Acyclic Edge Colouring: Bounds and Algorithms

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and the work has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution or University.

Rahul Muthu

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Abstract

An acyclic edge colouring of a graph is an assignment of colours to its edges in such a way that incident edges get distinct colours, and the edges of any cycle use at least three distinct colours. The latter condition is equivalent to the requirement that the subgraph induced by any pair of colour classes (set of edges receiving the same colour) is a forest. The minimum number of colours sufficient to acyclically colour the edges of a graph G, is called its acyclic chromatic index and is denoted by $a'(G)$.

In this thesis we study a problem called the *acyclic edge colouring* of graphs. For brevity, we often use the phrase "acyclic colouring", instead of "acyclic edge colouring". This notion is a more restricted version of the standard edge colouring notion, which in turn is the same as the standard vertex colouring notion on line graphs. This thesis studies this notion in the context of two different aims. The first aim is to get tight estimates on the minimum number of colours sufficient to achieve such colourings for any graph. The second is to actually produce such colourings using as few colours as possible.

The acyclic colouring problem can thus be viewed from a combinatorial perspective and also from an algorithmic perspective. It is interesting and challenging with respect to both perspectives. It is noteworthy that no good estimates on $a'(G)$ have been obtained for even highly structured classes like the complete graphs, or restricted families like bipartite graphs. From the algorithmic point of view it is NP-Hard to determine $a'(G)$ in general, and even when restricted to subcubic, 2-degenerate graphs. Its close relationship to standard vertex colourings indicates that it could be useful in modelling and solving problems involving conflict-free scheduling of activities. The acyclic edge colouring problem is also closely related to star and oriented colourings of line graphs which have applications in protocols for mobile communication.

In this thesis, we have contributed to improving the understanding of the acyclic colouring problem from a combinatorial as well as an algorithmic perspective. Specifically we have obtained the following results.

- We have improved the previously best known upper bound on $a'(G)$ for all graphs (16Δ) to 5.91Δ for graphs of girth at least 9. We get a further improvement to 4.52Δ when we restrict our attention to graphs of girth at least 220. Here, $\Delta(G)$ represents the maximum degree of the graph.

(This result which is joint work, forms a part of the Ph.D. thesis of my collaborator N. Narayanan, also.)

- We obtain a general relationship between the girth, $g(G)$ of a graph G and its acyclic chromatic index, which gives progressively better upper bounds on $a'(G)$, as the girth, $g(G)$ of the graph increases.

- We obtain exact estimates of $a'(G)$ for the well known graph classes: hypercubes, grids and tori. Also, we show that for each partial torus (a generalisation of grids and tori), its $a'(G)$ is always either Δ or $\Delta + 1$. Our proof also suggests an efficient algorithm to produce such colourings for these graphs.

- We also prove that $a'(G) \leq a'(G_1) + \cdots + a'(G_k)$ where $G_1, \ldots, G_k$ are the prime factors of G (with respect to cartesian product factorisation), provided $a'(G_i) > 1$, for some i. This generalises and extends the results we obtain for partial tori, mentioned above. The graph G can be efficiently coloured using $a'(G_1) + \ldots + a'(G_k)$ colours, provided we have acyclic colourings of each G_i using a disjoint set of $a'(G_i)$ colours.

- We prove that for the class of partial 2-trees, $a'(G) \leq \Delta + 1$. Our proof also yields a polynomial time algorithm for constructing an acyclic colouring of any partial 2-tree using $\Delta + 1$ colours.

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Chapter 1

Introduction

Many problems of practical importance can be studied by modelling them in a graph theoretic framework. Deciding the locations of retail outlets to meet the conflicting constraints of proximity to customers and minimising the total investment in setting-up costs is one such problem. Scheduling the repair of roads in a city while ensuring that important city centres are still accessible is another. Routing a cleaning vehicle through the roads of a city minimising the travel distance is another problem which can be solved using graph theoretic techniques. Graph theory is vast and each of these problems are modelled and solved using different properties of graphs. For an excellent introduction to graph theory and its applications see [Wes96],[Har69],[Rob93].

