Complexity Upper Bounds using Permutation Group theory

Thesis submitted in
partial fulfillment of the degree of
Doctor of Philosophy (Ph.D)

by

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Declaration

I declare that the thesis entitled *Complexity Upper Bounds using Permutation Group theory* submitted for the degree of Doctor of Philosophy is the record of the work carried out by me during *January 2003* to *January 2006* under the guidance of *V. Arvind* and has not formed the basis for the award of any degree, diploma, associateship, fellowship, titles in this University or any other University or other Institution of Higher learning.

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January 13, 2006

Certificate

I certify that the thesis entitled *Complexity Upper Bounds using Permutation Group theory* submitted for the degree of Doctor of Philosophy by *Piyush P Kurur* is the record of research carried out by him during *January 2003* to *January 2006* under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of higher learning.

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Contents

Notation

$\mathrm{C}_G(A)$	the centraliser of A in G, page 14.
$\mathrm{Diag}\,(G_1 \times G_2)$	diagonal subgroup of $G_1 \times G_2$, page 16.
$\mathrm{NCL}_G(A)$	the normal closure of A in G, page 14.
$\mathrm{Res}_T\,(G)$	for simple group T the T-residue of G, page 42.
$\mathrm{Soc}\,(G)$	Socle of G, page 39.
$G \ltimes H$	semidirect product of G and H, page 15.
$G \times H$	direct product of G and H, page 15.
$\mathrm{Sift}(g)$	The sift of g, page 43.
$H \hookrightarrow G$	H embeds into G, page 14.
$H \leq G$, $G \geq H$	H is a subgroup G.
$H < G$, $G > H$	$H \leq G$ and $H \neq G$.
$[G : H]$	the index of H in G for $H \leq G$, page 14.
$H \trianglelefteq G$, $G \trianglerighteq H$	H is a normal subgroup of G.
$H \triangleleft G$, $G \triangleright H$	$H \trianglelefteq G$ and $H \neq G$.
$[\Sigma : \Delta]$	index of the block Δ in Σ, page 19.
α^g	image of α under a permutation g.
$\mathcal{B}\,(\Sigma/\Delta)$	conjugate blocks of Δ contained in Σ, page 19.
$G\,(\Delta)$	pointwise stabiliser of Δ.
$\mathrm{Sym}\,(\Omega)$	symmetric group on the set Ω.
G_Δ	setwise stabiliser of Δ.
S_n	$\mathrm{Sym}\,(\{1, \ldots, n\})$.
$[L : K]$	degree of the extension L/K, page 74.
$\mathrm{H}\,(\alpha)$	height of the algebraic number α, page 77.
$\mathbb{O}_K$	ring of algebraic integers of number field K.
$\mathbb{F}_{p^r}$	unique finite field of cardinality p^r.
$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	field of rational, real and complex numbers respectively.
$\mathrm{N}\,(\mathfrak{a})$,$\mathrm{N}\,(\alpha)$	norm of the ideal $\mathfrak{a}$ and $\alpha\mathbb{O}_K$ respectively, page 78.
d_K	discriminant of the number field K.
$K[X]$	polynomials in X with coefficients from K.

K_f	splitting field of the polynomial $f(X)$ over K, page 75.
$\mathrm{Fix}\,(L, G)$	fixed field of L under G, page 76.

Chapter 1

Introduction

Considerable progress has been made recently in the design of efficient algorithms for computational problems in permutation group theory. Many of these results exploit the structure of permutation groups. As permutation groups arise naturally in many computational problems, algorithmic breakthrough in this area often led to progress in solving, at least partially, other computational problems; Graph Isomorphism being a striking example. It is reasonable to expect that these group-theoretic and algorithmic advances would lead to better insights into the complexity of computational problems which are connected to permutation group theory. In this thesis we study Graph Isomorphism and problems that arise in Galois theory. Our aim is to use the structural properties of permutation groups together with other algebraic techniques to prove complexity upper bounds.

Complexity theory is the study of resource bounded computations. Efficiency is measured in terms of the resource required to solve the problem as a function of the input size. Two of the most important measures are the time and space required to solve the problem on a Turing machine. Often problems have a trivial exponential time brute force algorithm that searches for a potential solution in the set of all possible solutions. Such exponential time algorithms are impractical as they take considerable time for solving instances of reasonable sizes. Following the suggestion of Edmonds [25] it is widely accepted that computational problems in P, i.e. problems that are solvable in polynomial time on a deterministic Turing machine, are those that are tractable. This assumption is called the *extended Church-Turing hypothesis*. The complexity class NP is the class of problems that can be solved on a nondeterministic Turing machine in polynomial time. It is exactly the class of decision problems for which yes instances have polynomial time verifiable certificates. Clearly NP contains the class P but whether this containment is strict is a central open problem in complexity theory.

An important concept in complexity theory is the notion of completeness. A problem P in a complexity class $\mathcal{C}$ is said to be complete for $\mathcal{C}$ if for any other P' in $\mathcal{C}$ instance of P' can efficiently reduced to P. Problems complete for a complexity class $\mathcal{C}$ are in some sense the hardest problems of $\mathcal{C}$. For the class NP starting with the work of Cook [23] and Levin [45] and subsequently Karp [31] many important computational problems have been show to be complete (see the book of Garey and Johnson [29]). If any of these problems have polynomial time algorithm then P = NP. Hence a problem being NP-complete is a strong evidence that it has no efficient (i.e. polynomial time) algorithm.

Classifying natural problems by showing it to be complete for a complexity class is an important goal in complexity theory. For a computational problem, proving complexity theoretic upper and lower bounds often requires novel insights into the mathematics underlying the problem. The tight classification of the complexity of the permanent [70], determinant [66, 72] are classic examples. Despite serious efforts many problems still elude such a tight classification.

In this thesis we study the complexity Graph Isomorphism and problems associated with Galois theory with an aim of classifying these in the frame work of complexity theory. A common thread that connects these two is the role of permutation group theory. The structure of permutation groups and the numerous efficient algorithms for permutation group problems play an important role in our results.

Permutation groups, apart from being a source of interesting computational problems, have played important role in algorithms for Graph Isomorphism like for example in the polynomial time algorithm of Luks [46] for bounded valence graphs. Group theory has played important role in various complexity theoretic results. Babai's [10] AM$\cap$co-AM upper bounds for matrix group problems and Barrington's [15] group theoretic characterisation of NC^1 are two classic examples.

1.1 Overview of this thesis

We now give an overview of the thesis. Chapter 2 is a brief survey of the complexity theory required for this thesis and Chapter 3 develops the required group theory. For our results on Galois theory we need some results from algebraic number theory. We describe these in Chapter 6. Our results on Graph Isomorphism and related problems are explained in Chapters 4 and 5. We describe our results on computational problems in Galois theory in Chapters 7, 8 and 9.

Graph Isomorphism

Given two undirected graphs $X_1 = (V_1, E_1)$ and $X_2 = (V_2, E_2)$ the Graph Isomorphism problem is to check whether X_1 and X_2 are isomorphic, i.e. to check whether there is a one-to-one map $f : V_1 \to V_2$ such that for every unordered pair $\{u, v\}$ from V_1, $\{u, v\} \in E_1$ if and only if $\{f(u), f(v)\} \in E_2$. In this thesis we also study a special case of Graph Isomorphism problem called the *bounded colour multiplicity Graph Isomorphism problem*, BCGI for short. Given two vertex-coloured graphs X_1 and X_2 such that the number of vertices with a given colour is less than a constant b, we want to check whether there is a colour preserving isomorphism, i.e. an isomorphism f from X_1 to X_2 such that $u \in V(X_1)$ and $f(u) \in V(X_2)$ are of the same colour. We call this problem *bounded colour multiplicity* graph isomorphism problem, BCGI_b for short.

In Chapter 4 we show that the Graph Isomorphism problem is in the complexity class SPP. In fact we prove a more general result: We show that the generic group theoretic problem FINDGROUP is in the complexity class $\mathrm{FP}^{\mathrm{SPP}}$. As a consequence many interesting problems in permutation group theory like Graph Isomorphism, Set stabiliser problem and the Hidden subgroup problem over permutation groups are in SPP (or $\mathrm{FP}^{\mathrm{SPP}}$ for functional problems). Computational problem in SPP (or $\mathrm{FP}^{\mathrm{SPP}}$ in case they are functional problems) are *low* for many important complexity classes like $\oplus$P (in fact $\mathrm{Mod}_k\mathrm{P}$ for all k), $\mathrm{C}_=\mathrm{P}$ etc. Hence by proving the Graph Isomorphism problem to be in SPP we have show it to be low for each of these classes. Earlier it was not even know whether GI was in $\oplus$P.

In Chapter 5 we prove that BCGI_b is in the $\mathrm{Mod}_k\mathrm{L}$-hierarchy where the constant k and the level of the hierarchy depends only on b. Recently Torán [68] has shown the Graph Isomorphism problem to be hard for various complexity classes. In particular he has proved that BCGI is hard for $\mathrm{Mod}_k\mathrm{L}$ for all k. The graph gadgets that he construct can be used to show the hardness of BCGI for the entire $\mathrm{Mod}_k\mathrm{L}$-hierarchy [8, Appendix], a stronger result. Our result on BCGI complements his result and gives a fairly tight classification of BCGI in terms of logspace counting classes.

Another consequence of our result is on the parallel complexity of BCGI. Our results improve the NC upper bound of Luks [47] to NC^2 (even TC^1).

Galois theory

Consider a number field K, a field extension of $\mathbb{Q}$ the field of rational numbers. The Galois group of K, denoted by $\mathrm{Gal}\,(K/\mathbb{Q})$, is the group of *field automorphisms* of K that when restricted to $\mathbb{Q}$ is identity. For a polynomial $f(X) \in \mathbb{Q}[X]$, the splitting field $\mathbb{Q}_f$ is the smallest extension of $\mathbb{Q}$ that contains all the roots of f. By the Galois group of f we mean the Galois group $\mathrm{Gal}\,(\mathbb{Q}_f/\mathbb{Q})$.

The Galois group of a degree d polynomial f can be thought of as a subgroup of S_d, the group of permutations on d objects. This follows from the fact that the Galois group of f is fully specified by giving its action on the roots of f.

Computing the Galois group of a polynomial is a fundamental problem in algorithmic number theory. Often one is interested in verifying whether the Galois group of a polynomial satisfies certain properties instead of actually computing the Galois group. Asymptotically, the best algorithm for computing the Galois group of a polynomial $f(X) \in \mathbb{Q}[X]$ is due to Landau [37] and runs in time polynomial in $\mathrm{size}\,(f)$ and the order of the Galois group of f. Since the Galois group of a polynomial $f(X)$ of degree n can have $n!$ elements, Landau's algorithm takes exponential-time in the worst case.

Besides being a natural computational problem, knowing the Galois group of a polynomial f or knowing certain properties of the Galois group of f gives information about the roots of f. A classic example is the seminal work of Galois showing that a polynomial f is solvable by radicals if and only if its Galois group is solvable. Thus checking whether a polynomial is solvable by radicals amounts to checking whether its Galois group is solvable and hence has an exponential time algorithm. Landau and Miller [39] gave a remarkable polynomial time algorithm for solvability checking. This algorithm manages to check solvability without actually computing the entire Galois group. This remarkable result gives hope that certain non-trivial properties of Galois groups can be tested efficiently. Chapter 7 deals with such efficiently testable properties of Galois group. We give polynomial time algorithms for nilpotence testing and Γ_d-testing.

We generalise the Landau-Miller algorithm and give a polynomial-time algorithm for testing whether the Galois group of a given polynomial is in Γ_d for constant d. The class of groups Γ_d often crops up in permutation group theoretic problems, e.g. Luks' polynomial-time algorithm [46] for testing isomorphism of bounded degree graphs.

Even though nilpotent groups are solvable the Landau-Miller solvability test does not give a polynomial time nilpotence test. The Landau-Miller algorithm gives a way to test whether all composition factors of the Galois group are abelian. Nilpotence however is a more "global" property in the sense that it cannot be inferred by knowing the composition factors alone. In Chapter 7 we give a characterisation of nilpotent permutation groups and this characterisation yields a polynomial time nilpotence test.

Many computational problems in algebraic number theory are hard. In the absence of non-trivial upper bounds, conditional results, i.e. results whose validity depends on widely believed yet unproven conjectures of number theory, are of great interest. We now look at complexity theoretic results of this thesis that depend on the validity of the generalised Riemann hypothesis. An important ingredient used in our results is the Chebotarev density theorem, a result on the distribution of primes. For the complexity theoretic applications of this thesis we need an effective version of Chebotarev density theorem due to Lagarias and Odlyzko [35] proved assuming the generalised Riemann hypothesis.

The problem of interest in Chapter 8 is order finding of Galois groups. Given a polynomial $f(X) \in \mathbb{Q}[X]$ we are interested in computing the order of $\mathrm{Gal}\,(f)$ (or equivalently the degree $[\mathbb{Q}_f : \mathbb{Q}]$ of the extension $\mathbb{Q}_f/\mathbb{Q}$). For permutation groups of degree n presented via a generating set, the order can be computed in time polynomial in n. Hence computing the order is no more difficult that computing the Galois group and there is an exponential time algorithm for it. We prove better upper bounds assuming generalised Riemann hypothesis.

Given a polynomial $f(X) \in \mathbb{Q}[X]$ we show that there is a polynomial time algorithm making one query to a #P oracle that computes the order of the Galois group of f [7]. Furthermore using Stockmeyer's result on approximating #P functions [64], we show that there is a randomised algorithm with NP oracle to approximate the order of the Galois group.

For polynomials with Galois group in Γ_d, d a constant, we give a polynomial time reduction from exact order finding to approximate order finding. Thus for polynomials with Galois group in Γ_d, d a constant, we have a randomised algorithm with an NP-oracle to compute the order assuming the generalised Riemann hypothesis.

Finally in Chapter 9 we give nontrivial upper bounds on computing the Galois group of some special polynomials. We show that given a polynomial $f(X) \in \mathbb{Q}[X]$ with abelian Galois group, there is a randomised algorithm for computing the Galois group. This we achieve by giving a polynomial time randomised algorithm for sampling almost uniformly from the Galois group of f. The effective version of the Chebotarev density theorem plays a crucial role here. The only nontrivial bound of non-abelian Galois group computation is the following. Given a polynomial $f(X) \in \mathbb{Q}[X]$ such that every irreducible factor g of f has non-abelian simple Galois group of small size, there is a polynomial time deterministic algorithm for computing the Galois group of f. This result uses a special property of non-abelian semi-simple groups called Scott's Lemma (Lemma 3.6) and is unconditional.

Chapter 2

Complexity theory

In this chapter we recall the complexity theory required for this thesis. A detailed presentation is available in any standard textbook on complexity theory ([13, 14]). The survey article of Fortnow and Homer [27] gives a historical perspective together with pointers to many important results of complexity theory.

By an *alphabet* we mean a finite set Σ of *letters*. A *string* of *length* n over an alphabet Σ is a finite sequence $x_1 \dots x_n$ of letters from Σ. For a string x we will use $|x|$ to denote the length of x. By Σ^* we mean the set of all strings over Σ. We will use ϵ to denote the *empty string*, the unique string of length 0. A *language* over Σ is a subset of Σ^*.

A *decision problem* is a computational problem where we expect a yes/no answer for e.g. the Graph Isomorphism problem. By suitably encoding instances of a problem, any decision problem can be seen as a language over $\{0, 1\}$; the language corresponding to a decision problem is the set of encodings of input instances which evaluate to "yes". We will use the terms language and decision problem interchangeably.

Often computational problem require more than a yes/no answer for e.g. consider the problem of sorting a list of numbers. The functions of interest for us are functions from Σ^* to Σ^*. Again for countable sets A and B by suitable encoding, functions from A to B can be thought of as functions from Σ^* to Σ^*. Computing such functions are called *functional problems.*

The complexity class P is the class of decision problems that can be solved in time bounded by a polynomial in the size of its inputs on a Turing machine. The class P is robust because Turing machines can simulate other reasonable models of computation with a polynomial time overhead. Moreover, most natural problems that have polynomial time algorithms are tractable in practice. These properties led Edmonds [25] to suggest P as the class of tractable problems and is now widely accepted as the *extended Church-Turing hypothesis*[1]. By FP we mean the class of functions from Σ^* to Σ^* that can be computed on a polynomial time bounded Turing machine.

There are certain problems for which a candidate solution can be verified in polynomial time. The complexity class NP captures exactly this. It is the class of problems that can be solved on a nondeterministic Turing machine in polynomial time. Clearly NP contains the class P but whether this containment is strict is a central open problem in complexity theory. Although widely believed that P $\neq$ NP, the P vs NP conjecture has successfully resisted attempts of resolution till date. This question gained importance after the concept of NP-completeness was formalised due to the seminal work of Cook [23] and Levin [45] which proved that checking satisfiability of boolean formulae, SAT, is NP-complete. Subsequently Karp [31] showed a number of combinatorial problems including clique problem and travelling salesman problem to be NP-complete. A problem being NP-complete is a strong evidence that it has no polynomial time algorithm. The class of NP-complete problems is particularly important in view of the large number of important problems that it contains. The book of Garey and Johnson [29] gives a through review of NP-completeness and intractability with a list of important NP-complete problems.

Are there problems that are of intermediate complexity in NP? Ladner [34] showed that if P $\neq$ NP then there are problems that are neither in P nor are NP-complete. It is of interest to know whether there are natural problems of this kind. Graph Isomorphism seems to be one such and is one of the topics of this thesis.

Analogous to the arithmetic hierarchy in computability, Stockmeyer defined the polynomial hierarchy [65]. However, unlike the arithmetic hierarchy, it is not known whether the polynomial hierarchy is infinite. Many interesting problems have been shown to be at different levels of the polynomial hierarchy. Like the working assumption that P $\neq$ NP, it is widely believed that the polynomial time hierarchy is infinite.

Important subclasses of P are the complexity classes L and NL. The class L consists of problems for which input instance of size n can be solved with $O(\log n)$ space on a deterministic Turing machine. Recently, Reingold [57] proved that L contains undirected s-t connectivity problem: given an undirected graph and two nodes s and t check whether there is a path from s to t. As a consequence many problems that involve connectivity in

[1]Quantum computing is a potential challenge to this hypothesis.

undirected graphs can be solved in logspace. We summarise these results here for use in later chapters.

Lemma 2.1 (Reingold)**.** *Given a undirected graph, computing the connected components, find a maximal spanning forest etc. can be solved in logspace.*

The class NL is the nondeterministic version of L consisting of languages that can be accepted by *nondeterministic* logspace bounded Turing machines. A complete problem for NL is the directed s-t connectivity problem: given a directed graph and two distinguished points s and t check if there is a path from s to t.

In order to capture complexity classes below P, we need to restrict the oracle access mechanism for nondeterministic and randomised logspace machines. A widely accepted oracle access mechanism is the "Ruzzo-Simon-Tompa" oracle access [58] mechanism in which the oracle machine is restricted to write oracle queries deterministically. In this thesis we will follow this mechanism when we deal with NL oracle machines.

Circuit depth and size gives an elegant way of capturing parallel complexity of a problem. The class AC^k consists of polynomial sized circuits of depth $O(\log^k n)$. If there is an additional constraint that each gate has bounded fanin we get the class NC^k. It is known that $\mathrm{NC}^k \subseteq \mathrm{AC}^k \subseteq \mathrm{NC}^{k+1}$. The class NC is the union $\cup_{k=1}^{\infty} \mathrm{NC}^k$ and captures problems that have efficient parallel algorithms: problems that can be solved in polylog time on a parallel machine with the number of processors bounded by a polynomial in the input size.

2.1 Counting complexity classes

Counting complexity classes are defined based on the number of accepting and rejecting paths of a nondeterministic computation. Consider the functional problem #SAT of counting the number of satisfying assignments of a boolean formula. Functional problems like #SAT are problems in the complexity class #P. The complexity class #P consists of all functions f from strings to non-negative integers for which there is a NP machine M_f such that $f(x)$ is the number of accepting paths of M_f on input x. The functions that are complete for #P are hard to compute functions as they directly give a way of solving NP-complete problems. Surprisingly, certain decision problems that have polynomial time algorithms have counting versions that are #P-complete. A classic example is the problem of counting the number of matchings of a bipartite graph. Counting the number of matching in a bipartite graph is equivalent to computing the permanent of a $(0,1)$-matrix which was shown to be #P complete by Valiant [70].

The class #P is closed under sum and product However it is not closed under subtraction. The closure of #P under subtraction is the class GapP. Alternatively, GapP can be defined as the class of all functions f for which there is a NP-machine M_f such that $f(x)$ is the difference of the accepting and rejecting paths of M_f on input x. Apart from being closed under subtraction GapP inherits all the nice closure properties of #P. We summarise these closure properties below (see [26]).

Theorem 2.2. *The class* #P *and* GapP *are closed under exponential summation and polynomial product, i.e. if $f(x, y)$ be a function in* #P *(*GapP *) then for any polynomial $r(.)$ the functions*

$$g(x) = \sum_{|y| \leq r(|x|)} f(x, y)$$

and

$$h(x) = \prod_{y \leq r(|x|)} f(x, y)$$

are in #P *(*GapP*).*

The functions in #P are hard to compute — Toda's [67] results shows that the entire polynomial hierarchy is contained in $\mathrm{P}^{\#\mathrm{P}}$. Nonetheless, certain #P functions can be efficiently approximated, for example #DNFSAT has polynomial time approximation algorithms. For approximating general #P function the best known result is due to Stockmeyer [64].

Theorem 2.3. *For every function f in* #P *and any fixed constant c there is a randomised polynomial time algorithm with* NP*-oracle that on input x computes a value $N_x \in \mathbb{N}$ such that*

$$\left(1 - \frac{1}{|x|^c}\right) N_x \leq f(x) \leq \left(1 + \frac{1}{|x|^c}\right) N_x$$

The class PP consists of all languages L for which there is a GapP function f such that $x \in L$ if and only if $f(x) > 0$. Surprisingly the entire polynomial hierarchy is contained in P^{PP} as shown by Toda [67]. The class $\mathrm{Mod}_k\mathrm{P}$ consists of all languages L for which there is a #P function f such that $x \in L$ if and only if $f(x)$ is not divisible by k. By $\oplus$P we mean the class $\mathrm{Mod}_2\mathrm{P}$.

UP and SPP

A language L is in UP if there is a #P function f such that x is in L if $f(x) = 1$ and x is not in L if $f(x) = 0$. The class UP was introduced by Valiant [69] to captures the complexity of one-way functions. One-way functions are functions that are easy to compute but hard to invert and their study is central to cryptography. The existence of one-way functions is equivalent to the complexity theoretic assumption that UP $\neq$ P. The class SPP is the UP analogue of GapP. A language L is in SPP if there is a function f in GapP such that for all strings x, $x \in L$ if $f(x) = 1$ and $x \notin L$ if $f(x) = 0$.

The class SPP is probably one of the most natural counting complexity class. An important property of SPP is that it is exactly the class of languages that are low for GapP. A language L is said to be *low* for a complexity class $\mathcal{C}$ if $\mathcal{C}^L = \mathcal{C}$. Schöning [59] introduced the concept of lowness as a tool for classifying complexity theoretic problems and showed that NP $\cap$ co-AM is low for Σ_2^p.

Due to the lowness of SPP for GapP, languages in SPP are in and low for all reasonable *gap-definable* complexity classes including itself [26]. Many interesting counting complexity classes like $\oplus$P, Mod$_k$P, PP,C$_=$P are gap definable and hence showing a language L to be in SPP in one stroke shows that it is in and low for each of these classes. Since SPP is low for itself, the class FP$^{\mathrm{SPP}}$ also share these interesting lowness properties. The class FP$^{\mathrm{SPP}}$ is essentially SPP as the bits of functions of FP$^{\mathrm{SPP}}$ can be computed in SPP. Computational problems that are NP-hard are not expected to share these lowness properties and hence languages in SPP (or functional problems in FP$^{\mathrm{SPP}}$) are unlikely to be NP-hard. In Chapter 4 we show that the Graph Isomorphism problem is in SPP.

We now describe an important technique that is used to give SPP-upper bounds. Let A be a language in NP. An polynomial time oracle machine M^A is said to make UP-like queries to A if there is an NP machine N accepting A such that for all inputs x and for all queries y made by M on input x, N has at most one accepting computation on y, i.e. for queries made by M the machine N behaves like a UP machine. Again due to the closure properties of GapP and the lowness properties of SPP we have the following important theorem [32].

Theorem 2.4. *Any language accepted by (function computed by) a polynomial time oracle machine M^A making* UP*-like queries to $A \in$* NP *is in* SPP *(*FP$^{\mathrm{SPP}}$*).*

Logspace Counting classes

Analogous to GapP and #P by considering NL machines we can define classes GapL and #L. The class #L consists of functions f for which there is an NL machine M_f such that $f(x)$ is the number of accepting paths of M_f on x. Similarly we say that a function $f(x)$ is in $\#\mathrm{L}^A$ for some language A if there is an oracle NL^A machine M_f^A such that $f(x)$ is the number of accepting paths of M_f^A on x. Recall that the oracle machine M_f^A follows the Ruzzo-Simon-Tompa access mechanism for making queries to A.

Logspace counting classes have played an important role in classifying natural problems in NC^2. For example it follows from the work of Toda [66] and Vinay [72] that the problem of computing the determinant of an integer matrix is complete for GapL (for a detailed study see the article of Mahajan and Vinay [51]). Also the complexity of perfect matching is now quite well characterised by Allender *et al* [5] using logspace counting classes and the isolation lemma.

The complexity class $\mathrm{Mod}_k\mathrm{L}$ is the logspace analogue of $\mathrm{Mod}_k\mathrm{P}$. The class $\mathrm{Mod}_k\mathrm{L}$ consists of languages L for which there is a function f in #L such that x is in L if and only if $f(x) \neq 0 \pmod{k}$. It is known that if $k_1 \mid k_2$ then we have $\mathrm{Mod}_{k_1}\mathrm{L} \subseteq \mathrm{Mod}_{k_2}\mathrm{L}$. For a prime p the complexity class $\mathrm{Mod}_p\mathrm{L}$ captures the complexity of determinant over $\mathbb{F}_p$ quite accurately (cf. [20]). Recently, Allender *et al* [4] showed that many important linear algebraic problems like finding the rank and checking feasibility of linear equations over $\mathbb{F}_p$ are intimately connected to the complexity class $\mathrm{Mod}_p\mathrm{L}$. A survey of important results in this area is given in the article of Allender [3]. We summarise these results in the following theorem.

Theorem 2.5 (Buntrock *et al*)**.** *Let p be a prime. Given a $m \times n$ matrix A and a $m \times 1$ column vector $\mathbf{b}$ over $\mathbb{F}_p$ the problem of testing whether the system of linear equations $A\mathbf{x} = \mathbf{b}$ is feasible is in $\mathrm{Mod}_p\mathrm{L}$. In case the system is feasible finding a nontrivial solution for the vector $\mathbf{x}$ of indeterminates is in $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$.*

We now define the $\mathrm{Mod}_k\mathrm{L}$ hierarchy. The first level of the $\mathrm{Mod}_k\mathrm{L}$-hierarchy is the class $\mathrm{Mod}_k\mathrm{L}$. A language L is said to be in the $l+1$th level of the $\mathrm{Mod}_k\mathrm{L}$-hierarchy if there is a function f in $\#\mathrm{L}^A$, A a language in the lth level of the $\mathrm{Mod}_k\mathrm{L}$-hierarchy, such that for all x, x is in L if and only if $f(x) \neq 0 \pmod{k}$.

The $\mathrm{Mod}_k\mathrm{L}$ hierarchy can also be seen as languages accepted by constant depth circuits with $\mathrm{Mod}_k\mathrm{L}$ oracle, i.e. the $\mathrm{Mod}_k\mathrm{L}$ hierarchy is exactly $\mathrm{AC}^0(\mathrm{Mod}_k\mathrm{L})$. It is not known whether the $\mathrm{Mod}_k\mathrm{L}$-hierarchy is infinite.

However for primes p the $\mathrm{Mod}_p\mathrm{L}$-hierarchy collapses to $\mathrm{Mod}_p\mathrm{L}$. In Chapter 5 we see the connections of BCGI with the $\mathrm{Mod}_k\mathrm{L}$-hierarchy.

Chapter 3

Group Theory

In this chapter we review the group theory in particular the theory of permutation groups required for this thesis. The groups we encounter here will all be finite. For a detailed presentation any standard text book on group theory (for example [30]) may be consulted. We follow the notation of Wielandt [74] for permutation groups.

We use the following notation: For groups G and H, $H \leq G$ means that H is a subgroup of G. By $H < G$ we mean that H is a strict subgroup of G i.e. $H \leq G$ and $H \neq G$. By $G \geq H$ and $G > H$ we mean $H \leq G$ and $H < G$ respectively. Similarly by $H \trianglelefteq G$ we mean H is a normal subgroup of G. When H is a strictly smaller normal subgroup we denote it by $H \triangleleft G$. As before we use $G \trianglerighteq H$ and $G \triangleright H$ to mean $H \trianglelefteq G$ and $H \triangleleft G$ respectively. Let G be a group and A be any subset of G. By the normal closure of A in G, denoted by $\mathrm{NCL}_G(A)$, we mean the smallest normal subgroup of G containing A. The centraliser $\mathrm{C}_G(A)$ is the subgroup of G that commutes with all the elements of A.

Let H be any subgroup of G. By the index of H in G, denoted by $[G : H]$, we mean the number of distinct H cosets in G. We have $[G : H] = \frac{\#G}{\#H}$. We say that H *embeds* into a group G, denoted by $H \hookrightarrow G$ if there is a one-to-one homomorphism from H to G. In other words H is isomorphic to a subgroup of G.

Consider a normal subgroup N of G. There is a canonical homomorphism from G to G/N that maps an element g in G to its coset Ng. The canonical homomorphism gives a one-to-one correspondences between subgroups of G/N and subgroups of G containing N. For a subgroup L of G/N, by the *pullback* of L in G we mean the unique subgroup of G that contains N under this correspondence. More generally suppose ψ is a homomorphism from G *onto* H then there is a one-to-one correspondence between subgroups of G containing $\ker(\psi)$ and subgroups of H given by $L \mapsto \psi(L)$. The *pullback* of a subgroup H' of H is the unique G' such that $\psi(G') = H'$.

Let K and H be subgroups of G then the set KH is also a subgroup if and only if $KH = HK$ and has order given by $\#KH = \frac{1}{\#K \cap H}.\#K.\#H$. If in addition H is a normal subgroup of KH and $K \cap H$ is trivial we say that KH is the *semidirect product* of K and H which we denote by $K \ltimes H$. The semidirect product $K \ltimes H$ is the *direct product* (or just product) $K \times H$ if both K and H are normal subgroup of KH.

Let G be any group. For a subgroup H, a series of groups $G = G_0 > \ldots > G_t = H$ is called a *tower* of groups between G and H. The subgroup H is *subnormal* if there exists a *subnormal tower of groups* between G and H, i.e. a tower of groups $G = G_0 \rhd \ldots \rhd G_t = H$ such that for all $0 \leq i < t$, G_{i+1} is a normal subgroup of G_i. For any group G the trivial group $\{1\}$ is subnormal and any subnormal tower of groups between G and $\{1\}$ is called a *subnormal series* for G. A *composition series* for G is a subnormal series $G = G_0 \rhd \ldots \rhd G_t = \{1\}$ such that each of the quotients G_i/G_{i+1} are simple.

Definition 3.1 (Solvable groups)**.** *A group G is said to be solvable if there is a subnormal series $G = G_0 \rhd \ldots \rhd G_t = \{1\}$ such that for all $0 \leq i < t$ the quotient G_i/G_{i+1} is abelian.*

We now define the class Γ_d of groups. The class Γ_d is a generalisation of the class of solvable groups; if G is solvable then G is in Γ_d for any d. Computational problems for groups in Γ_d occur in many permutation group theoretic algorithms for example in Luks' polynomial time algorithm for bounded degree graphs [46].

Definition 3.2. *A group G is said to be in Γ_d if there is a subnormal series $G = G_0 \rhd \ldots \rhd G_t = \{1\}$ such that for all $0 \leq i < t$ either G_i/G_{i+1} is abelian or is isomorphic to a subgroup of S_d, the group of permutations on d objects.*

For $d < 5$, since S_d is solvable, Γ_d is just the class of solvable groups. The Γ_d-testing, which we will describe in Chapter 7, will use the following closure properties of the class Γ_d.

Proposition 3.3. *Any subgroup of a group in Γ_d is also in Γ_d. For any group G and a normal subgroup H, G is in Γ_d if and only if the groups G/H and H are in Γ_d.*

An important subclass of the class of solvable group is the class of nilpotent groups which we define below.

Definition 3.4 (Nilpotent groups). *A group G is said to be nilpotent if all its Sylow subgroups are normal.*

The following lemma gives alternate characterisation of nilpotent groups (see Section 10.3 of Hall's book [30]).

Lemma 3.5. *Let G be a finite group then the following are equivalent.*

1. *G is nilpotent.*
2. *G is the product of all its Sylow subgroups.*
3. *For every prime p that divides the order of G there is a unique p-Sylow subgroup.*

Let G be a group and $H \trianglelefteq G$ be a normal subgroup of G. A *normal tower* between G and H is a subnormal series $G = G_0 \rhd \ldots \rhd G_t = H$ such that each G_i is normal in G. A *normal series* for G is a normal tower between G and the trivial normal subgroup $\{1\}$.

Let G_1 and G_2 be two isomorphic groups and let $\phi : G_1 \to G_2$ be an isomorphism. The *diagonal subgroup* with respect to ϕ, denoted by $\mathrm{Diag}_\phi (G_1 \times G_2)$, is the subgroup $\{\langle g, \phi(g)\rangle : g \in G_1\}$ of $G_1 \times G_2$. Even though the diagonal group $\mathrm{Diag}_\phi (G_1 \times G_2)$ depends on the isomorphism ϕ, it is isomorphic to G_1 (and G_2) and hence we will usually drop the isomorphism ϕ.

A group G is said to be *simple* if the only proper normal subgroup of G is the trivial group. Let T be a simple group. A group G is said to be T-semisimple if there is a positive integer k such that G is isomorphic to T^k. An important property of non-abelian semisimple group which we will use in many occasions is Scott's lemma [60] (see Luks' course notes for a proof[48, page 38]).

Lemma 3.6 (Scott's Lemma). *Let $T_1, \ldots, T_k$ be nonabelian finite simple groups. Let G be any subgroup of $\prod_{i=1}^{r} T_i$ that projects* onto *each T_i. Then G is a direct product of diagonal subgroups. More precisely, there is a partition $\cup_{j=1}^{s} I_s$ of $\{1, \ldots, r\}$ such that*

$$G = \prod_{j=1}^{s} \mathrm{Diag}\left(\prod_{i \in I_j} T_i\right).$$

The Scott's lemma is valid only for nonabelian simple groups. We give a counter example to illustrate this. Consider the vector space $\mathbb{F}_2^3$. Let W

be the subspace $\{(x_1, x_2, x_3)^{\mathrm{T}} | x_1 + x_2 + x_3 = 0\}$. The space W project onto each of the component $\mathbb{F}_2$ however it is easy to see that W is not a product of diagonal subgroups.

3.1 Permutation Groups

Let Ω be a finite set. The *symmetric group* $\mathrm{Sym}(\Omega)$ is the group of all permutations on Ω. By a *permutation group on* Ω we mean a subgroup of $\mathrm{Sym}(\Omega)$. By S_n we mean $\mathrm{Sym}(\{1, \ldots, n\})$. For a group G the action $g : a \mapsto ag$ makes G a permutation group on itself. This action is called the right *regular action*. Similarly the left regular action is the action $g : a \mapsto ga$.

While dealing with permutation groups over Ω we adopt the following convention: Lower case Greek letters will be used to denote elements of Ω where as upper case Greek letters will be used to denote subsets of Ω. Lower case Latin letters will be used to denote elements of $\mathrm{Sym}(\Omega)$ and upper case Latin letters will be used to denote subsets or subgroups of $\mathrm{Sym}(\Omega)$.

The image of $\alpha \in \Omega$ under the permutation $g \in \mathrm{Sym}(\Omega)$ will be denoted by α^g. The advantage of this notation is that group action behave similar to exponentiation, i.e. $(\alpha^g)^h = \alpha^{gh}$. For $A \subseteq \mathrm{Sym}(\Omega)$, α^A denotes the set $\{\alpha^g : g \in A\}$. In particular, for $G \leq \mathrm{Sym}(\Omega)$ the G-*orbit* containing α is α^G. The G-orbits form a partition of Ω. Given a generating set of G, a straight forward transitive closure algorithm can be used to compute all the orbits (cf. [49]).

Theorem 3.7. *Given $G \leq \mathrm{Sym}(\Omega)$ by a generating set A and $\alpha \in \Omega$, there is a polynomial-time algorithm to compute α^G. Moreover for each $\beta \in \alpha^G$ the above mentioned algorithm can compute a $g_\beta \in G$ such that $\alpha^{g_\beta} = \beta$.*

Let G be a permutation group action on Ω. For $\Delta \subseteq \Omega$ and $g \in \mathrm{Sym}(\Omega)$, Δ^g denotes $\{\alpha^g : \alpha \in \Delta\}$. The *set-wise stabiliser* of Δ, i.e. $\{g \in G : \Delta^g = \Delta\}$, is denoted by G_Δ. If Δ is the singleton set $\{\alpha\}$ we write G_α instead of $G_{\{\alpha\}}$. For any Δ by $G|_\Delta$ we mean G_Δ restricted to Δ. For a set $\Delta \subseteq \Omega$ the *pointwise stabiliser* will be denoted by $G(\Delta)$. Notice that $G(\Omega) = \{1\}$ and $G(\{\alpha\}) = G_\alpha$.

An often used result is the orbit-stabiliser formula stated below [74, Theorem 3.2].

Theorem 3.8 (Orbit-Stabiliser formula)**.** *Let G be a permutation group on $\mathrm{Sym}(\Omega)$ and let α be any element of Ω then the order of the group G is given by $\#G = \#G_\alpha . \#\alpha^G$.*

3.2 Strong generator set

Let G be a group and H be a subgroup of G. By a *right traversal* of H in G we mean a collection of coset representatives one from each right coset of H in G. Similarly we can define the left traversal of H in G. Let $G = G_0 \geq \ldots \geq G_t = \{1\}$ be a decreasing tower of subgroups of G. Let C_i denote the right traversal of G_i in G_{i-1} then the collection $\cup_{i=1}^{t} C_i$ is a generator set of G which we call a *strong generator set* of G with respect to the given tower. The strong generator set depends on the choice of the traversals C_i at each stage. Also $\#C_i = [G_i : G_{i-1}]$ and hence the order of G is given by $\prod_{i=1}^{t} \#C_i$.

We now describe a particularly useful strong generating set for permutation groups of degree n. Let G be a permutation group over Ω, a set of cardinality n. Without loss of generality we assume that Ω is the set $\{1, \ldots, n\}$. Let $G^{(i)}$ be the point-wise stabiliser of $\{1, \ldots, i\}$. The tower of groups $G = G^{(0)} \geq \ldots \geq G^{(n-1)} = \{1\}$ gives rise to a strong generator set called the Schreier-Sims strong generating set. For any permutation group $G \leq S_n$ note that $\#C_i \leq n - i$ and hence the Schreier-Sims strong generator set is a succinct presentation of G. There are polynomial time algorithm for computing the Schreier-Sims strong generator set [62, 63, 28]. Many algorithmic tasks involving permutation groups can be solved once a strong generator set is computed. We collect some of the useful computational results in the following theorem.

Theorem 3.9. *Let G a permutation group on Ω presented by giving a generator set of G. The following tasks can be done in polynomial time.*

1. *Computing the Schreier-Sims strong generator set.*
2. *Computing the order of G.*
3. *Given $g \in \mathrm{Sym}(\Omega)$ checking whether $g \in G$.*
4. *Given a subset Δ of Ω compute the pointwise stabiliser $G(\Delta)$.*

A detailed treatment of computational issues in permutation groups is available in the book by Seress [2].

3.3 Transitivity, Blocks and Primitivity

A permutation group G on Ω is *transitive* if there is only one G-orbit. Suppose $G \leq \mathrm{Sym}(\Omega)$ is transitive. Then $\Delta \subseteq \Omega$ is a *G-block* if for all $g \in G$

either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$. For every G, Ω is a block and each singleton $\{\alpha\}$ is a block. These are the *trivial blocks* of G. A transitive group G is *primitive* if it has only trivial blocks and it is *imprimitive* if it has nontrivial blocks. Examples for primitive groups are S_n and A_n. These are the "giants". However the following bound on primitive groups in Γ_d shows that they are small [11].

Theorem 3.10 (Babai, Cameron, Pálfy)**.** *Let $G \leq S_n$ be a primitive permutation group in Γ_d. Then $\#G \leq n^{O(d)}$.*

The above mentioned bound is a generalisation of Pálfy's bound [56] on the order of primitive solvable subgroups of S_n that was used in the Landau-Miller solvability test [39]. Bounds on sizes of primitive groups such as Theorem 3.10 are important in runtime analysis of various permutation group theory problems. In particular our Γ_d-testing algorithm depends on Theorem 3.10.

A G-block Δ is a *maximal subblock* of a G-block Σ if $\Delta \subset \Sigma$ and there is no G-block Υ such that $\Delta \subset \Upsilon \subset \Omega$. Let Δ and Σ be two G-blocks. A chain $\Delta = \Delta_0 \subset \ldots \subset \Delta_t = \Sigma$ is a *maximal increasing chain* of G-blocks between Δ and Σ if for all i, Δ_i is a maximal subblock of Δ_{i+1}.

If Δ is a G-block then Δ^g is also a G-block, for each $g \in G$. Two G-blocks Δ_1 and Δ_2 are *conjugates* (more precisely G-conjugates) if there is a $g \in G$ such that $\Delta_1^g = \Delta_2$. It is not difficult to see that the conjugate relation on the set of G-blocks forms a equivalence relation. Let Δ and Σ be two G-blocks such that $\Delta \subseteq \Sigma$. The Δ-*block system of* Σ, is the collection

$$\mathcal{B}(\Sigma/\Delta) = \{\Delta^g : g \in G \text{ and } \Delta^g \subseteq \Sigma\}.$$

The Δ-block system of Σ gives a partition of Σ. It follows that $\#\Delta$ divides $\#\Sigma$ and by *index* of Δ in Σ, which we denote by $[\Sigma : \Delta]$, we mean $\#\mathcal{B}(\Sigma/\Delta) = \frac{\#\Sigma}{\#\Delta}$. We will use $\mathcal{B}(\Delta)$ to denote $\mathcal{B}(\Omega/\Delta)$.

Blocks are fundamental structures associated with permutation groups and have intimate connections with subgroups of G. To illustrate this consider a finite group G as a permutation group on itself under the right regular action. A subset H of G is a subgroup if and only if H is a G-block containing the identity. For a subgroup H of G, which is a G-block under the right regular action, any other conjugate block of H is a right coset of H. More generally if G is a transitive permutation group on Ω, we have the following Galois correspondence between blocks and subgroups [74, Theorem 7.5].

Theorem 3.11 (Galois correspondence of blocks)**.** *Let $G \leq \mathrm{Sym}(\Omega)$ be transitive and $\alpha \in \Omega$. There is a one-to-one correspondence between G-blocks containing α and subgroups of G containing G_α given by $\Delta \rightleftharpoons G_\Delta$. Also for blocks $\Delta \subseteq \Sigma$ we have $[G_\Sigma : G_\Delta] = [\Sigma : \Delta]$.*

In particular the above theorem implies that G is primitive if and only if for all $\alpha \in \Omega$, G_α is a maximal proper subgroup of G.

