as products of absorptive parts as will be shown in a later lecture. The difference is that the generalized potential is energy dependent, becomes complex above the inelastic threshold and has relativsitic kinematic factors.

If the generalized potentials are given the elastic double spectral functions can be computed in terms of the absorptive parts, viz

$$A_{13}^{(el)}(s,t) = \frac{1}{\pi q\sqrt{s}}\left[\iint \frac{dt_2\, dt_3\, A_3^*(t_2,s)\, A_3(t_3,s)}{K^{\frac{1}{2}}(q^2,t,t_2,t_3)} + \iint \frac{du_2\, du_3\, A_2^*(u_2,s)\, A_2(u_3,s)}{K^{\frac{1}{2}}(q_2,t,u_2,u_3)}\right] \tag{4}$$

$$A_{12}^{(el)}(s,t) = \frac{1}{\pi q\sqrt{s}}\left[\iint dt_2\, du_3 \times \frac{\left\{A_3^*(s,t_2)\, A_2(s,u_3) + A_2^*(s,u_3) + A_3(s,t_2)\right\}}{K^{\frac{1}{2}}(q^2,u,t_2,u_3)}\right] \tag{5}$$

$$\frac{1}{K^{\frac{1}{2}}(q^2,t,t_2,t_3)} = 0 \text{ unless } t^{\frac{1}{2}} > t_2^{\frac{1}{2}} + t_3^{\frac{1}{2}} \tag{6}$$

For convenience we shall assume that $A_2 = 0$ which means $A_{12} = 0$ and only the first term remains in (4). We know that the absorptive part $A_3(s,t_i) = 0$ if $t_i < 4$. (i.e. below the threshold for scattering). Hence $A_{13}(s,t) = 0$ if $t < 16$ because of the inequality (6). Thus $A_3(s,t) = V_3(s,t)$ for $t < 16$. Now if V_3 (the direct potential) is given then $A_3(s,t)$ is known for $t < 16$. Again applying the inequality (6) we find that $A_{13}(s,t)$ can be calculated for $t < 36$

since $A_3(s,t)$ is known for $t < 16$. From this again we can calculate $A_3(s,t)$ for $t < 36$ and so on. Once A_3 is calculated to a sufficient degree of accuracy it can be fed into equation (1) and the complete matrix element is obtained.

The above procedure for computing the matrix element assumes that the potential V_3 is known which is not the case for the relativistic problem. This is where the strip approximation comes to our aid. In terms of the double spectral functions the approximation can be stated as follows. The scattering amplitudes in the physical regions are dominated by the strips of the double spectral functions.lying between each of the thresholds for the three reactions involved in a Mandelstam graph and the corresponding thresholds for inelastic processes (i.e. production of one more particle), therefore the double spectral (or strip) functions can be calculated using the elastic unitarity condition. More specifically, the strip approximation is the assumption that

$$A_{13}^{(el)}(s) = A_{13}^{(inel)}(t)$$

$$A_{13}^{(inel)}(s) = A_{13}^{(el)}(t)$$

with similar relations for the other double spectral functions. The approximation is based on the following experimental facts.

(i) The elastic scattering cross-sections tend to be large at low energies, with resonances in the low-angular momentum states.

(ii) At high energies all elastic cross-sections show a characteristic diffraction pattern consisting of a peak in the forward direction.

The first implies that for fixed energy the range of the corresponding potential must be short which means in the s channel that t, the momentum transfer should be large. The second implies that at high energies the momentum transfer is low. This complementary nature of the t and s variables have been expressed in terms of the double spectral functions.

The approximation can also be understood in term of the Cuthoshy diagrams. Fig 4a corresponds to a two-particle intermediate state in the s -channel while Fig. 4b corresponds not only to a two particle intermediate state in the t channel but also has the significance of a two-particle encharge potential in the s -channel, representing the effect of a sum over inelastic intermediate states containing definite numbers of particles in the s -channel.

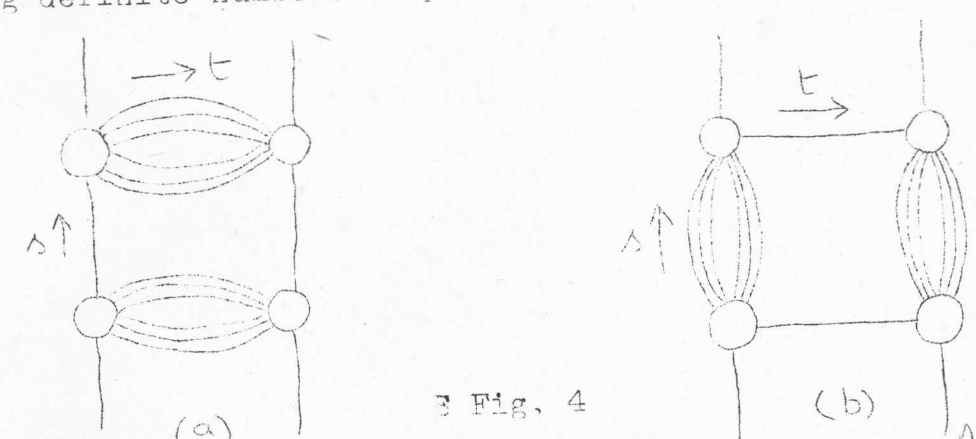

Fig. 4

Now by definition (and without assumption that $A_2 = 0$), the generalized potential V_3 is given by

$$V_3(s,t) = \int ds' \frac{A_{13}^{(inel)}(s',t)}{s'-s}$$

Since, according to the strip approximation

$$A_{13}^{(inel)}(s,t) = A_{13}^{(el)}(t,s)$$

this quantity can be calculated using the elastic unitarity condition in the third channel and $V_3(s,t)$ can thus be formally determined.

LECTURE III.

Before we proceed to application of the strip approximation to specific cases, we should point out a flaw in the argument of the last lecture that the complete matrix element is known once the generalized potentials are known and since these latter can be expressed in terms of the absorptive parts A_2 and A_3, these latter determine the amplitude A uniquely. But this is so only if there are no subtractions in the dispersion relation in the t (and u) variable. There are no subtractions if $A_3 \longrightarrow 0$ as $t \longrightarrow \infty$; one subtractions if $A_3/t \longrightarrow 0$ as $t \longrightarrow \infty$ and there are two subtractions if $A_3/t^2 \rightarrow 0$ as $t \longrightarrow \infty$ and so on. For the second case we can write

$$A(s,t) = A(s,t_0) + \frac{t-t_0}{\pi}\int dt' \frac{A_3(s,t')}{(t'-t_0)(t'-t)}$$

$A(s,t_0)$ is not determined by the double spectral functions and hence also not by the single spectral functions (the absorptive parts) either.

