STUDIES IN $SU(2)$ SAVVIDY MODEL OF QCD

By
ALOK KUMAR

The Institute of Mathematical Sciences, Chennai

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Jagannathan
Date: 24/10/2008
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Date: 24/10/2008
Convenor - Rahul Basu

Date: 24.10.2008
Member 1 - Manu Mathur

P. Saraf
Date: 24.10.2008
Member 2 - R. Parthasarathy

Date: 24.10.2008
Member 3 - R. Anishetty

D. Indumathi
Date: 24.10.8
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Guide - Rahul Basu

Declaration

I, hereby declare that the investigation presented in this thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part of a degree/diploma at this or any other Institution/University.

Alok Kumar

Alok Kumar

In Fond Memory of My Mother
and
those who taught me even a single word.

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CONTENTS

*This chapter is based on the publication, "Savvidy Vacuum in $SU(2)$ Yang-Mills Theory", D. Kay, A. Kumar and R. P.: Mod. Phys. Lett. **A20**, 1655-1662 (2005).

†This chapter is based on the publication, "SU(2) Yang-Mills Theory in the Savvidy Background at finite Temperature and Chemical Potential", R. Parthasarathy and A. Kumar: Phys. Rev. **D75**, 085007 (2007).

[‡]This chapter is based on "Chiral Symmetry Breaking in Gribov's Approach to QCD at Low Momentum," Alok Kumar, arXiv:0711.3970v1 [hep-th].

Abstract

$SU(2)$ Yang-Mills theory in the Savvidy background is studied at both zero temperature and finite temperature. The Savvidy vacuum in the Gaussian approximation is plagued by unstable modes both at zero temperature and at finite temperature. These unstable modes lead to an imaginary part in the one-loop effective energy density; therefore the Savvidy vacuum is unstable. With the motivation to get real one-loop effective energy density, the stable and unstable modes are separated. The stable modes are treated keeping terms only quadratic in the fluctuation. For the unstable modes we consider the full action including the cubic and quartic terms and the one-loop effective energy density of the unstable modes is added to that of the stable modes. The resulting energy density is real. This real one-loop effective energy density of the Savvidy vacuum has been used to calculate the bag constant B. The bag constant found to be $B^{\frac{1}{4}} = 188$ MeV for $N_f = 6$ for the gluon condensate 0.012 GeV^2 which compares well with the MIT bag model value of 145 MeV. It has been used for the calculation of the β-function and the calculated β-function is found to be in agreement with the known result.

The above procedure has been extended to finite temperature to get the real one-loop energy density as the temperature dependent imaginary part inhibits the analysis of phase transitions. We have introduced a chemical potential for gluons originating from the conservation of color charges. The resulting one-loop effective energy density at finite temperature and chemical potential in the Savvidy background is found to be real. The variation of the scaled temperature-dependent effective energy density with scaled temperature for different values of the chemical potential has been numerically evaluated. At high temperatures, the behaviour is like that of a noninteracting relativistic gas. A nonzero chemical potential triggers a possible deconfinement phase transition.

The issue of chiral symmetry breaking has been studied using Gribov's approach. The Schwinger-Dyson equation gets converted into a differential equation involving the quark propagator only in the Feynman gauge. This equation has been further modified with pion correction. A relation between dynamical mass function of the quark without pion correction, $M_0(q^2)$, and with pion correction, $M(q^2)$, at low momentum has been found. For low momentum, the variation of $M_0(q^2)$ and $M(q^2)$ with respect to q has been numerically studied. The result of this study led to the conclusion that pion back reaction has a small effect on the dynamical mass of quarks.

List of Publications:

1. Savvidy Vacuum in $SU(2)$ Yang-Mills Theory,
   Modern Physics Letters **A 20**, No.22 (2005), 1655-1662,
   Daniel Kay, A. Kumar and R. Parthasarathy.

2. $SU(2)$ Yang-Mills Theory in the Savvidy Background at Finite temperature and Chemical Potential,
   Physical Review **D75**, 085007 (2007),
   R. Parthasarathy and A. Kumar.

3. Chiral Symmetry Breaking in Gribov's Approach to QCD at Low Momentum,
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Chapter 1

Introduction

Gauge theories describe the interactions of elementary particles. The local Abelian $U(1)$ gauge theory provides the electromagnetic interactions. The unification of the weak and electromagnetic interactions is achieved by the $SU(2)_L \otimes U(1)$ gauge group by Weinberg, Salam and Glashow with $SU(2)_L \otimes U(1)$ symmetry broken to $U(1)_{e.m.}$. The strong interactions of quarks and gluons are described by the local $SU(3)_c$ gauge theory with the unbroken gauge group as color $SU(3)_c$. This theory is called 'Quantum Chromo Dynamics'. In order to understand the vacuum of QCD, physicists have studied the ground state of the non-Abelian $SU(2)$ Yang-Mills theory. Such a study provides a qualitative understanding of the ground state of QCD. In this thesis, we examine the ground state of $SU(2)$ Yang-Mills theory using the Background Field Method (BFM), the background being the classical covariantly constant chromomagnetic field in the third colour direction, known as the Savvidy Background—first at zero temperature and then at finite temperature. The motivations for such a study are given at the end of this chapter. Another important aspect of QCD, namely, the Spontaneous Breaking of Global Chiral Symmetry, is studied in Chapter V, using Gribov's approach.

1.1 QCD and its Birth

Quantum Chromo Dynamics (QCD) is accepted as the theory of strong interaction of quarks and gluons. It is a gauge theory with the gauge group as $SU(3)_c$, the unbroken color group. Quarks (spin-$\frac{1}{2}$) are in the fundamental representation and gluons (spin-1) are in the adjoint representation. Just as photons mediate the electromagnetic interaction between electrons in Quantum Electrodynamics (QED), the gluons mediate the strong interactions between the quarks in Quantum Chromo Dynamics (QCD). There are differences. While photons do not carry electric charge, gluons carry colour charge. QCD is a non-Abelian gauge theory and so the gluons interact among themselves (self-coupling). As the $SU(3)_c$ gauge symmetry is exact, the gluons are massless. We give a brief genesis of QCD now.

The quark model for hadrons, proposed by Gell-Mann [1] and independently by Ne'eman and Zweig [2], gave a neat classification scheme for hadrons with the underlying symmetry group being $SU(3)$. This is distinguished from $SU(3)_c$ as $SU(3)_f$, i.e., flavor $SU(3)$. In their original proposal, there were 3 flavors, u-quark ($\frac{2}{3}|e|$), d-quark (-$\frac{1}{3}|e|$) and s-quark (-$\frac{1}{3}|e|$), the quantities in the bracket referring to their electric charges. The Proton consists of two u quarks and one d quark. Mesons are made up of quark-antiquark pairs. Such a scheme explained the observed pattern of mesons and baryons [3] on the basis of unitary symmetry. The successful prediction of the baryon Ω^- (sss) with mass (1672 MeV) has been confirmed by experimental observation [4]. Besides describing the static properties of hadrons, the $SU(3)_f$ -quark model with current algebra techniques described the weak and electromagnetic interactions of hadrons [3,5]. However, the dynamics of the quarks inside the hadrons was not satisfactorily described by the quark model. A striking difficulty was the constitution of Δ^{++} (uuu) in which three identical fermions were in the same state ($\ell = 0$), violating the Pauli Exclusion principle. The latter issue led to the introduction of an additional quantum number, color, by Han and Nambu, Greenberg and Gell-Mann, carried by the quarks. Baryon wave functions must be totally antisymmetric in color quantum number. All observed hadrons are color singlet. In the simplest model, quarks are assigned the fundamental representation of a new, internal $SU(3)_c$ global symmetry. At present, we have six flavors of quarks, each coming in three colors (say, red, blue and green).

The three-color hypothesis received support from two important physical processes, namely, the total cross-section for $e^+e^- \to hadrons$ and in the decay rate of $\pi^0 \to 2\gamma$. The discrepancy between the experimental results and the theoretical predictions was resolved by endowing quarks with three colors. The next important step was to elevate color as a dynamical degree of freedom of the strong interactions with the color symmetry group as $SU(3)_c$, by Fritzsch, Gell-Mann and Leutwyler [6]. Thus QCD came into being as a theory of strong interactions by the confluence of non-Abelian gauge theory with gauge group $SU(3)_c$ and color as the dynamical degree of freedom. The Lagrangian density for QCD is thus,

$$\mathcal{L}_{QCD} \;=\; -\frac{1}{4}\, F^a_{\mu\nu}\, F^{a\mu\nu} + \bar{\psi}\, i\, \not{D}\, \psi, \tag{1.1}$$

where,

$$\left.\begin{array}{rcl} F^a_{\mu\nu} & = & \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc}\, A^b_\mu\, A^c_\nu\,, \\ \not{D} & = & \gamma^\mu\, D_\mu\,, \\ D_\mu & = & \partial_\mu - i\, g\, [A_\mu,]\,, \\ A_\mu & = & t^a\, A^a_\mu\,, \\ [t^a, t^b] & = & i\, f^{abc}\, t^c\,. \end{array}\right\} \tag{1.2}$$

In equation (1.1) and (1.2), g is the QCD coupling constant defining the self-interaction of the gluons (gauge fields) and also is the quark-gluon (minimal) coupling, γ^μ are the Dirac matrices, and t^a's are the generators of $SU(3)$. Here the f^{abc} are $SU(3)$ structure constants and in this thesis we focus on $SU(2)$ which has structure constants [3] given by the Levi-Civita symbol ϵ^{abc}. The sum over quark flavors is understood.

Now we summarise the important developments leading to the properties of the quark-gluon coupling. In the deep inelastic scattering of electrons or neutrinos by the nucleon, the two independent variables are the energy loss $\nu = E_e - E_e^{'}$ and the square of the momentum transfer $Q^2 = -q^2 > 0$. Introducing a dimensionless variable $x = Q^2/(2M\nu)$, where M is the nucleon mass, Bjorken [7] found that the structure functions describing the target, have the scaling property,

$$\begin{aligned} M\,W_1(x,Q^2) &\rightarrow F_1(x)\;, \\ \nu\,W_2(x,Q^2) &\rightarrow F_2(x)\;, \end{aligned}$$

when $\nu \rightarrow \infty$, $Q^2 \rightarrow \infty$, with $\frac{Q^2}{\nu}$ fixed. This property, known as "Bjorken scaling" tells that in this limit the scattering cross-section depends upon one variable which is the signature for elastic scattering of the electrons from one of the free and point-like constituents that carry a fraction x of the four-momentum of the proton. This implies that during a rapid scattering process the interactions among the constituents of the proton can be ignored. Bjorken [7] surmised that this feature may emanate from a new field theory. Feynman [8] interpreted the above scaling in terms of the parton model according to which the proton is a loosely bound assemblage of a small number of constituents (partons) and the virtual photon is elastically scattered by them. The deep inelastic cross-section is just the incoherent sum of the individual (electron-parton) cross-sections. So, a field theory of quarks should be such that at large momentum (short-distance) the quarks must be non-interacting inside the hadrons. This is precisely demonstrated by Gross and Wilczek [9] and Politzer [10], by calculating the one-loop β-function for $SU(3)_c$ Yang-Mills theory. The negativeness of the β-function resulted in a momentum dependence of g or $\alpha_s = \frac{g^2}{4\pi}$ such that as $q^2 \rightarrow \infty$, $\alpha_s \rightarrow 0$, the quarks are asymptotically free. Thus the field theory of the strong interaction of quarks and gluons is experimentally confirmed to be the $SU(3)_c$ Yang-Mills theory. The behavior of $\alpha_s(q^2)$ is such that at low momentum (large distance of separation for quarks), the coupling is very large. This region is the confining region. As the quark-gluon coupling (and the gluon-gluon coupling) is very large in this region, perturbative methods cannot be reliably used. The ground state of QCD in this region is identified with the "non-perturbative" QCD vacuum. This is non-trivial. The vacuum expectation value (VEV) $\langle 0|F^a_{\mu\nu}F^{a\mu\nu}|0\rangle$ is not zero and this is one of the ingredients of the sum rule approach to QCD [11]. There are experimental supports for the non-vanishing of the above condensate from Charmonium decay analysis [12]. The "non-perturbative" QCD vacuum is thus characterised by non-vanishing condensates $\langle 0|F^a_{\mu\nu}F^{a\mu\nu}|0\rangle$ and $\langle 0|\bar{\psi}\psi|0\rangle$. In the absence of a complete solution of QCD in this region, appropriate models of the QCD vacuum have been suggested and are useful in making quantitative estimates of the condensates. Such models are: the Dual Superconducting model [13], the Savvidy Vacuum Model [14], Bag models [15] and Instanton Gas Models [16].

1.2 Overview of the Thesis

In this thesis, we study the Savvidy vacuum with the emphasis on the unstable modes. Savvidy [14] considered $SU(2)$ Yang-Mills theory and found that the one-loop effective energy density in a covariantly constant chromomagnetic field has a minimum at non-zero magnetic field, lower than the classical minimum (at zero magnetic field). This feature gave rise to $\langle 0|F^a_{\mu\nu}F^{a\mu\nu}|0\rangle \neq 0$, i.e., the vacuum contains some nonzero field. However Nielsen and Olesen [17] pointed out that the one-loop effective energy density has an imaginary part and its presence implied that the Savvidy vacuum is unstable. In search of a lower energy state to which the above could decay, Nielsen and Ninomiya [18], using an ansatz for the unstable modes and adding this to the classical background, semi-classically obtained a lower energy density than the Savvidy value. Nielsen and Olesen [19] and Ambjorn and Olesen [20] studied quantum fluctuations in the Nielsen-Ninomiya approach and proposed a disordered "flux-tube" vacuum. The occurrence of the imaginary part in the one-loop energy density has been noted by Trottier [21], Parthasarathy, Singer and Viswanathan [22], Huang and Levi [23] in constant non-Abelian background as well.

In order to avoid the imaginary part in the one-loop effective energy density, various proposals have been made. Cho [24] used gauge invariance arguments to exclude the imaginary part. Kondo [26] introduced mass terms for the off-diagonal gluons to produce a stable vacuum. The generation of mass for the off-diagonal gluons is an additional input, not present in the original Lagrangian. Cho, Lee, Walker and Pak [25] have considered the unstable modes issue in the Gaussian approximation and by using a Wick rotation with causality were able to obtain a real effective energy density. Flory [27] considered a class of backgrounds (classical) having electric and magnetic fields and treated the unstable modes beyond the one-loop. This indicated the possibility of obtaining a stable vacuum. Thus, in order to represent the ground state of QCD by the Savvidy vacuum, it is necessary to realise a real minimum for the one-loop effective energy density. This requirement is further driven at finite temperature [28–31]. These studies show the effective energy density at finite temperature involve a temperature-dependent imaginary part. This severely inhibits the progress.

In these above studies, the Background Field Method (BFM) is used. This consists in expanding the action in (1.1) around a classical background for the gauge field A^a_μ, say $\bar{A}^a_\mu$, as

$$A^a_\mu = \bar{A}^a_\mu + a^a_\mu, \tag{1.3}$$

where a^a_μ's are the quantum fluctuations. The gauge chosen is the 'background gauge',

$$\bar{D}^{ab}_\mu a^b_\mu = (\partial_\mu \delta^{ab} + g\, f^{acb}\, \bar{A}^c_\mu)\, a^b_\mu = 0. \tag{1.4}$$

The corresponding gauge fixing term and Faddeev-Popov ghost terms are added. The calculation of the partition function Z involves a path integral over a^a_μ. At this stage, terms up to quadratic in a^a_μ alone are retained in most of these studies. This quadratic or Gaussian approximation eventually gives an effective energy density involving an imaginary part. The origin of the imaginary part is the negativeness of one of the eigenvalue of the fluctuation operator, the lowest Landau level in the infrared region.

In this thesis, in Chapter III, we study the Savvidy background field,

$$\begin{aligned}
\bar{A}^a_0 &= 0, \\
\bar{A}^a_i &= \delta^{a\,3}\left(-\frac{H\,y}{2}, \frac{H\,x}{2}, 0\right), \\
\bar{F}^3_{12} &= H,
\end{aligned} \tag{1.5}$$

that is, a covariantly constant color magnetic field in the third color direction of $SU(2)$. The action (1.1) is used and expanded using (1.3). Since the quark fields ψ are quantum, the Dirac term (1.1) becomes $i\bar{\psi}\bar{\not{D}}\psi$, neglecting term involving a^a_μ as higher order. Since there are no unstable modes for the Dirac part, it is enough to have the above term only, and the Dirac Lagrangian is $i\bar{\psi}\bar{\not{D}}\psi$. As this does not involve a^a_μ, this can be evaluated separately and added at the end. This leaves $-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$ to be expanded. This is done keeping the cubic and the quartic terms in a^a_μ. This expansion is exact. The terms quadratic in a^a_μ are then analyzed to find the stable and unstable modes. For the stable modes, it is sufficient to retain terms up to quadratic in a^a_μ as the functional integral is convergent. For the unstable modes, the functional integral becomes convergent only when the cubic and quartic terms are included. This is evaluated and added to the contribution from the stable modes. The resulting energy density has no imaginary parts and coincides with the real part of the earlier calculations. To this, the contribution from the quarks (N_f, number of flavors) is to be added. The energy density for pure $SU(2)$ Yang-Mills with one-loop correction is given by,

$$\mathcal{E} = \frac{H^2}{2} + \frac{11}{48\,\pi^2}\,(g^2\,H^2)\left\{\log\left(\frac{g\,H}{\mu^2}\right) - \frac{1}{2}\right\}. \tag{1.6}$$

The inclusion of fermions or quarks with N_f flavors changes 11 in (1.6) to $(11-N_f)$ [32]. The energy density with inclusion of N_f flavor fermions is given by,

$$\mathcal{E} \;=\; \frac{H^2}{2} + \frac{(11-N_f)}{48\,\pi^2}\,(g^2\,H^2)\left\{\log\left(\frac{g\,H}{\mu^2}\right) - \frac{1}{2}\right\}. \qquad (1.7)$$

This has minimum $\mathcal{E} = -\frac{(11-N_f)}{96\pi^2}\,(\mu^4)\,e^{-\frac{48\pi^2}{(11-N_f)}g^2}$ at $H_c = \frac{\mu^2}{g}e^{-\frac{24\pi^2}{(11-N_f)}}$. Inclusion of fermions raises the minimum energy density and hence fermions tend to destabilise the vacuum. Since the minimum energy density in field theory corresponds to vacuum, we have a non-trivial vacuum with energy $\mathcal{E}_{min}$. As $H_c^2 = \langle 0|\bar{F}^a_{\mu\nu}\,\bar{F}^{a\mu\nu}|0\rangle \neq 0$, the fact that the minimum occurs at $H_c \neq 0$ implies $\langle 0|F^a_{\mu\nu}F^{a\mu\nu}|0\rangle \neq 0$. The connection to the bag model is made by stating that the inside state of the bag is a perturbative vacuum where quarks and gluons are free and whose energy density can be, without any loss of generality, taken to be zero. The outside state of the bag is considered to be a non-perturbative vacuum whose energy density is $\mathcal{E} = -\frac{(11-N_f)}{96\pi^2}\,(\mu^4)\,e^{-\frac{48\pi^2}{(11-N_f)}g^2}$. Then the bag constant $B = \frac{(11-N_f)}{96\pi^2}\,(g^2H_c^2) = \frac{(11-N_f)}{96\pi^2}\,(\mu^4)\,e^{-\frac{48\pi^2}{(11-N_f)}g^2}$. The numerical value of the bag constant B can be calculated by equating the critical value of the magnetic field H_c with the gluon condensate as $\langle 0|\frac{g^2}{4\pi^2}F^a_{\mu\nu}F^{a\mu\nu}|0\rangle = H_c^2$. From (1.7), the one-loop β-function is found to be $\beta = -\frac{(11-N_f)}{24\pi^2}\,g^3$. The negativeness for $N_f < 11$ ensures asymptotic freedom and this result agrees with the general expression for the β-function (16.135) of Peskin [33], for $SU(2)$.

The above study of the Savvidy vacuum is extended in Chapter IV to finite temperature. Earlier studies of the Savvidy vacuum at finite temperature by Ninomiya and Sakai [28], Cabo, Kalashnikov and Shabad [29], Starinets, Vshivtsev and Zhukovsky [30] and Meisinger and Ogilvie [31] employed the quadratic approximation and invariably ended in a temperature dependent complex energy density. The presence of the temperature dependent imaginary part inhibits the analysis of phase transitions. Also, there is a discrepancy in the analytical expressions for the one-loop energy density between [30] and [31], an interchange of J_1 and Y_1, and a relative sign between two K_1 functions. In Chapter III, we carefully examine the one-loop energy density by identifying the stable and unstable modes and evaluating the Matsubara sum and the functional integrals, including the cubic and quartic terms for the unstable modes. We have introduced a chemical potential for gluons originating from the conservation of color charges; the details are given in Chapter IV. The resulting energy density is found to be real and coincides with the real part of [31], thereby resolving the discrepancy in favor of [31]. This allows us to analyse the phase structure of the theory and, at high temperatures, the behaviour is like that of a relativistic gas.

Thus in Chapters III and IV, we have shown the ground state energy of the

Savvidy model is real both at zero and finite temperature. The phenomenon of chiral symmetry breaking can be addressed in the same scenario, by introducing a mass term for quarks in (1.1) i.e., $m\bar{\psi}\psi$, and evaluating $\frac{\partial}{\partial m}(V_{1-loop})|_{m=0}$. This is always found to be zero. So, in order to understand chiral symmetry breaking in the low momentum region, we follow the approach of Gribov [34–36]. This is based on the use of Schwinger-Dyson equations for the inverse of the quark propagator with the gluon propagator in the Feynman gauge, and the assumption that at low momentum, the coupling g is a slowly varying function of momentum. Then, the Schwinger-Dyson equation is transformed to a differential equation. This is solved at high and low momentum region [34–36]. It is found that the dynamical mass function $M(q^2)$ for quark is such that $M(0) \neq 0$. This shows that chiral symmetry is broken. The resulting Goldstone mode is the pion. The Schwinger-Dyson equation is modified for the pion contribution. We have studied this improved Schwinger-Dyson equation at low momentum. The corresponding dynamical mass function for quarks is found. Its variation with the momentum is studied.

Chapter II describes briefly the Background Field Method. It is applied to $SU(2)$ Yang-Mills theory with the (classical) Savvidy background in Chapter III. The resulting effective density is real and its properties are discussed. Chapter IV contains the above study at finite temperature for gluons only. Gribov's approach to chiral symmetry breaking with pion corrections is studied in Chapter V.

1.3 Summary of the Main Results of the Thesis

We summarise our results:

1. In Chapter III, the one-loop effective energy density of a $SU(2)$ Yang-Mills in the Savvidy background is found to be

$$\mathcal{E} \;=\; \frac{H^2}{2} + \frac{(11-N_f)}{48\,\pi^2}\,(g^2\,H^2)\left\{\log\left(\frac{g\,H}{\mu^2}\right) - \frac{1}{2}\right\},$$

   where N_f is the number of fermion flavors. This energy density has no imaginary part. The β-function for this calculated to be $-\frac{(11-N_f)}{24\pi^2}\,g^3$. The bag constant B is calculated to be $\frac{(11-N_f)}{96\pi^2}(\mu^4)e^{-\frac{48\pi^2}{(11-N_f)g^2}}$. For $N_f = 6$, $B^{\frac{1}{4}} = 188\,MeV$, using the gluon condensate value of $0.012\,GeV^4$. This is to be compared with MIT bag model value of 145 MeV.

2. In Chapter IV, we calculate the one-loop effective energy density in the Savvidy background for pure $SU(2)$ Yang-Mills theory at finite temperature and chemical potential,

$$\mathcal{E} \;=\; \frac{H^2}{2} + \frac{11}{48\pi^2}\,(gH)^2\left\{\log\left(\frac{gH}{\Lambda^2}\right) - \frac{1}{2}\right\}$$

$$
\begin{aligned}
&+ \quad \frac{\pi^2}{45\beta^4} \\
&+ \quad \frac{(gH)^{\frac{3}{2}}}{\beta\pi^2}\sum_{\ell=1}^{\infty}\frac{cos(\mu\beta\ell)}{\ell}\left(-\frac{\pi}{2}Y_1(\beta\ell\sqrt{gH})\right. \\
&+ \quad \left. K_1(\beta\ell\sqrt{gH}) + 2\sum_{n=1}^{\infty}\sqrt{2n+1}\;K_1(\sqrt{2n+1}\beta\ell\sqrt{gH})\right),
\end{aligned}
$$

where $\beta = \frac{1}{kT}$, μ is the chemical potential, Y_1 is the Neumann and K_1 is the modified Bessel function. This energy density is real without an imaginary part. Using this energy density, the temperature dependent part of $\mathcal{E}$ i.e. $(\mathcal{E}_T)$ scaled by '$(gH)^2$' is plotted with respect to the scaled temperature $T(\frac{k}{(gH)^2})$ for four values of chemical potential. At high temperature $\frac{T}{\sqrt{gH}} > 1.4$, the behaviour is like that of non interacting relativistic gas. For the chemical potential $b = 0$, the variation is smooth but for $b = 1, 2, 3$, the variation shows a minimum and then rises smoothly. So, a nonzero chemical potential triggers a possible deconfinement phase transition.