Let $G \leq \mathrm{Sym}(\Omega)$ be transitive and Δ and Σ be two G-blocks such that $\Delta \subseteq \Sigma$. Let $G(\Sigma/\Delta)$ denote the group $\{g \in G : \Upsilon^g = \Upsilon \text{ for all } \Upsilon \in \mathcal{B}(\Sigma/\Delta)\}$. We write G^Δ for the group $G(\Omega/\Delta)$. For any $g \in G_\Sigma$, since g setwise stabilises Σ, g permutes the elements of $\mathcal{B}(\Sigma/\Delta)$. Hence for any $\Upsilon \in \mathcal{B}(\Sigma/\Delta)$ we have $\Upsilon^{g^{-1}G(\Sigma/\Delta)g} = \Upsilon$. Thus, $G(\Sigma/\Delta)$ is a normal subgroup of G_Σ. In particular, G^Δ is a normal subgroup of G. The following lemma lists important properties of $G(\Sigma/\Delta)$.

Theorem 3.12.

1. *For G-blocks $\Delta \subseteq \Sigma$, $G(\Sigma/\Delta)$ is the largest normal subgroup of G_Σ contained in G_Δ.*

2. *Let Σ be G-block then $G^\Sigma \hookrightarrow \prod_{\Upsilon \in \mathcal{B}(\Sigma)} G|_\Upsilon$.*

3. *Let Δ be a G-subblock of Σ then $\frac{G_\Sigma}{G(\Sigma/\Delta)}$ is a faithful permutation group on $\mathcal{B}(\Sigma/\Delta)$ and is primitive if and only if Δ is a maximal subblock.*

4. *The quotient group G^Σ/G^Δ can be embedded into the product group $\left(\frac{G_\Sigma}{G(\Sigma/\Delta)}\right)^l$ for some l.*

Proof. Let N be any normal subgroup of G_Σ contained in G_Δ. We have for $\Delta^N = \Delta$. Consider any $\Upsilon \in \mathcal{B}(\Sigma/\Delta)$. Since G_Σ acts transitively on $\mathcal{B}(\Sigma/\Delta)$ there is a $g \in G_\Sigma$ such that $\Upsilon = \Delta^g$. Since $gN = Ng$ we have $\Upsilon^N = \Delta^{gN} = \Delta^{Ng} = \Upsilon$. This proves that for all $\Upsilon \in \mathcal{B}(\Sigma/\Delta)$, $\Upsilon^N = \Upsilon$. Hence N is contained in $G(\Sigma/\Delta)$. This proves part 1.

The group G^Σ consists of all elements g of G that fixes setwise every block $\Upsilon \in \mathcal{B}(\Sigma)$ and hence we have the embedding of part 2.

That $\frac{G_\Sigma}{G(\Sigma/\Delta)}$ acts faithfully on $\mathcal{B}(\Sigma/\Delta)$ follows from the fact that for any two g and h in G_Σ, g and h has the same action on $\mathcal{B}(\Sigma/\Delta)$ if and only if $gG(\Sigma/\Delta)$ and $hG(\Sigma/\Delta)$ are equal. Any nontrivial $\frac{G_\Sigma}{G(\Sigma/\Delta)}$-block of $\mathcal{B}(\Sigma/\Delta)$ gives a nontrivial G-block between Δ and Σ and vice versa. Thus, $\frac{G_\Sigma}{G(\Sigma/\Delta)}$ is primitive if and only if Δ is a maximal subblock of Σ.

Finally for the last statement notice that we have the group isomorphism

$$\frac{G|_{\Upsilon}}{G\left(\Upsilon/\Delta_{\Upsilon}\right)|_{\Upsilon}} \cong \frac{G_{\Upsilon}}{G\left(\Upsilon/\Delta_{\Upsilon}\right)}.$$

Also since $G^{\Delta} = G^{\Sigma} \cap \prod G\left(\Upsilon/\Delta_{\Upsilon}\right)|_{\Upsilon}$ where Υ varies over $\mathcal{B}\left(\Sigma\right)$ and Δ_{Υ} is any conjugate of Δ contained in Υ we have

$$G^{\Sigma}/G^{\Delta} \hookrightarrow \prod_{\Upsilon\in\mathcal{B}(\Sigma)} \frac{G|_{\Upsilon}}{G\left(\Upsilon/\Delta_{\Upsilon}\right)|_{\Upsilon}} = \prod_{\Upsilon\in\mathcal{B}(\Sigma)} \frac{G_{\Upsilon}}{G\left(\Upsilon/\Delta_{\Upsilon}\right)}.$$

Let $g \in G$ be any element that maps Δ to Δ_{Υ} then it follows that $G_{\Upsilon} = g^{-1}G_{\Sigma}g$ and $G\left(\Upsilon/\Delta_{\Upsilon}\right) = g^{-1}G\left(\Sigma/\Delta\right)g$. Hence the quotient groups $\frac{G_{\Sigma}}{G(\Sigma/\Delta)}$ and $\frac{G_{\Upsilon}}{G(\Upsilon/\Delta_{\Upsilon})}$ are isomorphic. Thus, we see that G^{Σ}/G^{Δ} is isomorphic to a subgroup of $\left(\frac{G_{\Sigma}}{G(\Sigma/\Delta)}\right)^{l}$ for some l. $\square$

Lemma 3.13. *Let $G \leq \mathrm{Sym}\left(\Omega\right)$ be transitive and N be a normal subgroup of G. Let $\alpha \in \Omega$. Then the N-orbit α^{N} is a G-block and the collection of N-orbits is an α^{N}-block system of Ω under G action. If $N \neq \{1\}$ then $\#\alpha^{N} > 1$. Furthermore, if $G_{\alpha} \leq N \neq G$ then the α^{N}-block system is nontrivial implying that G is not primitive.*

Proof. Let $\alpha \in \Omega$ and $g \in G$. Then $(\alpha^{N})^{g} = \alpha^{Ng} = \alpha^{gN} = (\alpha^{g})^{N}$. Thus $(\alpha^{g})^{N}$ and α^{N} are identical if $\alpha^{g} \in \alpha^{N}$ and disjoint otherwise, since they are distinct N-orbits. Hence α^{N} is a G-block and the orbits of N is an α^{N}-block system of Ω under G action. If $\alpha^{N} = \{\alpha\}$ then for all $\beta \in \Omega$, $\beta^{N} = \{\beta\}$. Thus, $N = \{1\}$.

Finally, note that by the Orbit-Stabiliser formula $\#G = \#\Omega \cdot \#G_{\alpha}$ and $\#N = \#\alpha^{N} \cdot \#G_{\alpha}$. Thus, if $\{1\} \neq N \neq G$ then α^{N} is a proper G-block. This completes the proof. $\square$

3.4 Structure Tree and Structure Forest

An important structure associated with a transitive permutation group is its structure tree. Structure trees have proved useful in analysing various divide and conquer algorithms for permutation groups (see, e.g. [49]). Let G transitive permutation group acting on Ω. Consider a maximal chain of G-blocks $\{\alpha\} = \Delta_0 \subset \ldots \subset \Delta_t = \Omega$. For each such maximal chain we can associate a structure tree as follows:

Definition 3.14 (Structure Tree)**.** *Let G be a transitive permutation group acting on Ω and let $\Omega = \Delta_0 \supset \ldots \supset \Delta_t = \{\alpha\}$ be any maximal decreasing chain of G-blocks. A structure tree of G with respect to this maximal chain is a labelled rooted tree of depth t satisfying the following conditions.*

1. *The root is labelled Ω.*
2. *The leaves are labelled with singleton sets $\{\gamma\}$, $\gamma \in \Omega$.*
3. *For each node labelled Σ at level i the children of Σ are $\{\Delta \in \mathcal{B}(\Delta_{i+1}) : \Delta \subset \Sigma\}$.*

We will identify the nodes of the tree with its labels (which are G-blocks). Any root-to-leaf path $\Omega = \Delta_0 \supset \ldots \supset \Delta_t = \{\alpha\}$ in a structure tree is a maximal decreasing chain of G-blocks. Conversely any maximal decreasing chain $\Omega = \Delta_0 \supset \ldots \supset \Delta_t = \{\alpha\}$ of G-blocks gives a structure tree for which it is a root-to-leaf path. If two maximal chains $\Omega = \Delta_0 \supset \ldots \supset \Delta_t = \{\alpha\}$ and $\Omega = \Sigma_0 \supset \ldots \supset \Sigma_s = \{\beta\}$ gives the same structure tree then $t = s$ and there is a permutation $g \in G$ such that $\Delta_i^g = \Sigma_i$ for each i (pick any g that maps α to β). For two nodes Δ and Σ of a structure tree T, Σ is an ancestor of Δ if and only if $\Delta \subseteq \Sigma$.

Let G be a transitive permutation group on Ω and let T be any structure tree of G. There is a natural action of G on the nodes of T; a node Δ under the action of $g \in G$ goes to the node Δ^g. This action embeds G into the group of automorphisms of the labelled graph T. For any node Σ the subgroup of G that fixes Σ is the group G_Σ and for any child Δ of Σ the group $G(\Sigma/\Delta)$ is the subgroup of G that fixes all the siblings of Δ. It follows from Theorem 3.12 that G acts primitively on the children of the root of T.

For intransitive groups G on Ω instead of a structure tree we have a *structure forest* . Let $\Omega_1, \ldots, \Omega_k$ be the distinct G-orbits then G restricted to Ω_i is transitive on Ω_i. A structure forest of G is the collection of structure trees $\{T_1, \ldots, T_k\}$ where T_i is a structure tree of transitive group $G|_{\Omega_i}$ on Ω_i. Many permutation group algorithm uses a divide and conquer strategy by first computing the orbits and then finding the maximal blocks. The structure forest thus appear naturally in the analysis of such algorithms.

Given a permutation group G acting on Ω there is a polynomial time algorithm to compute a structure tree for G. A key step involved is to compute a minimal G-block. The polynomial time algorithm for finding the minimal G-block follows from the fact that the smallest block that contains α and β is exactly the connected component of α in the undirected graph X where the vertex set is Ω and the edge set is $\{\{\alpha, \beta\}^g : g \in G\}$. The graph X can be constructed in polynomial time as it amounts to finding the orbit of $\{\alpha, \beta\}$ under the action of G on the set of unordered pairs of Ω. To summarise the above discussion we have the following lemma [46, Lemma 1.3]

Lemma 3.15. *Given a permutation group G on Ω via a generating set A and a G-orbit Ω' there is a polynomial time algorithm for computing the G-block system of Ω' corresponding to a minimal G-block.*

Chapter 4

The Graph Isomorphism problem

In this chapter we study the Graph Isomorphism problem (GI for short). A *graph* X for us is an undirected graph i.e. a finite set of vertices $V(X)$ and a set $E(X)$ of unordered pairs of elements of $V(X)$. Two graphs X_1 and X_2 are isomorphic if there is a one-to-one map f from $V(X_1)$ onto $V(X_2)$ that preserves the edge relations i.e. for every unordered pair $\{u, v\}$, $u, v \in V(X_1)$, $\{u, v\} \in E(X_1)$ if and only if $\{f(u), f(v)\} \in E(X_2)$. The function f we will call an isomorphism between X_1 and X_2.

Definition 4.1 (Graph Isomorphism problem)**.** *Given two graphs X_1 and X_2 test whether they are isomorphic.*

The Graph Isomorphism problem is believed to be one of the natural problems that lie between the complexity classes P and NP. Even though no polynomial time algorithm is know, it is not expected to be NP-complete. In fact, under reasonable complexity theoretic assumptions, it appears that the graph isomorphism problem is unlikely to be NP-complete. It was shown by Boppana *et al* [19] that Graph Isomorphism is not NP-complete unless the polynomial hierarchy collapses to Σ_2^p. They also showed that graph non-isomorphism is in AM and hence GI is in NP $\cap$ co-AM. Schöning [59] generalised the result of Boppana *et al* [19] and proved that NP $\cap$ co-AM is low for Σ_2^p. It then follows that any language in NP $\cap$ co-AM cannot be NP-complete unless the polynomial hierarchy collapses to Σ_2^p. Lowness for PP is another property that NP-complete problems are unlikely to have. It was shown by Köbler *et al* [32] that GI is in LWPP and hence is low for the class PP. For a detailed study of the Graph Isomorphism problem from a complexity theoretic perspective see the book of Köbler *et al* [33].

The main result of this chapter is the SPP upper bound for Graph Isomorphism [6]. Our result is an improvement on the LWPP upper bound of Köbler *et al* [32]. As mentioned in Chapter 2 a problem in SPP (or $\mathrm{FP}^{\mathrm{SPP}}$) is low for many interesting complexity classes like $\oplus$P for example. Hence our result shows that GI is low for each of these complexity classes. Earlier it was not even known whether GI is in $\oplus$P.

To prove the SPP upper bound for GI it is sufficient to give an $\mathrm{FP}^{\mathrm{SPP}}$ algorithm for AUT. In fact in Section 4.4 we show that a generic permutation group theoretic problem which we will call FINDGROUP problem, has a $\mathrm{FP}^{\mathrm{SPP}}$ algorithm. Many interesting permutation group problems including AUT will be special cases of this generic problem.

4.1 Group theoretic formulation of Graph Isomorphism problem

We now formulate the Graph Isomorphism problem as a group theory problem. Many important upper bounds for GI were achieved by making use of this group theoretic formulation. The polynomial time algorithm for bounded valance graphs [46] and the fastest known algorithm for general graphs [75, 12] are group theoretic in nature.

Consider the family of n-vertex graph. With out loss of generality we fix their vertex set to be an n element set Ω, say $\{1, \ldots, n\}$ for example. Let $\mathcal{G}(\Omega)$ be the set of graphs with vertex set Ω. The permutations $g \in \mathrm{Sym}(\Omega)$ has a natural induced action on the set of unordered pairs $\binom{\Omega}{2}$ namely $\{u, v\}^g = \{u^g, v^g\}$. This natural action induced action on $\mathcal{G}(\Omega)$; a permutation $g \in \mathrm{Sym}(\Omega)$ maps the graph $X = (\Omega, E)$ to $X^g = (\Omega, E^g)$. The Graph Isomorphism problem can be formulated as follows: Given two graphs X_1 and X_2 in $\mathcal{G}(\Omega)$ check whether they are in the same orbit under the $\mathrm{Sym}(\Omega)$ action.

An *automorphism of a graph* X in $\mathcal{G}(\Omega)$ is an element of $\mathrm{Sym}(\Omega)$ such that $X^g = X$. The set of all automorphisms of a graph X, which we will denote by $\mathrm{Aut}(X)$, is but the stabiliser subgroup of the point X under $\mathrm{Sym}(\Omega)$ action. We now define the Graph Automorphism problem, AUT for short, which is closely related to GI.

Definition 4.2 (Graph Automorphism problem). *Given an undirected graph X compute a generator set of $\mathrm{Aut}(X)$ as a permutation group on $V(X)$.*

By #GI we mean the counting problem where given two graphs X_1 and X_2 we want to compute the number of isomorphisms between X_1 and X_2.

Similarly by #AUT we mean the counting version of AUT, i.e. given a graph X counting the number of automorphisms of X.

The problem #AUT is polynomial time Turing reducible to AUT as there are polynomial time algorithm for computing the order of a permutation group given by a generating set. Let X_1 and X_2 be isomorphic graphs in $\mathcal{G}(\Omega)$. Let $g \in \mathrm{Sym}\,(\Omega)$ be any isomorphism between X_1 and X_2 then the set of all isomorphism between X_1 and X_2 is the coset $\mathrm{Aut}\,(X_1)g$ and hence $\#\mathrm{Iso}(X_1, X_2) = \#\mathrm{Aut}\,(X_1) = \#\mathrm{Aut}\,(X_2)$. Mathon [52] proved that the Graph Isomorphism problem and Graph Automorphism problem are equivalent under polynomial time Turing reductions. As a result we have the following theorem.

Theorem 4.3 (Mathon)**.** *The computational problems GI, AUT, #GI and #AUT are all equivalent under polynomial-time Turing reductions.*

Mathon's result together with Toda's [67] theorem gives another reason to believe that GI is unlikely to be NP-complete. By Toda's theorem $\mathrm{P}^{\#\mathrm{P}}$ contains the entire polynomial hierarchy. Therefore the counting problem #GI is not #P-complete unless the polynomial hierarchy collapses to Δ_2^p. Counting versions of almost all known NP-complete problems are complete for #P. In fact counting versions of certain problems in P, like perfect matching, are also complete for #P.

4.2 Problems related to Graph Isomorphism

We look at problems that are closely related to GI. Many isomorphism problems of combinatorial structures are closely connected to the Graph Isomorphism problem. For a detailed account of these problems, their complexity and their relation to GI see the book of Köbler *et al* [33, Chapter 1].

First we consider slight variations of the Graph Isomorphism problem. A directed graph consists of a finite set of vertices V and a collection $E \subseteq V \times V$. Isomorphisms of directed graphs should also preserve the direction of edges, i.e. a bijection f from $V(X_1)$ to $V(X_2)$ is an isomorphism if for all u and v in $V(X_1)$ the ordered pair $(u, v) \in E(X_1)$ if and only if $(f(u), f(v)) \in E(X_2)$. The problem of directed Graph Isomorphism is to check whether two directed graphs X_1 and X_2 are isomorphic. A vertex coloured graph is a graph together with a colouring function, i.e. a map $\psi : V \to C$ where C is the set of colours. For coloured graphs X_1 and X_2 a map $f : V(X_1) \to V(X_2)$ is an isomorphism if f should preserve the edge relations and also the colours, i.e. for any $u \in V(X_1)$ both u and $f(u)$ should of the same colour. The problem of coloured Graph Isomorphism is to check whether two coloured graphs are isomorphic.

By attaching suitable graph gadgets one can show that each of these problems are polynomial time equivalent to GI (see [53]).

We now consider the Group Isomorphism problem. We are given a group G via its multiplication table, i.e. a two dimensional array indexed by elements of the group G where for each g and h in G the (g, h)th entry is gh. The Group Isomorphism problem, GrpI for short, is to check whether two groups presented via their multiplication table is isomorphic. It turns out that GrpI $\leq_m^p$ GI (see [53]) however a reduction in the other direction is not known. If instead of groups we consider semigroup Isomorphism, SemiGrpI we have and equivalence result, i.e. SemiGrpI $\equiv_m^p$ Gi [18]. We now define the Setwise stabiliser problem SetStab.

Definition 4.4 (Setwise Stabiliser Problem)**.** *Given a generator set for a permutation group G over Ω and a subset $\Delta \subseteq \Omega$ compute a generator set for G_Δ the set-wise stabiliser of Δ.*

There is a polynomial time many one reduction from AUT to SetStab. To see this consider a graph $X = (V, E)$ consider the action of $G = \mathrm{Sym}(V)$ on the set $\Omega = \binom{V}{2}$ of unordered pairs of V. Then $\mathrm{Aut}(X)$ is nothing but G_E (or $G_{\Omega \setminus E}$). However no reduction is known in the other direction. The Setwise stabiliser problem seems to be harder than the Graph Isomorphism problem.

We now define the *hidden subgroup problem*, Hsp for short. Many interesting computational problems for which there are efficient quantum algorithms are variants of the Hsp. This include the Shor's polynomial time algorithm for integer factoring and discrete logarithm [61]. Many other group theoretic problems including AUT can be cast as a Hidden subgroup problem.

For a group G and a set X, $\phi : G \to X$ is a *right hiding function* for a subgroup H of G if ϕ is constant and distinct on the right cosets of H, i.e. for any g_1 and g_2 in G, $\phi(g_1) = \phi(g_2)$ if and only if $Hg_1 = Hg_2$.

Definition 4.5 (Hidden Subgroup Problem)**.** *Given a group G by its generator set and a hiding function $\phi : G \to X$ for a subgroup H compute a generator set for H.*

4.3 Computing the lex-least element of a Coset

Consider a finite totally-ordered set $(\Omega, <)$. The order $<$ on Ω induces a natural order, the lexicographic order, on $\mathrm{Sym}(\Omega)$ as follows: $g < h$ if there is an $\alpha \in \Omega$ such that $\alpha^g < \alpha^h$ and for all $\beta < \alpha$, $\beta^g = \beta^h$. It is not difficult to see that $<$ is a total order on $\mathrm{Sym}(\Omega)$ and the least element is 1. In this section we give a polynomial time algorithm that computes the least element of Gg given g and a generating set for G. This algorithm plays a crucial role in our SPP algorithm for FINDGROUP.

Theorem 4.6. *Given a permutation group G on a totally ordered set $(\Omega, <)$ via a generator set. Let $g \in \mathrm{Sym}(\Omega)$ be any permutation of Ω. There is a polynomial-time algorithm that computes the lexicographically least element of Gg.*

Proof. Let g^* be the lex-least element of Gg. Let α be the least element of Ω. We first compute the set α^{Gg} in polynomial time — First compute the G-orbit $\Sigma = \alpha^G$ using Theorem 3.7 and then compute Σ^g. Furthermore we assume that we have actually computed for each $\eta \in \alpha^{Gg}$ an element $g_\eta \in Gg$ such that $\alpha^{g_\eta} = \eta$. Let β be the least element of α^{Gg} then we have $\alpha^{g^*} = \beta$. Otherwise $\alpha^{g_\beta} = \beta < \alpha^{g^*}$ and hence $g_\beta \in Gg$ is lesser that g^* in the lexicographic order which is a contradiction.

Every element of the coset $G_\alpha g_\beta \subseteq Gg$ maps α to β. Conversely consider any $h \in Gg$ that maps α to β. The elements hg_β^{-1} fixes α and hence $h \in G_\alpha g_\beta$. Therefore $G_\alpha g_\beta$ is exactly the set of elements that map α to β and hence contains g^*. We can use the above idea repeatedly as follows: Let $G^{(i)}$ be the subgroup of G that fixes pointwise the first i elements $\alpha_1, \ldots, \alpha_i$ of Ω. By Theorem 3.9 we can compute the strong generator set of G and hence compute the generator sets of each of the groups $G^{(i)}$ in polynomial time. Starting with $g_0 = g$, for $1 \leq i < n$ we compute an element $g_i \in Gg$ such that $g^* \in G^{(i)} g_i$. In the ith step we compute the least element β_i in the set $\alpha_i^{G^{(i-1)} g_{i-1}}$ and a permutation h that maps α_i to β_i. The permutation g_i is h. Since $G^{(n-1)} = \{1\}$, where $n = \#\Omega$, we have $g_{n-1} = g^*$. Algorithm 1 is the detailed presentation of the above discussion. $\square$

Input: An ordered set Ω, a generator set for $G \leq \mathrm{Sym}(\Omega)$, and a $g \in \mathrm{Sym}(\Omega)$
Output: Lexicographically least element in Gg
Let $\alpha_1 < \ldots < \alpha_n$ be the ordered list of element of Ω.
Let $G^{(i)}$ be the pointwise stabiliser of $\{\alpha_1, \ldots, \alpha_i\}$.
$g_0 = g$
for $i = 0$ **to** $n - 1$ **do**
  Find the element γ in $\alpha_i^{G^{(i)}}$ and $h \in G^{(i)}$ such that $\alpha_i^h = \gamma$ and $\beta = \gamma^{g_i}$ is minimum (Use Theorem 3.7);
  $g_{i+1} = hg_i$
end
return g_n

Algorithm 1: Lexicographically least in a Right Coset

We can easily extend the above result to show the following.

Theorem 4.7. *For an ordered set Ω there is a polynomial time algorithm that on input a generator set for permutation group G on Ω and $g_1, g_2 \in \mathrm{Sym}(\Omega)$, computes the lexicographically least element of g_1Gg_2. In particular there is a polynomial time algorithm to compute the lex-least element for a left coset gG.*

Proof. Since $g_1Gg_2 = g_1Gg_1^{-1}g_1g_2$ and the lex-least element of g_1Gg_2 is equal to the lex least element of Hg' where $H = g_1Gg_1^{-1}$ and $g' = g_1g_2$. If A is a generator set of G then $g_1Ag_1^{-1} = \{g_1hg_1^{-1} : h \in A\}$ is a generator set for H. The result then follows from Theorem 4.6. □

4.4 The FINDGROUP problem

In this section we study the generic group theoretic problem FINDGROUP. We give an $\mathrm{FP}^{\mathrm{SPP}}$ upper bound for FINDGROUP. Many permutation group theoretic problems AUT, HSP and SETSTAB special cases of FINDGROUP. Hence giving an $\mathrm{FP}^{\mathrm{SPP}}$ bound for FINDGROUP in one stroke gives SPP (or $\mathrm{FP}^{\mathrm{SPP}}$) upper bounds for each of these problems.

We define a generic permutation group problem called FINDGROUP. To each instance $\langle x, 0^n \rangle$ of FINDGROUP there is an associated an unknown subgroup $G_x \leq S_n$ for which there is polynomial time membership test, i.e. there is a polynomial-time function $mem(x, g)$ that takes x and $g \in S_n$ as input and evaluates to **true** if and only if $g \in G_x$. The FINDGROUP problem is to compute a generating set for G_x given $\langle x, 0^n \rangle$ as input. The rest of the section is devoted to the $\mathrm{FP}^{\mathrm{SPP}}$ upper bound for FINDGROUP

Fix an input instance $\langle x, 0^n \rangle$ be an input instance of FINDGROUP. Our goal is to compute a strong generator set for $G_x \leq S_n$ using the membership function $mem(.,.)$ as a subroutine. The input instance being fixed, we will sometimes drop the subscript and write G instead of the group G_x. Let $G^{(i)} \leq G$ denote pointwise stabiliser of $\{1, \ldots, i\}$. Starting from $i = n-1$ we compute a strong generator set for $G^{(i)}$ for decreasing values of i. Assuming that we have a generator set for $G^{(i)}$, we show that a generator set for $G^{(i-1)}$ can be computed in $\mathrm{FP}^{\mathrm{SPP}}$. We give a polynomial time deterministic algorithm making UP-like queries to a language L in NP which we now define. Consider the NP-machine M defined in Algorithm 2 and let L be the language accepted by M.

Input: $x \in \{0,1\}^*$, an integer $0 \leq i \leq n$, a subset $S \subseteq S_n$ and a partial permutation π

Verify using the membership test $mem(.)$ that $S \subseteq G^{(i)}$

1 Guess $g \in G^{(i-1)}$ i.e. guess $g \in S_n$ and verify using $mem(.)$.

Let H be the group generated by S.

Use Theorem 4.6 to compute the lexicographically least element g^* of Hg.

2 **if** $g \neq g^*$ **then Reject**.

3 **if** g^* *extends* π **then Accept**.

else Reject.

Algorithm 2: The NP machine for A

Here by a partial permutation we mean a partial function one-to-one function from $\{1, \ldots, n\}$ to itself. Let the $G^{(i-1)}$-orbit of i be $\{i_1, \ldots, i_k\}$. The set $\{g_1, \ldots, g_k\}$ where $g_s \in G^{(i-1)}$ is any permutation that maps i to i_s forms a right traversal of $G^{(i)}$ in $G^{(i-1)}$. Let g_s^* denote the lexicographically least element in the coset $G^{(i)}g_s$. We have the following proposition for the language L

Proposition 4.8. *Let S be a generator set for the group $G^{(i)}$. Consider a partial permutation π whose domain includes $\{1, \ldots, i\}$. Then the tuple $\langle x, i, S, \pi \rangle \in L$ if and only if π maps i to i_s and agrees with g_s^* for some s. For such an input, the machine M has only one accepting path.*

Proof. The nondeterminism in the definition of M is due to step 1 of Algorithm 2 where we guess an element g of $G^{(i-1)}$. Since $g \in G^{(i-1)}$, $i^g = i_s$ for some s. If S generates $G^{(i)}$ then only the path that guessed g_s^* survives (on all other paths step 2 rejects). Furthermore, if $\langle x, i, S, \pi \rangle \in L$ then π agrees with g_s^* as well (we verified this in step 3). The proposition follows. More generally if S generates a subgroup H of $G^{(i)}$ then the number of accepting paths on such a partial permutation π will be the index $[G^{(i)} : H]$. $\square$

We are ready to give the $\mathrm{FP}^{\mathrm{SPP}}$ algorithm for FINDGROUP. To begin with we already know $G^{(n-1)}$. Assume that a generating set D_i of $G^{(i)}$ is known. From D_i we will compute a right traversal C_i of $G^{(i)}$ in $G^{(i-1)}$ using the language L as oracle. The base algorithm will be a deterministic polynomial time algorithm that makes UP-like queries to L, i.e. for all queries that the machine makes to L the NP-machine of M described in Algorithm 2 will have at most one accepting path. To begin with we have a generating set $D_{n-1} = \{1\}$ of $G^{(n-1)}$. The complete algorithm is give below.

$C_i \leftarrow \emptyset$ for every $0 \leq i \leq n-2$.
$D_i \leftarrow \emptyset$ for every $0 \leq i \leq n-2$.
$D_{n-1} = 1$
for $i = n-1$ **down to** 1 **do**
  Let π_i be the partial permutation that fixes all elements from 1 to $i-1$.
  for $j = i+1$ **to** n **do**
    $\pi' \leftarrow \pi[i := j]$.
    if $\langle x, D_i, i, \pi' \rangle \in L$ **then**
      $C_i \leftarrow C_i \cup g$ where $g =$ `prefixSearch(`x, D_i, i, π'`)`.
    end
  end
  $D_{i-1} \leftarrow D_i \cup C_i$.
end
Result: D_0.
function `prefixSearch(`x, D_i, i, σ`)`
begin
  for $k \leftarrow i+1$ **to** n **do**
    Find the element l not in the range of π' such that $\langle x, 0^n, D_i, i, j, \pi'[k := l] \rangle \in L$ by making queries to L.
    $\sigma := \sigma[k := l]$.
  end
  return σ.
end

Algorithm 3: FP^L algorithm FINDGROUP

By $\pi[l := m]$ we mean the partial permutation σ that agrees with π except at l where its value is m. The function `prefixSearch(`x, i, D_i, π'`)` completes the partial permutation π' to an appropriate g_s^* using L as an oracle.

Proposition 4.9. *The Algorithm 3 computes the generator set for G and for all queries made to L the machine M described in Algorithm 2 has at most one accepting path.*

Proof. The invariant of the loop 1 is that D_i generates the subgroup $G^{(i)}$. In the beginning of the loop the invariant is true. Since inductively we have made sure that D_i generates $G^{(i)}$ by Proposition 4.8 there is at most one accepting path for any queries made, whether in step 3 or in step 4. Hence the polynomial time oracle machine makes only UP-like query to L whether in the main loop or in the subroutine `prefixSearch()`.

Proposition 4.8 also guarantees that the query in step 3 gives a "yes" answer if and only if j is in the orbit $i^{G^{(i-1)}}$. When j is indeed in the orbit $i^{G^{(i-1)}}$ then `prefixSearch()`, by making queries to L, returns the lexicographically least element in the coset $G^{(i)}g$ where g is some permutation in $G^{(i-1)}$ that maps i to j. Since we cycle through all $i < j \leq n$ in the loop 2, C_i will be a right traversal of $G^{(i)}$ in $G^{(i-1)}$ at the end of loop 2. As $D_{i-1} = D_i \cup C_i$ the loop invariant of loop 1 is maintained. Finally when $i = 0$, D_0 is the generator set for G. □

The following theorem is a direct consequence of Proposition 4.9 and Theorem 2.4.

Theorem 4.10. *The FINDGROUP problem is in* $\mathrm{FP}^{\mathrm{SPP}}$.

4.5 The complexity of Graph Isomorphism

We now give SPP upper bound for the Graph Isomorphism. Since GI $\equiv_T^p$ AUT it follows from the closure properties of SPP that it is sufficient to give an $\mathrm{FP}^{\mathrm{SPP}}$ algorithm for AUT. We show that AUT is a special case of the FINDGROUP problem. Without loss of generality assume that the vertex set of the graph is $\{1, \ldots, n\}$. Assume a suitable encoding of graphs say via adjacency matrix. For encodings x of n vertex graph X let G_x be the automorphism subgroup of X. There is a polynomial time membership test for G_x as given the encoding x of a graph X, there is a polynomial time algorithm to test whether a given permutation $g \in S_n$ is an automorphism of X. Hence AUT is a special case of FINDGROUP.

Similarly given permutation group G over Ω and a subset Σ of Ω there is a polynomial time membership test for elements of G_Σ. Hence the SetStab problem is a special case of FINDGROUP. For the hidden subgroup problem the hiding function ϕ gives a membership test: $h \in H$ if and only if $\phi(h) = \phi(1)$. Using Theorem 4.10 we have the main result of this chapter.

Theorem 4.11. *The Graph Isomorphism problem, the hidden subgroup problem over permutation groups, the set-wise stabiliser problem etc. are in* SPP *(or* $\mathrm{FP}^{\mathrm{SPP}}$ *in the case of functional problems).*

4.6 Discussion

We have shown that the Graph Isomorphism problem is in the complexity class SPP. We proved this by shown that given a graph X, a generator set for $\mathrm{Aut}\,(X)$ can be computed by a polynomial time deterministic machine making UP-like queries to an NP language. It is still open whether the Graph Isomorphism problem is in UP.

An approach to Graph Isomorphism is via Graph Canonisation. A function f from $\mathcal{G}(\Omega)$ to $\mathcal{G}(\Omega)$ is a *canonising function* on graphs if it satisfies the following properties: (1) for all $X \in \mathcal{G}(\Omega)$ $f(X)$ is isomorphic to X and (2) graphs X and Y in $\mathcal{G}(\Omega)$ are isomorphic if and only if $f(X) = f(Y)$. Intuitively f pick a *canonical* element from each equivalence class. It is not difficult to see that testing for isomorphism reduces to canonisation. In fact the asymptotically fastest algorithm [75, 12] for Graph Isomorphism is through Graph Canonisation. However the best known complexity theoretic upper bound for Graph canonisation is $\mathrm{FP}^{\mathrm{NP}}$. It would be interesting to show better upper bounds for this problem.

Chapter 5

Bounded colour multiplicity Graph Isomorphism problem

In this chapter we study the *bounded colour multiplicity Graph Isomorphism* problem, a restricted version of vertex coloured Graph Isomorphism problem. For a finite set C of *colours*, a C-coloured graph is a triple $X = (V, E, \psi)$ where V is the set of vertices, $E \subseteq \binom{V}{2}$ is the set of edges and $\psi : V \to C$ is the colouring that assigns to each vertex $v \in V$ a *colour* $\psi(v) \in C$. An isomorphism f between two C-coloured graphs $X_1 = (V_1, E_1, \psi_1)$ and $X_2 = (V_2, E_2, \psi_2)$ if it exists, is an isomorphism from the graph (V_1, E_1) to graph (V_2, E_2) that preserves the colours, i.e. for all $u \in V_1$, $\psi_1(u) = \psi_2(f(u))$. The Coloured Graph Isomorphism problem, for short, is to check whether two vertex coloured graphs X_1 and X_2 are isomorphic. Note that GI is the special case of CGI where every vertex has the same colour. On the other hand using suitable graph gadgets CGI is reducible to GI (details can be found in [33]).

We now define a restricted version of CGI, the bounded colour multiplicity Graph Isomorphism problem or BCGI for short. The colouring map ψ induces an equivalence relation on V; $u \sim_\psi v$ if $\psi(u) = \psi(v)$. A *colour class* is an equivalence class under this equivalence relation. For a colour $c \in C$, the *c-colour class* of X is the equivalence class $\{v \in V : \psi(v) = c\}$.

Definition 5.1 (BCGI_b)**.** *Given two C-coloured graphs X_1 and X_2 such that the size of each colour class of X_i, $i = 1, 2$, is bounded by a constant b, check whether $X_1 \cong X_2$.*

One of the first versions of Graph Isomorphism problem that was studied using group theoretic methods is the BCGI problem. Babai gave a randomised polynomial time algorithm for BCGI_b for each constant b [9]. This was improved to a deterministic polynomial time algorithm by Furst *et al* [28]. Subsequently, Luks [47] gave a remarkable NC algorithm.

Recently Torán [68] has proved various hardness results for Graph Isomorphism. In particular, he proved that BCGI_b is AC^0-many one hard for the logspace counting class $\mathrm{Mod}_k\mathrm{L}$ for each constant k. A key step in the hardness proofs is the construction of certain graph gadgets that enables the simulation of addition modulo k. In fact, these graph gadgets can be used to prove that BCGI is hard for the entire $\mathrm{Mod}_k\mathrm{L}$ hierarchy [8, Appendix].

In this chapter we prove that BCGI_b is in the $\mathrm{Mod}_k\mathrm{L}$ hierarchy, where the constant k and the level of the hierarchy depends on b [8]. Together with the hardness for the $\mathrm{Mod}_k\mathrm{L}$-hierarchy, we have a fairly tight classification. Though not explicitly mentioned, it appears that Luks' NC-algorithm puts BCGI_b in NC^k where the constant k depends on b (Luks solves a more general problem and as a consequence derives the NC algorithm for BCGI_b). Since the $\mathrm{Mod}_k\mathrm{L}$ hierarchy is contained in NC^2 (even TC^1) our result is an improvement on Luks' result.

We first prove that there is a logspace Turing reduction from BCGI_b to the pointwise stabiliser problem PWS_c (definition in Section 5.1) for some constant c that depends only on b. In this sequel we often say that a function f can be computed in the $\mathrm{Mod}_k\mathrm{L}$-hierarchy if there is a logspace bounded oracle machine M^A that computes f for some language A in the $\mathrm{Mod}_k\mathrm{L}$-hierarchy. We prove in this chapter that PWS_c is in the $\mathrm{Mod}_k\mathrm{L}$ hierarchy where k is the product of all primes less than c. This would imply that BCGI_b is the $\mathrm{Mod}_k\mathrm{L}$ hierarchy as $\mathrm{BCGI}_b \leq_T^{\log} \mathrm{PWS}_c$. We now define the problem PWS_c and give an outline of the $\mathrm{Mod}_k\mathrm{L}$ algorithm for it.

5.1 The Pointwise stabiliser problem

Given a permutation group G on Ω and a set $\Delta \subseteq \Omega$, recall that the pointwise stabiliser of Δ, $G(\Delta)$, is the subgroup $\{g \in G : \delta^g = \delta \text{ for all } \delta \in \Delta\}$. As opposed to the setwise stabiliser, the pointwise stabiliser can be computed in polynomial time (using Theorem 3.9 for example). However in this chapter we are interested in a restricted version where G-orbits are of bounded size which we show is in the $\mathrm{Mod}_k\mathrm{L}$-hierarchy. The polynomial time algorithm for the general case that uses Theorem 3.9 does not help us here because it is sequential.

Definition 5.2 (PWS_c)**.** *Let G be a permutation group on Ω such that each G-orbit is of cardinality at most c. Given a subset $\Delta \subseteq \Omega$ compute $G(\Delta)$.*

Recall the permutation group theoretic formulation of the Graph Isomorphism problem from Section 4.1. We generalise this to coloured graphs in a straight forward manner: Let $\mathcal{CG}(\Omega, C)$ denote the set of C-coloured graphs with vertex set Ω. As before there is a natural action of the group $\mathrm{Sym}\,(\Omega)$ on $\mathcal{CG}(\Omega, C)$: The graph (V, E, ψ) goes to (V, E^g, ψ^g) where ψ^g is the map $u \mapsto \psi(u^{g^{-1}})$. For a C-coloured graph $X \in \mathcal{CG}(\Omega, C)$, the automorphism subgroup $\mathrm{Aut}\,(X)$ is the stabiliser of the point X under this action. It is easy to verify that the set of C-coloured graphs in $\mathcal{CG}(\Omega, C)$ isomorphic to X is exactly the $\mathrm{Sym}\,(\Omega)$-orbit containing X. If X and Y are two isomorphic coloured graphs in $\mathcal{CG}(\Omega, C)$ and $g \in \mathrm{Sym}\,(\Omega)$ be such that $X^g = Y$ then, as before, the set of all isomorphisms between X and Y are exactly $\mathrm{Aut}\,(X)g$.

Definition 5.3 (AUT_b)**.** *Given a C-coloured graph X such that each colour class is of cardinality bounded by b compute a generator set for* $\mathrm{Aut}\,(X)$ *as a subgroup of* $\mathrm{Sym}\,(V(X))$.

Mathon's result (Theorem 4.3) generalises to coloured graphs as well and it can be show that the coloured graph isomorphism problem is logspace Turing reducible to coloured automorphism problem. In particular there is a logspace Turing reduction from BCGI_b to AUT_{2b}. To show that BCGI_b reduces to PWS_c for some constant c that depends on b it is therefore sufficient to give a logspace Turing reduction from AUT_b to PWS_c. We sketch the logspace reduction [47, Section 7][1].

Consider an instance $X = (V, E, \psi)$ in $\mathcal{CG}(V, C)$ of AUT_b. Recall that the colouring ψ of X partitions the vertex set V into disjoint colour classes $V_1, \ldots, V_m$ and for each i, $\#V_i \leq b$. Consider the group $G = \prod \mathrm{Sym}\,(V_i)$ acting on V. For $1 \leq i \leq j \leq m$ define the sets $\Omega_{ij} = 2^{V_{ij}}$, where V_{ij} denotes collection of all unordered pairs $\{u, v\}$, $u \in V_i$ and $v \in V_j$. The elements of set Ω_{ij} are subsets of unordered pairs $\{u, v\}$, $u \in V_i$ and $v \in V_j$. In particular consider the collection E_{ij} defined as follows

$$E_{ij} = \{\{u, v\} \in E : u \in V_i \text{ and } v \in V_j\}.$$

The subsets E_{ij} are points of Ω_{ij}. Define $\Omega = \cup\Omega_{ij}$. We have $\#\Omega_{ij} \leq 2^{b^2}$ and $\#\Omega \leq m^2.2^{b^2}$. The action of G on $V(X)$ extends naturally to Ω and hence G is a permutation group on Ω. If $\Delta = \{E_{ij} : 1 \leq i \leq j \leq m\}$ then the pointwise stabiliser of $\Delta \subseteq \Omega$, $G\,(\Delta)$, is the group $\mathrm{Aut}\,(X)$. Furthermore G maps a point in Ω_{ij} to another point in Ω_{ij} and hence G-orbits are of size bounded by $c = 2^{b^2}$. Given the instance $X = (V, E, \psi)$ of AUT_b, a generator

[1]Luks gives a NC-reduction but for a more general version.

set of G as a permutation group on Ω can be computed in FL. Hence the following proposition.

Proposition 5.4. *There is a logspace reduction from AUT_b to PWS_c where $c \leq 2^{b^2}$. Hence if for all constants c there is a constant k that depends only on c such that PWS_c is in the* $\mathrm{Mod}_k\mathrm{L}$*-hierarchy then for all constants b there is a constant k' that depends only on b such that $BCGI_b$ and AUT_b are in the* $\mathrm{Mod}_{k'}\mathrm{L}$*-hierarchy.*

In the rest of the chapter we prove that PWS_c is in the $\mathrm{Mod}_k\mathrm{L}$-hierarchy. We give an outline of the strategy. The key step in our algorithm is what we call "target reduction". Given an instance (G, Ω, Δ) of PWS_c we compute a subgroup G' of G such that

1. $G \geq G' \geq G(\Delta)$.
2. For every G-orbit Σ containing a point of Δ, $G'|_\Sigma$ is a proper subgroup of $G|_\Sigma$.

We prove that target reduction can be performed in the $\mathrm{Mod}_k\mathrm{L}$-hierarchy where k is the product of all primes less than c.