On perturbation theory, for potential scattering, $A(s,t) \longrightarrow 0$ as $t \longrightarrow \infty$ and for the relativistic case goes as a constant. For weak potentials, therefore, A is determined by A_3 alone. But as the strength of interaction increases we get bound states in $s, p, \ldots$ waves and we need one or more subtractions. (The bound states will be represented by pole terms of the form

$\dfrac{g^2 P_\ell(\cos\theta)}{s' - s}$ and for $\ell \geq 1$ $P_\ell(\cos\theta) \propto t$

so that a subtraction is necessistated for $t \longrightarrow \infty$). This situation seems paradomical since to start with A is determined completely by A_{13} but with increasing strength we are not able to get the terms corresponding to the bound states from the double spectrals functions. A further difficulty is that since with bound states for $\ell \geq 1$, $A \longrightarrow \infty$ as $t \longrightarrow \infty$, in the crossed channel where t is the energy there will be violation of unitarity. poles

The introduction of the Regge poles (i.e. poles in the complex angular momentum plane) resolves both these difficulties and enables one to calculate A form A_{13} and A_3 even when there are subtractions. Regge showed that $A(s, \ell)$ can be continued analytically from the physical values of ℓ (which can be only integers) both to non-integral and complex values of ℓ to the right of the line $\mathrm{Re}\,\ell = -1/2$ (mandelstam has recently removed this restriction and showed that analytic continuation is possible into the whole complex ℓ plane). Choosing a contour which encloses the poles of $A(s, \ell)$ also the showed that the matrix element can be written as

$$A(s,\ell) = \int_{\mathrm{Re}\,\ell = -1/2} d\ell \, \frac{(2\ell+1)(A(\ell,s)P_\ell(-z))}{\sin\pi\ell} + \sum_n \frac{(2\alpha_n(s)+1)\beta_n(s)P_{\alpha_n(s)}(-z)}{\sin\pi\alpha_n(s)} \qquad (1)$$

where β_n are the residues of the complex Regge poles. α_n and $z = \cos\theta$

Now for Legendre polynomial $P_\alpha(z)$ behaves as z^α for $z \longrightarrow \infty$

Hence in the complex z plane, the integral (first term on the right hand side of (1)) goes down like $|z|^{-1/2}$ as $z \longrightarrow \infty$ and is thus well-behaved. We shall call this the background term. The pole terms of (1) however behave like z^α as $z \longrightarrow \infty$ The pole that dominates is the one that has the largest real part and it is the pole further to the right.

Now we can fix the complete matrix element from A_3 or A_{13} For $P_\alpha(-z)$ is analytic in the z-plane except for a cut along the real axis from $z = 1$ to ∞ with discontinuity $P_\alpha(-z) \sin \pi\alpha$ (For $\alpha =$ an integer, P_α is completely analytic since the discontinuity is zero). Thus the spectral function $A_3(s,t)$ which is the discontinuity of the function A can be written as

$$A_3(s,t) = A_{3B}(s,t) + \sum_n \left(2\alpha_n(s)+1\right) \beta_n(s) P_{\alpha_n(s)}(z)$$

The first term on the right A_{3B} is obtained by taking the discontinuity across the cut in z in the background term. As already mentioned, the background term goes down like $t^{-1/2}$ so that there is no question of a subtraction and hence in this case A can be completely determined from A_3 and A_{13} . Now if we know A_3 numerically we can separate it into Regge pole terms and the background term and determine the α's and β's . These values can be substituted back to obtain the complete matrix element from A_3 without introducing arbitrary subtractions.

The second difficulty regarding the bad asymptotic behaviour in the crossed channel is settled as follows. According to the Regge formula, the asymptotic behaviour as $t \longrightarrow \infty$ depends on Δ. Because of this even if we have a p - wave resonance or bound state, the asymptotic behaviour is no longer proportional to t everywhere. We expect to get trouble if we have a bad asymptotic behaviour as $t \longrightarrow \infty$ when Δ is negative (i.e. is a momentum transfer). So long as we can keep $\alpha < 1$ when Δ is negative we would not expect to get into trouble even if $\alpha > 1$ for Δ positive so that we could have p or higher wave resonances. So the dependence of α on Δ can save the situation and this can be so only if the spectral function oscillates. (since the asymptotic behaviour is $t^{\alpha(\Delta)}$.) This follows from the dispersion relation.

$$A(\Delta, t) = \frac{1}{\pi} \int d\Delta' \, \frac{A_1(\Delta', t)}{\Delta' - \Delta}$$

If for a particular value of Δ', A_1, had bad asymptotic behaviour as a function of t and if there were no oscillations in it so that there could be no cancellation in origin, then even after integration the expression on the left (the complete scattering amplitude) will have same bad asymptotic behaviour, whatever the value of . So if the spectral function does not oscillate we cannot have an asymtotic behaviour for $A(\Delta, t)$ as a function of t which depends on Δ. If A_1, oscillates the bad asymptotic behaviour may cancel out in the dispersion integral and we can have the asymptotic behaviour depending on Δ as does happen in the Regge formula. (The fact that the single spectral functions oscillate means also that the double spectral functions do likewise)

LECTURE IV

In this lecture, we shall apply the strip approximation to the problem of the photoproduction of a pion from a nucleon, giving calculational details. The form of the matrix element can be argued out as follows. If k and q are the fourmomenta of the photon and pion and p_1 and p_2 those of the initial and final nucleons, then the energy-momentum conservation,

$$p_1 + k = p_2 + q$$

allows only three-independent momentum fourvectors which we can choose to be k, q and $P = \frac{p_1 + p_2}{2}$. In addition we have the polarization vector ϵ of the photon and the γ-matrices corresponding to the nucleons. Because of the Dirac equation, scalar products like $\gamma . p_1$ and $\gamma . p_2$ when acting on the corresponding spinors gives a constant (the mass) times the spinor. So only one invariant scalar, which we choose to be $\gamma . k$ is possible with the momentum vectors ($\gamma . q$ is not independent because of the conservation law) Further any higher powers or products of γ-matrices can be reduced to the lowest form. We have also the scalar product $\gamma . \epsilon$. As regards the scalar product of the polarization vector with the momentum vectors, we notice that the Lorentz condition gives $k . \epsilon = 0$ so that the independent scalars possible are $P . \epsilon$ and $q . \epsilon$ Thus, following Chew, Goldberger, Low and Nambu (CGLN), we can form the following gauge invariant forms