3. In Chapter V, in Gribov's approach at low momentum, we find a relation between the dynamical mass with pion correction $M(q^2)$ and the dynamical mass without pion correction $M_0(q^2)$,

$$
M(q^2) \;=\; M_0(q^2)\left[1 + \left(\frac{\beta q^2}{32\pi^2 f_\pi^2 p}\right)\left(\frac{1}{\alpha}\right)\left(1 - \frac{M_0^2(q^2)}{q^2}\right)\right], \tag{1.8}
$$

where

$$
\alpha \;=\; \left[\frac{\coth(\phi)}{2\dot{\phi}} + \frac{\beta^2}{4p} - \frac{\beta\, q^2}{16\pi^2 f_\pi^2 p}\right]
$$

and $\beta = 1 - g$, $f_\pi = 93\,MeV$, and p and ϕ are solutions to the parametric Schwinger-Dyson equation without pion correction. For low momentum, a graph has been plotted for $M(q^2)$ and $M_0(q^2)$ versus q; it is seen that the pion correction has a small effect on the quark mass $M_0(q^2)$ but makes it heavier.

Chapter 2

The Background Field Method (BFM)

2.1 Introduction

The Background Field Method (BFM) is useful in calculating the effective action of a Quantum Field Theory by expanding the field around a classical background field. The Green's functions are evaluated as a function of the background field. Historically, this method was introduced by DeWitt [37], then by Honerkamp [38] and was made popular by G. 't Hooft [39]. In the gauge theory, the classical Lagrangian is constructed to be gauge invariant but on quantisation the explicit gauge invariance is lost in the Feynman rules because of the necessity to add the gauge fixing and Faddeev-Popov ghost terms. To avoid the explicit breaking of gauge symmetry, the background field method (BFM) was developed in which one breaks only the invariance of the theory under quantum gauge transformations and there remains the gauge invariance of the classical fields, and it is this residual gauge invariance which serves as a useful book keeping device [40].

Briefly, the basic idea of the background field method is to write the gauge field appearing in the classical action as $\bar{A}^a_\mu + a^a_\mu$, where $\bar{A}^a_\mu$ is the classical background field and a^a_μ is the quantum field which is the variable of integration in the functional integral. Then, the gauge is chosen as the background field gauge, which breaks the gauge invariance in terms of the $a^a_\mu(x)$ field, but retains gauge invariance in terms of the $\bar{A}^a_\mu(x)$ field. Background field gauge invariance is further assured by coupling external sources only to the $\bar{A}^a_\mu(x)$ field. Thus, quantum calculations can be performed, yet explicit gauge invariance in the background field is not lost [41].

2.2 The Generating Functional in the Background Field Method

The generating functional for a non-Abelian gauge theory is given by

$$Z[J] = \int [dA]\, det \left[\frac{\delta G^a}{\delta \omega^l}\right] \exp i\left[S[A] - \frac{1}{2\,\alpha} G.G + J.A\right], \tag{2.1}$$

where A^a_μ is the $SU(2)$ gauge field, G^a's are the gauge fixing condition, ω^l are the infinitesimal gauge parameters and J is the external source and the classical action $S[A]$ is given by,

$$S = -\frac{1}{4}\int d^4x \left\{F^a_{\mu\nu}\, F^{a\mu\nu}\right\}, \tag{2.2}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc}\, A^b_\mu\, A^c_\nu.$$

For $SU(2)$, f^{abc} will be replaced by ϵ^{abc}.

In the 'background field method' we expand $A^a_\mu(x) = \bar{A}^a_\mu(x) + a^a_\mu(x)$ and the expanded generating functional is,

$$\tilde{Z}[J, \bar{A}^a_\mu(x)] = \int [da^a_\mu]\, det \left[\frac{\delta \tilde{G}^a}{\delta \omega^l}\right] \exp i\left[S[\bar{A} + a] - \frac{1}{2\alpha}\tilde{G}.\tilde{G} + J.\bar{A}\right], \tag{2.3}$$

where $\tilde{G}^a$ is the modified gauge fixing condition, the Background field gauge fixing condition, given by

$$\tilde{G}^a = \partial_\mu\, a^a_\mu + g\, f^{acb}\, \bar{A}^c_\mu\, a^b_\mu \equiv \bar{D}^{ab}_\mu\, a^b_\mu = 0.$$

The infinitesimal transformation are defined according to,

$$\begin{aligned}
\delta\, a^a_\mu &\equiv -g\, f^{abc}\, \omega^b\, a^c_\mu, \\
\delta\, \bar{A}^a_\mu &\equiv -g\, f^{abc}\, \omega^b\, \bar{A}^c_\mu + \partial_\mu\, \omega^a, \\
\delta\, J^a_\mu &\equiv -g\, f^{abc}\, \omega^b\, J^c_\mu,
\end{aligned} \tag{2.4}$$

so that the sum of both fields transforms in the usual way,

$$\delta\,(\bar{A}^a_\mu + a^a_\mu) \equiv -g\, f^{abc}\, \omega^b\, (\bar{A}^c_\mu + a^c_\mu) + \partial_\mu\, \omega^a,$$

and hence the classical action remains invariant [42]. It is to be noticed that in equation (2.4), the background gauge field $\bar{A}^a_\mu$ transforms like the gauge field while the quantum fields a^a_μ transform homogeneously. In the loop calculation of the effective action, the background field '$\bar{A}^a_\mu$' appears in external amputated legs, whereas the quantum gauge fields 'a_μ^a' and the ghost fields 'η^a' live only in internal lines.

2.3 The Background Field Gauge

One of the remarkable features of the background gauge is that it is free from the Gribov ambiguity. The background gauge fixing condition is given by,

$$\tilde{G}^a \equiv \bar{D}_\mu^{ab}\, a_\mu^b = (\partial_\mu \delta^{ab} + g\, f^{acb}\, \bar{A}_\mu^c)\, a_\mu^b = 0.$$

Now, consider the variation in this gauge $\tilde{G}^a$ under the infinitesimal transformation (2.4):

$$\begin{aligned}
\delta\, \tilde{G}^a &= \delta\,(\partial_\mu\, a_\mu^a) + g\, f^{acb}\, \delta(\bar{A}_\mu^c\, a_\mu^b), \\
&= \delta\,(\partial_\mu\, a_\mu^a) + g\, f^{abc}\, \delta(\bar{A}_\mu^b\, a_\mu^c), \\
&= \partial_\mu\,(\delta a_\mu^a) + g\, f^{abc}\, \left[(\delta \bar{A}_\mu^b)\, a_\mu^c + \bar{A}_\mu^b (\delta a_\mu^c)\right].
\end{aligned}$$

We use the infinitesimal transformation (2.4),

$$\begin{aligned}
\delta\, \tilde{G}^a &= \partial_\mu(-g f^{abc} \omega^b a_\mu^c) + g f^{abc} \left[-g f^{bpq} \omega^p \bar{A}_\mu^q a_\mu^c + (\partial_\mu \omega^b) a_\mu^c \right. \\
&\quad \left. - g f^{cpq} \omega^p \bar{A}_\mu^b a_\mu^q\right], \\
&= -g f^{abc} (\partial_\mu \omega^b) a_\mu^c - g f^{abc} \omega^b (\partial_\mu a_\mu^c) - g^2 f^{abc} f^{bpq} \omega^p \bar{A}_\mu^q a_\mu^c \\
&\quad + g f^{abc} (\partial_\mu \omega^b) a_\mu^c - g^2 f^{abc} f^{cpq} \omega^p \bar{A}_\mu^b a_\mu^q,
\end{aligned}$$

After cancellation of the first term with the fourth term, we have,

$$\begin{aligned}
\delta\, \tilde{G}^a &= -g f^{abc}\, \omega^b\, \partial_\mu (a_\mu^c) - g^2\, f^{abc}\, f^{bpq}\, \omega^p\, \bar{A}_\mu^q\, a_\mu^c - g^2\, f^{abc}\, f^{cpq}\, \omega^p\, \bar{A}_\mu^b\, a_\mu^q, \\
&= -g f^{abc}\, \omega^b\, \partial_\mu (a_\mu^c) - g^2\, \omega^p\, (f^{acq}\, f^{cpb} + f^{abc}\, f^{cpq})\, \bar{A}_\mu^b\, a_\mu^q.
\end{aligned}$$

Using the Jacobi identity,

$$f^{acq}\, f^{cpb} + f^{abc}\, f^{cpq} = -\, f^{apc}\, f^{bcq},$$

$$\begin{aligned}
\delta\, \tilde{G}^a &= -\, g f^{abc}\, \omega^b\, \partial_\mu (a_\mu^c) + g^2\, f^{apc}\, f^{bcq}\, \omega^p\, \bar{A}_\mu^b\, a_\mu^q, \\
&= -\, g f^{abc}\, \omega^b\, \bar{D}_\mu^{cq}\, (a_\mu^q).
\end{aligned}$$

Therefore, under the background gauge transformation $\tilde{G}^a = \bar{D}_\mu^{ab}(a_\mu^b)$ transforms like a_μ^a:

$$\begin{aligned}
\delta\, \tilde{G}^a &= -g\, f^{abc}\, \omega^b\, \bar{D}_\mu^{cq}\, (a_\mu^q), \\
&= -g\, f^{abc}\, \omega^b\, \tilde{G}^c.
\end{aligned}$$

Thus, $\tilde{G}^a$ transforms homogeneously and $\delta\tilde{G}^a$ is proportional to $\tilde{G}^a$. So on fixing the gauge $\tilde{G}^a = 0$, the change in the gauge $\delta\tilde{G}^a$ will be zero. So, the change in the gauge is free from the Gribov ambiguity. The change in the gauge fixing term $-\frac{1}{2\alpha}\tilde{G}^a.\tilde{G}^a$ is $-\frac{1}{2\alpha}\delta(\tilde{G}^a.\tilde{G}^a) = -\frac{1}{2\alpha}2\tilde{G}^a.\delta\tilde{G}^a = \frac{1}{\alpha}gf^{abc}\omega^b\tilde{G}^c.\tilde{G}^a$. Thus the change in gauge fixing term is proportional to itself, so that it is free from the Gribov ambiguity [42].

2.4 Quantum Yang-Mills Action in the Background Field

The original Lagrangian density depends on $\bar{A}^a_\mu$ and a^a_μ only through the sum $(\bar{A}^a_\mu + a^a_\mu)$. From (2.4) we have

$$\begin{aligned} \delta\,(\bar{A}^a_\mu + a^a_\mu) &= D^{ab}_\mu\,(\bar{A} + a)\,\omega^b \\ &= \partial_\mu\,\omega^a + g\,f^{acb}\,(\bar{A}^c_\mu + a^c_\mu)\,\omega^b\,. \end{aligned} \tag{2.5}$$

The gauge transformation in (2.4) should be distinguished from a true gauge transformation, by which we mean that the background field $\bar{A}^a_\mu$ is taken as *a fixed classical background with* $\delta\bar{A}^a_\mu = 0$, the fluctuations transforming as,

$$\begin{aligned} \delta\,a^a_\mu &= D^{ab}_\mu\,\omega^b = \partial_\mu\,\omega^a + g\,f^{acb}\,(\bar{A}^c_\mu + a^c_\mu)\,\omega^b, \\ &= \bar{D}^{ab}_\mu\,\omega^b + g\,f^{acb}\,a^c_\mu\,\omega^b. \end{aligned} \tag{2.6}$$

This transformation is in effect the same as (2.5) for $(\bar{A}^a_\mu + a^a_\mu)$. Now,

$$\delta\,\tilde{G}^a = \bar{D}^{ab}_\mu\,\bar{D}^{bd}_\mu\,\omega^d - g\,\bar{D}^{ab}_\mu\,(f^{bcd}\,\omega^c\,a^d_\mu), \tag{2.7}$$

such that

$$\frac{\delta\,\tilde{G}^a}{\delta\,\omega^l} = \bar{D}^{ab}_\mu\,\bar{D}^{bl}_\mu - g\,\bar{D}^{ab}_\mu\,(f^{bld}\,a^d_\mu). \tag{2.8}$$

Then the Faddeev-Popov determinant is given by,

$$\Delta_{FP} = det\{\bar{D}^{ab}_\mu\,\bar{D}^{bl}_\mu - g\,\bar{D}^{ab}_\mu\,(f^{bld}\,a^d_\mu)\}. \tag{2.9}$$

This is exponentiated using scalar anti-commuting variables (ghosts) η, η^* to give $\mathcal{L}_{ghost} = -(\bar{D}^{ab}_\mu\,\eta^*_b)(\bar{D}^\mu_{ac}\eta_c - g\,f^{abc}\,\eta_b\,a^\mu_c)$ and the gauge fixing term is $\mathcal{L}_{GF} = -\frac{1}{2\alpha}(\bar{D}^{ab}_\mu\,a^b_\mu)^2$. The full quantum action is

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{A_\mu} + \mathcal{L}_{ghost} + \mathcal{L}_{GF};$$

such that

$$\begin{aligned} \mathcal{L}_{A_\mu} &= -\frac{1}{4}\left(\bar{F}^a_{\mu\nu} + \bar{D}_\mu a^a_\nu - \bar{D}_\nu a^a_\mu + g\,f^{abc}\,a^b_\mu\,a^c_\nu\right)^2; \\ \mathcal{L}_{ghost} &= -(\bar{D}_\mu\,\eta^*_a)(\bar{D}^\mu\,\eta_a - g\,f^{abc}\,\eta_b\,a^\mu_c); \\ \mathcal{L}_{GF} &= -\frac{1}{2\,\alpha}\,(\bar{D}^\mu\,a^a_\mu)^2\,, \end{aligned} \tag{2.10}$$

where

$$
\begin{aligned}
\bar{F}^a_{\mu\nu} &= \partial_\mu \bar{A}^a_\nu - \partial_\nu \bar{A}^a_\mu + g\, f^{abc}\, \bar{A}^b_\mu\, \bar{A}^c_\nu, \\
\bar{D}^\mu\, a^a_\nu &= \partial_\mu\, a^a_\nu + g\, f^{abc}\, \bar{A}^b_\mu\, a^c_\nu, \\
\bar{D}^\mu\, \eta^a &= \partial_\mu\, \eta^a + g\, f^{abc}\, \bar{A}^b_\mu\, \eta^c.
\end{aligned}
$$

Then, the functional integral in (2.3) over $a^a_\mu(x)$ can be performed to obtain the effective energy density. This method is used in Chapters III and IV.

Chapter 3

Savvidy Vacuum in $SU(2)$ Yang-Mills Theory*

3.1 Introduction

The status of the vacuum in Quantum Field Theory (QFT) is like the physical medium in which creation and annihilation of various particles and their interactions are taking place in analogy with the ground state (vacuum) of condensed matter physics. QCD is a non-Abelian gauge theory and the source of its non-Abelian nature is interactions among gluons which are (unlike photon) color charged and this gluon-gluon interaction makes QCD ground state (vacuum) a complex and structured object. The non-trivial structure of the QCD vacuum manifests itself through the existence of various condensates. Two of the simplest are: the gluon condensate $\langle 0|\frac{g^2}{4\pi}F^a_{\mu\nu}F^{\mu\nu a}|0\rangle$ of '$mass^4$' dimension and the chiral condensate $\langle 0|\bar{\psi}\psi|0\rangle$ of '$mass^3$' dimension . These condensates are supposed to be of non-perturbative nature, which means that their finite numerical values are supposed to remain so after the divergent expressions obtained for the expectation values in perturbation theory are eliminated by renormalization. It is believed that the non-trivial vacuum structure of QCD forbids the propagation of quarks and gluons outside hadrons. Therefore, the question of confinement of quarks and gluons boils down to the study of the QCD vacuum. The first progress in this direction was made in 1975 by Polyakov [43], who discovered the non-perturbative "Instanton" solutions of the classical gauge-field equations. These Instantons are localised both in space and in imaginary time, and they have an energy lower than the perturbative vacuum. As a possibility, the structure of the ground state of QCD can contain a gas of such Instantons in Euclidean space-time.

The second progress in the direction to understand the QCD vacuum was made

*This chapter is based on the publication, "Savvidy Vacuum in $SU(2)$ Yang-Mills Theory", D. Kay, A. Kumar and R. P.: Mod. Phys. Lett. **A20**, 1655-1662 (2005).

by George Savvidy in 1977. He probed the vacuum by applying a background (external) chromomagnetic field and, for simplicity, he chose pure $SU(2)$ Yang-Mills theory in a covariantly constant magnetic field in the third color direction ($F_{12}^3 = H$) [14]. Savvidy found the one-loop effective energy density of the theory as,

$$\mathcal{E} = \frac{H^2}{2} + \frac{11}{48\pi^2}(gH)^2 \left\{ \log\left(\frac{gH}{\mu^2}\right) - \frac{1}{2} \right\}. \tag{3.1}$$

The minimum of $\mathcal{E}$ is $-\frac{11}{96\pi^2}(gH_c)^2$ for a non-zero value of the applied chromomagnetic field $H = H_c = \frac{\mu^2}{g^2} e^{-\frac{24\pi^2}{11 g^2}}$. The variation of the classical energy density ($\frac{H^2}{2}$) and the one-loop effective energy density (3.1) with the chromomagnetic field H is qualitatively shown in Figure 3.1 for $\alpha_s = 1$ and $\mu = 2.3$. From Figure 3.1 it can be seen that the one-loop effective energy density has minimum negative value for nonzero value of H unlike its classical part which has minimum value zero for $H = 0$. This study led to the conclusion that QCD vacuum with zero field strength is unstable, and decays into a state with a non-vanishing value of the field i.e., the energy density of perturbative vacuum gets lowered by the introduction of an external constant gauge field of magnetic type.

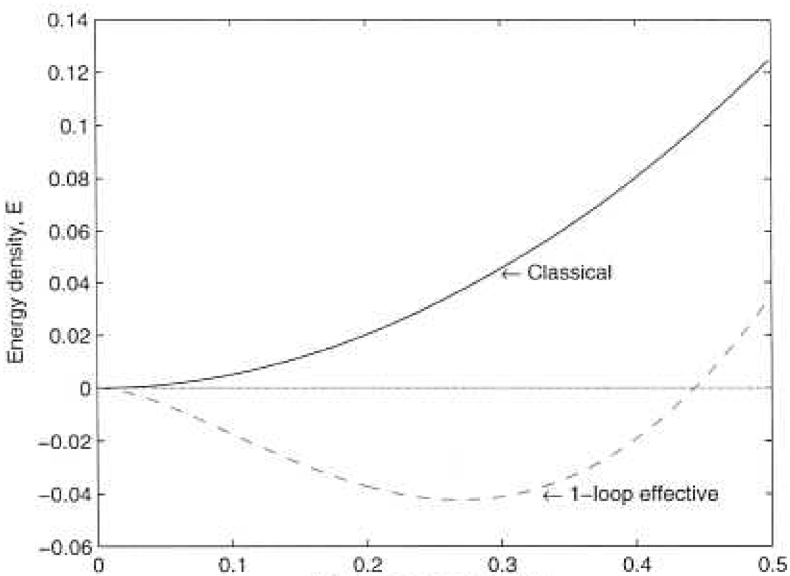

Figure 3.1: Variation of the classical energy density (solid line) and the one-loop effective energy density (broken line) with external applied chromomagnetic field H.

Further, Nielsen and Olesen [17] studied the Savvidy background carefully and observed that the one-loop effective energy density has an imaginary part. The presence of an imaginary part in the effective energy density is the signature of instability in the Savvidy vacuum and hence implies that the Savvidy vacuum would decay (dissipate) to a further lower energy state. In the search of a lower state to which the unstable Savvidy vacuum would decay, Nielsen and Ninomiya [18], using an ansatz for the unstable modes and adding this to the classical background, obtained semi-classically a lower energy density than the Savvidy value. Further Nielsen and Olessen [19] and Ambjorn and Olessen [20] studied quantum fluctuations in the Nielsen-Ninomiya approach and produced a disordered "flux-tube" vacuum. While the Savvidy background $[A_0^a = 0; A_i^a = \delta^{a3}(-\frac{Hy}{2}, \frac{Hx}{2}, 0)]$ for $SU(2)$ Yang-Mills theory is Abelian-like, the persistence of the imaginary part in the one-loop effective energy (hence instability) was shown even in a non-Abelian background by Anishetty [44] and Parthasarathy, Singer and Viswanathan [22].

Similar calculations in 3-d (Euclidean) $SU(2)$ Yang-Mills theory using the Savvidy background by Trottier [21] and the constant non-Abelian background of Ref. [22] by Huang and Levi [23] revealed the occurrence of the imaginary part in the one-loop effective energy density. In contrast, Leutwyler [45] considered a self-dual background and showed that there is no imaginary part. Parthasarathy, Basu and Anishetty [46] studied the Savvidy vacuum using the auxiliary field method and showed that the vacuum is stable for a range of coupling strengths.

All these studies of the QCD vacuum demonstrate its non-trivial structure. Although these studies are encouraging in the understanding of non-perturbative aspects of QCD, the persistence of the imaginary part in the effective energy density is disappointing and requires special attention. Cho used gauge invariance arguments to exclude the imaginary part so that the real part alone need be considered [24]. This result led to studies [25] on the unstable modes using a Wick rotation and causality. Further, Kondo [26] observed that the dynamically generated mass term for the off-diagonal gluons results in a stable vacuum. However, the gauge invariance arguments are qualitative and the generation of mass for the off-diagonal gluons is an additional input, not present in the original Lagrangian. Therefore, it is highly imperative to address this imaginary part of the one-loop effective energy density within the framework of the Background Field Method (BFM) without taking any resort to qualitative argument.

An effort in this direction was taken by Flory [27] by treating the unstable modes beyond the one-loop approximation i.e., Gaussian approximation, for a class of backgrounds having electric and magnetic fields. Subsequently, Kay [47] considered the Savvidy background following Flory. The results of Flory and Kay indicate a possibility of obtaining a stable Savvidy vacuum solution within the Background Field Method in a gauge invariant manner. With this motivation to study the Savvidy vacuum to obtain a real one-loop energy density, we consider the unstable modes in the Savvidy background carefully and separate them from the stable modes. Then for the unstable modes, we include the cubic and quartic terms in the expanded action. The corresponding partition function is evaluated and added to the effective one-loop energy density for the stable modes. The resulting energy density has no imaginary part and coincides with the real part of the earlier calculations [48].