We now argue that the target reduction procedure can be used to compute the pointwise stabiliser. Starting with G, by applying the target reduction procedure compute a subgroup G' which is strictly smaller than G on each of the orbits that contain points of Δ. Since $G \geq G' \geq G(\Delta)$, $G(\Delta) = G'(\Delta)$. Moreover for each G-orbit Σ such that $\Sigma \cap \Delta \neq \emptyset$, the projection $G'|_\Sigma$ is a proper subgroup of $G|_\Sigma$ and hence $\#G'|_\Sigma \leq \frac{1}{2}\#G|_\Sigma$. We then repeat the target reduction procedure with G replaced by G'. Since G-orbits are of size bounded by a constant c, after $O(c.\log c)$ iterations of the target reduction step we converge to $G(\Delta)$. Thus if the target reduction step in in the lth level of the $\mathrm{Mod}_k\mathrm{L}$-hierarchy then PWS_c can be solved in the $l.c.\log c$ level of the $\mathrm{Mod}_k\mathrm{L}$-hierarchy. The detailed description of the target reduction procedure is given in Section 5.5.

For the target reduction procedure we require a special strong generating set for G. We consider a special normal series $G = N_0 \rhd \ldots \rhd N_l = 1$ of length l bounded by a constant that depends only on c such that each of the quotient group N_i/N_{i+1} is T_i-semisimple for some simple group T_i. Using this normal series we compute a strong generator set C of G. The computation of the strong generator set C proceeds in l stages. Each of this stage involves solving a certain normal closure problem for which we give a $\mathrm{FL}^{\mathrm{Mod}_k\mathrm{L}}$ algorithm. The detailed procedure for computing the strong generator set C is described in Section 5.4.

For computing the strong generator set C and for the target reduction procedure we require some more group theory. The two important group theoretic concepts we require is (1) the socle and (2) residual series of a group. The normal series $G = N_0 \trianglerighteq \ldots \trianglerighteq N_l = 1$ which is used to compute the strong generator set C is obtained by patching up the residue series of each of the constant sized groups G_i. The target reduction procedure makes use of the O'Nan-Scott theorem (Theorem 5.9), a result on the structure of socles of primitive permutation groups. In the next two sections we develop the group theory required for this chapter.

5.2 Characteristic subgroups and Socles

In this section we develop some more group theory relevant for this chapter. Most of the group theory that we require, albeit in a slightly different form, is developed by Luks [47] for his NC-algorithm.

Definition 5.5 (Characteristic subgroup)**.** *A subgroup H of a finite group G is a* characteristic subgroup *if all automorphisms of G maps H to itself.*

In this context, notice that a normal subgroup of G is a subgroup that is invariant under inner automorphisms of G whereas a characteristic subgroup is invariant under all automorphisms. Hence characteristic subgroups are normal subgroups. For a characteristic subgroup R of G, the restriction of any G-automorphism to R is an R-automorphism. The following proposition directly follows from the above discussion.

Proposition 5.6.

1. *If R_1 is a characteristic subgroup of G and R_2 is a characteristic subgroup of R_1 then R_2 is also a characteristic subgroup of G.*

2. *If R_1 and R_2 be characteristic subgroups of G then so is R_1R_2 and $R_1 \cap R_2$.*

3. *Let R_1 and R_2 be normal subgroups of G such that $R_1 \cap R_2 = \{1\}$. If R_1R_2 and R_1 are characteristic subgroups of G then so is R_2.*

The entire group G and the trivial subgroup $\{1\}$ are characteristic subgroups of G. The centre $\mathrm{C}_G(G)$ of G, the subgroup of elements of G that commute with all elements of G, is also a characteristic subgroup of G.

Let G be a nontrivial finite group. By a *minimal normal subgroup* of G we mean a normal subgroup $N \trianglelefteq G$ different from $\{1\}$ which is minimal in the containment order, i.e. there is no proper subgroup of N other than $\{1\}$ that is normal in G. For a simple group T, the only minimal normal subgroup is T itself. We now state an important lemma about minimal normal subgroups of a group G ([24, Theorem 4.3A]).

Lemma 5.7. *Let G be any group and let K be a minimal normal subgroup of G. Then $K = T_1 \times \ldots \times T_n$ where T_i's are all isomorphic to a simple group T (i.e. K is T semisimple). Moreover for any i and j there is an element $g \in G$ such that $T_i = g^{-1}T_j g$.*

Having defined the minimal normal subgroup we define the *socle* of a group G, an important characteristic subgroup of G.

Definition 5.8 (Socle)**.** *For a finite group G the* socle $\mathrm{Soc}\,(G)$ *is the subgroup generated by the set of all minimal normal subgroups of G.*

Clearly any automorphism of G maps minimal normal subgroups to minimal normal subgroups and hence fixes the socle. Therefore $\mathrm{Soc}\,(G)$ is a characteristic subgroup of G. We now state a restricted version of the O'Nan-Scott theorem, a theorem on the structure of socles of primitive permutation groups, suitable for our purposes. A complete statement of the theorem (Theorem 4.1A of [24]), its proof and its applications to the study of permutation groups can be found in Chapter 4 of the book by Dixon and Mortimer [24].

Theorem 5.9 (O'Nan-Scott theorem)**.** *Let G be a primitive permutation group on Ω with socle $\mathrm{Soc}\,(G) = K$. Then K is transitive and T-semisimple for some simple group T. Furthermore exactly one of the following is true for K.*

1. *K is abelian in which case K is elementary abelian and regular on Ω. Also K is the unique minimal normal subgroup of G. For an $\alpha \in \Omega$, the group K_α is the trivial group $\{1\}$.*

2. *K is nonabelian and is the unique minimal normal subgroup of G. For $\alpha \in \Omega$, K_α is a proper subgroup of K.*

3. *K is nonabelian and is a product $K = K_1 \times K_2$, where K_1 and K_2 are isomorphic. The subgroups K_1 and K_2 are the only minimal normal subgroups of G and each K_i is regular on Ω. Furthermore the centraliser of K_1 in G, $\mathrm{C}_G(K_1)$, is K_2 and vice-versa. For $\alpha \in \Omega$, K_α is a diagonal subgroup of $K_1 \times K_2$.*

5.3 Residues and Residual Series

In the previous section we studied an important characteristic subgroup, the socle. Let G be a finite group. For any simple group T, we associate a characteristic subgroup of G called its T-residue.

Definition 5.10 (Residue subgroup)**.** *Let T be a finite simple group. For a group G we say that the normal subgroup N is a T-residue of G if G/N is T-semisimple and for all $H \trianglelefteq G$ contained in N, G/H is T-semisimple if and only if $H = N$.*

To prove that T-residues are unique, we require the following two lemmas on normal subgroups of semisimple groups.

Lemma 5.11. *Let G be a semisimple group with a normal subgroup H. Then $G = L \times H$ for some normal subgroup L of G. Moreover G/H is also semisimple.*

Proof. Let G be T-semisimple. Depending on whether T is abelian or not we have two cases.

T is abelian In this case $T = \mathbb{F}_p$ for some prime p. The group G is therefore a vector V over $\mathbb{F}_p$. The subgroup H corresponds to a subspace W of V. We can decompose V as the direct sum $V = W \oplus W'$. The required group L is the subspace W'. Clearly G/H is isomorphic to the subspace L and is hence $\mathbb{F}_p$-semisimple.

T is nonabelian Let $G = T_1 \times \ldots \times T_k$ where each T_i is isomorphic to T. Firstly the projection H_i of H on any of the group T_i is either trivial or the full group T_i. Otherwise H_i will be a nontrivial normal subgroup of T_i which contradicts the fact that T_i is simple. Thus we assume, with out loss of generality, that there is an integer $l \leq k$ such that H projects onto each of the group T_i for $1 \leq i \leq l$ and is trivial on T_j for $l < j \leq k$. By Scott's Lemma H is a product of diagonals of the groups $\{T_i\}_{1 \leq i \leq l}$. Consider any two indices $i, j \leq l$. Any diagonal group $\mathrm{Diag}\,(T_i \times T_j)$ is not a normal subgroup of $T_i \times T_j$. To see this consider an element $\langle a, \psi(a)\rangle \in \mathrm{Diag}_\psi\,(T_1 \times T_2)$ for some isomorphism $\psi : T_i \to T_j$. Let b be any element of T_i that does not commute with a then $\langle 1, \psi(b)\rangle^{-1}\langle a, \psi(a)\rangle\langle 1, \psi(b)\rangle = \langle a, \psi(b^{-1}ab)\rangle \notin \mathrm{Diag}_\psi\,(T_i \times T_j)$. Therefore H is exactly the subgroup $T_1 \times \ldots \times T_l$. The required group L is $T_{l+1} \times \ldots \times T_k$. Clearly $G/H = L$ is T-semisimple. $\square$

The next lemma follows directly from Lemma 5.11 (consider the semisimple group G/N and its normal subgroup H/N).

Lemma 5.12. *Let G be any group with a normal subgroup N such that G/N is semisimple. Let H be any subgroup of G containing N. Then there is a normal subgroup L of G containing N such that $G = LH$ and $L \cap H = N$.*

We now prove that for any simple group T, the T-residue is unique. This a slightly weaker version of Lemma 6.2 stated in Luks [47] and is sufficient for our purpose. The proof of the more general version [47, Lemma 6.2] is along similar lines.

Lemma 5.13 (Luks)**.** *Let G be any finite group. For any simple group T there is a unique T residue, i.e. there is a normal subgroup N of G such that G/N is T-semisimple and for any $H \trianglelefteq G$ such that G/H is T-semisimple, H contains N.*

Proof. The proof is via induction on the order of G. Firstly, if G itself is T-semisimple then lemma is clearly true; the unique T-residue is $\{1\}$. This is the base case of our induction.

We assume the assertion to be true for all groups of order less than k. Consider a group G of order k. If possible, let N_1 and N_2 be two distinct T-residues of G. Let $H = N_1 \cap N_2$ and $N = N_1N_2$. From the minimality of N_i's it follows that H is a strict subgroup of N_i for $i = 1, 2$. We have two cases.

Case 1 ($H \neq \{1\}$): In this case G/H is a group of smaller cardinality than G and hence, by induction hypothesis, has a unique T-residue L/H for some normal subgroup L of G containing H. Since $N_i/H \trianglelefteq G/H$ and $\frac{G/H}{N_i/H} \cong G/N_i$ is T-semisimple, N_i/H contains L/H. Therefore N_i contains L for $i = 1, 2$. Therefore $L \subseteq N_1 \cap N_2 = H$ and since L contains H, $L = H$. However $G/L \cong \frac{G/H}{L/H}$ and hence is T-semisimple. This contradicts the minimality of N_i's.

Case 2 ($H = \{1\}$): We prove that in this case G itself is T-semisimple. Firstly, $N = N_1N_2 = N_1 \times N_2$. Hence the subgroup N_1 is isomorphic to N_1N_2/N_2 and since $N_1N_2/N_2 \trianglelefteq G/N_2$, is itself T-semisimple (Lemma 5.11).

Consider the group G with normal subgroup N_2. The quotient group G/N_2 is T-semisimple and N is a normal subgroup of G containing N_2. Using Lemma 5.12 we have a normal subgroup L of G such that $L \cap N = N_2$ and $G = LN$. Since $N = N_1 \times N_2$ and $L \geq N_2$ it follows that $G = LN_1$. But $L \cap N_1 = \{1\}$ and hence $G = L \times N_1$.

Having proved that $G = N_1 \times L$ it is easy to see that L itself is T-semisimple. This is because L is isomorphic to G/N_1 which is T-semisimple. Hence $G = N_1 \times L$ is T-semisimple. This however contradicts the minimality of N_1 and N_2 as the unique T-residue of G is $\{1\}$. $\square$

In view of Lemma 5.13, we use $\mathrm{Res}_T(G)$ to denote the unique T-residue of G. For any simple group T since the T-residue of G is unique, any G-automorphism has to map $\mathrm{Res}_T(G)$ to itself. Hence $\mathrm{Res}_T(G)$ is a characteristic subgroup of G. Based on residues, we can define an important normal series called the *residual* series.

Definition 5.14 (Residual series)**.** *A* residual series *of G is a series $G = R_0 \trianglerighteq \ldots \trianglerighteq R_l = \{1\}$ where for all $1 \leq i \leq l$, $R_i = \mathrm{Res}_{T_i}(R_{i-1})$ for some simple group T_i.*

In fact from Proposition 5.6 it follows that the residual series is a series of characteristic subgroups. We now prove an important property of residual series of primitive permutation group due to Luks [47, Lemma 6.3].

Lemma 5.15 (Luks)**.** *Let G be a primitive permutation group acting on Ω and let $G = R_0 \triangleright \ldots \triangleright R_t = \{1\}$ be any residual series then the last nontrivial subgroup in the series, is the socle of G.*

Proof. We assume that $t > 1$ for otherwise G itself is T-semisimple, hence is its own socle and we are through.

Let S be the socle of G. The group G being primitive, it follows from the O'Nan-Scott theorem that the socle S and hence all the minimal normal subgroups of G are T-semisimple for some simple group T.

First let us suppose that R_{t-1} does not contain S. Since R_{t-1} is a normal subgroup of G there is a minimal normal subgroup K of G that is contained in R_{t-1}. This rules out cases 1 and 2 of the O'Nan-Scott theorem as in those cases G has a unique normal subgroup which is also the socle S. Thus G has exactly two minimal normal subgroups K_1 and K_2, $S = K_1 \times K_2$ and R_{t-1} contains one of them say K_1. Let s be the largest index i such that R_i contains S. Clearly $s < t-1$ and $R_{s+1} \geq R_{t-1} \neq 1$. Moreover R_{s+1} contains K_1 but not K_2.

The group R_s/R_{s+1} is semisimple and $R_{s+1}S$ is a normal subgroup of R_s containing R_{s+1}. Therefore by Lemma 5.12 we have a subgroup L of R_s such that $R_s = LR_{s+1}S$ and $L \cap R_{s+1}S = R_{s+1}$. However since L contains R_{s+1} and hence K_1 it follows that $R_s = LK_2$. Furthermore $L \cap K_2 = 1$. Thus $R_s = L \times K_2$ and every element of L commutes with K_2. This is possible only if $L = K_1$ as by the O'Nan-Scott theorem $\mathrm{C}_G(K_2) = K_1$.

For a T-semisimple group G, $\mathrm{Res}_{T'}(G)$ is either 1 or the whole of G depending on whether $T' = T$ or not. We have proved that if R_{t-1} does not contain the socle S then $R_s = S$ which is T-semisimple by the O'Nan-Scott theorem. Since R_{s+1} is a proper subgroup of R_s it follows that $R_{s+1} = \mathrm{Res}_T(R_s) = 1$. This however contradicts the fact that $R_{s+1} \geq R_{t-1} \neq 1$. Hence R_{t-1} contains the socle S.

Having proved that R_{t-1} contains S it is easy to see that R_{t-1} is indeed the socle. The group R_{t-1} being T-semisimple there is a subgroup L of R such that $L \times S = R_{t-1}$ (Lemma 5.11). It follows from Proposition 5.6 that L is a characteristic subgroup of G. This is not possible unless L is the trivial group otherwise there is a minimal normal subgroup K of G contained in L and $K \leq S \cap L$.

$\square$

5.4 Strong generator set revisited

Recall that for every decreasing tower of groups $G = G_0 \geq \ldots \geq G_t = \{1\}$ we can associate a generator set called the strong generator set. We now generalise this to relative strong generator set. Let H be a subgroup of G and let $G = G_0 \geq \ldots \geq G_t = H$ be a decreasing sequence of groups from G to H. Let C_i denote the coset representatives of G_i in G_{i-1}. Then the set $C = \cup C_i$ is called a *strong generator set* of G *relative* to H, SGS of G rel H for short. For any element $g \in G$ there is a unique h in H such that $g = g_1 \ldots g_t h$ where $g_i \in C_i$. By sift of g with respect to the strong generator set C we mean this h. We will use $\mathrm{Sift}(g)$ to denote the sift of g with respect the strong generator set C. The sift of an element is not unique and depends on the choice of the coset representatives C_i.

A *semisimple series* from G to a normal subgroup N is a normal series $G = N_0 \trianglerighteq \ldots \trianglerighteq N_t = N$ where the quotient groups N_i/N_{i+1} are T_i-semisimple for simple groups T_i. We associate a strong generator set for such a series. Let the quotient group N_i/N_{i+1} be $\prod_j T_{ij}$ where each T_{ij} is isomorphic to T_i. Consider a normal series (normal in N_i) given by $N_i = N_{i,0} \triangleright \ldots \triangleright N_{i,n_i} = N_{i+1}$ where $N_{i,s}/N_{i+1}$ is the group $\prod_{j>s} T_{ij}$. Let C_{ij} be the right (or left) traversal of $N_{i,j}$ over $N_{i,j+1}$. Then $C = \cup_{i,j} C_{i,j}$ forms a strong generator set for G rel N with respect to the subnormal series $\{N_{i,j}\}$.

We are interested in permutation group G over Ω with bounded orbits. The simple groups $\{T_i\}_{0 \leq i < t}$ that occur will all be of order bounded by a constant and the semisimple series which we construct for G will be of bounded length. Furthermore, the computation of strong generator set C is done inductively by computing the strong generator set of G relative to N_i starting with $i = 0$. The fact that the series $\{N_i\}_{i=1}^t$ is of bounded length is important for the $\mathrm{Mod}_k\mathrm{L}$-hierarchy upper bound. Hence in this context it is more natural to associate the semisimple series $\{N_i\}_{i=1}^t$ to the strong generator set C than the subnormal series $\{N_{i,j}\}$.

We now prove a property analogues to Proposition 3.2 of Luks and McKenzie [50].

Proposition 5.16. *Given a group G via a generator set A. Let $G = N_0 \trianglerighteq \ldots \trianglerighteq N_t = N$ be a semisimple series from G to N and let $C = \cup_{ij} C_{ij}$ be the associated strong generator set of G relative to N. Let S be the set containing the following elements:*

1. $\mathrm{Sift}(g)$ *for all* $g \in A$.
2. $\mathrm{Sift}(x^{-1}yx)$ *for all* $x \in C_{ij}$ *and* $y \in C_{lm}$, $(i,j) < (l,m)$.
3. $\mathrm{Sift}(xy)$ *for all* $x, y \in C_{ij}$ *for all* i *and* j.

Then the normal closure $\mathrm{NCL}_G(S)$ *of* S *in* G *is* N.

Proof. The proof is similar to that of Proposition 3.2 of Luks and McKenzie [50]. The set S is clearly a subset of N and since N is a normal subgroup of G we have $\mathrm{NCL}_G(S) \trianglelefteq N$. To prove the converse consider any element $h \in N$. There exists elements $y_1, \ldots, y_m$ in A such that $h = y_1 \ldots y_l$. For ease of notation we assume that $l = 2$ and $h = xy$ for $x, y \in A$. The general case is similar. Since S contains the sifts of all the elements of A there exists $x_{ij} \in C_{ij}$ and $y_{lm} \in C_{lm}$ such that $x = \prod_{i,j} x_{ij} s_1$ and $y = \prod_{lm} y_{ij} s_2$ where s_1 and s_2 are elements of S and hence $\mathrm{NCL}_G(S)$. Hence h is given by

$$h = \left(\prod_{ij} x_{ij}\right) s_1 \left(\prod_{lm} y_{lm}\right) s_2. \tag{5.1}$$

We prove that h can be written as $\prod_{ij} z_{ij} s$ where $s \in \mathrm{NCL}_G(S)$. The first task is to push down s_1 to the end. For any $y \in G$ since $\mathrm{NCL}_G(S)$ is normal subgroup of G we have $y\mathrm{NCL}_G(S) = \mathrm{NCL}_G(S)y$ and therefore whenever we have a product of the form $h = \ldots sy \ldots$, $s \in \mathrm{NCL}_G(S)$ and $y \in C$, we can replace it with $h = \ldots ys^* \ldots$ for some $s^* \in \mathrm{NCL}_G(S)$.

For products of the form $h = \ldots yx \ldots$ where $x \in C_{ij}$ and $y \in C_{lm}$ with $(i,j) < (l,m)$, since S contains $\mathrm{Sift}(x^{-1}yx)$ we can rewrite it as

$$h = \ldots yx \ldots = \ldots x \left(\prod_{(r,t) > (i,j)} u_{rt}\right) s \ldots, \; u_{rt} \in C_{rt}. \tag{5.2}$$

Similarly when $h = \dots xy \dots$ where $x, y \in C_{ij}$, since S contains $\mathrm{Sift}(xy)$ we can rewrite h as

$$h = \dots xy \dots = \dots z \left(\prod_{r>i} \prod_{t} u_{rt} \right) s \dots, \; u_{rt} \in C_{rt}. \tag{5.3}$$

By repeatedly rewriting the expression of h in Eq. 5.1 using Eqs. 5.2 and 5.3, we have $h = (\prod_i \prod_j z_{ij})s$ for some $s \in \mathrm{NCL}_G(S)$. However since h is in N, we have $z_{ij} = 1$ for all i and j. Therefore $h = s$ and hence $h \in \mathrm{NCL}_G(S)$. $\square$

5.4.1 Computing the strong generator set

We are give a generator set A for a permutation group G on Ω with orbits of size bounded by a constant c. We will find the strong generator set for G with respect to a semisimple series of length bounded by a constant that depends only on c. The semisimple series which we consider is similar to the residual series of G.

Consider a permutation group G on Ω with orbits $\Omega_1, \dots, \Omega_m$ all of size bounded by an constant c. Let G_i's be the projection of G onto Ω_i, Then G_i's are all of order bounded by $c!$ as $\#\Omega \leq c$. Let $\mathcal{T} = \{T_1, \dots, T_k\}$ be the collection of all simple groups of order at most $c!$ then $k = \#\mathcal{T}$ is a constant for us that depends only on c. For $1 \leq i \leq m$ define a k length normal series $G_i = R_{i,0} \trianglerighteq \dots \trianglerighteq R_{i,k}$ where $R_{i,s} = \mathrm{Res}_{T_s}(R_{i,s-1})$. The group $R_{i,k}$ is a proper subgroup of G_i as there exists a normal subgroup H_i of G_i such that G_i/H_i is simple and isomorphic to some T_s. Repeat this process starting with $R_{i,k}$ in place of G_i. We would have to repeat this at most $c.\log c$ times before we hit the trivial group $\{1\}$. Thus, for each $1 \leq i \leq m$, we have a residual series $G_i = R_{i,0} \trianglerighteq \dots \trianglerighteq R_{i,l} = \{1\}$ where the constant l depends only on c. Let R_s denote the product group $R_s = \prod_i R_{i,s}$ then $R_0 \trianglerighteq \dots \trianglerighteq R_l$ is a residual series for the product group $\prod_i G_i$. Since for $1 \leq i \leq m$ the group G_i is of order less that $c!$ in FL we compute the groups $R_{i,s}$ and hence the product groups R_s for each $1 \leq s \leq l$.

Let $N_s \trianglelefteq G$ be the normal subgroup $G \cap R_s$ then $G = N_0 \trianglerighteq \dots \trianglerighteq N_l = \{1\}$ is a semisimple series for G as $N_i/N_{i+1} = (G \cap R_i)/(G \cap R_{i+1}) \hookrightarrow R_i/R_{i+1}$ via the map $xN_{i+1} \mapsto xR_{i+1}$. We prove the following important property due to Luks [47, Lemma 6.4].

Proposition 5.17 (Luks). *Let $H \leq \mathrm{Sym}(\Omega)$ be any subgroup of the product $\prod_i G_i$. For all i, if $H|_{\Omega_i} = G_i$ then $H \cap R_s|_{\Omega_i} = R_{i,s}$, $1 \leq s \leq l$.*

Proof. Let ψ denote the homomorphism that restricts an element of the product group $R_0 = \prod_i G_i$ to its action on Ω_i. Fix an s. Let L and M be the groups $H \cap R_s$ and $H \cap R_{s+1}$ respectively. The groups $H \cap R_s|_{\Omega_i}$ and $H \cap R_{s+1}|_{\Omega_i}$ are $\psi(L)$ and $\psi(M)$ respectively.

First we prove that $\psi(M)$ is a normal subgroup of $\psi(L)$ and the quotient group $\psi(L)/\psi(M)$ is T-semisimple. As $L \leq R_s$ and $M = L \cap R_{s+1}$ the map $gM \mapsto gR_{s+1}$ is an embedding of L/M into R_s/R_{s+1}. The quotient group L/M is thus T-semisimple. Let K be the kernel of the map ψ in L. Consider the normal subgroup MK of L. Since ψ maps K to 1 it follows that $\psi(M) = \psi(MK)$. However MK is a normal subgroup of L containing K and hence $\psi(MK) = \psi(M)$ is a normal subgroup of $\psi(L)$. The quotient group $\psi(L)/\psi(M)$ is thus $\frac{L/K}{MK/K} = L/MK$. However $L \trianglerighteq MK \trianglerighteq M$ is a normal series with L/M being T-semisimple. The group MK/M is a normal subgroup of the semisimple group L/M. Hence by Lemma 5.11 $L/MK \cong \frac{L/M}{MK/M}$ is also T-semisimple. We have thus proved that $\psi(M)$ is a normal subgroup of $\psi(L)$ and the quotient group $\psi(L)/\psi(M)$ is T-semisimple. If $\psi(L)$ is $R_{i,s}$ then this is impossible unless $\psi(M)$ is $R_{i,s+1}$. Let $\psi(H) = \psi(H \cap R_1) = G_i = R_{i,0}$. Assume that $\psi(H \cap R_s) = R_{i,s}$ for some s. Then we have just proved that $\psi(H \cap R_{s+1}) = R_{i,s+1}$. Now repeat the argument with s replaced by $s+1$. As result we have $\psi(H \cap R_j) = R_{i,j+1}$ for all $1 \leq j \leq l$. This completes the proof. $\square$

In particular, Proposition 5.17 proves that for all s, $N_s|_{\Omega_i}$ is $R_{i,s}$. Thus for any G-orbit Σ, $G|_\Sigma = N_0|_\Sigma \trianglerighteq \ldots \trianglerighteq N_l|_\Sigma$ is a residual series for G_i. Hence we call this series a *locally residual* series. We show that a strong generator set for G with respect to this locally residual generator set can be computed in the Mod_kL-hierarchy. A property which we use repeatedly is the following:

Proposition 5.18. *Let N and K be two normal subgroups of G such that $N \geq K$. Let C and D be the strong generator set of G relative to N and N relative to K respectively. Then $C \cup D$ gives a strong generator set of G relative to K.*

Firstly, since each of the groups G_i are constant sized, the residual series $G_i = R_{i,0} \trianglerighteq \ldots \trianglerighteq R_{i,l} = \{1\}$ for each G_i can be computed separately in logspace. We prove by induction on i that an SGS A_i of G rel N_i can be computed in the ith level of the Mod_kL-hierarchy where k is the product of all primes less that c. In addition, we prove inductively that given $g \in G$, $\mathrm{Sift}(g)$ with respect to A_i can also be computed in ith level of the Mod_kL-hierarchy. This sifting procedure is required for our induction step.

To begin with we know the strong generator set of G relative to N_0. Assuming we have already computed the strong generator set A_i of G relative to N_i. Using the sifting procedure for A_i as an oracle, we compute a set S such that $\mathrm{NCL}_G(S) = N_i$ (Proposition 5.16). To complete the induction we give $\mathrm{FL}^{\mathrm{Mod}_k\mathrm{L}}$ algorithms for the following.

(1) Given S and the SGS of G rel N_s compute the strong generator set C of N_s rel N_{s+1}.

(2) Given $x \in N_s$ compute $\mathrm{Sift}(x)$ with respect to the SGS C.

Depending on whether N_s/N_{s+1} is abelian or not we have two cases. If N_s/N_{s+1} is abelian then it is $\mathbb{F}_p$-semisimple for some prime p. We prove that in this case both (1) and (2) can be done in $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$. On the other hand when N_s/N_{s+1} is non-abelian we prove that both (1) and (2) can be done in FL.

Computing the strong generating set: nonabelian case

Let L_i and M_i denote the group $R_{i,s}$ and $R_{i,s+1}$ respectively. Let L and M be the product groups $R_s = \prod_{i=1}^m L_i$ and $R_{s+1} = \prod_{i=1}^m M_i$. Then N_s and N_{s+1} are the $G \cap L$ and $G \cap M$ respectively. Our task is to compute the strong generator set of N_s rel N_{s+1} for which we give an FL algorithm.

The group L/M is T-semisimple as each L_i/M_i is T-semisimple. Consequently, L/M is of the form $T_1 \times \ldots \times T_r$ where $T_i \cong T$ for all $1 \le i \le r$. The quotient group N_s/N_{s+1} can be faithfully embedded into $\prod_{i=1}^m L_i/M_i$ via the map $xN_s \mapsto xM$ and hence can be seen as a subgroup of L/M. Furthermore since N_s projects onto L_i for $1 \le i \le m$ (Proposition 5.17), by Scott's Lemma we know that N_s/N_{s+1} is a product of diagonal groups of $T_1 \times \ldots \times T_r$, i.e. there is a partition $\mathcal{I} = \{I_1, \ldots, I_s\}$ of indices $\{1, \ldots, r\}$ such that

$$N_s/N_{s+1} = \prod_{i=1}^{s} \mathrm{Diag}\left(\prod_{j \in I_i} T_j\right).$$

Let $\phi_i : L \mapsto T_i$ be the homomorphism obtained by composing the natural quotient homomorphism from L to L/M and the projection map to T_i. Fix an index $i_j \in I_j$ for each I_j. Since ϕ_i restricted to N_s is onto (because N_s projects onto L_i) for each $x \in T_{i_j}$ one can associate a permutation x^* in N_s such that $\phi_{i_j}(x^*) = x$ and for all i not in I_j, $\phi_i(x^*)$ is identity. Let B_j be the set of such x^* one for each $x \in T_{i_j}$. The set $\cup_j B_j$ gives strong generator set of N_s rel N_{s+1}. We will show that this strong generator set for N_s rel

N_{s+1} can be computed in FL. To this end we prove that the following can be computed in FL.

1. The partition $\mathcal{I}$.

2. The collection of sets $\{B_j\}_j$

3. Sifts of $g \in N_s$ with respect to the SGS $\cup_t B_t$.

Computing $\mathcal{I}$: For indices i and j we say that i is *linked* to j if i and j falls in the same partition. Clearly i and j are linked if and only if N_s/N_{s+1} restricted to $T_i \times T_j$ is a diagonal group Diag $(T_i \times T_j)$. The relation $i \sim j$ if i is linked to j, is an equivalence relation and the equivalence classes give the partition $\mathcal{I}$. Consider an undirected graph $\mathcal{G}$ with vertex set $V = \{1, \ldots, r\}$ and edge set $\{\{i, j\} : i \text{ and } j \text{ are linked}\}$. Each connected component $\mathcal{C}_k$ in $\mathcal{G}$ corresponds to diagonal part of N_s/N_{s+1}. Hence to compute $\mathcal{I}$ it is sufficient to compute the connected components of $\mathcal{G}$.

To compute the graph $\mathcal{G}$, it is sufficient to give an algorithm to check whether T_i and T_j are linked. For this we compute N_s/N_{s+1} restricted to $T_i \times T_j$. Let ϕ_{ij} denote the projection of L/M to $T_i \times T_j$. We give an FL algorithm (Algorithm 4) that computes a subset $D_{i,j}$ of elements in N_s such that the projection from $D_{i,j}$ to $T_i \times T_j$ is $\phi_{ij}(N_s)$. Since $\phi_{ij}(N_s)$ is of order bounded by a constant that depends only on c, one can easily determine whether it is $T_i \times T_j$ or Diag $(T_i \times T_j)$ (by checking the order for example).

Initialise $D_{i,j}$ to be the set of x^* one for each $x \in \phi_{ij}(S)$.
repeat
  Let S' be the set $g^{-1}sg$ for each $g \in A$ and $s \in D_{i,j}$.
  Add to $D_{i,j}$ all elements s in S' such that no two elements of $D_{i,j}$ have the same image under ϕ_{ij}.
until $D_{i,j}$ *is not modified* ;
return $D_{i,j}$

Algorithm 4: Computing N_s/N_{s+1} restricted to $T_i \times T_j$

Using Algorithm 4 we compute the edges of the graph $\mathcal{G}$. The graph $\mathcal{G}$ is a disconnected set of cliques one for each diagonal component. In FL we compute its connected components. Let $\mathcal{C}_1, \ldots, \mathcal{C}_s$ be the connected components of $\mathcal{G}$. Then the vertices of $\mathcal{C}_k$ gives us I_k. Thus in FL we compute the partition $\mathcal{I}$.

Computing B_k: To compute the set B_k the main algorithmic step is the computation of elements $g_k \in N_s$, $1 \leq k \leq s$ such that $\phi_{i_k}(g_k) \neq 1$ and

$\phi_{i_j}(g_k) = 1$ for all j not equal to k. Given two i and j such that T_i and T_j are not linked, using Algorithm 4 one can compute an element g that is nontrivial on T_i and trivial on T_j. However we want elements g_k that is nontrivial on T_{i_k} and trivial on all other T_{i_j} simultaneously. We make use of the following proposition.

Proposition 5.19. *Let x and y be permutations in N_s. Let X denote the indices i such that $\phi_i(x)$ is trivial. Similarly let Y be the set of all j such that $\phi_j(y)$ is trivial. Then the commutator $[x, y]$ has the property that $\phi_j([x, y]) = 1$ for all $j \in X \cup Y$.*

Proof. For all $i \in X$ since $\phi_i(x) = 1$ we have $\phi_i([x, y]) = \phi_i(x^{-1}y^{-1}xy) = \phi_i(y^{-1}y) = 1$. Similarly for all $j \in Y$, $\phi_j([x, y]) = 1$. Therefore for all $k \in X \cup Y$ $\phi_k([x, y]) = 1$. □

We use Proposition 5.19 to compute the required permutations g_k. For this purpose we need *iterated commutators.* Let $[h_1, \ldots, h_k]$ be defined as

$$\begin{aligned} [h_1, h_2] &= h_1^{-1}h_2^{-1}h_1h_2, \\ [h_1, \ldots, h_i, h_{i+1}] &= [[h_1, \ldots, h_i], h_{i+1}]\,. \end{aligned}$$

To compute g_k we compute a sequence of elements $h_1, \ldots, h_s$ satisfying the following properties.

1. $\phi_{i_k}(h_1) \neq 1$,
2. $\phi_{i_k}([h_1, \ldots, h_j]) \neq 1$ for all j and
3. $\phi_{i_j}(h_j) = 1$ for all $1 \leq j \leq s$ and $j \neq k$.

It follows from Proposition 5.19 that given $h_1, \ldots, h_s$ with the above mentioned properties, $g_k = [h_1, \ldots, h_s]$ has the required properties: $\phi_{i_k}(g_k)$ is nontrivial and $\phi_{i_j}(g_k) = 1$ for $1 \leq j \leq s$ and $j \neq k$. We give the logspace algorithm (Algorithm 5) to find such a sequence $h_1, \ldots, h_s$.

In Algorithm 5 the step 1 is possible only because T_{i_k} is nonabelian and simple. The simplicity of T_{i_k} guarantees that its centre is trivial and hence for any nontrivial element g of T_{i_k} there is an $h \in T_{i_k}$ such that g and h do not commute. The loop invariant is that g's value is $\phi_{i_k}([h_1, \ldots, h_j]) \neq 1$. Step 1 ensures that (1) $\phi_{i_k}([h_1, \ldots, h_j]) \neq 1$ and (2) $\phi_{i_j}(h_j) = 1$. Therefore Algorithm 5 indeed computes a sequence $h_1, \ldots, h_s$ with the desired properties.

Having got the sequence $h_1, \ldots, h_s$ we show that the iterated commutator $[h_1, \ldots, h_s]$ can be computed in logspace. It is sufficient to compute the

Let h_i be any permutation such that $\phi_{i_k}(h_1) \neq 1$. Such an element has to exist in the set S itself.
$g \leftarrow \phi_{i_k}(h_1)$
for $j = 1$ **to** s *and* $j \neq k$ **do**
    Using Algorithm 4 find an h_j such that $\phi_{i_k}(h_j)$ does not commute with g and $\phi_{i_j}(h_j) = 1$.
    $h \leftarrow \phi_{i_k}(h_j)$
    $g \leftarrow [g, h]$
    output h_j
end

Algorithm 5: Computing h_i's

action of $[h_1, \ldots, h_s]$ separately for each G-orbit. The iterated commutator $[h_1, \ldots, h_s]$ is a formula over the h_i's, and since each G-orbit is of bounded size, the action of $[h_1, \ldots, h_s]$ restricted to a G-orbit Ω_i can be computed by a bounded width branching program. Hence the iterated commutator can be computed in FL (in fact ever in NC^1). Thus we have the following proposition.

Proposition 5.20. *Let G be a permutation group with bounded-size orbits. Given $h_1, \ldots, h_n \in G$, the iterated commutator $[h_1, \ldots, h_n]$ can be computed in deterministic logspace.*

Using Algorithm 5 and Preposition 5.20, for all $1 \leq k \leq s$, we compute in FL a permutation $g_k \in N_s$ such that $\phi_{i_k}(g_k) \neq 1$ and for all $1 \leq j \leq s$, $j \neq k$ $\phi_{i_j}(g_k) = 1$.

Finally from the permutations g_k, we now describe how the set B_k can be computed. Since T_{i_k} is simple, $T_{i_k} = \phi_{i_k}(\mathrm{NCL}_G(g_k))$. We compute a set B_k of distinct inverse images of $\phi_{i_k}(\mathrm{NCL}_G(g_k))$, $1 \leq k \leq l$. Start with $B_k = \{g_k\}$. The algorithm consists of $\#T$ stages in which we update B_k. At every stage update B_k by adding, for every element g in the generating set for G and $x \in B_k$, the elements $y = g^{-1}xg$ to B_k if $\phi_{i_k}(y) \notin \phi_{i_k}(B_k)$. We repeat this process till $\phi_{i_k}(B_k)$ generates T_{i_k}. Since $\#T_{i_k} = \#T \leq c!$ we require at most $c!$ stages each of which is in FL. Thus the sets B_k, $1 \leq k \leq s$ can be computed in FL.

Having computed the sets B_k, we compute the strong generator set $B = \cup_{k=1}^{s} B_k$ of N_s rel N_{s+1}.

Computing sifts:

Finally we explain how to compute $\mathrm{Sift}(x)$ for any $x \in N_s$ with respect to the computed strong generator set $B = \cup_{k=1}^{s} B_k$ of N_s rel N_{s+1}. Given $x \in N_s$ in FL we compute for each, $1 \le k \le s$, a permutation $y_k \in B_k$ such that $\phi_{i_k}(y_k) = (\phi_{i_k}(x))^{-1}$. The sift of x is given by $\mathrm{Sift}(x) = x\prod_{k=1}^{s} y_k$. This completes the nonabelian case of our induction step.

Computing the strong generating set: abelian case

We are given a set $S \subset N_s$ such that $\mathrm{NCL}_G(S) = N_s$. Our task is to compute the strong generator set of N_s rel N_{s+1}. Since N_s/N_{s+1} is semisimple and abelian, it is $\mathbb{F}_p$-semisimple for some prime $p \le c$. First we describe $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ algorithms for some basic linear algebraic problems over $\mathbb{F}_p$ that follows from the results of Buntrock *et al* [20] . These will be used as subroutines in our $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ algorithm for computing the strong generator set of N_s rel N_{s+1}.

Proposition 5.21. *For a prime p consider the vector space $V = \mathbb{F}_p^r$ then*

1. *Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of V. Given $\mathbf{v} \in V$, in* $\mathrm{Mod}_p\mathrm{L}$ *we can check whether $\mathbf{v}$ is contained in the subspace U of V spanned by $\mathcal{B}$. Furthermore, if $\mathbf{v} \in U$ then in* $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ *we can compute $a_1, \ldots, a_n \in \mathbb{F}_p$ such that $\mathbf{v} = \sum_{i=1}^{n} a_i\mathbf{v}_i$.*

2. *Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of V not necessarily linearly independent and let U be the subspace of V spanned by $\mathcal{B}$. Then in* $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ *we can compute a subset $\mathcal{B}' \subseteq \mathcal{B}$ such that $\mathcal{B}'$ is a basis for U.*

Proof. Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ denote the standard basis for $V = \mathbb{F}_p^r$ and let $\mathbf{v}_i = \sum_{j=1}^{m} v_{i,j}\mathbf{e}_j$ for $1 \le i \le n$. Let $\mathbf{v} = \sum_{j=1}^{m} v_j\mathbf{e}_j$. Let A be the matrix $(v_{i,j})$, $1 \le i \le n$ and $1 \le j \le m$. Let $\mathbf{b}$ the column vector $(v_1, \ldots, v_m)^T$. Then the vector $\mathbf{v}$ is in the span of $\mathcal{B}$ if and only if the system of linear equation $A\mathbf{x} = \mathbf{b}$ has a solution. Furthermore if $x_i = a_i$, $1 \le i \le n$ is a solution to $A\mathbf{x} = \mathbf{b}$ then $\mathbf{v} = \sum_{i=1}^{n} a_i\mathbf{v}_i$. Part 1 then follows from Theorem 2.5.

To prove part 2 consider the $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ algorithm that cycles over all $1 \le i \le n$ and outputs $\mathbf{v}_i$ if it is not in the span of the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}\}$. Clearly the output $\mathcal{B}'$ is a basis of the vector space spanned by $\mathcal{B}$. □

We fix some notations: Recall that $R_{i,s}/R_{i,s+1}$ is isomorphic to vector space over $\mathbb{F}_p$ which we denote by V_i. Let V be the direct sum $\oplus_{i=1}^{m} V_i$. Then R_s/R_{s+1} is isomorphic to V. For a permutation $x \in R_s$ let $\mathbf{v}_x$ denote the image of xR_{s+1} under the above mentioned isomorphism. If x and y are permutations in L then for integers a and b it follows that $\mathbf{v}_{x^a y^b} = \tilde{a}\mathbf{v}_x + \tilde{b}\mathbf{v}_y$ where $\tilde{a}$ and $\tilde{b}$ are the elements $a \pmod p$ and $b \pmod p$ of $\mathbb{F}_p$ respectively. Furthermore the vector space structure of R_s/R_{s+1} is obtainable effectively in logspace, i.e. for an element $x \in R_s$ one can compute the image $\mathbf{v}_x$ of the coset xR_{s+1} in V in FL. This is because each of the groups $R_{i,s}$ are constant sized. Since $\frac{N_s}{N_{s+1}} \hookrightarrow R_s/R_{s+1}$ it is isomorphic to a subspace of V which denote by W. We are given a subset $S \subseteq N_s$ such that $\mathrm{NCL}_G(S) = N_s$.

The group $\mathrm{NCL}_G(S)$ is the group generated by the set $\{g^{-1}sg | s \in Sg \in G\}$ and hence the conjugation action can be seen as a linear action of G on V as we now explain: For each element $g \in G$, g maps $\mathbf{v}_h$, $h \in R_s$, to the vector $\mathbf{v}_{h^*}$ where $h^* = g^{-1}hg$. Since both R_s and R_{s+1} are normalised by G, each $g \in G$ is an invertible linear transformation from V to V.

First $S \subseteq N_s$ and N_s is a normal subgroup of G. Therefore $\mathrm{NCL}_H(S)$ is a subgroup of N_s that is closed under conjugation by elements of H. Thus we have the following observation of Luks and McKenzie [50] about the normal closure $\mathrm{NCL}_H(S)$.