$$M_A = i\gamma_5 (\gamma . \epsilon)(\gamma . k)$$

$$M_B = 2i\gamma_5\left[(P\cdot\epsilon)(q\cdot k) - (q\cdot\epsilon)(P\cdot k)\right]$$

$$M_C = \gamma_5\left[(\gamma\cdot\epsilon)(q\cdot k) - (\gamma\cdot k)\, q\cdot\epsilon\right]$$

$$M_D = 2\gamma_5\left[(\gamma\cdot\epsilon)(P\cdot k) - (\gamma\cdot k)(P\cdot\epsilon) - i m(\gamma\cdot\epsilon)(\gamma\cdot k)\right] \qquad (1)$$

where m is the mass of the nucleon and the γ_5 matrix appears in each of the products to take care of the odd intrinsic parity of the pion. The above forms can be seen to be gauge invariant by replacing by [illegible] when the matrix element should vanish. The above choice is dictated by the simplicity of the corresponding centre of mass version.

The complete invariant matrix element for the process can be written as

$$T = M_A A + M_B B + M_C C + M_D D \qquad (2)$$

where A, B, C, D are each functions of the variables s, u, and t defined by

$$s = -(p_1+k)^2 = -(p_2+q)^2$$

$$= (E_1+k')^2 = (E_2+\omega)^2$$

$$u = -(p_1-q)^2 = -(p_2-k)^2$$

$$= m^2 - 2E_2k' - 2k'q'\cos\theta$$

$$t = -(p_2-p_1)^2 = -(k-q_1)^2$$

$$= 1 - 2k'\omega + 2kq\cos\theta$$

(3)

which represent respectively the square of the total energy in the centre of mass the photoproduction reaction and the two reactions

$$\gamma + \overline{N} \longrightarrow \pi + \overline{N} \qquad (\text{II})$$

$$\gamma + \pi \longrightarrow N + \overline{N} \qquad (\text{III})$$

respectively. The centre of mass values of these one also given in (3) where E_1, and E_2 are the energies of the initial and final nucleons and k' and q' are the centre of mass momenta of the photon and pion respectively.

Each of the amplitudes A,B,C,D, can be further subdivided into amplitudes corresponding to isotopic spin channels. Since isotopic spin is good only in parts for an electromagnetic interaction ($\Delta I = 0, \pm 1$), we must introduce in addition to the isotopic spin combinations used in pion-nucleon scattering (where isotopic spin is strictly conserved) also an isosclar part. Thus each of the amplitudes A, B, C, D can be written in the form

$$A_\alpha = \frac{1}{2}\left\{ \tau_\alpha , \tau_3 \right\} A^{+} + \frac{1}{2}\left[\tau_\alpha + \tau_3 \right] A^{(-)} + \tau_\alpha A^{(0)} \qquad (4)$$

etc. where the terms on the right represent respectively the symmetric, antisymmetric and isoscalar parts respectively. The τ's are the usual isospin matrices, τ_3 appearing because of the isovector part of the photon interaction which behaves like the 3rd component of an isovector. Thus we have to deal in all with tweleve invariant functions of s, t, and u which are supposed to obey the Mandelstam representation.

Next we notice a relation between the first and second channels which saves us the calculation of quantities for the second channel, once these have been calculated for the first channel. This is the crossing symmetry of the amplitudes under the interchange of s and u. This is so since going over from process I to II we are changing p_1 to $-p_2$ and p_2 to $-p_1$ i.e. P to $-P$ leaving the pion and photon variables uncharged. The γ-matrices and the i factor get hermitian and complex -conjugated el. Further under the intervhange only the isotopic antisymmetric combination charges sign. We therefore have

$$\left[A^{(+,0)}(s,u,t), B^{(+,0)}(s,u,t), C^{(-)}(s,u,t), D^{(+,0)}(s,u,t)\right]$$

$$\longrightarrow \left[A^{(+,0)}(u,s,t), B^{(+,0)}(u,s,t), C^{(-)}(u,s,t), D^{(+,0)}(u,s,t)\right]$$

$$\left[A^{(-)}(s,u,t), B^{(-)}(s,u,t), C^{(+,0)}(s,u,t), D^{(-)}(s,u,t)\right]$$

$$\longrightarrow \left[A^{(-)}(u,s,t), B^{(-)}(u,s,t), C^{(+,0)}(u,s,t), D^{(-)}(u,s,t)\right] \tag{5}$$

Thus the amplitudes for the second channel are connected in a simple way to those of the first.

Ball has examined the question whether the C G L N amplitudes are completely free from kinematric singularities (i.e. singularities other than those postulated by Mandelstam) and finds that only the amplitude, B, has an additional pôle in the t variable which can be taken into the account or removed by a suitable subtractions.

The next step is to go over from the invariant frame in which the matrix element (2) is defined. We notice that the product of the γ - matrices in (2) are to sandwiched between the initial and final nucleon spinors using the enplicit forms

$$u(p) = \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} E+m \\ \vec{\sigma}\cdot\vec{p} \end{pmatrix}$$

$$\gamma_5 = \begin{pmatrix} 0 & -I \\ -1 & 0 \end{pmatrix}$$

$$\gamma_j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix} \quad ; \; j = 1, 2, 3$$

$$\gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{6}$$

and the centre-of-mass quantities for the momentum and energy and reassembling the terms, the matrix element in the centre-of-mass frame can be written as

$$F(s,u,t) = i\vec{\sigma}.\vec{\epsilon}\ \mathcal{F}_1(s,u,t) + \frac{\vec{\sigma}.\vec{q}'\,\vec{\sigma}.(\vec{k}'\times\vec{\epsilon})}{q'k'}\mathcal{F}_2(s,u,t)$$

$$+ \frac{i\vec{\sigma}.\vec{k}'\,\vec{q}'.\vec{\epsilon}}{k'q'}\ \mathcal{F}_3(s,u,t) + \frac{i\vec{\sigma}.\vec{q}'\,\vec{q}'.\vec{\epsilon}}{q'^2}$$

(7)

where

$$c(W-m)\mathcal{F}_1 = \alpha = (W-m)A + (W-m)^2 D - \frac{(t-1)}{2}(C-D)$$

$$\frac{c(W+m)(m+E^2)}{q'}\mathcal{F}_2 = \beta = -(W+m)A$$

$$+ (W+m)^2 D - \frac{t-1}{2}(C-D)$$

$$\frac{C\mathcal{F}_3}{q'} = \gamma = (W-m)B + (C-D)$$

$$\frac{C(m+E_2}{q'^2}\mathcal{F}_4 = -(W+m)B + (C-D)$$

(8)

Which also defines the amplitudes $\alpha, \beta, \gamma, \delta$ which are proferable in the sense that in going from the amplitudes A,B,C,D to ($\alpha, \beta, \gamma, \delta$) does not involve any new singularities.

