MATSCIENCE REPORT 3

LECTURE COURSE ON
INTRODUCTION TO COMPLEX VARIABLE THEORY

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INTRODUCTION TO
COMPLEX VARIABLE THEORY

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Lectures on the Theory of Functions of a complex Variable.

CHAPTER I.

CONVERGENCE PROBLEMS.

Preliminary definitions and theorems.

By a neighbourhood of a point z_0 is meant the set of all points z such that $|z - z_0| < \epsilon$. The neighbourhood of the point at infinity is the set of all points outside the circle $|z| > R$

A point z_0 is said to be a limiting point of a set of points S if every neighbourhood of z_0 contains a point of S distinct from z_0. It follows that every neighbourhood of z_0 contains an infinite number of points of S.

A set is said to be closed if every limiting point of the set belongs to the set.

An interior limiting point z_0 is one for which there exists a neighbourhood composed entirely of points of the set. Limiting points which are not interior are called boundary points.

A set which consists entirely of interior limiting points is said to be open.

A set consisting of points $z_1, z_2, ----, z_n, --$ is said to be bounded if there exists a positive number K such that $|z_n| < K$ for all n.

A set of points is said to be convex if every pair of its points can be joined by a polygonal arc consisting entirely of points of the set. An open convex set is a domain. A closed convex set is a closed region.

The set of all real numbers x such that $a \leq x \leq b$ is called an interval. A sequence of intervals $I_1, I_2, ----, I_n, --$ forms a nest if I_{n+1} consists only of points I_n and if the length of I_n tends to zero as $n \rightarrow \infty$. One proves that there is one and only one point which belongs to all the intervals of the nest.

For complex numbers, a nest is defined as follows: A sequence of closed rectangles $R_1, R_2, \ldots, R_n, ---$ whose sides are parallel to the real and imaginary axes is called a nest if R_{n+1} consists only of points of R_n and if the lengths of the sides of R_n tend to zero as $n \rightarrow \infty$. Clearly there is one and only one point which lies in all the intervals of the nest.

The Bolzano Weierstrass Theorem.

If a set S is bounded and contains infinitely many points, then it possesses at least one limiting point.

In the two-dimensional case, this can be proved using the next theorem stated above.

The convergence of complex sequences.

A sequence $z_1, z_2, ----, z_n ---$ is convergent if there is only one limit point Z for the sequence which is finite. Thus Z is the limit of the sequence $z_1, z_2, ----, z_n, ---$ if, given any positive number ϵ, we can find an integer N such that $|z_n - Z| < \epsilon$ for all $n > N$

Cauchy's principle of convergence.

The necessary and sufficient condition for the convergence of a complex sequence $Z_1, Z_2, \ldots, Z_n$ is that, given any positive number ϵ, there should exist an integer N depending on ϵ, such that $|Z_{N+p} - Z_N| < \epsilon$ for all positive p.

This is necessary since $Z_n \to Z$, there exists a number N such that $|Z_n - Z| < \frac{1}{2}\epsilon$ for $n \geqslant N$

$$\therefore \quad |Z_{N+p} - Z_n| \leq |Z_{N+p} - Z| + |Z_n - Z| < \epsilon$$

<u>This is sufficient</u>: Since the neighbourhood ϵ of Z_N contains all but a finite number of points of the sequence, the sequence has at least one limiting point. We show now that there cannot be two limiting points. For let Z', Z'', be these points. Then N may be so chosen that Z_N is as near as you please to Z'. But this neighbourhood necessarily contains Z''. Thus $Z' \neq Z''$ can be brought as near as you please. But this is impossible since $Z' \neq Z'$. Thus there can be only one limiting point. More precisely, given ϵ, we have integers q and r such that

$$|Z' - Z_{N+q}| < \epsilon$$

$$|Z'' - Z_{N+r}| < \epsilon$$

$$|Z' - Z''| = |Z' - Z_{N+q}| - |Z_{N+q} - Z_N| - |Z_{N+r} - Z_N| - |Z'' - N_{N+r}|$$

$$\leq |Z' - Z_{N+q}| + \cdots$$

$$< \epsilon \qquad \therefore \quad Z' \equiv Z''$$

The maximum and minimum limits of a sequence of real numbers.

Theorem. An oscillatory bounded sequence $x_1, x_2, \ldots, x_n$ possesses a greatest and a least limiting point.

Proof: Since the sequence is bounded, $x_n <$ a real number K for all n. Now divide the real numbers into two classes L and R. A number x belongs to R if there are only a finite number of members of the sequence $> x$ and belongs to L if there are an infinity of such numbers. R clearly exists all numbers $> K$ belong to this class. L too exists for since if all numbers $\in R$ then no matter how large x is, all but a finite number of members of the sequence would be $< -x$ i e $x_n \to -\infty$. But by assumption, this is not so. Thus

1) every number $\in L$ or R

2) L and R exist.

3) any member of L is $<$ any member of R.

Thus by Dedekend's th., there exists a real number such that $L < \xi$ and $R > \xi$.

To prove ξ is a limiting point: $\xi + \frac{1}{2}\epsilon \in R$. Thus there are only a finite number of members of the sequence $> \xi + \frac{1}{2}\epsilon$. Thus $\xi + \epsilon$ is not a limiting point. But $\xi - \epsilon \in L$ so that there are an infinity of members of sequence in the neighbourhood ϵ of ξ. Thus ξ is a greatest limiting point.

$$\xi = \overline{\text{Limit}}\, x_n$$

The minimum limit of the sequence is defined throgh

$$\underline{\lim}\, x_n = -\overline{\lim}\,(-x_n)$$

Infinite series.

An infinite series

$$S = a_1 + a_2 + \cdots\cdots + a_n + \cdots\cdot$$

is to be understood as the limit of the sequence of the partial sums

$$S_n = a_1 + a_2 + \cdots\cdots\cdots + a_n$$

The convergence of the series can be checked by Cauchy test.

Absolute convergence. $S = \sum_0^\infty a_n$

is absolutely convergent if

$$\sigma = \sum_0^\infty |a_n|$$

is convergent. Absolute convergence implies convergence, but not conversely.

Proof: Since σ is convergent, given ϵ, we can find N such that

$$(\sigma_{N+p} - \sigma_N) < \epsilon \quad \text{for all } p \text{ +ve}$$

But

$$\begin{aligned} |S_{N+p} - S_N| &= |a_{N+1} + \cdots\cdots + a_{N+p}| \\ &\leq |a_{N+1}| \cdots\cdots + |a_{N+p}| \\ &= \sigma_{N+p} - \sigma_N = |\sigma_{N+p} - \sigma_N| < \epsilon \end{aligned}$$

therefore is a convergent sequence.

$\because$ all terms σ_N are +ve

The converse is not true since for instance $1 + i + \frac{i^2}{2!} + \cdots$ is convergent, but not absolutely.

Sufficient conditions for absolute convergence.

Cauchy test: The series $\sum_n a_n$ of complex numbers is absolutely convergent if $\overline{\lim}\, |a_n|^{\frac{1}{n}} < 1$ but is divergent if $\overline{\lim}\, |a_n|^{1/n} > 1$.

If $\overline{\lim} |a_n|^{1/n} < 1$ let $\overline{\lim} |a_n|^{1/n} = 1 - 2c$ $0 < c \leq \frac{1}{2}$. Therefore $|a_n|^{1/n} \leq 1 - c$ for all but a finite number of n. Thus $|a_n| \leq (1-c)^n$ for $n \geqslant$ some integer M. But for $n \geqslant M$

$$|\sigma_{N+p} - \sigma_n| \leq (1-c)^{n+1} + (1-c)^{n+2} + \dots + (1-c)^{n+p}$$
$$\leq \frac{1}{c}\left\{(1-c)^n - (1-c)^{n+p+1}\right\} < \frac{(1-c)^{n+1}}{c}$$

Thus, given ϵ, we can choose an N such that $|\sigma_{N+p} - \sigma_N| < \epsilon$ for all $+ve\ p$. Thus $\sum a_n$ is convergent.

On the other hand, if $\overline{\lim} |a_n|^{\frac{1}{n}} > 1 = 1 + 2d$ say, $(d > 0)$, then $|a_n|^{1/n} > 1 + d$ for an infinite number of n or $|a_n| > (1+d)^n$ for an infinite number of n. Thus a_n does not tend to zero as $n \longrightarrow \infty$ or $\sum a_n$ is divergent.

Further tests available are: D' Alambert's test and Raabe's test.

<u>Tests for non-absolute convergence.</u>

Lemma: If $A_n v_n$ tends to a finite limit as $n \to \infty$, then if one of the series

$$A_0 v_0 + \sum_1^{\infty} (A_n - A_{n-1}) v_n, \quad \sum_0^{\infty} A_n (v_n - v_{n+1})$$

is convergent, so is the other.

<u>Proof:</u>

$$A_0 v_0 + \sum_1^{N} (A_n - A_{n-1}) v_n - \sum_0^{N-1} A_n (v_n - v_{n+1})$$
$$= A_N v_n \qquad Q.E.D$$

1) Dedekiund's test

$\sum a_n v_n$ is convergent if

i) $\sum a_n$ has bounded partial sums

ii) $\sum (v_n - v_{n+1})$ is absolutely convergent

iii) v_n tends to zero as $n \longrightarrow \infty$

If $A_n = a_0 + a_1 + \cdots\cdots + a_n$ by (i) $|A_n| < K$ for all n

$$\sum A_n v_n < K|v_n|$$

By (iii), $|A_n v_n| \longrightarrow 0$ as $n \rightarrow \infty$

$$|A_n(v_n - v_{n+1})| < K|v_n - v_{n+1}|$$

$\therefore$ By (ii), $\sum A_n(v_n - v_{n+1})$ is absolutely cngt

Thus is convergent

$$A_0 v_0 + \sum_1^\infty (A_n - A_{n-1}) v_n$$

2) du Bois-Reymound's test.

$\sum a_n v_n$ is convergent if

i) $\sum a_n$ is convergent,

ii) $\sum (v_n - v_{n+1})$ is absolutely convergent.

Let $A_n = a_{n+1} + a_{n+2} + \cdots\cdots$

Then by (i), A_n is convergent and $\rightarrow 0$ as $n \rightarrow \infty$

Now

$$v_n = v_0 - (v_0 - v_1) - (v_1 - v_2) \cdots - (v_{n-1} - v_n)$$

$$|v_n| \leq |v_0| + \sum_{r=0}^{\infty} \left|(v_r - v_{r+1})\right|$$

$$\because |v_r| \leq |v_0| + |v_0 - v_1| + \cdots + |v_{n-1} - v$$

By (ii) $|v_n| < k$, i.e, $|A_n v_n| < k|A_n|$

so that $A_n v_n \rightarrow 0$ as $n \rightarrow \infty$

Lemma shows that

$$a_0 v_0 + \sum_1^\infty a_n v_n = a_0 v_0 - \sum_1^\infty (A_n - A_{n-1}) v_n$$

is convergent if $\sum A_n (v_n - v_{n+1})$ ' is convergent.

But since $A_n \rightarrow 0$ as $n \rightarrow \infty$, $|A_n| < K$

$$\text{i.e } |A_n (v_n - v_{n+1})| < K |v_n - v_{n+1}|$$

Thus

$\sum A_n (v_n - v_{n+1})$ is in fact absolutely convergent

Q.E.D.

Uniform Convergence.

Let $\{S_n(x)\}$ be a sequence of functions in a bounded closed region D and let the sequence be convergent at each point $Z \in D$. Then given an ϵ we can find an $N(\epsilon, Z)$ which is finite (but not necessarily bounded) such that for all $n \geq N(\epsilon, Z)$

$$|S(Z) - S_n(Z)| < \epsilon$$

If furthermore $N(\epsilon, Z)$ is bounded, the convergence is said to be uniform. In this case, $N(\epsilon, Z) \leq M(\epsilon)$ so that for all $n \geq M(\epsilon)$

$$|S(Z) - S_n(Z)| \leq \epsilon$$

Note that $M(\epsilon)$ is independent of Z.

The necessary and sufficient condition for the unirom convergence of a sequence $\{S_n(\epsilon)\}$ in a bounded closed region D is that, given ϵ, we can find an $m(\epsilon)$ independent of z such that for all $n \geqslant m(\epsilon)$

$$|S_{m+p}(z) - S_m(z)| < \epsilon$$

for any positive p.

<u>Theorem.</u> If $\{S_n(x)\}$ is a convergent sequence continuous in D, a sufficient condition for the continuity of $S(z)$ in D is that convergence be uniform

We have

$$|S(z) - S_n(z)| < \epsilon \text{ for } n \geqslant M$$

$$\therefore \ |S(z_1) - S(z_2)| = |\{S(z_1) - S_n(z_1)\} + \{S_n(z_1) - S_n(z_2)\} + \{S_n(z_2) - S(z_2)\}| \leq 2\epsilon + |S_n(z_1) - S_n(z_2)|$$

But $S_n(z)$ is continuous. Therefore we can find a δ such that for all $|z_1 - z_2| < \delta$

$$|S_n(z_1) - S_n(z_2)| < \epsilon$$

or

$$|S(z_1) - S(z_2)| < 3\epsilon$$

Q.E.D.

From this one shows that if $\{S_n(z)\}$ is a sequence uniformly convergent in D, and if a contour L ~~lies~~ in D

$$\lim_{n \to \infty} \int S_n(z)\,dz = \int_L S(z)\,dz$$

<u>Theorem.</u> If $\{S_n(z)\}$ converges uniformly in every closed region D within a closed contour C and $S_n(z)$ is an analytic functions regular in C, $S_n(z)$ is regular in C, $S(z)$ is regular in C and $S_n' \to S'(z)$

$S(z)$ is continuous in C. Now

$$\underset{n \to \infty}{Lt} \int_L S_n(z)\,dz = \int_L S(z)\,dz$$

where L lies within C. Thus —— since $S_n(z)$ is regular in C, R.H.S depends only on the affixes of the end points. Thus $S(z)$ is regular in C.

Now

$$S'(a) - S_n'(a) = \frac{1}{2\pi i}\int_\Gamma \frac{S(z) - S_n(z)}{(z-a)^2}\,dz$$

where Γ lies within C and outside D. Let δ be the shortest distance from D to Γ and l the length of Γ. Then for all $n \geqslant M$,

$$|S'(a) - S_n'(a)| \leq \frac{1}{2\pi}\frac{\epsilon l}{\delta^2}$$

But L.H.S. is independent of a. Therefore

$$\{S_n'(a)\} \to S'(a)$$

Uniformly Convergent Series.

Let $S_n(x) = u_0(z) + \ldots\ldots + u_n(z)$

Then if $\{S_n(z)\} \to S(z)$ uniformly, the infinite series

$$u_0 + u_1(z) + \ldots\ldots$$

is said to converge uniformly. The theorems we proved for uniform sequences are directly applicable here.

Wuerstrass's M - test.

A sufficient condition for the infinite series $\Sigma u_n(z)$ to be uniformly and absolutely convergent in D is that $|u_n(z)| \leq M_n$ for any n

and ... is convergent.

The condition for absolute convergence is obvious.

Now

$$\left|S_{m+p}(z) - S_m(z)\right| = \left|\sum_{m+1}^{m+p} u_n(z)\right| < \sum_{m+1}^{\infty} M_n$$

But $\because$ $\sum M_n$ is convergent, there is an m for which

$$\sum_{m+1}^{\infty} M_n < \epsilon$$

$\therefore$ $\left|S_{m+p}(z) - S_m(z)\right| < \epsilon$ where $z \in D$ | p is any $+ve$ integer

Q.E.D.

<u>Infinite Products.</u>

If

$$p_n = \prod_{n=1}^{n} (1 + a_n)$$

The limit of $\{p_n\}$ is called the infinite product and we write

$$= \prod_{r=1}^{\infty} (1 + a_r)$$

If $\{p_n\}$ tends to a finite non-zero limit, the product is said to be convergent while if it tends to zero or becomes infinite, it is said to be divergent.

For convergence, clearly none of the $(1 + a_r)$ should vanish.

The necessary and sufficient condition for convergence of $\{p_n\}$ is the convergence of $\sum \log(1 + a_n)$ where each logarithim has its principle value.

<u>Proof:</u>

Let $S_n = \sum_{1}^{n} \log(1 + a_n)$

$$S_n \to S, \quad p_n \to p$$

Now

$$S_n = \log p_n + 2q_n \pi i$$

we now show that q_n is const. for sufficiently large which will prove that the condition is necessary.

If α_n, β_n denote the principal values of $(1+a_n)$ and p_n, $\alpha_n \to 0$, $\beta_n \to \beta$ since $\{S_n\}$ is convergent

But

$$\alpha_1 + \dots + \alpha_n = \beta_n + 2q_n \pi i$$

$$\therefore \quad 2\{q_{n+1} - q_n\} \pi i = \alpha_{n+1} - (\beta_{n+1} - \beta_n) \xrightarrow[n\to\infty]{} 0$$

since q_n is an integer, $q_n = q$ for all sufficiently large n.

QUE.D.

Absolutely convergent infinite products. The necessary and sufficient condition for the absolute convergence of $\prod (1+a_n)$ is the absolute convergence of $\sum a_n$.

For since $a_n \to 0$ as $n \to \infty$, there is an integer N such that for all $n \geq N$, $|a_n| \leq 1/2$

Therefore

$$\left| 1 - \frac{\log(1+a_n)}{a_n} \right| = \left| \frac{a_n}{2} - \frac{a_n^2}{3} + \dots \right|$$

$$\leq 1/2 \{ |a_n| + |a_n|^2 + \dots \} \leq \frac{1}{2}$$

~~or~~

or $\frac{1}{2}|a_n| \leq \log(1+a_n) \leq 3/2\, |a_n|$

The result follows.

Uniformly convergent infinite products.

The infinite product $\prod \{1 + u_n(z)\}$ converges uniformly and absolutely in the bounded closed region if

$|u_n(z)| \leq M_n$ where M_n is independent of z and $\sum M_n$ is convergent. (Cf. Weirstrass's M. Test).

Absolute convergence under such conditions has al-readybeen proved.

Let $P_n = \prod_1^n (1 + M_r)$ for $n > m$

$$\left| f_n(z) - f_m(z) \right| = \left| f_m(z) \right| \left| \prod_{m+1}^{n} \{1 + u_r(z)\} - 1 \right|$$

But clearly

$$\left| \prod_{m+1}^{n} \{1 + u_r(z)\} - 1 \right| \leq \left\{ \prod_{m+1}^{n} \{1 + M_r\} - 1 \right\}$$

$$\left| f_n(z) - f_m(z) \right| \leq \prod_{r=1}^{m} (1 + M_r) \left| \prod_{m+1}^{n} (1 + M_r) - 1 \right|$$

$$= P_n - P_m$$

From the convergence of $\{P_n\}$, uniform convergence of $\{f_n(z)\}$ now follows.

We also have: if the infinite product $\prod \{1 + u_n(z)\}$ converges uniformly to $f(z)$ in every closed region within a closed contour C and if each factor C then $f(z)$ is an analytic function regular within C.

<u>Functions defined by infinite integrals.</u>

Let the function $f(z,t)$ be such that

i) It is a continuous function of both the variables z when z lies within the closed contour and $a \leq t \leq T$ for every finite T.

ii) For each such value of t, it is an analytic function of z, regular within C.

iii) The integral $f(z) = \int_a^\infty F(z,t)\,dt$

is convergent when Z lies within C and uniformly convergent when Z lies on any closed region D within C.

Then $f(z)$ is an analytic function of Z regular within C whose derivatives may be found by differentiating under the integral sign.

The last condition tells us that given an , there is an ϵ independent of Z such that for all $T \geqslant t_0$ to

$$\left| f(z) - \int_a^T F(z,t)\,dt \right| < \epsilon$$

when $Z \in D$

Consider

$$f_n(z) = \int_a^n F(z,t)\,dt$$

where n is an integer.

So by an analogue of one of our previous theorems, since $f_n(z)$ is regular within C and converges uniformly to $f(z)$ in any domain $D \in C$, $f(Z)$ is regular in D and $f_n'(z) \to f'(z)$ uniformly. But

$$f_n'(z) = \frac{d}{dZ}\int_a^n F(z,t)\,dt = \int_a^n \frac{\partial F(z,t)}{\partial z}\,d$$

and so

$$f'(z) = \int_a^\infty \frac{\partial F(z,t)}{\partial z}\,dt$$

Q.E.D.

The analogue of Weirstrass's M. test for the uniform convergence of infinite integrals is the following: Let $F(z,t)$ be a continuous function of t when $z \in D$ and $t \geq a$ such that $|F(z,t)| \leq M(t)$ at each z where $M(t)$ is independent of z. Then if $\int_a^\infty M(t)\,dt$ converges, $\int_a^\infty F(z,t)\,dt$ converges uniformly and absolutely in D.

Clearly since $F(z,t)$ continuous function of t, $\int_a^T F(z,t)\,dt$ exists for $T > a$. Since $\int_a^\infty M(t)\,dt$ is convergent, there is a T independent of z such that $\int_T^\infty M(t)\,dt$ is $< \epsilon$. Now such a T, with $T' > T$

$$\left| \int_T^{T'} F(z,t)\,dt \right| \leq \int_T^{T'} |F(z,t)|\,dt$$

$$\leq \int_T^\infty M(t)\,dt < \epsilon$$

Therefore $\int_a^\infty F(z,t)\,dt$ converges uniformly and absolutely in D.