Graph colouring is an important problem in graph theory and combinatorics as well. It arose famously in the context of the *four colour theorem*, which states that the regions of a map can be coloured using four colours so that regions sharing a boundary get distinct colours. A proper colouring of a graph assigns colours to its vertices in such a way that no pair of adjacent vertices get the same colour. Equivalently, this may be viewed as an assignment of colours to vertices such that the set of vertices receiving the same colour is an independent set. When viewed in this way we can speak of a proper colouring as a partition of the vertex set into independent sets. In general, graph colouring can be thought of as a partition of the elements of a graph (vertices or edges), with the requirement that the partition satisfies specified constraints.

Standard vertex colouring is used to schedule activities in a conflict-free manner. Suppose we need to schedule meetings of several committees at various time slots. We model it by a graph with one vertex for each committee and an edge between two vertices if the corresponding committees have a common member. A proper vertex colouring of this graph, indicates a schedule where no two committees with a common member are scheduled a meeting at the same time. Scheduling the allocation of processor registers to program variables by a compiler, to enhance performance, is a very important real-time application of proper vertex colourings. Proper edge colouring can be used in a similar fashion to schedule matches between teams in a sports league, so that, no team is assigned two matches to be played at the same time. Colouring problems are good tools for modelling scheduling problems of various types. In this thesis we consider an important variant of graph colouring called acyclic edge colouring. More details are given in Section 1.2.

1.1 Definitions and notations

Here, we present some definitions and notations which we use throughout the thesis. We also state some well known results. Where we have used a notation, concept or result not stated here, it can be taken to have its standard meaning in widespread use. We consider only finite, simple, undirected graphs $G = (V, E)$, where V is called the set of *vertices* and E is called the set of *edges*. Throughout, we use n to denote $|V|$. Each edge is a 2-element subset of V. Sometimes, we use the short notation (u, v) to denote $\{u, v\} \in E$. If $v \in e$, where $e \in E$, then we say that v is an *endpoint* of e. Two vertices v_1 and v_2 are said to be *adjacent* (or *neighbours*) if $(v_1, v_2) \in E$. A vertex v and an edge e are said to be *incident* to each other if $v \in e$. Two edges are said to be *incident* to each other if they share an endpoint.

A *path* in G is a sequence of vertices $v_0, v_1, \ldots, v_{k-1}, v_k$, such that the $v_i's$ are all distinct and for $0 \leq i < k, (v_i, v_{i+1}) \in E$. The length of a path is the number of edges it contains. A path starting at a vertex u and ending at a vertex v is called a u-v path. We denote a path on n vertices by P_n. G is *connected* if, for every pair of vertices u, v in V, there is a u-v path.

A *cycle* in G is a sequence of vertices $v_0, v_1, \ldots, v_{k-1}, v_k$, such that $v_0 = v_k$, $\{v_0, \ldots, v_{k-1}\}$ are all distinct and for $0 \leq i < k$, $(v_i, v_{i+1}) \in E$. The length of a cycle is the number of edges it contains. We use C_n to denote a cycle on n vertices. The length of a shortest cycle and a longest cycle in a graph are called its girth and circumference respectively. We will use girth frequently and for a graph G we denote it by $g(G)$. G is 2-*connected* if, for every pair of vertices u, v in V, there is cycle which passes through both u and v.