3.2 The Effective Energy Density in the Background Field Method (BFM)

For simplicity, we choose pure Yang-Mills theory to study the Savvidy background. We consider the Euclidean partition function for $SU(2)$ Yang-Mills theory. The Euclidean functional integral for a pure $SU(2)$ Yang-Mills theory is

$$Z = \int [dA_\mu^a]\, e^S, \tag{3.2}$$

where the classical action S is

$$S = \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a \right\}, \tag{3.3}$$

with $F_{\mu\nu}^a$, the field strength tensor, given as

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\,\epsilon^{abc} A_\mu^b A_\nu^c. \tag{3.4}$$

Under the spell of the background field method (BFM), we expand the gauge field A_μ^a around the classical background field $\bar{A}_\mu^a$, which satisfies the classical equation of motion $\bar{D}_\mu^{ab}\bar{F}_{\mu\nu}^b = 0$ with $\bar{D}_\mu^{ab} = \partial_\mu\delta^{ab} + g\epsilon^{acb}\bar{A}_\mu^c$ as the "background covariant derivative", and $\bar{F}_{\mu\nu}^a = \partial_\mu \bar{A}_\nu^a - \partial_\nu \bar{A}_\mu^a + g\,\epsilon^{abc}\bar{A}_\mu^b \bar{A}_\nu^c$.
Expanding, A_μ^a, as

$$A_\mu^a = \bar{A}_\mu^a + a_\mu^a,$$

we have,

$$F_{\mu\nu}^a = \bar{F}_{\mu\nu}^a + \bar{D}_\mu^{ac} a_\nu^c - \bar{D}_\nu^{ac} a_\mu^c + g\,\epsilon^{abc} a_\mu^b a_\nu^c. \tag{3.5}$$

The full expanded action is given by

$$\begin{aligned} S = \int d^4x \Big\{ & -\frac{1}{4}\bar{F}_{\mu\nu}^a \bar{F}_{\mu\nu}^a + (\bar{D}_\mu^{ac}\bar{F}_{\mu\nu}^c)\, a_\nu^a + \frac{1}{2} a_\nu^c\, (\bar{D}_\mu^{ca}\bar{D}_\mu^{ac'})\, a_\nu^{c'} \\ & -\frac{1}{2} a_\nu^c\, (\bar{D}_\mu^{ca}\bar{D}_\nu^{ac'})\, a_\mu^{c'} - \frac{1}{2} g\,\epsilon^{abc}\bar{F}_{\mu\nu}^a a_\mu^b a_\nu^c - g\,\epsilon^{ab'c'}(\bar{D}_\mu^{ac} a_\nu^c)\, a_\mu^{b'} a_\nu^{c'} \\ & -\frac{1}{4} g^2 \epsilon^{abc}\epsilon^{ab'c'} a_\mu^b a_\nu^c a_\mu^{b'} a_\nu^{c'} \Big\}. \end{aligned} \tag{3.6}$$

The first term is purely classical and the second term is linear in the quantum fluctuation 'a_μ^a'. The second term vanishes as the background field satisfies the classical equation of motion. The third , fourth and fifth terms are quadratic in the quantum fluctuation. The sixth term is cubic in the quantum fluctuation and the seventh term in the expanded action (3.6) is quartic in the quantum fluctuation. We consider the terms quadratic in the quantum fluctuations and denote them by T_2:

$$T_2 \;=\; \frac{1}{2}\,a_\nu^c\,(\bar{D}_\mu^{ca}\,\bar{D}_\mu^{ac'})\,a_\nu^{c'} \;-\; \frac{1}{2}\,a_\nu^c\,(\bar{D}_\mu^{ca}\,\bar{D}_\nu^{ac'})\,a_\mu^{c'} \;-\; \frac{1}{2}\,g\,\epsilon^{abc}\,\bar{F}_{\mu\nu}^a\,a_\mu^b\,a_\nu^c. \quad (3.7)$$

We use the relation

$$\bar{D}_\mu^{ca}\,\bar{D}_\nu^{ac'} \;-\; \bar{D}_\nu^{ca}\,\bar{D}_\mu^{ac'} \;=\; g\,\epsilon^{cdc'}\,\bar{F}_{\mu\nu}^d, \quad (3.8)$$

to simplify T_2 as,

$$T_2 \;=\; \frac{1}{2}\,a_\mu^a\,\Theta_{\mu\nu}^{ac}\,a_\nu^c \;-\; \frac{1}{2}\,a_\nu^a\,(\bar{D}_\nu^{ab}\bar{D}_\mu^{bc})\,a_\mu^c, \quad (3.9)$$

where

$$\Theta_{\mu\nu}^{ac} \;=\; (\bar{D}_\lambda^{ab}\bar{D}_\lambda^{bc})\,\delta_{\mu\nu} \;+\; 2\,g\,\epsilon^{aec}\,\bar{F}_{\mu\nu}^e. \quad (3.10)$$

The full expanded action then becomes,

$$S \;=\; \int d^4x\left\{-\frac{1}{4}\,\bar{F}_{\mu\nu}^a\,\bar{F}_{\mu\nu}^a \;+\; \frac{1}{2}\,a_\mu^a\,\Theta_{\mu\nu}^{ac}\,a_\nu^c \;+\; g\,\epsilon^{acd}\,(\bar{D}_\nu^{ae}\,a_\mu^e)\,a_\mu^c\,a_\nu^d \right.$$
$$\left. -\frac{g^2}{4}\left((a_\mu^a\,a_\mu^a)^2 - (a_\mu^a\,a_\mu^c\,a_\nu^a\,a_\nu^c)\right)\right\} \;-\; \frac{1}{2}\,a_\nu^a\,(\bar{D}_\nu^{ab}\bar{D}_\mu^{bc})\,a_\mu^c. \quad (3.11)$$

The integration measure is

$$[d\,A_\mu^a] \;=\; [d\,\bar{A}_\mu^a][d\,a_\mu^a]. \quad (3.12)$$

For constant background field $\bar{A}_\mu^a$, we have

$$[d\,A_\mu^a] \;\to\; [d\,a_\mu^a]. \quad (3.13)$$

For the quantisation of the theory, we have to fix the gauge. We choose the background field gauge (as explained in Chapter II, this gauge is free from the Gribov ambiguity):

$$\bar{D}_\mu^{ab}\,a_\mu^b \;=\; 0, \quad (3.14)$$

and employ the Faddeev-Popov method for quantisation in the background gauge. We have the Euclidean partition function,

$$Z \;=\; N\int [d\,a_\mu^a]\,\Delta_{FP}\,e^{\left(S - \frac{1}{2\alpha}\int d^4x\,(\bar{D}_\mu^{ab}\,a_\mu^b)^2\right)}, \quad (3.15)$$

where Δ_{FP} is the Faddeev-Popov determinant and gauge fixing (3.14) is implemented by the second term in the exponent. Now,

$$\begin{aligned} \frac{1}{2\alpha} \int d^4 x \, (\bar{D}^{ac}_{\mu} \, a^b_{\mu})^2 &= \frac{1}{2\alpha} \int d^4 x \, \bar{D}^{ab}_{\mu} \, a^b_{\mu} \, \bar{D}^{ac}_{\nu} \, a^c_{\nu}, \\ &= -\frac{1}{2\alpha} \int d^4 x \, a^a_{\mu} \, \bar{D}^{ab}_{\mu} \, \bar{D}^{bc}_{\nu} \, a^c_{\nu}, \end{aligned} \tag{3.16}$$

where a partial integration is performed. Then,

$$Z = N \int [d\, a^a_{\mu}] \, \Delta_{FP} \, e^{(S + \frac{1}{2\alpha} \int d^4 x \, a^a_{\mu} \, (\bar{D}^{ab}_{\mu} \, \bar{D}^{bc}_{\nu}) \, a^c_{\nu})} \, . \tag{3.17}$$

Choosing $\alpha = 1$ in (3.17), the gauge fixing term cancels the last term of the full expanded classical action (3.11). The ghost Lagrangian is given by (2.10) in Chapter II. The second term is cubic in the combined ghost and gauge field fluctuations and is omitted, so the $\mathcal{L}_{ghost} = -(\bar{D}^{ab}_{\mu} \eta^{*b})(\bar{D}^{ac}_{\mu} \eta^c)$. In Z, the functional integral over the ghost fields gives the Faddeev-Popov determinant, as

$$\Delta_{FP} = -\, det(-\bar{D}^{ab}_{\mu} \bar{D}^{bc}_{\mu}). \tag{3.18}$$

The complete Euclidean partition function is

$$Z = N \int [d\, a^a_{\mu}] \, e^{S'}, \tag{3.19}$$

with

$$\begin{aligned} S' = \int d^4 x \Big\{ &-\frac{1}{4} \bar{F}^a_{\mu\nu} \, \bar{F}^a_{\mu\nu} + \frac{1}{2} a^a_{\mu} \, \Theta^{ac}_{\mu\nu} \, a^c_{\nu} + g \, \epsilon^{acd} \, (\bar{D}^{ae}_{\nu} \, a^e_{\mu}) \, a^c_{\mu} \, a^d_{\nu} \\ &- \frac{g^2}{4} \left((a^a_{\mu} \, a^a_{\mu})^2 - (a^a_{\mu} \, a^c_{\mu} \, a^a_{\nu} \, a^c_{\nu}) \right) \Big\} - \log \, det(-\bar{D}^{ab}_{\mu} \bar{D}^{bc}_{\mu}), \end{aligned} \tag{3.20}$$

where $\Theta^{ac}_{\mu\nu}$ is given by (3.10). The last term of (3.20) is the result of integration over the ghost fields. Equation (3.20) is the full effective action after quantisation in the constant covariant background field $\bar{A}^a_{\mu}$.

In the Gaussian approximation, one ignores the cubic and quartic terms in the quantum fluctuations 'a^a_{μ}'. The linear term in the fluctuation drops out as the background field solves the classical equation of motion. Therefore, in the Gaussian approximation, the effective potential (action), is

$$\Gamma^{eff} = S^{cl} + \Gamma^{one-loop},$$

and

$$\Gamma^{one-loop} = \frac{1}{2} Tr \, \log \, det \, (-\Theta^{ac}_{\mu\nu}) - Tr \, \log \, det \, (-\bar{D}^{ab}_{\mu} \bar{D}^{bc}_{\mu}). \tag{3.21}$$

For four-dimensional SU(2) Yang-Mills theory, in the Savvidy background,

$$\begin{aligned} \bar{A}^a_0 &= 0\,; \\ \bar{A}^a_i &= \delta^{a3}\left(-\frac{H\,y}{2}, \frac{H\,x}{2}, 0\right), \end{aligned} \tag{3.22}$$

which solves the classical equation of motion, $\bar{D}^{ab}_\mu \bar{F}^b_{\mu\nu} = 0$, we notice

$$\begin{aligned} -\Theta^{ac}_{44} &= -\Theta^{ac}_{33} = -\,\bar{D}^{ab}_\lambda \bar{D}^{bc}_\lambda\,, \\ \Theta^{ac}_{41} &= \Theta^{ac}_{42} = \Theta^{ac}_{43} = \Theta^{ac}_{31} = \Theta^{ac}_{32} = 0. \end{aligned} \tag{3.23}$$

The contributions from $\mu, \nu = 3, 4$ cancel exactly the ghost contribution as can be seen from (3.21). Therefore, we need to consider eigenmodes and eigenvalues for further analysis of Θ^{ac}_{ij} for only $i, j = 1, 2$ and $a, c = 1, 2, 3$. The corresponding eigenmodes and eigenvalues are (see Appendix 3.2 for the eigenvalues and eigenmodes calculation):

$$\begin{aligned} a^3_1 \pm i\,a^3_2 &: \; k^2_3 + k^2_4 \text{ (plane waves)}, \\ (a^1_1 + i\,a^1_2) + i\,(a^2_1 + i\,a^2_2) &: \; (2n+1)\,gH - 2\,gH + k^2_3 + k^2_4, \\ (a^1_1 + i\,a^1_2) - i\,(a^2_1 + i\,a^2_2) &: \; (2n+1)\,gH + 2\,gH + k^2_3 + k^2_4, \\ (a^1_1 - i\,a^1_2) + i\,(a^2_1 - i\,a^2_2) &: \; (2n+1)\,gH + 2\,gH + k^2_3 + k^2_4, \\ (a^1_1 - i\,a^1_2) - i\,(a^2_1 - i\,a^2_2) &: \; (2n+1)\,gH - 2\,gH + k^2_3 + k^2_4. \end{aligned} \tag{3.24}$$

Therefore, within the Gaussian (quadratic) approximation, we then have,

$$\begin{aligned} \Gamma^{one-loop} = \; & \frac{1}{2}\left(\frac{gH}{2\pi}\right) 2 \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{d\,k_3}{2\pi} \int_{-\infty}^{\infty} \frac{d\,k_4}{2\pi} \\ & \left[\log\left\{\frac{1}{\mu^2}\left\{(2n+1)\,gH + 2\,gH + k^2_3 + k^2_4\right\}\right\}\right. \\ & \left. + \log\left\{\frac{1}{\mu^2}\left\{(2n+1)\,gH - 2\,gH + k^2_3 + k^2_4\right\}\right\}\right], \end{aligned} \tag{3.25}$$

where the '$\frac{gH}{2\pi}$' prefactor is from the harmonic oscillator density of states, the '2' prefactor is due to the multiplicity of each eigenvalue in (3.24) and μ^2 is the dimensionful constant to render the argument of the logarithm dimensionless. Expression (3.25) agrees with (32) of Flyvberg [32].

It is to be noticed that the integrals (3.25) have ultra-violet divergences as k_3 and k_4 tend to infinity and both logarithmic functions diverge. In order to regularise these ultra-violet divergences, we use the ϵ-regularization prescription by Salam and Strathdee [49]. This regularization prescription is explained in detail in Appendix (3.1). The argument of the first logarithmic function in (3.25) is positive for all n, k_3, k_4. The sum over n and the integration over k_3, k_4 with Salam's ϵ-regularization prescription give the finite part, as,

$$\frac{11}{96\pi^2}\left(g^2H^2\right)\left\{\log\left(\frac{gH}{\mu^2}\right)+C\right\}, \tag{3.26}$$

where C is a real constant.

The argument of the second logarithmic function in (3.25) is positive for all k_3, k_4 and $n \neq 0$ and a similar regularization gives the finite part for $n \neq 0$ as,

$$-\frac{1}{96\pi^2}\left(g^2H^2\right)\left\{\log\left(\frac{gH}{\mu^2}\right)+C'\right\}, \tag{3.27}$$

where C' is a real constant. The argument of the second logarithmic function will be negative for $n = 0$ and $k_3^2 + k_4^2 < gH$ and the logarithmic function with negative argument will produce an imaginary part. This imaginary part is the source of instability in the Savvidy vacuum in the quadratic approximation. Nevertheless, we employ the Salam's ϵ-regularization prescription for $n = 0$ in the second logarithmic function of (3.25) to calculate the finite part as,

$$\frac{1}{8\pi^2}\left(g^2H^2\right)\left\{\log\left(\frac{gH}{\mu^2}\right)+C''\right\} - i\,\frac{1}{8\pi^2}(g^2H^2)(\pi). \tag{3.28}$$

We collect all finite parts obtained using Salam's ϵ-regularization prescription (3.26), (3.27) and (3.28) and add to the classical background contribution, to obtain the resulting one-loop effective energy density as

$$\mathcal{E} \;=\; \frac{H^2}{2} + \frac{11}{48\pi^2}\,g^2H^2\left\{\log\left(\frac{gH}{\mu^2}\right)+R\right\} - i\,\frac{g^2H^2}{8\pi}, \tag{3.29}$$

where R is a real constant which can be fixed by the Coleman-Weinberg normalisation,

$$\left.\frac{\partial}{\partial H^2}\,Re(\mathcal{E})\right|_{gH=\mu^2} \;=\; \frac{1}{2}, \tag{3.30}$$

as $R = -\frac{1}{2}$. This result for $\mathcal{E}$ with $R = -\frac{1}{2}$ agrees with earlier calculations [32].

3.3 Zero Temperature Unstable Modes—Inclusion of the Cubic and Quartic Terms

In order to tame the instability of the Savvidy vacuum in the quadratic approximation we go beyond the quadratic approximation for unstable modes and consider the full action for the unstable modes. As seen from (3.24), there are two unstable modes,

$$
\begin{aligned}
a_+^+ &\equiv (a_1^1 + i\,a_2^1) + i\,(a_1^2 + i\,a_2^2), \\
a_-^- &\equiv (a_1^1 - i\,a_2^1) - i\,(a_1^2 - i\,a_2^2),
\end{aligned}
\tag{3.31}
$$

whose eigenvalues become negative and result in an imaginary part to the one-loop effective energy density, when $k_3^2 + k_4^2 < gH$ for $n = 0$. These unstable modes each have density of states '$\frac{gH}{2\pi}$'. These unstable modes are given in terms of the normalised eigenfunctions, as

$$
\phi_{k_3,k_4}(x) \;=\; \sqrt{\frac{gH}{2\pi}}\, e^{-\frac{gH}{4}(x_1^2+x_2^2)}\,\frac{1}{\sqrt{L_3 L_4}}\, e^{i\,(k_3 x_3 + k_4 x_4)},
\tag{3.32}
$$

where the first exponential is the ground state wave function ($n = 0$) of the two-dimensional harmonic oscillator in the (x_1-x_2) plane while the second exponential is the "box-normalised" plane wave in the x_3 and x_4 directions. It is easy to verify $\int \phi^*_{k_3,k_4}(x)\phi_{k_3,k_4}(x)\, d^4x = 1$. The unstable modes in (3.31) are expressed as $a_+^+ = c_+^+\phi_{k_3,k_4}(x)$ and $a_-^- = c_-^-\phi^*_{k_3,k_4}(x)$, its complex conjugate, with $c_+^+(c_-^-)$ as constant vectors. For these unstable modes we consider the full action in (3.20).

The unstable modes involve only the Lorentz indices 1 and 2 (as can be seen from (3.24)) and SU(2) indices 1 and 2 (because the classical background $\bar{A}^a_\mu$ is in the third color isospin direction). Therefore, the cubic term $\epsilon^{acd}\,(\bar{D}^{ac}_\nu\, a^c_\mu)\, a^c_\mu\, a^d_\nu$ vanishes because the first term, $\epsilon^{acd}\,(\partial_\nu a^a_\mu)\, a^c_\mu\, a^d_\nu$, reduces to $\epsilon^{acd}\{(\partial_1 a^a_2)\, a^c_2\, a^d_1 + (\partial_2 a^a_1)\, a^c_1\, a^d_2\}$, which is zero when a, c, d are restricted to 1 and 2, and the second term $\epsilon^{acd}\,\epsilon^{a3e}\,\bar{A}^3_\nu\, a^e_\mu\, a^c_\mu\, a^d_\nu = \bar{A}^3_\nu\,\{a^d_\mu\, a^3_\mu\, a^d_\nu - a^c_\mu\, a^c_\mu\, a^3_\nu\}$ vanishes as well, as the unstable modes do not involve the SU(2) index 3.

The term quartic in the fluctuations a^a_μ in (3.20) has two parts, $(a^a_\mu\, a^a_\mu)^2$ and $a^a_\mu\, a^c_\mu\, a^a_\nu a^c_\nu$. Again, the unstable modes involve the Lorentz and SU(2) color indices 1 and 2, so that the combined term $(a^a_\mu\, a^a_\mu)^2 - a^a_\mu\, a^c_\mu\, a^a_\nu\, a^c_\nu$ is found to be $\frac{1}{8}a_-^-\, a_-^-\, a_+^+\, a_-^-$, where a_+^+ and a_-^- are given in (3.31). From (3.19) and (3.20), the partition function for the unstable modes is

$$
\begin{aligned}
Z_{unstable} &= \int [d\,a^a_u]\; e^{\int d^4x \{a_u\,(gH - k_3^2 - k_4^2)\,a_u \,-\, \frac{g^2}{32}|a_u|^4\}}, \\
&= \int [d\,a_u]\; e^{-\int d^4x \{a_u\,(k_3^2 + k_4^2 - gH)\,a_u \,+\, \frac{g^2}{32}|a_u|^4\}},
\end{aligned}
\tag{3.33}
$$

where a_u stands for a_+^+ and a_-^-. It is clear from (3.24) that the unstable modes occur for $k_3^2 + k_4^2 < gH$. For this, the quadratic term in the exponent (3.33) diverges. However, the quartic term in (3.33) provides necessary and crucial convergence. The normalised unstable mode eigenfunctions can be obtained from

(3.32). The degeneracy factor D is given by

$$D = 2 \cdot \frac{gH}{2\pi} \frac{L_3}{2\pi} \frac{L_4}{2\pi} \equiv \frac{gH}{4\pi^3} \Omega, \tag{3.34}$$

where the factor 2 accounts for the two unstable modes and $\Omega = L_3 L_4$ is the two-dimensional volume. The quadratic term in the exponential of (3.33) is

$$\int d^4 x \, c^2 \, (k_3^2 + k_4^2 - gH) \, \phi^2_{k_3,k_4}(x), \tag{3.35}$$

and since $\phi^2_{k_3,k_4}(x)$'s are normalised to unity, (3.35) becomes

$$c^2 \, (k_3^2 + k_4^2 - gH). \tag{3.36}$$

The quartic term in the exponential of (3.33) is

$$\frac{g^2}{32} \int d^4 x \, c^2 \, \phi^2_{k_3,k_4}(x) \left\{ 2 \frac{L_3 L_4}{(2\pi)^2} \right\} \int d\,k_3' \, d\,k_4' \, c^2 \, \phi^2_{k_3',k_4'}(x), \tag{3.37}$$

where,

1. we have introduced primed states for the second pair of unstable modes and integrated them as is necessitated by the occurrence of $a_+^+ a_+^+ a_-^- a_-^-$ and the pair a_-^- are allowed to have momenta differing from those of a_+^+ by a small but bounded region $< gH$, and

2. we have introduced the density of states for the primed modes by $\{...\}$ in (3.37).

Using (3.32), (3.37) becomes,

$$\frac{g^2}{32} \int d^4 x \, c^2 \left(\frac{gH}{2\pi} \right)^2 \frac{1}{L_3^2 L_4^2} \left\{ 2 \frac{L_3 L_4}{(2\pi)^2} \right\} e^{-gH\,(x_1^2 + x_2^2)} \int d\,k_3' \, d\,k_4' \, . \tag{3.38}$$

Now $\int dx_1 \, dx_2 \, e^{-gH\,(x_1^2+x_2^2)} = \frac{\pi}{gH}$ and $\int d\,x_3 \, d\,x_4 = L_3 \, L_4$, so that (3.38) becomes

$$\frac{g^3 \, H}{256\pi^3} c^4 \int d\,k_3' \, d\,k_4'. \tag{3.39}$$

For the primed states differing from k_3, k_4 by a small amount bounded by gH, we have $\int d\,k_3' \, d\,k_4' = \pi g H$ and so (3.39) becomes

$$\frac{g^4 \, H^2}{256\pi^2} c^4. \tag{3.40}$$

Using (3.36) and (3.40) in (3.33), we get

$$Z_{unstable} = \left(\prod_{k_3,k_4} \int d\,c \, e^{-\left\{ c^2 (k_3^2 + k_4^2 - gH) + \frac{g^4 H^2}{256\pi^2} c^4 \right\}} \right)^D, \tag{3.41}$$

where we have introduced the degeneracy factor D in $Z_{unstable}$. Now we make the following substitutions: $c' = \sqrt{gH}\,c$, $k_3' = \frac{k_3}{\sqrt{gH}}$, $k_4' = \frac{k_4}{\sqrt{gH}}$,
so that (3.41) becomes,

$$Z_{unstable} = \left(\frac{1}{\sqrt{gH}} \prod_{k_3, k_4} \int d\,c'\, e^{-\left\{ c'^2 (k_3'^2 + k_4'^2 - 1) + \frac{g^2}{256\pi^2} c'^4 \right\}} \right)^D . \tag{3.42}$$

It is important to observe that the expression in the curly brackets in the exponent (3.42) is independent of H. The H dependence in $Z_{unstable}$ is only from the prefactor $\frac{1}{\sqrt{gH}}$, D and from the k_3 and k_4 integrations. This is very essential for our purpose. Next, we observe that the functional integral over $[d\,a_u]$ in (3.33) is converted to an ordinary integral over $d\,c'$ in (3.42). Now, the integral in (3.42) over $d\,c'$ is convergent, due to the c'^4 term in the exponent (coming from the quartic contribution), irrespective of whether $k_3'^2 + k_4'^2$ is < 1 or > 1. Denoting the value of this integral by I (which is independent of H), the contribution to the energy density from the unstable modes is,

$$\begin{aligned} \mathcal{E}_{unstable} &\equiv -\frac{1}{\Omega} \log\{Z_{unstable}\} = -\frac{D}{\Omega} \log\left(\frac{I}{\sqrt{gH}}\right) \int d\,k_3\, d\,k_4 \\ &= -\frac{D}{\Omega} (\pi g H) \log\left(\frac{I}{\sqrt{gH}}\right) + J, \end{aligned} \tag{3.43}$$

where we have used the unstable eigenvalue requirement $k_3^2 + k_4^2 < gH$ in performing the $d\,k_3\,d\,k_4$ integrations in (3.43) to get the first term in the second line of (3.43). The contribution from the $d\,k_3\,d\,k_4$ integration beyond the above requirement will be infinite and is denoted by J in (3.43). By considering the finite part (consistent with (3.28)), and using D in (3.34), the finite part of the unstable mode contribution to the energy density becomes,

$$-\frac{g^2 H^2}{4\pi^2} \left\{ \log(I) - \frac{1}{2} \log\left(\frac{gH}{\mu^2}\right) \right\} = \frac{g^2 H^2}{8\pi^2} \log\left(\frac{gH}{\mu^2}\right) - \frac{g^2 H^2}{4\pi^2} \log(I). \tag{3.44}$$

The first term in (3.44) is exactly the real part (3.28) as the second term in (3.44) can be absorbed into C'' in (3.28). In this procedure of including the quartic terms in the unstable modes, there is no imaginary part and so the effective energy density is

$$\mathcal{E} = \frac{H^2}{2} + \frac{11}{48\pi^2} (g^2 H^2) \left\{ \log\left(\frac{gH}{\mu^2}\right) - \frac{1}{2} \right\}, \tag{3.45}$$

which coincides with the real part of earlier calculations.

It is to be mentioned that in treating the unstable modes exactly, we have considered the contributions of the quadratic and quartic terms. Despite this, the calculations leading to the effective action are gauge invariant. In order to see this, notice that the mixing of the quadratic and the quartic terms occurs in the exponent in equation (3.42). The first term comes from the quadratic part and the second from the quartic part of the unstable modes. The second term is one order in g^2 higher. The crucial observation is that these two terms in (3.42) are independent of H. The H-dependence is in the prefactor $\frac{1}{\sqrt{gH}}$, D in (3.34) and the $d\,k_3\,d\,k_4$ integrations in (3.42) and all these involve the gauge invariant combination gH. The mixing effect is thus contained in the integral I which is independent of H. Then the calculation for the complete unstable modes leads to (3.44). The effect of mixing is contained in $-\frac{g^2\,H^2}{4\,\pi^2}\,\log(I)$ which is absorbed into C'' (3.28). Therefore R in (3.29) is fixed by Coleman-Weinberg normalisation as $-\frac{1}{2}$. So the calculations involve only gauge invariant quantities.