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CHAPTER II.

Some Theorems on the Properties of Analytic Functions.

A function which is one-valued and differentiable at every point of a domain D save possibly for a finite number of exceptional points is said to be <u>analytic</u> in D. If furthermore no point of D is a singularity of the analytic function, the function is said to be regular in D.

An inequality of frequent service follows from

$$f^{n}(a) = \frac{n!}{2\pi i}\int_{c}\frac{f(z)}{(z-a)^{n+1}}\,dz$$

where $f(z)$ is an analytic function regular within a circle c of radus R centred at a. Let $|f(z)| \leq M$ everywhere on c. We have

$$|f^{n}(a)| \leq \frac{n!}{2\pi i}\int_{c}\frac{|f(z)|}{|(z-a)^{n+1}|}\,dz$$

$$\leq \frac{n!}{2\pi}\;\frac{M.\,2\pi R}{R^{n+1}}$$

or $|f^{n}(a)| \leq \dfrac{M\,n!}{R^{n}}$

Thks inequality leads to Liouville's theorem:

If $f(z)$ is an integral function which satisfies the inequality $|f(z)| < M$ for all values of z, then $f(z)$ is a constant. This follows since at any a,

$$|f'(a)| < M/R$$

Making $R \to \infty$ we find $f'(a) = 0$

Thus $f(z)$ is a constant. We thus see that an integral function which is not a constant, cannot be bounded everywhere in the complex plane.

<u>Zeros of an analytic function.</u>

If $f(z)$ is an analytic function which vanishes when $z = a$ and is regular in a neighbourhood $|z-a| < R$ of a, a is said to be a zero of $\cancel{|z-a|<R}$ $f(z)$. We may write

$$f(z) = \sum^{\infty} a_n (z-a)^n$$

in a certain neighbourhood of <u>a</u>. If a_m is the first nonvanishing term in the expansion, the zero is said to be of order m.

We now show that there is a neighbourhood of <u>a</u> containing no other zero of $f(z)$ than $z = a$.

For let m be the order of the zero at $z = a$ Then

$$f(z) = (z-a)^m \sum_{n=0}^{\infty} a_{m+n} (z-a)^n$$

$$= (z-a)^m \phi(z)$$

where $\phi(z)$ is regular for $|z-a| < R$ and non-vanishing at $z = a$. Let $\phi(a) = 2c$. Then since $\phi(z)$ is continuous, there is a neighbourhood of a where $|\phi(z)| > c$

For, for some δ such that $|z-a| < \delta$, $|\phi(z) - \phi(a)| < c$

But $$\phi(z) = \phi(a) + \phi(z) - \phi(a)$$

or $$|\phi(z)| \geqslant \Big|\{|\phi(a)| - |\phi(a) - \phi(z)|\}\Big|$$

$> |c|$

It follows that the only point in $|z-a| < \delta$ where $f(z)$ vanishes is $z = a$.

It follows that if $\{Z_n\}$ is a sequence of zeros of $f(z)$ regular in D and the limit of the sequence is interior to D, $f(Z)$ is identically zero.

A point <u>a</u> is an isolated singularity of a function if there is a neighbourhood of <u>a</u> which contains no other singularity of $f(z)$. We now have, for $0 < |z-a| < R$ by Laurent's theorem,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

where $\sum_{1}^{\infty} b_n (z-a)^{-n}$ is called the principal part of $f(z)$ at the singular point $z = a$. If this series terminates and $b_n = 0$ for $n > m$, the point <u>a</u> is called a pole of order m. In such a case

$$f(z) = (z-a)^{-m} \{ b_m + b_{m-1}(z-a) + \dots + b_1 (z-a)^m \} + \sum_{0}^{\infty} a_n (z-a)^{m+}$$
$$= (z-a)^{-m} \phi(z)$$

where $\phi(z)$ is regular in a certain neighbourhood of <u>a</u> and $\phi(a) = b_m \neq 0$. Thus for some δ such that $|z-a| < \delta$

$$|\phi(z)| > \tfrac{1}{2} |b_m|$$

or

$$|f(z)| \geq \tfrac{1}{2} |b_m| (z-a)^{-m}$$

Thus if $z = a$ is a pole $f(z)$ tends to infinity as z tends to <u>a</u> in any manner.

If the principal series does not terminate, the point $z = a$ is called an <u>isolated essential singularity</u> of

If a is the limiting point of a sequence of zeroes of $f(z)$ regular in the open interval $0 < |z-a| < R$

then either $f(z) = 0$ or $z = a$ is a singularity of $f(z)$. Clearly it is not a pole since $f(z)$ does not tend to infinity as z tends to a in any manner. Thus except for the case when $f(z) = 0$ except at $z = a$, the point $z = a$ is an isolated essential singularity.

Similarly if a is the limit point of a sequence of poles of $f(z)$, a is not a pole since it is not isolated: it is called an <u>essential singularity</u>.

<u>Weierstrass's Theorem.</u>

In every neighbourhood of an isolated essential singularity, there exists a point at which the function differs by as little as we please from any previously assigned number.

It immediately follows that there are an infinity of points at which the function approaches as near as we please any assigned number.

<u>Proof</u>: Let $f(z)$ be an analytic function with an isolated essential singularity at $z = a$. Let C be any real or complex number. We have to show that given an ϵ and an η, there is a point z_1 in $|z-a| < r$ at which

$$|f(z) - c| < \epsilon$$

If <u>a</u> is the limiting point a sequence of zeros of $f(z) - C$ the result is clearly true since we have only to take z_1 to be a zero of $f(z) - C$ in the given region. If <u>a</u> is not a limiting points of zeros, there is a neighbourhood of a which contains no zeroes of $f(z) - C$ provided r is

$$g(z) = \frac{1}{\{f(z) - c\}}$$

is regular in $c < |z-a| < r$ We have to show that there is a point z_1, here such that $g(z_1) > \frac{1}{\epsilon}$

Let $g(z) \leq \frac{1}{\epsilon}$. If the principal part of $g(z)$ at a is $\sum_1^\infty b_n (z-a)^{-n}$

$$b_n = \frac{1}{2\pi i} \int_{|z|=\rho} g(z)(z-a)^{n-1} dz$$

where $0 < \rho < r$

or $b_n \leq \frac{\rho^n}{\epsilon}$

But b_n is independent of ρ. Making $\rho \to 0$, we find $b_n = 0$. But this is impossible since it would imply that $f(z)$ is regular at a or has a pole there. Therefore $|g(z)| > \frac{1}{\epsilon}$ for $z = z_1$. This proves Weierstrass's theorem.

A more remarkable theorem is due to Picard which states that in every neighbourhood of an isolated essential singularity, there exists a point at which the function actually attains any given value with atmost one exception.

e.g. $\sin \frac{1}{z}$ attains every value in the interval $0 < |z| < r$ no matter how small r is while $e^{1/z}$ attains every value except zero.

Picard's theorem will be proved later when we discuss elliptic anodular functions.

Theorem: If the only singularities of an analytic func-

function is a rational function.

Let $F(z)$ be a function with a finite number of poles at $a_1, a_2, \ldots\ldots, a_k, \ldots \infty$ of orders $n_1, n_2, \ldots, n_k, m$ respectively.

(An infinite number of poles is impossible since the limiting point of these poles is an essential singularity). Thus

$$G(z) = \sum_0^\infty b_n z^n = \frac{(z-a_1)^{n_1}(z-a_2)^{n_2}\cdots(z-a}{F(z)}$$

is an integral function . Write

$$G(z) = \sum_0^\infty b_n z^n$$

Clearly $G(z)$ has a pole of order $N = n_1 + n_2 + \cdots + n_k + m$ at infinity. Consequently the Taylor expansion terminates at $n = N$ or $G(z)$ is a polynomical. Thus $F(z) = G(z) / \{(z-a)^{n_1} (z-a_2)^{n_2} \cdots$ is a rational function.

The number of zeroes of an analytic function.

The Principle of the argument.

Let $f(z)$ be regular within and on a closed contour C save for poles $b_1, b_2, \ldots, b_n$ none of which lie on C. Furthermore let it have zeroes $a_1, a_2, \ldots, a_m$ none of which also lie on . Then the excess of the number of zeroes over the number of poles within C is $\frac{1}{2\pi}\left[\arg f(z)\right]_C$

For if a is a zero of order r,

$$f(z) = (z-a)^r \phi(z)$$

where $\phi(z)$ is regular and non-zero in a certain neighbourhood of a. Therefore

$$\frac{f'(z)}{f(z)} = \frac{r}{r-a} + \frac{\phi'(z)}{\phi(z)}$$

has a simple pole at $z = a$ with residue or similarly if $z = a$ is a pole of order s,

$$f(z) = \phi(z)/(z-a)^s$$

$$f'(z)/f(z) = \frac{(z-a)^s}{\phi(z)}\left[-\frac{\phi(z)\,s}{(z-a)^{s+1}} + \frac{\phi'(z)}{(z-a)^s}\right]$$

$$= -\frac{s}{(z-a)} + \phi'(z)/\phi(z)$$

which has thus a simple pole of residue $-s$ at $z = a$.

Thus

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{p=1}^{m} r_p - \sum_{p=1}^{n} s_p$$

Regarding a zero of order r as equivalent to r simple zeroes and similarly for s, this takes the form

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \text{excess of zeroes over poles within } C.$$

But $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\Big[\log f(z)\Big]_C$

$$= \frac{1}{2\pi i}\Big[\log|f(z)| + i\arg f(z)\Big]_C$$

$$= \frac{1}{2\pi}\arg f(z)\Big|_C$$

which proves the theorem (often called the <u>principle of the argument</u>).

<u>Rouche's theorem.</u>

If $f(z)$ and $g(z)$ are regular within and on a closed contour C on which $f(z)$ does not vanish and $|g(z)| < |f(z)|$ then $f(z)$ and $f(z) + g(z)$ have the same number of zeroes within C.

Let $f(z)$ and $f(z) + g(z)$ have m & n zeroes respectively within c. Then

$$F(z) = 1 + g(z)/f(z)$$

has n zeroes and m poles within c. and never vanishes on c. Thus arg. $F(z)\big|_c = 2\pi(n-m)$

But

$$Re\{F(z)\} = 1 + Re\, g(z)/f(z) \geq 1 - \left|\frac{g(z)}{f(z)}\right| >$$

Thus $[\arg F(z)]_c = 0$ which proves the theorem.

In particular, let $f(z) = a_0 z^m$

$$g(z) = a_1 z^{m-1} + a_2 z^{m-2} + \cdots + a_m$$

For sufficiently large $|z| = R$

$$\left|\frac{g(z)}{f(z)}\right| = \frac{1}{|a_0 z|}\left[a_1 + \frac{a_2}{z} + \cdots + \frac{a_m}{z^{m-1}}\right] <$$

Clearly $f(z)$ has a zero of order m within c.

Thus $f(z) + g(z)$ has m zeroes there. Therefore the function $f(z) + g(z) / (z-z_1)(z-z_2)\cdots(z-z_n)$ is an integral function which tends to a_0 as $|z| \to \infty$. Consequently by Liouville's th., it is indentically equal to .

Hence $a_0 z^m + a_1 z^{m-1} + \cdots + a_m = a_0(z-z_1)\cdots(z-z_m)$

This proves the fundamental theorem of the algebra of complex numbers. A polynomial of degree m has m zeroes and can be experessed as a product of m linear factors.

<u>Cauchey's Theorem on the Partial Fraction expansion of a Function.</u>

The theorem is due to Cauchy:

Let

1) F(z) be regular save for poles in any fninite region of the Z plane.

2) there exist an increasing sequence of positive numbers R_n such that $R_n \to \infty$ as $n \longrightarrow \infty$ and such that the circle C_n of radius R_n does not pass through a pole of $f(z)$ for any value of n.

3) the upper bound of $f(z)$ on C_n be bounded as $n \longrightarrow \infty$

4) $\left| f(R_n e^{i\theta}) \right| \longrightarrow 0$ as $n \longrightarrow \infty$ uniformly with respect to θ in $0 \leq \theta \leq 2\pi$ or more generally in every portion of this interval which does not include one of a finite number of exceptional values of θ.

Then if ξ is not a pole of $f(z)$

$$f(\xi) = \lim_{n \to \infty} S_n(\xi)$$

where $S_n(\xi)$ is the sum of the residues of $\dfrac{f(z)}{\xi - z}$ at the poles of $f(z)$ within C_n.

The theorem provides a means of expansion of $f(z)$ in partial fractions.

Let ξ be a point in a bounded closed region D in $|z| \leq R_n$ which contains no poles of $f(z)$. Then since $R_n \to \infty$, there is an integer N for which $R_N > R$ i.e ξ lies within all the circles C_n for which $n \geqslant N$. The singularities of $f(z)/\xi - z$ are ξ and poles of $f(z)$

Thus

$$\frac{1}{2\pi i}\int_{C_n}\frac{f(z)}{\xi - z}\,dz = S_n(\xi) - f(\xi)$$

We now show that the integral on the L.H.S. tends to zero as $n \longrightarrow \infty$.

For simplicity suppose there is only one exceptional value $\bar{\theta}$, α say (The proof is easily extended to the general case). By (3), there is a positive number of M independent of n such that $|f(z)| < M$ when Z lies on C_n.

By 4), if δ is any positive number, the upper bound ϵ_n of $|f(R_n e^{i\theta})|$ when $\alpha + \delta \leq \theta \leq 2\pi + \alpha - \delta$ tends to zero as $n \longrightarrow \infty$. It follows that for $n \geqslant N$,

$$\left|\int_{C_n}\frac{f(z)}{\xi - z}\,dz\right| \leq \frac{R_n}{R_n - R}\int_0^{2\pi}|f(R_n e^{i\theta})|\,d\theta$$

$$\leq \frac{R_n}{R_n - R}\left\{\int_{\alpha-\delta}^{\alpha+\delta} M\,d\theta + \int_{\alpha+\delta}^{2\pi+\alpha-\delta}\epsilon_n\,d\theta\right\}$$

$$\leq \frac{R_n}{R_n - R}\left[2M\delta + 2\pi\epsilon_n\right]$$

and so

$$\overline{\lim_{n\to\infty}}\left|\int_{C_n}\frac{f(z)}{\xi - z}\,dz\right| \leq 2M\delta$$

But since δ is arbitarary,

$$\lim \int \quad f(z) \quad dz = 0$$

$$f(\xi) = S_1(\xi) + \sum_{n=1}^{\infty} \left[S_{n+1}(\xi) - S_n(\xi) \right]$$

which converges uniformly when ξ lies in a bounded closed region containing no poles of $f(\xi)$. Notice that the partial fraction series is not necessarily absolutely convergent so that the terms have to be suitably bracketed as shown.

EXPANSION OF COSEC Z AS A SERIES OF PARTIAL FRACTIONS.

Cosec z = $\dfrac{1}{\sin z}$ has simple poles at $z = 0, \pm 2\pi, \ldots$ so that the circles C_n given by $|z| = (n+\frac{1}{2})\pi$ donot pass through any of these.

Draw with each pole as centre a circle of radius
We shall show that cosec z is bounded in the region T exterior to these circles.

Now $|\text{cosec } z| = \dfrac{2}{e^{i(x+iy)} - e^{-i(x+iy)}}$

$$\leq 2 / e^{-y} - e^{y} = \text{cosech } |y|$$

Thus $|\text{cosec } z| \leq \text{cosech } |a|$ outside the strip $Im|z| \leq a$

But cosec z is clearly bounded within that portion of T within the rectangle with verticles $\pm \frac{3}{2}\pi \pm ia$ in and consequently, by periodicity, bounded in $Im|z| \leq \alpha$. Thus $|\text{cosec } z| \leq$ M in T, M being clearly dependent on E. Since C_n are entirely within T, this inequality holds on C_n, independent of

On C_n, $z = (n+\frac{1}{2})\pi e^{i\theta}$. There-where $\delta \leq \theta \leq \pi - \delta$, or $\pi + \delta \leq \theta \leq 2\pi - \delta$, δ being any small positive number. Thus as $n \to \infty$ $|\operatorname{cosec} z|$ tends to zero uniformoy with respect to θ in all directions except $\theta = 0$ or π.

Thus the conditions of the theorem are satisfied. Therefore if ξ is not a pole of $\operatorname{cosec} z$,

$$\operatorname{cosec} \xi = \lim_{n \to \infty} \sum_{-n}^{n} \text{Residue of} \left[\frac{\operatorname{cosec} z}{\xi - z} \right] \text{ at } z = m\pi$$

$$= \lim_{n \to \infty} \sum_{-n}^{n} \frac{(-1)^m}{\xi - m\pi} = \frac{1}{\xi} + \sum_{1}^{\infty} \frac{2(-1)^m \xi}{\xi^2 - m^2\pi^2}$$

By Wuerstrass's M - test, this converges absolutely and uniformly when ξ lies in any bounded part of T and hence may be integrated term by term. This gives

$$\log \tan \xi/2 = \log A + \log \frac{\xi}{2} + \sum_{m=1}^{\infty} (-1)^m \left[\log\left(1 - \frac{\xi^2}{m^2\pi^2}\right) \right]$$

$$\tan \xi/2 = \frac{1}{2} A \xi \prod_{1}^{\infty} \left\{ \frac{1 - \xi^2/4m^2\pi^2}{1 - \xi^2/(2m-1)^2\pi^2} \right.$$

But $\tan \xi/2 \,/\, \xi/2 \to 1$ as $\xi \to 0$, Thus $A = 1$

Hence setting $\xi = 2z$ we get

$$\frac{\tan z}{z} = \prod_{1}^{\infty} \left[1 - z^2/m^2\pi^2 \right] \Big/ \prod_{1}^{\infty} \left[1 - \frac{4z^2}{(2m-1)^2\pi^2} \right]$$

which expresses $\tan z / z$ as a quotient of two integral functions each of which is an integral function.

AN EXTENSION OF CAUCHY'S THEOREM.

Let $f(z)$ satisfy the conditions (1), (2) and (3) but not (4) of the previous theorem. Then $F(z) = f(z)/z$ obviously satisfies (1) and (2).

On C_n, $z = R_n e^{i\theta}$ and so

$$|F(z)| = |f(z)|/R_n \leq M/R_n$$

Thus as $n \to \infty$, $F(R_n e^{i\theta})$ tends to zero uniformly with respect to θ in $0 \leq \theta \leq 2\pi$. Thus $F(z)$ satisfies all the conditions of the theorem and hence

$$\frac{f(z)}{z} = \lim_{n\to\infty} \left\{ \text{sum of residues of } \frac{f(t)}{t(z-t)} \text{ at poles} \right.$$

of $f(t)/t$ within C_n

Suppose that $f(z)$ is regular at the origina and has only simple poles, say at a_1, a_2 . . . . with resides b_1, b_2, ... Arranging α_i's in the sequence $0 \leq |a_1| \leq a_2 \leq \cdots$ . . . , we find that the residue at a_r of $f(t)/t(z-t)$ is

$$\frac{b_r}{a_r(z-a_r)} = \frac{b_r}{z}\left[\frac{1}{(z-a_r)} + \frac{1}{a_r}\right]$$

while the residue at $z = 0$ is $f(0)/z$. Thus,

$$\frac{f(z)}{z} = \frac{f(0)}{z} + \sum_{r=1}^{\infty} b_r\left[\frac{1}{(z-a_r)} + \frac{1}{a_r}\right]$$

or

More generally if as $n \to \infty$, $|z^{-p} f(z)|$ is bounded on C_n as $n \to \infty$, where p is a positive integer, $f(z)/z^{p+1}$ satisfies all the conditions of the theorem and the partial fraction expansion immediately follows.

THE REPRESENTATION OF AN INTEGRAL FUNCTION AS AN INFINITE PRODUCT.

Let $F(z)$ be an integral function which does not vanish at the origin and with simple zeroes at $z_1, z_2, \ldots z_n, \ldots$; $|z_n|$ must clearly tend to infinity as these zeroes can have no limiting points of finite affix.

Let $F(z) = (z - z_r)\, \Phi(z)$ then $\phi(z)$ is regular and non-zero in a certain neighbourhood of .

Hence

$$\frac{F'(z)}{F(z)} = \frac{1}{(z - z_r)} + \frac{\phi'(z)}{\phi(z)}$$

so that the only singularities of $F'(z)/F(z)$ are simple poles at z_r with residue 1.

Let us suppose that $F'(z)/F(z)$ satisfies the conditions of Mitlag - Leffler's theorem. Then,

$$F'(z)/F(z) = \sum_{n=1}^{\infty} \frac{1}{z - z_r}$$

the series being uniformly convergent in any bounded closed region which contains none of the zeroes of $F(z)$, provided the terms are suitably bracketed. We can thus integrate term by term. Therefore,

$$\log F(z) = \log F(0) + \sum_{n=1}^{\infty} \log\left(1 - \frac{z}{z_r}\right)$$

Or $$F(z) = F(0) \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right)$$

the infinite product being uniformly convergent in any bounded closed region which contains none of the zeroes of $F(z)$.