We are now in a position to evaluate the double spectral functions. Take the photon in the z — direction, $\vec{k}'$ and $\vec{q}'$ define the $z - x$ plane. The polarization vector $\vec{\epsilon}$ lies in the xy plane making an angle ψ with the x -axis. We shall now make use of the elastic unitarity condition, i.e. we shall retair in the complete set of intermediate states in the imaginary part of F , i.e. its absorptive part F' , only the state containing a pion and a nucleon so that unitarity condition.

$$2 \mathcal{I}m T = \Sigma \langle T^+ | n \rangle \langle n | T | i \rangle$$

can be written as the product of the matrix element for pion-nucleon scattering and the photoproduction matrix element itself. The centre-of-mass makrix element for the former process can be written in the form

$$I = a + \frac{\vec{\sigma}.\vec{q}_2 \, \vec{\sigma}.\vec{q}_1 \, b}{q_1 \, q_2}$$

(9)

where q_{V1} and q_{V2} are the momenta of the initial and final pions. The unitarity condition can therefore be written as

$$F' = \eta \int_{-1}^{1} dz_2 \int_0^{2\pi} d\varphi \, I^*(s, z_3) F(s, z_2) \tag{10}$$

η is a normalization factor; $z_1 = \cos\theta_1$, $z_2 = \cos\theta_2$ where θ_1 is the angle between the directions of the initial and final set of particles, θ_2, φ those between the initial and intermediate state particles and z_3 is the cosine of the angle between the directions of the intermediate and final state particles. Now fixing the energy variables we can write the one-variable dispersion relations

$$I^*(s, z_3) = \frac{1}{\pi} \int dz_3' \, \frac{I^{2*}(s, z_3') + I^{3*}(s, z_3')}{z_3' - z_3}$$

$$F(s, z_3) = \frac{1}{\pi} \int dz_2' \, \frac{F^2(s, s, z_2') + F^3(s, z_2')}{z_2' - z_2} \tag{11}$$

where we have placed both the absorptive parts I^2 and I^3 (as well as F^2 and F^3) under a single integral for convenience though they contribute to different regions of the variables of integration. Thus $I^2(s, z)$ will be non-zero if

$$z < 1 - \frac{\{s - (m-1)^2\}}{2 q_V'^2}$$

apart from a δ -function at $z_1 = 1 - \frac{(s-m^2-2)}{2q^2}$ at the nucleon pole and $I^3(s, z)$ will be monzero only if

$z_1 > 1 + \frac{t}{2q^2}$ (This follows from the different roles in the 2nd and 3rd channels) Similarly F^2 will be non-zero only if

$$z < \frac{k\omega - \frac{1}{2}(m^2-2m-1-s)}{kq}$$

again apart from a δ -function and F^3 will be non-zero only if

$$z > \frac{k\omega + 3/2}{kq}$$

Substitution the expressions (7) and (9) for F and I and using the properties of the σ -matrices like $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1, \sigma_x\sigma_y = i\sigma_z$ etc., we can write the unitarity condition can be written in the co-ordinate system chosen as

$$F' = i\mathcal{F}_1(s, z_1)(\sigma_x \cos\psi + \sigma_z \sin\psi)$$

$$- \mathcal{F}_2(s, z_1)(1 - z_1^2)^{1/2} \sin\psi$$

$$+ i\mathcal{F}_2(s, z_1)(-\sigma_x z_1 \cos\psi - \sigma_y z_1 \sin\psi$$

$$+ \sigma_z(1 - z_1^2)^{1/2} \cos\psi)$$

$$+ i\mathcal{F}_3(s, z_3)\,\sigma_z(1 - z_1^2)^{1/2} \cos\psi$$

$$+ i\mathcal{F}_4(s, z_3)(1 - z_1^2)^{1/2} \cos\psi\left[\sigma_z z_1 + \sigma_x(1 - z_1^2)^{1/2}\right]$$

$$= n \iiiint \frac{dz_1'\, dz_2\, dz_3'\, d\varphi}{}$$

$$\cdot\Big[a_2^*(s,z_3')+a_3^*(s,z_3')+b_2^*(s,z_3')+b_3^*(s,z_3')$$

$$\times\Big\{z_3-i\big(-\sigma_x z_1(1-z_2^2)^{1/2}\sin\varphi$$

$$+\sigma_y\big[z_1(1-z_2^2)^{1/2}\cos\varphi-(1-z_1^2)^{1/2}z_2\big]$$

$$+\sigma_z(1-z_1^2)^{1/2}(1-z_2^2)^{1/2}\sin\varphi\Big\}\Big]$$

$$\times\Big[\Big\{F_1^2(s,z_2')+F_1^3(s,z_2')\Big\}(\sigma_x\cos\psi+\sigma_y\sin\psi)$$

$$+\Big\{F_2^2(s,z_2')+F_2^3(s,z_2')\Big\}$$

$$\times\Big\{-(1-z_2^2)^{1/2}(\cos\varphi\sin\psi-\sin\varphi\cos\psi)$$

$$+i\big[-z_2\cos\psi\,\sigma_x-z_2\sin\psi\,\sigma_y$$

$$+(1-z_2^2)^{1/2}(\cos\varphi\cos\psi+\sin\varphi\sin\psi)\sigma_z\Big\}$$

$$+i\Big\{F_3^2(s,z_2')+F_3^3(s,z_2')\Big\}$$

$$\times(1-z_2^2)^{1/2}(\cos\varphi\cos\psi+\sin\varphi\sin\psi)\sigma_z$$

$$+i\Big\{F_4^2(s,z_2')+F_4^3(s,z_2')\Big\}$$

$$\times(1-z_2^2)^{1/2}(\cos\varphi\cos\psi+\sin\varphi\sin\psi)$$

$$\times\Big\{(1-z_2^2)^{1/2}\cos\varphi\,\sigma_x+(1-z_2^2)^{1/2}\sin\varphi\,\sigma_y+z_2\sigma_z\Big\}\Big] \qquad (12)$$