Proposition 5.22 (Luks and McKenzie)**.** *Let $H \leq G$ and let W be the subspace $\{\mathbf{v}_x | x \in \mathrm{NCL}_H(S)\}$. Then W is the smallest subspace of V containing $\{\mathbf{v}_s | s \in S\}$ and closed under the action of elements of H*

We compute the generator set of $\mathrm{NCL}_{N_j}(S)$ rel N_{s+1} inductively starting with $j = s$ down to $j = 0$ using the SGS of N_j rel N_s that is already computed. Let U_j denote the subspace of V associated to $\mathrm{NCL}_{N_j}(S)/N_{s+1}$ then it follows from 5.22 that U_j is the closure of $\{\mathbf{v}_s | s \in S\}$ under N_j. We compute a basis for U_j.

To begin with since N_s/N_{s+1} is commutative. It follows that $S \cup N_{s+1}$ is a generating set for $\mathrm{NCL}_{N_s}(S)$ and hence U_s is spanned by $\{\mathbf{v}_s | s \in S\}$. Assume that we have already computed a basis for U_{j+1}. Our task is to compute a basis for U_j using the basis for U_{j+1} and the strong generator set C of N_j rel N_{j+1}. The vector space U_j is the span of gU_{j+1} where g ranges over the distinct coset representative of N_{j+1} in N_j.

Proposition 5.23. *Given a basis for U_{j+1} we can compute a basis for U_j in* $\mathrm{Mod}_p\mathrm{L}$.

Proof. Recall that N_j/N_{j+1} is T-semisimple for some simple group T. Since U_{j+1} is stabilised by N_{j+1} we can assume that N_j/N_{j+1} is acting on U_{j+1}. Depending on whether T_j is abelian or not we have two cases.

T is non-abelian We prove that in this case it is sufficient to find the closure of U_{j+1} under all monomials M over C which are of degree bounded by a constant c' that depends only on c. Recall that $R_j/R_{j+1} = T_1 \times \ldots \times T_n$ for some integer n and there is a partition $\mathcal{I} = \{I_1 \ldots I_r\}$ such that

N_j/N_{j+1} is product of diagonal groups Diag $\left(\prod_{i\in I_k} T_i\right)$, $1 \leq i \leq r$. Each of the diagonal component Diag $\left(\prod_{i\in I_k} T_i\right)$ is isomorphic to T and the strong generator set C is the union $C = \cup C_k$ where C_k consists of one element $g \in N_j$ for each $gN_{j+1} \in$ Diag $\left(\prod_{i\in I_k} T_i\right)$.

Consider any element $g \in C$. We say that g is trivial on Ω_i if $g|_{\Omega_i} \in R_{i,j+1}$. For each $g \in C$ there exists $h_g \in N_{j+1}$ such that $g|_{\Omega_i} = h|_{\Omega_i}$ for all Ω_i for which g is trivial. Consider the elements $\mu_g = h_g - g$, $g \in C$. Then U_j is the space spanned by MU_{j+1} where M ranges over all (non-commutative) monomial in $\{\mu_g : g \in C\}$. If $g \in C_i$ and $h \in C_j$, $i \neq j$ then since $gh = hgx$ for some $x \in N_{j+1}$ we can assume that $P(g)$ and $Q(h)$ commutes for any two polynomials $P(X)$ and $Q(X)$. Any monomial M in μ_g's can therefore be assumed to be in the form $\mu_1 \ldots \mu_n$ where μ_i is either 1 or $h_g - g$ for some $g \in C_i$.

For any i if g is trivial on Ω_i then $\mu_g V_i = 0$. Since each orbit Ω_i is of cardinality at most c there exists a constant c' that depends only on c such that for any orbit Ω_i there are at most c' distinct μ_g's that are non-zero on V_i. Consider a monomial $M = \mu_1 \ldots \mu_n$ of degree $n > c'$. For any V_i there is a μ_k such that $\mu_k V_i = 0$. Therefore since V is the direct sum $V_1 \oplus \ldots \oplus V_m$, $MV = 0$. As a consequence to obtain U_j it is sufficient to take the closure of U_{j+1} with respect to monomials in $\{\mu_g | g \in C\}$ of total degree bounded by c'. In FL we can enumerate all monomials over C of degree bounded by c'. Hence U_j is the obtained by taking the span of MU_{j+1} where M is a monomial in C of degree at most c. A basis of for U_j can then be computed in $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ using Proposition 5.21.

T is abelian The vector space U_{j+1} is closed action of N_{j+1}. Therefore as far as computing the closure of U_{j+1} is concerned we assume that the group algebra of the quotient group N_j/N_{j+1} is acting on V. In this case the group algebra of N_j/N_{j+1} is abelian. Therefore elements g and h can be thought of as commuting linear transformations over V. Also there is a prime $q < c$ such that $g^q - 1 = 0$. In $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ we can find a set $\mathcal{T}$ of elements in the group algebra N_j/N_{j+1} such that the U_j is the span of $\{\tau U_{j+1} : \tau \in \mathcal{T}\}$. $\square$

We now give the $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ algorithm for computing the strong generator set of N_s rel N_{s+1} problem. Let W denote the subspace of $\{\mathbf{v}_x | x \in N_s\}$. It follows from Proposition 5.23 that a basis $\mathcal{B}$ for the space W. We can keep track of the entire permutations: Whenever we add the vector $g\mathbf{v}_x$ into $\mathcal{B}$ we add the corresponding permutation $g^{-1}xg$ into B. We thus have a subset B of N_s such that $\{\mathbf{v}_x | x \in B\}$ spans W. Let $B = \{x_1, \ldots, x_n\}$ then, for $1 \le i \le n$, define the set $C_i = \{x_i^a : 1 \le a \le p-1\}$. The set $\cup_{i=1}^n C_i$ is the strong generator set of N_s rel N_{s+1}. Clearly C can be computed in $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$.

Finally, we describe how to compute $\mathrm{Sift}(x)$ for any $x \in N_s$ with respect to the above mentioned strong generator set. In logspace compute the vector $\mathbf{v}_x \in V$ corresponding to the permutation x. Using Proposition 5.21 compute $a_i \in \mathbb{F}_p$ such that $\mathbf{v}_x = \sum a_i \mathbf{v}_{x_i}$. The sift of x is given by $\mathrm{Sift}(x) = x \prod_{i=1}^r x_i^{-a_i}$. This completes the abelian case of our induction step.

We have thus shown that the strong generator set of G rel N_{s+1} can be computed inductively starting from $s = 0$. Since the locally residual series $G = N_0 \trianglerighteq \ldots \trianglerighteq N_l$ is of length l bounded by a constant in c we have the following theorem.

Theorem 5.24. *Let G be a permutation group with orbits of size bounded by a constant c. Given a generator set A for G, we can compute the strong generator set for G with respect to the locally residual series in the* $\mathrm{Mod}_k\mathrm{L}$-*hierarchy. The constant k is the product of all primes less than c and the level of the hierarchy depends only on c.*

5.5 The target reduction procedure

Our goal is to show that PWS_c is in the $\mathrm{Mod}_k\mathrm{L}$-hierarchy. The heart of the algorithm is the target reduction procedure: Given a instance (G, Ω, Δ) of PWS_c, we compute a subgroup G' of G containing $G(\Delta)$ such that for each G-orbit Ω' that contains a point of Δ, $G'|_{\Omega'}$ is a proper subgroup of $G|_{\Omega'}$. In this section we show that target reduction can be performed in the $\mathrm{Mod}_k\mathrm{L}$-hierarchy.

Let $\Omega_1, \ldots, \Omega_m$ be the set of G-orbits. We fix some terminologies and conventions local to this section. Points in Δ will be called *target points*. *Target orbits* are G-orbits that contain target points, i.e. orbits Ω_i such that $\Omega_i \cap \Delta \neq \emptyset$.

Firstly, if $\Sigma \subseteq \Delta$ then $G(\Sigma) \ge G(\Delta)$. Let Σ be the subset of Δ that contains exactly one target point from each target orbit. Any target orbit will continue to be a target orbit even if we replace Δ by the subset Σ, i.e. if G' be the group obtained by performing target reduction on (G, Ω, Σ) then $G' \ge G(\Delta)$ and $G'|_{\Omega'}$ is a proper subgroup of $G|_{\Omega'}$ for all target orbit Ω'. Therefore as far as target reduction is concerned, we can assume that the instance (G, Ω, Δ) is such that each G-orbit contains at most one target point.

We make an additional assumption that G acts primitively on each target orbit which we justify now. Consider a structure forest $\mathcal{F} = \{\mathcal{T}_1, \ldots, \mathcal{T}_m\}$ of G where $\mathcal{T}_i$ is the structure tree of the transitive action of G on Ω_i. Let Ω^* denote the vertices of $\mathcal{F}$. We identify the set Ω with the set of leaf nodes of $\mathcal{F}$. Recall from Section 3.4 that G's action on Ω can be extended to Ω^* such that given the action of an element $g \in G$ on Ω^*, we can recover its action on Ω. Furthermore, all the G-orbits of Ω^* are of size bounded by c. In FL we can compute the structure forest $\mathcal{F}$ of G by separately computing the structure tree $\mathcal{T}_i$ for each $1 \leq i \leq m$. Furthermore for a given g in G, in FL we can compute the action of g on the Ω^*.

Let $\Omega_i^* \subseteq \Omega^*$, $1 \leq i \leq m$, denote the children of the root of $\mathcal{T}_i$. Then for every $1 \leq i \leq m$, the set Ω_i^* is an orbit and G acts primitively on it (Theorem 3.12). Let Δ^* be the ancestors of elements of Δ in $\cup_{i=1}^m \Omega_i^*$.

Proposition 5.25. *The group $G(\Delta^*)$ contains $G(\Delta)$ and for any subgroup H of G, if $H|_{\Omega_i^*} < G|_{\Omega_i^*}$ then $H|_{\Omega_i} < G|_{\Omega_i}$.*

Proof. Consider any $\delta^* \in \Delta^*$. There is a $\delta \in \Delta$ such that δ^* is the ancestor of δ in the structure forest of G. Let Σ be the leaves of the structure tree rooted at δ^* then Σ is a G-block that contains δ (Section 3.4). Thus for a $g \in G$ if $\delta^g = \delta$ then $\Sigma^g = \Sigma$ and hence $\delta^{*g} = \delta^*$. This proves that $G(\Delta^*) \geq G(\Delta)$.

Consider any subgroup H of G. Recall that the action of G on the structure tree $\mathcal{T}_i$ depends only on the action of $G|_{\Omega_i}$ on Ω_i. Hence if $H|_{\Omega_i} = G|_{\Omega_i}$ then $H|_{\Omega_i^*} = G|_{\Omega_i^*}$. □

Proposition 5.25 proves that target reduction for the instance (G, Ω, Δ) can be achieved by performing target reduction on (G, Ω^*, Δ^*). Summarising the above discussions, for target reduction we assume that the given instance (G, Ω, Δ) has the following properties.

1. All G-orbits are of size bounded by a constant c.
2. G acts primitively on each target orbit.
3. Each target orbit contains a unique target point.

In this section we use the following notation: X denotes the set of all i such that Ω_i is a target orbit. For each index $i \in X$, δ_i denotes the unique target point in Ω_i. Identifying the indices of X, the corresponding target orbits and the target points leads to no confusion. Hence for a subset of Z of X, by target orbits of Z we mean the collection $\{\Omega_i | i \in Z\}$. Similarly by target points of Z we mean the set $\{\delta_i | i \in Z\}$.

Overview of the target reduction step

First we use Theorem 5.24 to compute the strong generator set of G with respect to the locally residual series $G = N_0 \trianglerighteq \ldots \trianglerighteq N_l = \{1\}$. In fact in the $\mathrm{Mod}_k\mathrm{L}$-hierarchy we obtain the following.

1. For each i a residual series $G_i = R_{i,0} \trianglerighteq \ldots \trianglerighteq R_{i,l} = 1$ such that, $R_{i,s+1} = \mathrm{Res}_{T_s}(R_{i,s})$.
2. The product group $R_s = \prod_i R_{i,s}$ and
3. The groups $N_s = G \cap R_s$.

Consider any target orbit Ω_i. Recall that $N_s|_{\Omega_i} = R_{i,s}$ (Proposition 5.17) and hence $G_i = N_0|_{\Omega_i} \trianglerighteq \ldots \trianglerighteq N_l|_{\Omega_i} = \{1\}$ is a residual series for G_i. Since G_i acts primitively on Ω_i, the last nontrivial group in this series is $\mathrm{Soc}(G_i)$ (Lemma 5.15). Let X_s denote the set of all i such that $N_s|_{\Omega_i} = \mathrm{Soc}(G_i)$ and $N_{s+1}|_{\Omega_i} = \{1\}$. We have $\cup X_s = X$.

The target reduction is done inductively in l stages where in the sth stage we handle the target orbits in X_s. Inductively we compute the generator sets of the sequence of groups $G = H_0 \geq \ldots \geq H_{l+1}$ such that for all s, $H_s \geq G(\Delta)$ and $H_s|_{\Omega_i}$ is a proper subgroup of $G|_{\Omega_i}$ for each i in $\cup_{j=0}^{s-1} X_s$. In fact the group H_s that we compute in the sth stage will contain N_s. Since $X = \cup_{j=0}^{l} X_s$, the group H_{l+1} is the required group G'.

To begin with $H_0 = G$. Inductively assume that we have computed a generator set A_s of H_s. To compute H_{s+1} we first identify a subset $Y_s \subseteq X_s$ of critical indices. A subset Y of X is said to be *critical* if it has the following properties:

1. Let N be the subgroup of N_s that fixes all the target points in Y then $N|_{\Omega_i} < N_s|_{\Omega_i}$ for all $i \in X_s$.
2. For any $x \in G$ there is a $y \in N_s$ such that $x^* = xy$ fixes all the points of Y.

Proposition 5.26. *Let H be the subgroup of H_s that fixes all the target points in a critical subset Y of X_s then $H_s \geq H \geq G(\Delta)$ and for all $i \in X_s$ $H|_{\Omega_i} < G_i$.*

Proof. Let Δ' be the subset of Δ containing all the target points of Y. By induction hypothesis $H_s \geq G(\Delta)$ and hence $H = H_s(\Delta') \geq G(\Delta)$. The group $H_s \cap R_s = N_s$ we have $H \cap R_s = N$. Since $N|_{\Omega_i} < R_{i,s}$ for all $i \in X_s$, $H \cap R_s|_{\Omega_i} < R_{i,s}$. Therefore by Proposition 5.17 we have $H|_{\Omega_i} < G_i$ for all $i \in X_s$. □

Recall that our goal is to compute a subgroup H_{s+1} of H_s that contains $G(\Delta)$ and is strictly smaller that G_i on Ω_i for each $i \in X_s$. By Proposition 5.26 it is sufficient to choose H_{s+1} to be the subgroup of H_s that fixes all the target points in Y for some critical subset Y of X_s. In the sth stage of the algorithm we identify a critical subset Y_s which is *effective*, i.e. given any $x \in G$, in $\mathrm{FL}^{\mathrm{Mod}_k\mathrm{L}}$ we can compute a $y \in N_s$ such that $x^* = xy$ fixes all the points in Y_s. The subgroup H_{s+1} is the subgroup of H_s that fixes all the target points of Y_s.

We now show how a generator set for H_{s+1} can be computed. Let A_s be the a generator set for H_s. Since Y_s is critical for each $x \in A_s$ there is a $y \in N_s$ such that $x^* = xy$ fixes all the target points in Y_s. Let A_s^* denote the set $\{x^* | x \in A_s\}$.

Proposition 5.27. *Let N be the subgroup of N_s that fixes all the target points of Y_s and let B be a generator set for N. Then $A_s^* \cup B$ generates the subgroup H_{s+1}.*

Proof. First, we claim that $A_x^* N_s$ generates H_s. Consider any $g \in H_s$. Since A_s generates H_s, for some integer $t \geq 0$ there exists t elements $x_1, \dots, x_t$ in A_s such that g is the product $\prod_{i=1}^{t} x_i$. For any $x \in A_s$ there is an element $y \in N_s$ such that $x^* = xy$ is contained in A_s^*, Therefore we have for $1 \leq i \leq t$ elements $x_i^* \in A_s^*$ and $y_i \in N_s$ such that $g = x_1^* y_1 \dots x_t^* y_t$. This proves our claim.

The group N_s is a normal subgroup of G and hence is also a normal subgroup of H_s. Therefore every element $g \in H_s$ can be written as $g = g_1 g_2$ where g_1 is contained in the group generated by A_s^* and $g_2 \in N_s$. Furthermore, since every element of A_x^* fixes all the target points of Y_s, so does g_1. As a consequence any element of H_{s+1}, the subgroup of H_s that fixes the target points of Y_s, is of the form uv where u is in the group generated by A_s^* and $v \in N$. Hence $A_s^* N$ generates the group H_{s+1} and if B is a generator set of N, $A_s^* \cup B$ generates H_{s+1}. □

To complete the inductive procedure for target reduction it is thus sufficient to perform the following subtasks.

1. Compute a critical subset $Y_s \subseteq X_s$.
2. Given $x \in G$ compute a $y \in N_s$ such that xy fixes each of the target points of Y_s.
3. Compute a generator set of the subgroup N of N_s that fixes each of the target points of Y_s.

Depending on whether N_s/N_{s+1} is abelian or not we have two case. When N_s/N_{s+1} is abelian then N_s/N_{s+1} is $\mathbb{F}_p$-semisimple for some prime $p \leq c$. In this case we show that the steps 1, 2 and 3 can be done in $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$. On the other hand when N_s/N_{s+1} is nonabelian then the steps 1, 2 and 3 can be done in FL. We explain these two case in the next two subsections.

5.5.1 Computing the critical orbits: abelian case

In this case the quotient group N_s/N_{s+1} is $\mathbb{F}_p$-semisimple for some prime $p \leq c$. Let $\Omega_1, \ldots, \Omega_m$ be the G-orbits and let $G_i = G|_{\Omega_i}$. Then by the O'Nan-Scott theorem $\mathrm{Soc}\,(G_i)$ is regular on Ω_i, i.e. $\mathrm{Soc}\,(G_i)$ is transitive on Ω_i and for any $\delta \in \Omega_i$, the subgroup of $\mathrm{Soc}\,(G_i)$ that fixes δ is the trivial group. As a consequence we have the following property.

Proposition 5.28. *Let Y be any subset of X_s and let K be the subgroup of N_s that fixes the target points of Y. Then we have.*

1. *$K \trianglelefteq G$.*
2. *For any i in X_s either $K|_{\Omega_i}$ is trivial or is* $\mathrm{Soc}\,(G_i)$.
3. *For any $i \in X_s$ such that $K|_{\Omega_i}$ is nontrivial and for any two elements δ and δ' of Ω_i, there is an element h of K such that $\delta^h = \delta'$.*

Proof. The group K is a subgroup of N_s and hence for all $i \in X_s$, $K|_{\Omega_i}$ is a subgroup of $\mathrm{Soc}\,(G_i)$. Moreover K fixes the target point δ_i of Ω_i for each $i \in Y$. Therefore the projection of K onto Ω_i is the subgroup of $\mathrm{Soc}\,(G_i)$ that fixes δ_i. However since $\mathrm{Soc}\,(G_i)$ is regular on Ω_i, $K|_{\Omega_i}$ is trivial for all $i \in Y$. Thus K is the intersection of the groups N_s and $\prod_{i \notin Y} R_{i,s}$. The group G normalises the group $\prod_{i \notin Y} R_{i,s}$. Also $N_s \trianglelefteq G$. Hence their intersection K is a normal subgroup of G. This proves part 1.

Consider an $i \in X_s$. As argued before since $K \leq N_s$, $K|_{\Omega_i}$ is a subgroup of $\mathrm{Soc}\,(G_i)$. Suppose that $K|_{\Omega_i}$ is nontrivial. Then since K is a normal subgroup of G and since $G|_{\Omega_i} = G_i$, it follows that $K|_{\Omega_i}$ is a normal subgroup of G_i. However by the O'Nan-Scott theorem $\mathrm{Soc}\,(G_i)$ is the unique minimal normal subgroup of G_i. Therefore $K|_{\Omega_i} = \mathrm{Soc}\,(G_i)$ which proves part 2.

Finally consider an i in X_s such that $K|_{\Omega_i}$ is nontrivial. The group $\mathrm{Soc}\,(G_i)$ is regular on Ω_i (O'Nan-Scott theorem). Hence for any two elements δ and δ' of Ω_i we have an element $g \in \mathrm{Soc}\,(G_i)$ such that $\delta^g = \delta'$. Since $K|_{\Omega_i} = \mathrm{Soc}\,(G_i)$, part 3 then follows as there is an element $g^* \in K$ such that $g^*|_{\Omega_i} = g$. ∎

We fix the following notation for this subsection. Recall that N_s is the intersection of the product group $R_s = \prod_{i=1}^{m} R_{i,s}$ and G (Subsection 5.4.1). The quotient groups $R_{i,s}/R_{i,s+1}$ is $\mathbb{F}_p$-semisimple and hence is a vector space V_i over $\mathbb{F}_p$. Let V be the direct sum $\oplus_{i=1}^{m} V_i$ then R_s/R_{s+1} is isomorphic to V. We identify the vector space V with the quotient group R_s/R_{s+1} under this isomorphism, i.e. for every element $x \in R_s$ we associate isomorphic image $\mathbf{v}_x$ of x in V. Clearly for any x and y in R_s/R_{s+1} the vector $\mathbf{v}_{x+y}$ is $\mathbf{v}_x + \mathbf{v}_y$ and for any integer a $\mathbf{v}_{x^a} = \tilde{a}\mathbf{v}_x$, where $\tilde{a} \in \mathbb{F}_p$ is the element $a \pmod p$.

Proposition 5.29. *Given any element $x \in R_s$, in* FL *we can compute the vector $\mathbf{v}_x$.*

Proof. Let $\mathbf{w}_i$ denote the projection of $\mathbf{v}_x$ onto the vector space V_i. Since the vector space V is the direct sum of the subspaces V_i, $\mathbf{v}_x = \sum_{i=1}^{m} \mathbf{w}_i$. The order of the group $R_{i,s}$ is at most $c!$ as the size of the orbit Ω_i is less than c. Hence in FL we can compute the projection $\mathbf{w}_i$ and thus compute $\mathbf{v}_x$. □

The quotient group N_s/N_{s+1} is a subgroup of the R_s/R_{s+1} (more precisely $N_s/N_{s+1} \hookrightarrow R_s/R_{s+1}$) and hence is a subspace U of V.

Proposition 5.30. *Given the strong generator set C of N_s, in* $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ *a subset $B = \{x_1, \ldots, x_r\}$ of C can be computed such that the vectors $\mathbf{v}_{x_1}, \ldots, \mathbf{v}_{x_r}$ forms a basis of U.*

Proof. The subset $\mathcal{C} = \{\mathbf{v}_x | x \in C\}$ of V spans U. Using Proposition 5.21 in $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ compute a subset $\mathcal{B}$ of $\mathcal{C}$ that forms a basis of U. For each $\mathbf{v} \in \mathcal{B}$ pick a permutation $x \in C$, say the lexicographically least such that $\mathbf{v}_x = \mathbf{v}$ and form the subset B of C. Clearly B can be computed in $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$. □

Our goals are (1) to compute a critical subset Y_s of X_s, (2) compute a generator of the subgroup N of N_s that fixes all the points of Y_s and (3) give a $x \in G$ compute a $y \in N_s$ such that $x^* = xy$ fixes all the target points of Y_s. By Proposition 5.28 it follows that the subgroup K of N_s that fixes some of the target points of X_s is such that for all i either $K|_{\Omega_i} = 1$ or $K|_{\Omega_i} = \mathrm{Soc}\,(G_i)$. The subset Y_s of X_s that we choose will be a minimal subset of target points such that the subgroup of N_s that fixes the points of Y_s will be trivial on all the target orbits of X_s. First we prove the following proposition that will be used to identify Y_s and later on to compute the set N.

Proposition 5.31. *Given any subset Y of X_s there is a $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ algorithm to compute the generator set of the subgroup K of N_s that fixes all the target points of Y.*

Proof. Since N_{s+1} is trivial on all the target orbits it follows that K contains N_{s+1}. Let W denote the vector space associated with the quotient group K/N_{s+1}. From Proposition 5.28 it follows that K is trivial on all the orbits of Y. Therefore K is the intersection of the groups N_s and $\prod_{j \notin Y} R_{i,s}$. Hence W is the subspace $U \cap \oplus_{i \notin Y_s} V_i$.

In $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ we first compute the set $B = \{x_1, \ldots, x_r\}$ such that $\mathcal{B} = \{\mathbf{v}_{x_1}, \ldots, \mathbf{v}_{x_r}\}$ is a basis for U (Proposition 5.30). Consider the projection of U on to the space $\oplus_{i \in Y_s} V_i$. Then W is the kernel of this projection. Let A be the matrix associated to this projection with respect to the basis $\mathcal{B}$. The subspace W consists of all vectors $\sum a_i \mathbf{v}_{x_i}$ which are solutions of the linear equation $A\mathbf{x} = 0$. Using Theorem 2.5 we can compute a basis $\mathbf{u}_1, \ldots, \mathbf{u}_t$ for W. In fact the algorithm outputs the elements $a_{ij} \in \mathbb{F}_p$ such that $\mathbf{u}_i = \sum_{j=1}^{r} a_{ij} \mathbf{v}_{x_j}$. Let $g_i = \prod_{j=1}^{r} x_j^{a_{ij}}$ then clearly the set $D = \{g_1, \ldots, g_t\}$ generates the quotient group K/N_{s+1}. As part of the strong generator set of G we have already computed a strong generator set C' of N_{s+1}. The set $C' \cup D$ gives a generator set for K. $\square$

We now present the $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ algorithm (Algorithm 6) for computing Y_s.

$Y \leftarrow \emptyset$.
foreach $i \in X_s$ **do**
  Let K_i be the subgroup of N_s that fixes target points of Y.
1 **if** *K is nontrivial on Ω_i* **then** $Y \leftarrow Y \cup \{i\}$
end
Return the set $Y_s = Y$.

Algorithm 6: Computing Y_s

For the step 1 in $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ we first compute the generator set D_i of K_i (Proposition 5.31). The group K_i is trivial on i if and only if all the elements of D_i is trivial on Ω_i. Thus step 1 can be performed by making a query to a $\mathrm{Mod}_p\mathrm{L}$ oracle and hence Algorithm 6 is a $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ procedure. Having computed Y_s using Proposition 5.31 we compute in $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ the generator set of the subgroup N of N_s that fixes all the target points of Y_s. To complete the abelian case we show that for each $x \in G$, x^* can be computed in $\mathrm{FL}^{\mathrm{Mod}_k\mathrm{L}}$.

Proposition 5.32. *Given any $x \in G$ there is a $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ algorithm to compute an element y of N_s such that xy fixes all the points of Y_s.*

Proof. Without loss of generality assume that $Y_s = \{1, \dots, t\}$ for some integer t. Let δ_i denote the target point in Ω_i. Let $K_0 = N_s$ and for $1 \leq i \leq t$ let K_i denote the subgroup of N_s that fixes the target points $\delta_1, \dots, \delta_i$.

First, we prove by induction that there are elements $h_i \in K_i$, $0 \leq i < t$, such that $xh_0 \dots h_i$ fixes every element of the set $\{\delta_1, \dots, \delta_{i+1}\}$. Let x map δ_1 to δ'_1. Since $K_0|_{\Omega_1}$ is transitive on Ω_1 there is an element $h_0 \in K_0$ that maps δ'_1 to δ_1. Hence xh_0 fixes δ_1. Inductively assume that there exists elements h_j, $0 \leq j < i$ such that $xh_0 \dots, h_{i-1}$ fixes the target points $\delta_1, \dots, \delta_i$. Let $xh_0 \dots h_{i-1}$ map δ_{i+1} to δ'_{i+1}. Since K_i is nontrivial on Ω_{i+1}, it follows from part 3 of Proposition 5.28 that there is an element $h \in K_i$ that maps δ'_{i+1} back to δ_{i+1}. Let $h_i = h$. The group K_i is trivial on all the G-orbits $\Omega_1, \dots, \Omega_i$ and therefore $xh_1 \dots h_i$ fixes all the points in the set $\{\delta_1, \dots, \delta_{i+1}\}$. The element $y = h_0 \dots h_{t-1} \in N_s$ is such that xy fixes all the target points of Y_s. We now give the $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ algorithm for computing y.

Let x map δ_i to ν_i. We want to compute an element $y \in N_s$ that maps ν_i to δ_i for all $i \in Y_s$. To this end consider the vector space $V' = \oplus_{i \in Y_s} V_i$ and let U' be the projection of U onto V'. Recall that the groups $R_{i,s}$ and $N_{s+1}|_{\Omega_i}$ are trivial for all $i \in Y_s$. Therefore the vector spaces V' and U' are isomorphic to the groups R_s and N_s restricted to the target orbits of Y_s. For an element $x \in R_s$ let $\mathbf{u}_x$ be the image of x in V' under this restriction. In fact $\mathbf{u}_x$ is the projection of $\mathbf{v}_x$ onto V'. Analogues to Proposition 5.30, using Proposition 5.21 we compute in $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ a subset $B' = \{x_1, \dots, x_t\}$ of B such that $\mathcal{B}' = \{\mathbf{u}_{x_1}, \dots, \mathbf{u}_{x_t}\}$ forms a basis for U'. First we show that $\mathbf{u}_y$ can be computed in FL and then we recover the permutation y in $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$.

Consider an $i \in Y_s$. Since $R_{i,s}$ is a constant sized transitive permutation group on Ω_i, in $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$ we can compute an element $y_i \in R_{i,s}$ that maps ν_i back to δ_i. The group $N_s|_{\Omega_i} = \mathrm{Soc}\,(G_i)$ and by the O'Nan-Scott theorem $\mathrm{Soc}\,(G_i)$ has a regular action on Ω_i. The element $yy_i^{-1} \in R_s$ fixes the point ν_i and therefore restricted to Ω_i is trivial. As a result if $\mathbf{w}_i$ denotes the projection of $\mathbf{u}_y$ onto V_i then $\mathbf{w}_i = \mathbf{v}_{y_i}$. Since $R_{i,s}$ is a group of order bounded by $c!$, in FL we can compute the vector $\mathbf{w}_i$. The vector $\mathbf{u}_y$ is given by $\sum_{i \in Y_s} \mathbf{w}_i$ which can also be computed in FL.

To complete the algorithm we need to recover y from the vector $\mathbf{u}_y$. Using Proposition 5.21 we compute, in $\mathrm{FL}^{\mathrm{Mod}_p\mathrm{L}}$, elements $a_1, \dots, a_t \in \mathbb{F}_p$ such that $\mathbf{u}_y = \sum_{i=1}^{t} a_i \mathbf{u}_{x_i}$. The permutation $y = \prod_{i=1}^{t} x_i^{a_i}$ is the required element of N_s. $\square$

5.5.2 Computing the critical orbits: nonabelian case

Consider any $i \in X_s$. By O'Nan-Scott theorem Soc (G_i) is either K (type 2) for some minimal normal subgroup K of G_i or is of the form $K_1 \times K_2$ (type 3) where K_1 and K_2 are the only minimal normal subgroups of G_i. For each $i \in X_s$, by a *socle part* associated to i we mean a minimal normal subgroup of G_i. For a T-semisimple group $L = T_1 \times \ldots \times T_r$ by a *simple part* we mean one of the subgroup T_i.

The quotient group N_s/N_{s+1} is a subgroup of the T_s-semisimple group R_s/R_{s+1}. Hence by Scott's Lemma (Lemma 3.6), N_s/N_{s+1} is a product of diagonals of simple parts of R_s/R_{s+1}. Consider two simple parts T' and T'' of R_s/R_{s+1}. As before we say that T' and T'' are *linked* if in N_s/N_{s+1}, T' and T'' are in the same diagonal component. We now extend the "linking" relation to socle parts. Any socle part K is the product of certain subset of simple parts of R_s/R_{s+1}. We say that the socle parts K' and K'' are *linked* if $K' = \ldots \times T' \times \ldots$ and $K'' = \ldots \times T'' \times \ldots$ such that T' and T'' are linked. For socle parts K' and K'' we prove that either they are fully linked or are unlinked.

Proposition 5.33. *Let $K' = T'_1 \times \ldots \times T'_u$ and $K'' = T''_1 \times \ldots \times T''_v$ be two socle parts. If K' and K'' are linked then $u = v$ and there is a permutation $\pi \in S_u$ such that T'_i is linked to T''_{i^π}.*

Proof. Let K' and K'' be socle parts corresponding to orbits Ω' and Ω''. Let us assume without loss of generality that T'_1 is linked to T''_1. Since K' is the minimal subgroup of $G|_{\Omega'}$, for any i there is an element $g \in G$ such that $g^{-1}T'_1 g = T'_i$ (Lemma 5.7). The element g maps via conjugation T''_1 to some T''_i. Thus for any simple part T'_i in K' there is a simple part T''_i in K'' such that T'_i and T''_i are linked. However no two simple parts of K' are linked. Each simple part of K' therefore, is linked to distinct simple part of K''. By interchanging the role of K' and K'' we can prove the converse. As a result, we have $u = v$ and $\pi \in S_u$ is the permutation that maps i to j if T'_i is linked to T''_j. $\square$

Recall that for target reduction our goal is to (1) compute the set Y_s of critical orbits, (2) compute a generator set of subgroup N of N_s that fixes all the points of Y_s and (3) for each $x \in G$ an element $y \in N_s$ such that $x^* = xy$ is trivial on all the target points of Y_s. We now show that each of these three tasks can be achieved by an FL algorithm.

Computing Y_s

Let $\mathcal{K}$ be the collection of socle parts of orbits of X_s. To construct critical subset Y_s of X_s consider the graph $\mathcal{G} = (\mathcal{K}, \mathcal{E})$ where the edge set $\mathcal{E}$ is partitioned into the set of red edges $\mathcal{R}$ and the set of blue edges $\mathcal{B}$. The red edges $\mathcal{R}$ consists of all unordered pairs $\{K_1, K_2\}$ where K_1 and K_2 are linked. On the other hand the blue edges consists of all unordered pairs $\{K_1, K_2\}$ such that K_1 and K_2 are distinct socle parts of the same G-orbit. We have the following proposition about the structure of the graph $\mathcal{G}$.

Proposition 5.34. *The red subgraph, i.e. $\mathcal{G}_{\mathrm{red}} = (\mathcal{K}, \mathcal{R})$, consists of disconnected cliques and any blue edge is between two disconnected red cliques.*

Proof. The "linking" relation is an equivalence relation and hence the red subgraph consists of disconnected red cliques. Any blue edge is between two socle parts of the same G-orbit. Hence they cannot be linked. Therefore blue edges are always between two disconnected red cliques in the red subgraph. □

In logspace we compute the set $\mathcal{C}$ of red cliques in the red subgraph $\mathcal{G}_{\mathrm{red}}$. We partition the set $\mathcal{C}$ into subsets $\mathcal{C}'$ and $\mathcal{C}''$, where a red clique C is put in $\mathcal{C}'$ if C contains an element $K = \mathrm{Soc}\,(G_i)$ for some $i \in X_s$. The remaining cliques are put in $\mathcal{C}''$. We now construct the subset of critical orbits Y_s as the union of Y_s' and Y_s''.

The set Y_s' consists of one index i per clique $C \in \mathcal{C}'$ such that $K = \mathrm{Soc}\,(G_i) \in C$. Shrink all the red cliques in $\mathcal{G}$ and delete all vertices (and blue edges incident on them) that corresponds to cliques in $\mathcal{C}'$. Call the new graph $\mathcal{G}'$. In $\mathcal{G}'$, compute the lexicographically first spanning forest of blue edges. Let $\mathcal{B}'$ be the blue edges in the spanning forest. Recall that each $e \in \mathcal{B}'$ corresponds to the orbit Ω_i where $\mathrm{Soc}\,(G_i) = K \times K'$. The subset Y_s'' of the critical subset X_s'' consist of such indices i corresponding to edges in $\mathcal{B}'$. We prove the following proposition

Proposition 5.35. *The set $Y_s = Y_s' \cup Y_s''$ can be computed in logspace. Let N be the subgroup of N_s that fixes all the target points of Y_s then $N|_{\Omega_i} \leq R_{i,s}$ for all $i \in X_s$.*

Proof. The sets Y_s' and Y_s'' can be computed in logspace as this involves reachability in undirected graphs (Lemma 2.1).

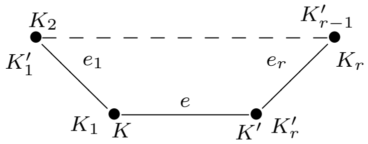
Let Δ' be the subset containing all the target points of Y_s and let $N = N_s\,(\Delta')$. Depending on whether $\mathrm{Soc}\,(G_i)$ has one or two socle parts we have the following two cases.

Case 1: The socle $\mathrm{Soc}\,(G_i)$ is itself a socle part K. Consider the red clique $C \in \mathcal{C}'$ that contains K. By the construction of Y'_s there is a $j \in Y'_s$ such that $\mathrm{Soc}\,(G_j) = K' \in C$. Since K' and K are linked, any $h \in N_s$ when restricted to $\Omega_i \cup \Omega_j$ is of the form $\langle \phi(h'), h' \rangle$, $h' \in K'$, for some isomorphism ϕ from K' to K. Therefore N when restricted to Ω_i is $\phi(K'_{\delta_j})$. By O'Nan-Scott's theorem K' is transitive and hence K'_{δ_j} is a proper subgroup of K'. Thus N restricted to Ω_i is also a proper subgroup of $K = R_{i,s}$.

Case 2: The socle $\mathrm{Soc}\,(G_i)$ is of the form $K \times K'$. Then there is blue edge $e = \{K, K'\}$ in the graph $\mathcal{G}$. Firstly if e is one of the edges in the maximal spanning forest $\mathcal{B}'$ then N fixes the target point corresponding to e. The group $N|_{\Omega_i}$ is a diagonal group $\mathrm{Diag}\,(K \times K')$ (O'Nan-Scott theorem) and hence a proper subset of $K \times K'$. Thus we have disposed the case when e is an edge of the spanning forest $\mathcal{B}'$.

We now handle the case when e is not an edge of the spanning forest $\mathcal{B}'$. Suppose that the edge $e = \{K, K'\}$ connects the distinct red cliques C_1 and C_2 with $K \in C_1$ and $K' \in C_2$. If C_1 (or C_2) is a clique in $\mathcal{C}'$ then there is a $j \in Y'_s$ such that $\mathrm{Soc}\,(G_j) = K_j \in C_1$ (or C_2). By an argument similar to the Case 1 it follows that N restricted to K is a strict subgroup K'' isomorphic to the subgroup of K_j that fixes δ_j. Hence N restricted to Ω_i is $K'' \times K'$ which is a strict subgroup of $R_{i,s} = K \times K'$.

Suppose that both C_1 and C_2 are cliques of $\mathcal{C}''$. Then since $\mathcal{B}'$ forms a maximal spanning forest, adding edge e to $\mathcal{B}'$ gives a cycle $e_1, \ldots, e_r, e$ (see the figure below).



Let Ω_{j_t} be the orbit that corresponds to the edge e_t and let $\mathrm{Soc}\,(G_{j_t}) = K_t \times K'_t$. The group N fixes all the points $\delta_{j_t} \in \Omega_{j_t}$. By case 3 of O'Nan-Scott theorem it follows that $N|_{\Omega_{j_t}}$ is the diagonal group $\mathrm{Diag}\,(K_t \times K'_t)$. Note that K_1 and K'_t are linked to K and K' respectively in N_s. Hence the group N restricted to Ω_i is a diagonal group $\mathrm{Diag}\,(K \times K')$ which is a strict subgroup of $R_{s,t} = K \times K'$. $\square$

Computing x^*

Given an $x \in G$ we give an FL algorithm to compute a $y \in N_s$ such that $x^* = xy$ fixes all target points in Y_s. For any $i \in Y_s$ let the target point δ_i

of Ω_i be mapped to ν_i. Then we want to find a y in N_s that maps ν_i back to δ_i.

Proposition 5.36. *Given an $i \in X_s$. There is an* FL *algorithm to compute a subset $D_i \subseteq N_s$ elements such that (1) the projection of D_i to* $\mathrm{Soc}\,(G_i)$ *is one-to-one and (2) for any socle part K' of* $\mathrm{Soc}\,(G_j)$*, $j \in X_s$, D_i projected to K' is trivial if K' is not linked to any of the socle parts of* $\mathrm{Soc}\,(G_i)$*.*

Proof. Let $R_s/R_{s+1} = T_1 \times \ldots \times T_u$ where each T_i is isomorphic to T. Since $N_s/N_{s+1} \hookrightarrow R_s/R_{s+1}$ there exists a partition $\mathcal{I} = \{I_1, \ldots, I_t\}$ of indices $1, \ldots, u$ such that N_s/N_{s+1} is the product $\prod_{k=1}^{t} \mathrm{Diag}\left(\prod_{j \in I_k} T_j\right)$. We have computed the strong generator set C of N_s rel N_{s+1} as part of the SGS of G. Recall that the strong generator set C of N_s rel N_{s+1} consists of subset $C_1, \ldots, C_t$ where the subset C_k corresponds to the diagonal group $\mathrm{Diag}\left(\prod_{j \in I_k} T_j\right)$, i.e. the projection of C_k on T_j is the group T_j if $j \in B_i$ and 1 otherwise.

Let $x_1, \ldots, x_r$ be the elements of C whose action on Ω_i is nontrivial. Then for any other socle part K_j that is not linked to any of the socle parts of $\mathrm{Soc}\,(G_i)$, x_i's are trivial on K_j. Furthermore if z_i denotes the projection of x_i onto $\mathrm{Soc}\,(G_i)$, then $z_1, \ldots, z_r$ generates $\mathrm{Soc}\,(G_i)$. For each element $z \in K$ we express z as a product $z = z_{i_1} \ldots z_{i_k}$. Include into D_i the element $x_z = x_{i_1} \ldots x_{i_k}$. Since $\mathrm{Soc}\,(G_i)$ is a constant sized group and each x_i's are elements of the group G with constant sized orbits, D_i can be computed in FL.

□

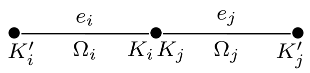
Remark 5.37. Consider the SGS $C = \cup_{k=1}^{t} C_k$ of N_s rel N_{s+1} where C_k corresponds to the diagonal component $\mathrm{Diag}\left(\prod_{j \in I_k} T_j\right)$. For a $j \in X_s$ the elements of C_k is nontrivial on Ω_j if and only if a $\mathrm{Soc}\,(G_j) = \ldots \times T_r \times \ldots$ and $r \in I_k$. It follows from the proof of Proposition 5.36 that we can ensure $C_k \subseteq D_j$ for any j such that C_k is nontrivial on Ω_j. Therefore the set $\cup_{i \in X_s} D_i \cup C'$ is a generator set for N_s rel N_{s+1} where C' denotes the elements of C that are trivial on all the target orbits.

We now prove that given any $x \in G$ we can compute in FL an element $y \in N_s$ such that $x^* = xy$ fixes all the target points of Y_s. Intuitively we want to choose an element y in N_s that "negates" the effect of x on δ_i for all $i \in Y_s$

First we handle the target points in Y'_s. Each $i \in Y'_s$ can be handled independent of the other target points in Y_s, i.e. we can compute elements

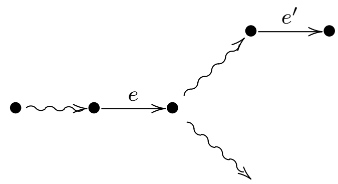
y_i such that xy_i fixes δ_i and for all j in $Y_s \setminus \{i\}$, $y_i|_{\Omega_j}$ is 1. That such an element exists follows from the fact that (1) $\mathrm{Soc}\,(G_i)$ is transitive and (2) none of the socle parts of $\mathrm{Soc}\,(G_j)$ is linked to $K_i = \mathrm{Soc}\,(G_i)$ for all $j \in Y_s \setminus \{i\}$. We compute y_i in FL using Proposition 5.36.