If however condition (4) is not satisfied, we find

$$\frac{F'(z)}{F(z)} = \frac{F'(0)}{F(0)} + \sum_{1}^{\infty} \frac{1}{z - z_n} + \frac{1}{z_n}$$

and so

$$\log F(z) = \log F(0) + \frac{F'(0)}{F(0)} z + \sum_{1}^{\infty} \left[\log\left(1 - \frac{z}{z_n}\right) + z/z_n\right]$$

Thus

$$F(z) = F(0)\, e^{z\, F'(0)/F(0)} \prod_{n=1}^{\infty} \left\{\left(1 - \frac{z}{z_n}\right) e^{z/z_n}\right\}$$

This shows that under such conditions, $F(z)$ is uniquely determined if $F(0)$ & $F'(0)$ are known.

CHAPTER III.

INTEGRAL FUNCTIONS.

An analytic function which is regular in every finite region in the Z plane is called an integral function.

Lemma. The most general integral function with no zeroes is of the form $e^{g(z)}$ where $g(z)$ is an integral function. For if $f(z)$ is an integral which never vanishes,

$$F(z) = f'(z)/f(z)$$

is an integral function. But

$$\log F(z) = \log f(z_0) + \int_{z_0}^{z} F(z)\,dz$$

the integration from z_0 to Z being any path. Since the R.H.S. is an integral function, the result follows. Notice that if $F(z)$ is an integral function,

$$G(z) = \int_{z_0}^{Z} F(z)\,dz$$

is an integral function. Since the integral on the R.H.S. is a function only of the end-points of integration, $G(z)$ is one-valued. Furthermore, it is differentiable since

$$G(z_2) - G(z_1) = \int_{z_0}^{z_2} F(z)dz - \int_{z_0}^{z_1} F(z)dz$$

$$= \int_{z_1}^{z_1+\delta} F(z)\,dz \quad \text{if} \quad z_2 = z_1 + \delta$$

Since $F(z)$ is differentiable, given an ϵ, there is a δ such that

$$|F(z) - F(z_1)| < \epsilon \quad \text{for} \quad |z - z_1| \leq \delta$$

But
$$F(z) = F(z_1) + F(z) - F(z_1)$$
Therefore
$$|F(z)| < |F(z_1)| + \epsilon \quad \text{for } |z - z_1| < \delta$$
$$\therefore \left[|G(z_2) - G(z_1)|\right] \leq \{|F(z_1)| + \epsilon\}\delta$$

Since $F(z)$ is bounded for finite z, the differentiability of $G(z)$ follows.

The construction of an integral function within given zeroes.

If $F(z)$ has only a finite number of zeroes, say $z_1, z_2, \ldots, z_n$, the problem is simple. Since $f(z)/(z-z_1)(z-z_2)\ldots(z-z_n)$ is an integral function with no zeroes and hence of the form $e^{g(z)}$ where $g(z)$ is an integral function. Thus
$$f(z) = (z - z_1) \ldots (z - z_n)\, e^{g(z)}$$
We have however integral functions with an infinite number of zeroes whose limiting point is at infinity. For these, we have

Weierstrass's Theorem. If $z_1, z_2, \ldots z_n, \ldots$ be any sequence of numbers whose only limiting point is at infinity, it is possible to construct an integral function which vanishes at $z_1, z_2, \ldots$ and nowhere else.

Let us definite Weierstrass's primary factors
$$E(z,0) = 1 - z, \quad E(z,p) = (1-z)\, e^{z + z^2/2 + \cdots z^p/p} \quad (p > 0)$$
each of which is an integral function with but one zero, a simple

Now when $|z| < 1$

$$E(z,p) = \exp\left\{\log(1-z) + z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right\}$$

$$\exp\left\{-\frac{z^{p+1}}{p+1} - \frac{z^{p+2}}{p+2} - \cdots\right\}$$

or log $E(z,p)$

$$= -\frac{z^{p+1}}{p+1} - \frac{z^{p+2}}{p+2} - \cdots$$

Therefore for $|z| \leq 1/2$

$$\left|\log E(z,p)\right| \leq |z|^{p+1}/_{p+1}\left[1 + \frac{1}{2} + \frac{1}{4} + \cdots\right] \leq 2|z|^{p+1}$$

Let us suppose that the origin is not a zero of the integral function to be constructed, for if we wish it to be a zero, we have merely to multiply the function determined below by an appropriate power of z, Let the zeroes be arranged in a sequence of non-decreasing modulus, multiple zeros being appropriately repeated. Since $r_n = |z_n| \to \infty$ with n, there exist $p_1, p_2, \cdot p_n,$ integers such that

$$\sum_1^{\infty} \left(r/r_n\right)^{p_n}$$

converges for all $+ve$ r. In fact we need choose only $p_n = n$, since for sufficiently large r_r, $r/r_n < \frac{1}{2}$ for any given r.

We next assign abbitrarily $+ve$ n_0, R and then choose

$$r_n \leq 2R \leq r_{N+1}$$

Hence when $n > N$ and $|z| > R$

$$\left|\frac{z}{z_n}\right| \leq \frac{R}{r_n} < \frac{R}{r_{n+1}} < \frac{1}{2}$$

and so

$$\left|\log E\left(\frac{z}{z_n}, p_{n-1}\right)\right| \leq 2\left(\frac{R}{r_n}\right)^{p_n} < 2\left(\frac{1}{2}\right)^{p_n}$$

Thus by Weierstrass's M test, the series

$$\sum_1^{\infty} \log E\left(\frac{z}{z_n}, p_{n-1}\right)$$

converges uniformly and absolutely where $|z| \leq R$.

This implies that the infinite product

$$\prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p_{n-1}\right)$$

converges uniformly and absolutely no matter now large is. Since each factor of the product is an integral function, it follows that the infinite product is regular for every z such that $|z| \leq R$ i.e. it is an integral function.

With the same value of R , choose an integer M such that $r_M \leq R \leq r_{M+1}$. Then all the functions of the sequence

$$\prod_{n=1}^{m} E\left(\frac{z}{z_n}, p_{n-1}\right) \quad \left[m = M+1, M+2, \cdots\right.$$

vanish at $z_1, z_2, \cdots, z_M$ and nowhere else in $|z| \leq R$. Thus by Hurwitz's theorem, the only zeroes of $G(z)$ in $|z| \leq R$ are $z_1, z_2, \cdots, z_M$. But since R is arbitrary, it follows that the only zeros of $G(z)$ are the points of the sequence $z_1, z_2, \cdots$ which completes the proof the theorem.

Since theee are many sequences p_n possible, $G(z)$ is not uniquely fixed. However, when such a sequence is fixed, the most general integral function with the given zeros is $G(z)\, e^{H(z)}$ where $H(z)$ is an integral function.

The Principle of the maximum modulus.

Let $f(z)$ be an analytic function continuous within and on a closed contour C and regular within C. Let M be the upper bound of $f(z)$ on C. Then the inequality $|f(z)| \leq M$ holds everywhere within C. Furthermore $|f(z)| = M$ at a point within C if and only if $f(z)$ is a constant.

If n is any +ve integer and a point within C,

$$\{f(a)\}^n = \frac{1}{2\pi i}\int_C \{f(z)\}^n \frac{dz}{z-a}$$

If a is at a distance δ from C, it follows that

$$|f(a)|^n \leq M^n l / 2\pi\delta$$

where l is the length of C. Hence

$$|f(a)| \leq M\left(l/2\pi\delta\right)^{1/n}$$

Since $|f(a)|$ is independent of n we find, letting $n \to \infty$

$$|f(a)| \leq M$$

Again

$$n\, f'(a)\{f(a)\}^{n-1} = \frac{1}{2\pi i}\int_C \{f(z)\}^n \frac{dz}{(z-a)^2}$$

or

$$n\,|f'(a)|\,|f(a)|^{n-1} \leq \frac{1}{2\pi}\, M^n l/\delta^2$$

Now if there is a point a within C such that $|f(a)| = M$

$$|f'(a)| \leq Ml/2\pi\delta^2 \cdot \frac{1}{n}$$

Making $n \to \infty$, we get $f'(a) = 0$. If we have shown in this fashion that $f^{n-1}(a) = 0$ if $|f(a)| = M$, for the m^{th} derivative, we find

$$\left[\{f(a)\}^n\right]^{[m]} = \frac{m!}{2\pi i}\int_C \frac{\{f(z)\}^n}{(z-a)^{m+1}}\,dz$$

$$= \frac{d}{dz^{m-1}}\left[n f(z)^{n-1} f'(z)\right] = \frac{d}{dz^{m-2}}\left[n(n-1) f(z)^{n-2} f'(z) + n f(z)^{n-1} f''(z)\right]$$

$$= n f(z)^{n-m} f^{(m)}(z)$$

Therefore

$$\left|f^{(m)}(a)\right| \le \frac{m!\,l}{2\pi\delta^2}\,\frac{M^n}{M^{n-m}}\,\frac{1}{n} = \frac{m!\,l}{2\pi\delta^2}\,\frac{M}{n}$$

Thus making $r \to \infty$, $f^{(m)}(a) = 0$

By induction, it follows that $f^n(a) = 0$ for all n. Thus $f(z) = M$ within C in such a case.

A similar theorems holds in an annuals within which $f(z)$ is regular, M being the upper bound of $f(z)$ on the bounding curves of the annulus.

<u>The maximum modulus of an integral function.</u>

If $f(z)$ is an integral function, its maximum modulus $M(r)$ on $|z| = r$ is a steadily increasing unbounded function. For by the principle of maximum modulus,

$$|f(r_1 e^{i\theta})| < M(r_2) \quad \text{for } r_1 < r_2$$

i.e. $$M(r_1) < M(r_2)$$

save when $f(z)$ is a constant. Further $M(r)$ cannot be bounded, for if it were, $f(z)$ would be a constant, by Louville's theorem. This proves the statement.

The order of an integral function.

As integral function is of finite order if there is a real no. k independent of r such that its maximum modulus $M(r)$ on the circle $|z| = r$ satisfies

$$\log M(r) < r^k$$

for all sufficiently large r. Otherwise it is said to be of infinite order. k clearly is not zero or $-ve$ since in such a case, by Louville's theorem $f(z)$ would be a constant.

By means of a Dedekind section of real numbers, we can find ξ' such that for $k > \xi$

$$\log M(r) < r^k \quad \text{for sufficiently large}$$

and $$\log M(r) \geq r^{k_1}$$

Thus

$$\frac{\log \log M(r)}{\log (r)} < K$$

$$\Omega = \lim \quad \log \log M(r)$$

ρ is called the order of the integral function and is infinite and when the integral function is of infinite order.

For example, a polynomial has $\rho = 0$, while e^z, $\sin z$, $\cos z$ have $\rho = 1$, $\cos\sqrt{z}$ has $\rho = \frac{1}{2}$ and e^{e^z} is of infinite order.

<u>Theorem</u>. $e^{H(z)}$ is an integral function of finite order with no zeros if and only if $H(z)$ is a polynomial.

Clearly if $H(z)$ is a polynomial order ρ is an integral function of order ρ. To complete the proof of the theorem, we notice that

$$M(r) = \left| e^{\operatorname{Re} H(z)} \right|$$

so that $\operatorname{Re} H(z)$ is a $+$ve monotonically increasing function.

Thus
$$\log M(r) = \operatorname{Re} H(z) < r^{\rho+\epsilon}$$

for $\epsilon > 0$ and sufficiently large r. We have thus to show that $H(z)$ is a polynomial of degree not exceeding ρ.

Now
$$H(z) = \sum_{0}^{\infty} a_n z^n$$

where

$$a_n = \frac{1}{2\pi i} \int_C H(z) \frac{dz}{z^{n+1}}$$ where C is the circle $|z| = r$. When $n > 0$,

$$\int_c \overline{H}(z)\,\frac{dz}{z^{n+1}} = \int_c \sum_0^\infty \overline{a}_m \overline{z}^m \frac{dz}{z^{n+1}}$$
$$= \sum_0^\infty \int_0^{2\pi} \overline{a}_m r^{m-n} e^{-i(m+n)\theta}\, i\, d\theta = 0$$

the term by term integration being valid by the uniform convergence of $\sum \overline{a}_m \overline{z}^m$. Thus

$$a_n = \frac{1}{\pi i}\int_c Rl\ H(z)\ dz/_{z^{n+1}} = \frac{1}{\pi}\int_0^{2\pi} Re\, H(re^{i\theta}) \cdot \frac{d\theta}{r^n e^{in\theta}}$$

Thus

$$|a_n| r^n < \frac{1}{\pi}\int_0^{2\pi} |Re\, H|\, d\theta \quad \text{for } n > 0$$
$$\text{i.e} \quad |a_n| < 2 r^{\epsilon + (\rho - n)}$$

which gives, on making $r \to \infty$ $a_n = 0$ for $n > \rho$

Thus $H(z)$ is a polynomial of degree not exceeding ρ.

Jensen's inequality.

Let $f(z)$ be an integral function which does not vanish at the origin. Let its zeros, arranged in order of increasing modulus, be $z_1, z_2, z_3, \ldots\ldots$ multiple zeros being repeated. Then if

$$|z_N| < R \le |z_{N+1}|$$

we have

$$R^N f(0) \le M(R)\ |z_1 z_2 \ldots\ldots z_N|$$

For, the function

$$F(z) = f(z) \prod^N \frac{R^2 - z z_n}{}$$

is also an integral function and

$$|F(z)| = |f(z)| \prod_{1}^{N} \left| \left\{ \frac{(R^2 - z\bar{z}_n)(R^2 - \bar{z} z_n)}{R^2(z - z_n)(\bar{z} - \bar{z}_n)} \right\}^{1/2} \right|$$

$$= |f(z)| \prod_{1}^{N} \left\{ \frac{R^4 - R^2 2 \operatorname{Re} z\bar{z}_n + R^2|z_n|^2}{R^4 - 2R^2 z\bar{z}_n + R^2|z_n|^2} \right.$$

when $|z| = R$

$$= |f(z)| \leq M(R)$$

where $M(R)$ is the maximum modulus of $f(z)$ on $|z| = R$

Putting $z = 0$, we find

$$R^N|f(0)| \big/ |z_1 z_2 \dots z_N| \leq M(R)$$

Clearly the result is true provided only that $f(z)$ is regular in $|z| \leq R$.

Let $n(r)$ be the number of zeros of $f(z)$ in $|z| \leq r$

$n(r)$ is a non-decreasing function of r which is constant in any interval which does not contain the modulus of a zero of $f(z)$.

Then

$$\log \frac{R^N}{|z_1 z_2 \dots z_n|} = \log \left|\frac{R}{z_1}\right| \left|\frac{R}{z_2}\right| \dots \left|\frac{R}{z_N}\right|$$

$$= \sum_{1}^{N} \int_{|z_n|}^{R} \frac{dx}{x}$$

Let

$$|z_1| < r_1 < |z_2| < r_2 \dots |z_N| \leq R$$

$$\int^{|z_1|} \frac{n(x)}{} dx = 0, \quad \int^{|z_2|} n(x)\, dx = \log|z_2|$$

$$\int_0^R \frac{n(x)}{x}\,dx = \log \frac{|z_2|}{|z_1|}\,\frac{|z_3|^2}{|z_2|^2}\cdots\cdots\frac{|z_5|^4}{|z_4|^4}\cdots\cdots\frac{R^N}{(z_N)^{N-1}}$$

$$= \log \frac{R^N}{|z_1 z_2 \cdots\cdots z_N|}$$

Thus Jensen's inequality becomes

$$\int_0^R \frac{n(x)}{x}\,dx \leq \log M(r) - \log f(0)$$

<u>Lemma.</u> The above result shows that if $f(z)$ is an integral function of finite order ρ, $n(r) = O(r^{\rho+\epsilon})$ for every positive value of ϵ and all sufficiently large.

For if we put $R = 2r$ we have

$$\int_0^{2r} \frac{n(x)}{x}\,dx \leq \log M(2r) - \log|f(0)| < A r^{\rho+\epsilon}$$

for $\epsilon > 0$ and all sufficiently large r, A being a finite constant independent of r. But since $n(x)$ is non-decreasing,

$$n(r)\log 2 \leq \int_r^{2r} \frac{n(x)}{x}\,dx \leq \int_0^{2r} \frac{n(x)}{x}\,dx < A$$

and so

$$n(r) = O(r^{\rho+\epsilon})$$

<u>NOTE.</u> <u>Definition of the symbols O and o:</u> $f(z) = O(1)$ as $z \to \alpha$ means that there exists +ve numbers A and δ such that $|f(z)| < A$ when $|z-\alpha| < \delta$ i.e., $f(z)$ is bounded in the neighbourhood of the point α

If $\dfrac{f(z)}{\phi(z)} = O(1)$ as $z \to \alpha$, we write

If however $f(z)/\phi(z) \to 0$ as $z \to \alpha$ we write $f(z) = o(|\phi(z)|)$. Thus $z^2 = o(|z|)$ as but $z \to 0$ but $z = o(z^2)$ as $z \to \infty$.

The theorem $n(r) = O\, r^{\rho+\epsilon}$ tells us that $n(r)$ cannot increase much too fast with r. This is to be expected for if we look at a polynomial for example for which $\rho = 0$, the theorem tells us $n(r)$ becomes constant for sufficiently large r. Thus the rate of increase of $n(r)$ is correlated with the order of the integral function. This too one anticipates since rapidly increasing $n(r)$ implies an integral function whose $M(r)$ increases faster i.e. whose ρ is greater.

<u>The exponent of convergence of zeros of $f(z)$</u>

Let the integral function $f(z)$ have its zeros $z_1, z_2, \ldots$ arranged in order of increasing modulus. Then the exponent of convergence ρ_1 of these zeros is defined throgh

$$\rho_1 = \overline{\lim}_{n\to\infty} \log n / \log r_n$$

where $r_n = |z_n|$

<u>Theorem</u>. If ρ_1 is finite, the series $\sum r_n^{-\tau}$ converges when $\tau > \rho_1$ and diverges when $\tau < \rho_1$; but if ρ_1 is infinite, the series diverges for every real τ.

For if ρ_1 is finite and $\tau > \rho_1$,

$$\log n / \log r_n < \frac{1}{2}(\tau + \rho_1)$$

for all sufficiently large n. This gives, i.e.

$$r_n^{\frac{1}{2}(T+\rho_1)} > n$$

or

$$r_n^{T} > n^{1+p}$$

where $p = 2T/_{T+\rho_1} - 1 = \frac{T-\rho_1}{T+\rho_1} > 0.$ Thus $r_n^{-T} > \frac{1}{n^{1+p}}$

and since $\sum \frac{1}{n^{1+p}}$ converges for $p > 0$, r_n^{-T} converges.

On the other hand if ρ_1 is finite and $T < \rho_1$ or ρ_1 is infinite, there exist a sequence of integers such that $r_n^{T} < n$. Let N be such a value of n and m be the least integer greater than $\frac{1}{2}N$. Then since r_n increases with n,

$$\sum_{N-m}^{N} r_n^{-T} > \frac{m}{r_N^{T}} > \frac{m}{N} > \frac{1}{2}$$

But as there are values of N as large as we please, $\sum r_n^{-T}$ diverges.

If ρ_1 is finite, $\sum r_n^{-\rho_1}$ may converges or diverges. For example, $r_n = n$ has $\rho_1 = 1$ and the series $\sum 1/n$ diverges while $r_n = n/(\log n)^2$ also has $\rho_1 = 1$ while series $\frac{1}{n(\log n)^2}$ converges.

Theorem. If $f(z)$ is an integral function of finite order ρ, ρ_1 is finite and does not exceed ρ. For

$$\rho_1 = \overline{\lim_{n\to\infty}} \frac{\log n}{\log r_n} = \overline{\lim_{r\to\infty}} \frac{\log n(r)}{\log(r)}$$

$$\leq \overline{\lim_{r\to\infty}} \frac{\log\left(A r^{\rho+\epsilon}\right)}{\log r} = \rho + \epsilon$$

for every $\epsilon > 0$ and hence $\rho_1 \leq \rho$.

Canonical Products

If $f(z)$ is an integral function of finite order ρ with an infinite number of zeros $z_1, z_2 \ldots$ we saw that $\sum r_n^{-1-[\rho_1+\epsilon-1]}$ converges. Thus there is a least integer p for which $\sum r_n^{-1-p}$ converges. Clearly

$$\rho_1 - 1 \leq p \leq \rho_1 \leq \rho$$

Since $\sum r_n^{-1-p}$ converges,

$$G(z) = \prod_1^\infty E\left(z/z_n, p\right)$$

converges uniformly $\sim$ absolutely in any bounded closed region which contains none of the z_n and represents an integral function which vanishes only at z_n, i.e. the zeroes of $f(z)$. We call it the <u>canonical product</u> formed with the zeros of $f(z)$; the integer p is called its <u>genus</u>.