3.4 Inclusion of Fermions

Inclusion of fermions will not change the form of the energy density; the only change will be the replacement of the prefactor 11 in (3.45) by (11-N_f) for N_f fermion flavors. In order to calculate the contribution to the one-loop effective energy density of the inclusion of N_f fermion flavors in the Savvidy background, we consider the partition function for fermions [32],

$$\Delta\,F \;=\; \int [d\,\bar{\psi}]\,[d\,\psi]\,exp\left[\int d^4\,x\,\bar{\psi}\,i\,\gamma_\mu\,\bar{D}_\mu\psi\right]. \tag{3.46}$$

The sum over quark flavors is understood. Upon carrying out the functional integral, (3.46) becomes

$$\Delta\,F \;=\; det\,[i\,\gamma_\mu\,\bar{D}_\mu]. \tag{3.47}$$

The fermion contribution to the classical action is

$$\Gamma \;=\; -\,Tr\,ln\,[i\,\gamma_\mu\,\bar{D}_\mu]. \tag{3.48}$$

In order to compute Γ, we need to solve the Dirac equation,

$$\begin{aligned} i\,\gamma_\mu\,\bar{D}_\mu\,\psi &= \lambda\,\psi, \\ \bar{D}_\mu &= \partial_\mu + i\,[\bar{A}_\mu,], \\ \bar{A}_\mu &= \bar{A}^a_\mu\,t^a. \end{aligned}$$

For the Savvidy background, $i\gamma_\mu\bar{D}_\mu$ is found to be diagonal in $SU(2)$ space and each Dirac equation describes a fermion moving in a constant (color) magnetic field. The eigenvalues for massless fermions are given by

$$\begin{aligned} \lambda^2 &= k_3^2 + k_4^2 + \frac{g\,H}{2}\,(2\,n + 1 + \lambda); \qquad (3.49) \\ n &= 0, 1, 2, 3, ..., \\ \lambda &= \pm 1. \end{aligned}$$

Unlike the gauge field fluctuation, for $k^2 \to 0, n = 0$, the eigenvalues do not become negative. Evaluating 'Tr log' using ϵ-regularization procedure, we find

$$\Delta F = -\frac{1}{48\pi^2}(g^2H^2)N_f \log\left(\frac{gH}{\mu^2}\right) + g^2H^2\left(O\left(\frac{1}{\epsilon}\right) + RFC\right), \quad (3.50)$$

where RFC is a real finite constant and $O(\frac{1}{\epsilon}) \to \infty$ as $\epsilon \to 0$. After renormalization like the gauge field case,

$$\Delta F = -\frac{1}{48\pi^2}(g^2H^2)N_f \left\{\log\left(\frac{gH}{\mu^2}\right) - \frac{1}{2}\right\}. \quad (3.51)$$

Therefore, the one-loop effective potential in the Savvidy background with the inclusion of N_f fermion flavors is

$$\mathcal{E} = \frac{H^2}{2} + \frac{(11-N_f)}{48\pi^2}(g^2H^2)\left\{\log\left(\frac{gH}{\mu^2}\right) - \frac{1}{2}\right\}. \quad (3.52)$$

The one-loop effective energy density is free from imaginary part i.e., it is real and can be used for further calculation like the bag constant.

Let us calculate the minimum for the one-loop energy density, in the Savvidy background,

$$\frac{\partial\mathcal{E}}{\partial H} = H + \frac{(11-N_f)}{24\pi^2}(g^2H)\left\{\log\left(\frac{gH}{\mu^2}\right)\right\}. \quad (3.53)$$

For the minimum of $\mathcal{E}$, $\frac{\partial\mathcal{E}}{\partial H} = 0$, gives,

$$H\left[1 + \frac{(11-N_f)}{24\pi^2}(g^2)\log\left(\frac{gH}{\mu^2}\right)\right] = 0. \quad (3.54)$$

The above equation has two solutions,

$$\begin{aligned} H &= 0, \qquad \text{and} \\ \frac{(11-N_f)}{24\pi^2}(g^2)\log\left(\frac{gH_c}{\mu^2}\right) &= -1. \end{aligned} \quad (3.55)$$

Therefore, the minimum for the energy density $\mathcal{E}_{min}$ is for $H = H_c = \frac{\mu^2}{g}e^{-\frac{24\pi^2}{(11-N_f)g^2}}$ and not for $H = 0$ as it can be verified from the sign of second derivative $\frac{\partial^2\mathcal{E}}{\partial H^2}$. The minimum energy is given by

$$\mathcal{E}_{min} = -\frac{(11-N_f)}{96\pi^2}\mu^4 e^{-\frac{48\pi^2}{(11-N_f)g^2}}, \quad (3.56)$$

at

$$\begin{aligned} H &= H_c, \\ &= \frac{\mu^2}{g} e^{-\frac{24\pi^2}{(11-N_f)g^2}}. \end{aligned} \tag{3.57}$$

The bag constant, B, which is the difference of the energy density of the perturbative vacuum and the non-perturbative vacuum, is

$$\begin{aligned} B &= 0 - \mathcal{E}_{min}, \\ &= \frac{(11 - N_f)}{96\,\pi^2}\,\mu^4\,e^{-\frac{48\pi^2}{(11-N_f)g^2}}. \end{aligned} \tag{3.58}$$

The numerical value of the bag constant can be calculated by finding $H = H_c$ where the energy density would be minimum and equating H_c^2 with the gluon condensate, $\langle 0|\frac{g^2}{\pi}F^a_{\mu\nu}F^{\mu\nu a}|0\rangle = H_c^2$. The value of the gluon condensate has been obtained using Charmonium decay analysis. Using this value of the gluon condensate, $G = 0.012\,GeV^4$, (3.58) gives $B^{\frac{1}{4}} = 188\,MeV$ for $N_f = 6$, $B^{\frac{1}{4}} = 196\,MeV$ for $N_f = 5$, and $B^{\frac{1}{4}} = 205\,MeV$ for $N_f = 4$. This value for $B^{\frac{1}{4}}$ should be compared with the phenomenological MIT estimate of 145 Mev [32]. Although our estimate for $N_f = 6$ is close to the MIT value, as the gluon condensate value came form Charmonium decay, it is appropriate to use the estimate for $N_f = 4$. This is in qualitative agreement with the MIT value. We have considered SU(2) QCD here and this can be improved by considering SU(3) QCD.

Now, we use the one-loop effective energy density (3.52) to calculate the beta function by the Callan-Symanzik equation,

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \gamma H\frac{\partial}{\partial H}\right]\Gamma(g, H, \mu) = 0, \tag{3.59}$$

where $\beta = g\,\gamma$ [47],

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\left\{\frac{\partial}{\partial g} - \frac{H}{g}\frac{\partial}{\partial H}\right\}\right]\Gamma(g, H, \mu) = 0. \tag{3.60}$$

Now recognising Γ in (3.60) as $\mathcal{E}$ and solving the equation (3.60), we have,

$$\frac{(11 - N_f)}{24\,\pi^2}\,g^2 + \frac{\beta(g)}{g} = 0. \tag{3.61}$$

Therefore the beta function is given by

$$\beta(g) = -\frac{(11 - N_f)}{24\,\pi^2}\,g^3. \tag{3.62}$$

The one-loop expression for β-function for SU(N) theory given in Peskin (equation (16.135)) [33] agrees with (3.62) for $N = 2$. The above result obtained for SU(2) can be extended to SU(3). It is discussed in detail in [50] that the result is the same as (3.45) for the one-loop effective energy density with $(11 - N_f)$ replaced by $\frac{33-2N_f}{2}$.

Chapter 4

$SU(2)$ Yang-Mills Theory in the Savvidy Background at Finite Temperature and Chemical Potential*

4.1 Introduction

We study in this chapter the Savvidy vacuum at finite temperature and chemical potential as a continuation of Chapter III. The Savvidy vacuum at finite temperature has been studied by several authors [51–52,28–31] in the quadratic approximation. The results of these studies show the occurrence of an imaginary part in the effective energy density even at finite temperature like in the zero temperature case. This persistence of the imaginary part as a function of temperature is a serious issue in the view of the stability of the Savvidy vacuum at finite temperature. It has been observed by Meisinger and Ogilvie [31] that with the introduction of a Polyakov loop, specified by ϕ, it is possible to stabilise the vacuum, if $\beta\sqrt{gH} < \phi < 2\pi - \beta\sqrt{gH}$, where $\beta = 1/kT$, H is the chromomagnetic background in the third color direction, as for this range of ϕ, the imaginary part becomes zero. However, the imaginary part is nonzero at the global minimum. Therefore, in the understanding of the finite temperature behaviour, the temperature-dependent imaginary part poses difficulty and hence it is required to be addressed in the finite temperature field theory using the background field method. The main purpose of this chapter is to extend the method of Chapter III and [48] to finite temperature. We also introduce a chemical potential for

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gluons. The one-loop corrected effective energy density is found to be real when we include the cubic and quartic terms for the unstable modes.

We introduce a chemical potential for massless non-Abelian gauge bosons, although the notion of chemical potential for bosons is not defined in the sense that particle number is not conserved. However, in the case of non-Abelian gauge bosons, the notion of chemical potential is possible as argued by Anishetty [53]. It is based on the observation that in non-Abelian SU(N) gauge theories, the local gauge invariance of the action $S = \int F^a_{\mu\nu} F^a_{\mu\nu} d^4x$ under the infinitesimal gauge transformation $A^a_\mu \to A^a_\mu + D^{ab}_\mu \omega^b$ where D^{ab}_μ is the standard covariant derivative and $\omega^a \in SU(N)$, gives conserved color charges, $Q^a = \int d^3x j^a_0$, where $j^a_\mu = f^{abc} A^b_\nu F^c_{\nu\mu}$ is the divergence less Noether current, since using $D^{ab}_\mu F^b_{\mu\nu} = 0$, we have $j^a_\mu = -\frac{1}{g}\partial_\nu F^a_{\nu\mu}$ and so $\partial_\mu j^a_\mu = 0$. The color charge of the physical states obeying Gauss law $D^{ab}_i E^b_i \,|\psi\rangle = 0$, $E^a_i = F^a_{i0}$, is invariant up to global gauge transformations. The above conserved color charges satisfy the algebra, $[Q^a, Q^b] = f^{abc} Q^c$ and so one cannot define charge eigenstates with respect to all color directions. Hence, one chooses eigenstates of $Q^a Q^a$ and Q^c where the index c specifies the Cartan sub-algebra. In the case of $SU(2)$ it will be Q^3. Further, as the quadratic charge operator in the partition function generates a non-locality in the action density, one chooses Q^3 in $SU(2)$ gauge theory. The grand canonical partition function will now have μQ^3 in the Hamiltonian, where μ is space-time independent. This leads to the result (see Anishetty in [53] for details) of using $A^a_0 = \lambda^a - i\mu\delta^{a3}$, where λ^a vanishes over the space boundary. Thus in massless gauge theories, the chemical potential μ arises from conserved color charges. This procedure will not be applicable to Abelian gauge bosons. It can be interpreted as the constant term which the A^a_0 field approaches at the space boundary. This conclusion has also been obtained by Actor [53].

The presence of the chemical potential does not change the short-distance properties of the fields. The role of the chemical potential as a constant term in the A^a_0 field is similar to the [53] constant color background A^a_0 field of Belyaev or to the Polyakov loop specified by a constant A^a_0 field in the third color direction as in [31]. As we will see, the chemical potential will appear as $(k_4 \pm \mu)$ in the eigenvalues and its effect is to keep the diagonal gluons $(a = 3)$ massless. The off-diagonal gluons acquire an effective mass in the infrared region. Another purpose of this chapter, is to resolve a discrepancy in the analytical expressions for the finite temperature one-loop energy density between [30] and [31]. (The discrepancy is an interchange of J_1 and Y_1 and also a relative sign between the two K_1 functions in the energy density). Our results show that the finite part of the effective energy density is real. The real energy density coincides with the real part of [31] which therefore resolves the discrepancy as mentioned above, in favor of [31].

4.2 The Effective Energy Density in the Background Field Method (BFM) at Finite Temperature

The Euclidean functional integral for an $SU(2)$ pure Yang-Mills theory is

$$Z = \int [dA_\mu^a] e^S, \tag{4.1}$$

where

$$S = \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \right\}, \tag{4.2}$$

and

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} A_\mu^b A_\nu^c. \tag{4.3}$$

We expand $A_\mu^a = \bar{A}_\mu^a + a_\mu^a$ with $\bar{A}_\mu^a$ as the classical background field satisfying the equation of motion $\bar{D}_\mu^{ab} \bar{F}_{\mu\nu}^b = 0$, with $\bar{D}_\mu^{ab} = \partial_\mu \delta^{ab} + g\epsilon^{acb} \bar{A}_\mu^c$ as the *background covariant derivative*; $\bar{F}_{\mu\nu}^a$ is the same as (4.3) with $\bar{A}_\mu^a$. We choose the background field gauge to satisfy

$$\bar{D}_\mu^{ab} a_\mu^b = 0. \tag{4.4}$$

In order to quantise the non-Abelian gauge theory, we have to fix the gauge and we use the Faddeev-Popov method of path integral quantisation in the background field gauge. Using the background gauge (4.4) and calculating the Faddeev-Popov ghost determinant as we have done in Chapter III, we have

$$Z = \int [da_\mu^a] e^{S'}, \tag{4.5}$$

with

$$\begin{aligned} S' = \int d^4x \Big(& -\frac{1}{4} \bar{F}_{\mu\nu}^a \bar{F}_{\mu\nu}^a + \frac{1}{2} a_\mu^a \Theta_{\mu\nu}^{ac} a_\nu^c + g\epsilon^{acd} (\bar{D}_\nu^{ae} a_\mu^e) a_\mu^c a_\nu^d \\ & - \frac{g^2}{4} \{ (a_\mu^a a_\mu^a)^2 - a_\mu^a a_\mu^c a_\nu^a a_\nu^c \} - \log\, det(-\bar{D}_\mu^{ab} \bar{D}_\mu^{bc}) \Big), \end{aligned} \tag{4.6}$$

where, the differential operator $\Theta_{\mu\nu}^{ac}$ is given by

$$\Theta_{\mu\nu}^{ac} = (\bar{D}_\lambda^{ab} \bar{D}_\lambda^{bc}) \delta_{\mu\nu} + 2g\epsilon^{aec} \bar{F}_{\mu\nu}^e. \tag{4.7}$$

The expansion of (4.6) is exact. Now, in the quadratic (Gaussian) approximation, one ignores the cubic and the quartic terms in the quantum fluctuation a_μ^a, and then the one-loop effective potential is,

$$\Gamma^{one-loop} = \frac{1}{2} Tr \log\, det(-\Theta_{\mu\nu}^{ac}) - Tr \log\, det(-\bar{D}_\mu^{ab} \bar{D}_\mu^{bc}). \tag{4.8}$$

The Savvidy background is the chromomagnetic field in the third color direction and is given by

$$\bar{A}_0^a = 0 \;\; ; \;\; \bar{A}_i^a \;=\; \delta^{a3}\left(-\frac{Hy}{2}, \frac{Hx}{2}, 0\right), \tag{4.9}$$

which gives $\bar{F}_{12}^3 = H$ and solves the classical equation of motion, $\bar{D}_\mu^{ab}\bar{F}_{\mu\nu}^b = 0$. One of the ways to introduce a chemical potential, μ, is via a background field and we introduce it as in [53],

$$\bar{A}_\mu^a \;=\; \frac{i\mu}{g}v_\mu\delta^{a3}, \qquad v_\mu = (1,0,0,0). \tag{4.10}$$

We combine both backgrounds (4.9) and (4.10) to get,

$$\bar{A}_\mu^a \;=\; \delta^{a3}\left(\frac{i\mu}{g}, -\frac{Hy}{2}, \frac{Hx}{2}, 0\right), \tag{4.11}$$

which again gives $\bar{F}_{12}^3 = H$ as the non-vanishing background field strength and *which solves the classical equation of motion.* In this method of introducing the chemical potential via (4.10) and (4.11), it will play the role as the Polyakov loop specified by a constant $\bar{A}_4^a$ field in the third color direction as in [31] with the identification $\phi = \beta\mu$. As in Chapter III and in Ref. [48] equation (10), we notice that (4.7) gives,

$$\Theta_{44}^{ac} = \Theta_{33}^{ac} \;=\; \bar{D}_\lambda^{ab}\bar{D}_\lambda^{bc}, \tag{4.12}$$

where

$$\bar{D}_\lambda^{ab} \;=\; \partial_\lambda\delta^{ab} + g\epsilon^{a3b}\bar{A}_\lambda^3 + \mu\epsilon^{a3b}v_\lambda, \tag{4.13}$$

where $\bar{A}_\lambda^3$ is given by (4.9) and v_λ is defined in (4.10). Using (4.11) in (4.7), it is found that

$$\Theta_{41}^{ac} = \Theta_{42}^{ac} = \;\Theta_{43}^{ac} \;= \Theta_{31}^{ac} = \Theta_{32}^{ac} = 0. \tag{4.14}$$

Therefore, the ghost contributions cancel exactly the contributions of $\Theta_{\mu\nu}^{ac}$ corresponding to the indices μ, ν = 3, 4. The only eigenvalues of $\Theta_{\mu\nu}^{ac}$ which are relevant correspond to $\mu\,,\nu$ = 1, 2. The eigenmodes and the corresponding eigenvalues of $\Theta_{\mu\nu}^{ac}$ for i, j = 1, 2 are found to be (see Appendix 3.2):

$$\left.\begin{array}{ccc}
(a_1^1 + ia_1^2) + i(a_2^1 + ia_2^2) & : & (k_4+\mu)^2 + k_3^2 + (2N+1)gH - 2gH, \\
(a_1^1 + ia_1^2) - i(a_2^1 + ia_2^2) & : & (k_4+\mu)^2 + k_3^2 + (2N+1)gH + 2gH, \\
(a_1^1 - ia_1^2) + i(a_2^1 - ia_2^2) & : & (k_4-\mu)^2 + k_3^2 + (2N+1)gH + 2gH, \\
(a_1^1 - ia_1^2) - i(a_2^1 - ia_2^2) & : & (k_4-\mu)^2 + k_3^2 + (2N+1)gH - 2gH, \\
a_i^3 & : & k_4^2 + k_3^2 + k_2^2 + k_1^2;\; i = 1,2.
\end{array}\right\} \tag{4.15}$$

where $N = 0, 1, 2, \cdots$ is the harmonic oscillator quantum number (in the x-y plane). If we take $\mu = 0$ in (4.15), the spectrum of (4.15) will match with the spectrum of equation (3.24) in Chapter III, which is the case of the Savvidy background without chemical potential.

One of the formal ways to take finite temperature into account in quantum field theoretic descriptions due to Matsubara [55], known as imaginary time formalism, is explained in detail in Appendix 4.1. In this formalism, we replace 'k_4' by '$\frac{2\pi n}{\beta}$' and $\int_{-\infty}^{\infty} d\,k_4$ by $\sum_{-\infty}^{\infty}$ [54]. We use the eigenvalues for the eigenmodes of Θ_{ij}^{ac} as in (4.15) for the calculation of $\Gamma^{one-loop}$ (4.8) and employ the Matsubara formalism to get the effective potential in the "quadratic approximation" at finite temperature and chemical potential, which is given by

$$\begin{aligned}
\Gamma^{one-loop} &= 2 \times \frac{1}{2} \times \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \log\left[\frac{1}{\Lambda^2}\left(\frac{4\pi^2 n^2}{\beta^2} + \vec{k}^2\right)\right] \\
&+ \frac{1}{2} \times \frac{1}{\beta}\left(\frac{gH}{2\pi}\right) \sum_{n=-\infty}^{\infty} \sum_{N=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \\
&\left[\log\left\{\frac{1}{\Lambda^2}\left\{\left(\frac{2\pi n}{\beta} + \mu\right)^2 + k_3^2 + (2N+1)gH - 2gH\right\}\right\}\right. \\
&+ \log\left\{\frac{1}{\Lambda^2}\left\{\left(\frac{2\pi n}{\beta} + \mu\right)^2 + k_3^2 + (2N+1)gH + 2gH\right\}\right\} \\
&+ \log\left\{\frac{1}{\Lambda^2}\left\{\left(\frac{2\pi n}{\beta} - \mu\right)^2 + k_3^2 + (2N+1)gH + 2gH\right\}\right\} \\
&\left.+ \log\left\{\frac{1}{\Lambda^2}\left\{\left(\frac{2\pi n}{\beta} - \mu\right)^2 + k_3^2 + (2N+1)gH - 2gH\right\}\right\}\right],
\end{aligned} \tag{4.16}$$

where the first term corresponds to the last eigenvalues in (4.15) and the remaining correspond, respectively, to the first four eigenvalues in (4.15). The prefactor '$\frac{gH}{2\pi}$' is the harmonic oscillator degeneracy and the prefactor 2 in the first term accounts for the two modes in the last term in (4.15) for $i = 1, 2$. Here Λ is the dimensionful parameter to render the argument of the logarithmic function in (4.16) dimensionless. It is convenient to suppress 'Λ^2' hereafter.

This expression for the one-loop effective potential at finite temperature and chemical potential (4.16) agrees with equation (8) of Ref. [31] with their ϕ replaced by $\mu\beta$ and with equation (2.16) of Ninomiya and Sakai [28] with $\mu = 0$. It can be noticed from (4.15), that for $k_3 = 0$ and $n = 0$, $N = 0$, the first and the fourth eigenvalues become $\mu^2 - gH$ and therefore to avoid negative eigenvalues $\mu > \sqrt{gH}$. However, for $n = 1$, the fourth eigenvalue becomes $(\frac{2\pi}{\beta} - \mu)^2 - gH$ (with $N = 0$ and $k_3 = 0$) and if this is to remain positive, then $\mu < \frac{2\pi}{\beta} - \sqrt{gH}$. With this, the first eigenvalue remains positive. So for $\sqrt{gH} < \mu < \frac{2\pi}{\beta} - \sqrt{gH}$, it is possible to avoid the negative eigenvalues and hence the instability. However, the instability enters at the global minimum [31]. So, this kind of restriction on μ is not enough.

In order to see the origin of the instability, we consider the second logarithmic of (4.16) (excluding prefactor)

$$L_2 = - \sum_{n=-\infty}^{\infty} \sum_{N=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \log\left\{ \left(\frac{2\pi n}{\beta} + \mu \right)^2 + k_3^2 + A \right\}, \tag{4.17}$$

where $A = 2NgH - gH$. The integral L_2 is regularised using the ϵ-regularization prescription of Salam and Strathdee [49] (and explained in Appendix 3.1):

$$L_2 = - \sum_{n=-\infty}^{\infty} \sum_{N=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \int_0^{\infty} dt \; t^{-1} \; e^{-t\left\{ \left(\frac{2\pi n}{\beta} + \mu \right)^2 + k_3^2 + A \right\}}.$$

After performing the k_3-integration and the sum over n, we have,

$$L_2 = -\frac{1}{2\sqrt{\pi}} \sum_{N=0}^{\infty} \int_0^{\infty} dt \; t^{-\frac{3}{2}} \; \theta_3\left(\frac{2\mu t i}{\beta}, \frac{4\pi i t}{\beta^2} \right) \; e^{-t(\mu^2 + A)},$$

where

$$\theta_3(z, \tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau n^2} \; e^{2\pi i n z},$$

is the Jacobi theta function. We use the property of the θ_3-function (see the Appendix 4.2 for the θ_3-function) [56],

$$\theta_3(z, i\tau) = \tau^{-\frac{1}{2}} \, e^{\left(\frac{-\pi z^2}{\tau} \right)} \; \theta_3\left(\frac{z}{i\tau}, \frac{i}{\tau} \right),$$

to rewrite the above expression for L_2 as

$$L_2 = -\frac{\beta}{4\pi} \sum_{N=0}^{\infty} \int_0^{\infty} dt \, t^{-2} \, \theta_3\left(\frac{\mu\beta}{2\pi}, \frac{i\beta^2}{4\pi t} \right) e^{-tA}.$$

Including the prefactor in (4.16), the contribution to $\Gamma^{one-loop}$ from L_2 is

$$\Gamma^{one-loop}(L_2) = -\frac{gH}{16\pi^2} \sum_{N=0}^{\infty} \int_0^{\infty} dt \; t^{-2} \; \theta_3\left(\frac{\mu\beta}{2\pi}, \frac{i\beta^2}{4\pi t} \right) e^{-tA}.$$

The unstable mode corresponds to $N = 0$. Splitting the above sum over N for $N = 0$ and $N = 1, 2, \cdots, \infty$, and carrying out the sum over N (1 to ∞), we find,

$$\begin{aligned} \Gamma^{one-loop}(L_2) &= -\frac{gH}{16\pi^2} \int_0^{\infty} dt \; t^{-2} \; \theta_3\left(\frac{\mu\beta}{2\pi}, \frac{i\beta^2}{4\pi t} \right) \\ &\quad \times \left(e^{tgH} + \frac{e^{-tgH}}{1 - e^{-2tgH}} \right). \end{aligned} \tag{4.18}$$

The first term in $(\cdots)$ of (4.18) corresponds to the unstable mode ($N = 0$) contribution and the second term is the result of the sum over N from 1 to ∞.