The target points in Y_s'' however cannot be handles independently, i.e. the choice of a y_i for some $i \in Y_s''$ will have a nontrivial action on some of the other target orbits in $j \in Y_s''$ as illustrated below. Recall that for each $i \in Y_s'$ there is an edge $e_i \in \mathcal{B}'$. The difficulty arises for i and j in Y_s'' for which the edges e_i and e_j share a common vertex in $\mathcal{G}'$ (see figure below).



In such a case $\mathrm{Soc}\,(G_i) = K_i \times K_i'$ and $\mathrm{Soc}\,(G_j) = K_j \times K_j'$ and K_i and K_j are linked. Thus any nontrivial element chosen from K_i will have a nontrivial action on Ω_j. Any element y_j that we choose for the orbit Ω_j has to negate this "propagated" effect. The main idea therefore is to systematically choose elements y_e for each edge in $\mathcal{B}'$ keeping in view this propagated effect.

Recall that we have computed a lexicographically least maximal spanning forest $\mathcal{F}$ of $\mathcal{G}'$ with edge set $\mathcal{B}' \subseteq \mathcal{B}$. Each tree $\mathcal{T}$ in the forest $\mathcal{F}$ can be considered as a rooted tree with root at the lexicographically least vertex of $\mathcal{T}$. Thus each edge in $\mathcal{B}$ acquires a direction: the tail at the vertex closer to the root (see figure below). This gives a partial order on $\mathcal{B}'$: edges $e < e'$ if e and e' belong to the same tree and the unique path from the root to the tail of e' contains the edge e.

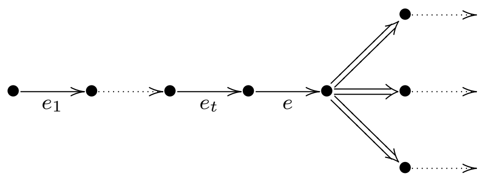


Since the vertices of $\mathcal{G}'$ corresponds to red-cliques in $\mathcal{G}$ which in turn corresponds to linked socle parts the following proposition is direct.

Proposition 5.38. *Let e be any edge with tail at K_1 and head at K_2. Fix an edge $e' \not> e$ and let Ω_i be the orbit corresponding to e'. Then the socle part K_2 is not linked to any of the socle parts of* $\mathrm{Soc}\,(G_i)$.

Consider any edge $e \in \mathcal{B}'$ and let $\mathcal{T}_e$ be rooted tree in the forest $\mathcal{B}'$ containing e. We give an FL algorithm (Algorithm 7) to compute a permutation

$y_e \in N_s$ that is trivial on all orbits corresponding to edges $e' \not> e$ and for the unique path $e_1, \ldots, e_t$ from the root of $\mathcal{T}_e$ to the tail of e, negates the action of $xy_{e_1} \ldots y_{e_t}$ on the target point corresponding to e. In fact y_e will be trivial on all orbits other than those that corresponds to e and edges going out of the head of e (indicated by a double arrow in the figure).



The main idea behind algorithm is that once the path $e_1, \ldots, e_t$ is computed (which can be done in FL using Reingold's algorithm [57]), the propagated effect of y_{e_i}'s on the orbit associated to e can be kept track of using constant amount of space. This is because for each i since $e_i < e_j$ the permutation y_{e_i} is trivial on all the orbits associated to e_j for $j > i+1$. Let e be the directed edge (K_1, K_2) then since K_2 is transitive on the orbit associated to e, the permutation y_e will be chosen such that y_e does not effect any of the orbits associated to edges $e' \not> e$. Proposition 5.38 guarantees that such a y_e exists.

Proposition 5.39. *Algorithm 7 is a* FL *algorithm.*

Proof. The algorithm can be seen as the composition of two stages (1) computing the paths $e_1, \ldots, e_t$ (step 1 of Algorithm 7) and (2) computing y_e (loop 2 of Algorithm 7). Computing the path involves applying Reingold's s-t connectivity algorithm [57] (also Lemma 2.1) repeatedly and hence is in FL. The permutation y_i (step 3 of Algorithm 7) is computed by first computing the set D_i (using Proposition 5.36). Recall that D_i projects onto $\mathrm{Soc}(G_i) = K_1 \times K_2$ and K_2 is transitive (O'Nan-Scott). Hence in FL by examining all the $\#\mathrm{Soc}(G_i)$ elements of D_i we can find a desired y_i. All these can be achieved by logspace bounded computations. □

Let $e_1, \ldots, e_r$ denote a topological sorting of edges in $\mathcal{B}'$ and let $y'' = y_{e_1} \cdots y_{e_r}$ then y'' is trivial on all orbits of Y_s' and xy'' fixes all the target points in Y_s''. Recall that we have already computed a y' such that y' is trivial on all the orbits in Y_s'' and xy' fixes all points of Y_s'. Let $x^* = xy'y''$.

Proposition 5.40. *The element $x^* \in G$ fixes all the points in Y_s.*

Proof. We prove by induction. Let $z = xy'$ then as argued before z fixes all the target points on Y_s'. Consider any topological ordering $e_1, \ldots, e_r$ of

Input: The permutation $x \in G$ and an edge $e \in \mathcal{B}'$
Output: The permutation y_e

1 Compute a path $e_1, \ldots, e_t = e$ from the corresponding root of the tree containing e to e.
Let Σ_i be the G-orbit associated to e_i.
Let δ_i be the target point associated with the edge e_i.
$y^* \leftarrow 1$.
2 **for** $i = 1$ **to** t **do**
  $\delta \leftarrow \delta_i^{xy^*}$.
  if $\delta = \delta_i$ **then** $y_i \leftarrow 1$.
  else
    Let e_i be the directed edge (K_1, K_2) (tail at K_1 and head at K_2).
3   Using Proposition 5.36 compute y_i such that $\delta^{y_i} = \delta_i$ with the additional property that y_i is trivial and all socle parts not linked to K_2.
  end
  if $i < t$ **then** $y^* = y_i|_{\Sigma_{i+1}}$.
end
return y_t

Algorithm 7: Computing the permutations y_e.

edges in $\mathcal{B}'$. Let Ω_i denote the G-orbit corresponding to e_i. Let $0 \leq k \leq r+1$ be the largest index such that the permutation z fixes the target points of $\cup_{j=1}^{k-1}\Omega_j$. If $k = r+1$ then we are through. Otherwise Algorithm 7 computes a y_{e_k} such that zy_{e_k} fixes δ_k, the target point of Ω_k. Let e_k be the directed edge (K_1, K_2) with the head at K_2. By step 3 we have ensured that y_{e_k} projected onto Ω_k is in K_2. Since $e_1, \ldots, e_r$ is a topological sort, $e_i \not> e_k$ for every $i < k$. Therefore for any $i < k$, the socle parts of $\mathrm{Soc}\,(G_i)$ is not linked to K_2 (Proposition 5.38). Hence y_i is trivial on all Ω_i for $1 \leq i < k$ and $z' = zy_{e_k}$ fixes all the target points in $\cup_{j=1}^{k}\Omega_j$. We now repeat the argument with z replaced with $z' = zy_{e_k}$. At each stage k increase by 1 and hence after r steps $k = r+1$ are we are through. □

Remark 5.41. Consider any $x \in G$ and $x^* = xy$ where $y \in N_s$ be the permutation computed by our algorithm such that x^* fixes all points in Y_s. Let $e = (K_1, K_2)$ be a minimal edge in the $<$ order (or in other words an edge going out of a root node) with head at K_2. Then it is straight forward to verify that y restricted to K_1 is trivial. This follows from the choice of

y_i's in Algorithm 7.

Given an element $x \in G$ we have associated for each $i \in Y_s$ an element y_i such that $x \prod_{i \in Y_s} y_i$ (note the order in which the indices of Y_s'' are taken does matter; the corresponding edges should be topologically sorted). We can ensure that the choice of y_i's depended only on the action of x on the target points of Y_s. In other words for two elements x_1 and x_2 in G and let $\{y_i\}_{i \in Y_s}$ and $\{y_i'\}_{i \in Y_s}$ be the elements chosen. Then if $\delta^{x_1} = \delta^{x_2}$ for all target point δ of Y_s, $y_i = y_i'$ for all $i \in Y_s$ and

$$\prod_{i \in Y_s} y_i = x_1^{*-1} x_1 = x_2^{*-1} x_2 = \prod_{i \in Y_s} y_i'.$$

Furthermore we can assume that if x fixes all the target points of Y_s then each of the $y_i = 1$ for all $i \in Y_s$. We will make this additional assumption which will be helpful in computing the generator set for N.

Computing N

Finally we give the FL algorithm to compute the generator set of N, the subgroup of N_s that fixes all the target points in Y_s. To this end we examine the strong generator C of N_s rel N_{s+1} which we have computed as part of the generator set of G.

Let $R_s/R_{s+1} = T_1 \times \ldots \times T_u$ where each T_i is isomorphic to a non-abelian simple group T. We have a partition $\mathcal{I} = \{I_1, \ldots, I_t\}$ of indices $\{1, \ldots, u\}$, such that the quotient group N_s/N_{s+1} is the product of diagonals $\prod_{k=1}^{t} \mathrm{Diag}\left(\prod_{j \in I_k} T_j\right)$. Recall that the strong generator set C of N_s rel N_{s+1} consists of subset $C_1, \ldots, C_t$ where each C_k corresponds to the diagonal group $\mathrm{Diag}\left(\prod_{j \in I_k} T_j\right)$.

Without loss of generality we assume that $Y_s = \{1, \ldots, r_1, r_1 + 1, \ldots, r\}$ where $Y_s' = \{1, \ldots, r_1\}$ and $Y_s'' = \{r_1 + 1, \ldots, r\}$. We assume further without loss of generality that the ordering $r_1 + 1, \ldots, r$ of elements of Y_s'' is compatible with the ordering of edges in the forest $\mathcal{F}$, i.e. if e_j is the edge associated to $j > r_1$ then $e_{r_1+1}, \ldots, e_{r_2}$ is a topological sort of edges of $\mathcal{F}$.

Using Proposition 5.36 in FL for each $i \in Y_s$ compute the sets D_i such that (1) the projection of D_i on to $\mathrm{Soc}\,(G_i)$ is one-to-one and (2) for any socle part K of $\mathrm{Soc}\,(G_j)$ not linked to socle parts of $\mathrm{Soc}\,(G_i)$ the projection of D_i is 1.

Recall that for each $x \in G$ we gave an FL to compute $x^* = xy$, $y \in N_s$ such that x^* fixes all the target points of Y_s. This we achieved by computing for each $i \in Y_s$ an element $y_i \in N_s$ such that $xy_1 \dots y_{r_1+r_2} = x^*$. Furthermore we assume that the y_i's are canonical as described in Remark 5.41. For each $i \in Y_s$ let $D_i^* = \{x^* | x \in D_i\}$.

Proposition 5.42. *Let A be a generator set of N_{s+1} and let C' be the elements of C that is trivial on the target orbits of Y_s. Then $D^* = (\cup_{i \in Y_s} D_i^*) \cup C' \cup A$ is a generator set of N, the subgroup of N_s that fixes all the target points of Y_s.*

Proof. Clearly every element of D^* is contained in N therefore N^*, the group generated by D^*, is contained in N. We now prove the converse.

It follows from Remark 5.37 that $D = (\cup_{i \in Y_s} D_i) \cup C'$ forms a generator set of N_s. Hence any element $x \in N$ can be written as $x = x_1 \dots x_r yz$ where $x_i \in D_i$, y is in the group generated by C' and $z \in N_{s+1}$.

For any $i \in Y_s'$ notice that x_i is trivial on all target orbits in $Y_s \setminus \{i\}$. Hence x_i fixes all the target points of Y_s. By Remark 5.41 it follows that $x_i^* = x_i \in D_i^*$. Thus to prove that x is in N^* it is sufficient to prove that $z = x_{r_1+1} \dots x_r \in N^*$. Let e_j denote the edge corresponding to the point $j \in Y_s''$. Let $k \leq r+1$ be the largest integer such that x_j restricted to Ω_j is 1 for all $r_1 < j < k$. By the construction of D_k^* we have $x_k^* \in D_k^*$ such that $x_k^* = x_k y$ fixes all the target points of Y_s where y satisfies the properties of Remark 5.41.

Consider the permutation $z' = x_k^{*-1} z = y x_{k+1} \dots x_r$. Since both z and x_k^{*-1} fixes the target points of Y_s so does z'. We show that z' is trivial on all orbits Ω_j, $1 \leq j \leq k$. First for any $1 \leq i < k$, x_j is trivial for all $j > k$. Recall that if e_k be the directed edge (K_1, K_2) with the head at K_2 then y is trivial on Ω_j for all $j < k$ and projected to Ω_k, is an element of the subgroup K_2 (this follows from Remark 5.41). Therefore $z' = y x_{k+1} \dots x_r$ is trivial on Ω_j for all $r_1 < j < k$. Furthermore note that none of the socle parts of $\mathrm{Soc}\,(G_j)$ is linked to K_1 for $k < j \leq r$ and hence x_j projected on Ω_k is also an element of K_2. This proves that z' projected to Ω_k is an element of K_2. However by the O'Nan-Scott theorem K_2 is regular. This is possible if and only if z' restricted to Ω_k is 1 as z' fixes the target point of Ω_k.

We repeat this argument with z replaced with z' and in each step k increases by at least 1. Thus it follows that there exists element $x_j^* \in D_j^*$, $r_1 < j \leq r$, such that $z \prod_{j=r_1+1}^{r} x_j^*$ is 1 on all the target orbits of Y_s and hence is of the form gh where g is in the group generated by C' and h is in N_{s+1}. It follows that x is in the group generated by D^*.

□

Given $x \in G$ since x^* can be computed in FL it follows that a generator set for N can be computed in FL. This completes the algorithm for target reduction in the nonabelian case.

Theorem 5.43. *Given an instance (G, Ω, Δ) of PWS$_c$ subgroup G' of G satisfying the properties (1) $G \geq G' \geq G(\Delta)$ and (2) for all G-orbit Σ such that $\Sigma \cap \Delta \neq \emptyset$, $G|_\Sigma > G'|_\Sigma$ can be computed in the* $\mathrm{Mod}_k\mathrm{L}$ *hierarchy where k is the product of all primes less than c and the level of the hierarchy is a constant that depends only on c.*

5.6 Complexity of BCGI$_b$

We now show that PWS$_c$ is in the $\mathrm{Mod}_k\mathrm{L}$-hierarchy. The complete algorithm is given below (Algorithm 8). Each iteration of the loop 1 is in the $\mathrm{Mod}_k\mathrm{L}$-hierarchy: step 2 uses Theorem 5.24 and step 3 uses Theorem 5.43. Since the G-orbits are of size bounded by c, we will have to iterate through the loop 1 at most $c.\log c$ times before each point in Δ is a G-orbit in itself (i.e. H is $G(\Delta)$).

Input: An instance (G, Δ) of PWS$_c$
Output: A generator set for $G(\Delta)$.
$H \leftarrow G$.
1 **repeat**
2 Compute a strong generator set for H with respect to a locally residual series.
3 Compute the generator set of H' such that $H \geq H' \geq H(\Delta)$ and such that $H'|_\Sigma < H|_\Sigma$ for each H-orbit Σ containing a point of Δ.
$H \leftarrow H'$
until *H fixes all points in Δ* ;
return *the generator set for H*

Algorithm 8: Complete algorithm for PWS$_c$

Using Proposition 5.4 we have the main theorem of this chapter.

Theorem 5.44. *The PWS$_c$, AUT$_b$ and BCGI$_b$ are in the* $\mathrm{Mod}_k\mathrm{L}$*-hierarchy.*

5.7 Discussion

We have proved the BCGI is in the $\mathrm{Mod}_k\mathrm{L}$-hierarchy. In fact we proved this by proving that PWS$_c$ is in the $\mathrm{Mod}_k\mathrm{L}$-hierarchy. The algorithm involved two stages; computing the strong generator set and the target reduction. Both these stages handled the abelian and non-abelian quotients separately.

It is surprising that even though the group theory is more involved the non-abelian quotients could be handled in logspace where as the abelian quotient required $\mathrm{Mod}_p\mathrm{L}$ as an oracle for some appropriate prime p. Thus if the composition series of G had only nonabelian simple groups then PWS_c for (G, Ω, Δ) can be solved in logspace. In view of the hardness result of Torán [68], we cannot improve on the complexity of handling the abelian quotients unless $\mathrm{Mod}_p\mathrm{L}$ is in L.

There is a gap between our upper bound for BCGI and the lower bound that follows from Torán's results [68]. It follows from Torán's result that BCGI_b is hard for the lth level $\mathrm{Mod}_k\mathrm{L}$ where b is exponential in k and l. Our upper bound however places BCGI_b is a higher lever of the $\mathrm{Mod}_k\mathrm{L}$-hierarchy.

Chapter 6

Computational Galois theory

We now move on to the next part of this thesis where we show upper bounds on certain computational problems in Galois theory. Given a polynomial $f(X)$ of degree n over $\mathbb{Q}$ we are interested in the following three fundamental tasks.

1. Compute the Galois group as a permutation group on the roots of $f(X)$,
2. Compute the order of the Galois group of $f(X)$ or equivalently the degree $[\mathbb{Q}_f : \mathbb{Q}]$ of the splitting field extension of $f(X)$.
3. Check whether the Galois group of $f(X)$ satisfies certain properties.

Given a polynomial $f(X)$ over $\mathbb{Q}$, in Chapter 7 we give polynomial time algorithms for (1) checking whether the Galois group of $f(X)$ is nilpotent and (2) checking whether the Galois group of $f(X)$ is in Γ_d. Chapter 8 deals with computing the order of the Galois group of a polynomial. We prove certain upper bounds assuming the generalised Riemann hypothesis. Finally in Chapter 9 we give some algorithms for computing the Galois group of certain special polynomials.

For a polynomial $f(X)$, the Galois group G can be seen as a permutation group on the set of roots of $f(X)$. Combinatorial structures like orbits and blocks associated with the Galois group G play an important role in our results. The Galois correspondence between blocks and subgroups on one hand (Theorem 3.11) and subgroups and subfields on the other hand (Theorem 6.1) gives us a Galois correspondence (Theorem 7.1) between subfields, subgroups and blocks. This interplay between fields and permutation group theoretic structures is crucial for our upper bounds. Apart from the permutation group theory we require, for our conditional results of Chapter 8 and 9, an effective version of the Chebotarev density theorem proved assuming the generalised Riemann hypothesis (Section 8.1).

In this chapter we give a brief description of the Galois theory and algebraic number theory required for our results in Chapters 7, 8 and 9. In Section 6.1 we describe the required Galois theory and in Section 6.3 some algebraic number theory. In Section 6.4 we explain some fundamental algorithmic results that we require in this thesis. To measure the complexity of various algorithms we need a precise formulation of sizes of various algebraic entities. This is explained in Section 6.4. Finally, in Section 6.5, we prove some bounds that will be needed in analysing the complexity of various algorithms in this thesis.

6.1 Galois theory

We recall some basic facts from Galois theory required for this thesis. A detailed account is available in any standard text book on Galois theory or Algebra for example Lang [40, Chapter VI]. By $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ we mean the field of rational, real and complex numbers respectively. The ring of integers will be denoted by $\mathbb{Z}$. For primes p, $\mathbb{F}_{p^r}$ denotes the unique finite field of p^r elements.

Let K be a field. A field L is said to be a *field extension* of K, denoted by L/K, if $L \supseteq K$. For a field extension L/K, L is a vector space over K and its dimension, denoted by $[L : K]$, is the *degree* of L/K. An extension L/K is *finite* if its degree $[L : K]$ is finite. If L/M and M/K are finite extensions then $[L : K] = [L : M].[M : K]$.

Let K be any field. By $K[X]$ we mean the ring of polynomials in X with coefficients from K. The ring $K[X]$ is a *unique factorisation domain.* A polynomial $f(X) \in K[X]$ is *irreducible* if it has no nontrivial factor.

For a field K the smallest positive integer n such that $n.1 = 0$, if it exists, is called the characteristic of K. If no such integer exists then we say that K is of characteristic 0. For example the fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are of characteristic 0 where as the field $\mathbb{F}_{p^r}$ is of characteristic p. For any field K, the characteristic is either 0 or a prime p. If L/K is an extension then the characteristic of L is same as the characteristic of K.

Let L/K be an extension. Then $\alpha \in L$ is *algebraic* over K if there is an $f(X) \in K[X]$ such that $f(\alpha) = 0$. For α algebraic over K, the *minimal polynomial* of α over K is the unique monic polynomial $\mu_\alpha[K](X)$ of least degree in $K[X]$ for which α is a root. When K is clear from the context, we simply write $\mu_\alpha(X)$ instead of $\mu_\alpha[K](X)$. Elements $\alpha, \beta \in L$ are *conjugates* over K if they have the same minimal polynomial over K.

Let L/K be an extension and let $\alpha \in L$ then $K(\alpha)$ is the smallest subfield of L containing K and α. If α is algebraic over K and if $\mu_\alpha(X)$ is the minimal polynomial of α over K then $K(\alpha)$ is isomorphic to $K[X]/\mu_\alpha(X)$, the ring of polynomials over K modulo $\mu_\alpha(x)$. If L/K is a finite extension then by the primitive element theorem [40, Theorem 4.6, Chapter V] there is an $\alpha \in L$ such that $L = K(\alpha)$. Such an element α is called a *primitive element* of L. A *primitive polynomial* of an extension L/K is the minimal polynomial of some primitive element of L/K. Thus if $T(X)$ is a primitive polynomial of L/K then the field L is isomorphic to $K[X]/T(X)$.

The *splitting field* K_f of $f \in K[X]$ is the smallest extension of K containing all the roots of f. An extension L/K is *normal* if for all irreducible polynomials $f(X) \in K[X]$, either $f(X)$ splits completely into linear factors or has no root in L. Any finite normal extension over K is the splitting field of a polynomial in $K[X]$. Let L/K be any extension. By *normal closure* of L over K we mean the smallest normal extension of K that contains L. For a finite extension L/K, let $T(X)$ be any primitive polynomial. The normal closure of L over K is the splitting field over K of $T(X)$.

An extension L/K is *separable* if for all irreducible polynomials $f(X) \in K[X]$ there are no multiple roots in L. In particular all characteristic 0 fields are separable and so are all finite fields. A normal and separable extension L/K is called a *Galois extension*.

A field K is *algebraically closed* if every polynomial in $K[X]$ splits over K. For example the field of complex numbers $\mathbb{C}$ is algebraically closed. Let K be any field. The *algebraic closure* of K, denoted by $\overline{K}$, is the smallest field containing K that is algebraically closed. For every field there is a unique algebraic closure up to isomorphism.

An *automorphism* of a field L is a field isomorphism $\sigma : L \to L$. The *Galois group* $\mathrm{Gal}\,(L/K)$ of a field extension L/K is the subgroup of automorphisms of L that leaves K fixed, i.e. for all $\alpha \in K$, $\sigma(\alpha) = \alpha$. The Galois group of a polynomial $f \in K[X]$ is $\mathrm{Gal}\,(K_f/K)$. Let $f(X)$ be a polynomial over K of degree n. If α is a root of $f(X)$ and $\sigma \in \mathrm{Gal}\,(K_f/K)$ then $\sigma(\alpha)$ is also a root of $f(X)$. Each $\sigma \in \mathrm{Gal}\,(K_f/K)$ is thus completely determined by $\sigma(\alpha_i)$, $1 \le i \le n$, where the $\alpha_1, \ldots, \alpha_n$ are the roots of f. Thus the Galois group of a polynomial $f(X)$ can be seen as a permutation group on the set of roots of $f(X)$ and hence has order at most $n!$.

For a subgroup G of automorphisms of L, the *fixed field* $\mathrm{Fix}\,(L, G)$ is the largest subfield K of L such that every element of G restricted to K

gives the identity automorphism. We now state the fundamental theorem of Galois theory [40, Theorem 1.1, Chapter VI] which, given a finite Galois extension L/K with Galois group G, gives a *Galois correspondence* between subgroups of G and subfields of L containing K.

Theorem 6.1. *Let L/K be a Galois extension with Galois group G. There is a one-to-one correspondence between subfields E of L containing K and subgroups H of G, given by $E \rightleftharpoons \mathrm{Fix}\,(L, H)$. The Galois group $\mathrm{Gal}\,(L/E)$ is H and E/K is a Galois extension if and only if H is a normal subgroup of G. If H is a normal subgroup of G and $E = \mathrm{Fix}\,(L, H)$ then $\mathrm{Gal}\,(E/K)$ is the quotient group G/H.*

6.2 Finite Fields

A finite field is a field of finite cardinality. An example for a finite field is $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, the field of integers modulo a prime p with addition and multiplication defined modulo p. For any prime p and an integer r there is a unique field of cardinality p^r which we denote by $\mathbb{F}_{p^r}$. We have $[\mathbb{F}_{p^r} : \mathbb{F}_p] = r$ and $\mathbb{F}_{p^r}$ is the splitting field of $f(X)$ for any irreducible polynomial $f(X) \in \mathbb{F}_p[X]$ of degree r. For integers n and r, $\mathbb{F}_{p^n}$ is an extension of $\mathbb{F}_{p^r}$ if and only if r divides n in which case the degree $[\mathbb{F}_{p^n} : \mathbb{F}_{p^r}]$ is given by $\frac{n}{r}$. Also in this case $\mathbb{F}_{p^n}/\mathbb{F}_{p^r}$ is a Galois extension.

Consider the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. The map $\sigma : a \mapsto a^p$ is an automorphism of $\overline{\mathbb{F}}_p$ that is identity on $\mathbb{F}_p$. The automorphism σ is called the *Frobenius* automorphism. The Galois group $\mathrm{Gal}\,(\mathbb{F}_{p^n}/\mathbb{F}_{p^r})$ is a cyclic group of order $\frac{n}{r}$ and is generated by σ^r. In terms of the Frobenius we can give a different characterisation of the field $\mathbb{F}_{p^r}$. The field $\mathbb{F}_{p^r}$ is the fixed field of $\overline{\mathbb{F}}_p$ under the group of automorphisms generated by σ^r. Equivalently, $\mathbb{F}_{p^r}$ is the set of roots of the polynomial $X^{p^r} - X$ in $\overline{\mathbb{F}}_p$.

Let $f(X)$ be any polynomial in $\mathbb{F}_q$, q a power of prime p. Let $f(X)$ factorise as $f_1 \dots f_r$ over $\mathbb{F}_q$ and let d_i denote the degrees of f_i. The splitting field of f is $\mathbb{F}_{q^m}$ where m is the least common multiple of the integers $d_1, \dots, d_r$.

6.3 Algebraic numbers and number fields

We now recall some algebraic number theory. A detailed presentation is available in any standard textbook on algebraic number theory like for example the one due to Neukirch [55].

Algebraic numbers are roots of polynomials over $\mathbb{Q}$ and *algebraic integers* are roots of monic polynomials in $\mathbb{Z}[X]$. The set of rational algebraic integers, i.e. algebraic integers in $\mathbb{Q}$, is exactly $\mathbb{Z}$. For an algebraic number α there is an integer $m \in \mathbb{Z}$ such that $m\alpha$ is an algebraic integer. A *number field* is a finite extension of $\mathbb{Q}$.

Let α be an algebraic number and let K be the number field $\mathbb{Q}(\alpha)$. Since $\mathbb{C}$ is algebraically closed and contains $\mathbb{Q}$, K can be seen as a subfield of $\mathbb{C}$, i.e. there is an isomorphism from K to a subfield of $\mathbb{C}$. Such an isomorphism from K to $\mathbb{C}$ is called an embedding of K. If K is of degree n then there exists n distinct embeddings of K into $\mathbb{C}$. An embedding σ of K is a *real embedding* if the image of K under σ is contained in $\mathbb{R}$, otherwise it is a *complex embedding*. The *height* of α, denoted by $\mathrm{H}(\alpha)$, is $\max\{|\sigma(\alpha)|^{c_\sigma}\}$, where σ varies over all embeddings of K and c_σ is either 1 or 2 depending on whether σ is a real or complex embedding. Let $\mu_\alpha(X) \in \mathbb{Q}[X]$ be the minimal polynomial of α then $\mathrm{H}(\alpha)$ is $\max\{|\eta|^{c_\eta}\}$ where η runs over all roots of μ_α in $\mathbb{C}$ and c_η is 1 if η is a real root and 2 otherwise. If α' is a conjugate of α then $\mathrm{H}(\alpha) = \mathrm{H}(\alpha')$. The height of an algebraic number is a measure of its size. Using Cauchy-Schwartz the following inequalities can be derived.

Lemma 6.2. *For any two algebraic numbers α and β:*

1. $\mathrm{H}(\alpha+\beta) \leq \mathrm{H}(\alpha) + \mathrm{H}(\beta)$.
2. $\mathrm{H}(\alpha\beta) \leq \mathrm{H}(\alpha)\,\mathrm{H}(\beta)$.

6.3.1 Ring of Algebraic Integers

Let K be a number field of degree n and let $\mathbb{O}_K$ denote the *ring of algebraic integers* of K. There exist $\omega_1, \ldots, \omega_n \in \mathbb{O}_K$ such that $\mathbb{O}_K = \mathbb{Z}\omega_1 + \ldots + \mathbb{Z}\omega_n$. Such a set of elements in $\mathbb{O}_K$ is a *basis* for $\mathbb{O}_K$. If $\omega_1, \ldots, \omega_n$ is a basis for $\mathbb{O}_K$ then $K = \mathbb{Q}\omega_1 + \ldots + \mathbb{Q}\omega_n$, i.e. the set $\{\omega_1, \ldots, \omega_n\}$ is a basis of K as a vector space over $\mathbb{Q}$. For two bases $\theta_1, \ldots, \theta_n$ and $\omega_1, \ldots, \omega_n$ of $\mathbb{O}_K$ there is a unimodular matrix $A = (a_{ij})$ such that $\omega_i = \sum a_{ij}\theta_j$ for all $1 \leq i \leq n$..

Let K be a number field of degree n. Recall that K has n distinct embedding $\sigma_1, \ldots, \sigma_n$ into $\mathbb{C}$. Let $\omega_1, \ldots, \omega_n$ be a basis for $\mathbb{O}_K$. Then the *discriminant* d_K of K is the positive integer $|\det(\sigma_j(\omega_i))|^2$. The discriminant is independent of the basis chosen for $\mathbb{O}_K$.

An ideal $\mathfrak{a}$ of $\mathbb{O}_K$ is an additive subgroup of $\mathbb{O}_K$ such that for every $\alpha \in \mathbb{O}_K$ and $\beta \in \mathfrak{a}$ $\alpha\beta \in \mathfrak{a}$. Let $\mathfrak{a}$ be an ideal of $\mathbb{O}_K$. For an algebraic integer $\alpha \in \mathbb{O}_K$, the set $\alpha\mathbb{O}_K = \{\alpha\beta | \beta \in \mathbb{O}_K\}$ is an ideal. Such ideals are called *principal ideals*. Often we will denote the principal ideal $\alpha\mathbb{O}_K$ as α.

A principal ideal domain is a ring where all ideals are principal. An example for a principal ideal domain is $\mathbb{Z}$.

We define the sum and product of ideals of $\mathbb{O}_K$. For ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathbb{O}_K$, by $\mathfrak{a}+\mathfrak{b}$ we mean $\{\alpha+\beta : \alpha \in \mathfrak{a}, \beta \in \mathfrak{b}\}$. Similarly by $\mathfrak{a}\mathfrak{b}$ we mean $\{\sum_i \alpha_i\beta_i : \alpha_i \in \mathfrak{a}, \beta_i \in \mathfrak{b}\}$. Furthermore $\mathfrak{a}+\mathfrak{b}$ is the smallest ideal that contains $\mathfrak{a}$ and $\mathfrak{b}$ and $\mathfrak{a}\mathfrak{b}$ is the ideal $\mathfrak{a} \cap \mathfrak{b}$.

We say that $\mathfrak{a}$ *divides* $\mathfrak{b}$, denoted by $\mathfrak{a} \mid \mathfrak{b}$, if $\mathfrak{a} \supseteq \mathfrak{b}$. Unlike $\mathbb{Z}$, for number fields K, $\mathbb{O}_K$ need not be a unique factorisation domain (for example in the ring $\mathbb{Z}[\sqrt{-5}]$, 21 has two factorisations [55, Chapter I, §3]). However ideals of $\mathbb{O}_K$ have the unique factorisation property, i.e. any ideal $\mathfrak{a}$ has a unique factorisation into prime ideals as $\mathfrak{a} = \mathfrak{p}_1^{a_1} \ldots \mathfrak{p}_r^{a_r}$, where a_i is the highest power k such that $\mathfrak{p}_i^k$ divides $\mathfrak{a}$ ($\mathbb{O}_K$ is a *Dedekind domain*).

For any ideal $\mathfrak{a}$, the ring $\mathbb{O}_K/\mathfrak{a}$ is a finite ring. The *norm* of $\mathfrak{a}$, denoted by $\mathrm{N}(\mathfrak{a})$, is the number of elements in $\mathbb{O}_K/\mathfrak{a}$. Consider a number field K of degree n. Let $\sigma_1, \ldots, \sigma_n$ be the n distinct embeddings of K into $\mathbb{C}$. For any $\alpha \in \mathbb{O}_K$, the norm of the principal ideal $\alpha\mathbb{O}_K$, which we denote by $\mathrm{N}(\alpha)$, is equal to the product $\prod_i \sigma_i(\alpha)$.

Let $p \in \mathbb{Z}$ be any prime. For a number field K, the principal ideal $p\mathbb{O}_K$, which we denote by p, need not be a prime ideal. Knowing how the principal ideal p factorise is important and Kummer-Dedekind theorem is algorithmically useful for this purpose (see [22, Theorem 4.8.13] for a proof).

Theorem 6.3 (Kummer-Dedekind)**.** *Let $K = \mathbb{Q}(\theta)$, where θ is an algebraic integer with minimal polynomial $T(X) \in \mathbb{Z}[X]$. Let $p \in \mathbb{Z}$ be a prime that does not divide the index $[\mathbb{O}_K : \mathbb{Z}[\theta]]$. Suppose $T = T_1^{e_1} \ldots T_k^{e_k} (mod\ p)$ is the factorisation of T over $\mathbb{F}_p$ into its irreducible factors. Then $p\mathbb{O}_K$ factors into prime ideals as $p\mathbb{O}_K = \mathfrak{p}_1^{e_1} \ldots \mathfrak{p}_k^{e_k}$. Moreover the prime ideals $\mathfrak{p}_i$ are given by $\mathfrak{p}_i = p\mathbb{O}_K + T_i(\theta)\mathbb{O}_K$ and $\mathbb{O}_K/\mathfrak{p}_i \cong \mathbb{Z}[\theta]/(p, T_i(\theta))$.*

6.4 Basic algorithms

In this section we give an overview of the algorithmic results required for this thesis. For a detailed presentation of various algorithmic aspects of algebraic number theory we refer the reader to the textbook of Cohen [22]. The algorithms we describe take various algebraic entities like algebraic numbers and number fields as inputs. We need to encode these algebraic entities over a finite alphabet Σ typically $\{0, 1\}$. The complexity of these algorithms are measured in terms of the size of these encodings. Our first goal is to make this precise.

6.4.1 Encoding algebraic entities

For integers c we use the standard binary encoding. The size of an integer c is therefore $\lceil \lg c \rceil$. A rational number r is given by a pair of coprime integers (a, b) such that $r = \frac{a}{b}$. Thus, $\mathrm{size}\,(r) = \mathrm{size}\,(a) + \mathrm{size}\,(b)$. Elements of the finite field $\mathbb{F}_p$, for prime p, will be represented as integers in between 0 and p. Hence an element of $\mathbb{F}_p$ is of size $\lg p$.

The fields that we encounter in this thesis are either finite fields or number fields. Recall that any field K is a vector space over the associated base field which is either $\mathbb{Q}$, if the characteristic is 0, or $\mathbb{F}_p$, if the characteristic is p. We follow the approach of Lenstra [43, 44] for encoding fields. Here we describe how number fields are presented. A similar approach can be taken for finite fields for which we refer to the article of Lenstra [43].

There are two algorithmically equivalent ways of presenting a number field K, (1) by explicit data and (2) by presenting a primitive polynomial for K. Let K be a number field of degree n. By *explicit data* we mean a linearly independent basis $e_1, \ldots, e_n$ for K as a vector space over $\mathbb{Q}$ together with n^3 rationals $\{c_{ijk}\}_{1 \leq i,j,k, \leq n}$ such that $e_i e_j = \sum_k c_{ijk} e_k$. In addition by multiplying each e_i's by suitable rational integers we assume, with out loss of generality, that e_i's are algebraic integers. Thus the field can be presented by giving the list $\{c_{ijk}\}_{1 \leq i,j,k \leq n}$ and by size of K we mean $\sum \mathrm{size}\,(c_{ijk})$.

Any $\alpha \in K$ can be expressed uniquely as a summation $\alpha = \sum a_i e_i$, $a_i \in \mathbb{Q}$. By $\mathrm{size}\,(\alpha)$ we mean $\sum \mathrm{size}\,(a_i)$. A polynomial of degree d over K is presented by giving the ordered list of all its d coefficients and hence for $f(X) = a_0 + \ldots + a_d X^d$ in $K[X]$ by $\mathrm{size}\,(f)$ we mean $\sum \mathrm{size}\,(a_i)$.

Recall that $K = \mathbb{Q}(X)/\mu(X)$ for some primitive polynomial $\mu(X)$ over $\mathbb{Q}$. Thus a number field K can be presented by giving a primitive polynomial $\mu(X) \in \mathbb{Q}[X]$. If $\mu(X) = c_0 + c_1 X + \ldots + c_{n-1} X^{n-1} + X^n$ then by $\mathrm{size}\,(K)$ we mean $\sum \mathrm{size}\,(c_i)$. As before we can ensure that $\mu(X)$ is a monic polynomial with coefficients from $\mathbb{Z}$.

A primitive polynomial $\mu(X)$ for K directly gives explicit data for K: choose e_i to be $X^{i-1} \pmod{\mu(X)}$. Conversely we show that given explicit data, one can compute a primitive polynomial. We first prove the following lemma.

Lemma 6.4. *Let K be a number field of degree n presented via explicit data $\{c_{ijk}\}_{1 \leq i,j,k \leq n}$. Let $e_1, \ldots, e_n$ denote the corresponding basis for K. Given an algebraic number $\alpha = \sum_{i=1}^n a_i e_i$ there is an algorithm to compute the minimal polynomial of α that runs in time bounded by a polynomial in* $\mathrm{size}\,(K)$ *and* $\sum \mathrm{size}\,(a_i)$

Proof. Recall that K is a vector space over $\mathbb{Q}$ with a basis $\{e_i\}_{i=1}^n$ and any algebraic number α in K is a vector $\sum a_i e_i$. The degree d of α is the largest i such that the set $\{1, \ldots, \alpha^{i-1}\}$ is a linearly independent set of vectors. It follows that $-\alpha^d$ can be written as a linear combination $-\alpha^d = c_0 + \ldots + c_{d-1}\alpha^{d-1}$. Using the explicit data we can compute the vectors $\alpha^i = \sum_j a_{ij} e_j$ in time polynomial in $\mathrm{size}(K)$ and $\sum \mathrm{size}(a_i)$. Furthermore in polynomial time we can compute d and the rationals $\{c_i : 0 \leq i < d\}$ as it involves solving linear equations over $\mathbb{Q}$. The minimal polynomial of α is therefore $c_0 + c_1 X + \ldots + c_{d-1}X^{d-1} + X^d$. $\square$

We require the following effective version of primitive element theorem (see Section 6.10 of van der Waerden's book [71]).

Lemma 6.5. *Let α and β be algebraic numbers of degrees m and n respectively. Let $\{\alpha_i\}_{i=1}^m$ and $\{\beta_j\}_{j=1}^n$ be their $\mathbb{Q}$-conjugates. Let c be any integer such that $\alpha_i + c\beta_j \neq \alpha_r + c\beta_s$ for all $(i,j) \neq (r,s)$. Then $\alpha + c\beta$ is a primitive element of $\mathbb{Q}(\alpha, \beta)$.*

Consider the set $A = \{\frac{\alpha_i - \alpha_r}{\beta_s - \beta_j} | s \neq j\}$ of $\binom{m}{2}.\binom{n}{2} + 1$ algebraic numbers. It follows from Lemma 6.5 that if $c \notin A$ then $\alpha + c\beta$ is a primitive element of $\mathbb{Q}(\alpha, \beta)$. Therefore there exists an integer c, $1 \leq c \leq m^2 n^2 + 1$ such that $\alpha + c\beta$ is primitive. We summarise this in the following proposition.

Proposition 6.6. *Let α and β be algebraic numbers of degree m and n respectively. There exists an integer $c \in \{1, \ldots, m^2 n^2 + 1\}$ such that $\alpha + c\beta$ is a primitive element of $\mathbb{Q}(\alpha, \beta)$.*

Computing the primitive polynomial for K is now straight forward. Let K be presented via explicit data $\{c_{ijk}\}_{1 \leq i,j,k \leq n}$ and let $e_1, \ldots, e_n$ be the corresponding basis. We compute constants $1 \leq c_i \leq n^4 + 1$, $1 \leq i \leq n$, such that $\sum_{i=1}^n c_i e_i$ is a primitive element for K.

Let K_r denote the field $\mathbb{Q}(e_1, \ldots, e_r)$. We compute the primitive element γ_i of K_i inductively. To begin with $\gamma_1 = e_1$. Assume that we have computed γ_{i-1}. We choose an integer c_i from the set $\{1, \ldots, n^4 + 1\}$ such that the minimal polynomial of $\gamma_{i-1} + c_i e_i$ is of maximal degree. The algebraic number $\gamma_i = \gamma_{i-1} + c_i e_i$ is a primitive element for K_i. Having computed γ_n we can compute the minimal polynomial for γ_n using Lemma 6.4. This gives a primitive polynomial for K.

We have thus proved that presenting number fields via explicit data or via a primitive polynomial are polynomial time equivalent. The size of K in each of these presentation might differ but only up to a polynomial factor. Thus we can assume without loss of generality either of the two presentation.

6.4.2 Factoring polynomials and related problems

Recall that for a field K, the ring $K[X]$ is a unique factorisation domain. Polynomials $f(X)$ over $\mathbb{Q}$ can be factored into irreducible factors in polynomial time using the celebrated Lenstra-Lenstra-Lovász [42] algorithm. A key step in this algorithm is lattice basis reduction. A. K. Lenstra [41] generalised this basis reduction to give a polynomial time algorithm for factoring polynomials over number fields. Using norms of polynomials, Landau [38] gave a polynomial time reduction from factoring over K to factoring over $\mathbb{Q}$. We summarise these results in the following theorem.