6 Borel's theorems on canonical products.

<u>Lemma</u> If $G(z)$ is a canonical product of genus p with zeros $z_1, z_2 \ldots$, and if N is a positive integer such that $|z_n| \leq 2|z| < |z_{N+1}|$ then $\log \left|\prod_1^N \left[1 - \frac{z}{z_n}\right]\right| - I \leq \log|G(z)| \leq I$

where $I = \sum_1^N \log\left[1 + |z/z_n|\right] + 2\sum_{N+1} |z/z_n|$ $(p = 0)$

and

$$A\sum_1^N \left|\frac{z}{z_n}\right|^p + 2\sum_{N+1}^\infty \left|\frac{z}{z_n}\right|^{p+1} \quad (p > 0)$$

A being independent of z.

The canonical product is

$$G(z) = \prod_1^N E\left(\frac{z}{z_n}, p\right) \prod_{N+1}^\infty E\left(\frac{z}{z_n}, p\right) = \Pi_1 \Pi_2 \text{ (say)}$$

Let $|z| = r$, $|z_n| = r_n$, $\left|\frac{z}{z_n}\right| = u_n$ in Π_2

$u_n < \frac{1}{2}$

and so

$$\left|\log|\Pi_2|\right| \le |\log \Pi_2| \le \left|\log E\left(\frac{z}{z_n}, p\right)\right| \le 2\sum_{N+1}^{\infty} a^{p+1}$$

Now when $p > 0$ and $|z/z_n| > \frac{1}{2}$,

$$\left|E\left(\frac{z}{z_n}, p\right)\right| \le \left(1 + \left|\frac{z}{z_n}\right|\right) \exp\left[\left|\frac{z}{z_n}\right| + \frac{1}{2}\left|\frac{z}{z_n}\right|^2 + \cdots + \frac{1}{p}\left|\frac{z}{z_n}\right|^p\right]$$

and

$$\log\left|E\left(\frac{z}{z_n}, p\right)\right| \le \log\left[1 + \left|\frac{z}{z_n}\right|\right] + \left|\frac{z}{z_n}\right| + \frac{1}{2}\left|\frac{z}{z_n}\right|^2 + \cdots + \frac{1}{p}\left|\frac{z}{z_n}\right|^p$$

where A is independent of z. Similarly

$$\log\left|E\left(\frac{z}{z_n}, p\right)\right| \geq \log\left|1 - z/z_n\right| - \left|\frac{z}{z_n}\right| - \left|\frac{z^2}{z_n}\right|^2 \cdot \frac{1}{2} \cdots \left|\frac{z}{z_n}\right|^p \cdot \frac{1}{p}$$

$$\geq \log\left|1 - \frac{z}{z_n}\right| - A\left|\frac{z}{z_n}\right|^p$$

Since $u_n \geq \frac{1}{2}$ in Π_1, we have

$$\sum_1^N \log\left|1 - \frac{z}{z_n}\right| - A\sum_1^N u_n^p \le \log|\Pi_1| \le A\sum_1^N u_n^p$$

Also

$$-2\sum_{N+1}^{\infty} u_r^{p+1} \le \log|\Pi_2| \le 2\sum_{N+1}^{\infty} u_n^{p+1}$$

Thus

$$\log \prod_1^N \left[\left|1 - \frac{z}{z_n}\right|\right] - I \le \log|G(z)| \le I \quad (p > 0)$$

<u>When $p = 0$</u>

$$\log|G(z)| = \log|\Pi_1| + \log|\Pi_2| \le \sum_1^N \log\left|1 - \frac{z}{z_n}\right|$$

$$+ 2\sum^{\infty} u_n$$

$$\leq \sum_1^N \log(1+u_n) + 2\sum_{N+1}^{\infty} u_n = I$$

$$\log|G(z)| \geqslant \sum_1^N \log\left|1-\frac{z}{z_n}\right| - 2\sum_{N+1}^{\infty} u_n$$

$$= \sum_1^N \log\left|1-\frac{z}{z_n}\right| + \sum_1^N \log(1+u_n) - I$$

$$\geqslant \sum_1^N \log\left|1-\frac{z}{z_n}\right| - I$$

This proves the lemma.

<u>Borel's first Theorem.</u>

The order of a canonical product is equal to the exponent of convergence of its zeros.

Let μ be a no. such that $p < \mu \leq \mu + 1$ and also such that $r_n^{-\mu}$ converges. Then when $p > 0$,

$$I = A\sum_1^N u_n^{\mu} u_n^{p-\mu} + 2\sum_{N+1}^{\infty} u_n^{\mu} u_n^{p+1-\mu}$$

$$\leq 2^{\mu-p} A \sum_1^N u_n^{\mu} + 2^{\mu-p}\sum_{N+1}^{\infty} u_n^{\mu}$$

since $u_n \geqslant \frac{1}{2}$ when $n \leq N$ and $u_n < \frac{1}{2}$ when $n > N$

This gives

$$I \leq 2^{\mu-p} r^{\mu}\left\{A\sum_1^N r_n^{-\mu} + \sum_{N+1}^{\infty} r_n^{-\mu}\right\} = O(r^{\mu})$$

One finds the same result to be true when $p = 0$.

If ρ_1 is the exponent of convergences of zeros of $G(z)$ when $\mu > \rho_1$, $|G(z)| < e^{r^{\mu}}$

for all sufficiently large r and hence $\rho \leq \mu$. But since μ is any no. greater that, ρ_1, $\rho \leq \rho_1$. However we have already shown that $\rho \geq \rho_1$. Thus $\rho = \rho_1$.

Borel's second Theorem

Given a canonical product of $G(z)$ of order and an arbitrary positive no. ϵ, there exist an infinite no. of circles of arbitarily large . on each of which holds the inequality

$$|G(z)| > e^{-|z|^{\rho+\epsilon}}$$

This theorem sets a lower bound on the minimum value $m(r)$ of $G(z)$ on $|z| = r$

Describe a circle $|z - z_n| = r_n^{-h}$ about each zero z_n for which $r_n = |z_n| > 1$, h being any real no. $> \rho$. Since Σr_n^{-h} converges, these circles do not cover the whole plane and hence there exist an infinite number of circles $|z| = r$ of arbitrarily large rad. which do not intersect any of these small circles. We have thus to prove the inequality for sufficiently large $|z|$, provided z lies outside these small circles. Now

$$\log |G(z)| > \log \prod_1^N \left|1 - \frac{z}{z_n}\right| - I$$

But $I < r^{\rho + \frac{1}{2}\epsilon}$ for sufficiently large r.

Let N be such that $r_N \leq 2r < r_{N+1}$ when $r_n \leq 1$, $\left|1 - \frac{z}{z_n}\right| > 1$ provided $r > 2$ and so

$$\log \prod_{r_n \leq 1} |1 - z/z_n| > 0$$

But when $1 < r_n \leq 2r$ and z lies outside the small

$$\left|1-\frac{z}{z_n}\right| = \frac{|z-z_n|}{|z_n|} \geqslant \frac{1}{r_n^{1+h}} \geqslant \frac{1}{(2r)^{1+h}}$$

Hence

$$\log \prod_{1<r_n\leq 2r} \left|1-\frac{z}{z_n}\right| \geqslant -N(1+h)\log 2r$$

But $N < A r^{\rho+\frac{1}{2}\epsilon}$ for sufficiently large r.

Hence

$$\log \prod^{N} \left|1 - z/z_n\right| > -A(1+h)\log(2r)\, r^{\rho+\frac{1}{2}\epsilon}$$

Thus when z lies outside the small circles and is sufficiently large,

$$\log |G(z)| > -r^{\rho+\frac{1}{2}\epsilon}\left\{1+A(1+h)\log 2r\right\}$$

$$> -r^{\rho+\epsilon}$$

This proves Borel's second Theorem.

Hadmard's Factorization Theorem

If $f(z)$ is an integral function of finite order ρ which has zeros $z_1, z_2, \ldots$ and does not vanish at the origin, it can be factorized in the form

$$f(z) = G(z)e^{H(z)}$$

where $G(z)$ is the canonical product formed with the zeros of $f(z)$ and $H(z)$ is a polynomial of degree not exceeding ρ.

$H(z)$ should be shown as a a polynomial of degree $\leq \rho$

Let ρ_1 be the exponent of convergence of zeros of $f(z)$

Then $G(z)$ is of order ρ_1 and $\rho_1 \leq \rho$.

Then

$$|G(z)| > e^{-r^{\rho_1+\epsilon}}$$

on an infinite number of circles of sufficiently large radius.

But since $f(z)$ is of order ρ,

$$|f(z)| < e^{r^{\rho+\epsilon}}$$

for all sufficiently large r. Thus

$$\left|e^{H(z)}\right| = \left|\frac{f(z)}{G(z)}\right| < e^{r^{\rho+\epsilon} + r^{\rho_1+\epsilon}} < e^{2r^{\rho+\epsilon}}$$

or

$$\operatorname{Re} H(z) < 2\, r^{\rho+\epsilon}$$

for an indefinitely increasing sequence of values of r.
Thus $H(z)$ as a polynomial of degree not exceeding ρ. Q.E.D.

When $G(z)$ is of genus p and $H(z)$ of degree q, the greater of the integers p or q is called the genus ~~fx~~of $f(z)$. Since $p \geq \rho$ and $q \leq \rho$, the genus of an integral function does not exceed its order. When ρ is not an integer, it may be shown that the genus is the greatest integer less than ρ while if it is an integer, it is either ρ or $\rho - 1$.

<u>An alternative Proof of the above Theorem</u>

<u>Lemma</u>

If $f(z)$ is an integral function and $f(0) \neq 0$, then $f(z) = f(0)\, p(z)\, e^{g(z)}$ where $p(z)$ is a product of primary factors and $g(z)$ is an integral function

Let $p(z)$ be formed with the zeros of $f(z)$.

Then

$$\phi(z) = \frac{f'(z)}{f(z)} - \frac{p'(z)}{p(z)}$$

is an integral function since the poles of one are cancelled by the poles of the other. Hence

$$g(z) = \int_0^z \phi(t)\,dt = \log f(z) - \log f(0)$$

is an integral function and the result stated follows. If $f(z)$ has a zero of order p at the origin, a factor z^p has to be inserted. Clearly the factorization is not unique.

If we take $p(z)$ to be the canonical product formed with the zeros of $f(z)$, we have now to prove that in $f(z)=p(z)\,e^{Q(z)}$ $Q(z)$ is a polynomial of degree not exceeding ρ. Let ν be the greatest integer smaller than ρ. Then

$$\frac{d}{dz}\left\{\frac{f'(z)}{f(z)}\right\} = Q^{\nu+1}(z) - \nu!\sum_1^\infty \frac{1}{(z_n-z)^\nu}$$

Let

$$g_R(z) = \frac{f(z)}{f(0)} \prod_{|z_n|\le R}\left(1-\frac{z}{z_n}\right)^{-1}$$

$|1-z/z_n| \geq 1$ for $|z|=2R$, $|z_n|\le R$, $|g_R(z)| \le \dfrac{f(z)}{f(0)}$

Since g_R is an integral function, this is true for

$$= O\left[e^{2R^{\rho+\epsilon}}\right]$$

$2R$ also.

Let $h_R(z) = \log g_R(z)$ where the logarithm is on that branch for which $h_R(0) = 0$. Then $h_R(z)$ is regular for and $|z| \le R$ Rand

$$\operatorname{Re} h_R(z) < K R^{\rho+\epsilon}$$

Then by the Borel - Caratheodory theorem,

$$|h^{\nu+1}(z)| < 2^{\nu+2}(\nu+1)!\,R\;K\,R^{\rho+\epsilon}$$

for $|z| = r < R$ and for $|z| = \frac{R}{2}$ this gives

$$h_R^{\nu+1}(z) = O\left(R^{\delta+\epsilon+\nu-1}\right)$$

Therefore

$$Q^{(\nu+1)}(z) = h_R^{\nu+1}(\nu) + \nu! \sum_{|z_n|>R} \frac{1}{(z_n - z)^{\nu+1}}$$

$$= O\left(R^{\delta+\epsilon+\nu-1}\right) + O\left(\sum_{|z_n|>R} |z_n|^{-\nu-1}\right)$$

for $|z| = \frac{1}{2}R$ and so also for $|z| < \frac{1}{2}R$. Since $\nu+1 > \rho$, the first term in R.H.S. tends to zero as $R \to \infty$ if ϵ ~~w~~here small enough. The second term tends to zero since $\sum |z_n|^{-\nu-1}$ is convergent. Since the L.H.S. is independent of R, this proves Hadmard's theorem.

An application of the above theorem is in expressing $\frac{\sin \pi z}{\pi z}$ as an infinite product. $\frac{\sin \pi z}{\pi z}$ is an integral function of order one and has zeros at $z = \pm 1, \pm 2, \ldots$

The exponent of convergence is given by

$$\delta_1 = \overline{\lim_{n\to\infty}} \frac{\log n}{\log r_n} = 1$$

The genus of the canonical product is $p = 1$ since if $p = 0$ $\sum n^{-1-p} = \sum \frac{1}{n}$ diverges. Thus the primary factors read

Thus,

$$\frac{\sin \pi z}{\pi z} = \prod_{-\infty}^{\infty}{}' \left(1 - \frac{z}{n}\right) e^{z/n} \, e^{g(z)}$$

Thus

where the accent indicates that the term $n = 0$ is omitted in the infinite product, $g(z)$ is a polynomial of degree not exceeding 1. But as $z \to 0$, L.H.S. $= 1$.

Thus $g(0) = 0$. Similarly taking derivatives on L. H. S. and comparing values at $z = 0$, we find that in fact $g(z) = 0$.

Thus

$$\frac{\sin \pi z}{\pi z} = \prod_{-\infty}^{\infty}{}' \left(1 - \frac{z}{n}\right) e^{-z/n}$$

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CHAPTER IV.

CONFORMAL REPRESENTATION

Let $\omega = f(z)$ where $f(z)$ is an analytic function, map a domain S of the z-plane isogonally on a domain Σ of the ω- plane. If c is a continuous curve $z = x(t) + iy(t)$ in S, C mapped to Γ in Σ such that

$$\omega = f\{x(t) + iy(t)\}$$

Then if ϕ and ψ are the angles of the tangents to C in S and Σ for the same t,

$$\phi = \tan^{-1}\frac{\dot{y}}{\dot{x}} = \arg\{\dot{x}(t) + i\dot{y}(t)\}$$

$$\psi = \arg\frac{d}{dt}f\{x(t) + iy(t)\}$$

$$= \arg\left[f'(z)\{\dot{x}(t) + i\dot{y}(t)\}\right]$$

$$= \phi + \arg f'(z)$$

Since $f'(z)$ depends only on z and not on the curve through a , this implies that the transformation conserves not only the magnitude, but also the sense of the angle of two curves. A conformal transformation is an isogonal transformation which preserves the magnitude as well as the sense of the anlge. Now if s and σ are the lengths of C and Γ up to the point of parameter t,

$$\frac{ds}{dt} = \left[\dot{x}^2 + \dot{y}^2\right]^{\frac{1}{2}} = \left[\dot{x}(t) + i\dot{y}(t)\right]$$

so that

Thus such a transformation maps a small neighbourhood of a point. z_0 on a neighbourhood of the corresponding ω_0 which correct to first order is obtained by a magnification on the ratio $|f'(z_0)|$ and a roatation throgh the angle $\arg f'(z_0)$

Note that since an isogonal maping should set up a one - one correspondence between Σ and C, $f(z)$ is analytic, one-valued and does not take any value more than once in C. Such a function is called a <u>simple</u> function (the French German words are 'univalente' and 'schticht().

If $f(z)$ is simple in C, $f'(z) \neq 0$ in C For if $f'(z_0) = 0$, then $f(z) - f(z_0)$ has a zero of order n $(n > 2)$ at z_0. Since $f(z)$ is not a constant, we can find a circle $|z - z_0| = \delta$ on which $f(z) - f(z_0)$ does not vanish. Let m be the lower bound of $f(z) - f(z_0)$ on this cifcle and let $0 < |a| < m$
Then by

Rouches' theorem $f(z) - f(z_0) - a$ has n zeros within this circle which contradicts the hypothesis that is simple in C.

Now let us consider a transformation with an $f(z)$ for which $f'(z)$ has a zero order of order n at a pt. z_0. Then in the neighbourhood of this point,

$$f(z) = f(z_0) + a\,(z - z_0)^{n+1} + \dots\dots$$

where $a \neq 0$. Hence if $\omega_0 = f(z_0)$, $\omega_1 = f(z_1)$ and $\omega_2 = f(z_2)$ where z_1 and z_2 near to z_0

$$\omega_1 - \omega_0 = a(z_1 - z_0)^{n+1}$$

If

$$\omega_1 - \omega_0 = \rho_1 e^{i\phi}, \quad \delta = \arg a,$$

$$\rho_1 e^{i\phi_1} = |a|\, r^{n+1} e^{i\{\delta + (n+1)\theta_1\}} + \ldots$$

Hence as $z_1 \to z_0$

$$\lim \phi_1 = \lim \{\delta + \overline{n+1}\,\theta_1\} = \delta + (n+1)\alpha_1$$

$$\lim \phi_2 = \delta + (n+1)\alpha_2$$

so that

$$(\phi_1 - \phi_2) = (n+1)(\alpha_2 - \alpha_1)$$

Further the linear magnification is zero. Thus the transformation is not conformal. This shows why the property $f'(z) \neq 0$ of simple functions is important.

Riemann's Theorem on Conformal Mapping.

The fundamental problem here is to transform a given domain S of the z-plane into any given domain Σ of the w-plane. It is however clearly sufficient to discuss whether it is possible to map conformally any given domain on the interior of a circle.

We cannot however map any S on the interior of a circle, e.g. a domain with a simple boundary point which may be taken to be the point at infinity. In such a case $\zeta = f(z)$ would be an integral function and hence a constant by Liouville's theorem.

Riemann's Theorem: If S is a domain bounded by a simple closed Jordan curve C , there exists a unique analytic function $f(z)$ regular in S such that $\omega = f(z)$ maps S conformally on $|\omega| < 1$ and also transforms a point $z = a$ within C into the origin and a given direction at $z = a$ into the positive direction of the x axis.

Clearly $f(z)/z-a$ is regular and non-zero in S and hence $f(z) = (z-a)e^{\phi(z)}$ where $\phi(z)$ as regular in C. Since $|f(z)| = 1$ on C

$$\log|z-a| + Re\,\phi(z) = 0$$

Now $\log f = V$ is regular in C except at $z = a$.

Also

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Thus we have to find a V which is real solution of Laplace's equation which vanishes on C, regular within S except near $z = a$ where it behaves like $\log(z-a)$. In such a case

$$V = \log(z-a) + Re\,\phi(z)$$

The imaginary part is found viz the Cauchy - Riemann equations so that $f(z)$ is known up to an additive constant.

Homographis Transformations.

Consider the conformal mapping of the complete z-plane on the complete w - plane save finite number of exceptional points. $\omega = f(z)$, $f(z)$ cannot have an essential singularity since in its neighbourhood, $f(z)$ takes on any assigned value an infinity of times $f(z)$ thus has only poles including the point at infinity and hence is a rational function. Since a rational function takes on any value p times, we require p to be one. If the singular point is at a finite distance, the transformation is

$$\omega = \frac{az+b}{cz+d}$$

such that $ad - bc \neq 0$ while if it is at infinity, we have

$$\omega = Az + B \quad (A \neq 0)$$

In either case, the transformation is homographic. Let us now consider such transformations for which $|\omega| = 1$ for $|z| = 1$

Then.

$$(az+b)(\bar{a}\bar{z}+\bar{b}) = (cz+d)(\bar{c}\bar{z}+\bar{d})$$

Thus

$$a\bar{a} + b\bar{b} = 1 = c\bar{c} + d\bar{d}, \quad a\bar{b} = c\bar{d}$$

Also $ad \neq bc$, If $a \neq 0$, $\bar{b} = \frac{c\bar{d}}{a}$ so that

$$(a\bar{a} - c\bar{c})(a\bar{a} - d\bar{d}) = 0$$

so that

$$|a| = |c| \quad \text{or} \quad |a| = |d|, \quad c = ae^{\gamma i}$$

where γ where is real. Since $a\bar{b} = c\bar{d}$, $d = be^{\gamma i}$ so that $\omega = e^{-\gamma i}$ which is not a homographic transformation.

If $|a| = |d|$, $d = \bar{a}e^{-2\delta}$, δ real, so that $c = \bar{b}\, e^{-2\delta}$. This gives

$$\omega = e^{2\delta} \frac{az+b}{\bar{a}+\bar{b}z}$$

where $|a| \neq |b|$. Similarly one shows that when $a = 0$, this formula is still true with $\omega = \frac{e^{2\alpha}}{z}$ α real. Clearly

$$(1 - \omega\bar{\omega})\left[a + b\bar{z}\right]^2 = (1 - z\bar{z})(a\bar{a} - b\bar{b})$$

when $|z|<1$, $|\omega|<1$ or $|\omega|>1$ according as $|a|>|b|$ or $|a|<b$. Thus the transformation maps $|z|\le 1$ conformally on $|\omega|\le 1$ only if $|a|>|b|$. We show later that no other conformal transformation has this property.

Schwarz's Lemma.

If $f(z)$ is regular in $|z|<R$, where it satisfies the inequality $|f(z)|\le M$ and if $f(0)=0$, then $|f(z)|\le \dfrac{M|z|}{R}$ whenever $|z|<R$. Also equality can occur only if $f(z)=\dfrac{Mze^{i\alpha}}{R}$ where α is a real constant.