We now perform the φ and z_2 integrations the simplest of these integrals will be the one in which the φ dependence is completely contained in the factor in the denominator $z_3' - z_3 =$

$$z_3' - z_1 z_2 - (1-z_1^2)^{1/2}(1-z_2^2)^{1/2}\cos\varphi$$

integration over φ will then gives the factor

$$\frac{2\pi}{\sqrt{z_3'^2 + z_1^2 + z_2^2 - 1 - 2z_1 z_2 z_3'}}$$

and if the only other factor certaining z_2 is the denominator $z_2' - z_2$, integration over z_2 will a give a function of the type

$$\frac{1}{\sqrt{k}} \log \frac{z_1 - z_2' z_3' + \sqrt{k}}{z_1 - z_2' z_3' - \sqrt{k}} \tag{13}$$

where

$$k = z_1^2 + z_2'^2 + z_3'^2 - 1 - 2z_1 z_2' z_3'$$

The numerator will contain factors of the type

$$\left[I^{(2)*}(s, z_3') + I^{(3)*}(s, z_3') \right] \left[F^{(2)}(s, z_2') + F^{(2)}(s, z_2') \right] \tag{14}$$

Now the function (13), considered as function z_1 has both left and right hand branch cuts which is seen from the identify

$$\frac{1}{\sqrt{k}} \log \frac{z_1 - z_2' z_3' + \sqrt{k}}{z_1 - z_2' z_3' - \sqrt{k}} = 2 \int \frac{dz_1'}{z_1' - z_1} \cdot \frac{1}{\sqrt{k}}$$

This was in fact the way (13) was obtained after the z_2 integration. The discontinuity $\frac{1}{\sqrt{k}}$. The cuts are determined by the points at which the charges sign namely

$$z_1 = z_2 z_3 \pm (z_2^2 - 1)^{1/2} (z_3^2 - 1)^{1/2}$$

Now the dispersion relations obeyed by the absorptive parts (equation (2) and (3) of Lecture II) sush as

$$F' = \frac{1}{\pi} \int \frac{F_{13}\, dt'}{t' - t} - \frac{1}{\pi} \int \frac{F_{12}\, dt'}{t' - t} \tag{15}$$

where we have rewritten the second integral over u as a cut on the negative axis of the t -variable clearly shows that and the double spectral functions are respectively twice the discontinuities across the right and left-hand branch cuts in the t (or equivalently the z_1) plane. We also notice that for the right hand cut where $z_1 > z_2 z_3 \pm (z_2^2 - 1)^{1/2} (z_3^2 - 1)^{1/2}$ is to be positive the z_2 and z_3 must be both positive or both negative, whereas for the left hand cut with $z_1 < z_2 z_3 \pm (z_2^2 - 1)^{1/2} (z_3^2 - 1)^{1/2}$ to be negative, one of them must be positive and one of them negative. These requirements are met by taking in (14), the combination to go to the positive cut i.e. to F_{13} and the combination $\left[I^{(2)*}(s,\gamma_3')F^{(3)}(s,\gamma_2')\right] + I^{(3)*}(s,\gamma_3')F^{(2)}(s,\gamma_2')$ to go to the negative cut i.e. F_{12} (These follows from the region where $I^{(2)}, I^{(3)}$ and $F^{(2)}, F^{(3)}$ are nonvanishing). Thus for the simple type of considered we could write

$$F_{13} = \eta' \int d\gamma_2' \int d\gamma_3' \, K_1(\gamma_1, \gamma_2', \gamma_3')$$

$$\times \left\{ I^{3*}(s,\gamma_3') F^{3}(s,\gamma_2') + I^{2*}(s,\gamma_3') F^{2}(s,\gamma_2') \right\}$$

$$F_{12} = \eta' \int d\gamma_2' \int d\gamma_3' \, K_2(\gamma_1, \gamma_2', \gamma_3')$$

$$\times \left\{ I^{3*}(s,\gamma_3') F^{2}(s,\gamma_2') + I^{(2)*}(s,\gamma_3') F^{(2)}(s,\gamma_2') \right\} \qquad (16)$$

where

$$K_1 = -\frac{1}{\sqrt{k}} \; ; \; \gamma_1 > \gamma_2\gamma_3 + (\gamma_2^2-1)^{1/2}(\gamma_3^2-1)^{1/2}$$

$$= 0 \quad ; \; \gamma_1 < \gamma_2\gamma_3 + (\gamma_2^2-1)^{1/2}(\gamma_3^2-1)^{1/2}$$

$$K_2 = \frac{1}{\sqrt{k}} \; ; \; \gamma_1 < \quad '' \quad ''$$

$$= 0 \; ; \; \gamma_1 > \quad '' \quad ''$$

(17)

Those were the facts referred to and made use of in lecture II

Even for the more complicated integrals in (12), the common feature is that the cuts in the Z_1-plane are still identified by those of the function (13). A further complication is that we are not having one but four amplitudes and four double spectral functions and hence require four relations between them. These are obtained by identifying the terms independent of the σ matrix and the coefficients of $i\sigma_x$, $i\sigma_y$ and $i\sigma_z$ on both sides of (12) and then identifying the cuts as in (16). For instance, the σ- independent terms involve only the double spectral function F_2^{13} which are identifying the corresponding terms and combinations on the right hand side give.

$$(1-z_1^2)^{1/2} \; F_2^{13}(s, z) = \iint dz_2' \, dz_3' \, \frac{K_1}{(1-z_1^2)^{1/2}}$$

$$\times \Big\{ (a_2^* F_2^2 + a_3^* F_2^3)(z_3 - z_1 * z_2)$$

$$- (b_2^* F_2^2 + b_3^* F_2^3)(z_2^2 - z_3^2)$$

$$+ (b_2^* F_1^2 + b_3^* F_1^3)(z_2 - z_1 z_3) \Big\} \qquad (18)$$

The identification of the coefficients of the σ_x, σ_y and σ_z will give simultaneous equations for the double spectral functions $F^{(13,\,12)}_{1,2,3,4}$ from which they can be individually determined. We have further to reexpress them as functions of the t variable instead of Z and this can be done using the relation between them (equation (3))