The third term in (4.16) has no negative eigenvalue and writing this as in (4.17), the sum over $N = 0, 1, 2, \cdots, \infty$ is performed to give

$$\begin{aligned}\Gamma^{one-loop}(L_3) &= -\frac{gH}{16\pi^2}\int_0^\infty dt\, t^{-2}\,\theta_3\left(\frac{\mu\beta}{2\pi},\frac{i\beta^2}{4\pi t}\right)\sum_{N=0}^{\infty} e^{-((2N+1)gH+2gH)} \\ &= -\frac{gH}{16\pi^2}\int_0^\infty dt\, t^{-2}\,\theta_3\left(\frac{\mu\beta}{2\pi},\frac{i\beta^2}{4\pi t}\right)\frac{e^{-3tgH}}{1-e^{-2tgH}}. \qquad (4.19)\end{aligned}$$

The fourth term in (4.16) is the same as the third term except for μ replaced by $-\mu$. So also the fifth term and the second term are same but μ is replaced by $-\mu$. We use the property of the Jacobi theta function, $\theta_3(z,\tau) = \theta_3(-z,\tau)$ [56]; therefore, we have,

$$\begin{aligned}\sum_{j=2}^{5}\Gamma^{one-loop}(L_j) &= -\frac{gH}{8\pi^2}\int_0^\infty dt\; t^{-2}\;\theta_3\left(\frac{\mu\beta}{2\pi},\frac{i\beta^2}{4\pi t}\right) \\ &\quad\times\left(e^{tgH} - e^{-tgH} + \frac{2\,e^{-tgH}}{1-e^{-2tgH}}\right), \qquad (4.20)\end{aligned}$$

Now we can rewrite $\theta_3\left(\frac{\mu\beta}{2\pi},\frac{i\beta^2}{4\pi t}\right)$ using the definition of θ_3,

$$\theta_3(z,\tau) = \sum_{\ell=-\infty}^{\infty} e^{i\pi\tau\ell^2}\, e^{2\pi i z\ell}.$$

Isolating the $\ell = 0$ part and carrying out the sum over ℓ from 1 to ∞, we have,

$$\theta_3\left(\frac{\mu\beta}{2\pi},\frac{i\beta^2}{4\pi t}\right) = 1 + 2\sum_{\ell=1}^{\infty} cos(\mu\beta\ell)\; e^{-\left(\frac{\beta^2\ell^2}{4t}\right)}.$$

The exponentials in (4.20) can be combined to give $\frac{\cosh(2tgH)}{\sinh(tgH)}$ i.e.,

$$e^{-tgH} - e^{tgH} + \frac{2\,e^{-tgH}}{1-e^{-2tgH}} = \frac{\cosh(2tgH)}{\sinh(tgH)}.$$

Therefore, (4.20) becomes

$$-\tfrac{gH}{8\pi^2}\int_0^\infty dt\; t^{-2}\;\tfrac{cosh(2tgH)}{sinh(tgH)}\left(1 + 2\sum_{\ell=1}^{\infty}\cos(\mu\beta\ell)\; e^{-\frac{\beta^2\ell^2}{4t}}\right). \qquad (4.21)$$

The first term in (4.16) is given by

$$\Gamma^{one-loop}(L_1) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^3\,k}{(2\pi)^3}\log\left[\frac{4\pi^2\,n^2}{\beta^2} + \vec{k}^2\right].$$

We use the spherical co-ordinate of $\vec{k}$ and perform the summation from $n = -\infty$ to ∞ with the use of the relation

$$\sum_{n=1}^{\infty} \frac{y}{y^2+n^2} = -\frac{1}{2y} + \frac{\pi}{2}\coth(\pi y).$$

This familiar term $\Gamma^{one-loop}(L_1)$ has a finite part which is found to be $-\pi^2/(45\beta^4)$. Thus the one-loop potential is

$$\begin{aligned}\Gamma^{one-loop} &= -\frac{\pi^2}{45\beta^4} \\ &\quad -\frac{(gH)^2}{8\pi^2}\int_0^{\infty} d\tau\, \tau^{-2}\frac{\cosh(2\tau)}{\sinh\tau}\left(1+2\sum_{\ell=1}^{\infty}\cos(\mu\beta\ell)e^{-\frac{\beta^2\ell^2 gH}{4\tau}}\right),\end{aligned} \tag{4.22}$$

where $\tau = gHt$. It is to be noticed in (4.22) that the first term of the second term is independent of $\beta = 1/kT$ and reproduces the temperature independent part (i.e. zero temperature, $T = 0$,) term. This term was evaluated in Chapter III and in [48] treating the unstable modes carefully including the cubic and showing that the quartic terms and the effective energy density have no imaginary part. It is given by,

$$\mathcal{E}_{one-loop}(T=0) = \frac{H^2}{2} + \frac{11(gH)^2}{48\pi^2}\left\{\left(\frac{gH}{\Lambda^2}\right) - \frac{1}{2}\right\}. \tag{4.23}$$

We consider only the temperature-dependent part of (4.22),

$$\begin{aligned}\Gamma^{one-loop}(T) &= -\frac{\pi^2}{45\beta^4} \\ &\quad -\frac{(gH)^2}{4\pi^2}\int_0^{\infty} d\tau\,\tau^{-2}\left(\frac{\cosh(2tgH)}{\sinh(tgH)}\right)\left(\sum_{\ell=1}^{\infty}\cos(\mu\beta\ell)e^{\left(-\frac{\beta^2\ell^2 gH}{4\tau}\right)}\right).\end{aligned}$$

Rewriting $\frac{\cosh(2tgH)}{\sinh(tgH)} = e^{-tgH} - e^{tgH} + \frac{2e^{-tgH}}{1-e^{-2tgH}}$, we have

$$\begin{aligned}\Gamma^{one-loop}(T) &= -\frac{\pi^2}{45\beta^4} - \frac{(gH)^2}{4\pi^2}\times \\ &\quad \int_0^{\infty} d\tau\,\tau^{-2}\left(e^{\tau} - e^{-\tau} + \frac{2e^{-\tau}}{1-2e^{-2\tau}}\right)\left(\sum_{\ell=1}^{\infty}\cos(\mu\beta\ell)e^{\left(-\frac{\beta^2\ell^2 gH}{4\tau}\right)}\right).\end{aligned} \tag{4.24}$$

The first term in $\int_0^{\infty}$ corresponds to the contribution from the unstable modes. Now, we expand

$$\frac{1}{1-e^{-2\tau}} = \sum_{n=0}^{\infty} e^{-2n\tau} = 1 + \sum_{n=1}^{\infty} e^{-2n\tau},$$

and we introduce the notation for the integrals as follows,

$$I_1 = \int_0^\infty d\tau\, \tau^{-2}\, e^{\tau}\, e^{-\frac{\beta^2\ell^2 gH}{4\tau}}, \tag{4.25}$$

$$I_2 = \int_0^\infty d\tau\, \tau^{-2}\, e^{-\tau}\, e^{-\frac{\beta^2\ell^2 gH}{4\tau}}, \tag{4.26}$$

$$I_3 = 2\sum_{n=1}^{\infty} \int_0^\infty d\tau\, \tau^{-2}\, e^{-(2n+1)\tau}\, e^{-\frac{\beta^2\ell^2 gH}{4\tau}}. \tag{4.27}$$

The finite temperature part of (4.24) is written as

$$\begin{aligned} \Gamma^{one-loop} &= -\frac{\pi^2}{45\beta^4} \\ &- \frac{(gH)^2}{4\pi^2}\sum_{\ell=1}^{\infty}\cos(\mu\beta\ell)\,\Big(I_1 + I_2 + I_3\Big), \end{aligned} \tag{4.28}$$

in which I_1 is from the $N = 0$ *unstable mode*.

In I_1 (4.25), we perform a Wick rotation to arrive at

$$\begin{aligned} I_1 &= i^{-1}\int_0^\infty dt\, t^{-2}\, e^{i\left(t+\frac{\beta^2\ell^2 gH}{4t}\right)}, \\ &= \frac{2\pi i}{\beta\ell\sqrt{gH}}\, H_1^{(1)}\big(\beta\ell\sqrt{gH}\big), \end{aligned} \tag{4.29}$$

where $H_1^{(1)}$ is the Hankel function of the first kind [57]. I_2 and I_3 integrals are evaluated using [57]

$$K_\nu(xz) = \frac{z^\nu}{2}\int_0^\infty t^{-\nu-1}\, e^{-\frac{x}{2}\left(t+\frac{z^2}{t}\right)}\, dt, \tag{4.30}$$

as

$$I_2 = \frac{4}{\beta\ell\sqrt{gH}}\, K_1\big(\beta\ell\sqrt{gH}\big), \tag{4.31}$$

$$I_3 = 8\sum_{n=1}^{\infty}\frac{\sqrt{(2n+1)}}{\beta\ell\sqrt{gH}}\, K_1\Big(\sqrt{(2n+1)}\beta\ell\sqrt{gH}\Big), \tag{4.32}$$

so that the finite part of (4.28) becomes,

$$\begin{aligned} \Gamma^{one-loop}(T) &= \frac{\pi^2}{45\beta^4} - \frac{(gH)^{\frac{3}{2}}}{\beta\pi^2}\sum_{\ell=0}^{\infty}\frac{\cos(\mu\beta\ell)}{\ell}\left\{\frac{i\pi}{2}H_1^{(1)}\left(\beta\ell\sqrt{gH}\right)\right. \\ &\left. +K_1\left(\beta\ell\sqrt{gH}\right) + 2\sum_{n=1}^{\infty}\sqrt{(2n+1)}K_1\left(\sqrt{2n+1}\beta\ell\sqrt{gH}\right)\right\}. \end{aligned} \tag{4.33}$$

Now, using $H_1^{(1)}(\beta\ell\sqrt{gH}) = J_1(\beta\ell\sqrt{gH}) + iY_1(\beta\ell\sqrt{gH})$, we find that our result (4.33) agrees with Meisinger and Ogilvie [31] who used a different method to evaluate (4.16). The interchange of J_1 and Y_1 and a relative sign between the two K_1 functions in Starinets, Vshivtsev, and Zhukovskii [30] are incorrect. From (4.28) and (4.33), it is seen that the unstable mode ($N = 0$) contributions are contained in I_1 and explicitly

$$\Gamma^{one-loop}_{unstable}(T) = -\frac{(gH)^{\frac{3}{2}}}{\beta\pi^2}\sum_{\ell=1}^{\infty}\frac{\cos(\mu\beta\ell)}{\ell}\times \left\{-\frac{\pi}{2}Y_1\left(\beta\ell\sqrt{gH}\right) + \frac{i\pi}{2}J_1\left(\beta\ell\sqrt{gH}\right)\right\}. \quad (4.34)$$

The imaginary part above is reminiscent of the zero temperature situation in Chapter III and in Ref. [48]. In Ref. [48] we have treated the unstable modes by including the cubic and quartic terms in (3.6) and showed that the contribution is real. In view of the important difficulties arising from the imaginary part at *finite temperature*, we treat the unstable modes at finite temperature including the cubic and the quartic terms in the expansion (4.6).

4.3 Finite Temperature Unstable Modes—Inclusion of the Cubic and Quartic Terms

There are two unstable modes for the harmonic oscillator quantum number $N = 0$, as seen from (4.15). They are the first and the fourth eigenvalues in (4.15) with $N = 0$. In order to consider unstable modes in the full expansion i.e., including the cubic and quartic terms, we consider directly (4.5) for Z but confine ourselves only to the unstable modes. These two unstable modes are

$$\left.\begin{array}{rcl} a_+^+ & \equiv & (a_1^1 + i\,a_1^2) + i\,(a_2^1 + i\,a_2^2) \\ a_-^- & \equiv & (a_1^1 - i\,a_1^2) - i\,(a_2^1 - i\,a_2^2) \end{array}\right\} \quad (4.35)$$

with eigenvalues $(k_4+\mu)^2 + k_3^2 - gH$ and$(k_4-\mu)^2 + k_3^2 - gH$ respectively.

The normalised unstable mode eigenfunctions are

$$\phi^{\pm}_{k_3,n}(x) = \sqrt{\frac{gH}{2\pi}}e^{-\frac{gH}{4}(x_1^2+x_2^2)}\frac{1}{\sqrt{L_3\beta}}e^{-i\left(k_3x_3+\left(\frac{2\pi n}{\beta}\pm\mu\right)x_4\right)}, \quad (4.36)$$

where the '+' sign and the '−' sign over $\phi^{\pm}_{k_3,n}(x)$ are corresponding to a_+^+ and a_-^- respectively and hereafter it will be followed. The first exponential is the ground state wave function ($N = 0$) of the harmonic oscillator in the (x_1-x_2) plane, the second exponential is the box-normalised plane wave in the x_3 direction and the

S^1-harmonics in the x_4-direction. The index n is the Matsubara index and is not the same n in (4.33) as the latter index originated in the expansion of $\frac{1}{1-e^{-2\tau}}$ in the expression above (4.27). The unstable eigenmodes in (4.15) are given by

$$\left.\begin{array}{rcl} a_+^+ &=& c^+(k_3,n)\phi^+_{k_3,n}(x) \\ a_-^- &=& c^-(k_3,n)\phi^-_{k_3,n}(x) \end{array}\right\} \tag{4.37}$$

where the summation over k_3,n is understood in the sense that (4.35) forms a complete orthonormal set. Since $(a_+^+)^\dagger = a_-^-$, so $(c^+\phi^+)^\dagger = c^-\phi^-$.

The unstable modes involve Lorentz indices 1 and 2 (as can be seen in (4.15) and (4.35)) and the $SU(2)$ color indices 1 and 2, since the classical background in (4.11) is in the third color direction. Therefore, we restrict Lorentz indices to 1 and 2 and color indices to 1 and 2, when the cubic term $\epsilon^{acd}(\bar{D}^{ae}_\nu a^e_\mu)\, a^c_\mu a^d_\nu$ vanishes as in Chapter III. The quartic terms in a^a_μ in (4.5) is found to be $\frac{1}{8}\ a_+^+\ a_-^-\ a_+^+\ a_-^-$ as in Chapter III. The full partition function for the unstable modes, from (4.5) and (4.6), is

$$Z_{unstable} = \int [da^a_\mu]\ e^{-\int d^4x \{a_u(k_3^2+(k_4\pm\mu)^2-gH)a_u + \frac{g^2}{32}a_u^4\}}, \tag{4.38}$$

where a_u stands for a_+^+ and a_-^-. From (4.38), it can be seen, replacing k_4 by $\frac{2\pi n}{\beta}$ (Matsubara frequency), that the unstable modes for $k_3^2 + (\frac{2\pi n}{\beta} \pm \mu)^2 < gH$ render the quadratic term in a_u in the exponent of (4.38) divergent. *However, the quartic term in a_u in the exponent of (4.38) provides the necessary and the crucial convergence.* Thus the overall integral over a_u will be convergent. Now we expand a_u in terms of the eigenfunctions (4.36) and carry out the d^4x integration in the exponent. The quadratic term becomes

$$\{k_3^2 + (\tfrac{2\pi n}{\beta} \pm \mu)^2 - gH\}c^2(k_3,n), \tag{4.39}$$

and the quartic term becomes

$$\frac{g^2}{32}\ \frac{(gH)^2}{4\pi^2}\ \frac{\pi}{gH}c^4(k_3,n) = \frac{g^3H}{128\pi}c^4(k_3,n), \tag{4.40}$$

where we have taken all the four $\phi_{k_3,n}(x)$ having the same k_3 , n and $|c^+|^2 = |c^-|^2 = |c|^2$. Then, we obtain

$$Z_{unstable} = \left(\int \prod_{k_3,n} dc(k_3,n)\ e^{\left\{\left(gH-k_3^2-(\frac{2\pi n}{\beta}\pm\mu)^2\right)c^2 - \frac{g^2(gH)}{128\pi}c^4\right\}}\right)^D, \tag{4.41}$$

where D is the degeneracy factor, $D = \frac{gH}{2\pi^2}V$, with V as the spatial volume. It is to be observed in (4.41) that the index n (from from the Matsubara replacement of k_4 by $2\pi n/\beta$) takes values from $-\infty$ to ∞. Now, introducing $\hat{c} = \sqrt{gH}c$ and $\hat{k}_3 = k_3/\sqrt{gH}$, (4.41) is written as

$$Z_{unstable} = \left(\frac{1}{\sqrt{gH}} \int \prod_{k_3, n} d\hat{c}\, e^{\left\{ \left(1 - \hat{k}_3^2 - \left(\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}} \right)^2 \right) \hat{c}^2 - \frac{g^2}{128\pi(gH)} \hat{c}^4 \right\}} \right)^D . \quad (4.42)$$

The contribution of the complete unstable modes to the energy density, $((-1/\beta V) \log Z_{unstable})$, is then

$$C_u = -\frac{gH\sqrt{gH}}{2\pi^2\beta} \int d\hat{k}_3 \sum_n \log \left(\frac{J(\hat{k}_3, n)}{\sqrt{gH}} \right), \quad (4.43)$$

where

$$J(\hat{k}_3, n) = \int_{-\infty}^{\infty} d\hat{c}\, e^{\left\{ \left(1 - \hat{k}_3^2 - \left(\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}} \right)^2 \right) \hat{c}^2 - \frac{g^2}{128\pi(gH)} \hat{c}^4 \right\}} . \quad (4.44)$$

In (4.44), if the quartic term is neglected, then (4.44) will give (4.34). We retain the crucial quartic term contribution. The integral over $d\hat{c}$ is convergent *irrespective* of the sign of the coefficient of the $\hat{c}^2$ term. It is evaluated using [57],

$$\int_0^\infty e^{-(\beta^2 x^4 + 2\gamma^2 x^2)} dx = 2^{-\frac{3}{2}} \left(\frac{\gamma}{\beta} \right) e^{\left(\frac{\gamma^4}{2\beta^2} \right)} K_{\frac{1}{4}} \left(\frac{\gamma^4}{2\beta^2} \right),$$

with

$$\begin{aligned} \beta^2 &= \frac{g^2}{128\pi(gH)}, \\ \gamma^2 &= \frac{1}{2} \left(\hat{k}_3^2 + \left(\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}} \right)^2 - 1 \right), \end{aligned}$$

Then (4.44) becomes,

$$\begin{aligned} J(\hat{k}_3, n) &= \frac{8\sqrt{\pi gH}}{\sqrt{2}g} \left(\hat{k}_3^2 + \left(\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}} \right)^2 - 1 \right)^{\frac{1}{2}} e^{\frac{16\pi(gH)}{g^2} \left\{ \hat{k}_3^2 + \left(\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}} \right)^2 - 1 \right\}^2} \\ &\times K_{\frac{1}{4}} \left(\frac{16\pi(gH)}{g^2} \left\{ \hat{k}_3^2 + \left(\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}} \right)^2 - 1 \right\}^2 \right). \end{aligned} \quad (4.45)$$

When $\hat{k}_3^2 + (\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}})^2 < 1$ (that is where the instability arises: the left hand side never becomes zero at finite temperature since n starts from 1 and so the argument of $K_{\frac{1}{4}}$ will be small), the above expression is approximated using [58]

$$K_\nu(x) \to \frac{2^{\nu-1}\Gamma(\nu)}{x^\nu}; \quad \text{small } x,$$

as

$$
\begin{aligned}
J(\hat{k}_3, n) &\simeq \frac{8\sqrt{\pi g H}}{\sqrt{2g}} \left(\hat{k}_3^2 + \left(\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}} \right)^2 - 1 \right)^{\frac{1}{2}} \\
&\quad \times \frac{2^{-\frac{3}{4}}\Gamma(\frac{1}{4})}{\left(\hat{k}_3^2 + \left(\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}} \right)^2 - 1 \right)^{\frac{1}{2}}} \frac{\sqrt{g}}{2(\pi g H)^{\frac{1}{4}}}, \\
&= \frac{2\sqrt{2}}{\sqrt{g}} (\pi g H)^{\frac{1}{4}} 2^{-\frac{3}{4}} \Gamma\left(\frac{1}{4}\right). \qquad (4.46)
\end{aligned}
$$

In particular the radical $\{\hat{k}_3^2 + \left(\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}}\right)^2 - 1\}^{\frac{1}{2}}$ *gets cancelled.* When $\hat{k}_3^2 + \left(\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}}\right)^2 < 1$, the exponential in (4.45) is approximated to unity. *This result is real. The imaginary part coming from the radical gets cancelled by the contribution from $K_{\frac{1}{4}}$. This is made possible by the inclusion of the quartic term.* When (4.46) is used in (4.43), we have

$$
C_u = -\log \left\{ \frac{2\sqrt{2}}{g} \frac{(\pi g H)^{\frac{1}{4}}}{\sqrt{gH}} 2^{-\frac{3}{4}} \Gamma\left(\frac{1}{4}\right) \right\} \frac{gH\sqrt{gH}}{2\pi^2\beta} \int d\hat{k}_3 \sum_n \qquad (4.47)
$$

The $\hat{k}_3$-integration and the sum over n are constrained by $\hat{k}_3^2 + (\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}})^2 < 1$, and so $n < \frac{\beta\sqrt{gH}}{2\pi}(1 - \hat{k}_3^2)^{\frac{1}{2}} \mp \frac{\mu\beta}{2\pi}$, with $n_{max} = \frac{\beta\sqrt{gH}}{2\pi}(1 - \hat{k}_3^2)^{\frac{1}{2}} \mp \frac{\mu\beta}{2\pi}$. So,

$$
\int d\hat{k}_3 \sum_{n=1}^{n_{max}} = 2\int_0^1 d\hat{k}_3\, n_{max} = \beta\left(\frac{\sqrt{gH}}{4} \mp \frac{\mu}{2\pi}\right).
$$

Then, the above expression (4.47) becomes independent of β. For studying finite temperature effects, this term does not contribute. Then, the expression (4.47) becomes,

$$
\begin{aligned}
C_u &= -\log \left\{ \frac{2\sqrt{2}}{g} (\pi g H)^{\frac{1}{4}} 2^{-\frac{3}{4}} \Gamma\left(\frac{1}{4}\right) \right\} \frac{gH\sqrt{gH}}{2\pi^2\beta} \times \beta \left(\frac{\sqrt{gH}}{4} \mp \frac{\mu}{2\pi} \right) \\
&= -\frac{gH\sqrt{gH}}{2\pi^2} \log \left\{ \frac{2\sqrt{2}}{g} (\pi g H)^{\frac{1}{4}} 2^{-\frac{3}{4}} \Gamma\left(\frac{1}{4}\right) \right\} \left(\frac{\sqrt{gH}}{4} \mp \frac{\mu}{2\pi} \right) \qquad (4.48)
\end{aligned}
$$

which is independent of β.

When $\hat{k}_3^2 + (\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}})^2 > 1$, the integration for $\hat{k}_3$ will be from 1 to ∞ [If however the positive value of this is close to unity, then the argument of the $K_{\frac{1}{4}}$ function will be small and the small x approximation can be used to get expressions similar to (4.48) which will produce uninteresting (β independent) terms], the argument of $K_{\frac{1}{4}}$ function becomes large and the expression (4.45) is approximated, using [58],

$$K_\nu(x) \;\rightarrow\; \sqrt{\frac{\pi}{2x}}\, e^{-x}; \quad \textit{large } x,$$

as

$$J(\hat{k}_3, n) \;=\; \sqrt{\pi}\left(\hat{k}_3^2 + \left(\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}}\right)^2 - 1\right)^{-\frac{1}{2}}. \tag{4.49}$$

Then (4.43) is evaluated using

$$\int_1^\infty d\hat{k}_3 \sum_n \log J(\hat{k}_3, n) \;=\; -\frac{1}{2}\int_1^\infty d\hat{k}_3 \sum_n \log\left\{\hat{k}_3^2 + \left(\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}}\right)^2 - 1\right\}, \tag{4.50}$$

after omitting $-\int_1^\infty \sum_n d\hat{k}_3 \log\sqrt{gH}$ and $\int_1^\infty d\hat{k}_3 \sum_n \log(\sqrt{\pi})$ as these give an infinite contribution. Now, we consider the R.H.S. of (4.50) without $\hat{k}_3$ integration, to perform the summation over n:

$$\begin{aligned}
S \;&=\; \sum_{n=-\infty}^{\infty} \log\left\{\hat{k}_3^2 - 1 + \left(\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}}\right)^2\right\},\\
&=\; \sum_{n=-\infty}^{\infty} \log\left[\left(\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}}\right)^2 \left\{1 + \left(\frac{\beta\sqrt{gH(\hat{k}_3^2-1)}}{2\pi n \pm \mu\beta}\right)^2\right\}\right],\\
&=\; 2\sum_{n=-\infty}^{\infty} \log\left(\frac{2\pi n}{\beta\sqrt{gH}} \pm \frac{\mu}{\sqrt{gH}}\right) + \sum_{n=-\infty}^{\infty} \log\left\{1 + \left(\frac{\beta\sqrt{gH(\hat{k}_3^2-1)}}{2\pi n \pm \mu\beta}\right)^2\right\}.
\end{aligned}$$

We neglect the first term, as it will give an uninteresting infinite term when the $\hat{k}_3$-integration is performed. So

$$\begin{aligned}
S \;&=\; \log \prod_{n=-\infty}^{\infty} \left\{1 + \left(\frac{\beta\sqrt{gH(\hat{k}_3^2-1)}}{2\pi n \pm \mu\beta}\right)^2\right\},\\
&=\; \log\left[\frac{\cosh\left(\beta\sqrt{gH(\hat{k}_3^2-1)}\right) - \cos(\mu\beta)}{1-\cos(\mu\beta)}\right],\\
&=\; \log\left\{\cosh\left(\beta\sqrt{gH(\hat{k}_3^2-1)}\right) - \cos(\mu\beta)\right\} - \log\left(1-\cos(\mu\beta)\right).
\end{aligned}$$