Clearly $\phi(z)=f(z)/z$ is regular in $|z|<R$. Let $|a|<R$ and $|a|<r<R$. Then in the region $|z|\le r$, $\phi(z)$ attains its maximum value on $|z|=r$. Hence

$$\phi(a)\le \max_{|z|=r}\left|\frac{f(z)}{z}\right| = \frac{1}{r}\max|f(z)|$$

and so

$$\phi(a)\le M/r$$

since $\phi(a)$ is independent of r, making $r\to R$, we find

$$\phi(a)\le M/R$$

If $|\phi(a)|=M/R$ we know that $\phi(z)$ is constant and of modulus M/R. Thus $f(z)=\dfrac{M}{R}\,z\,e^{i\alpha}$

Riemann's Theorem for a circle.

Let $\omega = f(z)$ map $|z| \leq 1$ into $|\omega| \leq 1$ and turn the interior point $z = c$ into the origin. We shall show that the transformation is necessarily homographic.

Now $\zeta = z-c/1-\bar{c}z$ maps $|z| \leq 1$ conformally. on $|\omega| \leq 1$ and turns $z = c$ into the origin. Thus $\omega = \phi(\zeta)$ maps $|\zeta| \leq 1$ into $|\omega| \leq 1$ and observes the origins. Thus $|\omega| \leq |\zeta|$ for $|\zeta| \leq 1$. But we also write $\zeta = \Phi(\omega)$ which now gives $|\zeta| \leq |\omega|$ for $|\omega| \leq 1$. Hence $|\zeta| = |\omega|$ Therefore $\omega = \zeta e^{i\alpha}$, α being a real constant. Thus

$$\omega = e^{i\alpha} \frac{z-c}{1-\bar{c}z}$$

which is honographic.

If we enquire the direction $\arg(z-c) = \beta$ at c to be transformed into the $+ve$ real axis, we should take $\alpha = -\beta$.

We can now show that there is a unique function $\omega = f(z)$ which maps $|z-a| \leq R$ conformally on $|\omega| \leq 1$, transforms an interior point $z = c$ into the origin and a given direction at c into $+ve$ real axis. For $z' = \frac{z-a}{R}$ transforms $|z-a| \leq R$ on $|z'| \leq 1$, transforms $z = c$ into the interior point $z' = c'$ and leaves the prese cribed direction unaltered. Thenext step is as before. This proves Riemann's theorem for a circle.

The conformal representation of a half-plane on a circle.

We now consider how to transforms the half-plane ~~in~~ $\operatorname{Im} z > 0$ conformally on $|\omega| < 1$ and a point C of that half-plane into $\omega = 0$. If $|z - ik)| = R$ with C lying within the circle for sufficiently large R, we notice that as $R \to \infty$, this circle fills up the entire upper half-plane. Now

$$\omega = \frac{Re^{i\alpha}(z-c)}{R^2 - (z - iR)(\bar{c} + iR)} = \frac{i e^{i\alpha}(z-c)}{\left[1 - \frac{i\bar{c}}{R}\right](z - \bar{c})}$$

maps $|z - ik| \leq R$ conformally on $|\omega| \leq 1$. Thus as $R \longrightarrow \infty$,

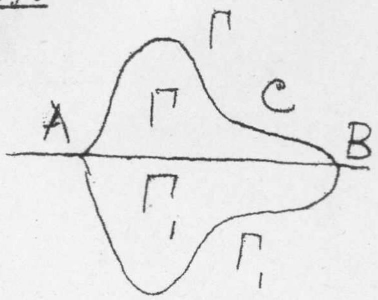
$$\omega = \lambda \, {z-c}/{z-\bar{c}}$$

where $|\lambda| = 1$ maps $\operatorname{Im} z > 0$ conformally on $|\omega| < 1$ and turns $z = c$ into $\omega = 0$. For suitable λ, any prescribed direction at $z = c$ can be made to correspond to the real axis at $\omega = 0$.

Riemann's theorem may be proved in this case as before.

Clearly by a preliminary translation and rotation, any half-plane can be mapped on .

Schwarz's principle of symmetry.



Let $f(z)$ within Γ and continuous within and on Γ and real on the portion of the real axis AB. Then if Γ_1 is obtained by reflecting Γ on real axis, $f(z)$ can be analytically continued across AB

by the relation $f(z) = \overline{f}(\overline{z})$

Let $\phi(z) = f(z)$ when $z \in \Gamma$ and $= \overline{f}(\overline{z})$ when $z \in \Gamma_1$, $\phi(z)$ is regular in Γ as well as in Γ_1, the latter since

$$\lim_{h \to 0} \frac{\phi(z+h) - \phi(z)}{h} = \lim_{h \to 0} \frac{\overline{f}(\overline{z+h}) - f(\overline{z})}{h}$$

$$= \lim_{h \to 0} \left[\overline{\frac{f(\overline{z}+\overline{h}) - f(\overline{z})}{\overline{h}}} \right] = 0$$

Evidently $\phi(z)$ is continuous within and on C. Hence

$$F(z) = \frac{1}{2\pi i} \int_C \phi(t)/_{t-z}\, dt$$

is regular within C when $z \in \Gamma$

$$F(z) = \frac{1}{2\pi i} \int \frac{\phi(t)\,dt}{t-z} + \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\phi(t)\,dt}{t-z}$$

$$= \phi(z)$$

Similarly $F(z) = \phi(z)$, $z \in \Gamma$. By continuity, this is true if z is an interior point of the segment AB but not an end point.

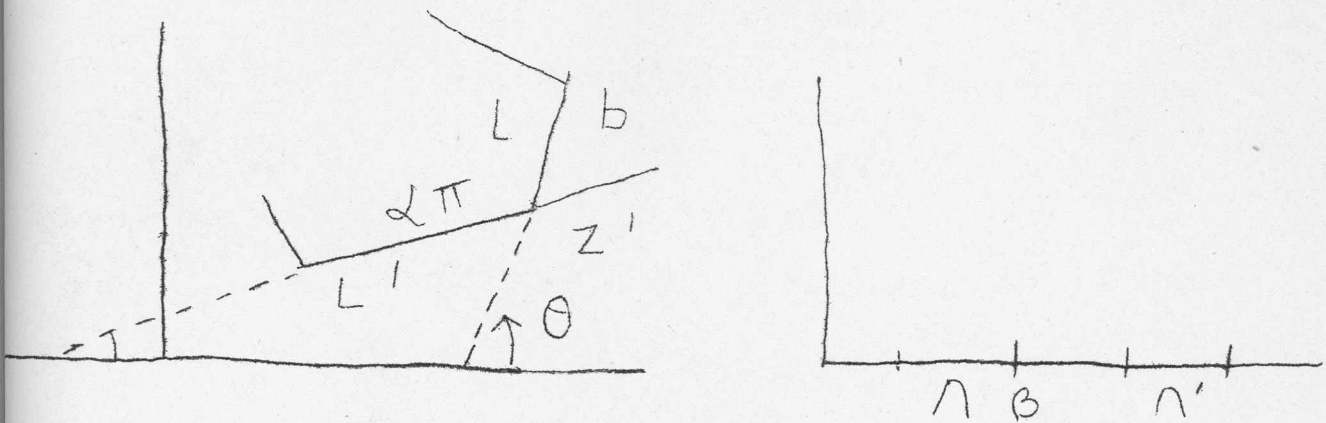
Thus we find that $\phi(z)$ provides an analytical continuation of $f(z)$ into Γ_1.

<u>The conformal representation of a polygon on a half-plane.</u>

Let $z = F(\omega)$ map $\operatorname{im} \omega > 0$ into the interior of a polygon in the z - plane. Thus $F'(\omega) \neq 0$ for $\operatorname{im} \omega > 0$ and regular there. Hence

$$d/_{d\omega} \log F'(\omega) = F''(\omega)/_{F'(\omega)}$$

is regular there.



Let L be a side of the polygon Γ making an angle θ with the z - axis and b a point on it. Let Λ be the segment on the real ω axis which corresponds to L and let β correspond to b. Then if z lies on L,

$$|z-b|\, e^{-i\theta} = \{F(\omega)-b\}\, e^{-i\theta}$$

is continuous and real on Λ and by Schwarz's principle can be continued down. Thus $\{F(\omega)-b\}\, e^{-i\theta}$ is regular in a neighbourhood of β and may be written

$$\{F(\omega)-b\}\, e^{-i\theta} = \sum_{1}^{\infty} a_r(\omega-\beta)^r$$

Where $a_1 \neq 0$

Hence $\frac{d}{d\omega} \log F'(\omega)$ is regular in a neighbourhood of β and real when ω is real. Similarly if b corresponds to the point at infinity,

$$\{F(\omega) - b\} e^{-i\theta} = \sum_{1}^{\infty} c_r \omega^{-r}$$

where $c_1 \neq \infty$. This gives

$$\frac{F''(\omega)}{F'(\omega)} = -\frac{2}{\omega} + \sum_{2}^{\infty} \frac{c_r'}{\omega^r} \qquad \text{(A)}$$

Again $\frac{d}{d\omega} \log F'(\omega)$ is regular in the neighbourhood of infinity and real when ω is real.

Now let L and L' be two consecutive sides of intersecting at an angle $\alpha\pi$ at z' with $z' = F(\omega')$. Clearly $F(\omega)$ is not regular in the neighbourhood of ω' since its argument changes discontinuously at ω'. However $\left\{\{z' - F(\omega)\} e^{-i\theta}\right\}^{1/\alpha}$ is continuous and real when ω lies on Λ' Since $F(\omega) \neq z'$ for $\omega > 0$, it is also regular in the upper halfplane. Thus as before we have

$$\left[\{z' - F(\omega)\} e^{-i\theta}\right]^{1/\alpha} = \sum_{1}^{\infty} b_r (\omega - \omega')^r$$

where $b_1 \neq 0$. Hence

$$F(\omega) = z' + e^{i\theta} (\omega - \omega')^{\alpha} \sum_{1}^{\infty} b_r' (\omega - \omega')^r$$

where $b_0' \neq 0$

i.e. $$F'(\omega) = (\omega - \omega')^{\alpha - 1} \phi(\omega)$$

where $\phi(\omega)$ is regular and non-zero in a certain neighbourhood of ω'. Hence

$$\frac{d}{d\omega} \log F'(\omega) = -\frac{(\alpha-1)}{\omega+\omega'} + \sum_{0}^{\infty} \frac{\phi'(\omega)}{\phi(\omega)}$$

has a simple pole with residue $(\alpha-1)$ there. Similarly if Z' corresponds to the point at infinity of the ω plane,

$$\frac{d}{d\omega} \log F'(\omega) = -\frac{(\alpha+1)}{\omega} + \sum_{2}^{\infty} \frac{d_r}{\omega^r}$$

and hence is regular there.

Now let a,b,c,..... (all of finite affix) on the real ω axis correspond to verticles of Γ of angles $\alpha\pi, \beta\pi, \gamma\pi, \ldots$ We notice that $\frac{d}{d\omega} \log F'(\omega)$ can be analytically continued from $\omega > 0$ to $\operatorname{im} \omega < 0$ which shows us that it has poles at a,b,c,.... and nowhere else. Hence

$$\frac{d}{d\omega} \log F'(\omega) = \frac{(\alpha-1)}{\omega-a} + \frac{\beta-1}{\omega-b} + \cdots + \frac{(\lambda-1)}{(\omega-l)} + \phi$$

where $\phi(\omega)$ is a polynomial. For large ω,

Since $\alpha+\beta+\gamma+\cdots+\lambda-n=-2$ (A) then shows us that $\phi(\omega)=0$ ($\because$ a polynomial small at infinity is identically zero). Thus

$$F(\omega) = A\int(\omega-a)^{\alpha-1}(\omega-b)^{\beta-1}\cdots(\omega-l)^{\lambda-1}\,d\omega +$$

This result is due to Schwarz and Christoffel.

In the integration, we take some definite branch of the integrand and never allow the path of integration to pass below real axis. $F(\omega)$ is hence regular for $\operatorname{im}\omega \geq 0$ except at a,b,c,..... l .

We can now fix a.b, ... l in the following fashion. The polygon $F(\omega)$ describes how the same angles as , for the two to be similar, $\Lambda - 3$ other conditions must be fulfilled, Thus having arbitrarily fixed 3 of the a,b,c, ... l, We use the rest to make the 2 polygons similar. New suitable choice of $|A|$, arg A and B will fix the scale, orientation and position of the polygon $F(\omega) = Z$ in the Z plane so that it can ~~be~~ finally be made to coincide with Γ.

When ω describes the contour shown in the figure, Des-cribes the polygon Γ indeed at the verticles. When $R \to \infty$ and the indentations at a,b,.... $l \to 0$, the region $\omega > 0$ gets mapped conformally on the interior of Γ.

If a corresponds to the point at infinity in the w - plane,

$$\frac{d}{d\omega} \log F'(\omega) = \frac{\beta-1}{\omega-b} + \frac{\gamma-1}{\omega-c} + \dots + \frac{\lambda-1}{\omega-l} + \phi($$

For large ω,

$$\frac{d}{d\omega} \log F'(\omega) \simeq - \frac{\beta+\gamma+\dots+\lambda+(n-1)}{\omega} + \phi($$

$$= -\frac{(\alpha+1)}{\omega} + \phi(\omega)$$

so that by (B), $\phi(\omega) = 0$. Hence,

$$F(\omega) = A \int (\omega-b)^{\beta-1}(\omega-c)^{\gamma-1} \dots (\omega-l)^{\lambda-1} + E$$

Examples.

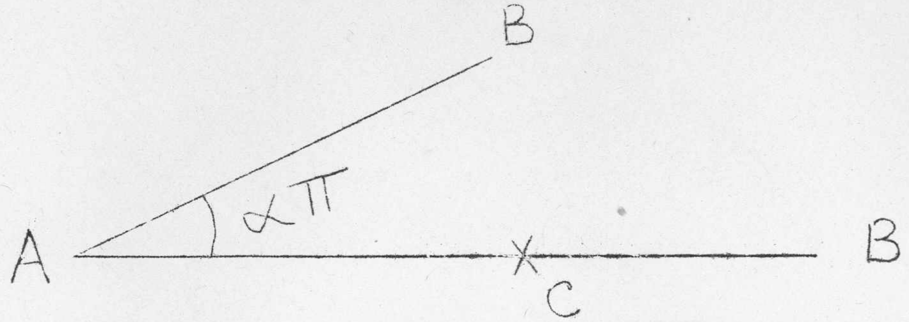
1) Let ABC be a triangle with angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$ with $\alpha+\beta+\gamma = 1$. This is conformally mapped on $\operatorname{Im} \omega > 0$ by

$$z = D \int (\omega-a)^{\alpha-1}(\omega-c)^{\gamma-1} d\omega + E$$

where b has been taken to correspond to point at infinity in the W-plane.

Case. (i)

Keep A and C and the direction AB fixed and let $\gamma \to 1$. The Δ becomes



Now

$$Z = D\int(\omega - a)^{\alpha - 1}\,d\omega + E$$

Let $\quad a = 0, \quad D = 1, \quad E = 0$

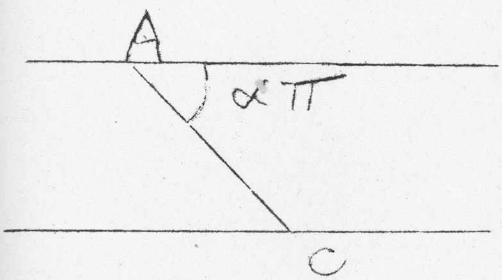
$$\therefore \quad Z = \omega^{\alpha} = r^{\alpha} e^{i\theta\alpha}$$

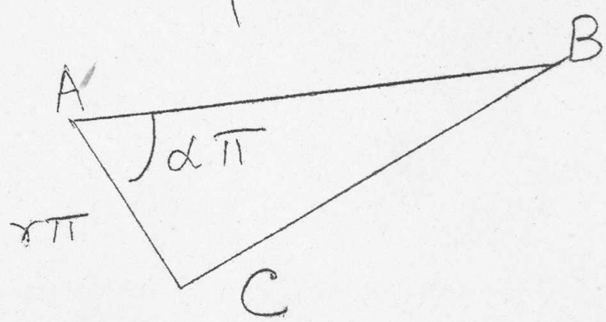
Fixing r , as θ vari8s from 0 to , π, $|z| = r$ and Z covers an angle $\alpha\pi$. When varies from 0 to ∞ , the interior of the region between the two straight lines get mapped on an .

Case KK.

Keep A and C fixed and let $\beta \longrightarrow 0$

Then the triangle becomes





and

$$z = D\int(\omega - a)^{\alpha - 1}(\omega - c)^{\gamma - 1}\,d\omega + E$$

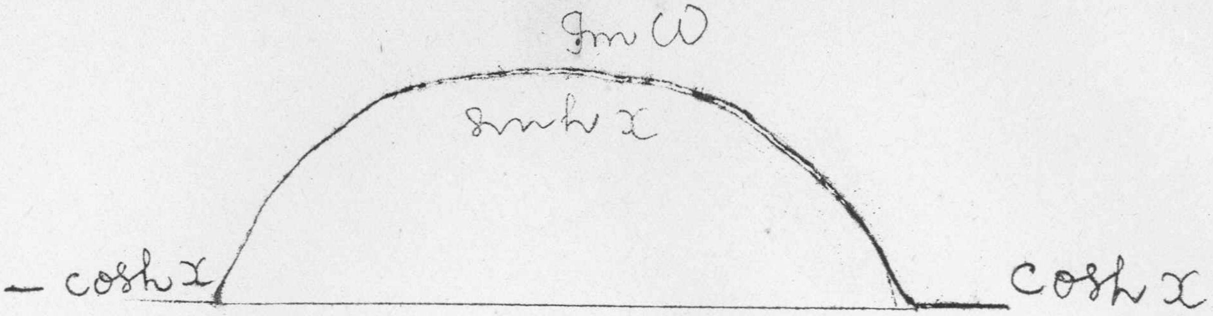
with $\alpha + \gamma = 1$. Let $\alpha = \gamma = \frac{1}{2}$ $\quad c = -1 = -a$

(for similplicty). Then $\quad D = 1, E = 0$

$$Z = \int \frac{d\omega}{\sqrt{\omega^2 - 1}} = \cosh^{-1}\omega$$

or

$$\omega = \cosh x \cos y + i \sinh x \sin y$$

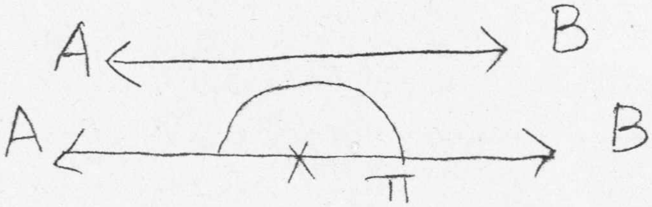


Fixing x_0 at some positive value, as y varies from 0 to π , , sweeps out the upper half of an ellipse. As x varies from 0 to ∞ , this ellipse sweeps out the entire region

$\omega > 0$.

<u>Case III.</u>

Keep C and the direction AB fixed and let α , and $\beta \to 0$ and $\gamma \to 1$. Then the trangle becomes

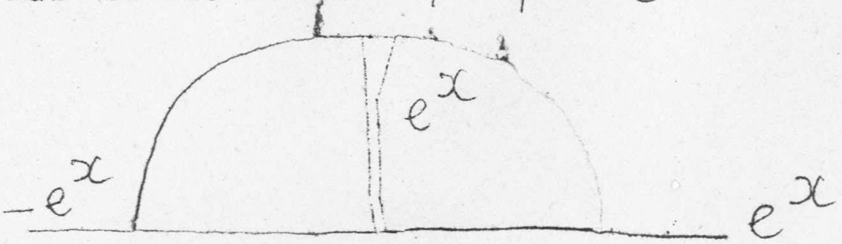


In this case,

$$Z = D \int \frac{d\omega}{\omega - a} \, d\omega + E$$

Set $D = 1, \ E = a = 0$ Then $\omega = e^z = e^x e^{iy}$

If x is kept fixed and y varies from 0 to π, ω moves around the upper half of the circle $|\omega| = e^x$.



As x varies from $-\infty$ to $+\infty$, this circular arc sweeps out the upper half of the w- - plane.