We have not introduced the isotopic spin decomposition of the amplitudes and consequently of the double spectral functions. This is easily done by identifying the coefficients of $\frac{1}{2}\{\tau_\alpha, \tau_\beta\}, \frac{1}{2}[\tau_\alpha, \tau_\beta]$ and τ_α on both sides of the unitarity condition after the amplitudes on both sides have been decomposed into those corresponding to the three isotopic spin possibilities. On the right hand side we will be multiplying the complete matrix element for pion nucleon scattering.

$$\frac{1}{2}\{\tau_\alpha, \tau_\beta\} \underline{I}^{(+)} + \frac{1}{2}[\tau_\alpha, \tau_\beta] \underline{I}^{-} \qquad (19)$$

by the photoproduction matrix element

$$\frac{1}{2}\{\tau_\beta, \tau_3\} F^{(+)} + \frac{1}{2}[\tau_\beta, \tau_3] F^{(-)} + \tau_\beta F^{(0)} \qquad (20)$$

where β is the isotopic spin index of the intermediate pion. Now we can obtain $\frac{1}{2}\{\tau_\alpha, \tau_\beta\}$ from this product by multiplying $\frac{1}{2}\{\tau_\alpha, \tau_\beta\}$ by $\frac{1}{2}\{\tau_\beta, \tau_3\}$

and also $\frac{1}{2}[\tau_\alpha, \tau_\beta]$ by $\frac{1}{2}[\tau_\beta, \tau_3]$

The antisymmetric part $\frac{1}{2}[\tau_\alpha, \tau_3]$ is obtained by multiplying the symmetric as well as the antisymmetric part in (19) by the antisymmetric part in (20) and vice versa. The isoscalar part τ_α is obtained by multiplying both the symmetric and antisymmetric part of (19) by the isoscalar part (20). More specifically the isotopic symmetric part of a double spectral function will have terms of the type

$$\underline{I}^{i*(+)} F^{i(+)} - 2\underline{I}^{i*(-)} F^{i(-)}$$

The antisymmetric part will have terms of the type

$$I^{i*(+)} F^{i(-)} - I^{i*(-)} \left[F^{i(+)} + F^{i(-)} \right]$$

The isoscalar part will have

$$\left[I^{i*(+)} - 2 I^{*(-)} \right] F^{i(0)}$$

i can take the two values 2 and 3

e.g.

$$\mathcal{F}_2^{13(+)} = \iint d\eta_2' \, d\eta_3' \frac{K_1}{1-\eta_1^2}$$
$$\times \Big[\left\{ a^{i*(+)} \mathcal{F}_2^{i(+)} - 2a_i^{*(-)} \mathcal{F}_2^{i(-)} \right\}^{\eta_1} (\eta_3 - \eta_1 \eta_2)$$
$$- \left\{ b^{i*(+)} \mathcal{F}_2^{i(+)} - 2b^{i*(-)} \mathcal{F}_2^{i(-)} \right\} (\eta_2^2 - \eta_3^2)$$
$$+ \left\{ b^{i*(+)} \mathcal{F}_1^{i(+)} - 2b^{i*(-)} \mathcal{F}_1^{i(-)} \right\} (\eta_2 - \eta_1 \eta_3) \Big]$$

Lecture V

As already mentioned, we need not explicitly calculate the double spectral functions for the second channel (photoproduction of pions from the anti-nucleon) since these will follow from the crossing relations (equation(5) of last lecture), let us now consider the third channel.

$$\gamma + \pi \rightarrow N + \overline{N}$$

The matrix element for this is obtained from that for photoproduction, by observing that in going over from channel I to III, the following transformations take place:

$$p_1 \longrightarrow p_1 \; ; \quad q \rightarrow -q$$

and the positive energy spiner Bor the antinucleon has to be used. Specifically,

$$M_A \rightarrow i\gamma_5\, \gamma\cdot\epsilon\,\gamma\cdot k$$

$$M_B \rightarrow i\gamma_5\big[-(p_2-p_1)\cdot\epsilon\, q\cdot k + (p_2-p_1)\cdot k\, q\cdot\epsilon$$

$$M_C \rightarrow -\gamma_5(\gamma\cdot\epsilon\, q\cdot k - \gamma\cdot k\, q\cdot\epsilon)$$

$$M_D \rightarrow 2\gamma_5\Big[\gamma\cdot\epsilon\,\frac{(p_2-p_1)}{2}\cdot k - \gamma\cdot k\,\frac{(p_2-p_1)}{2}\cdot\epsilon - im\gamma\cdot\epsilon\,\gamma\cdot k\Big]$$

Using the explicit representations for the spinors and the γ- matrices (equation (6) of previous lecture) and noting that the antinucleon spinor is represented by

$$v(p) = \begin{pmatrix} -\sigma\cdot p \\ E+m \end{pmatrix} \frac{1}{\sqrt{(E+m)2m}}$$

the centre of mass version of (1) will be

$$F_{III} = iL\,\frac{p\cdot\epsilon}{p} + M\,\frac{\sigma\cdot(\vec{p}\times\vec{\epsilon})}{p} + N\,\frac{\vec{\sigma}\cdot\vec{p}\;p\cdot(\vec{\epsilon}\times\vec{\ell})}{p^2\ell} + P\,\frac{\sigma\cdot(\vec{\epsilon}\times\vec{\ell})}{\ell} \qquad (2)$$

where

$$L = -\frac{\ell p}{m}(A + 4E^2 B)$$

$$M = \frac{2\ell E p}{}\, C$$

$$N = \frac{p^2 l}{m(E+m)}(A+2ED)$$

$$P = l\left(A - \frac{2E^2D}{m}\right) \qquad (3)$$

l and p are the centre of mass momenta of the initial and final particles respectively and E is the energy of the nucleon (or antinucleon)

The lowest mass two particles intermediate state in the unitarity condition will be the two pion state so that we will have the product of the matrix elements for the processes

$$\pi + \pi \rightarrow N + \bar{N} \text{ and } \gamma + \pi \rightarrow \pi + \pi$$

The centre of mass matrix element of the first of these processes is of the form.

$$K = G\frac{\vec{\sigma}.\vec{p}}{p} + H\frac{\sigma.\vec{\pi}}{\pi}$$

Where π and p are momenta of the pions and the nucleons respectively. There is only one amplitude for the second process, because of G invariance which forbids three (or any odd number of) pion intermediate states. Thus only the isoscalar part of the interaction contributes to this matrix element. As a consequence only the isoscalar part of the amplitude of the process.

$\gamma + \pi \rightarrow N + \bar{N}$ survives, since on multiplying both the symmetric and antisymmetric combination of the isospin matrices in the matrix element K by the isoscalar amplitude for the matrix element of photoproduction of a pion on a pion, we get only an isoscalar amplitude.