In general, the *connectivity* of a simple graph, is the smallest number of vertices which need to be deleted to disconnect the graph. Analogously, the *edge connectivity* of a graph is the minimum number of edges which need to be deleted to disconnect the graph.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We say G_2 is a *subgraph* of G_1 if $V_2 \subseteq V_1$ and $E_2 \subseteq E_1$. If, in addition, for each $v_1, v_2 \in V_2$, $(v_1, v_2) \in E_2$ iff $(v_1, v_2) \in E_1$, we say G_2 is an *induced subgraph* of G_1. The subgraph of a graph $G = (V, E)$ induced by the vertex set V' is denoted by $G[V']$. A pair of graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same set of vertices V are *complementary* if for all $u, v \in V$, $(u, v) \in E_1$ iff $(u, v) \notin E_2$. The complement of a graph G is denoted by $\overline{G}$. $G = (V, E)$ is said to be a *complete graph* if for each $u, v \in V$, $(u, v) \in E$. The complete graph on n vertices is denoted by K_n. A graph and its complement constitute a partition of the edge set of the complete graph. A maximal connected subgraph of a graph is called a *component* of the graph. Similarly, a maximal 2-connected subgraph is called a *block* of the graph.

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* to each other if there exists a bijective function $f : V_1 \to V_2$ such that $(u, v) \in E_1$ iff $(f(u), f(v)) \in E_2$. The function f is said to be an *isomorphism* between the graphs. A pair of isomorphic graphs have identical structural properties.

A graph $G = (V, E)$ is said to be bipartite if V can be partitioned into two non-empty parts V_1 and V_2 (called *partite sets*) such that each edge $e \in E$ joins a vertex of V_1 to a vertex of V_2. The concept of bipartite graphs can be extended naturally to r-partite graphs for each $r \geq 2$. A complete bipartite graph is a bipartite graph in which there is an edge joining each pair of vertices lying in different partite sets. A complete bipartite graph with partite sets of size m and n is denoted by $K_{m,n}$.

The *degree* of a vertex v in G, denoted by $d_G(v)$, is the number of edges incident to v. If G is clear from the context, we use the shorter notations d_v or $d(v)$. The *maximum degree* of a graph G, denoted by Δ_G or $\Delta(G)$, is the maximum degree of any vertex in G. Similarly, the *minimum degree* of a graph G, denoted by δ_G *or* $\delta(G)$, is the minimum degree of any vertex in G. The average degree of a vertex in G, denoted by $d_{avg}(G)$, is defined to be $2|E(G)|/|V(G)|$. The symbol G is often dropped in these notations when the graph under consideration is clear from the context.

A subset $M \subseteq E$ is a matching if no pair of edges in M are incident to each other and its size is the number of edges in it. A *maximal matching* of a graph G is a matching of G which is not contained in any other matching of G. A *maximum matching* of G is a matching of maximum cardinality. The size of a maximum matching of G is an important parameter. It is denoted by $\alpha'(G)$.

The parameters $\Delta(G)$, $g(G)$ and $\alpha'(G)$ have a strong bearing on acyclic edge colouring and the acyclic chromatic index which is the focus of this thesis.

A *clique* in a graph G is a complete subgraph of G. A *maximal clique* K of a graph G is one which is not contained within any other clique of G. The *size* of a clique of G is the number of vertices it contains; a clique on k vertices is also called a k-clique. A *maximum clique* of G is a clique whose size is at least as large as the size of any other clique of G. The *clique number* of G, denoted $\omega(G)$, is the size of a maximum clique in G. Similarly, an *independent set* I in G is $I \subseteq V$ such that there is no edge in G joining two vertices in I. The *independence number* of G, denoted $\alpha(G)$, is the number of vertices in a maximum independent set of G.

1.1.1 Graph colourings

First, we define vertex and edge colourings.

Definition 1.1.1. A *proper vertex colouring* of a graph G is a function $f : V \rightarrow C$, where C is any finite set of labels (called colours) such that adjacent vertices are mapped to different colours.

Definition 1.1.2. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colours (i.e. $|C|$) sufficient to properly colour the vertices of G.

The set of vertices receiving the same colour in a colouring, is called a *colour class*. Clearly, each colour class forms an independent set. Any proper vertex colouring of a graph, partitions V into independent sets.