We omit the last term, which will not give a finite contribution on performing the $\hat{k}_3$ integration. So

$$
\begin{aligned}
S &= \log\left\{\cosh\left(\beta\sqrt{gH(\hat{k}_3^2-1)}\right)-\cos(\mu\beta)\right\},\\
&= \log\left\{e^{\beta\sqrt{gH(\hat{k}_3^2-1)}}+e^{-\beta\sqrt{gH(\hat{k}_3^2-1)}}-2\cos(\mu\beta)\right\}-\log(2),\\
&= \log\left\{e^{\beta\sqrt{gH(\hat{k}_3^2-1)}}\left(1+e^{-2\beta\sqrt{gH(\hat{k}_3^2-1)}}-2\cos(\mu\beta)e^{-\beta\sqrt{gH(\hat{k}_3^2-1)}}\right)\right\},\\
&= \beta\sqrt{gH(\hat{k}_3^2-1)}+\log\left(1+e^{-2\beta\sqrt{gH(\hat{k}_3^2-1)}}-2\cos(\mu\beta)e^{-\beta\sqrt{gH(\hat{k}_3^2-1)}}\right),\\
&= \beta\sqrt{gH(\hat{k}_3^2-1)}+\log\left(1-e^{\left(-2\beta\sqrt{gH(\hat{k}_3^2-1)}+i\mu\beta\right)}\right)+\log\left(1-e^{\left(-2\beta\sqrt{gH(\hat{k}_3^2-1)}-i\mu\beta\right)}\right),
\end{aligned}
$$

where we have omitted terms which does not give a finite result. We use the following expansion,

$$
\log(1-x) \;=\; \sum_{m=1}^{\infty}\frac{x^m}{m},
$$

to write

$$
\begin{aligned}
S &= \beta\sqrt{gH(\hat{k}_3^2-1)}+\sum_{m=1}^{\infty}\frac{e^{\left(-m\beta\sqrt{gH(\hat{k}_3^2-1)}+im\mu\beta\right)}}{m}+\sum_{m=1}^{\infty}\frac{e^{\left(-m\beta\sqrt{gH(\hat{k}_3^2-1)}-im\mu\beta\right)}}{m},\\
&= \beta\sqrt{gH(\hat{k}_3^2-1)}+\sum_{m=1}^{\infty}\frac{e^{-m\beta\sqrt{gH(\hat{k}_3^2-1)}}}{m}\left(e^{im\mu\beta}+e^{-im\mu\beta}\right),\\
&= \beta\sqrt{gH(\hat{k}_3^2-1)}+2\sum_{m=1}^{\infty}\frac{\cos(m\mu\beta)}{m}\,e^{-m\beta\sqrt{gH(\hat{k}_3^2-1)}}.
\end{aligned}
$$

Using this S in (4.43), we find

$$
C_u=\frac{(gH)^{\frac{3}{2}}}{2\pi^2\beta}\left[\beta\int_1^{\infty}d\,\hat{k}_3\sqrt{gH(\hat{k}_3^2-1)}+2\sum_{m=1}^{\infty}\int_1^{\infty}d\,\hat{k}_3\frac{\cos(m\mu\beta)}{m}\,e^{-m\beta\sqrt{gH(\hat{k}_3^2-1)}}\right].
$$

From the Table of Integrals, [57],

$$
\int_1^{\infty}d\,y\,e^{-m\beta\sqrt{(y^2-1)gH}} \;=\; -\frac{\pi}{2}Y_1(m\beta\sqrt{gH})-\int_0^{\frac{\pi}{2}}\cos(m\beta\sqrt{gH}\cos\theta)\cos\theta\,d\,\theta,
$$

so we have,

$$
\begin{aligned}
C_u &= \frac{(gH)^{\frac{3}{2}}}{2\pi^2}\int_1^{\infty}d\,\hat{k}_3\sqrt{gH(\hat{k}_3^2-1)}\\
&\quad+\frac{(gH)^{\frac{3}{2}}}{\beta\pi^2}\sum_{\ell=1}^{\infty}\frac{\cos(\mu\beta\ell)}{\ell}\left(-\frac{\pi}{2}Y_1\left(\beta\ell\sqrt{gH}\right)\right)+I_1, \qquad (4.51)
\end{aligned}
$$

where

$$I_1 = -\frac{(gH)^{\frac{3}{2}}}{\beta\pi^2}\sum_{\ell=1}^{\infty}\int_0^{\frac{\pi}{2}}\cos\left(\ell\beta\sqrt{gH}\cos\theta\right)\cos\theta\, d\theta. \tag{4.52}$$

The first term of (4.51) is β independent and can be neglected while discussing finite temperature effects. The integral I_1 in (4.52) is evaluated [57] to be

$$I_1 = -\frac{gH\sqrt{gH}}{\beta\pi^2}\sum_{\ell=1}^{\infty}\left(-\left\{-1+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(n\beta\sqrt{gH})^{2k}}{1\cdot(2^2-1)\cdot(4^2-1)\cdots((2k)^2-1)}\right\}\right),$$

which does not contribute to the finite part of the energy density. Thus, the unstable mode contributions to the finite β-dependent part of the energy density is found to be

$$\frac{(gH)^{\frac{3}{2}}}{\beta\pi^2}\sum_{\ell=1}^{\infty}\frac{\cos(\mu\beta\ell)}{l}\left(-\frac{\pi}{2}Y_1\left(\ell\beta\sqrt{gH}\right)\right), \tag{4.53}$$

which is just the real part of (4.34). There is no imaginary part. This is due to the inclusion of the cubic and the quartic terms in the unstable modes. Thus, the difficulties associated with the imaginary part are not due to the intrinsic property of the $SU(2)$ chromomagnetic ground state but due to the use of the Gaussian approximation. This reaffirms our earlier study of Chapter III and Ref. [48] of the same system at zero temperature.

The complete expression for the energy density of the $SU(2)$ chromomagnetic state, including the zero contribution from Ref. [48] is,

$$\begin{aligned}\mathcal{E} \;=\; & \frac{H^2}{2}+\frac{11(gH)^2}{48\pi^2}\left\{\log\left(\frac{gH}{\Lambda^2}\right)-\frac{1}{2}\right\} \\ & +\frac{\pi^2}{45\beta^4} \\ & +\frac{(gH)^{\frac{3}{2}}}{\beta\pi^2}\sum_{\ell=1}^{\infty}\frac{\cos(\mu\beta\ell)}{\ell}\left[-\frac{\pi}{2}Y_1\left(\beta\ell\sqrt{gH}\right)\right. \\ & \left.+K_1\left(\beta\ell\sqrt{gH}\right)+2\sum_{n=1}^{\infty}\sqrt{2n+1}\,K_1\left(\sqrt{2n+1}\beta\ell\sqrt{gH}\right)\right].\end{aligned} \tag{4.54}$$

The finite temperature part agrees with the real part of [31].

4.4 Numerical Results and Discussion

The expression for the effective energy density (4.54) involves summation over ℓ and in most studies, the high and low temperature behaviours have been examined. First, it can be seen from Ref. [59] that the K_1 functions fall-off to zero when the argument is greater than 5. On the other hand the Y_1 function is oscillatory with decreasing amplitude. Second, we use the "polynomial approximations" for these functions as given in Ref. [59] and verify that they are good by computing these functions for various values of x from 0.05 to large values and comparing them with the tables of these functions. Then, MATLAB was used to find the values of $Y_1(x)$ and $K_1(x)$ for various values of x; these agree with the previous method. Third, we need to evaluate the sums in (4.54).

For the Y_1 function appearing in (4.54), we found the typical sum $\sum_{\ell=1}^{\infty} \frac{Y_1(\ell x)}{\ell}$ converged to a steady value for ℓx up to 200 whereas for the sum $\sum_{\ell=1}^{\infty} \frac{K_1(\ell x)}{\ell}$, a steady value is reached when $\ell x = 20$. These allow us to choose $\ell_{max} = 200/x$ for the Y_1 sum while $\ell_{max} = 20/x$ for the sum involving K_1. Keeping ℓ_{max} as $20/x$, in the last K_1 sum, the n sum was carried out from $n = 1$ to n_{max} with $n_{max} = \frac{1}{2}\{\frac{\ell_{max}^2}{\ell^2} - 1\}$. In order to evaluate the temperature variation of (4.54), we first set $\beta = \frac{a}{\sqrt{gH}}$ and $\mu = b\sqrt{gH}$. Then the temperature dependent part of (4.54) becomes

$$\begin{aligned}\frac{\mathcal{E}_T}{(gH)^2} &= \frac{\pi^2}{45a^4} + \frac{1}{\pi^2 a}\sum_{\ell=1}^{\infty} \frac{\cos(ab\ell)}{\ell}\Big(-\frac{\pi}{2}Y_1(a\ell) \\ &\quad + K_1(a\ell) + 2\sum_{n=1}^{\infty} \sqrt{2n+1}K_1(a\ell\sqrt{2n+1})\Big). \qquad (4.55)\end{aligned}$$

In Figure 4.1, we have plotted $\frac{\mathcal{E}_T}{(gH)^2}$ with $T = \frac{\sqrt{gH}}{k}\frac{1}{a}$, that is, T in units of $\frac{\sqrt{gH}}{k}$, for $b = 0, 1, 2, 3$. For $b = 0$, zero chemical potential, the variation is smooth apart from small oscillatory behaviour at low temperatures. For $b = 2, 3$, the variation shows a minimum and then rises smoothly. At high temperatures, the behaviour is like that of a non-interacting relativistic gas. In [31], the Polyakov loop is measured in terms of ϕ which in our notation is $\mu\beta$ and that is ab. In the sense that a is a variable, it is not possible to relate directly our results to [31]. However, the importance of the chemical potential is seen in Figure 4.1. A non-zero chemical potential or non-zero ϕ triggers a possible deconfinement phase transition. Our variation is qualitatively in agreement with [31] for their "real part".

Now, we wish to examine the inclusion of the cubic and quartic terms for all the modes. It can be seen from (4.15) that the stable eigenvalues are distinctly different from the unstable eigenvalues for a given N. So it is justifiable to consider the corresponding eigenmodes as orthogonal. Then, from (4.15), it follows that the cubic terms will vanish for the stable modes as well. The resulting full expression can be evaluated as in (4.45) with explicit N appearing. When the logarithm is taken, as in (4.43), the finite part will remain unaltered. The situation for the unstable modes is different in the sense the troublesome imaginary part does not appear in (4.46). At high temperatures, the behaviour of $\mathcal{E}_T$ vs T is like that of a

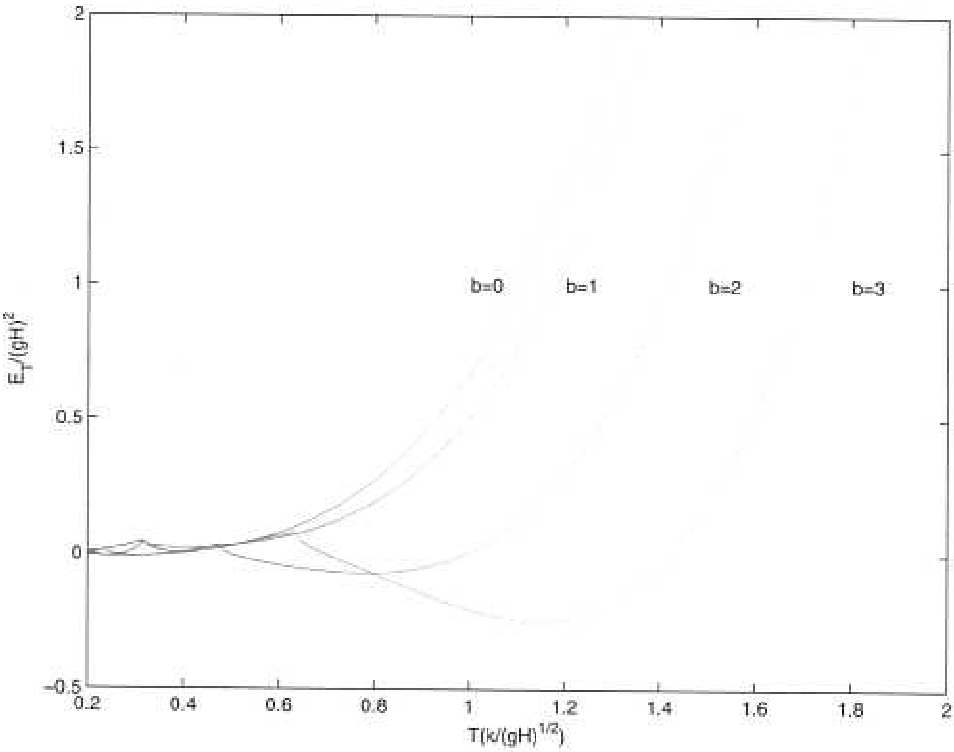

Figure 4.1: Variation of scaled energy density with scaled temperature.

relativistic gas. Recently [82], there are speculations that the behaviour at high T could be similar to that of a fluid. These studies use hydrodynamical equations to include the correlations among hadrons. Since correlations require some energy, the behaviour of $\mathcal{E}_T$ at large T will show a downward trend. In this thesis, we have considered gluons and quarks. Hadronization has not been studied. In future, we will try to include correlation among quarks and gluons, besides the condensates.

4.5 Conclusion

We have considered the one-loop effective energy density of a pure $SU(2)$ Yang-Mills theory in the Savvidy background at finite temperature and chemical potential. The unstable modes are treated by keeping the cubic and the quartic terms in the fluctuations. This result is added to the contribution from the stable modes. There is no imaginary part. The variation of the energy density for a given chromomagnetic background with temperature is studied numerically. When the chemical potential is non-zero, the variation shows a minimum which is (roughly) interpreted as indicating a deconfinement phase transition. At high temperatures, the behaviour is like that of a relativistic gas.

Chapter 5

Spontaneous Breaking of Chiral Symmetry (SBCS) using Gribov's Method*

5.1 Introduction

Spontaneous breaking of chiral symmetry and confinement are the two generic non-perturbative features of QCD in the infrared region and they are supposed to be related in the sense that both have the same origin in the non-Abelian nature of the theory [60]. A study of these features requires non-perturbative tools. The Schwinger-Dyson equations (DSEs) are one of these non-perturbative tools. It is an "infinite tower" of coupled integral equations. They are difficult to solve exactly. Chiral symmetry is the invariance of the Lagrangian with respect to independent rotation of the left-handed, $(\frac{1-\gamma_5}{2})\psi$ and the right-handed, $(\frac{1+\gamma_5}{2})\psi$ spinors in the chiral limit (i.e., when the bare mass of the fermions set to zero in the Lagrangian). Whenever a mass term $(m\bar{\psi}\psi)$ is added to the model an interaction between the left-handed and the right-handed spinors is introduced, giving the fermion a mass, and chiral symmetry is broken explicitly. But in the chiral limit ($m_{bare} = 0$), it may happen that strong interactions between massless fermions give rise to the formation of fermion-anti-fermion bound states and generation of a fermion mass and we say the global symmetry is broken spontaneously. Spontaneous breaking of global chiral symmetry is a kind of phase transition with order parameter $\langle 0|\bar{\psi}\psi|0\rangle \neq 0$, which is called the chiral condensate.

*This chapter is based on "Chiral Symmetry Breaking in Gribov's Approach to QCD at Low Momentum," Alok Kumar, arXiv:0711.3970v1 [hep-th].

5.2 The Schwinger-Dyson Equation

One of the relevant Schwinger-Dyson equations for the study of spontaneous breaking of chiral symmetry is the 'Gap equation'. This is called the gap equation for quarks because for values of the coupling constants above some critical value, this equation has nonzero solution for the fermion mass even in the absence of an explicit external mass [61]. The Schwinger-Dyson equation provides a non-perturbative analysis of the quark propagator and is given by,

$$S^{-1}(p) = Z_2\,(i\gamma\cdot p + m_b) + Z_1 \int \frac{d^4q}{(2\pi)^4}\, g^2\gamma^\mu \frac{\lambda^a}{2}\, S(q)\, \Gamma^{\nu b}\, D^{ab}_{\mu\nu}(p-q), \qquad (5.1)$$

where $S(p)$ is the full quark propagator, $D^{ab}_{\mu\nu}$ is the dressed gluon propagator, $\Gamma^{\nu b}$ is the dressed quark-gluon vertex, m_b is the bare mass of the quark (which would be put to zero for avoidance of explicit breaking of chiral symmetry), Z_1 is the quark-gluon vertex renormalization and Z_2 is the quark wave function renormalization. A complete solution of (5.1) is difficult to realize fully. We invoke some approximations and assumptions following Gribov [36] :

1. QCD coupling g^2 in the infrared region, will be taken to be constant following Gribov [36]. This kind of 'freezing' of g^2 has been pointed out earlier [62,63] and discussed by Aguilar Mihara and Natale [64];
2. We will take $Z_1 = Z_2 = 1$ and $m_b = 0$;
3. We will invoke the rainbow approximation (Ladder approximation). The Ladder or planar approximation is a replacement of the full gluon-quark vertex Γ^μ by the bare vertex γ^μ (gluon lines don't cross) so that $\Gamma^{\nu b} = \gamma^\nu \frac{\lambda^b}{2}$;
4. We will take $D^{ab}_{\mu\nu}(p-q) = \delta^{ab}\, D_{\mu\nu}(p-q)$ for the gluon propagator.

Then, (5.1) becomes,

$$S^{-1}(p) \;=\; i\gamma\cdot p + g^2 \int \frac{d^4\, q}{(2\pi)^4}\, \gamma^\mu\, S(p-q)\, \gamma^\nu\, D_{\mu\nu}(p-q). \qquad (5.2)$$

The standard procedure is to assume a solution of (5.2) in terms of functions $A(p^2)$ and $M(p^2)$ so as to have

$$S(p) \;=\; \frac{1}{i\gamma.p\, A(p^2) + M(p^2)} \equiv i\gamma.p\, C(p^2) + B(p^2), \qquad (5.3)$$

where

$$C(p^2) \;=\; -\frac{A(p^2)}{p^2\, A^2(p^2) + M^2(p^2)},$$
$$B(p^2) \;=\; \frac{M(p^2)}{p^2\, A^2(p^2) + M^2(p^2)}.$$

Equation (5.3) is used for solution of (5.2) and this leads to two coupled equations involving unknowns $B(p^2)$ and $C(p^2)$ upon taking traces on both sides of (5.2), as

$$\frac{B(p^2)}{p^2\,C^2(p^2) + B^2(p^2)} = g^2 \int \frac{d^4 q}{(2\pi)^2}\, D^{\mu}_{\mu}\, B((p-q)^2), \tag{5.4}$$

$$-\frac{p^2\,C(p^2)}{p^2\,C^2(p^2) + B^2(p^2)} = p^2 + g^2 \int \frac{d^4 q}{(2\pi)^2}\, C((p-q)^2) \times \{2\,p^{\mu}\cdot(p-q)^{\nu} - p.(p-q)\,\delta^{\mu\nu}\}\; D_{\mu\nu}(q). \tag{5.5}$$

In Ref. [65], it is demonstrated that the gluon propagator has '$\frac{1}{q^4}$' behaviour in the infrared region which is faster than the Coulombic-like '$\frac{1}{q^2}$'. This demonstration is based on the dual-Meissner effect and it is also supported by phenomenological quark-antiquark confining linear potentials. Therefore, in the Feynman gauge,

$$D^{ab}_{\mu\nu} = -\frac{m^2}{q^4}\,\delta_{\mu\nu}\,\delta^{ab}, \tag{5.6}$$

where m is a parameter of mass dimension. We consider $B(k^2)$ as an analytic function of k^2 and use the integral representation of $B(k^2)$ as,

$$B(k^2) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\alpha\, \frac{\bar{B}(\alpha)}{k^2 - \alpha}. \tag{5.7}$$

Such a representation is justifiable on the ground that quark fields are not observable, the observability criteria being taken as the vanishing of the commutator of the BRST charge Q_B and the corresponding field [66], the physical subspace being annihilated by Q_B. As the commutator of Q_B with quark fields is not vanishing, quarks are not observable. This ensures that the quark fields are not asymptotic states. Such a representation is used in Ref. [67], a theory of QCD with string tension. We use (5.6), (5.7) in order to simplify (5.4) and (5.5). With the help of the Feynman parametrisation technique to solve the integration, it has been found [68],

$$\frac{4\,p^2}{9} + M^2(p^2) = \frac{m^2\,g^2}{2\pi^2}, \tag{5.8}$$

which gives

$$M(0) = \lim_{p\to 0} M(p^2) = \frac{mg}{\sqrt{2}\pi}. \tag{5.9}$$

This nonzero value of the dynamical mass function in the zero momentum limit shows that chiral symmetry is broken spontaneously in the confining region of QCD described by the model of Ref. [65]. The results consist of two parameters,

m and the QCD coupling constant, g in the infrared region. Denoting $\frac{m^2 g^2}{2\pi^2}$ by D, we have found that [68],

$$\langle \bar{q}q \rangle \;=\; -\frac{81 D\sqrt{D}}{80\pi^2}.$$

Using the lattice QCD estimate [69] of $\langle 0|\bar{\psi}\psi|0\rangle$ as $(-250\ \text{MeV})^3$ for light quarks, $\sqrt{D}$ is determined to be 534 MeV. This gives $M(0)$ (5.9) as 0.534 GeV which compares well with the lattice estimate of ≈ 0.6 GeV [70]. The determination of D using light quark condensate fixes $\frac{m^2 g^2}{2\pi^2}$ as 0.28 GeV^2. We need to know g^2 for the determination of m. We use the expression of Alkofer and Fischer [71] for $\alpha(p^2)$ which is $\frac{g^2}{4\pi}$ after normalising the parameters of $\alpha(M_Z^2)$. So m depends upon p this its numerical values are tabulated in Ref. [68]. We find for $g^2 \approx 8$, the dual gluon mass is ≈ 828 MeV. This value agrees well with the estimate of Baker et al. [72] who used the above value of g^2.

As a summary, the '$\frac{1}{q^4}$' behaviour of the infrared gluon propagator obtained from the dual Meissner effect description of QCD in the confining region as in Ref. [66] is used in the Schwinger-Dyson equation for the quark propagator in the rainbow approximation in the Feynman gauge. Chiral symmetry breaking is seen from $M(0) \neq 0$. Using the light quark condensate value, the numerical value of the combination $m^2 g^2$ is determined as $\frac{m^2 g^2}{2\pi^2} = 0.28$ GeV^2. The expression of Alkofer and Fischer [71] is used to evaluate g^2 and this depends upon the momentum p. The corresponding m value for $g^2 = 8$ is found to be 828 MeV as a prediction which agrees with [72]. These results were obtained in the rainbow approximation, in which the full quark-gluon vertex had been replaced by the bare vertex. *We now consider an approach by Gribov, in which the full quark-gluon vertex is used, along with the Ward identity.*

5.3 Gribov's Concept of Chiral Symmetry Breaking

In order to address the issues of confinement and chiral symmetry, Gribov developed a different picture of infrared region of QCD. His first observation is that the existence of light (almost massless) quarks might be playing a fundamental role in making the QCD vacuum a nontrivial and highly polarizable medium forbidding quark propagation outside hadrons [34,74,75]. He suggested that the mechanism of the confinement is due to the existence of light quarks. The motivation behind this is the phenomenon of supercritical binding of electrons by heavy nuclei in Quantum Electrodynamics (QED) [73]. He applied the phenomenon of supercritical binding of electrons by heavy nuclei in QED to QCD. In the strong static central electric field (Coulombic potential) of heavy nuclei, the electron would be relativistic and its energy spectrum would be given by the Dirac equation. The ground state energy of the electron in the Coulombic field of nucleus with charge Ze^+ is found to be $E_0 \propto \sqrt{(1-(\alpha Z)^2)}$ where $\alpha = \frac{1}{137}$ is the fine structure constant. It can be seen from the ground state energy relation that when $Z > Z_{crit} = 137$, the ground state energy becomes imaginary—this is an indication of instability of the physical system. Hence an isolated point-like nucleus with charge $Z > 137$ would be unstable and capture an electron from the vacuum to form a supercritical bound state while a positron is emitted; thus the stability with charged vacuum is achieved. Outside the length of this bound state the nucleus would be seen with an effective charge of $(Z-1)e^+$ and this process continues until the effective charge of the nucleus becomes $Z = Z_{crit} = 137$ [74]. Gribov's idea is that a similar mechanism is responsible for the confinement of quarks in QCD due to the existence of very light quarks.

This process applies even to the color charged single quark, where the role of (αZ) is played by the quark-gluon strong running coupling constant at large distance. The vacuum structure of light quarks is drastically modified due to this process, and the quark becomes a resonance that cannot be observed as an asymptotic state. In Ref. [35] Gribov calculated that a pair of light fermions interacting in a Coulomb-like manner develops super-critical behaviour much earlier when the running coupling constant attains a definite value $\alpha_{crit}/\pi = C_F^{-1}(1-\sqrt{2/3}) = 0.137$ where $C_F = \frac{N_c^2-1}{2N_c}$, N_c is the number of color charges and $C_F = \frac{4}{3}$ for $N_c = 3$ and C_F is the Casimir operator. It is to be noted that the critical value for QCD is 0.137 whereas it is 1 for QED. Therefore, the phenomenon of supercritical binding happens earlier in the case of QCD than QED.