Another feature in evaluating the double spectral functions (which can be done exactly as in the previous case for channel I) is that there are no double spectral functions of the type $L_{(13),\ (23)}$. This is due to the particular choice of the matrix elements and without further significance.

After having obtained the "elastic" double spectral functions for the three channels, we now use the strip approximation in the dispersion relations for the absorptive parts. For channel I.

$$\rho_1^{(\pm,0)}(s,u,t) = \frac{1}{\pi}\int_4^\infty \frac{dt'}{t'-t} \times\left[\rho_{13}^{el\,(\pm,0)}(s,t') + \rho_{13}^{inel\,(\pm,0)}(s,t')\right]$$

$$+ \frac{1}{\pi}\int_{(m+1)^2}^\infty \frac{du'}{u'-u}\left[\rho_{12}^{el\,(\pm,0)}(s,u') + \rho_{12}^{inel\,(\pm,0)}(s,u')\right]$$

where ρ stands for any one of the amplitudes $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ or $\mathcal{F}_4$. Now $\rho^{el}_{13\,/12}$ are precisely the "elastic" double spectral functions evaluated in the previous lecture. For the inelastic parts $\rho^{inel}_{13,\,12}$ we use the strip approximation

$$\rho_{13}^{inel}(s,t) = \rho_{13}^{el}(t,s)$$

$$\rho_{12}^{inel}(s,u) = \rho_{12}^{el}(u,s)$$

Now $\rho^{el}_{13}(t,s)$ is the elastic double spectral function obtained from the third channel and $\rho^{el}_{12}(u,s)$ is got from the crossing relations. Noting that $\rho^{el}_{13}(t,s)$ will themselves involve the absoptive parts ρ_1 we see that the strip approximation leads to an integral equation for the absorptive parts which has to be solved in successive stages, as mentioned in Lecture II.

For the III channel the integral equation will be

$$\rho^{(0)}_3(t,u,s) = \frac{1}{\pi}\int_{(m+1)^2}^{\infty} \frac{ds'}{s'-s}\left[\rho^{(0)}_{13}(t,s') + \rho^{(0)}_{13}(s',t)\right] + \frac{1}{\pi}\int_{(m+1)}^{\infty} \frac{d\omega'}{\omega'-n}\left[\rho^{(0)}_{23}(t,u') + \rho^{(0)}_{23}(u',t)\right]$$

Finally we notice that a complete solution of the photoproduction problem involves a knowledge of the matrix elements for the processes

$$\pi + N \longrightarrow \pi + N$$

$$\pi + \pi \longrightarrow N + \bar{N}$$

and $$\gamma + \pi \longrightarrow \pi + \pi$$

We shall end by mentioning that a similar evaluation of the double spectral functions can be carried out for nucleon-nucleon scattering which involves two isotopic spin combinations (0 and 1) and five scattering amplitudes for each value of the isospin. The latter can be argued out as follows.

For a given value of the total isospin and that of the total angular momentum, J, we have initially and finally three values of the relative orbital angular momentum, $\ell = J, J \pm 1$ between which transitions might take place. Singlet to triplet transition is forbidden. Parity conservation prevents transitions between states with $\ell = J$ and $\ell = J \pm 1$ for the triplet states. Thus we have two amplitudes for the transition $J \to J$ (one arising from the singlet and the other from the triplet transition). Further for the triplet case we have the three transitions. We have $J+1 \to J+1$, $J-1 \to J-1$ and $J+1 \to J-1$ (which by time reversal invariance is equivalent to $J-1 \to J+1$) So that we have in all five amplitudes necessary to represent nucleon-nucleon scattering for a prescribed isospin channel.

The double spectral functions can be evaluated for the problem as before starting either with the Goldberger, Nambu, Oehme (GNO) set of amplitudes or the recent one of Goldberger, et al. But there is an ambiguity in the labelling of the particles in the 2nd and 3rd channels. Since in the first channel the lowest 2 particle intermediate state is the two-nucleon state itself, there would be no ambiguity in labelling (in the centre of mass matrix element) the σ matrices corresponding to the two nucleons. We could follow the path of the two nucleons separately and label the σ operators operating at the vertices in the two paths by 1 and 2 respectively. But for the second and third channels, $N + \overline{N} \longrightarrow N + \overline{N}$ there would be an ambiguity in labelling the σ_- matrices according to the intermediate state chosen. If we choose the lowest - mass intermediate state, viz, the two pions, we have perforce to operate the σ_- matrices between the nucleon and antinucleon both in the initial and finall states, so that we could lable these as 1 and 2 (of course this labelling is different from that for the first channel, so that there would be no simple way of ariving at the matrix element of channels II and III, given that for channel I) If on the other hand, we choose a nucleon - antinucleon intermediate state there would be two possibilities for labelling the σ operators depending on whether they operate between the two nucleons or a nucleon and an antinuvleon at a vertex.

LECTURE VI

We shall study the consequences of the strip approximation by taking two simple cases, — the photoproduction of pions on pions and pion-pion scattering. The first-mentioned process is the simplest that one can think of in that it involves only a single scattering amplitude, — the isoscalar amplitude. In the centre of mass frame the matrix element is of the form.

$$\frac{\vec{\epsilon} \cdot \vec{k} \times \vec{q} \; \epsilon_{\alpha\beta\gamma} F(s,u,t)}{k\,q} \qquad (1)$$

where k and q are the centre of mass momenta of the initial photo (or pion) and the final pions respectively. α is the isotopic spin lable. of the initial pion and β and r those of the final pions, $\epsilon_{\alpha\beta\gamma}$ is a third rank antisymmetric tensor.

Since there is only one amplitude involved the double-spectral functions will be particularly simple. e.g.

$$F_{13}(s,t) = 4\iint dz_2\,dz_3\,K_1(z_1,z_2,z_3) \times \left[I_3^*(s,z_3)F_3(s,z_2) + I_2^*(s,z_3)F_2(s,z_2) \right]$$

where I represents the matrix element for pion-pion scattering. We notice further that the elastic double spectral function for the third channel (which by the strip approximation will be the inelastic double spectral function for channel I) can be written down simply without explicit calculation by the simple change $s \longleftrightarrow t$, since the problem is completely symmetric, each channel representing the same process.