It is clear that for any graph G, $\chi(G) \leq \Delta(G)+1$. In fact, there is a very simple linear time algorithm which, given an arbitrary ordering of $V(G)$, iteratively produces a proper colouring of $V(G)$ using at most $\Delta(G)+1$ colours. Also, by Brooks' Theorem ([Bro41]), $\chi(G) \leq \Delta(G)$ for any connected graph G unless G is a complete graph or an odd cycle.

Definition 1.1.3. A *proper edge colouring* of a graph G is a function $f : E \to C$, where C is any finite set of colours such that incident edges are mapped to different colours.

Definition 1.1.4. The *chromatic index* of a graph G, denoted by $\chi'(G)$, is the minimum number of colours (i.e. $|C|$) sufficient to properly edge colour G.

The set of edges receiving the same colour in an edge colouring, is called a colour class. It follows that the edges in any colour class form a matching. We have $\chi'(G) \geq |E(G)|/\alpha'(G)$. Any proper edge colouring of a graph partitions its edge set into matchings.

The study of edge parameters of a graph can be thought of as the study of the corresponding vertex parameters of an associated graph, called its *line graph*. The line graphs are defined below.

Definition 1.1.5. The *line graph*, denoted by $L(G) = (V', E')$, of a graph $G = (V, E)$ is the graph in which $V' = E$ and $(v_1', v_2') \in E'$ if v_1' and v_2' are incident to each other in G.

The class of line graphs has been well characterised and there are very efficient algorithms [Kra43] to determine whether a given input graph is a line graph.

In the light of this definition, it is clear that a proper edge colouring of any graph G is a proper vertex colouring of $L(G)$ and *vice versa*. Thus, $\chi'(G) = \chi(L(G))$. This relationship *does not*, however, hold for all problems. For example, an *eulerian tour* of G is a *hamiltonian cycle* in $L(G)$, but the converse is not true. Since incident edges must receive different colours, it is clear that $\chi'(G) \geq \Delta(G)$. Since $\Delta(L(G)) \leq 2\Delta(G) - 2$, it follows that $\chi(L(G)) \leq 2\Delta(G) - 1$. This trivial upper bound on the chromatic index was significantly improved by Vizing [Viz64] and his result is stated in the theorem below.

Theorem (Vizing)**.** *For any graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

Vizing's proof is constructive and the resulting algorithm is an efficient one for producing an edge colouring of any graph G using at most $\Delta(G) + 1$ colours. The running time of a straightforward implementation of Vizing's idea is $O(n^4)$. There have subsequently been several improvements and refined algorithms. See [GNK$^+$85] for better results.

We now define notions central to the problem which is addressed in this thesis.

Definition 1.1.6. An *acyclic edge colouring* of a graph $G = (V, E)$ is a proper edge colouring of G such that there is no two coloured cycle. Equivalently, the subgraph induced by any pair of colour classes is a forest.

Definition 1.1.7. The *acyclic chromatic index* (or *acyclic edge chromatic number*) of a graph G, denoted by $a'(G)$, is the least number of colours sufficient to acyclically colour the edges of G.

The analogous notions with vertices are also defined and studied (see [AMR91], [FGR03], [Gru73], [Sku04]). An acyclic vertex colouring of a graph using k colours, is a proper colouring of its vertices such that every cycle uses at least three colours. The acyclic chromatic number, $a(G)$, is the smallest k for which such a colouring exists.

In this thesis, we study the acyclic edge colouring problem which is the same as the acyclic vertex colouring problem restricted to the class of line graphs. In Section 1.2, we look at some known trivial and non-trivial bounds on $a'(G)$.

1.1.2 Graph classes

In this section, we define the graph classes we study. Some classes of graphs we study are defined on the basis of an operator called the *cartesian product* which is used to generate complicated graphs from simpler ones. We define this operation below as also some of the graphs we consider.