2) Let us now consider the conformal representation of a rectangle on the half-plane $\Im\, \omega > 0$. The transformation is

$$Z = D \int \left[(\omega - a)(\omega - b)(\omega - c)(\omega - d)\right]^{-\frac{1}{2}} d\omega$$

In particular, let us take

$$Z = \int_0^{\omega} \left[(1-\omega^2)(1-k^2\omega^2)\right]^{-\frac{1}{2}} d\omega$$

$$0 < k < 1$$

Thus

$$Z = \frac{1}{k} \int_0^{\omega} \frac{d\omega}{\left[(1-\omega^2)\left(\frac{1}{k^2} - \omega^2\right)\right]^{\frac{1}{2}}}$$

$$= \int_0^{\omega} \frac{1}{k} \frac{d\omega}{(1+\omega)^{\frac{1}{2}}(1-\omega)^{\frac{1}{2}}\left(\frac{1}{k}+\omega\right)^{\frac{1}{2}}\left(\frac{1}{k}-\omega\right)^{\frac{1}{2}}}$$

Let the integrand be $+ve$ when $-1 < \omega < 1$ Then if we continue it along a path in the upper half-plane, it is equal to $i\left[(\omega^2-1)(1-k^2\omega^2)\right]^{-\frac{1}{2}}$ when $1 < \omega < \frac{1}{k}$ (This is since $\frac{1}{(1-\omega)^{\frac{1}{2}}}$ $= \frac{1}{|1-\omega|^{\frac{1}{2}} e^{i\theta/2}}$ with $\theta = 0$ for $-1 < \omega < 1$ and θ increases by $-\pi$ when $1 < \omega < \frac{1}{k}$ Thus in this region

$$\frac{1}{(1-\omega)^{\frac{1}{2}}} = \frac{1}{|1-\omega|^{\frac{1}{2}} e^{-i\pi/2}} = \frac{i}{|1-\omega|^{\frac{1}{2}}}$$

It is also equal to $-\left[(\omega^2-1)(k^2\omega^2-1)\right]^{-\frac{1}{2}}$ when $\omega > 1/k$. Similarly it is equal to

$$-i \frac{1}{[(\omega^2-1)(1-k^2\omega^2)]^{1/2}}$$

when

$$-\frac{1}{k} < \omega < -1$$

and equal to

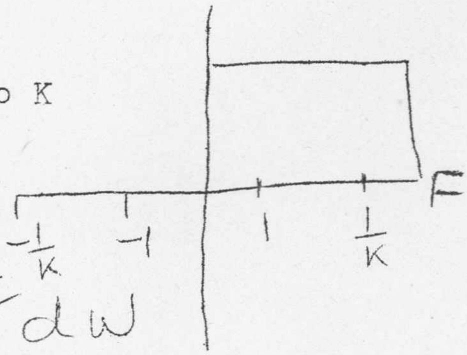
$$-\frac{1}{[(\omega^2-1)(k^2\omega^2-1)]^{\frac{1}{2}}}$$

when $\omega < -\frac{1}{k}$

Let us now consider how Z behaves as ω describes the positive quadrant $|\omega| \leq R,\ R = 1/k$

when ω moves from 0 to 1 along the real axis Z moves from 0 to K

where



$$K = \int_0^1 [(1-\omega^2)(1-k^2\omega^2)]^{-\frac{1}{2}} d\omega$$

Since the integral converges, K is finite and positive (since the integrand is positive).

when ω moves from 1 to $\frac{1}{k}$, the path of Z is given by

$$Z = \int_0^1 [(1-\omega^2)(1-k^2\omega^2)]^{-\frac{1}{2}} d\omega + i\int_1^\omega [(\omega^2-1)(1-k^2\omega^2)]^{-\frac{1}{2}} d\omega$$

Thus Z moves from K to $K + iK'$ where K' is positive and given by

$$K' = \int_0^{1/k} [(\omega^2-1)(1-k^2\omega^2)]^{-\frac{1}{2}} d\omega$$

K' is evidently finite. Since the integral

$$\int_0^{\omega}\left[(\omega^2-1)(1-k^2\omega^2)\right]^{-1/2} d\omega$$

monotonically increases in this region, Z moves along the straight line $\operatorname{Re} Z = K$ from K to $K+iK'$.

When $\omega > 1/k$,

$$=(K+iK') - \int_{1/k}^{\omega}\left[(\omega^2-1)(1-k^2\omega^2)\right]^{-\frac{1}{2}} d\omega$$

Let $k\omega = \frac{1}{t}$. Then,

$$\int_{1/k}^{\omega}\left[(\omega^2-1)(1-k^2\omega^2)\right]^{-1/2} d\omega = \int_{1/k\omega} \cancel{\frac{1}{t}}\,(1-t^2)(1-k^2t^2)^{-1/2}$$

$$= K - \int_0^{1/k\omega}\left[(1-t^2)\ (1-k^2t^2)\right]^{-1/2} dt$$

and so $$Z = iK' + \int_0^{\frac{1}{k\omega}}\left[(1-t^2)\ (1-k^2t^2)\right]^{-\frac{1}{2}} dt \tag{A}$$

Thus as ω moves from $\frac{1}{k}$ to R along the real axis, Z moves along $\operatorname{Im} Z = iK'$ from the point $K+iK'$ to the point $iK' + \int_0^{1/kR}\left[(1-t^2)\,(1-k^2t^2)\right]^{-\frac{1}{2}} dt$

By analytical continuation, (A) is true for $|\omega| > 1/k$

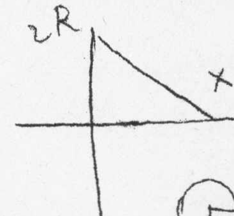
Thus Z is regular there and we may write,

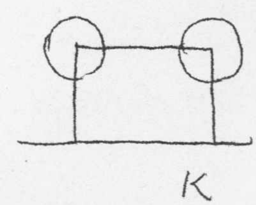
$$Z = iK' + \frac{1}{K\omega} - \frac{(1+k^2)}{6k^2\omega^3}$$

expanding the integrand and integrating by term. Thus

$$Z - iK' = \frac{1}{K\omega}\left[1 + O\left(\frac{1}{|\omega|^2}\right)\right]$$

when ω is large. Thus as ω moves from R to iR and $R \to \infty$, Z lies within a circle centred at iK' whose radius tends to zero. Also $\arg\left[Z - iK'\right]$ decreases by $\frac{\pi}{2}$ when ω moves along this path.





When $\omega = iv$, $v = +ve$,

$$Z = i\int_0^v \left\{(1+v^2)(1+k'^2v^2)\right\}^{-\frac{1}{2}} dv$$

where the integral is real $+ve$ and decreases as v decreases. Thus as ω moves from iR to Z moves from a point near iK' to 0. Thus we see that ω moves along the boundary of the $+ve$ quadrant $|\omega| \leq R$, Z moves along the rectangle with vertices $0, K, K+iK', iK'$, indented at iK' and that the radius of identation $\to 0$ as $R \to \infty$. Thus the region $0 < \arg\omega < \frac{\pi}{2}$ mapped informally on the interior of this rectangle.

Since Z is purely imaginary when $Re\,\omega = 0$, by Schwarz's principle of symmetry, we see that the region is mapped conformally on the interior of the rectangle with vertices $0, iK', -K+iK', -K$, Thus we see that

$$Z = \int_0^\omega \left[(1-\omega^2)(1-k^2\omega^2)\right]^{-\frac{1}{2}} d\omega$$

maps $Im\,\omega > 0$ conformally on the interior of the rectangle

CHAPTER V.

The Elliptic Functions of Weierstrass.

Doubly periodic meromorphic functions are commonly called elliptic functions.

It is obvious that an analytic function which is not a constant cannot have arbitrarily small periods. This immediately implies that if an analytic function $f(z)$ other than a constant has a set of periods $2\omega, 2\lambda\omega$ where λ is ~~xx~~ real, each of these periods is a multiple of a single fundamental period.

Proof: If λ is our integer, there is nothing to be proved. Therefore let λ be non-integral. Let us write $2\lambda\omega = 2m\omega + 2\alpha\omega$ where m is an integer and $0 < \alpha < 1$.

There cannot be an infinity of such periods $2\alpha\omega$. Let the least of these $2\alpha\omega$ be equal to $2\alpha_1\omega$. Then all other periods must be multiples of $2\alpha_1\omega$. Otherwise we can again find periods. $2\alpha_2\omega < 2\alpha_1\omega$. Thus since this process of arriving at smaller and smaller periods must have a stop, we evidently have a least period such that all other periods are its multiples. This proves the theorem.

We have incidentally also shown that if 2ω and ~~xxe~~ $2\lambda\omega$ are periods λ where is real, λ is rational.

Now let us consider the case where $f(z)$ has two fundamental periods $2\omega_1$ and $2\omega_2$ whose ratio is not real. We shall show that every other period is a sum of multiples of a certain pair of fundamental periods, not necessarily $2\omega_1$ and $2\omega_2$. For let 2ω be any other period of $f(z)$. We may then

$$\operatorname{Re} \omega = \lambda \operatorname{Re} \omega_1 + \mu \operatorname{Re} \omega_2$$
$$\operatorname{Im} \omega = \lambda \operatorname{Im} \omega_1 + \mu \operatorname{Im} \omega_2$$
$$\text{i.e} \quad \omega = \lambda \omega_1 + \mu \omega_2$$

where λ and μ are real. If λ and μ are integral, there is nothing more to be proved. So assume they are non-integral. Then we may write

$$2\omega = 2l\omega_1 + 2m\omega_2 + 2\alpha\omega_1 + 2\beta\omega_2$$

where $0 < \alpha < 1$, $0 < \beta < 1$. Neither α nor β can be zero since if e.g. $\alpha = 0$ $2\omega_2$ and $2\beta\omega_2$ would be periods which contradicts our hypothesis that $2\omega_2$ is a fundamental period. Thus we now have a new period $2\alpha\omega_1 + 2\beta\omega_2$ lying within the parallelogram with vertices 0, $2\omega_1$, $2\omega_1 + 2\omega_2$, $2\omega_2$. The no. of such periods has to be evidently finite. Among these therefore, there is a period with a least β, say $2\alpha\omega_1 + 2\beta'\omega_2 = 2\omega_1'$ say. Then clearly all other periods are the sum of multiples of $2\omega_1'$ and $2\omega_1$. Tis proves the theorem. It follows that if an analytic function $f(z)$ other than a constant has periods whose ratio is not real, it is necessarily doubly periodic.

Note that from a pair of primitive periods $2\omega_1$, $2\omega_2'$ an unlimited number of other primitive periods can be found. For let

$$\Omega_1 = a\omega_1 + b\omega_2', \quad \Omega_2 = c\omega_1 + d\omega_2'$$

which gives

$$\omega_1 = \frac{d\Omega_1 - b\Omega_2}{ad - bc}, \quad \omega_2' = \frac{-c\Omega_1 + a\Omega_2}{ad - bc}$$

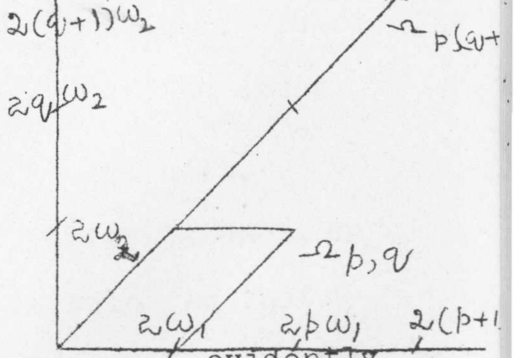
Thus if a, b, c, d are integers with $ad - bc = 1$ we find that $2\Omega_1$ and $2\Omega_2$ are also a pair of primitive periods.

Let $2\omega_1$ and $2\omega_2$ form a pair of primitive periods of an elliptic function. We may conveniently choose $\mathrm{Im}\, \omega_2/\omega_1 > 0$ then $0, 2\omega_1, 2\omega_1 + 2\omega_2, 2\omega_2$ are the vertices of a parallelogram (the privitive period - parallelogram") described in the positive sense. There are evidently an unlimited number of primitiv e period - parallelograms. Also vertices of these parallelograms are the only points whose affixes are periods.

Let

$$\Omega_{m,n} = 2m\omega_1 + 2n\omega_2$$

$m, n = 0, \pm 1, \pm 2, ---$ then $\Omega_{p,q}, \Omega_{p+1,q}, \Omega_{p+1,q+1}, \Omega_{p,q+1}$ are the vertices of a parallelogram obtained from the primitive one by a translation without a relation. It is called a period-parallelogram or a mesh. The complex plane is completely covered by a system of non-overlapping meshes.



evidently

The points $Z + \Omega_{p,q}$, $Z + \Omega_{r,s}$ lie in different meshes and $f(z)$ has the same value at these ~~p~~ points. When the meshesare translated so as to coincide with each other, these points coincide. These are called congruent points.

It is evident that behaviour of an elliptic function $f(z)$ is completely determined by its behaviour in a primitive period-parallelogram. This immediately implies: 1) $f(z)$ must posses poles in every mesh. 2) It can have only a finite number of poles in any mesh. 3) It can have only a finite number of zeros in any mesh.

It is convenient often to have meshes on whose boundaries, $f(z)$ has no poles or zeros. Since $f(z)$ has only a finite number of poles or zeros, this can always be achieved by translating a mesh without rotation. The resultant parallelogram is called a cell. The set of poles or zeros in a given cell is called an irreducible set.

Theorem: The sum of the residues of an elliptic function as its poles in any cell is zero.

The proof proceeds along obvious lines. The number of poles of an elliptic function in a cell is called its order. Clearly the least order an elliptic function can ~~xx x~~ have is two. An elliptic function of order n has m zeros in a cell. This is so since $f'(z)/f(z)$ is an elliptic function of the same period as $f(z)$ with a residue $n-m$ in the cell.

The sum of the affixes of the zeros of an elliptic function in any cell exceeds the sum ofthe affixes of its poles in that cell by a period.

Proof: This excess is given by

$$\int_C \frac{z f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_t^{t+2\omega_1}\left\{\frac{z f'(z)}{f(z)} - \frac{(z+2\omega_2)f'(z)}{f(z)}\right\}dz$$

$$-\ \frac{1}{\quad}\int^{t+2\omega_2}\left\{\frac{z f'(z)}{\quad} - \frac{(z+2\omega_1)f'(z)}{\quad}\right\}dz$$

$$= \left\{ 2\omega_1 \int_t^{t+2\omega_2} \frac{f'(z)}{f(z)}\,dz - 2\omega_2 \int_t^{t+2\omega_1} \frac{f'(z)}{f(z)}\,dz \right\}$$

$$= \frac{1}{2\pi i} \left\{ 2\omega_1 \Big[\log f(z) \Big]_t^{t+2\omega_2} - 2\omega_2 \Big[\log f(z) \Big]_t^{t+2\omega_1} \right.$$

$$= 2m\,\omega_1 - 2n\,\omega_2$$

where

m and n

are integers Q.E.D

Weierstrass's Sigma Function.

If $f(z)$ is a simply periodic analytic function with period π with zeros $z_1, \ldots\ldots, z_m$ poles $p_1, p_2, \ldots\ldots p_n$, in $0 < \mathrm{Re}\, z < \pi$

$$f(z) = g(z)\, e^{ikz} \frac{\prod_1^m \sin(z - z_r)}{\prod_1^p \sin(z - p_r)}$$

where $g(z)$ is an integral function of period π and $k = 0$ or 1 according as $(m-n)$ given or odd.

In order to exhibit in a similar fashion how an elliptic function depends on its zeros and poles, we first construct an integral function with simple zeros at $\Omega_{m,n}$.

If $\arg \omega_2/\omega_1 = \theta$

$$|m\omega_1 + n\omega_2|^2 = m^2|\omega_1|^2 + n^2|\omega_2|^2 + 2mn\cos\theta\,|\omega_1\omega_2|$$

Since $0 < \theta < \pi$, $\cos\theta = \pm\mu$, $0 \leq \mu \leq 1$ thus

$$\begin{aligned}|m\omega_1 + n\omega_2|^2 &= m^2|\omega_1|^2 + n^2|\omega_2|^2 \pm 2mn\mu\,|\omega_1\omega_2| \\ &= (1-\mu)\left[m^2|\omega_1|^2 + n^2|\omega_2|^2\right] \\ &\quad + \mu\left[m|\omega_1| \pm n|\omega_2|\right]^2 \\ &\geqslant (1-\mu)\,a^2(m^2+n^2), \quad a = \min\left[|\omega_1|, |\omega_2|\right]\end{aligned}$$

$|||^{ly}$ $$|m\omega_1 + n\omega_2|^2 \leq (1+\mu)\,b^2(m^2+n^2), \quad b = \max\left[|\omega_1|, |\omega_2|\right]$$

Thus $\sum'_{m,n} |\Omega_{m,n}|^{-\alpha}$ converges if and only if $\sum'_{m,n} (m^2+n^2)^{-\alpha/2}$. The latter however converges if $\alpha > 2$ and diverges for $\alpha \leq 2$. Thus the exponent of convergence of $\Omega_{m,n}$ is 2. Hence the doubly-infinite product

$$\sigma(z|\omega_1, \omega_2) = z\prod'_{m,n}\left\{\left(1 - \frac{z}{\Omega_{m,n}}\right)\exp\left(\frac{z}{\Omega_{m,n}} + \frac{z^2}{2\Omega^2_{m,n}}\right)\right.$$

converges uniformaly and absolutely when z lies in any bounded closed region which contains none of the $\Omega_{m,n}$.
This function is called Weierstrass's sigma function and is an integral function of order 2 with simple zeros at $\Omega_{m,n}$.
Since $\Omega_{m,-n} = -\Omega_{m,n}$, $\sigma(z)$ is an odd function of z .

Weierstrass's Elliptic Function

Now

$$\log \sigma(z) = \log z + \sum{}' \left\{ \log\left(1 - \frac{z}{\Omega_{m,n}}\right) + \frac{z}{\Omega_{m,n}} + \frac{z^2}{\Omega^2_{m,n}}\right.$$

Define Weierstrass's zeta function

$$\zeta(z) = + \frac{d}{dz} \log \sigma(z) = \frac{1}{z} + \sum{}' \left\{ \frac{1}{z - \Omega_{m,n}} + \frac{1}{\Omega_{m,n}} + \frac{z}{\Omega^2_{m,n}}\right.$$

The general term of the series is $\dfrac{1}{\Omega^2_{m,n}(z - \Omega_{m,n})} = O(|\Omega_{m,n}$

so that the series converges uniformly and absolutely ~~none of the~~ in any bounded closed region containing none of the $\Omega_{m,n}$; $\zeta(z)$ is thus regular save for simple poles at $\Omega_{m,n}$ with residue 1 . $\zeta(z)$ is evidently an odd function of z .

Weierstrass's elliptic function is defined by

$$\wp(z) = -\frac{d}{dz}\zeta(z) = \frac{1}{z^2} + \sum{}' \left\{ \frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{\Omega^2_{m,n}} \right\}$$

$$= \wp(-z)$$

Now

$$\wp'(z) = -\sum \frac{2}{(z-\Omega_{m,n})^3} = -\wp'(-z)$$

$$\wp'(z+2\omega_1) = -\sum \frac{2}{(z+2\omega_1-\Omega_{m,n})^3}$$

$$= -\sum \frac{2}{(z-\Omega_{m-1,n})^3} = \wp'(z)$$

$\wp'(z)$ is thus an elliptic function integrating,

$$\wp(z+2\omega_1) = \wp(z) + C$$

If $z = -\omega_1$

$$\wp(\omega_1) = \wp(-\omega_1) + C$$

since $\wp(\omega_1) = \wp(-\omega_1), C = 0.$ Thus $\wp(z)$ is also an elliptic function with a pair of primitive periods $2\omega_1, 2\omega_2$.

The Pseudo - periodicity of $\zeta(z)$ and $\sigma(z)$

Since $\wp(z+2\omega_1) = \wp(z)$

$$\zeta(z+2\omega_1) = \zeta(z) + 2\eta_1$$

where $2\eta_1 = \zeta(\omega_1) - \zeta(-\omega_1) = 2\zeta(\omega_1)$

$|||^{ly}$ $\zeta(z+2\omega_2) = \zeta(z) + 2\eta(\omega_2) = \zeta(z) + 2\zeta($

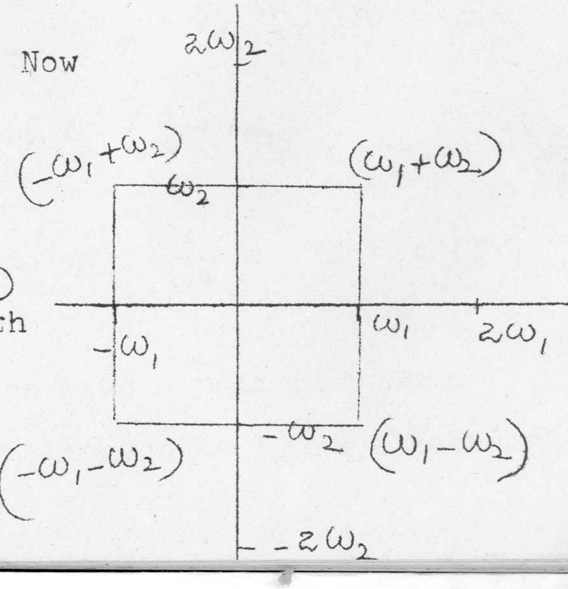
$\zeta(z)$ thus has pseudo - periodicity. Now

$$\eta_1\omega_2 - \eta_2\omega_1 = \frac{1}{2}\pi i$$

For, within the cell C with verticeces

$(\omega_1-\omega_2), (\omega_1+\omega_2), (-\omega_1+\omega_2), (-\omega_1-\omega_2)$

$\zeta(z)$ has a simple pole at the orgion with residue 1. Thus



$$2\pi i = \int_C \zeta(z)\,dz = \int_{(\omega_1-\omega_2)}^{(\omega_1+\omega_2)} \left[\zeta(z) - \zeta(z-2\omega_1)\right] dz$$

$$+ \int_{(-\omega_1-\omega_2)}^{(\omega_1-\omega_2)} \left\{\left[\zeta(z) - \zeta(z+2\omega_2)\right]\right\} dz$$

$$= 2\eta_1\, 2\omega_2 - 2\eta_2\, 2\omega_1$$

$$= 4\left[\eta_1\omega_2 - \eta_2\omega_1\right] \quad \text{or} \quad \eta_1\omega_2 - \eta_2\omega_1 = \tfrac{1}{2}\pi i$$

Let us define a new period $2\omega_3$ via the relation $\omega_1 + \omega_2 + \omega_3 = 0$. Then,

$$\zeta(z + 2\omega_3) = \zeta(z) + 2\eta_3 = \zeta(z - 2\omega_1 - 2\omega_2)$$

$$= \zeta(z - 2\omega_1) - 2\eta_2 = \zeta(z) - 2\eta_1 - 2\eta_2$$

so that $\eta_1 + \eta_2 + \eta_3 \equiv 0$

Now, $\zeta(z + 2\omega_1) = \zeta(z) + 2\eta_1$ so that

$$\sigma(z + 2\omega_1) = 2\eta_1 | (z) | = A e^{2\eta_1 z} \sigma(z)$$

where $A = \dfrac{\sigma(\omega_1)}{\sigma(-\omega_1)} e^{2\eta_1\omega_1} = -e^{+2\eta_1\omega_1}$

Hence, $\sigma(z + 2\omega_1) = -e^{2\eta_1(z+\omega_1)}\sigma(z)$

with ~~perpendicular~~ similar equations for ω_2 and ω_3. Thus $\sigma(z)$ is also pseudoperiodic.