For the quantitative description of the physical picture of supercritical binding of color charges, Gribov derived a nonlinear differential equation for the Green function of the light quarks using the Schwinger-Dyson equation. In deriving this equation, he chose Feynman gauge for gluons and collected only the most singular (logarithmically enhanced) terms in the infrared momentum region as these contributions are expected to cause the chiral symmetry breaking and confinement [36]. The second term in (5.2), using $\alpha_s/(4\pi) = g^2$, $D_{\mu\nu}(p-q) = -\frac{i g_{\mu\nu}}{(p-q)^2}$ (in Feynman gauge) and letting $p - q = k$, becomes,

$$-i\, C_F \frac{\alpha_s}{\pi} \int \frac{d^4 k}{4\pi^2} \gamma^\mu G(k) \gamma_\mu \frac{1}{(q-k)^2}, \tag{5.10}$$

where we denoted the external momentum by q_μ, introduced $C_F = \frac{(N_c^2-1)}{2N_c}$ which is $\frac{4}{3}$ for $N_c = 3$ and denoted S by G. Then (5.2) becomes

$$G^{-1}(q) \;=\; i\gamma \cdot q - C_F \frac{\alpha_s}{\pi} \int \frac{d^4 k}{4\pi^2} \gamma^\mu G(k) \gamma_\mu \frac{1}{(q-k)^2}. \tag{5.11}$$

Differentiating (5.11) twice with respect to the external momentum, q_μ, and using,

$$\partial^2 \frac{1}{(q-k)^2 + i\epsilon} \;=\; 4\pi^2\, i\, \delta^{(4)}(q-k), \tag{5.12}$$

we find,

$$\partial^2 G^{-1}(q) \;=\; C_F \frac{\alpha_s}{\pi} \gamma^\mu G(q) \gamma_\mu. \tag{5.13}$$

Equation (5.13) describes the lowest order diagram with the quark-gluon vertex as the bare vertex. Higher order diagrams can be effected by replacing γ_μ in (5.11) by $\Gamma^\mu(q,k,q-k)$. When (5.11) is differentiated (twice) with respect to the external momentum q_μ, it is found [77] that in each diagram, the most singular contribution from the infrared is obtained when both the derivatives act in the same gluon line which gives the delta-function. Other terms, which have derivatives on Γ, are less singular and are omitted. Then for the most singular terms, we have $\Gamma^\mu(q,q,0)$. It is to be noted that while the original Schwinger-Dyson equation involves one bare and one full vertex, in Gribov's approach we have two full vertices. The use of the Ward identity leads to

$$\Gamma_\mu(q,q,0) \;=\; \partial_\mu G^{-1}(q) \tag{5.14}$$

to write (5.13) as

$$\partial^2 G^{-1}(q) \;=\; g\,(\partial^\mu G^{-1})\, G\, (\partial_\mu G^{-1}) + \ldots, \tag{5.15}$$

where $g = C_F \frac{\alpha_s}{\pi}$ and the dots in (5.15) stand for less infra-red singular terms which are neglected here. *In this way, the integral Schwinger-Dyson equation is converted into a partial differential equation for* $G^{-1}(q)$ and this has been made possible by the choice of the Feynman gauge. The remarkable feature is that (5.15) involves only the quark Green function.

5.4 Solutions of Gribov's Equation for Quark Green Function

We use the general form of the inverse quark Green function,

$$G^{-1}(q) \;=\; a(q^2)\; \not{q} - b(q^2), \tag{5.16}$$

where a and b are two unknown scalar functions of q^2. A polar parametrisation of $G^{-1}(q)$ is given by

$$G^{-1}(q) \;=\; -\rho \exp\left(-\frac{1}{2}\phi \frac{\not{q}}{q}\right), \tag{5.17}$$

where ρ and ϕ are functions of q^2 ($q = \sqrt{q^\mu q_\mu}$). We use the relation $\not{q}\not{q} = q^2$ which is due to the anti-commutation relation of gamma matrices, to expand the R.H.S. exponential function of (5.17), as

$$\begin{aligned}
e^{-\frac{1}{2}\phi\frac{\not{q}}{q}} &= 1 - \frac{1}{2}\phi\frac{\not{q}}{q} + \frac{1}{2!}\left(\frac{\phi}{2}\right)^2 - \frac{1}{3!}\left(\frac{\phi}{2}\right)^3\frac{\not{q}}{q} + \frac{1}{4!}\left(\frac{\phi}{2}\right)^4\cdots, \\
&= \left(1 + \frac{1}{2!}\left(\frac{\phi}{2}\right)^2 + \frac{1}{4!}\left(\frac{\phi}{2}\right)^4 + \cdots\right) \\
&\quad + \frac{\not{q}}{q}\left(-\frac{1}{2}\phi\frac{\not{q}}{q} - \frac{1}{3!}\left(\frac{\phi}{2}\right)^3\frac{\not{q}}{q} - \cdots\right).
\end{aligned}$$

The expansion of cosh and sinh are given by,

$$\begin{aligned}
\sinh\left(\frac{\phi}{2}\right) &= \frac{e^{\frac{\phi}{2}} - e^{-\frac{\phi}{2}}}{2}, \\
&= \frac{\phi}{2} + \frac{1}{3!}\left(\frac{\phi}{2}\right)^3 + \cdots. \\
\cosh\left(\frac{\phi}{2}\right) &= \frac{e^{\frac{\phi}{2}} + e^{-\frac{\phi}{2}}}{2}, \\
&= 1 + \frac{1}{2!}\left(\frac{\phi}{2}\right)^2 + \cdots.
\end{aligned}$$

Therefore, we can write $e^{-\frac{1}{2}\phi\frac{\not{q}}{q}}$ as,

$$e^{-\frac{1}{2}\phi\frac{\not{q}}{q}} = \cosh\left(\frac{\phi}{2}\right) - \frac{\not{q}}{q}\sinh\left(\frac{\phi}{2}\right). \tag{5.18}$$

Hence, from (5.16) and (5.17), we find

$$G^{-1}(q) = -\rho\left\{\cosh\left(\frac{\phi}{2}\right) - \frac{\not{q}}{q}\sinh\left(\frac{\phi}{2}\right)\right\} = a(q^2)\,\not{q} - b(q^2). \tag{5.19}$$

The comparison of the two equivalent forms of $G^{-1}(q)$ in (5.19), gives,

$$a(q^2) = \frac{1}{q}\rho\sinh\left(\frac{\phi}{2}\right), \tag{5.20}$$

$$b(q^2) = \rho\cosh\left(\frac{\phi}{2}\right). \tag{5.21}$$

The dynamical mass function $M_0(q^2)$ of the quark is defined as

$$M_0(q^2) = \frac{b(q^2)}{a(q^2)} = q\coth\left(\frac{\phi}{2}\right), \tag{5.22}$$

which involves only ϕ. The subscript '0' on M will be explained later. From (5.19), we have

$$G^{-1}(q) \;=\; \frac{\not{q}}{q}\rho \sinh\left(\frac{\phi}{2}\right) - \rho\cosh\left(\frac{\phi}{2}\right) .$$

We introduce ξ as

$$\xi \;\equiv\; \ln q = \ln\sqrt{q^{\mu}q_{\mu}} \tag{5.23}$$

and denote $\partial_\xi f(q) = \dot{f}(q)$. Substituting this in (5.15), we obtain a pair of coupled differential equations for ϕ and ρ as [76],

$$\dot{p} \;=\; 1 - p^2 - \beta^2\left(\frac{1}{4}\dot{\phi}^2 + 3\sinh^2\left(\frac{\phi}{2}\right)\right); \tag{5.24}$$

$$\ddot{\phi} + 2\,p\,\dot{\phi} - 3\sinh(\phi) \;=\; 0, \tag{5.25}$$

where

$$\begin{aligned} p &= 1 + \beta\frac{\dot{\rho}}{\rho}, \\ \text{with } \beta &= 1 - g = 1 - C_F\frac{\alpha_s}{\pi}. \end{aligned} \tag{5.26}$$

By solving (5.24) and (5.25) for ϕ and ρ for large and small q, it was found [35,36] that the dynamical mass function, $M_0(q^2)$ in (5.22) behaved such that $M_0(0) \neq 0$. This is taken as the signature for chiral symmetry breaking since on general grounds the Green function has the form $G^{-1}(q^2) = Z_2(\not{q} - M_0)$ to be identified with $a(q^2)\not{q} - b(q^2)$.

5.5 Solutions of Gribov's Equation for Quark Green Function with Pion Correction

In the spontaneous breaking of chiral symmetry, massless pions appear as the Goldstone mode in the physical spectrum and they make corrections to the quark propagator. Gribov took into account this back-reaction of the pions on quarks, and obtained a 'pion corrected' equation for $G^{-1}(q)$ [35,36]. The coupling of the pion to the quark can be related to the pion decay constant f_π via the Goldberger-Trieman relation, by taking into account the proper isospin factors for the light quark flavors. The pion corrected equation for quark Green function, see Gribov [79] and Ref. [78] for a review, is

$$\partial^2 G^{-1} \;=\; g\,(\partial^{\mu}G^{-1})\,G\,(\partial_{\mu}G^{-1}) - \frac{3}{16\pi^2 f_\pi^2}\,\{i\gamma_5, G^{-1}\}\,G\,\{i\gamma_5, G^{-1}\}, \tag{5.27}$$

where f_π is the pion decay constant, $f_\pi = 0.093GeV$. It is to be noted that the pion corrected Gribov's equation is still a differential equation involving only light quark Green functions.

We use the same parametrisation as (5.17) for (5.27), with (ρ, ϕ) replaced by (ρ', ϕ'), i.e.,

$$G^{-1}(q) \;=\; -\rho' \exp\left(-\frac{1}{2}\phi' \frac{\not q}{q}\right). \qquad (5.28)$$

Writing (5.28) as

$$G^{-1}(q) \;=\; \frac{\not q}{q}\rho' \sinh\left(\frac{\phi'}{2}\right) - \rho' \cosh\left(\frac{\phi'}{2}\right),$$

and substituting in (5.27), we obtain,

$$\dot{p}' \;=\; 1 - p'^2 - \beta^2\left(\frac{1}{4}\dot{\phi}'^2 + 3\sinh^2\left(\frac{\phi'}{2}\right)\right) + \frac{3\beta q^2}{4\pi^2 f_\pi^2}\cosh^2\left(\frac{\phi'}{2}\right), \qquad (5.29)$$

$$\ddot{\phi}' + 2\,p'\,\dot{\phi}' - 3\,\sinh(\phi') \;=\; 0. \qquad (5.30)$$

Equations (5.29) and (5.30) reduce to (5.24) and (5.25) respectively when the pion correction is neglected. In the spirit of (5.22), the dynamical mass with pion correction is

$$M(q^2) \;=\; q \coth\left(\frac{\phi'}{2}\right). \qquad (5.31)$$

We are interested in the infrared region. For low momentum, $|\overrightarrow{q}| \to 0$, we linearise the pair of equations (5.29) and (5.30) around (ρ, ϕ) as

$$\phi' = \phi + \delta\phi \quad \& \quad p' = p + \delta p$$

and keep only terms linear in $\delta\phi$ and δp. This is a reasonable procedure as we are interested in the low momentum region. Then, the ϕ-equations (5.25) and (5.30) give the relation

$$2\,\delta p\,\dot{\phi} - 3\,\cosh(\phi)\,\delta\phi \;=\; 0, \qquad (5.32)$$

and the p-equations (5.24) and (5.29) give

$$\delta p \;=\; \left(-\frac{3\beta^2}{4p} + \frac{3\beta q^2}{16\pi^2 {f_\pi}^2 p}\right)\sinh(\phi)\,\delta\phi + \frac{3\beta q^2}{8\pi^2 {f_\pi}^2 p}\cosh^2\left(\frac{\phi}{2}\right). \qquad (5.33)$$

From (5.32) and (5.33), we find

$$\delta\phi \left[\frac{\coth(\phi)}{2\dot{\phi}} + \frac{\beta^2}{4p} - \frac{\beta\, q^2}{16\pi^2 {f_\pi}^2 p}\right] = \frac{\beta\, q\, M_0(q^2)}{16\pi^2 {f_\pi}^2 p}, \tag{5.34}$$

where we have used $\coth\left(\frac{\phi}{2}\right) = \frac{M_0(q^2)}{q}$ from (5.22). The dynamical mass with pion correction (5.31) is

$$\begin{aligned} M(q^2) &= q\coth\left(\frac{\phi+\delta\phi}{2}\right), \\ &\approx q\left[\frac{\coth\left(\frac{\phi}{2}\right)+\frac{\delta\phi}{2}}{1+\frac{\delta\phi}{2}\coth\left(\frac{\phi}{2}\right)}\right], \\ &\approx q\left[\coth\left(\frac{\phi}{2}\right)+\frac{\delta\phi}{2}\right]\left[1-\frac{\delta\phi}{2}\coth\left(\frac{\phi}{2}\right)\right], \\ &\approx q\left[\coth\left(\frac{\phi}{2}\right)+\frac{\delta\phi}{2}\left(1-\coth^2\left(\frac{\phi}{2}\right)\right)\right], \end{aligned}$$

where we have kept the terms linear in $\delta\phi$. We use the relation $\coth\left(\frac{\phi}{2}\right) = \frac{M_0(q^2)}{q}$ from (5.22), so $M(q^2)$ becomes,

$$M(q^2) = M_0(q^2) + q\left(\frac{\delta\phi}{2}\right)\left(1-\frac{M_0^2(q^2)}{q^2}\right). \tag{5.35}$$

Substituting $\delta\phi$ from (5.34) in (5.35), we find

$$M(q^2) = M_0(q^2)\left[1+\left(\frac{\beta q^2}{32\pi^2 f_\pi^2 p}\right)\left(\frac{1}{\alpha}\right)\left(1-\frac{M_0^2(q^2)}{q^2}\right)\right], \tag{5.36}$$

where

$$\alpha = \left[\frac{\coth(\phi)}{2\dot{\phi}} + \frac{\beta^2}{4p} - \frac{\beta q^2}{16\pi^2 f_\pi^2 p}\right].$$

Equation (5.36) gives a relationship between the dynamical mass of quarks with pion correction, $M(q^2)$, and without pion correction, $M_0(q^2)$, at low momentum. This is our main result [81]. Further the expression in (5.36) and α involve solutions to (5.24) and (5.25). It can be seen from equation (5.36) that in the limit $f_\pi \to \infty$ (i.e., no pion correction), $M(q^2) \to M_0(q^2)$.

5.6 Results and Discussion

Now we consider the solutions of (5.24) and (5.25) in the infrared region $q \to 0$. In Ref. [76], one possible solution when $|\overrightarrow{q}| \to 0$ is $p \to p_0$ with $p_0^2 = 1$ and

$\phi = C\,e^{\xi}$ for $p_0 = 1$, where the arbitrary constant C has the dimension inverse of length. We use the expansion for $\coth(x)$ and keep the first three terms only [80],

$$\coth(x) \;\approx\; \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45}. \tag{5.37}$$

Using the solution for ϕ at low momentum $\phi = Cq$ and $\dot{\phi} = Cq$, the dynamical mass without pion correction, $M_0(q^2)$, is given by,

$$M_0(q^2) \;=\; \frac{2}{C} + \frac{C\,q^2}{6} - \frac{C^3\,q^4}{360}, \tag{5.38}$$

while α is given by

$$\alpha \;=\; \frac{1}{2}\left(\frac{1}{C^2\,q^2} + \frac{1}{3} - \frac{C^2\,q^2}{45}\right) + \frac{\beta^2}{4} - \frac{\beta\,q^2}{16\,\pi^2\,f_\pi^2}, \tag{5.39}$$

and the dynamical mass with pion correction, $M(q^2)$, is given by

$$M(q^2) \;=\; M_0(q^2)\left[1 + \frac{\beta C^2 q^2 \frac{\left(q^2 - M_0^2(q^2)\right)}{16\pi^2 f_\pi^2}}{1 + (\frac{1}{3} + \frac{\beta^2}{2})C^2\,q^2 - \frac{C^4\,q^4}{45} - \frac{\beta\,C^2\,q^4}{8\,\pi^2\,f_\pi^2}}\right]. \tag{5.40}$$

This is valid in the low momentum region only. In the limit $q \to 0$, we find $M(0) \to M_0(0) = \frac{2}{C} \neq 0$. For space-like momenta we replace 'q^2' by '$-q^2$', and (5.38) and (5.40) change to,

$$M_0(q^2) \;=\; \frac{2}{C} - \frac{C\,q^2}{6} - \frac{C^3\,q^4}{360}, \tag{5.41}$$

$$M(q^2) \;=\; M_0(q^2)\left[1 + \frac{\beta\,C^2\,q^2 \frac{\left(q^2 + M_0^2(q^2)\right)}{16\pi^2 f_\pi^2}}{1 - (\frac{1}{3} + \frac{\beta^2}{2})C^2\,q^2 - \frac{C^4\,q^4}{45} - \frac{\beta\,C^2\,q^4}{8\,\pi^2\,f_\pi^2}}\right]. \tag{5.42}$$

We use (5.41) and (5.42) to exhibit the behaviour of $M_0(q^2)$ and $M(q^2)$ at low momentum. We use $f_\pi = 0.093$ GeV [33] and the arbitrary constant C is taken to reproduce the numerical value of $M_0(0) = M(0)$ as estimated in Ref. [76]: $M_0(0) = M(0) = 0.1$ GeV and so from (5.41), $C = 20$ GeV^{-1}. At low momenta we take the strong coupling constant to be constant and use the supercritical value $\alpha_c = 0.43$ as found by Gribov [36]. For this value of α_s, $\beta = 1 - g = 0.8175$. We show the variation of $M_0(q^2)$ and $M(q^2)$ for a range of q from $q = 0$ to 0.045 GeV using Matlab in Figure 5.1. Here the solid line corresponds to the variation of $M_0(q^2)$ and the broken line is that for $M(q^2)$. It is seen from Figure 5.1 that in the low momentum region the pion correction to the mass of quarks is small. This feature is similar to the study of Ref. [76] at large momentum.

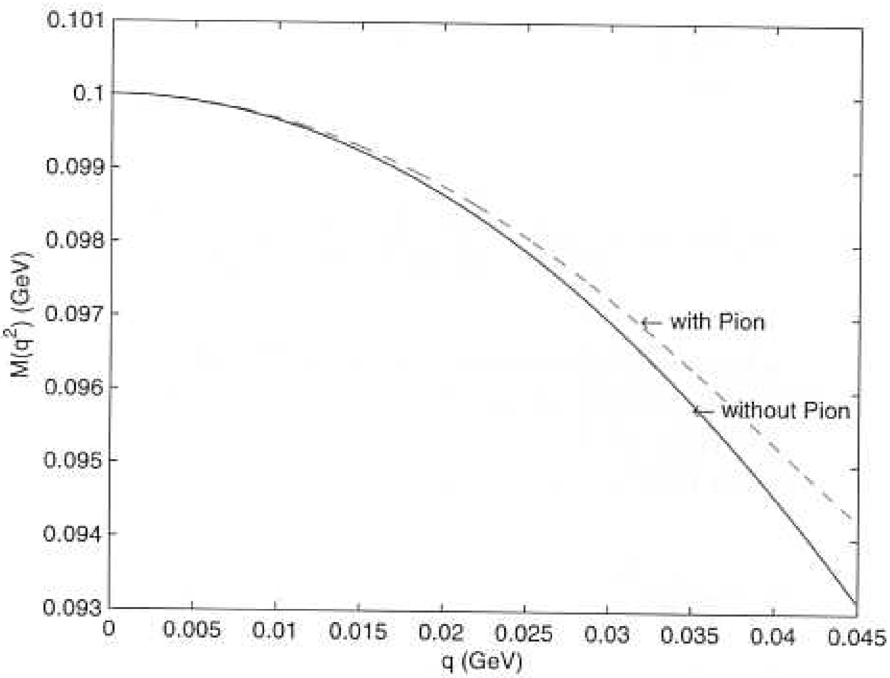

Figure 5.1: Variation of dynamical mass function with pion correction, $M(q^2)$ (broken line), and without pion correction, $M_0(q^2)$ (solid line), with momentum scale.

Appendix 3.1

ϵ - Regularization Prescription[†]

For the illustration of Salam's ϵ-regularization prescription, which is the regularization of the ultraviolet divergence in logarithmic functions, let us choose a typical integral, for example, the first logarithmic integral of equation (3.25) of Chapter III :

$$I = \left(\frac{gH}{2\pi}\right)\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{dk_3}{2\pi}\int_{-\infty}^{\infty}\frac{dk_4}{2\pi}\log\left\{\frac{1}{\mu^2}\{(2n+1)gH+2gH+k_3^2+k_4^2\}\right\}. \tag{1}$$

Let us change the Cartesian coordinates (k_3,k_4) to polar coordinates (k,θ) and take the gH factor outside the inner curly bracket. The angular integration over θ produces an overall 2π factor. We re-scale k by $\sqrt{gH}k$ as this rescaling produces an overall factor (gH) but does not affect the limit of integration of k i.e., 0 to ∞. Then (1) becomes

$$I = \frac{(gH)^2}{8\pi^2}\sum_{n=0}^{\infty}\int_0^{\infty}dk^2\log\left\{\frac{gH}{\mu^2}\{(2n+3)+k^2\}\right\}. \tag{2}$$

Let us suppose $E = \frac{gH}{\mu^2}(2n+3+k^2)$. The integral is ultra-violet divergent. To avoid the cut in the logarithm we add the Feynman condition '$-i\delta$' i.e. $\log(E) \to \log(E-i\delta)$. Next, the integrand is regularized by observing that up-to an infinite uninteresting constant,

$$\begin{aligned}\log(E-i\delta) &= \lim_{\epsilon\to 0}\frac{(E-i\delta)^\epsilon - 1}{\epsilon},\\ &= \lim_{\epsilon\to 0}\frac{(E-i\delta)^\epsilon}{\epsilon} - \frac{1}{\epsilon}.\end{aligned} \tag{3}$$

The infinite $\frac{1}{\epsilon}$ term is omitted. Then

$$\log(E-i\delta) \;\to\; \lim_{\epsilon\to 0}\frac{(E-i\delta)^\epsilon}{\epsilon},$$

[†]A. Salam and J. Strathdee Nucl. Phy. **B90**, 203 (1975).

$$= \lim_{\epsilon\to 0} -\frac{(E-i\delta)^{-\epsilon}}{\epsilon}.$$

Let us consider the Gamma function,

$$\Gamma(\epsilon) = \int_0^\infty dt\, t^{\epsilon-1}\, e^{-t}.$$

Substituting $t = i\,y\,(E-i\delta)$ and $dt = i\,dy\,(E-i\delta)$, we have

$$\begin{aligned}\Gamma(\epsilon) &= \int_0^\infty dy\, i(E-i\delta)\, y^{\epsilon-1}\,(i)^{\epsilon-1}\,(E-i\delta)^{\epsilon-1}\, e^{-iy(E-i\delta)};\\ &= i^\epsilon\,(E-i\delta)^\epsilon \int_0^\infty dy\, y^{\epsilon-1}\, e^{-iy(E-i\delta)}.\end{aligned}$$

Therefore,

$$(E-i\delta)^{-\epsilon} = \frac{i^\epsilon}{\Gamma(\epsilon)} \int_0^\infty dt\, t^{\epsilon-1}\, e^{-it(E-i\delta)}. \tag{4}$$

So, the integral I can be evaluated, using this prescription, as

$$I = \frac{(gH)^2}{8\pi^2} \lim_{\epsilon\to 0} \sum_{n=0}^{\infty} \int_0^\infty dk^2 \left(-\frac{i^\epsilon}{\epsilon\,\Gamma(\epsilon)}\right) \int_0^\infty dt\, t^{-1+\epsilon}\, e^{-it(\frac{gH}{\mu^2})(2n+3+k^2)}.$$

Under Wick rotation we perform the following substitution,

$$t \to \frac{t}{i}; \qquad dt \to \frac{dt}{i},$$

so that

$$I = \frac{(gH)^2}{8\pi^2} \lim_{\epsilon\to 0} \sum_{n=0}^{\infty} \int_0^\infty d\,k^2 \left(-\frac{i^\epsilon}{\epsilon\,\Gamma(\epsilon)}\right) \int_0^\infty \frac{dt}{i}\, \frac{t^{-1+\epsilon}}{i^{-1+\epsilon}}\, e^{-t(\frac{gH}{\mu^2})(2n+3+k^2)}.$$

We integrate out k and perform the sum over n to get,

$$I = \frac{(gH)^2}{8\pi^2} \lim_{\epsilon\to 0} \left(-\frac{\Gamma(-1+\epsilon)}{\epsilon\Gamma(\epsilon)}\right) 2^{-\epsilon+1} \left(\frac{gH}{\mu^2}\right)^{-\epsilon} \xi\left(-1+\epsilon, \frac{3}{2}\right), \tag{5}$$

where ξ is Riemann Zeta function. We use the property of the Gamma function, $\Gamma(n+1) = n\Gamma(n)$ to write $\Gamma(\epsilon) = (-1+\epsilon)\Gamma(-1+\epsilon)$ and $\frac{\Gamma(-1+\epsilon)}{\Gamma(\epsilon)} = \frac{1}{\epsilon-1}$. We expand the Zeta function around $\epsilon = 0$ as $\xi(-1+\epsilon, \frac{3}{2}) = \xi(-1, \frac{3}{2}) + \epsilon\xi'\left(-1, \frac{3}{2}\right) + \cdots$. Only the first two terms of the Zeta function expansion will contribute. The contribution of the other terms to I will be zero as $\epsilon \to 0$. Therefore, (5) can be written as

$$\begin{aligned} I = \lim_{\epsilon\to 0} \Bigg[&-\frac{(gH)^2}{8\pi^2} \frac{2^{(-\epsilon+1)}}{(-1+\epsilon)} \frac{e^{-\epsilon\log\left(\frac{gH}{\mu^2}\right)}}{\epsilon} \xi\left(-1, \frac{3}{2}\right) \\ &- \frac{(gH)^2}{8\pi^2} \frac{2^{(-\epsilon+1)}}{(-1+\epsilon)} \left(\frac{gH}{\mu^2}\right)^{-\epsilon} \xi'(-1, \frac{3}{2}) \Bigg]. \end{aligned} \tag{6}$$

In the limit $\epsilon \to 0$, $\frac{e^{-\epsilon \log\left(\frac{gH}{\mu^2}\right)}}{\epsilon} \to -\log\left(\frac{gH}{\mu^2}\right)$ and we use the value $\xi(-1, \frac{3}{2}) = -\frac{11}{24}$. The second term produces a real finite constant (RFC) which corresponds to C. Putting everything in (6), we get the desired result of Chapter III, equation (3.26),

$$I \;=\; \frac{11\, g^2\, H^2}{96\, \pi^2} \left\{ \log\left(\frac{gH}{\mu^2}\right) + C \right\} \, . \tag{7}$$

This procedure is used for the regularization of ultraviolet divergences of the logarithmic functions in Chapter III and Chapter IV.