Theorem 6.7. *Given a number field K and a polynomial $f(X)$ in $K[X]$ there is an algorithm that computes the irreducible factors of $f(X)$ in time bounded by a polynomial in* $\operatorname{size}(f)$ *and* $\operatorname{size}(K)$*.*

Let K be a finite field of characteristic p. Berlekamp [16] gave a deterministic polynomial time algorithm for factoring polynomials over K for small primes p. However for large characteristic only randomised algorithms are known. Given a polynomial $f(X) \in \mathbb{F}_q$, there are randomised algorithms that run in time polynomial in $\operatorname{size}(f)$ and $\lg q$ [17, 21] for factoring $f(X)$. We summarise these results in the following theorem.

Theorem 6.8. *Given a polynomial $f(X)$ over the finite field K of characteristic p there is a deterministic algorithm that runs in time polynomial in* $\operatorname{size}(f)$ *and p for factoring $f(X)$. Given a polynomial $f(X) \in \mathbb{F}_q$ there is a randomised algorithm that runs in time polynomial in* $\operatorname{size}(f)$ *and* $\lg q$ *to factor $f(X)$.*

Even though factoring polynomials over finite fields do not have efficient deterministic algorithms, there are efficient deterministic irreducibility tests. More generally given a polynomial $f(X) \in \mathbb{F}_q[X]$ and an integer d there is a polynomial time deterministic algorithm to compute the product of all irreducible factors of $f(X)$ of degree d (see Section 14.2 of [73]). In particular, for a give number d one can compute the number of irreducible factor of $f(X)$ of degree d and thus we have an efficient irreducibility test. We summarise this result in the following theorem.

Theorem 6.9. *Given a polynomial $f(X) \in \mathbb{F}_q[X]$ of degree n and an integer $d \leq n$, there is a deterministic algorithm that runs in time polynomial in* $\operatorname{size}(f)$ *and* $\lg q$ *that computes the product of all the irreducible factors of $f(X)$ of degree d. In particular there is a deterministic algorithm that runs in time polynomial in* $\operatorname{size}(f)$ *and* $\lg q$ *that computes for each d the number of irreducible factors of $f(X)$ of degree d.*

6.4.3 Algorithms for Galois group computation

Let L/K be a field extension. Recall that the Galois group $\mathrm{Gal}(L/K)$ is the group of automorphisms of L that are identity when restricted to K. In this section we describe known algorithms for computing the Galois group of a polynomial and related problems.

Firstly from a computational point of view if K is a finite field then the problem is trivial as the Galois group is generated by an appropriate power of the Frobenius (see Section 6.2). Let $q = p^r$. Given a polynomial $f(X)$ over a finite field $\mathbb{F}_q$ recall that if $d_1, \ldots, d_k$ are the set of degrees of irreducible factors of $f(X)$ then the Galois group of $f(X)$ is a cyclic group of order m where m is the least common multiple of $d_1, \ldots, d_k$ and is generated by the rth power of the Frobenius. By Theorem 6.9 we can compute the degrees $d_1, \ldots, d_k$ in time polynomial in $\mathrm{size}(f)$ and $\lg q$.

For polynomials $f(X)$ over number fields the best known algorithm for computing the Galois group is due to Landau [37]. Given a polynomial $f(X) \in K[X]$ Landau's algorithm computes the Galois group $\mathrm{Gal}(K_f/K)$ in time polynomial in $\mathrm{size}(f)$ and $[K_f : K]$. Since $[K_f : K]$ could be as large as $n!$, this is an exponential time algorithm. We give a brief sketch of this algorithm.

Let K be any number field and let L/K be an extension, not necessarily Galois. There exists a primitive element α such that $L = K(\alpha)$. Let $\mu(X) \in K[X]$ be the minimal polynomial of α over K. Using Theorem 6.7 we first factorise $\mu(X)$ over L. Any root of $\mu(X)$ can be expressed as a polynomial in α. Let $P_1(\alpha), \ldots, P_r(\alpha)$, $P_i(X) \in K[X]$, be the distinct roots of $\mu(X)$ over L then the Galois group $\mathrm{Gal}(L/K)$ consists of exactly r elements given by the linear maps $\alpha \mapsto P_i(\alpha)$. Thus the Galois group L/K can be computed in time polynomial in $\mathrm{size}(L)$. Thus the problem of computing the Galois group amounts to computing a primitive polynomial for the splitting field of $f(X)$.

Given a polynomial $f(X) \in K[X]$ of degree n. Let $\alpha_1, \ldots, \alpha_n$ be the roots of $f(X)$. Using Theorem 6.7 we compute the fields $L_i = K(\alpha_1, \ldots, \alpha_i)$ inductively. Start with $L_0 = K$. Assume that we have already computed the explicit data for L_i. If $f(X)$ splits completely over L_i we stop otherwise let $g(X)$ be an irreducible factor of $f(X)$ of degree greater than 1 over L_i which we obtain using Theorem 6.7. Then the field $L_{i+1} = L_i[X]/g(X)$ contains at least one more root of $f(X)$ than L_i and the explicit data for L_{i+1} can be computed. Having computed the splitting field L of f we can compute the Galois group $\mathrm{Gal}(L/K)$ as described above. We summarise the above discussion in the following theorem.

Theorem 6.10 (Landau). *Given a polynomial $f(X)$ over a number field K and a positive integer N. There is a deterministic algorithm running in time bounded by a polynomial in N and* $\text{size}(f)$ *that checks whether the splitting field of f is of degree less than N and if yes computes the entire multiplication table for* $\text{Gal}(K_f/K)$. *In particular given a number fields L/K and an integer N, there is an algorithm running in time polynomial in* $\text{size}(L)$, $\text{size}(K)$ *and N that decides whether the normal closure $\tilde{L}$ of L is of degree less than N over K and if yes computes the explicit data for $\tilde{L}$.*

The algorithm of Theorem 6.10 outputs the entire multiplication table of the Galois group of $f(X)$. There is a much more succinct way of presenting the Galois group. Recall that the Galois group of $f(X)$ is completely specified by its action on the roots of $f(X)$. If n is the degree of $f(X)$, by suitably naming the roots of $f(X)$ the Galois group of $f(X)$ can be seen as a subgroup of S_n. In other words there is a faithful representation of $\text{Gal}(K_f/K)$ as a permutation group over a cardinality n set Ω. Recall that subgroups of S_n can be presented succinctly via a generator set. Moreover several natural algorithmic tasks for a permutation group G given a generator set for it can be accomplished efficiently. For example it is possible to determine if G is solvable in polynomial time, or to determine a composition series for G in polynomial time among several other tasks (a survey of important results is available in the article of Luks [49]). Thus determining the Galois group by its action on the roots of f is a reasonable way of describing the output.

For polynomials $f(X)$ with small Galois group Theorem 6.10 gives an efficient algorithm for Galois group computation. For example if $f(X)$ is irreducible and $f(X)$ splits in $K[X]/f(X)$ then we have a polynomial time algorithm for computing the Galois group of $f(X)$. In particular Theorem 6.10 gives a polynomial time algorithm to compute the Galois group of an irreducible polynomial $f(X)$ with abelian Galois group. This is because an abelian transitive subgroup of S_n is of size n.

Proposition 6.11. *Let G be a transitive abelian permutation group on Ω then for any $\alpha \in \Omega$ $G_\alpha = 1$ and $\#G = \#\Omega$.*

Proof. Fix any $\alpha \in \Omega$. Since G is transitive for any $\beta \in \Omega$ there is a $g_\beta \in G$ such that $\alpha^{g_\beta} = \beta$. The group G_β is given by $g_\beta^{-1} G_\alpha g_\beta$ and since G is abelian is equal to G_α. This implies any element that fixes α fixes all elements β pointwise and hence is identity. Therefore $G_\alpha = 1$. By Orbit-Stabiliser formula (Theorem 3.8) we have $\#G = \#\Omega.\#G_\alpha = \#\Omega$. □

Proposition 6.11 and Theorem 6.10 can be used to give a polynomial time algorithm to test whether the Galois group of a polynomial $f(X) \in K[X]$ is abelian [37]. Given a polynomial $f(X)$ we first compute all its irreducible factors over $K[X]$. For each irreducible factor $g(X)$ we to compute the Galois group $\mathrm{Gal}(K_g/K)$ using Theorem 6.10. If $\mathrm{Gal}(K_g/K)$ is too large, i.e. $\mathrm{Gal}(K_g/K)$ is of order greater than the degree of $g(X)$ then clearly it is not abelian (Proposition 6.11). Having computed the Galois group $\mathrm{Gal}(K_g/K)$ explicitly we can check whether it is abelian. The Galois group of $f(X)$ is abelian if the Galois groups of each of its irreducible factor is abelian.

Theorem 6.12 (Landau)**.** *Let $f(X) \in K[X]$ be any polynomial. There is an algorithm that runs in time polynomial in* $\mathrm{size}(f)$ *and* $\mathrm{size}(K)$ *that checks whether the Galois group of $f(X)$ is abelian. If in addition the polynomial $f(X)$ is irreducible, there is an algorithm that runs in time polynomial in* $\mathrm{size}(f)$ *and* $\mathrm{size}(K)$ *that computes the Galois group of f.*

Even though the Galois group of an irreducible polynomial $f(X) \in \mathbb{Q}[X]$ with abelian Galois group can be computed efficiently, there are no efficient algorithms when $f(X)$ is reducible. In fact even when $f(X)$ is a product of quadratic polynomials nothing better than the exponential algorithm is known ([44, Problem 3.4]). In Chapter 9, assuming the generalised Riemann hypothesis, we give a randomised polynomial time algorithm for this problem.

We now consider another important task, computing the fixed field of a field L given a set of automorphism of L. Given an automorphism σ of L, the fixed field of L under automorphisms generated by σ is the kernel of the map $\sigma - 1$. Let L be any field of degree n presented via explicit data $\{c_{ijk}\}_{1 \leq i,j,k \leq n}$. Let $e_1, \ldots, e_n$ be the corresponding basis for L. Then any automorphism σ is a linear map on L as a vector space over $\mathbb{Q}$. Having fixed a basis $e_1, \ldots, e_n$, each σ can be represented by an $n \times n$ matrix A_σ over $\mathbb{Q}$. The subspace of solutions for the linear equation $A_\sigma \mathbf{x} = \mathbf{x}$ is exactly the fixed field of L under σ. A basis for this field can be computed in polynomial time and thus we have its explicit data.

To find the fixed field of L under the automorphisms generated by $S = \{\sigma_1, \ldots, \sigma_r\}$, consider the fixed field L_i of L under the automorphisms generated by $\{\sigma_1, \ldots, \sigma_i\}$. Starting from $L_0 = L$ we compute the fields L_i inductively. Having computed a basis for L_{i-1}, we compute L_i by computing a basis of the kernel of $\sigma_i - 1$ over L_i as described above. This inductive algorithm is evidently polynomial time. Thus we have the following theorem.

Theorem 6.13. *Given a field L via explicit data $\{c_{ijk}\}_{1 \leq i,j,k \leq n}$ and a set S of automorphisms of L. There is a deterministic algorithm running in* *time polynomial in $\#S$ and* $\mathrm{size}(L)$ *to compute the fixed field of L under the group generated by S.*

6.5 Some useful bounds

In this section we prove certain bounds that will be useful in analysing the algorithms of this thesis. Given a number field K of degree n presented via explicit data. Our first goal is to give a bound on the height of algebraic numbers of K in terms of its size.

Lemma 6.14. *Let K be a number field of degree n presented via explicit data $\{c_{ijk}\}_{1\leq i,j,k\leq n}$. Let $e_1,\ldots,e_n$ be the corresponding basis for K. Then for each i, $\mathrm{H}(e_i) \leq n.2^{\mathrm{size}(K)}$. It follows that for any α in K, $\mathrm{H}(\alpha) \leq n^2.2^{\mathrm{size}(\alpha)+\mathrm{size}(K)}$.*

Proof. Let $1 \leq i \leq n$ be such that $\mathrm{H}(e_i)$ is maximum. We have $e_ie_i = \sum_{k=1}^{n} c_{iik}e_k$. Each c_{ijk} is less than $2^{\mathrm{size}(K)}$. Hence it follows that $\mathrm{H}(e_i)^2 \leq n\mathrm{H}(e_i)\,2^{\mathrm{size}(K)}$ (Lemma 6.2) and $\mathrm{H}(e_i) \leq n.2^{\mathrm{size}(K)}$.

Let $\alpha = \sum_{k=1}^{n} a_ie_k$. Recall that $\mathrm{size}(\alpha) = \sum \mathrm{size}(a_k)$ and hence $a_k \leq 2^{\mathrm{size}(\alpha)}$. Therefore $\mathrm{H}(\alpha) \leq n.2^{\mathrm{size}(\alpha)}.\mathrm{H}(e_i) = n^2.2^{\mathrm{size}(\alpha)+\mathrm{size}(K)}$. $\square$

Conversely, in many cases we would like to bound the sizes of algebraic numbers given a bound on its height. The following lemma serves this purpose.

Lemma 6.15. *Let K be a number field of degree n and $\alpha \in \mathbb{O}_K$. Then* $\mathrm{size}(\alpha) \leq n^3(\lg \mathrm{H}(\alpha) + \mathrm{size}(K))$.

Proof. Let $\{c_{ijk}\}_{1\leq i,j,k\leq n}$ be the explicit data for K and let $e_1,\ldots,e_n$ be the corresponding basis. Let $\sigma_1,\ldots,\sigma_n$ be the n distinct embeddings of K into $\mathbb{C}$ and let $e_{ij} = \sigma_i(e_j)$. Consider the algebraic integer $\alpha = \sum_j c_je_j$. If $\alpha_i = \sigma_i(\alpha) = \sum_j c_je_{ij}$ then $|\alpha_i| \leq \mathrm{H}(\alpha)$.

Consider the system of linear equations $\sum c_je_{ij} = \alpha_i$, $1 \leq i \leq n$. By Cramer's rule, we have $c_j = \frac{\det(A_j)}{\det(A)}$ where A is the $n\times n$ matrix $(e_{ij})_{1\leq i,j\leq n}$ and A_j is the matrix obtained by replacing the j^{th} column of A by α_i's. The number $\det(A)^2$ is a symmetric function on the algebraic integers e_{ij}'s. It follows that $\det(A)^2$ is in $\mathbb{Z}$ and therefore $|c_j| \leq \det(A_j)$ for each i.

For an $n\times n$ matrix M, the determinant of M is bounded by $n^n\lambda^n$ where λ is the largest entry of M. Therefore $|\det(A_i)| \leq n^n.\lambda^n$ where $\lambda =$

$\max(\mathrm{H}(e_i), \mathrm{H}(\alpha))$ $(\mathrm{H}(e_{ij}) = \mathrm{H}(e_i))$. We have thus proved that $\mathrm{size}(c_i) \leq n.(\lg n + \lg \mathrm{H}(\alpha) + \lg \mathrm{H}(e_i)) \leq n.(\lg \mathrm{H}(\alpha) + \mathrm{size}(K) + 2\lg n)$. We then obtain

$$\mathrm{size}(\alpha) = \sum_{i=0}^{n-1} \mathrm{size}(c_i) \leq n^3(\lg \mathrm{H}(\alpha) + \mathrm{size}(K)).$$

□

We now prove a bound on the size of the minimal polynomial of an algebraic integer α in terms of its height.

Proposition 6.16. *Let α be an algebraic integer of degree n and let $\mu_\alpha(X)$ be the minimal polynomial of it over $\mathbb{Q}$. Then* $\mathrm{size}(\mu_\alpha(X)) \leq n^2(1 + \lg \mathrm{H}(\alpha))$.

Proof. Let $\alpha_1, \ldots, \alpha_n$ be the conjugates of α then the minimal polynomial is give by $\mu(X) = \sum_{i=0}^{n} s_i X^i$ where s_i is the ith symmetric function over $\alpha_1, \ldots, \alpha_n$, i.e. s_i is the sum of all possible products of elements in $\{\alpha_1, \ldots, \alpha_n\}$ taken i at a time. Since $\mathrm{H}(\alpha_i) = \mathrm{H}(\alpha)$ we have $|s_i| \leq \binom{n}{i}\mathrm{H}(\alpha)^i$. Therefore

$$\mathrm{size}(\mu_\alpha(X)) \leq \sum_i \lg\left[\binom{n}{i}\mathrm{H}(\alpha)^i\right] \leq n^2(1 + \lg \mathrm{H}(\alpha)).$$

□

Conversely, given and algebraic number α, we often need a bound on $\mathrm{H}(\alpha)$ in terms of the size of its minimal polynomial μ_α over $\mathbb{Q}$. For this purpose we state an inequality due to Landau [36] (a proof of this inequality is available in [73, Theorem 6.31]). Consider a polynomial $f(X) = \sum a_i X^i \in \mathbb{C}[X]$. Define $\|f\|_2$ as $\sqrt{\sum |a_i|^2}$. We use the following inequality to bound the sizes of algebraic numbers.

Lemma 6.17 (Landau)**.** *Let $f(X) = a_0 + \ldots + a_d X^d \in \mathbb{C}[X]$ of degree d, and let $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$ be its roots. Then,*

$$|a_d| \prod_{i=1}^{d} max(1, |\alpha_i|) \leq \|f\|_2.$$

Let α be an algebraic integer of degree n and let $\mu_\alpha(X) = c_0 + c_1 X + \ldots + c_n X^{n-1} + X^n$ be its minimal polynomial. For all i, $c_i \leq 2^{\mathrm{size}(\mu_\alpha)}$ and therefore $\|\mu_\alpha\|_2 \leq \sqrt{n}2^{\mathrm{size}(\mu_\alpha)}$. Recall that $\mathrm{H}(\alpha)$ is the maximum over $|\eta|^{c_\eta}$

where η ranges over the roots of μ_α and c_η is either 1 or 2 depending on whether η is real or complex. Together with Landau's inequality we thus have the following bound.

Proposition 6.18. *Let α be an algebraic integer with minimal polynomial μ_α over $\mathbb{Q}$. Then $\mathrm{H}(\alpha) \leq n.4^{\mathrm{size}(\mu_\alpha)}$.*

We now prove a bound on the discriminant of a number field. For a polynomial $T(X) \in \mathbb{Q}[X]$ with roots $\theta_1, \ldots, \theta_n$ by *discriminant*, which we denote by d_T we mean the product $\prod_{i \neq j}(\theta_i - \theta_j)$. For an algebraic number α, by the discriminant of α, denoted by d_α, we mean the discriminant of the minimal polynomial $\mu_\alpha(X)$ of α over $\mathbb{Q}$. Consider a number field $K = \mathbb{Q}(\theta)$ where θ is an algebraic integer. The discriminant d_K of K divides d_θ and $\frac{d_\theta}{d_K} = [\mathbb{O}_K : \mathbb{Z}[\theta]]^2$. Therefore to bound d_K it is sufficient to give a bound on the discriminant d_θ. We use of this to show the following bound on the discriminant.

Theorem 6.19. *Let $f(X)$ be a monic polynomial over $\mathbb{Z}$ of degree n with roots $\alpha_1, \ldots, \alpha_n$. There exists an algebraic integer $\theta = \sum c_i \alpha$, $c_i \in \mathbb{Z}$ such that the θ is a primitive element for the extension $\mathbb{Q}_f/\mathbb{Q}$ and $\lg d_\theta \leq (n!)^3.\mathrm{size}(f)$. As a result we have $\lg d_{\mathbb{Q}_f} \leq (n!)^3.\mathrm{size}(f)$.*

Proof. Let $N \leq n!$ be the degree of the splitting field $K = \mathbb{Q}_f$. Let $\alpha_1, \ldots, \alpha_n$ be the roots of $f(X)$ then $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$. By Proposition 6.6 there are integer constant $1 \leq c_i \leq N^4 + 1$ such that $\theta = \sum_i c_i \alpha_i$ is a primitive element of K. Since $f(X)$ is a monic polynomial over $\mathbb{Z}$ it follows that α_i's are algebraic integers and so is θ.

By Landau's inequality we have $|\alpha_i| \leq \sqrt{n} 2^{\mathrm{size}(f)}$ and therefore we have $\mathrm{H}(\theta) \leq n.N^4 4^{\mathrm{size}(f)}$. Therefore we have

$$\lg d_K \leq \lg d_\theta \leq N^2(1 + 2\lg n + 4\lg N + 2\mathrm{size}(f)).$$

Since $N \leq n!$ we have the required bound. □

6.6 Discussion

Computing the Galois group of $f(X)$ is hard both in theory and in practice. No polynomial time algorithm is known. Current computer algebra systems can compute typically the Galois group of polynomials of degree in the range 20 to 25. Apart from their mathematical significance many computational problems that arise in algebraic number theory have wide range of application especially in cryptography. Hence algorithms that run efficiently in practice are of utmost importance. For a detailed presentation of algorithms from a practical point of view we refer to the book of Cohen [22].

However, in this thesis our focus is not directed on these issues. A polynomial time algorithm will be considered efficient although practical implementations might turn out to be too slow. In this sense our approach is similar to the approach of Lenstra in his survey article [44]. In the absence of efficient algorithms our attempt would be to give nontrivial complexity upper bounds.

As mentioned before, to prove nontrivial complexity theoretic results one often require novel techniques. A striking example is the recent AKS algorithm for primality testing [1]. Although the algorithm runs in polynomial time, as far as testing primality of large numbers in practical contexts like for example in cryptographic application, the randomised tests are still preferred. However the techniques developed could lead to solutions of other interesting questions.

Often complexity theoretic classification of natural problems have been fruitful in understanding the inherent intractability of these problems. For example showing hardness results for a problem say for NP in some sense shows that the problem is computationally hard. Even though computing Galois groups are hard in practice, no hardness results (in the complexity theoretic sense) is yet known. Showing such hardness results could be challenging and probably need considerable mathematical techniques.

Attempts to understand the complexity of natural problems have led to considerable progress in complexity theory as well. For example study of one-way functions led Valiant to define the complexity class UP [69]. Algebraic number theory and Galois theory is a rich source of natural computational problems and studying the complexity of these problems might lead to considerable progress in complexity theory as well. What makes these problems particularly attractive is the availability of powerful mathematical tools. Unfortunately even though the mathematics is fairly well understood virtually nothing is known about the computational complexity of many of the fundamental problems in this area. On one hand even for polynomials with abelian Galois group no efficient algorithms are know unconditionally. On the other hand there is no hardness result known despite the fact that the best known algorithm for computing the Galois group is exponential time.

Chapter 7

Testing nilpotence of Galois group

Given a polynomial $f(X)$ over $\mathbb{Q}$ in this chapter we study the following two problems (1) checking whether the Galois group is nilpotent and (2) checking whether the Galois group is in Γ_d. As mentioned before knowing certain properties of the Galois group of a polynomial $f(X)$ gives information about its roots. For example the seminal work of Galois shows that a polynomial $f(X)$ is solvable by radicals if and only if its Galois group is solvable. However algorithmically this does not give a satisfactory answer as computing the Galois group is hard. Landau and Miller [39] achieved a remarkable breakthrough by giving a polynomial time algorithm for checking solvability.

First, we show that given a polynomial $f(X) \in \mathbb{Q}[X]$, we can test whether the Galois group of $f(X)$ is nilpotent in polynomial time. Recall that a group G is nilpotent if all its Sylow subgroups are normal. Even though every nilpotent group is solvable, the Landau-Miller solvability test does not give a nilpotence test. This is because knowing the composition factors of a group G alone is not enough to decide whether G is nilpotent.

We generalise the Landau-Miller test and show that Γ_d-testing for constant d is in polynomial time. Recall that a group G is in Γ_d if there is a composition series $G = G_0 \triangleright \ldots \triangleright G_t = 1$ such that G_i/G_{i+1} is either abelian or is isomorphic to a subgroup of S_d.

An important idea used in both these tests is the Galois correspondence between blocks, fields and groups that we now explain. Let $f(X)$ be an irreducible polynomial in $\mathbb{Q}[X]$ and let G be the Galois group of $f(X)$. Since $f(X)$ is irreducible, G is a transitive permutation group on Ω, the set of roots of $f(X)$. For a block Δ recall that G_Δ is the subgroup of G that setwise stabilises Δ. Let $\mathbb{Q}_\Delta$ denote the fixed field $\mathrm{Fix}\,(\mathbb{Q}_f, G_\Delta)$.

Theorem 7.1. *Let $f(X) \in \mathbb{Q}[X]$ be an irreducible polynomial with Galois group G. Let Ω be the roots of $f(X)$ and let $\alpha \in \Omega$ be any particular root. There is a one-to-one correspondence between G-blocks containing α, subgroups of G containing G_α and subfields between $\mathbb{Q}(\alpha)$ and $\mathbb{Q}$ given by*

$$\Delta \rightleftharpoons G_\Delta \rightleftharpoons \mathbb{Q}_\Delta.$$

Furthermore if $\{\alpha\} = \Delta_0 \subseteq \ldots \subseteq \Delta_m = \Omega$ is an increasing chain of blocks then $\mathbb{Q}(\alpha) = \mathbb{Q}_{\Delta_0} \supseteq \ldots \supseteq \mathbb{Q}_{\Delta_m} = \mathbb{Q}$ is a decreasing tower of number fields between $\mathbb{Q}(\alpha)$ and $\mathbb{Q}$.

The first is the Galois correspondence between G-blocks and subgroups of G containing G_α (Theorem 3.11) and the second correspondence is via the fundamental theorem of Galois theory (Theorem 6.1). The crucial observation of Landau and Miller is that even though the Galois group G is unknown, the field $\mathbb{Q}_\Delta$ can be computed in polynomial time. Knowing the structure of the fields $\mathbb{Q}_\Delta$ gives us valuable information about the groups G_Δ. Consider a permutation group G on the set Ω. Let $\Delta \subseteq \Sigma$ be two G-blocks recall that $\mathcal{B}\,(\Sigma/\Delta)$ denotes the set of blocks $\{\Delta^g | g \in G \;\Delta^g \subseteq \Sigma\}$. The group $G\,(\Sigma/\Delta)$ is the subgroup of G_Σ that fixes all the blocks in $\mathcal{B}\,(\Sigma/\Delta)$ and $G^\Delta = G\,(\Omega/\Delta)$.

Proposition 7.2. *Let $\Delta \subseteq \Sigma$ be two G blocks then.*

1. *The normal closure of the field $\mathbb{Q}_\Delta$ over $\mathbb{Q}_\Sigma$ is exactly the fixed field* $\mathrm{Fix}\,(\mathbb{Q}_f, G\,(\Sigma/\Delta))$. *In particular the normal closure of $\mathbb{Q}_\Delta$ is the fixed field* $\mathrm{Fix}\,(\mathbb{Q}_f, G^\Delta)$.
2. *The index of G-blocks $[\Sigma : \Delta]$ is equal to the degree $[\mathbb{Q}_\Delta : \mathbb{Q}_\Sigma]$.*

Proof. Recall that the normal closure of $\mathbb{Q}_\Delta$ over $\mathbb{Q}_\Sigma$ is the smallest field containing $\mathbb{Q}_\Delta$ that is normal over $\mathbb{Q}_\Sigma$. The field $\mathbb{Q}_f$ is a Galois extension of $\mathbb{Q}_\Sigma$ with Galois group G_Σ. Since $\mathbb{Q}_\Delta$ is contained in $\mathbb{Q}_f$ it follows that the normal closure of $\mathbb{Q}_\Delta$ over $\mathbb{Q}_\Sigma$ is contained in $\mathbb{Q}_f$. Let $L \subseteq \mathbb{Q}_f$ be any normal extension of $\mathbb{Q}_\Sigma$ containing $\mathbb{Q}_\Delta$. By the fundamental theorem of Galois theory (Theorem 6.1), L is the fixed field $\mathrm{Fix}\,(\mathbb{Q}_f, H)$ for some subgroup H of G_Δ that is normal in G_Σ. The group $G\,(\Sigma/\Delta)$ is the largest subgroup of G_Δ that is normal in G_Σ (Theorem 3.12). Therefore $\mathrm{Fix}\,(\mathbb{Q}_f, G\,(\Sigma/\Delta))$ is the smallest normal extension of $\mathbb{Q}_\Sigma$ that contains $\mathbb{Q}_\Delta$ and is thus the normal closure of $\mathbb{Q}_\Delta$ over $\mathbb{Q}_\Sigma$. Let $\Sigma = \Omega$ then we have $\mathbb{Q}_\Sigma = \mathbb{Q}_\Omega = \mathbb{Q}$.

Hence the normal closure of $\mathbb{Q}_\Delta$ is $\mathrm{Fix}\left(\mathbb{Q}_f, G^\Delta\right)$. This completes the proof of part 1.

For G-blocks $\Delta \subseteq \Sigma$, by the Galois correspondence of blocks (Theorem 3.11), the index $[\Sigma : \Delta] = [G_\Sigma : G_\Delta]$. Consider any G-block Ψ. The extension $\mathbb{Q}_f/\mathbb{Q}_\Psi$ is Galois with Galois group G_Ψ. Therefore $[\mathbb{Q}_f : \mathbb{Q}_\Psi] = \#G_\Psi$. It follows that $[G_\Sigma : G_\Delta] = \frac{[\mathbb{Q}_f:\mathbb{Q}_\Sigma]}{[\mathbb{Q}_f:\mathbb{Q}_\Delta]} = [\mathbb{Q}_\Delta : \mathbb{Q}_\Sigma]$. This proves part 2. $\square$

Proposition 7.2 will play an important role in our algorithms for nilpotence and Γ_d testing. We give polynomial time algorithm for computing the fields $\mathbb{Q}_\Delta$ in Section 7.1. In Section 7.2 we study the block structure of transitive nilpotent permutation groups. Using these properties we give a nilpotence test. Finally the Γ_d-test is given in section 7.3.

7.1 Computing the fields $\mathbb{Q}_\Delta$

The goal of this section is to prove the following theorem that plays an important role in both the property testing algorithms we are going to describe.

Theorem 7.3. *Let $f(X)$ be an irreducible polynomial over $\mathbb{Q}$ with Ω as its set of roots. Let $G \leq \mathrm{Sym}(\Omega)$ be its Galois group. Let Δ be any G-block of Ω such that $\alpha \in \Delta$ for some $\alpha \in \Omega$. There is an algorithm that given the field $\mathbb{Q}_\Delta$ as a subfield of $\mathbb{Q}(\alpha)$, runs in time polynomial in* $\mathrm{size}(f)$ *and* $\mathrm{size}(\mathbb{Q}_\Delta)$ *and computes the field $\mathbb{Q}_\Sigma$ for all G-blocks Σ such that Δ is a maximal G-subblock of Σ. Moreover* $\mathrm{size}(\mathbb{Q}_\Sigma)$ *is at most a polynomial in* $\mathrm{size}(f)$ *and is independent of the size of the presentation of $\mathbb{Q}_\Delta$.*

Although stated differently, this algorithm is due Landau and Miller [39] and is used in their polynomial-time solvability test. Through a sequence of lemmas we prove this theorem in the rest of this section.

For a G-block Δ let $T_\Delta(X)$ be the polynomial defined by

$$T_\Delta(X) = \prod_{\eta \in \Delta} (X - \eta).$$

Proposition 7.4. *If $T_\Delta(X) = \delta_0 + \delta_1 X + \ldots + \delta_{r-1}X^{r-1} + X^r$ then field $\mathbb{Q}_\Delta$ is the field $\mathbb{Q}(\delta_0, \ldots, \delta_{r-1})$.*

Proof. For any automorphism $\sigma \in G$ we have $\sigma(T_\Delta) = T_{\Delta^\sigma}$. Therefore if σ is in G_Δ then $\sigma(T_\Delta) = T_\Delta$. Let $T_\Delta(X) = \delta_0 + \delta_1 X + \ldots + \delta_{r-1}X^{r-1} + X^r$.

Comparing the coefficients of X^i we have $\sigma(\delta_i) = \delta_i$ for all $0 \le i < r$. Conversely if for some $\sigma \in G$, if $\sigma(\delta_i) = \delta_i$, $0 \le i < r$, then $\sigma(T_\Delta) = T_\Delta$ and hence $\sigma \in G_\Delta$. Thus we have the following proposition. $\square$

In view of Proposition 7.4, to compute $\mathbb{Q}_\Delta$ it is sufficient to compute the polynomial $T_\Delta(X)$. The algebraic integers δ_i's are symmetric functions on the roots of $f(X)$ in Δ and hence using Lemma 6.15 and Proposition 6.18, size (δ_i) is bounded by a polynomial in size (f). Having computed the polynomial T_Δ, one can compute the field $\mathbb{Q}_\Delta$ in time polynomial in size (f).

We prove the following important lemma on the irreducible factors of $f(X)$ over $\mathbb{Q}_\Delta$.

Lemma 7.5. *Let Δ be a G-block containing α. There is a one-to-one correspondence between irreducible factors of $f(X)$ over $\mathbb{Q}_\Delta$ and orbits of G_Δ given by*

$$\Omega' \rightleftharpoons \prod_{\eta \in \Omega'} (X - \eta), \ \ \Omega' \text{ a } G\text{-orbit.}$$

Proof. Let $g(X)$ be an irreducible factor of $f(X)$ over $\mathbb{Q}_\Delta$. Then $G_\Delta = \mathrm{Gal}\,(\mathbb{Q}_f/\mathbb{Q}_\Delta)$ acts transitively on the roots of $g(X)$. Hence for any two roots η and η' of $g(X)$ there is an element $\sigma \in G_\Delta$ such that $\sigma(\eta) = \eta'$. Therefore η and η' belong to the same G_Δ-orbit. Conversely if η and η' belong to the same G_Δ orbit then there is a $\sigma \in G_\Delta$ such that $\sigma(\eta) = \eta'$ and they are $\mathbb{Q}_\Delta$-conjugates. This is possible if and only if η and η' are the roots of the same irreducible factor g of $f(X)$ over $\mathbb{Q}_\Delta$. $\square$

The above lemma has the following important corollary.

Lemma 7.6. *Let Δ be any G-block containing α. The polynomial T_Δ is the irreducible factor of f over $\mathbb{Q}_\Delta$ which has α as its root. Let Σ be any G-block such that $\Sigma \supseteq \Delta$. If g is an irreducible factor of f over $\mathbb{Q}_\Delta$ then Σ contains a root of g if and only if it contains all the roots of g.*

Proof. In the correspondence of Lemma 7.5, T_Δ corresponds to the orbit of α under G_Δ. Hence T_Δ is the factor of $f(X)$ that has α as a root.

Let $\Delta \subseteq \Sigma$ be any two G-blocks and let $g(X)$ be an irreducible factor of $f(X)$ over $\mathbb{Q}_\Delta$. Suppose that Σ contains a root η of g. Any other root η' of g is in the same G_Δ orbit, i.e. $\eta' \in \eta^{G_\Delta}$. However since $\Delta \subseteq \Sigma$ we have $G_\Delta \le G_\Sigma$. Hence $\eta' \in \eta^{G_\Sigma} = \Sigma$. $\square$

The above theorem gives a polynomial time algorithm to identify the polynomial $T_\Delta(X)$. Recall that $\mathbb{Q}_\Delta$ is a subfield of $\mathbb{Q}(\alpha)$. The polynomial

$T_\Delta(X)$ is that irreducible factor $g(X)$ of $f(X)$ for which $g(\alpha) = 0$. We now prove an important lemma from which the proof of Theorem 7.3 is more or less direct.

Lemma 7.7. *Let Δ be a G-block containing α. Given the field $\mathbb{Q}_\Delta$ as a subfield of $\mathbb{Q}(\alpha)$ and an irreducible factor g of f over $\mathbb{Q}_\Delta$, we can compute in polynomial time T_Σ as a polynomial in $\mathbb{Q}(\alpha)[Y]$, where Σ is the smallest G-block containing Δ and the roots of g.*

Proof. Let the factorisation of f over $\mathbb{Q}_\Delta$ be $f = g_0 \dots g_r$, where $g_0 = T_\Delta$ and $g = g_1$. Denote the set of roots of g_i by Φ_i, for each i. Then by Lemma 7.5, Φ_i's are the orbits of G_Δ and the polynomial $T_\Sigma(X)$ is precisely the product of g_i such that $\Phi_i \subseteq \Sigma$. Let β be any root of $g(X)$, and $\sigma \in \mathrm{Gal}\,(\mathbb{Q}_f/\mathbb{Q})$ be an automorphism that maps α to β. The map σ is in fact an isomorphism between the fields $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$. Let Σ be the smallest G-block containing Δ and Φ_1. It follows from the Galois correspondence of blocks (Theorem 3.11) that G_Σ is generated by $G_\Delta \cup \{\sigma\}$. If we knew the permutation σ and the orbits Φ_i explicitly then the following transitive closure procedure would give us Σ.

$S \leftarrow \{\Delta, \Phi_1\}$
repeat
  $S' \leftarrow \{\Phi^\sigma \mid \Phi \in S\}$
  foreach *orbit* Φ_j **do**
1   **if** $\Phi_j \cap \Phi^\sigma \neq \emptyset$ *for some* $\Phi^\sigma \in S'$ **then** $S \leftarrow S \cup \{\Phi_j\}$
  end
until S *is unchanged* ;
Output $\bigcup\{\Phi \mid \Phi \in S\})$

Algorithm 9: Computing Σ

Our goal is to "simulate" Algorithm 9. The key ideas is that the orbit Φ_i corresponds to the irreducible factor g_i of $f(X)$ over $\mathbb{Q}_\Delta$ and testing whether $\Phi_j \cap \Phi_i^\sigma \neq \emptyset$ (step 1 of the Algorithm 9) amounts to checking whether the g.c.d of g_j and g_i^σ is nontrivial. We give a polynomial time algorithm for computing the g.c.d of g_j and g_i^σ.

First, we compute the extension field $\mathbb{Q}(\alpha, \beta)$. We factor the polynomial $g(X)$ over the field $\mathbb{Q}(\alpha)$ into irreducible factors. Let h be any irreducible factor of g over $\mathbb{Q}(\alpha)$ then $\mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\alpha)[X]/h(X)$. Since $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] \leq n^2$ and the heights $\mathrm{H}\,(\alpha) = \mathrm{H}\,(\beta)$ is bounded by $2^{O(\mathrm{size}(f))}$ (Proposition 6.18), we can compute in polynomial time the explicit data of $\mathbb{Q}(\alpha, \beta)$. Furthermore, in polynomial time we compute a primitive element $\gamma = \alpha + c\beta$,

$1 \leq c \leq n^8+1$, of the field $\mathbb{Q}(\alpha, \beta)$ and polynomials $a(X)$ and $b(X)$ in $\mathbb{Q}[X]$ such that $\alpha = a(\gamma)$ and $\beta = b(\gamma)$.

Any irreducible factors $g_i(X)$ can be written as a bivariate polynomial $g_i(X, \alpha)$. Hence symbolically $g_i^\sigma(X)$ is the bivariate polynomial $g_i(X, \beta)$. In $g_i(X, \alpha)$ and $g_i(X, \beta)$ we replace α and β by $a(\gamma)$ and $b(\gamma)$ respectively to get the polynomials $g_i(X)$ and $g_i^\sigma(X)$ as polynomials of over $\mathbb{Q}(\gamma) = \mathbb{Q}(\alpha, \beta)$. Having computed the polynomials g_i^σ and g_j as polynomials over the same field $\mathbb{Q}(\gamma)$, one can compute their g.c.d in polynomial time. The complete algorithm to compute the polynomial T_Σ is given below (Algorithm 10). Clearly Algorithm 10 runs in time bounded by a polynomial in the input size. The correctness of the algorithm follows from the correctness of Algorithm 9.

$S \leftarrow \{T_\Delta, g\}$
repeat
  $S' \leftarrow \{g_i^\sigma \mid g_i \in S\}$.
  foreach *factor* g_j **do**
    if $gcd(g_j, h^\sigma)$ *is nontrivial for some* $h^\sigma \in S'$ **then**
    $S \leftarrow S \cup \{g_j\}$
  end
until S *is unchanged* ;
Output $T_\Sigma = \prod_{g_i \in S} g_i$

Algorithm 10: Computing T_Σ

□

We now complete the proof of Theorem 7.3. By Proposition 7.4, it suffices to compute the set $\mathcal{S}$ of polynomials T_Σ such that Σ is a minimal G-block properly containing Δ. Let $f(X)$ factor as $f(X) = g_0 \dots g_r$ over $\mathbb{Q}_\Delta$ with $g_0 = T_\Delta$.

Let Σ_i be the smallest G-block containing Δ and all the roots of g_i. For any G-block Σ such that Δ is a maximal G-subblock of Σ, there is an i, $1 \leq i \leq r$ such that $\Sigma = \Sigma_i$. Using Lemma 7.7 we compute T_{Σ_i}'s for each $1 \leq i \leq r$. The G-block $\Sigma_j \subseteq \Sigma_i$ if and only if T_{Σ_j} divides T_{Σ_i} and hence Σ_i is a minimal G-block properly containing Δ if and only if T_{Σ_i} is not divisible by T_{Σ_j} for all $j \neq i$. The set $\mathcal{S}$ is the collection of all the polynomials T_{Σ_i} such that for all $j \neq i$, $T_{\Sigma_j} \nmid T_{\Sigma_i}$. Clearly computing $\mathcal{S}$ is in polynomial time.

Having computed the set $\mathcal{S}$ we compute the fields $\mathbb{Q}_{\Sigma_i}$ for all polynomials $T_{\Sigma_i}(X) \in \mathcal{S}$. Recall that $\mathbb{Q}_{\Sigma_i}$ is obtained by adjoining the coefficients of the polynomial T_{Σ_i} each of which are symmetric functions of roots of $f(X)$

in Σ_i. Thus although computing $\mathbb{Q}_{\Sigma_i}$ takes time proportional to size(f) and size$(\mathbb{Q}_\Delta)$, the size of the explicit data computed for the field $\mathbb{Q}_{\Sigma_i}$ is polynomial in size(f) and is independent of the size of presentation of $\mathbb{Q}_\Delta$. This completes the proof of Theorem 7.3.

Remark 7.8. That the size of the computed presentation of $\mathbb{Q}_\Sigma$ is bounded by a polynomial in size(f) and is independent on the size of $\mathbb{Q}_\Delta$ is important because Theorem 7.3 will be used repeatedly in our algorithms to compute a tower of fields $\mathbb{Q}_{\Delta_0} \supset \ldots \supset \mathbb{Q}_{\Delta_m}$ for a maximal chain of G-blocks $\Delta_0 \subset \ldots \subset \Delta_m$. The length m of such a chain of G-blocks could be as large as $\lg n$ where n is the degree of f. If the size of $\mathbb{Q}_{\Delta_i}$ depended on the size of presentation of $\mathbb{Q}_{\Delta_{i-1}}$ then the presentation of $\mathbb{Q}_{\Delta_m}$ could be as large as $n^{\lg n}$.

7.2 Nilpotence testing for Galois groups

In this section we give a polynomial time algorithm for testing whether the Galois group of a given polynomial is nilpotent. We give a characterisation of transitive nilpotent groups which can be tested in polynomial time. Recall that a finite group G is *nilpotent* if and only if every Sylow subgroup of G is normal (see Lemma 3.5 for other equivalent definitions). For a nilpotent group G and a prime p that divides $\#G$, there is a unique p-Sylow subgroup which we denote in this section by G_p. In fact G_p is the set of all element of G that has order a power of p. Moreover any subgroup H of G is also nilpotent and the p-Sylow subgroup of H is $G_p \cap H$. If $\{p_1, \ldots, p_k\}$ are the set of prime factors of $\#G$ then $G = G_{p_1} \times \ldots \times G_{p_k}$.