Theorem. If two elliptic functions have a pair of common periods whose ratio is not real, they are connected by an algebraic relation.

Let $2\omega_1$ & $2\omega_2$ be a pair of primitive periods whose ratio is not real such that every period of $f(z)$ and $g(z)$ are expressible as sums of multiples of these. Let $a_1, \ldots, a_m$ be the poles of $f(z)$ or $g(z)$ in the parallelogram with vertices $0, 2\omega_1, (2\omega_1 + 2\omega_2), 2\omega_2$. Let μ_r be the order of the pole at a_r, μ_r being the greater order if both f and g have poles there.

Let $F(\xi, \eta)$ be a polynomial of degree n in ξ and η with no constant term so that it involves $\frac{1}{2}n(n+3)$ arbitrary constants.

$$a_0 + a_1\xi + \ldots\ldots + a_n \xi^n$$

$$b_0 + b_1\eta + \ldots\ldots + b_n \eta^n$$

$$nf(n) + (n-1) + \ldots \quad = n + \frac{n(n+1)}{2} + n(n+3)/2 -$$

Then $\phi(z) = F\{f(z), g(z)\}$ is an elliptic function with primitive periods $2\omega_1, 2\omega_2$ with poles at a_r of order $\leq n\mu_r$. Thus the number of constants in the principal part of $\phi(z)$ about all the a_r is $n(\mu_1 + \cdots + \mu_r)$. These may thus be made to identically vanish by subjecting the $\frac{1}{2}n(n+3)$ coefficients to satisfy a set of ~~xx xxxx~~ $n(\mu_1 + \cdots + \mu_m)$ homogeneous linear equations if only

$$(n+3) > 2(\mu_1 + \ldots + \mu_m)$$

which obtains for sufficiently large n. But then $\phi(z) = F\{f(z), g(z)\}$ is an elliptic function with no singularities.

This proves the result.

This theorem shows that $\wp(z)$ & $\wp'(z)$ must be connected by an algebraic relation.

Now,

$$\wp(z) - \frac{1}{z^2} = \sum_0^\infty a_n z^n$$

Where $a_2 = 3 \sum' \Omega_{m,n}^{-4}$; $a_4 = 5 \sum' \Omega_{m,n}^{-6}$

Since $\wp(-z) = \wp(z)$, $a_{2n+1} \equiv 0$. Also $a_0 = 0$ since $\wp(z) - \frac{1}{z^2} = 0$ at $z = 0$. Thus

$$\wp(z) = z^{-2} + a_2 z^2 + a_4 z^4 + \text{etc } O(z^6)$$

where $O(z^n)$ has a zero of order n at the origin. This gives

$$\wp'(z) = -2z^{-3} + 2a_2 z + 4a_4 z^3 + O(z^5)$$

Observe that

$$\wp^3(z) = z^{-6} + 3a_2 z^{-2} + 3a_4 + O(z^2)$$

$$\wp'^2(z) = 4z^{-6} - 8a_2 z^{-2} - 16a_4 + O(z^2)$$

Whence

$$\wp'^2(z) - 4\wp^3(z) = -20a_2 \wp(z) - 28a_4 + O(z^2)$$

Hence

$$\phi(z) = \wp'^2(z) - 4\wp^3(z) + 20a_2 \wp(z) + 28a_4$$

is regular in the neighbourhood of the origin has a double zero at $z = 0$. Since $\phi(z)$ is an elliptic function of periods $2\omega_1$, $2\omega_2$ whose only possible singularities are $\Omega_{m,n}$ we find $\Phi(z) \equiv 0$.

Hence

$$\wp'^2(z) = 4\wp^3(z) - 20a_2 \wp(z) - 28a_4$$

Let $g_2 = 20a_2 = 60 \Sigma' \Omega_{m,n}^{-4}$, $g_3 = 28a_4 = 140 \Sigma' \Omega_{m,n}^{-6}$

where g_2 and g_3 are called the invariants of $\wp(z)$. It follows that $\wp(z)$ ~~sat~~satisfies the differental equation

$$\left(\frac{dw}{dz}\right)^2 = 4w^3 - g_2 w - g_3$$

We now show that the roots e_1, e_2, e_3 of R. H. S. above are all distinct. Clearly e_r are the values of $\wp(z)$ at the zeros of $\wp'(z)$. But

$$\wp'(\omega_1) = \wp'(\omega_1 - 2\omega_1) = \wp'(-\omega_1) = -\wp'(\omega_1)$$

Thus $\wp(\omega_1)$ is a root. Similarly, $\wp(\omega_2)$ is a root, Since $\wp'(z)$ is an elliptic function of order 3 with a triple pole at $\Omega_{m,n}$, the sum $(\omega_1 + \omega_2)$ of the affixes of its irreducible zeros is a period. Hence $\wp(-\omega_1-\omega_2) = \wp(\omega_3)$ is a root. Hence $e_1 = \wp(\omega_1)$, $e_2 = \wp(\omega_2)$, $e_3 = \wp(\omega_3)$ But $\wp(z) - e_1$ is an elliptic function of order 2 with a double zero at $z = \omega_1$ and hence cannot vanish anywhere else in the cell. Thus $e_1 \neq e_2$. Similarly $e_1 \neq e_3 \neq e_2$ Thus the roots e_1, e_2, e_3 are all distinct.

Now if we are given that

$$\left(\frac{dw}{dz}\right) = \left[4w^3 - g_2 w - g_3\right]^{\frac{1}{2}} \quad \ldots\ldots (a)$$

and the roots of $4w^3 - g_2 w - g_3$ are all distinct (i.e. that $g_2^3 - 27g_3^2 \neq 0$), then it is possible to find $\Omega_{m,n} = 2m\omega_1 + 2n\omega_2$ such that

$$g_2 = 60 \Sigma' \Omega_{m,n}^{-4}; \quad g_3 = 140 \Sigma' \Omega_{m,n}^{-6}$$

This we shall show later. Then if in

$$z + \alpha = \int \frac{dw}{[4w^3 - g_2 w - g_3]^{\frac{1}{2}}},$$

We set $w = \wp(s \mid \omega_1, \omega_2)$, evidently $z + \alpha = \pm s$

i.e. $w = \wp(z + \alpha \mid \omega_1 \omega_2)$. This gives a method of evaluating such integrals.

The addition-theorem for $\wp(z)$

Let $f(z) = \wp'(z) + A\wp(z) + B$

where A & B are constants. $f(z)$ is thus an elliptic function of order 3 with a triple pole at $\Omega_{m,n}$. Let u and v be such that u, v & $u \pm v$ are not periods of $\wp(z)$.

Let

$$\wp'(u) + A\wp(u) + B = 0$$

$$\wp'(v) + A\wp(v) + B = 0$$

Since $f''(u) = \wp''(u) + A\wp'(u) \neq 0$, $f(z)$ has simple zeros at u and v and $-u-v$ is also a zero of $f(z)$

Thus

$$\wp'(-u-v) + A\wp(-u-v) + B$$
$$= -\wp'(u+v) + A\wp(u+v) + B = 0$$

Hence

$$\begin{vmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(u+v) & \wp'(u+v) & 1 \end{vmatrix} \equiv 0$$

Since $\wp'(z)$ is an algebraic function of $\wp(z)$, this gives an addition theorem for $\wp(z)$.

With the same A & B as above, where

$$A = -\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}.$$

We may also write

$$\begin{aligned} F(z) &= [\wp'(z) - A\wp(z) - B][\wp'(z) + A\wp(z) + B] \\ &= 4\wp^3(z) - A^2\wp^2(z) - (2AB + g_2)\wp(z) \\ &\quad - (g_3 + B^2) \end{aligned}$$

This has roots $z = \pm u, \pm v, \pm(u+v)$. Evidently,

$$\wp(u) + \wp(v) + \wp(u+v) = +\frac{1}{4}A^2 = \frac{1}{4}\left\{\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right\}^2$$

This is an alternative form of the additionthat setting $u = z$, $v = \omega_1$, we find

$$\wp(z + \omega_1) = e_1 + \frac{(e_1 - e_2)(e_1 - e_3)}{\wp(z) - e_1}$$

$$\wp(z + \omega_2) = e_2 + \frac{(e_2 - e_1)(e_2 - e_3)}{\wp(z) - e_2}$$

$$\wp(z + \omega_3) = e_3 + \frac{(e_3 - e_1)(e_3 - e_2)}{\wp(z) - e_3}$$

Where we use the fact that $e_1 + e_2 + e_3 = 0$ since they are the roots of the cubic $4\omega^3 - g_2\omega - g_3 = 0$.

The expression of an elliptic function in terms of σ functions.

Let $f(z)$ be an elliptic function with periods $2\omega_1, 2\omega_2$ a set of irreducible zeros $z_1, z_2, \ldots\ldots, z_n$ and poles $p_1, p_2, \ldots, p_n$ with

$$z_1 + z_2 + \ldots\ldots + z_n = p_1 + p_2 + \ldots\ldots + p_n' + \Omega$$

where Ω is a period. Calling $p_n = p_n' + \Omega$,
$p_1, p_2, \ldots p_n$ form a set of irreducible ~~xxxxx~~ poles with

$$z_1 + z_2 + \ldots\ldots + z_n = p_1 + p_2 + \ldots\ldots + p_n .$$

Let

$$F(z) = \prod_{r=1}^{n} \frac{\sigma(z - z_r)}{\sigma(z - p_r)} .$$

Then

$$F(z + 2\omega_1) = \prod_{1}^{n} \frac{e^{2\eta_1(z - z_r + \omega_1)}}{e^{2\eta_1(z - p_r + \omega_1)}} \frac{\sigma(z - z_r)}{\sigma(z - p_r)}$$

$$= \prod_{r=1}^{n} \frac{\sigma(z - z_r)}{\sigma(z - p_r)} = F(z)$$

so that $2\omega_1$ is a period. Similarly $2\omega_2$ is a period. $F(z)$ is thus an elliptic function. Thus $\frac{f(z)}{F(z)}$ is an elliptic function with no singularities and is hence a constant A say.

Therefore,

$$f(z) = A \prod_{r=1}^{n} \frac{\sigma(z - z_r)}{\sigma(z - p_r)}$$

which gives an expression for an elliptic function as the quotient of product of σ functions.

As an application of the above formula, consider

$$F(z) = \wp(z) - \wp(\alpha)$$

where α is not a period. Assume in the first instance that 2α is also not a period. $F(z)$ then has irreducible zeros at $z = \pm\alpha$ and a double pole at the origin so that

$$F(z) = A \frac{\sigma(z-\alpha)\,\sigma(z+\alpha)}{\sigma^2(z)}$$

For small z,

$$\wp(z) - \wp(\alpha) = \frac{1}{z^2} - \wp(\alpha) + O(z^2)$$

$$\sigma(z+\alpha) = \sigma(\alpha) + z\,\sigma'(\alpha) + O(z^2)$$

$$\sigma(z-\alpha) = -\sigma(\alpha) + z\,\sigma'(\alpha) + O(z^2)$$

$$\sigma(z) = z + O(z^3)$$

Thus

$$\frac{\sigma(z-\alpha)\,\sigma(z+\alpha)}{\sigma^2(z)} = \frac{z^2 O(1) - \sigma^2(\alpha)}{z^2[1 + O(z^2)]}$$

$$= -\frac{\sigma^2(\alpha)}{z^2}\left[1 + O(z^2)\right] + z^2 O(1)$$

$$= -\frac{\sigma^2(\alpha)}{z^2} + O(1)$$

Hence, $A = -\dfrac{1}{\sigma^2(\alpha)}$ it follows that

$$\wp(z) - \wp(\alpha) = -\frac{\sigma(z-\alpha)\,\sigma(z+\alpha)}{\sigma^2(z)\,\sigma^2(\alpha)}$$

provided 2α is not a period. Bit L. H. S. and R. H. S. are analytic functions regular when $\alpha \neq \Omega_{m,n}$. Therefore by analytical continuation, the formula is true provided only that α . is not a period of $\wp(z)$.

<u>The function $\{\wp(z) - e_r\}^{\frac{1}{2}}$.</u>

$\{\wp(z) - e_r\}^{\frac{1}{2}}$ is defined to be that square root with a simple pole with residue $+1$ at the origin.

Since $\wp'(z) = -2/z^3 + O(1)$

$$\wp'(z) = -2\left[\wp(z)-e_1\right]^{\frac{1}{2}}\left[\wp(z)-e_2\right]^{\frac{1}{2}}\left[\wp(z)-e_3\right]^{\frac{1}{2}}$$

But,

$$\wp(z) - e_r = -\frac{\sigma(z-\omega_r)\,\sigma(z+\omega_r)}{\sigma^2(z)\,\sigma^2(\omega_r)}$$

$$= e^{-2\eta_r z}\,\frac{\sigma^2(z+\omega_r)}{\sigma^2(z)\,\sigma^2(\omega_r)}$$

Hence,

$$\{\wp(z)-e_r\}^{\frac{1}{2}} = e^{-\eta_r z}\,\frac{\sigma(z+\omega_r)}{\sigma(z)\,\sigma(\omega_r)}$$

the $+ve$ root being taken since near $z=0$ $\sigma(z) = z + O(z^3)$. $[\wp(z)-e_r]^{1/2}$ has simple poles at $\Omega_{m,n}$ and simple zeros at $\omega_r + \Omega_{m,n}$. We notice that

$$[e_r - e_s]^{\frac{1}{2}} = e^{-\eta_s\omega_r}\,\frac{\sigma(\omega_r+\omega_s)}{\sigma(\omega_r)\,\sigma(\omega_s)}$$

Defining

$$\sigma_r(z) = e^{-\eta_r z}\,\frac{\sigma(z+\omega_r)}{\sigma(\omega_r)}.$$

We find

$$\{\wp(z)-e_r\}^{\frac{1}{2}} = \frac{\sigma_r(z)}{\sigma(z)},$$

$$[e_r-e_s]^{\frac{1}{2}} = \frac{\sigma_s(\omega_r)}{\sigma(\omega_r)}$$

Now $\sigma_r(z)$ is páseudo-periodic since

$$\sigma_r(z+2\omega_r) = e^{-\eta_r(z+2\omega_r)} \cdot (-) e^{2\eta_r(z+2\omega_r)} \frac{\sigma(z+\omega_r)}{\sigma(\omega_r)}$$

$$= e^{-2\eta_r(z+\omega_r)} \sigma_r(z)$$

$$\sigma_r(z+2\omega_s) = e^{-\eta_r(z+2\omega_s)} (-) e^{2\eta_s(z+\omega_r+\omega_s)} \cdot \frac{\sigma(z+\omega_r)}{\sigma(\omega_r)}$$

$$= (-) e^{2[\eta_s\omega_r-\eta_r\omega_s]} \cdot e^{2\eta_s(z+\omega_s)} \sigma_r(z)$$

$$= e^{2\eta_s(z+\omega_s)} \sigma_r(z)$$

since $\eta_s\omega_r-\eta_r\omega_s = \frac{1}{2}\pi i$. Notice that $\sigma_r(z)$ is an integral function with simple zeros at $\omega_r+\Omega_{m,n}$.

The expansion of an elliptic function of terms of ζ Functions.

Although ζ is not an elliptic function, we can form a linearcombination

$$\varphi(z) = \sum_{r=1}^{n} a_r \zeta(z-p_r)$$

with suitable a_r such that $\phi(z)$ is elliptic. For,

$$\phi(z+2\omega_1) = 2\eta_1 \sum^{n} a_r + \phi(z)$$

Thus $\phi(z)$ is an elliptic function if $\sum_1^n a_r = 0$

Let $f(z)$ be an elliptic function of primitive periods $2\omega_1, 2\omega_2$ and irreducible poles $p_1, p_2, \dots, p_n$. The principal part of $f(z)$ in the neighbourhood of p_k is

$$\sum_{s=1}^{m_k} a_{k,s} (z-p_k)^{-s}$$

with $\sum_{k=1}^{n} a_{k,s} \equiv 0$. Hence $\sum_{k=1}^{n} a_{k,1} \zeta(z-p_k)$ is elliptic. Also since $\zeta^{(1)}(z) = -\wp(z)$, $\sum_{k=1}^{n} a_{k,s} \zeta(z-p_k)$

is elliptic. Thus

$$F(z) = f(z) - \sum_{k=1}^{n} \sum_{s=1}^{m_k} (-1)^{s-1} \frac{a_{k,s}}{(s-1)!} \zeta^{(s-1)}(z-p_k)$$

is an integral elliptic function and hence a constant A.

Thus

$$f(z) = A + \sum_{k=1}^{n} \sum_{s=1}^{m_k} (-1)^{s-1} \frac{a_{k,s}}{(s-1)!} \zeta^{(s-1)}(z-p_k)$$

This gives an easy means of intergrating an elliptic function.

$$\int f(z)\,dz = \sum_{k=1}^{n} a_{k,1} \log \sigma(z-p_k)$$
$$+ \sum_{k=1}^{n} \sum_{s=2}^{m_k} (-1)^{s-1} \cdot \frac{a_{k,s}}{(s-1)!}$$
$$\times \zeta^{(s-2)}(z-p_k) + Az + B$$

Where B is a constant of integration.

The expression of an elliptic function in terms of p(z)

Let $f(z)$ be an even elliptic function of primitve periods $2\omega_1, 2\omega_2$ which is regular and non-zero at $\Omega_{m,n}$ $f(z)$ is of even order, $2k$ say.

If z_r is a zero of $f(z)$ in a certain cell, the point congruent to $-z_r$ is also a zero of the same order as z_r. Thus we can choose our $2k$ zeros as $z_1, z_2 \ldots, z_k$ and the points in the cell congruent to $-z_1, -z_2, \ldots, -z_k$

Similarly the poles may be chosen as $p_1, p_2, \ldots p_k$, and the points congruent to $-p_1, -p_2, \ldots\ldots, -p_k$,

Hence the function

$$F(z) = \prod_{r=1}^{k} \frac{p(z) - p(z_r)}{p(z) - p(p_r)}$$

is an elliptic function with the same poles and zeros as $f(z)$.

Thus,

$$f(z) = A \prod_{r=1}^{k} \frac{p(z) - p(z_r)}{p(z) - p(p_r)} \qquad (1)$$

If however $f(z)$ has poles or zeros at $\Omega_{m,n}$, they must be of even order, Hence $\{f(z)[p(z)]\}^s$ where s is a $+ve$ or $-ve$ integer, is regular and non-zero at $\Omega_{m,n}$ and hence is expressible in the form (1)

It is clear now that ~~they~~ any elliptic function $f(z)$ of primitive periods $2\omega_1, 2\omega_2$ can be expressed in the form

$$F\left(\wp(z)\right) + \wp'(z)\, G\left\{\wp(z)\right\} \ldots\ldots \tag{2}$$

where $\wp(z)$ has the same primitive periods $2\omega_1, 2\omega_2$ and $F(\lambda)$ and $G(\lambda)$ are rational function of λ.

For,
$$f(z) = \frac{1}{2}\left[f(z)+f(-z)\right] + \frac{1}{2}\left[f(z)-f(-z\right.$$

$$\frac{1}{2}\left[f(z)+f(-z)\right]$$

is an even elliptic function and hence can be expressed as a rational function $F\left\{\wp(z)\right\}$ of $\wp(z)$. Similarly since $\dfrac{\frac{1}{2}\left\{f(z)-f(-z)\right\}}{\wp'(z)}$ is also an even elliptic function, it can be written as a rational function $G\left\{\wp(z)\right\}$ of $\wp(z)$. The result (2) follows:

We notice now the important result that every elliptic function must possess an addition theorem.