Appendix 3.2

Eigenfunctions and Eigenvalues of $\Theta_{ij}^{ac}(x)$

$\Theta_{ij}^{ac}(x)$ is the differential operator which occurs in the determinant as $(-\Theta_{ij}^{ac}(x))$ when the quadratic approximation was used in Chapter III and Chapter IV. In order to evaluate the determinant, we find the eigenfunctions and eigenvalues in this appendix which have been used directly in Chapter III and Chapter IV in the Savvidy background. It is to be noted that $\Theta_{ij}^{ac}(x)$ consists of indices of three types : color indices (a, c), Lorentz group indices (i, j), and x which is a continuous space-time index. Color indices take values (1,2,3) as the color symmetry group is $SU(2)$ and Lorentz indices are restricted to (1,2) due to cancellation of the terms involving the (3,4) indices from the ghost contribution. In order to calculate the eigenfunctions and the eigenvalues of $\Theta_{ij}^{ac}(x)$ in the Savvidy background, we need to solve the following eigenvalue equation,

$$\Theta_{ij}^{ac}(x)a_j^c \;=\; \lambda\, a_i^a, \tag{1}$$

where

$$\Theta_{ij}^{ac}(x) \;=\; (-\bar{D}_\lambda^{ab}\bar{D}_\lambda^{bc})\delta_{ij} \;-\; 2g\epsilon^{aec}\,\bar{F}_{ij}^e, \tag{2}$$

and the overall negative sign is put in the definition of $\Theta_{ij}^{ac}(x)$ by hand as it appears as '$-\Theta_{ij}^{ac}(x)$' in the determinant. The Savvidy background is given by

$$\begin{aligned}\bar{A}_i^a &= \delta^{a3}\left(-\frac{Hx_2}{2}, \frac{Hx_1}{2}, 0\right),\\ \bar{A}_4^a &= 0,\end{aligned}$$

where $i = 1, 2, 3$ and $a = 1, 2, 3$. In this background, $\bar{F}_{12}^3 = -\bar{F}_{21}^3 = H$ and $\bar{F}_{ij}^a = 0$ for other combination of a, i, j. In this background, the relevant quantities for our purpose are,

$$\begin{aligned}\bar{A}_\lambda^3\partial_\lambda &= \frac{H}{2}(x_1\partial_2 - x_2\partial_1);\\ \bar{A}_\lambda^3\bar{A}_\lambda^3 \;. &= \frac{H^2}{4}(x_1^2 + x_2^2),\end{aligned}$$

with

$$
\begin{aligned}
\bar{D}^{ab}_{\lambda} &= \partial_{\lambda}\delta^{ab} + g\,\epsilon^{a3b}\,\bar{A}^{3}_{\lambda}, \\
\bar{D}^{bc}_{\lambda} &= \partial_{\lambda}\delta^{bc} + g\,\epsilon^{b3c}\,\bar{A}^{3}_{\lambda}, \\
(-\bar{D}^{ab}_{\lambda}\bar{D}^{bc}_{\lambda}) &= -(\partial_{\lambda}\partial_{\lambda}\delta^{ac} + 2g\epsilon^{a3c}\bar{A}^{3}_{\lambda}\partial_{\lambda} + g^{2}\,\epsilon^{a3b}\epsilon^{b3c}\bar{A}^{3}_{\lambda}\bar{A}^{3}_{\lambda}), \\
(-\bar{D}^{ab}_{\lambda}\bar{D}^{bc}_{\lambda}) &= -\partial_{\lambda}\partial_{\lambda}\delta^{ac} - gH\epsilon^{a3c}(x_1\partial_2 - x_2\partial_1) - \frac{g^2H^2}{4}\epsilon^{a3b}\epsilon^{b3c}(x_1^2+x_2^2), \qquad (3) \\
\Theta^{ac}_{ij} &= \left(-\partial_{\lambda}\partial_{\lambda}\delta^{ac} - gH\epsilon^{a3c}(x_1\partial_2 - x_2\partial_1) - \frac{g^2H^2}{4}\epsilon^{a3b}\epsilon^{b3c}(x_1^2+x_2^2)\right)\delta_{ij} \\
&\quad -2g\epsilon^{a3c}\bar{F}^{3}_{ij}, \qquad (4)
\end{aligned}
$$

where $a, c = 1, 2, 3$; $\lambda = 1, 2, 3, 4$, and $i, j = 1, 2$. It is observed that,

$$
\begin{aligned}
\Theta^{33}_{ij} &= -\partial_{\lambda}\partial_{\lambda}\delta_{ij}, \\
\Theta^{11}_{ij} &= \Theta^{22}_{ij} = \left(-\partial_{\lambda}\partial_{\lambda} + \frac{g^2H^2}{4}(x_1^2+x_2^2)\right)\delta_{ij}, \qquad (5) \\
\Theta^{12}_{ij} &= -\Theta^{21}_{ij} = (gH(x_1\partial_2 - x_2\partial_1)\delta_{ij}) + 2g\bar{F}^{3}_{ij}, \\
\Theta^{13}_{ij} &= \Theta^{31}_{ij} = \Theta^{32}_{ij} = \Theta^{23}_{ij} = 0.
\end{aligned}
$$

Let us consider equation (1).

1. For $a = 3$,

$$
\begin{aligned}
\lambda\, a^3_i &= \Theta^{3c}_{ij}\, a^c_j = (-\partial_{\lambda}\partial_{\lambda})\delta_{ij}a^3_j, \\
\lambda\, a^3_i &= (-\partial_{\lambda}\partial_{\lambda})a^3_i, \qquad (6)
\end{aligned}
$$

so, a^3_i are plane waves. The eigenfunctions and eigenvalues are a^3_1, a^3_2 : $k_1^2 + k_2^2 + k_3^2 + k_4^2$.

2. For $a = 1, 2$,

$$
\begin{aligned}
\lambda\, a^1_i &= \Theta^{11}_{ij}\, a^1_j + \Theta^{12}_{ij}\, a^2_j, \qquad (7) \\
\lambda\, a^2_i &= \Theta^{21}_{ij}\, a^1_j + \Theta^{22}_{ij}\, a^2_j, \qquad (8)
\end{aligned}
$$

so (8) can be rewritten, with the help of (5), as

$$
\lambda\, a^2_i = -\Theta^{12}_{ij}\, a^1_j + \Theta^{11}_{ij}\, a^2_j. \qquad (9)
$$

Now, we perform the operation (7)±i(9) to give

$$
\lambda\,(a^1_i \pm i a^2_i) = \Theta^{11}_{ij}\,(a^1_j \pm i a^2_j) + \Theta^{12}_{ij}\,(a^2_j \mp i a^1_j). \qquad (10)
$$

On simplifying (10),

$$
\lambda\,(a^1_i \pm i a^2_i) = (\Theta^{11}_{ij} \mp i\Theta^{12}_{ij})\,(a^1_j \pm i a^2_j). \qquad (11)
$$

Equation (11) is diagonalised in color indices.

Let us introduce the notation

$$a_i^+ \equiv a_i^1 + i a_i^2;$$
$$a_i^- \equiv a_i^1 - i a_i^2.$$

Equation (11) can be rewritten with the help of these a_i as,

$$\lambda\, a_i^+ = (\Theta_{ij}^{11} - i\Theta_{ij}^{12}) a_j^+, \tag{12}$$
$$\lambda\, a_i^- = (\Theta_{ij}^{11} + i\Theta_{ij}^{12}) a_j^-. \tag{13}$$

Now, we consider equation (12)

$$\begin{aligned}
\lambda\, a_1^+ &= (\Theta_{1j}^{11} - i\Theta_{1j}^{12}) a_j^+, \\
&= (\Theta_{11}^{11} - i\Theta_{11}^{12}) a_1^+ + (\Theta_{12}^{11} - i\Theta_{12}^{12}) a_2^+, \qquad (14)\\
\lambda\, a_2^+ &= (\Theta_{21}^{11} - i\Theta_{21}^{12}) a_1^+ + (\Theta_{22}^{11} - i\Theta_{22}^{12}) a_2^+. \qquad (15)
\end{aligned}$$

It is observed that

$$\begin{aligned}
\Theta_{21}^{11} &= \Theta_{12}^{11} = 0, \\
\Theta_{21}^{12} &= -\Theta_{12}^{12} = -2gH, \\
\Theta_{11}^{11} &= \Theta_{22}^{11} = -\partial_\lambda \partial_\lambda + \frac{g^2 H^2}{4}(x_1^2 + x_2^2), \\
\Theta_{22}^{12} &= \Theta_{11}^{12} = gH(x_1\partial_2 - x_2\partial_1).
\end{aligned}$$

Therefore, (14) and (15) become,

$$\lambda\, a_1^+ = (\Theta_{11}^{11} - i\Theta_{11}^{12}) a_1^+ + (-i\Theta_{12}^{12}) a_2^+; \tag{16}$$
$$\lambda\, a_2^+ = (i\Theta_{12}^{12}) a_1^+ + (\Theta_{11}^{11} - i\Theta_{12}^{12}) a_2^+. \tag{17}$$

We perform the operation (16) $\pm$ i(17) and take the complex conjugate. We introduce the notation,

$$\begin{aligned}
a_+^+ &= a_1^+ + i a_2^+, \\
a_-^+ &= a_1^+ - i a_2^+, \\
a_+^- &= a_1^- + i a_2^-, \\
a_-^- &= a_1^- - i a_2^-,
\end{aligned}$$

and we find

$$\left.\begin{aligned}
\lambda\, a_+^+ &= (\Theta_{11}^{11} - i\Theta_{11}^{12} - \Theta_{12}^{12}) a_+^+, \\
\lambda\, a_-^+ &= (\Theta_{11}^{11} - i\Theta_{11}^{12} + \Theta_{12}^{12}) a_-^+, \\
\lambda\, a_+^- &= (\Theta_{11}^{11} + i\Theta_{11}^{12} + \Theta_{12}^{12}) a_+^-, \\
\lambda\, a_-^- &= (\Theta_{11}^{11} + i\Theta_{11}^{12} - \Theta_{12}^{12}) a_-^-,.
\end{aligned}\right\} \tag{18}$$

Now we consider the first equation in (18) for finding the eigenvalues and eigenvectors,

$$\begin{aligned}
\lambda a^+_+ &= (\Theta^{11}_{11} - i\Theta^{12}_{11} - \Theta^{12}_{12})a^+_+, \\
&= (-\partial_1^2 - \partial_2^2 - \partial_3^2 - \partial_4^2) - igH(x_1\partial_2 - x_2\partial_1) \\
&\quad -2gH + \frac{g^2H^2}{4}(x_1^2 + x_2^2). \qquad (19)
\end{aligned}$$

As the 3,4 indices are free, these correspond to plane waves with eigenvalue $k_3^2+k_4^2$. We need to consider the eigenvalue of the part consisting of indices 1 and 2 only which are mixed. In order to consider the eigenvalues of the operator $-\partial_1^2 - \partial_2^2 - igH(x_1\partial_2 - x_2\partial_1) + \frac{g^2H^2}{4}(x_1^2 + x_2^2)$, we suppose the following,

$$\begin{aligned}
a_1 &= \partial_1 + \frac{gH}{2}x_1, \\
a_1^\dagger &= -\partial_1 + \frac{gH}{2}x_1, \\
a_2 &= \partial_2 + \frac{gH}{2}x_2, \\
a_2^\dagger &= -\partial_2 + \frac{gH}{2}x_2, \\
D &= a_1 - ia_2 = \partial_1 - i\partial_2 + \frac{gH}{2}(x_1 - ix_2), \\
D^\dagger &= a_1^\dagger + ia_2^\dagger = -\partial_1 - i\partial_2 + \frac{gH}{2}(x_1 + ix_2).
\end{aligned}$$

It can be seen from the above notation that

$$\begin{aligned}
DD^\dagger &= -\partial_1^2 - \partial_2^2 + gH - igH(x_1\partial_2 - x_2\partial_1) + \frac{g^2H^2}{4}(x_1^2 + x_2^2), \\
D^\dagger D &= -\partial_1^2 - \partial_2^2 - gH - igH(x_1\partial_2 - x_2\partial_1) + \frac{g^2H^2}{4}(x_1^2 + x_2^2), \\
\left[D, D^\dagger\right] &= 2gH,
\end{aligned}$$

so the eigenvalue of a^+_+ is given by $D^\dagger D + gH - 2gH + k_3^2 + k_4^2$ where $D^\dagger$ and D are like harmonic oscillator creation and annihilation operators. This number operator can be replaced by $n(2gH)$ and we have the eigenvalue $(2n+1)gH - 2gH + k_3^2 + k_4^2$. Similarly, we can find the remaining three eigenvalues by complex conjugation and following the above procedure. Thus all eigenmodes and eigenvalues are given by,

$$\begin{aligned}
a_1^3 \pm i\,a_2^3 &: \; k_3^2 + k_4^2 \text{ (plane waves)}, \\
(a_1^1 + i\,a_2^1) + i\,(a_1^2 + i\,a_2^2) &: \; (2n+1)\,gH - 2\,gH + k_3^2 + k_4^2, \\
(a_1^1 + i\,a_2^1) - i\,(a_1^2 + i\,a_2^2) &: \; (2n+1)\,gH + 2\,gH + k_3^2 + k_4^2, \qquad (20) \\
(a_1^1 - i\,a_2^1) + i\,(a_1^2 - i\,a_2^2) &: \; (2n+1)\,gH + 2\,gH + k_3^2 + k_4^2, \\
(a_1^1 - i\,a_2^1) - i\,(a_1^2 - i\,a_2^2) &: \; (2n+1)\,gH - 2\,gH + k_3^2 + k_4^2.
\end{aligned}$$

Appendix 4.1

Matsubara Frequency[‡]

We cannot use directly the results of zero-temperature at finite temperature. The reason is that at finite temperature, we have to average over all excited states of the system, not only its ground state. While the ground state (which state corresponds to zero temperature) is unique, the excited states are highly degenerate (infinitely degenerate in the thermodynamic limit). Matsubara used the analogy between the evolution operator in conventional time, $U = e^{-i\hat{H}t}$, and the (non-normalised) equilibrium statistical mechanics operator $\hat{\rho} = e^{-\beta\hat{H}}, \beta = \frac{1}{kT}$. The idea of Matsubara was to use this analogy to define some new Matsubara, or thermal, Green's functions, closely related to conventional causal Green's functions in real time.

If we introduce the variable τ, $0 < \tau < \beta$, we see that $\hat{\rho}$ satisfies the Bloch equation,

$$\frac{\partial}{\partial\tau}\hat{\rho}(\tau) \;=\; -\mathcal{H}\,\hat{\rho}(\tau), \tag{1}$$

with the initial condition $\hat{\rho}(0) = I$. If we perform the transformation $t \leftrightarrow -i\tau$ this equation transforms into the Schrodinger equation for $\hat{\rho}(it)$ on the imaginary interval $0 > t > -i\tau$:

$$\frac{\partial}{\partial t}\hat{\rho}(it) \;=\; \mathcal{H}\,\hat{\rho}(it). \tag{2}$$

The general definition of the causal, retarded, and advanced one-particle Green's functions, is

$$G_{\alpha\beta}(x_1, t_1; x_2, t_2) \;=\; -i\,Tr\left(\hat{\rho}\,T\left(\psi_\alpha(x_1, t_1)\psi^\dagger_\beta(x_2, t_2)\right)\right). \tag{3}$$

We can define the thermal Green's function in analogy with the above,

$$\begin{aligned} G_{\alpha\alpha'}(\hat{x}, \tau; \hat{x}', \tau') \;&=\; -\langle \Upsilon_\tau\left(\psi^M_\alpha(\hat{x}, \tau)\;\bar{\psi}^M_{\alpha'}(\hat{x}', \tau')\right)\rangle \\ &\equiv -Tr\left\{e^{-\beta(\hat{H}-\hat{\Omega})}\;\Upsilon_\tau\left(\psi^M_\alpha(\hat{x}, \tau)\;\bar{\psi}^M_{\alpha'}(\hat{x}', \tau')\right)\right\}, \end{aligned} \tag{4}$$

[‡]T. Matsubara, Prog. Theo. Phys., 14 351 (1955).

where Υ_τ is the time-ordered product with respect to the time variable τ and ψ^M is the Matsubara operators, obtained by Wick's rotation of Heisenberg operators:

$$\begin{aligned}\psi(\hat{x},t) &= e^{i\hat{H}t}\psi(\hat{x})\,e^{-i\hat{H}t} \to \psi^M(\hat{x},\tau),\\ &= e^{\hat{H}\tau}\psi(\hat{x})\,e^{-\hat{H}\tau},\end{aligned}$$

and it satisfies the analytic continued Heisenberg equations,

$$\frac{\partial}{\partial\tau}\psi^M(\hat{x},\tau) = [\hat{H},\psi^M(\hat{x},\tau)]\,.$$

Now, if $\tau > \tau'$, then

$$\begin{aligned}G(\hat{x},\tau;\hat{x}',\tau') &= -Tr\left\{e^{-\beta(\hat{H}-\hat{\Omega})}\,\psi^M(\hat{x},\tau)\,\bar{\psi}^M(\hat{x}',\tau')\right\},\\ &= -e^{\beta\Omega}Tr\left\{e^{-\beta\hat{H}}e^{\hat{H}\tau}\psi(\hat{x})e^{-\hat{H}\tau}e^{\hat{H}\tau'}\bar{\psi}(\hat{x})e^{H\tau'}\right\},\\ &= -e^{\beta\Omega}Tr\left\{e^{-(\beta-\tau+\tau')\hat{H}}\,\psi(\hat{x})e^{-\hat{H}(\tau-\tau')}\,\bar{\psi}(\hat{x})\right\}.\end{aligned} \tag{5}$$

So, thermal Green's functions depend on a variable $\tau-\tau'$, which changes from -β to β. Because of the trace cyclicity property, G is a periodic function of $\Delta\tau = \tau-\tau'$ on the whole real axis. Therefore, it can be expanded in a Fourier series,

$$G(\tau) = \frac{1}{\beta}\sum_{-\infty}^{\infty} G(\omega_n)\,e^{-i\omega_n\tau},$$

where the Matsubara frequencies are $\omega_n = \frac{\pi n}{\beta}$. Let us take some $\tau < 0$:

$$\begin{aligned}G(\tau) &= \pm Tr\left\{e^{\beta(\hat{\Omega}-\hat{H})}\,\psi^\dagger\,e^{\hat{H}\tau}\psi\,e^{\hat{H}\tau}\right\},\\ &= \pm e^{\beta\Omega}\,Tr\left\{e^{-\hat{H}(\tau+\beta)}\,\psi^\dagger\,e^{\hat{H}\tau}\psi\right\},\end{aligned}$$

$$\begin{aligned}G(\tau+\beta) &= -Tr\left\{e^{\beta(\hat{\Omega}-\hat{H})}\,e^{\hat{H}(\tau+\beta)}\,\psi\,e^{\hat{H}(\tau+\beta)}\psi^\dagger\right\},\\ &= e^{\beta\Omega}\,Tr\left\{e^{\hat{H}\tau}\,\psi\,e^{-\hat{H}(\tau+\beta)}\psi^\dagger\right\},\\ &= \mp G(\tau).\end{aligned} \tag{6}$$

The negative sign is for Fermi statistics. Thus thermal Green's functions are periodic (for bosons) and anti-periodic (for fermions) with respect to the period β. Therefore, the Matsubara frequencies are,

$$\begin{aligned}\omega_n &= \frac{2\pi n}{\beta},\ \text{for Bosons},\ \ n = 0,\pm1,\cdots\\ \omega_n &= \frac{(2\pi n+1)}{\beta},\ \text{for Fermions},\ \ n = 0,\pm1,\cdots\end{aligned}$$

Appendix 4.2

The Jacobi θ_3 Function and its Properties

The Jacobi θ functions are four basic functions of Jacobi's theory of elliptic functions. They are functions of two complex variables, z and τ, known as the argument and the half period ratio, respectively. It is often convenient to use the quantity $q = e^{\pi i \tau}$. In Chapter IV we used the Jacobi theta function of the third kind: $\theta_3(z, \tau)$ which is defined as,

$$\begin{aligned} \theta(z,\tau) &= \sum_{n=-\infty}^{\infty} e^{\pi i \tau n^2} e^{2\pi i n z}, \\ &= 1 + 2\sum_{n=1}^{\infty} e^{\pi i \tau n^2} \cos(2\pi i n z). \end{aligned} \tag{1}$$

Various properties of $\theta_3(z,\tau)$ are as follows:

- It can be seen from equation (1) that $\theta_3(z,\tau)$ is an even function with respect to z i.e., if we replace z by $-z$ we get the same value because 'cosine' and 'n^2' are even functions.

- $\theta_3(z,\tau)$ is a periodic function with respect to the variable z with period 1,

$$\begin{aligned} \theta_3(z+1,\tau) &= \sum_{n=-\infty}^{\infty} e^{\pi i \tau n^2} e^{2\pi i n (z+1)}, \\ &= \sum_{n=-\infty}^{\infty} e^{\pi i \tau n^2} e^{2\pi i n z} e^{2\pi i n}, \\ &= \theta_3(z,\tau) \; [\text{as}\, e^{2\pi i n}\, \text{is 1 for integer value of n}]. \end{aligned} \tag{2}$$

- $\theta_3(z + q\tau, \tau) = e^{-i\pi\tau q^2} e^{-2\pi i q z} \theta_3(z,\tau)$; this can be seen as,

$$\begin{aligned} \theta_3(z+q\tau,\tau) &= \sum_{n=-\infty}^{\infty} e^{\pi i \tau n^2} e^{2\pi i n (z+q\tau)}, \\ &= \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n^2+2qn)} e^{2\pi i n z}, \end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+q)^2 - i\pi\tau q^2} e^{2\pi i n z},$$

$$= \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+q)^2} e^{-i\pi\tau q^2} e^{2\pi i n z}. \tag{3}$$

Substituting $n + q = N$, (4.3) becomes,

$$\begin{aligned} \theta_3(z + q\tau, \tau) &= e^{-i\pi\tau q^2} \sum_{N=-\infty}^{\infty} e^{i\pi\tau N^2} e^{2\pi i(N-q)z}, \\ &= e^{-i\pi\tau q^2} \sum_{N=-\infty}^{\infty} e^{i\pi\tau N^2} e^{2\pi i(N-q)z}, \\ &= e^{-i\pi\tau q^2} e^{-2\pi i q z} \theta_3(z, \tau). \end{aligned} \tag{4}$$

One of the important Jacobi identities used in Chapter IV is,

$$\theta_3(z, i\tau) = (\tau)^{-\frac{1}{2}} e^{-\frac{\pi z^2}{\tau}} \theta_3\left(\frac{z}{i\tau}, \frac{i}{\tau}\right).$$

For the proof of this identity, we use Poisson's summation formula. Let $f : R \to R$ be an integrable function and let $\hat{f}(\xi) = \int_R e^{-2\pi i \xi x} f(x) dx, \quad \xi \in R$ be its Fourier transform. The Poisson summation formula is the assertion that $\sum_{n\in Z} f(n) = \sum_{n\in Z} \hat{f}(n)$, whenever f is such that both the infinite sums are absolutely convergent. Let

$$f(x) = e^{\pi i \tau x^2} e^{2\pi i x z}. \tag{5}$$

The Fourier transform of $f(x)$ is given by

$$\int_{-\infty}^{\infty} e^{i\pi\tau x^2 + 2\pi i x z} e^{-2\pi\, i x y} dx = (-i\tau)^{-\frac{1}{2}} e^{-i\pi\frac{(z-y)^2}{\tau}}. \tag{6}$$

We apply the Poisson summation formula, as explained above,

$$\begin{aligned} \sum_{-\infty}^{\infty} e^{i\pi\tau n^2 + 2\pi i n z} &= (-i\tau)^{-\frac{1}{2}} \sum_{-\infty}^{\infty} e^{-i\pi\frac{(z-n)^2}{\tau}}, \\ &= (-i\tau)^{-\frac{1}{2}} e^{-i\pi\frac{z^2}{\tau}} \theta_3\left(\frac{z}{\tau}, -\frac{1}{\tau}\right). \end{aligned}$$

Substituting $\tau \to i\tau$, we get

$$\theta_3(z, i\tau) = (\tau)^{-\frac{1}{2}} e^{-\frac{\pi z^2}{\tau}} \theta_3\left(\frac{z}{i\tau}, \frac{i}{\tau}\right). \tag{7}$$

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