A further point of interest is when we introduce the isotopic spin labels. For pion-pion scattering the matrix element will have the form.

$$A\,\delta_{\beta\gamma}\delta_{\xi\eta} + B\,\delta_{\beta\xi}\delta_{\gamma\eta} + C\,\delta_{\beta\eta}\delta_{\gamma\xi} \qquad (3)$$

ξ, η are the isospin labels of the initial pions and β, γ those of the final pions. This has to be multiplied by the matrix element for photoproduction of pions on pions which involves the combination $\epsilon_{\alpha\xi\eta}F$ multiplication gives the coefficient of $\epsilon_{\alpha\beta\gamma}$ on the right hand side of the unitarity condition as $(B-C)F^1$ But $B - C = A^{(1)}$ the pion-pion scattering matrix element in the isotopic spin state $I = 1$. By the Bose symmetry for the pions this is also the state with $J = 1$ (if we keep to low energies). Now this is precisely the state in which a reasonance has been observed in pion-pion scattering. The fact that only this part of the pion-pion scattering matrix element is non-vanishing in our problem, is fortunate since we will be making use of the dominance of the resonance.

We require some simple forms for the absorptive parts occurring on the right hand side of (2). A solution obtained from the one-dimensional approximations would be a good starting point. Solovev studied the photoproduction of pions on pions in such an approximation. By taking the nucleon-antinucleon intermediate state in the matrix element for $\gamma + \pi \rightarrow N + \bar{N}$ one of the solutions he obtained was

$$F = \frac{\Lambda}{s-1}\,\Phi(\nu)\,e^{i\delta(\nu)} \quad (4)$$

where Λ is the residue at the pion pole at $s = 1$ and $\Phi(\nu)$ is a function of $\nu = q^2$, the square of the momentum. $\delta(\nu)$ is the pion-pion phase shift. For the pion-pion scattering amplitude we make use of the resonance in the $I = 1$, $J = 1$ state to write a Breit-Wigher type of amplitude

$$\frac{3\Gamma\,(2t + s - 4)/(s-4)}{(s_r - s) - i\Gamma\left(\frac{s}{4} - 1\right)^{3/2}} \quad (5)$$

where s corresponds to the position of the resonance and Γ is the width.

Substituting these corresponding absorptive parts in (2) we obtain.

$$F_{13}(s,t) = \frac{1}{6}\iint \frac{dt_2\, dt_3}{(s-4)^{3/2}(s-1)} K_1(s,t,t_2,t_3)\, s^{1/2}$$

$$\times \Gamma^2 \Lambda \left[\frac{(t_2-4)^{1/2}}{t_3-1} \frac{(2s+t_2-4)\sin\delta\left(\frac{t_3}{4}-1\right)\Phi\left(\frac{t_3}{4}\ldots\right)}{(s-t_2)^2+\Gamma^2\left(\frac{t_2}{4}-1\right)^3} \right.$$

$$+\frac{1}{6}\iint \frac{du_2\, du_3\, K_1'(s,t,u_2,u_3)\, s^{1/2}}{(s-4)^{3/2}(s-1)} \Gamma^2 \Lambda$$

$$\left[\frac{(u_2-4)^{1/2}}{u_3-4} \cdot \frac{(4-2s-u_2)\sin\delta\left(\frac{u_3}{4}-1\right)\Phi\left(\frac{u_3}{4}-1\right)}{(s-u_2)^2+\Gamma^2\left(\frac{u_2}{4}-1\right)^3} \right. \qquad (6)$$

where

$$K_1(s,t,t_2,t_3)$$

$$= -\left[\frac{s}{(s-1)^2(s-4)}\left\{(s+2t-3)^2+(s+2t_2-3)^2\right\}\right.$$

$$+\frac{(2t_2+s-4)^2}{(s-4)^2} - 1$$

$$\left. - \frac{2s(s+2t-3)(s+2t_2-4)(s+2t_3-3)}{(s-4)^2(s-1)^2}\right]^{-1/2} \qquad (7)$$

K_1' is a similar expression in terms of u_2 and u_3. We have reexpressed the angle variables in terms of the t and u variables by using the relation

$$\left.\begin{aligned} t &= 1 - 2k\omega_q - 2kq\cos\theta \\ u &= 1 - 2k\omega_q + 2kq\cos\theta \end{aligned}\right\} \qquad (8)$$

for the photoproduction of pion on pions and

$$t = -2q^2(1-\cos\theta);\; u = -2q^2(1+\cos\theta)$$

for pion-pion scattering. (9)

Now if the dispersion relations for the absorptive part of our matrix element in the first channel, viz

$$F_1(s,t) = \frac{1}{\pi}\left[\int_4^{16} \frac{F_{13}(s,t')}{t'-t}\,dt' + \int_4^{16} \frac{F_{13}(t',s)}{t'-t}\,dt'\right] \; * \qquad (10)$$

the first integral representing the contribution from the elastic region should give a value for F_1 the absorptive part, which should not differ appreciably from the value. We started with, viz, the absorptive part of F, if the one-dimensional solution is at all reliable. So we shall content ourselves with estimating the contribution of the inelastic part F_1, arising from the second integral By making the change $s \longleftrightarrow t$ in (6) we see that the s dependence is completely contained in the

* We can forget the integration u containing F12 since when $s \to \infty$ (an approximation we will be presently making), for t fixed in a finite range, u becomes large and negative so that the denominator $u'-u$ never vanishes.

function K_1 , Explicitly $K_1(t, s, t_2, t_3)$ can be approximated for very large values of the variables by the expression

$$\frac{(t-1)(t-4)}{t^{\frac{1}{2}}}\left\{4s^2(t-4)+4s(t-3)(t-4)\right.$$
$$\left.-4s(t+2t_2-4)(t+2t_3-3)\right\}^{-\frac{1}{2}} \qquad (11)$$

to $\frac{1}{s}$ Remembering that t is confined to the values 4 to 16 and t_2 and t_3 are restricted by the inequality $t^{1/2} > t_2^{1/2} + t_3^{1/2}$

We thus see that for very large values of s , the inelastic contribution to the absorptive part goes down like $1/s$.

We can consider pion-pion scattering in a similar way since if we assume the dominance of the $I = 1, J = 1$, state for the problem, the number of independent scattering amplitudes which are three for the problem (corresponding to $I = 0, 1, 2$) become just one. Using the Breit - Wigner form for the absorptive parts, we find that the dependence in the inelastic part of the absorptive part is again contained in the function K_1 and the behaviour for large s is again $1/s$

Those results give us hope that the strip approximation may give sensible answers at high energies.

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