For if $f(z)$ is an elliptic function. Let

$$f(z) = F\left\{\wp(z)\right\}$$

where F denotes a rational function. Let

$$\wp(u) = p_1, \quad \wp(v) = p_2, \quad \wp'(u) = p_1', \quad \wp'(v) = p_2'$$

Then

$$f(u) = F\left[p_1, p_1'\right] \qquad \text{(a)},$$

$$f(v) = F\left[p_2, p_2'\right] \qquad \text{(b)},$$

$$f(u+v) = F\left\{\wp(u+v), \wp'(u+v)\right] \qquad \text{(c)}.$$

Using the addition theorem for $\wp(z)$,

$$f(u+v) = G\left[\wp_1, \wp_1', \wp_2, \wp_2'\right] \qquad \text{(d)}$$

However, since $\wp_1'^2 = 4\wp_1^3 - g_2\wp_1 - g_3$,
$\wp_2'^2 = 4\wp_2^3 - g_2\wp_2 - g_3$ (a) and (b) can be used to solve for $\wp_1$ and $\wp_2$ interms of $f(u)$ & $f(v)$. Finally (d) gives an algebraic relation connecting $f(u+v)$, $f(u)$ and $f(v)$. Thus proves the theorem.

<u>The evaluation of elliptic integrals.</u>

Let $I = \int F(t,u)\,dt$ where F is a rational function of t and u and

$$u^2 = a_0t^4 + 4a_1t^3 + 6a_2t^2 + 4a_3t + a_4$$

is a quartic or cubic function of t without a repeated factor. We shall show now that I can be evaluated in terms of Weierstrass's periodic and pseudo-periodic functions coupled with the elementary functions of analysis. Let

$$f = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3$$

If we make the transformation $x = lX + mY$, $y = l'X + m'Y$, where $\Delta = lm' - l'm \neq 0$ the quantities $g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2$

$$g_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$g_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}$$

become $G_2 = \Delta^4 g_2$, $G_3 = \Delta^6 g_3$. Further since f has no repeated factor, $g_2^3 - 27 g_3^2 \neq 0$. Let us make the transformation such that $A_0 = 0$, $A_2 = 0$. We see that $A_0 = 0$ if $l = l' t_0$ where t_0 is the root of

$$\varphi(t) = a_0 t^4 + 4 a_1 t^3 + 6 a_2 t^2 + 4 a_3 t + a_4 = 0$$

Thus A_2 vanishes if

$$\varphi_0'' m^2 + (6\varphi_0' - 2t_0\varphi_0'') m m' + (t_0^2 \varphi_0'' - 6 t_0 \varphi_0') m'^2 = 0$$

This gives

$$m = m' t_0 \quad \text{or} \quad m \varphi_0'' = m' (t_0 \varphi_0'' - 6 \varphi_0')$$

If $m = m' t_0$, we find that $\Delta = 0$. Writing $6\lambda = \dfrac{\varphi_0''}{\varphi_0'}$ and choosing $l' = 1$, $m' = \lambda$ we find the transformation matrix in the second case to be

$$\begin{bmatrix} t_0 & \lambda t_0 - 1 \\ 1 & \lambda \end{bmatrix}$$

with $\Delta = 1$. Thus

$$x = t_0(X + \lambda Y) - Y, \quad y = X + \lambda Y$$

Hence

$$f = 4A_1 X^3 Y + 4A_3 X Y^3 + A_4 Y^4$$

with $A_1 = -\frac{1}{4}\varphi_0'$, $A_3 = -\frac{1}{4}\frac{g_2}{A_1}$, $A_4 = \frac{-g_3}{A_1^2}$

i.e.

$$f = 4A_1 X^3 Y - \frac{g_2}{A_1} X Y^3 - \frac{g_3}{A_1^2} Y^4$$

Set therefore,

$$X = wY/A_1$$

$$f = Y^4/A_1^2\left[4w^3 - g_2 w - g_3\right]$$

But if $x = ty$,

$$y^4\varphi(t) = f = \frac{Y^4}{A_1^2}\left(4w^3 - g_2 w - g_3\right)$$

where $y = X + \lambda Y = Y\left[\frac{w}{A_1} + \lambda\right]$

$$= Y\left[\frac{w}{-\frac{1}{4}\varphi_0'} + \frac{\varphi_0''}{6\varphi_0'}\right]$$

Hence

$$\varphi(t) = \frac{16\varphi_0'^2}{\left(w - \frac{1}{24}\varphi_0''\right)^4}\left\{4w^3 - g_2 w - g_3\right\}$$

where

$$t = t_0 + \underline{\tfrac{1}{4}\varphi_0'}$$

Thus with this change of variables, I is brought to the canonical form $\int G(w,v)\,dw$ where G is a rational function of w and v where

$$v^2 = 4w^3 - g_2 w - g_3$$

Finding ω_1 and ω_2 whose ratio is not real such that

$$g_2 = 60 \sum_{m,n}' \Omega_{m,n}^{-4} \; ; \; g_3 = 140 \sum_{m,n}' \Omega_{m,n}^{-6}$$

We set $w = \wp(z/\omega_1, \omega_2)$ so that $v = \wp'(z/\omega_1,\omega_2)$ (We show later such an $\omega_1 \neq \omega_2$ can always be found if $g_2^3 - 27 g_3^2 \neq 0$ as is the case in our problem.) We have

$$I = \int G(w,v)\,dw = \int G\{\wp(z), \wp'(z)\}\wp'(z)\,dz$$

As the integrand is an elliptic function of periods $2\omega_1, 2\omega_2$ we can evaluate I in terms of Weirstrass's function as illustrated earlier. This proves the theorem.

CHAPTER VI

Jacobi's Elliptic Functions.

We defined the functions

$$\sigma_r(z) = e^{-\eta_r z} \frac{\sigma(z+\omega_r)}{\sigma(\omega_r)}$$

with

$$\sigma_r(z+2\omega_r) = -e^{2\eta_r(z+\omega_r)} \sigma_r(z)$$

$$\sigma_r(z+2\omega_s) = e^{2\eta_s(z+\omega_s)} \sigma_r(z)$$

$$(r \neq s)$$

in the last chapter. Let

$$S(z) = (e_1-e_2)^{\frac{1}{2}} \frac{\sigma(z)}{\sigma_2(z)}, \quad C(z) = \frac{\sigma_1(z)}{\sigma_2(z)}, \quad D(z) = \frac{\sigma_3(z)}{\sigma_2(z)}$$

Thus

$$S(z+2\omega_{1,3}) = -S(z), \quad S(z+2\omega_2) = S(z)$$

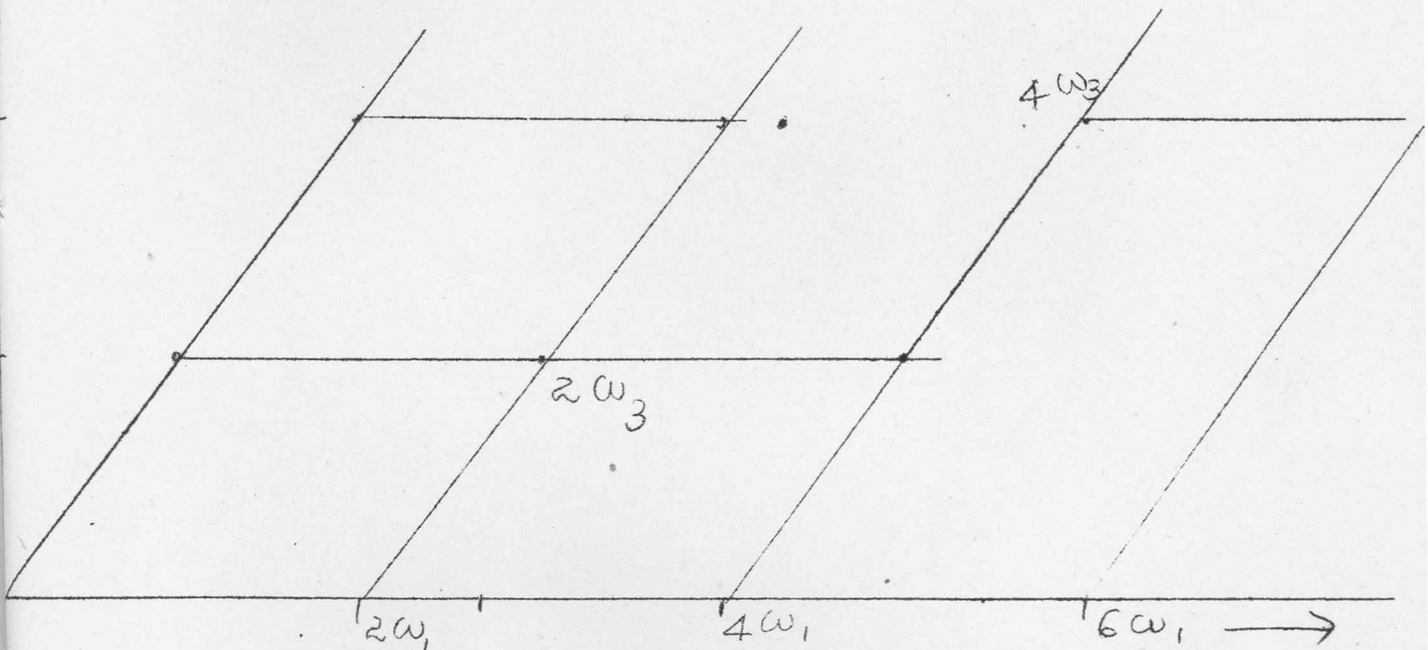
$$C(z+2\omega_{1,2}) = -C(z), \quad C(z+2\omega_3) = C(z),$$

$$D(z+2\omega_1) = D(z), \quad D(z+2\omega_{2,3}) = -D(z)$$

Thus the 3 functions are doubtly periodic and the primitive periods may be choosen as

$S(z)$	$4\omega_1, 2\omega_2$
$C(z)$	$4\omega_2, 2\omega_3$
$D(z)$	$4\omega_3, 2\omega_1$

Each has thus a different primitive period-parallelogram.



$S(z)$ has simple poles at $\omega_2 + \Omega_{m,n}$ and simple zeros at $\Omega_{m,n}$. $C(z)$ and $D(z)$ also have simple poles at $_2 + \Omega_{m,n}$. The zeros of $C(z)$ are at $\omega_1 + \Omega_{m,n}$ those of $D(z)$ at $\omega_3 + \Omega_{m,n}$.

Since $S(z), C(z), D(z)$ have a pair of common periods $4\omega_1, 4\omega_2$ they must be connected by an algebraic relation. Now,

$$S(z) = \left\{\frac{e_1 - e_2}{\wp(z) - e_2}\right\}^{\frac{1}{2}}, \quad C(z) = \left\{\frac{\wp(z) - e_1}{\wp(z) - e_2}\right\}^{\frac{1}{2}}$$

$$D(z) = \left\{\frac{\wp(z) - e_3}{\wp(z) - e_2}\right\}^{\frac{1}{2}}$$

Hence

$$C(z) = \left[1 - S^2(z)\right]^{\frac{1}{2}}, \quad D(z) = \left\{1 - \frac{e_3 - e_2}{e_1 - e_2} S^2(z)\right\}$$

$$= \left[1 - k^2 S^2(z)\right]^{\frac{1}{2}}$$

where

$$k = -\left[\frac{e_3 - e_2}{e_1 - e_2}\right]^{\frac{1}{2}}$$

is called the modulus of S, C and D is finite and $\neq 0$ or 1. Clearly $S'(z)$ is an algebraic function of S, C and D. Now

$$S'(z) = -\frac{(e_1 - e_2)^{\frac{1}{2}} \wp'(z)}{2\left[\wp(z) - e_2\right]^{3/2}}$$

and so,

$$S'(z) = \frac{(e_1 - e_2)^{\frac{1}{2}} \left[\wp(z) - e_1\right]^{\frac{1}{2}} \left[\wp(z) - e_3\right]^{\frac{1}{2}}}{\wp(z) - e_2}$$

$$= (e_1 - e_2)^{\frac{1}{2}} C(z) D(z)$$

Similarly,

$$C'(z) = -(e_1-e_2)^{\frac{1}{2}} \, D(z)\, S(z)$$

$$D'(z) = -(e_1-e_2)^{\frac{1}{2}} k^2 S(z)\, C(z)$$

$S(z)$, $C(z)$, ~~&~~ $D(z)$ is an elliptic function of order 2. $S(z)$ an odd function of Z while $C(z)$ and $D(z)$ are even functions since $\sigma(z)$ is an odd and $\sigma_r(z)$ an even function of z. The residue of $S(z)$ at ω_2 is

$$R = \lim_{z \to \omega_2} \left[(e_1-e_2)^{\frac{1}{2}} \sigma(z) \left[\frac{z-\omega_2}{\sigma_2(z)}\right]\right] = (e_1-e_2)^{\frac{1}{2}} \frac{\sigma(\omega_2)}{\sigma_2'(\omega_2)}$$

But

$$\sigma_2(z) = e^{\eta_2 z} \frac{\sigma(\omega_2 - z)}{\sigma(\omega_2)}$$

and so

$$\sigma_2'(\omega_2) = -e^{\eta_2\omega_2} / \sigma(\omega_2)$$

Also

$$(e_1-e_2)^{\frac{1}{2}} = -e^{-\eta_2\omega_1} \frac{\sigma(\omega_3)}{\sigma(\omega_1)\sigma(\omega_3)}$$

Hence $R = -(e_3-e_2)^{-\frac{1}{2}}$

The residue of $C(z)$ at points congruent to ω_2 is

$$\lim_{z \to \omega_2} \frac{(z-\omega_2)\,\sigma_1(z)}{\sigma_2(z)} = \frac{\sigma_1(\omega_2)}{\sigma_2'(\omega_2)}$$

$$= e^{-\eta_3 \omega_2} \frac{\sigma(\omega_2)\,\sigma(\omega_3)}{\sigma(\omega_2+\omega_3)}$$

$$= i\,(e_3-e_2)^{-\frac{1}{2}}$$

Thus the residue at points congruent to $3\omega_2$ is

$$-i\,(e_3-e_2)^{-1/2}$$

Similarly $D(z)$ has a residue $-i(e_1-e_2)^{-\frac{1}{2}}$ at ω_2 and $i(e_1-e_2)^{-\frac{1}{2}}$ at $\omega_2+2\omega_3$.

The complementary modulus ~~of~~ k' of S, C, & D is defined through

$$k' = \frac{(e_1-e_3)^{\frac{1}{2}}}{(e_1-e_2)^{\frac{1}{2}}}$$

so that
$$k^2 + k'^2 = +1$$

k' is finite and $\neq$ 0 or 1.

Jacobi's elliptic functions.

Let
$$\operatorname{sn} u = S\left[(e_1-e_2)^{-\frac{1}{2}}u\right],$$
$$\operatorname{cn} u = C\left[(e_1-e_2)^{-\frac{1}{2}}u\right],$$
and
$$\operatorname{dn} u = D\left[(e_1-e_2)^{-\frac{1}{2}}u\right]$$

Then

$$\mathrm{cn}\,u = \left[1 - \mathrm{sn}^2 u\right]^{\frac{1}{2}}, \quad \mathrm{dn}\,u = \left[1 - k^2 \mathrm{sn}^2 u\right]$$

$$\frac{d}{du}\,\mathrm{sn}\,u = \mathrm{cn}\,u\;\mathrm{dn}\,u,$$

$$\frac{d}{du}\,\mathrm{cn}\,u = -\,\mathrm{dn}\,u\;\mathrm{sn}\,u$$

$$\frac{d}{du}\,\mathrm{dn}\,u = -k^2\,\mathrm{sn}\,u\;\mathrm{cn}\,u$$

As a function of u, these have periods $4\omega_1(e_1 - e_2)^{\frac{1}{2}}$, $4\omega_2\,(e_1 - e_2)^{\frac{1}{2}}$. Hence the quarter periods of these functions are defined through

$$K = \omega_1 (e_1 - e_2)^{\frac{1}{2}}, \quad iK' = \omega_2 (e_1 - e_2)^{\frac{1}{2}}$$

Finally, the equations defining Jacobi's elliptic functions take the form

$$\mathrm{sn}\left[\frac{Kz}{\omega_1}\right] = \frac{K}{\omega_1}\,\frac{\sigma(z)}{\sigma_2(z)}, \quad \mathrm{cn}\left[\frac{Kz}{\omega_1}\right] = \frac{\sigma_1(z)}{\sigma_2(z)}$$

$$\mathrm{dn}\left(\frac{Kz}{\omega_1}\right) = \frac{\sigma_3(z)}{\sigma_2(z)}$$

Now

$$S(z) = -\,\frac{1}{(e_3 - e_2)^{1/2}\,(z - \omega_2)} + \varphi(z)$$

$\varphi(z)$ being regular at $z = \omega_2$. Hence

$$\mathrm{sn}\,u = -\,\frac{(e_1 - e_2)^{\frac{1}{2}}}{(e_3 - e_2)^{1/2}\,(\ldots)} + \varphi = \frac{1}{\ldots} + \ldots$$

Thus $sn\,u$ is an odd elliptic function of order two with primitive periods

$$4K, \quad 2iK'$$

with simple poles at points congruent to

$$iK', \quad 2K+iK'$$

with residues $\frac{1}{k}$, $-1/k$ respectively and simple zeros at points congruent to

$$0, \quad 2K$$

$cn\,u$ is an even elliptic function of order two with primitive periods

$$4K, \quad 2K+i2K'$$

with simple poles at points congruent to

$$iK', \quad 2K+iK'$$

with residues $-i/k$, $+i/k$ respectively and simple zeros at points congruent to

$$K \text{ or } 3K$$

$dn\,u$ is an even elliptic function of order two with primitive periods

$$2K, \quad 4iK'$$

with simple poles at points congruent to

$$iK', \quad 3iK'$$

with residues $-i$, $+i$ respectively and simple zeros at points congruent to

$$K+iK', \quad K+3iK'$$

The addition theorem.

Let us now derive the addition theorems for $sn\,u$, $cn\,u$ and $dn\,u$. Let

Let

$$F(u) = cn\,u\; cn(u-\alpha) + A\, sn\,u\; sn(u-\alpha)$$

$F(u)$ is an even elliptic function with primitive periods $2K,\ 2iK'$. If α is not congruent to K, iK' or $K+iK'$, $F(u)$ has simple poles at points congruent to $iK',\ \alpha+iK'$. Let us choose A so that the pr. part of $F(u)$ at iK' is zero. Then $F(u)$ by Liouville's theorem is a constant. Putting $u=0$, we find $F(u) = cn\,\alpha$. When $|u|$ is small,

$$F(u) = cn\,\alpha + u\, dn\,\alpha \cdot sn\,\alpha - uA\, sn\,\alpha + O(u^2)$$

so that $A = dn\,\alpha$ and

$$cn\,u\; cn(u-\alpha) + sn\,u\; sn(u-\alpha)\, dn\,\alpha = cn\,\alpha$$

By an analytical continuation, this formulas ~~is~~ α is not congruent to iK'.

Similarly $dn\,u\; dn(u-\alpha) + k^2 sn\,u\; sn(u-\alpha)\, cn\,\alpha = dn\,\alpha$

Set $\alpha = u+v,\ s_1 = sn\,u,\ d_2 = dn\,v$ etc. ~~etc.~~ Then,

$$c_1 c_2 - s_1 s_2\, dn(u+v) = cn(u+v)$$

$$d_1 d_2 - k^2 s_1 s_2\, cn(u+v) = dn(u+v)$$

Hence $$cn(u+v) = \frac{c_1 c_2 - d_1 d_2 s_1 s_2}{1 - k^2 s_1^2 s_2^2}$$

$$dn(u+v) = \frac{d_1 d_2 - k^2 s_1 s_2 c_1 c_2}{1 - k^2 s_1^2 s_2^2}$$

If we set $\alpha = -v$

$$s_1 d_2 \, sn\,(u+v) = c_2 - c_1 \, cn(u+v)$$

and so

$$sn\,(u+v) = \frac{s_1 c_2 d_2 + s_2 c_1 d_1}{1 - k^2 s_1^2 s_2^2}$$

$$= \frac{s_1(1-s_2^2)^{\frac{1}{2}}(1-k^2 s_2^2)^{\frac{1}{2}} + s_2(1-s_1^2)^{\frac{1}{2}}(1-k^2 s_1^2)^{\frac{1}{2}}}{1 - k^2 s_1^2 s_2^2}$$

which is the addition theorem for $sn\,u$. Similar theorems hold of $cn\,u$ and $dn\,u$. Finally,

$$sn\,(u+v) = \frac{sn\,u \, cn\,v \, dn\,v + sn\,v \, cn\,v \, dn\,u}{1 - k^2 \, sn^2 u \, sn^2 v},$$

$$cn\,(u+v) = \frac{cn\,u \, cn\,v - dn\,u \, dn\,v \, sn\,u \, sn\,v}{1 - k^2 \, sn^2 u \, sn^2 v}$$

$$dn\,(u+v) = \frac{dn\,u \, dn\,v - k^2 \, sn\,u \, sn\,v \, cn\,u \, cn\,v}{1 - k^2 \, sn^2 u \, sn^2 v}$$