Definition. Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *cartesian product* of G_1 and G_2, denoted by $G_1 \Box G_2$, is defined to be the graph $G = (V, E)$ where $V = V_1 \times V_2$ and E contains the edge joining (u_1, u_2) and (v_1, v_2) if and only if *either* $u_1 = v_1$ and $(u_2, v_2) \in E_2$ *or* $u_2 = v_2$ and $(u_1, v_1) \in E_1$.

This operation is associative and commutative. Thus, the cartesian product operator can be applied to any finite series of graphs in any order, without affecting the resultant graph.

Below are defined some well known classes of graphs definable in terms of cartesian product. We use PATHS and CYCLES to donote the set of all paths and all cycles on three or more vertices respectively. We sometimes use the notation EDGE when talking about the graph P_2.

Definition. A *hypercube* is any graph obtained as the cartesian product of a number of P_2's. Its dimension is the number of P_2's in the product.

Definition. A *grid* or *mesh* is any graph obtained as the product of graphs from PATHS. Its dimension is the number of paths in the product.

Definition. A *torus* is any graph obtained as the cartesian product of graphs from CYCLES. Its dimension is the number of cycles in the product.

The above three types of graphs are all special cases of the class of *partial tori* defined below. We also refer to these graphs as *grid-like* graphs.

Definition. A *d-dimensional partial torus* is a connected graph G, where G is of the form $G \cong G_1 \Box G_2 \cdots \Box G_d$, where $G_i \in$ EDGE $\cup$ PATHS $\cup$ CYCLES for each $i \leq d$.

Definition. A k-tree is any graph obtained from the complete graph K_{k+1}, by a sequence of zero or more operations of adding a new vertex adjacent to the vertices of an existing k-clique in the graph.

Definition. A *partial k-tree* is any subgraph of a k-tree.

1.1.3 Algorithms and complexity

Here, we give a brief introduction to algorithms and the complexity of problems. See [CLR89], for an excellent and comprehensive introduction to the the theory of algorithm design.

We define the efficiency of an algorithm as the number of elementary steps it uses in order to solve a problem instance of a given size. For an algorithm A, we denote by $T_A(n)$ the maximum running time (measured by the number of elementary operations) taken by the algorithm on any input of size n. We compare the relative merits of two algorithms for a problem on the basis of these functions, focussing on the function values asymptotically, as the input size grows arbitrarily large.

We say an algorithm for a problem is efficient if its running time is bounded by some polynomial function of its input size, n. A problem is said to be tractable, if it has an efficient algorithm to solve it and otherwise it is said to be intractable. We also use the notion of NP-hard optimisation problems. For an introduction to this notion, see [GJ79]. An NP-hard optimisation problem is not likely to admit an efficient algorithm for solving it exactly. In that case, we overcome the difficulty by designing an approximation algorithm.

1.2 Acyclic edge colouring

The notions of acyclic vertex and edge colourings were introduced by Grunbaum in [Gru73]. For the edge version, it was conjectured by Alon, Sudakov and Zaks (see [ASZ01]), and independently by Fiamcik (see [Fia78]), that $a'(G) \leq \Delta + 2$ for every G and they also provided examples of graphs (complete graphs on an even number of vertices) requiring $\Delta + 2$ colours in any acyclic edge colouring.

Conjecture 1.2.1. For any graph G, $a'(G) \leq \Delta(G) + 2$.

As we have seen, many graph theoretic problems concerning edges can be viewed as equivalent vertex problems in the corresponding line graph. Solutions to one problem yield solutions to the other and vice versa. However, this statement is not true in general as explained in Section 1.1.1. In the case of acyclic colouring though, the above observation holds. An acyclic vertex colouring of $L(G)$ yields an acyclic edge colouring of G and conversely also, an acyclic edge colouring of G yields an acyclic vertex colouring of $L(G)$. This is true even for the standard vertex and edge colouring problems. The number of colours used in each case is identical.

We obtain bounds on $a'(G)$ in terms of its maximum degree Δ. The reason for this choice is that Δ is a lower bound and there exists a function $f(\Delta)$ such that $a'(G) \leq f(\Delta)$ for all graphs. Here, the function has been shown to be linear. No other parameter better satisfies both the criteria of being easily computable and closely tied to the value. This is also a parameter commonly used to bound the chromatic number and chromatic index of graphs.