Lemma 7.9. *Let G be a transitive nilpotent permutation group on Ω then*

1. *For all primes p, p divides $\#G$ if and only if p divides $\#\Omega$.*
2. *For any prime $p \mid \#G$ and $\alpha \in \Omega$ there is a block Σ_p^α containing α such that $\#\Sigma_p^\alpha$ is the highest power of p that divides $\#\Omega$.*
3. *Let Δ be any G-block containing α such that $\#\Delta = p^l$ for some prime p dividing $\#G$. Then $\Delta \subseteq \Sigma_p^\alpha$. Also for all q different from p the q-Sylow subgroup of G_Δ is same as the q-Sylow subgroup of G_α, i.e. $G_q \cap G_\Delta = G_q \cap G_\alpha$.*

Proof. As G is transitive on Ω, $\#\Omega$ divides $\#G$ by Orbit-Stabiliser formula (Theorem 3.8). Hence, each prime factor of $\#\Omega$ divides $\#G$. Conversely let

p be a prime factor of $\#G$. For $\alpha \in \Omega$, the set $\Sigma_p^\alpha = \alpha^{G_p}$ is a nontrivial G-block as G_p is a normal subgroup of G (Lemma 3.13). Since G_p is transitive on Σ_p^α, it follows from the Orbit-Stabiliser formula that $\#\Sigma_p^\alpha$ divides $\#G_p$. Hence $\#\Sigma_p^\alpha$ is p^l for some $l > 0$. Since p divides the cardinality of a G-block Σ_p^α, p must divide $\#\Omega$. This proves part 1.

Next, we prove (2). From the Galois correspondence of G-blocks (Theorem 3.11) we have $[\Omega : \Sigma_p^\alpha] = [G : G_{\Sigma_p^\alpha}]$. The prime p does not divide $[G : G_p]$ as G_p is the p-Sylow subgroup of G. Therefore p does not divide $[G : G_{\Sigma_p^\alpha}]$ either as G_p is a subgroup of $G_{\Sigma_p^\alpha}$. Hence p is not a factor of $[\Omega : \Sigma_p^\alpha]$ and $\#\Sigma_p^\alpha$ is the highest power of p that divides $\#\Omega$.

To prove part 3 notice that G_Δ is a nilpotent group with the unique normal q-Sylow subgroup $G_q \cap G_\Delta$. Therefore we have $G_\Delta = \prod_q (G_q \cap G_\Delta)$. By the Galois correspondence (Theorem 3.11) of blocks we have

$$\#\Delta = [G_\Delta : G_\alpha] = \prod_q [G_q \cap G_\Delta : G_q \cap G_\alpha]. \tag{7.1}$$

Since $G_q \cap G_\Delta$ is a q-group, the prime p divides the index $[G_q \cap G_\Delta : G_q \cap G_\alpha]$ if and only if $q = p$. However, in Equation 7.1 $\#\Delta$ is a power of p. This is possible if and only if $[G_q \cap G_\Delta : G_q \cap G_\alpha] = 1$ for all $q \neq p$. Thus $G_q \cap G_\Delta = G_q \cap G_\alpha$ for all q different from p.

The group G_Δ is therefore the product group $G_p \cap G_\Delta \times \prod_{q \neq p} G_q \cap G_\alpha$. Since the group $G_{\Sigma_p^\alpha}$ contains both G_p and G_α we have $G_{\Sigma_p^\alpha} \geq G_\Delta$. Thus by Galois correspondence of blocks (Theorem 3.11), Δ is a G-subblock of Σ_p^α. $\square$

Nilpotent groups behave almost like p-groups. Let G be a transitive nilpotent permutation group on Ω and let p be a prime dividing $\#G$. We prove that as far as G-blocks contained in Σ_p^α are concerned, G behaves like G_p. The following lemma makes this precise.

Lemma 7.10. *Let G be a transitive nilpotent permutation group acting on Ω. Let p be any prime that divides $\#G$ and let G_p be the corresponding p-Sylow subgroup. Consider any element $\alpha \in \Omega$ and let Σ_p^α be the G-block α^{G_p}. A set $\Delta \subseteq \Sigma_p^\alpha$ is a G-block if and only if Δ is a G_p-block under the transitive action of G_p on Σ_p^α.*

Proof. Clearly any G-block contained in Σ_p^α is a G_p-block as G contains G_p. Conversely consider a G_p-block Δ of Σ_p^α. The group $G_p \cap G_\Delta$ contains $G_p \cap G_\alpha$. To see this consider the transitive action of G_p restricted to Σ_p^α. The restriction action is a homomorphism $\psi : G_p \to \mathrm{Sym}\left(\Sigma_p^\alpha\right)$. Let H

denote the image $\psi(G_p) = G_p|_{\Sigma_p^\alpha}$. The groups $G_p \cap G_\Delta$ and $G_p \cap G_\alpha$ are the pullbacks $\psi^{-1}(H_\Delta)$ and $\psi^{-1}(H_\alpha)$ respectively. Since the subset Δ is a H-block of Σ_p^α and contains α, $H_\Delta \geq H_\alpha$. Therefore $G_p \cap G_\Delta \geq G_p \cap G_\alpha$.

Consider the group $G' = (G_p \cap G_\Delta) \times \prod_{q \neq p} G_q \cap G_\alpha$. The group G_α is nilpotent and hence $G_\alpha = (G_p \cap G_\alpha) \times \prod_{q \neq p} G_q \cap G_\alpha$. Since $G_p \cap G_\Delta \geq G_p \cap G_\alpha$ we have $G' \geq G_\alpha$. Therefore by the Galois correspondence of blocks (Theorem 3.11) we have $\Delta = \alpha^{G'}$ is a G-block between $\{\alpha\}$ and Σ_p^α. □

We now study the structure of blocks of a p-group. We state the following result due to Luks [46, Lemma 1.1].

Lemma 7.11 (Luks)**.** *Let G be a p-group acting transitively on Ω and let Δ be a maximal G-block. Then the index $[\Omega : \Delta]$ is exactly p and $G_\Delta = G(\Omega/\Delta) = G^\Delta$ is a normal group of index p in G.*

Proof. Let Δ be a maximal G-block. By Galois correspondence of blocks we have $[\Omega : \Delta] = [G : G_\Delta]$. Suppose that $[G : G_\Delta] = p^l$ for $l \geq 1$. The group G being a p-group, it follows that there is a subnormal series $G = G_0 \rhd G_1 \ldots \rhd G_l = G_\Delta$ such that $[G_i : G_{i+1}] = p$ [30, Theorem 4.3.2]. Let α be any element of Δ. Since $G_\Delta \geq G_\alpha$, $(G_i)_\alpha = G_\alpha$. Therefore by Orbit-Stabiliser formula $\frac{\#\alpha^G}{\#\alpha^{G_1}} = \frac{\#G}{\#G_1} = [G : G_1] = p$. However G_1 is a normal subgroup of G and $\alpha^{G_1} \neq \Omega$. Therefore α^{G_1} is the maximal block Δ and $G_\Delta = G_1$.

Recall that $G^\Delta = G(\Omega/\Delta)$ is the largest normal subgroup of $G = G_\Omega$ contained in G_Δ (Theorem 3.12. However $G_\Delta = G_1$ itself is normal. Hence $G_\Delta = G^\Delta$. □

Applying Lemma 7.11 repeatedly we have the following lemma.

Lemma 7.12. *Let G be a transitive p-group acting on Ω and $\alpha \in \Omega$. Let $\{\alpha\} = \Delta_0 \subset \ldots \subset \Delta_t = \Omega$ be any maximal chain of G-blocks. Then*

1. *$[\Delta_{i+1} : \Delta_i] = p$ for all $0 \leq i < t$.*
2. *$G(\Delta_{i+1}/\Delta_i) = G_{\Delta_i}$.*
3. *The group G_{Δ_i} is a normal subgroup of $G_{\Delta_{i+1}}$ and the quotient group $G_{\Delta_{i+1}}/G_{\Delta_i}$ is cyclic of order p.*

In particular any minimal G-block is of cardinality p.

We now prove the following important property of transitive nilpotent permutation groups.

Lemma 7.13. *Let G be a transitive nilpotent permutation group on Ω. Let p be any prime dividing $\#G$. Let Δ be any G-block such that $\#\Delta = p^l$ for some integer $l \geq 0$. Let m be the highest power of p that divides $\#\Omega$. If $l < m$ then we have*

1. *There exists a G-block Σ such that Δ is a maximal G-subblock of Σ and $[\Sigma : \Delta] = p$.*

2. *For all G-blocks Σ such that Δ is a maximal G-subblock of Σ and $[\Sigma : \Delta] = p$, G_Δ is a normal subgroup of G_Σ.*

Proof. Let Σ_p^α as before denote the G-blocks α^{G_p}. Since $\#\Delta$ is a power of p it follows that Δ is a G-subblock of Σ_p^α (Lemma 7.9). The subset Δ is a G_p-block on the transitive action of G_p on Σ_p^α (Lemma 7.10). Consider the action of the p-group G_p on Σ_p^α. If $l < m$ there is a G_p-block Σ such that $\Sigma_p^\alpha \supseteq \Sigma \supset \Delta$ and $[\Sigma : \Delta] = p$. By Lemma 7.10 it follows that Σ is a G-block contained in Σ_p^α. This proves part 1.

Let α be any element of Δ. It follows from Lemma 7.9 that for all $q \neq p$ the q-Sylow subgroup of G_Σ and G_Δ are both $G_q \cap G_\alpha$. Let $\widehat{G}_p$ be the product group $\prod_{q \neq p} G_q$. The groups G_Σ and G_Δ are $(G_p \cap G_\Sigma) \times (\widehat{G}_p \cap G_\alpha)$ and $(G_p \cap G_\Delta) \times (\widehat{G}_p \cap G_\alpha)$ respectively. Moreover the groups $G_p \cap G_\Sigma$ and $G_p \cap G_\Delta$ are p-groups with index $[G_p \cap G_\Sigma : G_p \cap G_\Delta] = [G_\Sigma : G_\Delta] = [\Sigma : \Delta] = p$. Therefore $G_p \cap G_\Delta$ is a normal subgroup of $G_p \cap G_\Sigma$. As a result $G_\Delta = (G_p \cap G_\Delta) \times (\widehat{G}_p \cap G_\alpha)$ is a normal subgroup of $G_\Sigma = (G_p \cap G_\Sigma) \times (\widehat{G}_p \cap G_\alpha)$ and the quotient group $\frac{G_\Sigma}{G_\Delta} = \frac{G_p \cap G_\Sigma}{G_p \cap G_\Delta}$ is isomorphic to $\mathbb{F}_p$.

□

We give the following characterisation of transitive nilpotent groups.

Theorem 7.14. *Let G be a transitive permutation group on Ω then the following are equivalent.*

1. *G is nilpotent.*

2. *For all primes p dividing $\#G$, p divides $\#\Omega$ and there exists a maximal chain of G-block $\{\alpha\} = \Delta_0 \subset \ldots \subset \Delta_m$ such that*

    (a) *m is the highest power of p dividing $\#\Omega$.*

    (b) *G_{Δ_i} is a normal subgroup of $G_{\Delta_{i+1}}$.*

    (c) *$[\Delta_{i+1} : \Delta_i] = p$ for all $0 \leq i < m$.*

    (d) *$p \nmid [G : G^{\Delta_m}]$.*

Proof. If G is nilpotent then condition 2 holds. The required maximal chain of G-blocks is any maximal chain between $\{\alpha\}$ and Σ_p^α. We now prove the converse.

Consider any prime p dividing $\#G$. The prime p divides $\#\Omega$ and let $m > 0$ be the highest power of p dividing $\#\Omega$. Let $\{\alpha\} = \Delta_0 \subset \ldots \subset \Delta_m$ be a maximal chain of G-blocks satisfying the conditions 2a–2d. We prove that G^{Δ_m} is the unique p-Sylow subgroup for G.

Recall that $G\left(\Delta_{i+1}/\Delta_i\right)$ is the largest subgroup of G_{Δ_i} that is normal in $G_{\Delta_{i+1}}$ (Theorem 3.12). However since G_{Δ_i} itself is a normal subgroup of $G_{\Delta_{i+1}}$ it follows that $G_{\Delta_i} = G\left(\Delta_{i+1}/\Delta_i\right)$. Moreover $[G_{\Delta_{i+1}} : G_{\Delta_i}] = [\Delta_{i+1} : \Delta_i] = p$ and therefore $[G_{\Delta_{i+1}} : G\left(\Delta_{i+1}/\Delta_i\right)] = p$.

The group $\frac{G^{\Delta_{i+1}}}{G^{\Delta_i}}$ is a subgroup of the l_i-fold product of $\frac{G_{\Delta_{i+1}}}{G(\Delta_{i+1}/\Delta_i)}$ (Theorem 3.12). Hence $\frac{G^{\Delta_{i+1}}}{G^{\Delta_i}}$ is of order p^l for some l. As a result we have

$$\#G^{\Delta_m} = [G^{\Delta_m} : G^{\Delta_{m-1}}] \ldots [G^{\Delta_1} : G^{\Delta_0}] = \text{ a power of } p.$$

The group G^{Δ_m} is thus a p-group. Furthermore $p \nmid [G : G^{\Delta_m}]$ (condition 2d). Therefore G^{Δ_m} is a p-Sylow subgroup of G. Moreover the group $G^{\Delta_m} = G\left(\Omega/\Delta_m\right)$ is also a normal subgroup of $G = G_\Omega$ (part 1 of Theorem 3.12). Thus we have shown that for every prime p that divides $\#G$ the p-Sylow subgroup is normal. This proves that G is nilpotent. $\square$

7.2.1 The nilpotence test

Given $f(X) \in \mathbb{Q}[X]$ we want to test if the Galois group of $f(X)$ is nilpotent. If f is reducible then the Galois group of f is nilpotent if and only if the Galois group of each of its irreducible factors is nilpotent. This is because nilpotent groups are closed under products and subgroups. Since in polynomial time one can factor polynomials over $\mathbb{Q}$ (Theorem 6.7), without loss of generality we assume that $f(X)$ is irreducible. Let G be the Galois group of $f(X)$ considered as a subgroup of $\mathrm{Sym}\left(\Omega\right)$, where Ω is the set of roots of $f(X)$. Since f is irreducible, G is transitive on Ω.

We describe the main idea behind the algorithm. It follows from Theorem 7.14 that G is nilpotent if and only if for all primes p that divide the order of G, there is a maximal chain of G-blocks $\{\alpha\} = \Delta_0 \subset \ldots \subset \Delta_m$ satisfying the conditions of part 2 of Theorem 7.14. We do not have access to the sets Δ_i and the groups G_{Δ_i}. However we prove that conditions in part 2 of Theorem 7.14 can be verified once the fields $\mathbb{Q}(\alpha) = \mathbb{Q}_{\Delta_0} \supset \ldots \supset \mathbb{Q}_{\Delta_m}$ are known. Recall that for a G-block Δ, $\mathbb{Q}_\Delta$ is the fixed field of the splitting field $\mathbb{Q}_f$ under the automorphisms of G_Δ. Algorithm 11 is the complete algorithm.

Input: A polynomial $f(X) \in \mathbb{Q}[X]$ of degree n.
Result: *Accepts* f if Galois group of $f(X)$ is nilpotent, *Rejects* otherwise.

```
  Verify whether f(X) is solvable.
1 Compute the set P of all the prime factors of #Gal(f).
  Let G denote the Galois group of f thought of as a permutation
  group on Ω, the set of roots of f.
2 foreach p ∈ P do
3     if p does not divide n then Reject.
      Let m be the highest power of p dividing n.
      Q_{Δ_0} ← Q[X]/f(X).
4     for i ← 1 to m do
          Using Theorem 7.3 compute the set of fields

              F = {Q_Σ | Δ is a maximal G-block of Σ}.

5         Let Q_Σ be any field of F such that [Q_Σ : Q_{Δ_{i-1}}] = p. If no
          such field exists then Reject.
6         if Q_Σ/Q_{Δ_{i-1}} is not normal then Reject.
          else Q_{Δ_i} ← Q_Σ.
      end
      Let μ_{Δ_m}(X) be the primitive polynomial for Q_{Δ_m}.
7     if p divides #Gal(μ_{Δ_m}) then Reject.
  end
  Accept.
```

Algorithm 11: Nilpotence test

Given a polynomial $f(X)$ with solvable Galois group, as a by product of the Landau-Miller test [39], there is a polynomial time algorithm to compute the prime factors of $\#\mathrm{Gal}\,(f)$ (see also Theorem 7.23). Therefore the steps 1 and 7 of Algorithm 11 can be performed in polynomial time. All other steps can clearly be performed in polynomial time. This gives us the following proposition.

Proposition 7.15. *Algorithm 11 runs in time polynomial in* $\mathrm{size}\,(f)$.

We now argue the correctness of the algorithm in the following two propositions.

Proposition 7.16. *Algorithm 11 accepts $f(X)$ if the Galois group of f is nilpotent.*

Proof. Let G be the Galois group of $f(X)$ and let p be any prime that divides $\#G$. Let G_p be the p-Sylow subgroup of G and let Σ_p^α be the G-block α^{G_p}. The loop in step 4 in fact constructs the tower of fields $\mathbb{Q}_{\Sigma_p^\alpha} = \mathbb{Q}_{\Delta_m} \subset \ldots \subset \mathbb{Q}_{\Delta_0} = \mathbb{Q}(\alpha)$ for a maximal chain of G-blocks $\{\alpha\} = \Delta_0 \subset \ldots \Delta_m = \Sigma_p^\alpha$. Lemma 7.13 guarantees that the step 5 will never fail.

The extension $\mathbb{Q}_{\Delta_i}/\mathbb{Q}_{\Delta_{i-1}}$ is normal because $G_{\Delta_{i-1}}$ is a normal subgroup of G_{Δ_i}. Let K be the normal closure of $\mathbb{Q}_{\Delta_m}$ then it follows form Proposition 7.2 that $\mathrm{Gal}\,(\mathbb{Q}_f/K)$ is G^{Δ_m}. The Galois group of $\mu_{\Delta_m}(X)$ is the quotient group $\frac{G}{G^{\Delta_m}}$. Since $G^{\Delta_m} = G^{\Sigma_p^\alpha} = G_p$, p does not divide the order of the Galois group of $\mu_{\Delta_m}(X)$.

Thus no step in the loop 4 will reject the input if the Galois group of f is nilpotent. This completes the proof. □

We now prove the converse.

Proposition 7.17. *If Algorithm 11 accepts then the Galois group of $f(X)$ is nilpotent.*

Proof. Let Ω be the roots of $f(X)$ and let G be the Galois group of $f(X)$ as a permutation group on Ω. Since the algorithm has accepted $f(X)$ we have the following conditions of the Galois group G of $f(X)$.

1. Every prime p that divides $\#G$ also divides $n = \#\Omega$. This is verified in step 3.

2. For any prime p dividing $\#G$ let m be the highest power of p dividing n. There is a maximal chain $\{\alpha\} = \Delta_0 \subset \ldots \subset \Delta_m$ of G-blocks such that for all $0 \leq i < m$

   (a) G_{Δ_i} is a normal subgroup of $G_{\Delta_{i+1}}$. We verified this in step 6 by checking that the extension $\mathbb{Q}_{\Delta_{i+1}}/\mathbb{Q}_{\Delta_i}$ is a normal.

   (b) $[\Delta_{i+1} : \Delta_i] = p$. This is because $[\Delta_{i+1} : \Delta_i] = [\mathbb{Q}_{\Delta_{i+1}} : \mathbb{Q}_{\Delta_i}] = p$ (Proposition 7.2).

   (c) The prime p does not divide $[G : G^{\Delta_m}]$. As argued before the Galois group of the polynomial μ_{Δ_m}, a primitive polynomial of $\mathbb{Q}_{\Delta_m}$, is the Galois group $\frac{G}{G^{\Delta_m}}$. Thus in step 7 we have verified that p does not divide $\#\frac{G}{G^{\Delta_m}} = [G : G^{\Delta_m}]$.

Hence from Theorem 7.14, G is nilpotent. □

Combining Propositions 7.15, 7.16 and 7.17 we have the main theorem of this section.

Theorem 7.18. *Given a polynomial $f(X) \in \mathbb{Q}[X]$, there is an algorithm that runs in time polynomial in* size (f) *that decides whether the Galois group of f is nilpotent.*

7.3 Γ_d-testing for Galois groups

In this section we show that the technique underlying the Landau-Miller solvability test can be adapted to efficiently solve a more general problem, the problem of testing whether the Galois group of a polynomial $f(X) \in \mathbb{Q}[X]$ is in Γ_d for constant d. Recall that a group G is in Γ_d if there is a composition series $G = G_0 \rhd \ldots \rhd G_t = \{1\}$ such that G_i/G_{i+1} is either abelian or isomorphic to a subgroup of S_d. Given a polynomial $f(X)$ over $\mathbb{Q}$ of degree n, we give an algorithm that runs in time polynomial in size (f) and n^d to check whether the Galois group of f is in Γ_d. For constant d this yields a polynomial time Γ_d-test. As a byproduct of our polynomial time Γ_d-testing, we obtain a polynomial time algorithm to compute the prime factors of $\#\mathrm{Gal}\,(f)$ for any polynomial f with Galois group in Γ_d. Note that for $d < 5$, Γ_d is the class of solvable groups and hence our result is a generalisation of the result of Landau-Miller [39].

We are given a polynomial $f(X)$ over $\mathbb{Q}$. Since the class Γ_d is closed under subgroups and quotients and products, without loss of generality assume that $f(X)$ is irreducible of degree n. For describing the Γ_d test we fix the following notation for the rest of this section. Let G be the Galois group of f. Consider the faithful action of G as a permutation group on Ω, the set of roots of f. Let $\{\alpha\} = \Delta_0 \subset \ldots \subset \Delta_m = \Omega$ be any maximal chain of G-blocks. Recall that for all $0 \leq i < m$ the group $G\,(\Delta_{i+1}/\Delta_i)$ is a normal subgroup of $G_{\Delta_{i+1}}$ (Theorem 3.12). We have the following proposition.

Proposition 7.19. *The group G is in Γ_d if and only if the quotient groups $\frac{G_{\Delta_{i+1}}}{G(\Delta_{i+1}/\Delta_i)}$, $0 \leq i < m$, are all in Γ_d.*

Proof. The series $G = G^{\Delta_t} \rhd \ldots \rhd G^{\Delta_0} = 1$ gives a normal series for G. Hence G is in Γ_d if and only if for each $0 \leq i < m$ the quotient $\frac{G^{\Delta_{i+1}}}{G^{\Delta_i}}$ is in Γ_d. Consider the subgroups $G_{\Delta_{i+1}}$ and $G\,(\Delta_{i+1}/\Delta_i)$ of G. If G is in Γ_d so are $G_{\Delta_{i+1}}$ and $G\,(\Delta_{i+1}/\Delta_i)$ and hence their quotient $\frac{G_{\Delta_{i+1}}}{G(\Delta_{i+1}/\Delta_i)}$ (Proposition 3.3). On the other hand, $\frac{G^{\Delta_{i+1}}}{G^{\Delta_i}}$ is isomorphic to a subgroup of $\left(\frac{G_{\Delta_{i+1}}}{G(\Delta_{i+1}/\Delta_i)}\right)^l$ for some l (Theorem 3.12) and therefore $\frac{G^{\Delta_{i+1}}}{G^{\Delta_i}}$ is in Γ_d if $\frac{G_{\Delta_{i+1}}}{G(\Delta_{i+1}/\Delta_i)}$ is in Γ_d. Hence G is in Γ_d if and only if for each $0 \leq i < m$ the quotient group $\frac{G_{\Delta_{i+1}}}{G(\Delta_{i+1}/\Delta_i)}$ is in Γ_d. □

We have no access to the groups $G_{\Delta_{i+1}}$ and $G(\Delta_{i+1}/\Delta_i)$. However using Theorem 7.3 we can compute the field $K_i = \mathbb{Q}_{\Delta_i}$, $0 \leq i \leq m$, for some maximal chain of G-blocks $\{\alpha\} = \Delta_0 \subset \ldots \subset \Delta_m = \Omega$. Let L_i be the normal closure of K_{i-1} over K_i. Using Proposition 7.2 and 7.19 we have the following proposition.

Proposition 7.20. *The Galois group G is in Γ_d if and only the Galois groups* $\mathrm{Gal}(L_i/K_i)$*, $1 \leq i \leq m$, is in Γ_d. Furthermore, if G is in Γ_d then the degree $[L_i : \mathbb{Q}] = n^{O(d)}$.*

Proof. The field L_i is the fixed field $\mathrm{Fix}(\mathbb{Q}_f, G(\Delta_i/\Delta_{i-1}))$ (Proposition 7.2) and hence the Galois group $\mathrm{Gal}(\mathbb{Q}_f/L_i)$ is $G(\Delta_i/\Delta_{i-1})$. Moreover the Galois group of $\mathbb{Q}_f/\mathbb{Q}_{\Delta_i}$ is G_{Δ_i} and hence by the fundamental theorem of Galois theory (Theorem 6.1), the Galois group $\mathrm{Gal}(L_i/K_i)$ is the quotient group $\frac{G_{\Delta_i}}{G(\Delta_i/\Delta_{i-1})}$. It then follows from Proposition 7.19 that G is in Γ_d if and only if each of the Galois groups $\mathrm{Gal}(L_i/K_i)$ is in Γ_d.

The block Δ_{i-1} is a maximal G-subblock of Δ_i. Recall that the group $\frac{G_{\Delta_i}}{G(\Delta_i/\Delta_{i-1})}$ acts faithfully as a primitive permutation group on the set of G-blocks $\mathcal{B}(\Delta_i/\Delta_{i-1})$ (Theorem 3.12). Moreover if G is in Γ_d then so is $\frac{G_{\Delta_i}}{G(\Delta_i/\Delta_{i-1})}$ and hence by the Babai-Cameron-Pálfy bound (Theorem 3.10) we have

$$[L_i : K_i] = \#\mathrm{Gal}(L_i/K_i) = \#\frac{G_{\Delta_i}}{G(\Delta_i/\Delta_{i-1})} \leq [\Delta_i : \Delta_{i-1}]^{O(d)} \leq n^{O(d)}$$

Therefore $[L_i : \mathbb{Q}] \leq n^{O(d)}$. □

The above proposition in particular implies that if G is in Γ_d then the fields L_i can be computed in time polynomial in $\mathrm{size}(f)$ and n^d. To see this note that we have computed the explicit data of the fields K_i and K_{i-1} which are of size at most a polynomial in $\mathrm{size}(f)$. Since the degree of the normal closure L_i of K_{i-1} over K_i is bounded by a polynomial in n^d, we can use Landau's algorithm (Theorem 6.10) to compute the field L_i. Thus we have the following proposition.

Proposition 7.21. *If the Galois group G is in Γ_d then there is an algorithm that runs in time polynomial in* $\mathrm{size}(f)$ *and n^d to compute the fields L_i.*

We now describe the polynomial time algorithm for Γ_d-testing. The algorithm first computes the fields K_i in time polynomial in $\mathrm{size}\,(f)$. Let $b(n)$ be the bound on the size of primitive subgroups of S_n that are in Γ_d. By the Babai-Cameron-Pálfy bound we have $b(n) = n^{O(d)}$. For each i, using Landau's algorithm (Theorem 6.10), checks whether the degree $[L_i : K_i]$ is at most $b(n)$ and if yes computes it. If any of the degrees $[L_i : K_i]$ is greater than $b(n)$ then clearly G is not in Γ_d.

Having computed the fields L_i and K_i, the Galois groups $\mathrm{Gal}\,(L_i/K_i)$ are explicitly computed using Landau's algorithm. In time bounded by a polynomial in n^d we verify whether each of the groups $\mathrm{Gal}\,(L_i/K_i)$ is in Γ_d (this is sufficient because of Proposition 7.20). Algorithm 12 is the complete algorithm.

Input: An irreducible polynomial $f(X)$ of degree n over $\mathbb{Q}$.
Result: *Accept* if the Galois group of $f(X)$ is in Γ_d, *Reject* otherwise.
Let G be the Galois group of $f(X)$ as a permutation group on Ω, the roots of $f(X)$.
Using Theorem 7.3 compute the fields $K_i = \mathbb{Q}_{\Delta_i}$ for a maximal chain of G-blocks $\{\alpha\} = \Delta_0 \subset \ldots \subset \Delta_m = \Omega$.
foreach $1 \leq i \leq m$ **do**
  if $[L_i : K_i] > b(n)$ **then** **Reject**.
  else **if** $\mathrm{Gal}\,(L_i/K_i)$ *is not in* Γ_d **then Reject**.
end
Accept.

Algorithm 12: Γ_d-testing

The main theorem of this section follows.

Theorem 7.22. *Given a polynomial $f(X) \in \mathbb{Q}[X]$, there is an algorithm running in time polynomial in $\mathrm{size}\,(f)$ and $n^{O(d)}$ that decides whether the Galois group of f is in Γ_d.*

For any $1 \leq i \leq m$, we have $\#G = [\mathbb{Q}_f : \mathbb{Q}] = [\mathbb{Q}_f : L_i].[L_i : K_i].[K_i : \mathbb{Q}]$. Therefore any prime factor of $[L_i : K_i]$ divides $\#G$. Conversely $G^{\Delta_i}/G^{\Delta_{i-1}}$ is a subgroup of l_i-fold product of $\frac{G_{\Delta_i}}{G(\Delta_i/\Delta_{i-1})}$ (Theorem 3.12) for some integer $l_i \geq 0$. However by Proposition 7.20 $\frac{G_{\Delta_i}}{G(\Delta_i/\Delta_{i-1})} = \mathrm{Gal}\,(L_i/K_i)$. It follows that any prime factor of $\#G$ is a prime factor of $[L_i : K_i]$ for some $1 \leq i \leq m$. Therefore the set of primes dividing $\#G$ is exactly the set $\{p | p \text{ prime and } \exists\ 1 \leq i \leq m\ p \text{ divides } [L_i : K_i]\}$. If the Galois group G is in Γ_d, in time polynomial in $\mathrm{size}\,(f)$ and n^d we can compute the fields L_i and K_i. As a result we have the following theorem.

Theorem 7.23. *Given $f(X) \in \mathbb{Q}[X]$ with Galois group in Γ_d there is an algorithm running in time polynomial in* size (f) *and n^d that computes all the prime factors of* $\#\mathrm{Gal}\,(f)$.

7.4 Discussion

We saw that even though computing the Galois group of a polynomial is hard, certain properties of Galois groups can be efficiently tested. Landau and Miller showed that solvability is one such property. We have added nilpotence testing and Γ_d testing to this list. A group being solvable is in some sense a "local property". The solvability of a group G can be established by looking at the composition series of G. The composition series considered for G was $G = G^{\Delta_m} \rhd \ldots \rhd G^{\Delta_0} = 1$ for a maximal chain of G-blocks $\{\alpha\} = \Delta_0 \subset \ldots \subset \Delta_m = \Omega$. The two-way Galois correspondence of Theorem 7.1 and Theorem 3.12 ensured that it was sufficient to compute the fields $\{\mathbb{Q}_{\Delta_i}\}_{0 \leq i \leq m}$, to infer the solvability of $G^{\Delta_i}/G^{\Delta_{i-1}}$ and hence G. Nilpotence testing cannot be inferred from the composition series. However Theorem 7.14 together with Theorem 7.1 ensured that the nilpotence of G can be tested once the tower of fields $\{\mathbb{Q}_{\Delta_i}\}_{0 \leq i \leq m}$ for a suitable maximal chain of G-blocks $\{\alpha\} = \Delta_0 \subset \ldots \subset \Delta_m$ is computed. In this context an interesting open problem is to test whether the Galois group of a polynomial is supersolvable. A group G is *supersolvable* if there is a *normal series* $G = G_0 \rhd \ldots \rhd G_t = 1$ such that each of the quotient group G_i/G_{i+1} is cyclic. (see Chapter 10 of Hall's book [30]). Super solvable groups are a proper subclass of solvable groups and contain nilpotent groups. However, it is not clear whether the Landau-Miller solvability test or our nilpotence test can be adapted to an efficient supersolvability test. Even conditional results, for example assuming the generalised Riemann hypothesis, would be interesting.

What about nilpotence and Γ_d testing $\mathrm{Gal}\,(f)$ for polynomials $f(X)$ over a number field K? It is not difficult to see that our algorithms can be generalised. This is because our test require only certain basic algorithms like factoring of univariate polynomials and gcd computations and efficient algorithms for these basic tasks over arbitrary number field are known.

Chapter 8

Chebotarev density theorem and Order finding

In this chapter we study the problem of finding the order of the Galois group of a degree n polynomial $f(X) \in \mathbb{Q}[X]$ [7]. There is a polynomial time Turing reduction from computing the order to computing the Galois group because given a permutation group $G \leq S_n$ via its generators, the order of G can be computed in time polynomial in n (Theorem 3.9). In this chapter we show some conditional results. Assuming the generalised Riemann hypothesis we show better upper bounds for computing the order than the direct exponential time algorithm that follows from Landau's algorithm (Theorem 6.10).

Assuming the generalised Riemann hypothesis, we prove that there is a polynomial time deterministic algorithm that makes one query to a #P oracle to compute the order of the Galois group $\mathrm{Gal}(f)$ In particular, this shows that the order can be computed in PSPACE, which was not known before. Recall that computing the Galois group of a polynomial is not known to be in PSPACE as nothing better than the EXP upper bound is known. From the above result, by an application of Stockmeyer's result on approximating #P functions, we prove that there is a randomised algorithm with an NP oracle to approximate the order of the Galois group of $f(X)$.

Our next result is on computing the order of polynomials with Galois group in Γ_d. We give a polynomial time reduction from exact computation to approximate computation of order of $\mathrm{Gal}(f)$ for polynomials $f(X)$ with Galois group in Γ_d. Therefore assuming the generalised Riemann hypothesis, there is a randomised algorithm with NP-oracle for computing the order of $\mathrm{Gal}(f)$ exactly for polynomials $f(X)$ with Galois group in Γ_d.

We can assume that the given polynomial $f(X)$ is a monic polynomial over $\mathbb{Z}$. Otherwise by clearing denominator we can assume that $f(X) = a_0 + \ldots + a_n X^n$, $a_i \in \mathbb{Z}$. Consider the polynomial $g(X) = a_0 a_n^n + \ldots + a_i a_n^{n-i} X + \ldots + X^n$. Clearly $g(X)$ is a monic polynomial over $\mathbb{Z}$. Moreover $g(a_n X) = a_n^n f(X)$. Hence every root of $g(X)$ is of the form $a_n \alpha$ where α is a root of $f(X)$. Therefore $\mathbb{Q}_g = \mathbb{Q}_f$. Give $f(X)$ we can compute $g(X)$ in polynomial time and hence from now on, with out loss of generality, we will assume that the input polynomial $f(X)$ is a monic polynomial of $\mathbb{Z}$.

The main idea underlying these results is the following: For a positive integer x let $S^f(x)$ denote the number of primes $p \leq x$ such that $f(X) \pmod p$ splits completely over $\mathbb{F}_p$. It follows from the Chebotarev density theorem, which we describe in Section 8.1, that $S^f(x)$ is asymptotically $\frac{x}{\#G \ln x}$. Thus for large enough x, $\#G$ is close to $\frac{x}{S^f(x) \ln x}$. We prove that the function $x \mapsto S^f(x)$ is a #P function. The polynomial time algorithm makes a query to and #P oracle and computes $S^f(x)$. The effective version of Chebotarev density theorem guarantees that the order $\#G$ is then the nearest integer to $\frac{x}{S^f(x) \ln x}$. We now describe the Chebotarev density theorem which plays a crucial role in our complexity theoretic results.

8.1 Chebotarev density theorem

Let K be any number field and L/K be an extension of K. Recall that the ring of integers of L, $\mathbb{O}_L$, is a Dedekind domain and ideals of $\mathbb{O}_L$ has the unique factorisation property. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{O}_K$. The ideal $\mathfrak{p}\mathbb{O}_L$, which will also be denoted by $\mathfrak{p}$, need not be a prime ideal of $\mathbb{O}_L$. Let $\mathfrak{p}$ factorise as $\mathfrak{p} = \mathfrak{P}_1^{e_1} \ldots \mathfrak{P}_g^{e_g}$ over $\mathbb{O}_L$. If L/K is a Galois extension then all the exponent e_i are the same, i.e. $e_1 = \ldots = e_g = e$. A prime ideal $\mathfrak{p}$ of $\mathbb{O}_K$ is *ramified* over the extension L/K if $e > 1$ and *unramified* otherwise.

We now consider Galois extensions L/K. Let G be the Galois group of L/K. Consider a prime $\mathfrak{p}$ of $\mathbb{O}_K$ that is unramified in L. Let $\mathfrak{P}$ be any prime ideal of $\mathbb{O}_L$ that divides $\mathfrak{p}$. Since $\mathbb{O}_L$ and $\mathbb{O}_K$ are Dedekind domains it follows that $\mathbb{O}_L/\mathfrak{P}$ and $\mathbb{O}_K/\mathfrak{p}$ are finite fields of cardinality $\mathrm{N}(\mathfrak{P})$ and $\mathrm{N}(\mathfrak{p})$ respectively. Furthermore the field $\mathbb{O}_L/\mathfrak{P}$ is an extension of $\mathbb{O}_K/\mathfrak{p}$ and the corresponding Frobenius element is given by $\alpha (\mathrm{mod}\ \mathfrak{P}) \mapsto \alpha^{N(\mathfrak{p})} (\mathrm{mod}\ \mathfrak{P})$. For $\mathfrak{P} \mid \mathfrak{p}$ there is an element $\left(\frac{L/K}{\mathfrak{P}}\right)$ of $\mathrm{Gal}(L/K)$ such that

$$\left(\frac{L/K}{\mathfrak{P}}\right) \alpha = \alpha^{N(\mathfrak{p})} \pmod{\mathfrak{P}},$$

for all α in $\mathbb{O}_L$. This element is called the *Frobenius* element associated with

$\mathfrak{P}$ as its action modulo $\mathfrak{P}$ matches with the Frobenius element of the finite field extension $\frac{\mathbb{O}_L}{\mathfrak{P}}/\frac{\mathbb{O}_K}{\mathfrak{p}}$.

Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_g$ be the primes of $\mathbb{O}_L$ that divide $\mathfrak{p}$. The Galois group $\mathrm{Gal}\,(L/K)$ fixes the ideal $\mathfrak{p}$ and act transitively on the set $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_g\}$. In particular, if $\sigma \in \mathrm{Gal}\,(L/K)$ maps $\mathfrak{P}_1$ to $\mathfrak{P}_2$ then $\left(\frac{L/K}{\mathfrak{P}_2}\right) = \sigma \left(\frac{L/K}{\mathfrak{P}_1}\right) \sigma^{-1}$. Thus $\left(\frac{L/K}{\mathfrak{P}_i}\right)$ are all conjugates in $\mathrm{Gal}\,(L/K)$ and the subset $\mathrm{Frob}_{L/K}(\mathfrak{p})$ of $\mathrm{Gal}\,(L/K)$ defined by

$$\mathrm{Frob}_{L/K}(\mathfrak{p}) = \left\{ \left(\frac{L/K}{\mathfrak{P}}\right) : \mathfrak{P} | \mathfrak{p} \right\}$$

is a conjugacy class of $\mathrm{Gal}\,(L/K)$. Let C be any conjugacy class of G and let $\pi_C(x)$ denote the function

$$\pi_C(x) = \#\left\{\mathfrak{p} : \mathrm{Frob}_L(\mathfrak{p}) = C \text{ and } N(\mathfrak{p}) \leq x\right\}.$$

A remarkable result on the asymptotic value of $\pi_C(x)$ is the *Chebotarev density theorem* which states that $\pi_C(x) \sim \frac{\#C}{\#G} \frac{x}{\ln x}$. To apply this result in a complexity-theoretic setting we need the following effective version of the Chebotarev density theorem due to Lagarias and Odlyzko proved assuming the generalised Riemann Hypothesis [35].

Theorem 8.1 (Lagarias and Odlyzko)**.** *Let L/K be a Galois extension and C be any conjugacy class of* $\mathrm{Gal}\,(L/K)$*. Assuming the generalised Riemann hypothesis we have the following bound for $\pi_C(x)$:*

$$\left|\pi_C(x) - \frac{\#C}{\#G} \frac{x}{\ln x}\right| \leq O\left(\sqrt{x}.\ln x.\ln d_L + \#C\sqrt{x}\right).$$

An unramified prime ideal $\mathfrak{p}$ of K is said to be *completely split* if the number of prime ideals $\mathfrak{P}$ of L that divide $\mathfrak{p}$ is $[L : K]$. In this case $\mathrm{Frob}_L(\mathfrak{p})$ is the singleton conjugacy class containing the identity element. The number of completely split primes $\mathfrak{p}$ such that $\mathrm{N}\,(\mathfrak{p}) \leq x$ is denoted by $\pi_1(x)$. A direct consequence of Theorem 8.1 is the following.

Proposition 8.2. *Assuming the generalised Riemann Hypothesis we have*

$$\left|\pi_1(x) - \frac{1}{\#G} \frac{x}{\ln x}\right| \leq O\left(\sqrt{x}.\ln x.\ln d_L\right).$$

We are given a monic polynomial $f(X)$ over $\mathbb{Z}$. For an integer x, using the Chebotarev density theorem, we estimate the number of primes $p \leq x$ for which $f(X) \pmod{p}$ splits completely over $\mathbb{F}_p$.

Theorem 8.3. *Given a monic polynomial $f(X)$ over $\mathbb{Z}$ with Galois group G let $S^f(x)$ denote the number of primes $p \leq x$ such that $f(X) \pmod{p}$ splits completely over $\mathbb{F}_p$. Assuming generalised Riemann hypothesis we have*

$$\left| S^f(x) - \frac{1}{\#G}\frac{x}{\ln x} \right| \leq O\left(\sqrt{x}.\ln x.(n!)^3.\mathrm{size}\,(f)\right).$$

Proof. Let $\mathcal{S}^f(x)$ denote the set of all primes p such that $f(X) \pmod{p}$ splits completely over $\mathbb{F}_p$. Then $S^f(x) = \#\mathcal{S}^f(x)$. Let L be the splitting field $\mathbb{Q}_f$. Roots of $f(X)$ are algebraic integers and hence are contained in $\mathbb{O}_L$. Consider any prime $\mathfrak{p}$ of $\mathbb{O}_L$ and let p be the prime in $\mathbb{Z}$ such that $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$. Then for any root $\alpha \in \mathbb{O}_L$ of $f(X)$, $\alpha \pmod{\mathfrak{p}}$ is a root of $f(X) \pmod{p}$ in the finite field $\mathbb{O}_L/\mathfrak{p}$. Therefore $\mathbb{O}_L/\mathfrak{p}$ is the splitting field of $f(X) \pmod{p}$. If p is unramified and splits completely over L then $\mathbb{O}_L/\mathfrak{p} = \mathbb{F}_p$ for all $\mathfrak{p} \mid p$ and hence $f(X)$ splits completely over $\mathbb{F}_p$. Therefore all unramified primes $p \leq x$ that split completely over $L/\mathbb{Q}$ are contained in the set $\mathcal{S}^f(x)$.

We now prove that the number of primes p in $\mathcal{S}^f(x)$ that are not completely split are $\leq (n!)^3.\mathrm{size}\,(f)$. Let $\alpha_1, \ldots, \alpha_n$ denote the roots of $f(X)$. Then there exists an algebraic integer $\theta = \sum c_i\alpha_i$ such that θ is a primitive element of L and $\lg d_\theta \leq (n!)^3\mathrm{size}\,(f)$ (Theorem 6.19). Let $\mu_\theta(X)$ be the minimal polynomial of θ. Since $\theta = \sum c_i\alpha_i$, for any p if $f(X)$ splits completely over $\mathbb{F}_p$ then so does $\mu_\theta(X)$. If in addition p does not divide the discriminant d_θ then $\mu_\theta(X)$ splits completely into distinct linear terms. It follows from the Kummer-Dedekind theorem (Theorem 6.3) that p is unramified and splits completely over $L/\mathbb{Q}$. Therefore the primes $p \in \mathcal{S}^f(x)$ that are not completely split divide the discriminant d_θ. The number of primes that divide d_θ is bounded by $\lg d_\theta \leq (n!)^3.\mathrm{size}\,(f)$ (Theorem 6.19). Hence the number of primes in $\mathcal{S}^f(x)$ that are not completely split over $L/\mathbb{Q}$ is less that $(n!)^3.\mathrm{size}\,(f)$

We have thus proved that $\pi_1(x) \leq \#\mathcal{S}^f(x) = S^f(x) \leq \pi_1(x) + \lg d_\theta$. Also $d_L \leq d_\theta$. Thus

$$\begin{aligned}
\left| S^f(x) - \frac{1}{\#G}\frac{x}{\ln x} \right| &\leq \left| S^f(x) - \pi_1(x) \right| + \left| \pi_1(x) - \frac{1}{\#G}\frac{x}{\ln x} \right| \\
&\leq O\left(\sqrt{x}.\ln x.(n!)^3.\mathrm{size}\,(f)\right) \text{ (Proposition 8.2).}
\end{aligned}$$

□

8.2 Computing the order of the Galois group

In this section we prove our first result on order computation. We are given a monic polynomial $f(X)$ over $\mathbb{Z}$. As in the previous section let $S^f(x)$ denote the number primes $p \leq x$ such that $f(X)$ splits completely over $\mathbb{F}_p$.

Proposition 8.4. *Assuming generalised Riemann hypothesis there exists a constant c such that for $x \geq c.(n!)^{10}\mathrm{size}\,(f)^{2k}$*

$$\left| \#G - \frac{1}{S^f(x)} \frac{x}{\ln x} \right| \leq \frac{1}{n!.\mathrm{size}\,(f)^{k-1}}.$$

Therefore if $x \geq c(n!)^{10}\mathrm{size}\,(f)^2$ and $n \geq 2$, the integer closest to $\frac{1}{S^f(x)}\frac{x}{\ln x}$ is $\#G$.

Proof. Let $N(x) = \frac{1}{S^f(x)}\frac{x}{\ln x}$. From Theorem 8.3 we have

$$(1 - \varepsilon(x))\frac{1}{\#G}\frac{x}{\ln x} \leq S^f(x) \leq (1 + \varepsilon(x))\frac{1}{\#G}\frac{x}{\ln x}$$

where $\varepsilon(x)$ is $O\left(\frac{\ln^2 x.n!^3.\mathrm{size}(f)}{\sqrt{x}}\right)$. Therefore $(1 - \varepsilon(x))N(x) \leq \#G \leq (1 + \varepsilon(x))N(x)$. It follows that $N(x) \leq \frac{\#G}{1-\varepsilon(x)} \leq \frac{n!}{1-\varepsilon(x)}$. For $x = \Omega(n!^6.\mathrm{size}\,(f)^2)$, $\frac{1}{1-\varepsilon(x)} \leq 1 + 2\varepsilon(x)$. Therefore

$$\left| \#G - \frac{1}{S^f(x)}\frac{x}{\ln x} \right| = |\#G - N(x)| \leq n!\varepsilon(x)(1 + 2\varepsilon(x)).$$

There is a constant c such that for $x \geq c.n!^{10}\mathrm{size}\,(f)^{2k}$, $\varepsilon(x) \leq \frac{1}{4n!^2.\mathrm{size}(f)^{k-1}}$. It follows that for $x \geq c.n!^{10}\mathrm{size}\,(f)^{2k}$, $\left|\#G - \frac{1}{S^f(x)}\frac{x}{\ln x}\right|$ is bounded by $\frac{1}{n!.\mathrm{size}(f)^{k-1}}$. $\square$

Consider the machine M that on input $\langle f(X), x\rangle$ guesses a prime $p \leq x$ and checks whether $f(X)$ splits over $\mathbb{F}_p$. Since in time polynomial in $\mathrm{size}\,(f)$ and $\mathrm{size}\,(p)$ one can verify whether $f(X)$ completely splits over $\mathbb{F}_p$ (Theorem 6.9), M is an NP machine. The function $S^f(x)$ is the number of accepting paths of M on input $\langle f(X), x\rangle$ and therefore is in #P.

Proposition 8.5. *The function $\langle f, x\rangle \mapsto S^f(x)$ is in* #P.

We now give the $\mathrm{FP}^{\#\mathrm{P}}$ machine M to compute the order of the Galois group. Given the polynomial $f(X)$ the machine M makes a single query to the #P function of Proposition 8.5 and computes $S^f(x)$ for $x = c.(n!)^{10}.\mathrm{size}\,(f)^2$, where c is the constant of Proposition 8.4. Having computed $S^f(x)$ the machine M in polynomial time finds the integer N closest to $\frac{1}{S^f(x)}\frac{x}{\ln x}$. It follows from Proposition 8.4 that N is the order of $\mathrm{Gal}\,(f)$. Thus we have the following theorem.

Theorem 8.6. *Given a polynomial $f(X)$ over $\mathbb{Q}$, assuming the generalised Riemann hypothesis there is a polynomial time deterministic algorithm with a #P oracle that computes the order of* $\mathrm{Gal}\,(\mathbb{Q}_f/\mathbb{Q})$.

For an arbitrary function in #P, Stockmeyer proved the following theorem [64].

Theorem 8.7 (Stockmeyer)**.** *For every function F in #P and any fixed constant c there is a randomised polynomial time algorithm with* NP *oracle that on input string x of length n computes a value N_x such that*

$$\left(1 - \frac{1}{n^c}\right) N_x \leq F(x) \leq \left(1 + \frac{1}{n^c}\right) N_x.$$

Using the above theorem we show that there is a randomised polynomial time algorithm with NP oracle to approximate the order of the Galois group.

Theorem 8.8. *Given a polynomial $f(X)$ over $\mathbb{Q}$ there is a randomised algorithm with an* NP *oracle that runs in time polynomial in* $\mathrm{size}\,(f)$ *and approximates the order of the Galois group of f with a error of at most* $\frac{1}{\mathrm{size}(f)^{O(1)}}$.

Proof. Since $S^f(x)$ is in #P, using the randomised procedure of Theorem 8.7, for any constant k, we can compute a $\frac{1}{\mathrm{size}(f)^k}$-approximation $\tilde{S}^f(x)$ of $S^f(x)$, i.e. compute $\tilde{S}^f(x)$ such that $(1-\varepsilon)\tilde{S}^f(x) \leq S^f(x) \leq (1+\varepsilon)\tilde{S}^f(x)$ where $\varepsilon = \frac{1}{\mathrm{size}(f)^k}$. Therefore we have

$$\frac{1-\varepsilon}{S^f(x)} \leq \frac{1}{\tilde{S}^f(x)} \leq \frac{1+\varepsilon}{S^f(x)}.$$

By Proposition 8.4 there is a constant c such that for $x \geq c.n!^{10}\mathrm{size}\,(f)^{2(k+1)}$ $\left|\frac{1}{S^f(x)}\frac{x}{\ln x} - \#G\right|$ is bounded by $\frac{1}{n!.\mathrm{size}(f)^k}$. Choosing $x = c.n!^{10}\mathrm{size}\,(f)^{2(k+1)}$

we have

$$\left.\begin{aligned} \left|\frac{1}{\tilde{S}^f(x)}\frac{x}{\ln x} - \#G\right| &\leq \left|\frac{1}{S^f(x)}\frac{x}{\ln x} - \#G\right| + \varepsilon . \frac{1}{S^f(x)}\frac{x}{\ln x} \\ &\leq \frac{2}{n!.\mathrm{size}(f)^k} + \#G\frac{1}{\mathrm{size}(f)^k} \\ &\leq \#G\frac{2}{\mathrm{size}(f)^k}. \end{aligned}\right\} \tag{8.1}$$

The above inequality proves that the integer closest to $\frac{1}{\tilde{S}^f(x)}\frac{x}{\ln x}$ is a $\frac{2}{\mathrm{size}(f)^k}$-approximation of $\#G$.

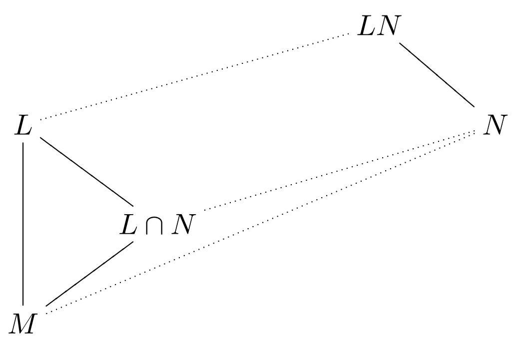
We now give the randomised algorithm with NP-oracle to compute an $\frac{1}{\mathrm{size}(f)^k}$ approximation of $\#G$. For $x = c.n!^{10}\mathrm{size}\,(f)^{2(k+1)}$, using Theorem 8.7, the algorithm first computes the approximation $\tilde{S}^f(x)$ of the #P function $S^f(x)$. Then compute the integer N closest to $\frac{1}{\tilde{S}^f(x)}\frac{x}{\ln x}$. It follows from inequality 8.1 that N is a $\frac{2}{\mathrm{size}(f)^k}$-approximation of $\#G$. □

8.3 Computing the order of Galois groups in Γ_d

Given a polynomial $f(X)$ over $\mathbb{Q}$ with $\mathrm{Gal}\,(\mathbb{Q}_f/\mathbb{Q})$ in Γ_d, in this section we show that $\#\mathrm{Gal}\,(\mathbb{Q}_f/\mathbb{Q})$ can be computed by a randomised polynomial-time algorithm with access to an NP oracle. The algorithm can be seen as a polynomial time Turing reduction from exact order finding to approximate order finding. The result then follows from Theorem 8.8.

First we state an important Lemma from Lang's book [40, Chapter VI, Theorem 1.12].

Lemma 8.9. *Let L/M be a Galois extension and let N be any field that contains M. Then LN/N is Galois and* $\mathrm{Gal}\,(LN/N) \cong \mathrm{Gal}\,(L/L\cap N)$. *Moreover the map that sends $\tau \in \mathrm{Gal}\,(LN/N)$ to its restriction on L is an isomorphism between the Galois groups* $\mathrm{Gal}\,(LN/N)$ *and* $\mathrm{Gal}\,(L/L\cap N)$.



Using Lemma 8.9 we prove the following theorem on simple Galois extensions, i.e. Galois extensions L/K such that $\mathrm{Gal}(L/K)$ is simple.

Theorem 8.10. *Let $L/M/N$ be finite extensions such that L/M is a simple Galois extension. Let E be a finite Galois extension of N containing M and let K be the normal closure of EL over N. Then $[K : E] = [L : M]^l$ for some integer $l \geq 0$.*

Proof. Let $L_1, \ldots, L_r$ be the conjugate fields of L over N. Fix r automorphisms $\{\sigma_i\}_{1 \leq i \leq r}$ of $\mathrm{Gal}\left(\overline{N}/N\right)$ such that $L_i = \sigma_i(L)$ and let $M_i = \sigma_i(M)$.

First we prove that the Galois group $\mathrm{Gal}(K/E)$ embeds into the product group $\prod_{i=1}^{r} \mathrm{Gal}(L_i/M_i)$. Let G_i be the Galois group $\mathrm{Gal}(L_i/M_i)$. For any $\tau \in \mathrm{Gal}(K/E)$ let τ_i denote the element of G_i obtained by restricting the action of τ on L_i. The homomorphism ψ that maps τ to $\langle \tau_1, \ldots, \tau_r \rangle$ is an embedding from $\mathrm{Gal}(K/E)$ to $\prod_{i=1}^{r} G_i$. This is because K is the field $EL_1 \ldots L_r$ and hence for any $\tau \in \mathrm{Gal}(K/E)$ if τ_i fixes L_i for all $1 \leq i \leq r$ then it fixes K as well.

Having proved that $\mathrm{Gal}(K/E)$ embeds into the product group $\prod G_i$ we now prove that the degree $[K : E]$ is a power of $[L : M]$. We dispose of the case when E contains one of the fields L_i. Since E is Galois over N, if E contains L_i, it contains all the other conjugate fields L_j and hence $K = E$. Therefore when E contains one of the L_i, $[K : E] = 1 = [L : M]^0$.

We now consider the case when E contains none of the fields $L_1, \ldots, L_r$. We prove that in this case the projection map $\tau \mapsto \tau_i$ from $\mathrm{Gal}(K/E)$ to G_i is onto. It is sufficient to show that for all $\sigma \in G_i = \mathrm{Gal}(L_i/M_i)$ there is an automorphism τ in $\mathrm{Gal}(K/E)$ such that $\tau_i = \sigma$, $1 \leq i \leq r$.

Since K/E is Galois, every element $\tau \in \mathrm{Gal}(EL_i/E)$ can be extended to an element $\tilde{\tau} \in \mathrm{Gal}(K/E)$ such that $\tilde{\tau}$ restricted to EL_i is τ ([40, Theorem 2.8, Chapter V]). Therefore, it is sufficient to prove that for any element $\sigma \in \mathrm{Gal}(L_i/M_i)$ there is an element $\tau_i \in \mathrm{Gal}(EL_i/E)$ such that τ_i restricted to L_i is σ.

Consider the extension EL_i/E. Since E is Galois and contains M, $E \supseteq M_i$. By Lemma 8.9, $\mathrm{Gal}(EL_i/E)$ is isomorphic to $\mathrm{Gal}(L_i/L_i \cap E)$ via the map that send an automorphism $\mathrm{Gal}(EL_i/E)$ to its restriction on L_i. The extensions L_i/M_i and E/M_i are Galois and hence $L_i \cap E/M_i$ is also Galois. Therefore $\mathrm{Gal}(L_i/L_i \cap E)$ is a normal subgroup of $\mathrm{Gal}(L_i/M_i)$ (Theorem 6.1). But $\mathrm{Gal}(L_i/M_i)$ is simple and $L_i \cap E \neq L_i$. Therefore $L_i \cap E = M_i$ and hence $\mathrm{Gal}(EL_i/E) \cong \mathrm{Gal}(L_i/M_i)$.

For any $\sigma \in \mathrm{Gal}(L_i/M_i)$, there is an element σ_i in $\mathrm{Gal}(EL_i/E)$ such that σ_i restricted to L_i is σ. Let $\tau \in \mathrm{Gal}(K/E)$ be any automorphism such that τ restricted to EL_i is σ_i. Then $\tau_i = \sigma$. As a result we have $\mathrm{Gal}\,(K/E)$ embeds *onto* the product group $\prod_{i=1}^{r} \mathrm{Gal}\,(L_i/M_i)$ via the map $\tau \mapsto \tau_i$.

If L_i/M_i is a simple abelian extension then $\mathrm{Gal}\,(L_i/M_i) \cong \mathbb{F}_p$ and therefore $\mathrm{Gal}\,(K/E)$ is isomorphic to a vector space over $\mathbb{F}_p$. Hence $[K : E] = \#\mathrm{Gal}\,(K/E) = p^l = [L : M]^l$ for some l. Otherwise if L_i/M_i is a nonabelian simple extension then using Scott's Lemma (Lemma 3.6), $\mathrm{Gal}\,(K/E)$ is a product of diagonal subgroups of $\mathrm{Gal}\,(L_i/M_i)$ and hence $[K : E] = [L : M]^l$. □

Given a polynomial $f(X)$ over $\mathbb{Q}$ of degree n. If the Galois group of f is in Γ_d then we show that the order of the Galois group of f can be computed by a randomised algorithm with an NP oracle. We first give a sketch of the algorithm here and defer the detailed description to Algorithm 13. For simplicity we assume that $f(X)$ is irreducible. Algorithm 13 handles reducible $f(X)$ as well.

For a number field K we denote the normal closure of K over $\mathbb{Q}$ by $\tilde{K}$. Let G be the Galois group of $f(X)$ thought of as a permutation group over Ω the set of roots of $f(X)$. For a G-block Δ recall that $\mathbb{Q}_\Delta$ denotes the fixed field $\mathrm{Fix}\,(\mathbb{Q}_f, G_\Delta)$. Using Theorem 7.3 repeatedly we can compute the number fields $K_i = \mathbb{Q}_{\Delta_i}$ for maximal chain of G-blocks $\{\alpha\} = \Delta_0 \subseteq \ldots \subseteq \Delta_m = \Omega$. Recall that the normal closure $\tilde{K}_m$ and $\tilde{K}_0$ are respectively $\mathbb{Q}$ and $\mathbb{Q}_f$ and $\#\mathrm{Gal}\,(f) = [\mathbb{Q}_f : \mathbb{Q}]$. For i decreasing from m to 0 we compute the degree $[\tilde{K}_i : \mathbb{Q}]$ inductively. To begin with the degree $[\tilde{K}_m : \mathbb{Q}] = 1$. Assuming we have computed the degree $[\tilde{K}_i : \mathbb{Q}]$ we show how the degree $[\tilde{K}_{i-1} : \mathbb{Q}]$ can be computed. Let L_i denote the normal closure of K_{i-1} over K_i. Recall that $\tilde{L}_i = \tilde{K}_{i-1}$ and hence it is sufficient to compute the degree $[\tilde{L}_i : \mathbb{Q}]$. Recall that L_i/K_i is a Galois extension with small ($\leq O(n^d)$) Galois group (Proposition 7.20). Hence we can compute the Galois group $H = \mathrm{Gal}\,(L_i/K_i)$ using Landau's algorithm. Furthermore in time polynomial in n^d we compute a composition series $H = H_0 \rhd \ldots \rhd H_t = 1$ for H where each of the quotient group H_i/H_{i+1} is simple. Let F_i denote the fixed field $\mathrm{Fix}\,(L_i, H_i)$. Consider the tower of extensions $\tilde{K}_i = \tilde{F}_0 \subseteq \ldots \subseteq \tilde{F}_t = \tilde{K}_{i-1}$. We have the following proposition

Proposition 8.11. *The extension F_{j+1}/F_j is a simple Galois extension and the degree $[\tilde{F}_{j+1} : \tilde{F}_j]$ is a power of the degree $[F_{j+1} : F_j]$.*

Proof. The group H_{j+1} is a normal subgroup of H_j such that H_j/H_{j+1} is simple. Hence by fundamental theorem of Galois theory, the extension F_{j+1}/F_j is a simple Galois extension. Using Theorem 8.10 for the field extensions $F_{j+1}/F_j/\mathbb{Q}$, the degree $[\tilde{F}_{j+1} : \tilde{F}_j]$ is a power of the degree $[F_{j+1} : F_j]$. $\square$

We compute the degree $[\tilde{F}_j : \mathbb{Q}]$ inductively for increasing j. To begin with $[\tilde{F}_0 : \mathbb{Q}] = [\tilde{K}_i : \mathbb{Q}]$ which we have already computed. Assume that we already know the degree $[\tilde{F}_j : \tilde{K}_i]$. We can compute a primitive polynomial $h(X)$ of F_{j+1} over $\mathbb{Q}$ in time polynomial in $\mathrm{size}\,(f)$ and n^d. Using Theorem 8.8 for a suitable small ε (say $\varepsilon = 0.1$) we compute an approximation A of the degree $[\tilde{F}_{j+1} : \mathbb{Q}]$ such that $(1-\varepsilon)A \leq [\tilde{F}_{j+1} : \mathbb{Q}] \leq (1+\varepsilon)A$. We have already computed the degrees $[\tilde{F}_j : \tilde{K}_i]$ and $[\tilde{K}_i : \mathbb{Q}]$ and therefore can compute $[\tilde{F}_i : \mathbb{Q}] = [\tilde{F}_j : \tilde{K}_i][\tilde{K}_i : \mathbb{Q}]$. Therefore $A' = \frac{A}{[\tilde{F}_j:\mathbb{Q}]}$ gives an ε-approximation of $[\tilde{F}_{j+1} : \tilde{F}_j]$.

Let r denote the degree $[F_{j+1} : F_j]$ which we have already computed. Then by Proposition 8.11, $[\tilde{F}_{j+1} : \tilde{F}_j]$ is a power of r. Let r^l be the power of r that is closest to A'. Since A' is an ε-approximation of $[\tilde{F}_{j+1} : \tilde{F}_j]$, if $\varepsilon < 0.1$ then $[\tilde{F}_{j+1} : \tilde{F}_j] = r^l$.

Having computed A', r and $[\tilde{F}_j : \mathbb{Q}]$, it is easy to find $[\tilde{F}_{j+1} : \tilde{F}_j]$ and thus $[\tilde{F}_{j+1} : \mathbb{Q}]$. This completes our description of the algorithm. Algorithm 13 is a detailed presentation.

Algorithm 13 can be seen as a polynomial time Turing reduction from exact order finding to approximate order finding. The step 1 can be seen as an oracle query to a function that gives an approximation of the order of the Galois group. We thus have the following theorem.

Theorem 8.12. *For polynomials $f(X)$ with Galois group in Γ_d there is a polynomial time (polynomial in* $\mathrm{size}\,(f)$ *and n^d) Turing reduction from exact order finding to approximate order finding. Hence there is a randomised algorithm with an* NP*-oracle to compute the order of* $\mathrm{Gal}\,(f)$ *for polynomials $f(X)$ with Galois group in Γ_d.*

8.4 Discussion

In this section we proved upper bounds on order finding for Galois group assuming the generalised Riemann hypothesis. We proved that computing the order of the Galois group of a polynomial $f(X)$ is in $\mathrm{FP}^{\#\mathrm{P}}$. In addition if the Galois group of $f(X)$ is in Γ_d, a fact that can be checked efficiently using Theorem 7.22, then the order of $\mathrm{Gal}\,(f)$ can be computed within the polynomial hierarchy. We can prove similar results for $f(X) \in K[X]$ where K is give via explicit data.

Input: A polynomial $f(X)$.
Output: The order of $\mathrm{Gal}\,(f)$.
if $f(X)$ *is a constant polynomial* **then return** 1
Let f factorise as gh where g is an irreducible polynomial over $\mathbb{Q}$.
Let G be the Galois group $\mathrm{Gal}\,(\mathbb{Q}_g/\mathbb{Q})$.
Using Theorem 7.3 compute the fields $K_i = \mathbb{Q}_{\Delta_i}$ for a maximal chain of G-blocks $\{\alpha\} = \Delta_0 \subseteq \ldots \subseteq \Delta_m = \Omega$.
Recursively compute $N_m = [\mathbb{Q}_h : \mathbb{Q}]$. N_i will denote the degree $[\mathbb{Q}_h\tilde{K}_i : \mathbb{Q}]$.
for $i \leftarrow m$ **downto** *0* **do**
    Compute the normal closure L_i of K_{i-1} over K_i.
    Compute $H = \mathrm{Gal}\,(L_i/K_i)$.
    Compute a composition series $H = H_0 \rhd \ldots \rhd H_t$.
    for $j \leftarrow 0$ **to** t **do**
        $F_j \leftarrow \mathrm{Fix}\,(L_i, H_j)$
        Compute the primitive polynomial $f_j(X)$ of F_j
    end
    $M_0 \leftarrow 1$, M_j will be the degree $[\mathbb{Q}_h\tilde{F}_j : \mathbb{Q}_h]$.
    for $j \leftarrow 1$ **to** t **do**
        Compute a 0.1-approximation A of $\#\mathrm{Gal}\,(hf_j)$.
        Let $r = [F_j : F_{j-1}]$.
        Compute the power r^l closest to $\frac{A}{M_{j-1}N_i}$.
        $M_j \leftarrow M_{j-1}.r^l$.
    end
    $N_{i-1} \leftarrow N_i.M_t$
end
return N_0.

Algorithm 13: Computing order of Galois group in Γ_d.

An interesting open problem is to give nontrivial upper bound unconditionally. Another interesting problem is to give better upper bounds for order finding for special polynomials, like for example polynomials with solvable Galois groups. One way to achieve this is to give better upper bounds for approximating the order of the Galois group. Certain #P-complete functions like #DNF can be approximated efficiently (Chapter 11 of the book by Motwani and Raghavan [54] gives a detailed presentation of such #P complete problems). It would be interesting to know whether the number of completely split primes less than a given number x can be approximated efficiently in which case we would have efficient order finding algorithm for polynomials with Galois group in Γ_d.

At present computing the Galois group looks harder than computing the order. It would be interesting to know for example whether the Galois group can be computed in PSPACE. Even conditional results will be interesting. For polynomials with solvable Galois group are there better upper bounds ?

Chapter 9

Computing Galois groups

In this chapter we give some upper bounds on computing the Galois group of certain special polynomials. Our first result is a randomised algorithm to compute the Galois group of polynomials with abelian Galois group [7]. This result makes use of the effective version of the Chebotarev density theorem and hences is conditional on the validity of the generalised Riemann hypothesis. We then consider polynomials $f(X)$ that are product of polynomials $\{f_i\}_{1\leq i\leq m}$ having the following properties (1) $\mathbb{Q}_{f_i} = \mathbb{Q}[X]/f_i(X)$ and (2) $\mathrm{Gal}\,(f_i)$ is simple and nonabelian. We show that in this case there is deterministic algorithm that runs in time polynomial in $\mathrm{size}\,(f)$ to compute the Galois group of f. This result is unconditional and Scott's Lemma plays a crucial role in the proof of this result. In particular, for this result the assumption that $\mathrm{Gal}\,(f_i)$ is nonabelian is crucial as Scott's lemma is not true for abelian simple groups.

Recall that if $f(X)$ is irreducible and has abelian Galois group then $\mathrm{Gal}\,(f)$ can be computed in polynomial time using Landau's algorithm (Theorem 6.12). However, when $f(X)$ is reducible with abelian Galois group, the Galois group can be exponentially large. Hence Landau's algorithm cannot be used directly. In fact even when the polynomial is a product of quadratic polynomial nothing better than the exponential time algorithm is known (cf. Lenstra [44]).

For polynomials $f(X)$ with abelian Galois group we give a polynomial time almost uniform sampling algorithm for elements of $\mathrm{Gal}\,(\mathbb{Q}_f/\mathbb{Q})$. It is easy to see that for a group G a random sample of $O(\lg G)$ elements from G is a generator set with high probability.

9.1 Computing abelian Galois groups

Given a polynomial $f(X)$ with abelian Galois group. Our task is to compute the Galois group G of $f(X)$. Let $f = f_1, \ldots, f_r$ be the factorisation of f into irreducible factors. Let G_i be the Galois group of f_i. Each of the groups G_i can be computed explicitly using Landau's algorithm. The group G is a subgroup of the product group $\prod_{i=1}^{r} G_i$ and projects onto each G_i, i.e. G embeds into the product group $\prod G_i$. Hence any $\sigma \in G$ can be considered as a tuple $\sigma = \langle \sigma_1, \ldots, \sigma_r \rangle$ where $\sigma_i \in G_i$.

There are two important properties of abelian extensions that we require. Firstly, each conjugacy class of G is a singleton set. Secondly, by factoring each of the irreducible factors f_i over $\mathbb{F}_p$ we can recover the Frobenius element associated to p (Proposition 9.2).

Let L denote the splitting field $\mathbb{Q}_f$. Recall that for each prime p we can associate a conjugacy class $\mathrm{Frob}_{L/\mathbb{Q}}(p)$ (see Section 8.1). Since G is abelian the conjugacy class $\mathrm{Frob}_{L/\mathbb{Q}}(p)$ is a singleton set $\{\sigma_p\}$. We show that for a given $\sigma \in G$ the probability that $\sigma_p = \sigma$ for a random prime is close to $\frac{1}{\#G}$. This follows from the Chebotarev density theorem. Hence picking primes p at random and recovering the corresponding Frobenius gives us an almost uniform sampler for elements of G. A polynomial size sample will then generate G.

Let p be any prime. To recover the Frobenius σ_p, we recover the corresponding Frobenius' $\sigma_{p,i}$ of G_i. Then $\sigma_p = \langle \sigma_{p,1}, \ldots, \sigma_{p,r} \rangle$. The following important property of polynomials with abelian Galois group is useful in recovering the Frobenius element σ_p.

Lemma 9.1. *Let $g \in \mathbb{Q}[X]$ be an irreducible polynomial of degree d with abelian Galois group. Let θ be any root of g and let $g(X) = \prod_{i=1}^{d}(X - A_i(\theta))$ be the factorisation of g over $\mathbb{Q}(\theta)$ where $A_i(X)$ are polynomial over $\mathbb{Q}$. For any $\sigma \in \mathrm{Gal}(\mathbb{Q}_g/\mathbb{Q})$ there is a unique index i such that σ maps η to $A_i(\eta)$ for any root η (not necessarily θ) of g.*

Proof. Let G be the Galois group of $g(X)$. Since g is irreducible and G is abelian, $\mathbb{Q}_g = \mathbb{Q}(\theta)$ and there is a unique automorphism σ_i that maps θ to $A_i(\theta)$. The automorphisms $\{\sigma_i\}_{i=1}^{d}$ constitutes the group G. Consider any root η of g. Since G is transitive there is a $\tau \in G$ such that $\tau(\theta) = \eta$. Now $\sigma_i(\eta) = \sigma_i\tau(\theta) = \tau\sigma_i(\theta)$ since G is abelian. Therefore $\sigma_i(\eta) = \tau(A_i(\theta)) = A_i(\tau(\theta)) = A_i(\eta)$. Therefore σ_i maps η to $A_i(\eta)$. □

We now show that given a prime p that does not divide the discriminant d_f, the automorphism σ_p can be recovered efficiently.

Proposition 9.2. *Given a prime p that does not divide d_f, there is a randomised algorithm running in time polynomial in* $\text{size}(f)$ *and* $\lg p$ *that computes the Frobenius σ_p as an r-tuple $\langle \sigma_{p,1}, \ldots, \sigma_{p,r} \rangle$ where $\sigma_{p,i} \in G_i$ is the Frobenius element corresponding to p for the extension $\mathbb{Q}_{f_i}/\mathbb{Q}$.*

Proof. Fix a root θ_i of $f_i(X)$ over the extension $\mathbb{Q}[X]/f_i(X)$. Let $f_i(X)$ factorise as

$$f_i(X) = \prod_{j=1}^{n_i} (X - A_{ij}(\theta_i)).$$

Compute the Galois group G_i of f_i using Landau's algorithm. Let σ_{ij} denote the unique automorphism of G_i that maps θ_i to $A_{ij}(\theta_i)$. Our task is to identify which of these is $\sigma_{p,i}$.

For each i we find the splitting field $\mathbb{F}_{q_i}$ of f_i over $\mathbb{F}_p$. Since f_i is irreducible over $\mathbb{Q}$ the order of the Frobenius $\sigma_{p,i}$ divides n_i, the degree of f_i. Therefore $[\mathbb{F}_{q_i} : \mathbb{F}_p]$ divides n_i and hence the splitting field is a small extension (of degree less than the degree of f) over $\mathbb{F}_p$. Let α be any root of $f_i(X)$ in $\mathbb{F}_{q_i}$. In polynomial time find the index j such that $\alpha^p = \tilde{A}_{ij}(\alpha)$ where $\tilde{A}_{ij}(X)$ is the polynomial $A_{ij}(X) \bmod p$. Since $p \nmid d_f$ the index j is unique as there are no multiple roots for $f_i(X)$ over $\mathbb{F}_p$. The Frobenius $\sigma_{p,i} = \sigma_{ij}$.

Having computed $\sigma_{p,i}$ for all $1 \leq i \leq r$ we have $\sigma_p = \langle \sigma_{p,1}, \ldots, \sigma_{p,r} \rangle$. $\square$

For our almost uniform sampler we study the distribution of σ_p for random primes p. We show that for a random prime p, the distribution of σ_p is almost uniform over G.

Proposition 9.3. *Let σ be any automorphism in* $\text{Gal}(\mathbb{Q}_f/\mathbb{Q})$. *Let $P_\sigma(x)$ denote the probability that for an unramified prime $p \leq x$ picked uniformly at random $\sigma_p = \sigma$. Assuming the generalised Riemann hypothesis, there exists a constant c independent of $f(X)$ such that*

$$\frac{1}{\#G}\left(1 - \frac{1}{n!}\right) \leq P_\sigma(x) \leq \frac{1}{\#G}\left(1 + \frac{1}{n!}\right)$$

for all $x \geq c.(n!)^{10}.\text{size}(f)^2$.

Proof. Let L be the splitting field $\mathbb{Q}_f$. For an automorphism $\sigma \in G$ let $\pi_\sigma(x)$ denote the number of unramified primes $p \leq x$ such that $\sigma_p = \sigma$. By the effective version Chebotarev density theorem (Theorem 8.1) we have $\left|\pi_\sigma(x) - \frac{1}{\#G}\frac{x}{\ln x}\right| \leq O(\sqrt{x}.\ln x.\ln d_L)$. Recall that $d_L \leq (n!)^3\text{size}(f)$. Also by the prime number theorem, the number of primes less than x is given by $\pi(x) = \frac{x}{\ln x}$. Therefore $P_\sigma(x) = \frac{\pi_\sigma(x)}{\pi(x)}$. It follows that

$$\left| P_\sigma(x) - \frac{1}{\#G} \right| \leq O\left(\frac{\ln x^2 . n!^3 \text{size}\,(f)}{\sqrt{x}} \right).$$

Therefore there is a constant c such that for $x \geq c.(n!)^{10}.\text{size}\,(f)^2$

$$\frac{1}{\#G}\left(1 - \frac{1}{n!}\right) \leq P_\sigma(x) \leq \frac{1}{\#G}\left(1 + \frac{1}{n!}\right).$$

□

Proposition 9.3 shows that picking random primes and computing σ_p gives an almost uniform sampling procedure. That σ_p can be computed given p follows from Proposition 9.2. The only missing result is to show that a polynomial sized sample generates G which we do now.

Lemma 9.4. *Let G be any group. Consider a sampling procedure that produces each element $g \in G$ with probability at least $\frac{1}{\lambda \#G}$, for some $\lambda > 1$. A sample set of size $4.\lambda.\lg \#G$ where each element is obtained by running the sampling procedure independently will generate G with probability at least $\frac{1}{4\lambda}$.*

Proof. Let $N = 4.\lambda.\lg \#G$ and let $g_1, \ldots, g_N$ be the group elements sampled by running the procedure N times. Let $G_0 = \{1\}$ and let G_i denote the group generated by $\{g_1, \ldots, g_i\}$. Define the random variable X_i as follows.

$$X_i = \begin{cases} 1 \text{ if } G_{i-1} \neq G \text{ and } g_i \in G_{i-1} \\ 0 \text{ otherwise} \end{cases}$$

We have $\text{Prob}[X_i = 1 \mid G_{i-1} = G] = 0$. If $G_{i-1} \neq G$ then $\#G_{i-1} \leq \frac{1}{2}\#G$. Therefore the probability $\text{Prob}[g_i \notin G_{i-1} \mid G_{i-1} \neq G]$ is at least $\frac{1}{2\lambda}$. We now compute the expectation of the variable X_i.

$$\begin{aligned}
\mathbf{E}[X_i] &= \text{Prob}[X_i = 1] \\
&= \text{Prob}[G_{i-1} \neq G_i \text{ and } g_i \in G_{i-1}] \\
&= \text{Prob}[g_i \in G_{i-1} \mid G_{i-1} \neq G_i].\text{Prob}[G_{i-1} \neq G_i] \\
&= 1 - \text{Prob}[g_i \notin G_{i-1} \mid G_{i-1} \neq G] \\
&\leq 1 - \frac{1}{2\lambda}.
\end{aligned}$$

Let X be the random variable $\sum_{i=1}^{N} X_i$. The random variable X is always positive with expectation $\mathbf{E}[X] = \sum \mathbf{E}[X_i] \leq N.(1 - \frac{1}{2\lambda})$. By Markov's inequality $\text{Prob}[X \geq t] \leq \frac{\mathbf{E}[X]}{t}$ for all t. Using $t = N - \lg \#G$ we have

$$\begin{aligned} \text{Prob}[X \geq N - \log \#G] &\leq \frac{1 - \frac{1}{2\lambda}}{1 - \frac{1}{4\lambda}} \\ &\leq 1 - \frac{1}{4\lambda}. \end{aligned}$$

Consider any sample $g_1, \ldots, g_N$ such that random variable X is less than $N - \lg \#G$. Assume that the random group G_N generated by $g_1, \ldots, g_N$ is different from G. Then $G_i \neq G$ for all $1 \leq i \leq n$. As a result there are at least $\lceil \lg \#G \rceil$ different indices i such that $g_i \notin G_{i-1}$. At each such i, $\#G_i \geq 2\#G_{i-1}$. Hence $\#G_N \geq G$. But G_i's are all subgroup of G. This contradicts the assumption that $G_N \neq G$. Therefore if $X \leq N - \lg \#G - 1$ then $G_N = G$. Thus

$$\begin{aligned} \text{Prob}[G_N = G] &\geq \text{Prob}[X < N - \lg \#G] \\ &= 1 - \text{Prob}[X \geq N - \lg \#G] \\ &\geq \frac{1}{4\lambda}. \end{aligned}$$

□

We are ready to give a randomised algorithm to compute the Galois group of $f(X)$. The idea is to pick a prime $p \leq x$ for some sufficiently large x at random and recover σ_p using Proposition 9.2. It follows from Proposition 9.3 that if $x \geq c.(n!)^{10}.\text{size}\,(f)^2$, an element σ will be obtained by this sampling procedure with probability at least $\frac{1}{2\#G}$. Therefore an $8 \lg \#G \leq 8n^2$ sized sample set will generate G with probability at least $\frac{1}{8}$. Algorithm 14 is the detailed presentation.

We now prove the main result of this section.

Theorem 9.5. *Given a polynomial $f(X)$ over $\mathbb{Q}$ of degree n with abelian Galois group. Assuming the generalised Riemann hypothesis there is a randomised algorithm that runs in time polynomial in* $\text{size}\,(f)$ *and outputs a strong generator set for Galois group of f with probability* $1 - \frac{1}{2^{n.\text{size}(f)}}$

Proof. Algorithm 14 gives a generator set of G with probability at least $\frac{1}{8}$. To improve the probability we run Algorithm 14 independently s times to get subsets $A_1, \ldots, A_s$ each of size $8n^2$. Since A_i's are picked independently

Input: A polynomial $f(X)$ over $\mathbb{Q}$.
Output: Galois group of $f(X)$.
Factorise f into irreducible factors $f_1, \dots f_r$.
Let $S \leftarrow \emptyset$
for $i = 1$ *to* $8n^2$ **do**
  Pick a prime $p \leq c.(n!)^{10}\text{size}\,(f)^2$ at random.
  Recover the σ_p using Proposition 9.2
  $S \leftarrow S \cup \{\sigma_p\}$
end
return S.

Algorithm 14: Computing abelian Galois group

at random, the probability that none of A_i's generate G is a at most $\left(\frac{7}{8}\right)^s$. Hence $A = \cup_{i=1}^{k} A_k$ is a generating set for G with at least $1 - (\frac{7}{8})^s$. Choosing $s = \frac{n.\text{size}(f)}{\lg 8 - \lg 7}$ we have the desired result. We can reduce the size of the set A to n^2 by computing a strong generator set for G. □

9.2 Computing simple Galois groups

We consider an interesting special case of nonabelian Galois groups computation for which we have a polynomial-time algorithm. Let $f(X)$ be a polynomial such that $f(X)$ factors as $f = \prod_{i=1}^{r} f_i(X)$ over $\mathbb{Q}$. Suppose the Galois group of $f_i(X)$ is small (of order bounded by a polynomial in $\text{size}\,(f)$), simple and nonabelian. Then there is a polynomial time algorithm to compute the Galois group of f.

Firstly using Landau's algorithm the groups $G_i = \text{Gal}\,(K_{f_i}/K)$ can be computed in time polynomial in $\text{size}\,(f)$ as G_i is of size bounded by a polynomial in $\text{size}\,(f)$. The Galois group $G = \text{Gal}\,(\mathbb{Q}_f/\mathbb{Q})$ is a subgroup of $\prod_{i=1}^{r} G_i$. Moreover since the splitting field $\mathbb{Q}_f$ contains the splitting field $\mathbb{Q}_{f_i}$, the projection from G to G_i is onto. Each of the groups G_i is simple and non-abelian. Therefore, by Scott's Lemma (Lemma 3.6), there is a partition on the set $\{1, \dots, r\}$ into subsets $I_1, \dots, I_s$ such that G is given by

$$G = \prod_{k=1}^{s} \text{Diag}\left(\prod_{j \in I_k} G_j\right).$$

As in Chapter 5 we say that i and j are *linked* if G_i and G_j belong to the same partition. In this case G projected to $G_i \times G_j$ is the diagonal group. This implies that i and j are linked if and only if the splitting fields f_i and

f_j are the same. This give a polynomial time algorithm to check whether i and j are linked: Compute the explicit data for the splitting field $L_i = \mathbb{Q}_{f_i}$ and factorise $f_j(X)$ over L_i. The indices i and j are linked if and only if $f_j(X)$ splits completely over L_i

The partitions $I_1, \ldots, I_s$ are the equivalence classes of the equivalence relation $\sim$ defined by $i \sim j$ if i linked to j. Since $i \sim j$ can be checked in time polynomial in $\mathrm{size}(f)$ the equivalence classes $\{I_k\}_{1 \leq k \leq s}$ can be computed in polynomial time. Putting it all together we have the following theorem.

Theorem 9.6. *Let $f(X) \in \mathbb{Q}[X]$ be a polynomial such that $f = f_1 f_2 \ldots f_r$ where each f_i has a non-abelian simple Galois group of size at most N. Then there is an algorithm that runs in time polynomial in* $\mathrm{size}(f)$ *and N to compute the Galois group of $f(X)$. In particular if N is bounded by a polynomial in* $\mathrm{size}(f)$*, there is a polynomial time algorithm for finding the Galois group of f.*

Proof. First factorise the polynomial f into $f_1, f_2 \ldots, f_n$. Compute the Galois groups $G_i = \mathrm{Gal}(\mathbb{Q}_{f_i}/\mathbb{Q})$ for each $1 \leq i \leq n$ in time polynomial in $\mathrm{size}(f)$ and N using Landau's algorithm (Theorem 6.10). As described before equivalence classes $\{I_k\}_{1 \leq k \leq s}$ can be computed in time polynomial in $\mathrm{size}(f)$ and N. For i and j that are linked, in order to compute the diagonal group, we need to find the right isomorphism between G_i and G_j. This can be computed by factoring f_j over $\mathbb{Q}_{f_i}$. We then output the group

$$G = \prod_{k=1}^{t} \mathrm{Diag}\left(\prod_{j \in I_k} G_j \right),$$

which is the required Galois group. □

9.3 Discussion

As all our results on computational Galois theory, we can prove similar results for polynomials $f(X)$ over a number field K given by explicit data. It still remains open whether there is a polynomial time deterministic algorithm to compute the Galois group of a polynomial with abelian Galois group. Even when $f(X)$ is a product of quadratic polynomials we do not have polynomial time deterministic algorithm.

For polynomials $f(X)$ with abelian Galois group, each conjugacy class of G was singleton. Also any prime p that does not divide the discriminant d_f, using Lemma 9.1 we could recover the Frobenius associated to the prime p.

These two properties gave us the uniform sampling procedure. However if the Galois group of $f(X)$ is not abelian we do not have a method to recover the action of the Frobenius. By factoring $f(X)$ over $\mathbb{F}_p$ for different primes we get only the cyclic structure of element of $\mathrm{Gal}\,(f)$.

Finally, in the absence of any good algorithms, it is of interest to prove hardness results for Galois group computations.

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