We have seen that $\chi(G) \leq \Delta(G)+1$, while $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$ for any graph. However, $\Delta+1$ is not always tight for $\chi(G)$, as seen in the case of bipartite graphs, where $\chi(G) = 2$, while $\Delta(G)$ is unbounded. We infer that in the case of ordinary vertex and edge colourings, we could have $\Delta(G)$ significantly away from $\chi(G)$, but $\Delta(G)$ and $\chi'(G)$ differ by at most 1. A similar phenomenon holds for acyclic colouring as the following bounds indicate. For any graph, $\chi(G) \leq a(G) \leq O(\Delta^{\frac{4}{3}})$, while $a'(G) = \theta(\Delta)$ (see [AMR91]). Thus, for planar graphs $a(G) \leq 5$ always, but $\Delta(G)$ is unbounded. This phenomen of the edge version being better concentrated is observed in list colouring problems also, in addition to the standard proper colouring and several other problems as well.

In view of Conjecture 1.2.1 and the trivial lower bound of $\Delta(G)$ on $a'(G)$, we often use the term *gap* to refer to the absolute difference in value between these two parameters of a graph.

The acyclic edge colouring problem is interesting and challenging because an exact estimate of $a'(G)$ is not yet known even if $G = K_n$, while usually graph invariants are easily determined for complete graphs. This also demonstrates the difficulty inherent in estimating this parameter. Another class of graphs for which there are no good estimates on $a'(G)$ are bipartite graphs. This gives further indication of the difficulty of the problem. A possible explanation for the problem being difficult even on bipartite graphs could be the fact that even cycles are in some sense are the *core* of the problem. Any proper colouring automatically ensures that odd cycles receive at least three colours.

When considering the acyclic colouring problem, we can always assume without loss of generality that the graph is connected, since the independent acyclic colourings of the different components using the same sets of colours yield an acyclic edge colouring of the whole graph. In fact, we often assume the graph is 2-connected, because there can never be a cycle contained in more than one block. Thus independent colourings of each block can be combined by renaming some of the colours and removing conflicts at cut vertices to ensure the colouring is proper. Another property is that Δ-regular graphs ($\Delta > 1$) need at least $\Delta+1$ colours for an acyclic colouring. To see this, observe that in any proper colouring using exactly Δ colours, each colour class is a perfect matching and hence, the union of any two colour classes is a 2-regular graph, which is always a collection of vertex disjoint cycles.

From the point of view of applicability, the acyclic edge colouring problem is closely related to standard versions of vertex colouring and edge colouring which are used to model conflict-free scheduling of stand-alone activities and interactions respectively. Acyclic edge colouring can be used in conflict-free scheduling of interactions in an environment where cyclic deadlocks may arise. Acyclic chromatic index is also related to star chromatic number and oriented chromatic number (see [KSZ97],[GTMP07]), both of which have applications in protocols for mobile communication. Any improved estimate on the acyclic chromatic index leads to an improvement in estimating the oriented chromatic number of line graphs.

1.3 Other/earlier results

While the acyclic edge colouring problem still provides a lot of scope for research, some work has already been done in the area and interesting results have been obtained. We now give below a survey of some of the prominent results already obtained in this area. In addition to results elaborated below, see also [FGR03],[Bur79],[MR98] for further work on this problem.

See [GP05] and [GGW06] for work on a generalised form of the acyclic colouring problem. In the afore-mentioned works, the authors define the r-*acyclic edge colouring* as a colouring in which any cycle in the graph of length k uses at least $min\{r, k\}$ colours. They define the associated parameter r-*acyclic chromatic index*, $a'_r(G)$ for a graph G as the minimum number of colours sufficient to obtain such a colouring. They obtain bounds on the value for random regular regular graphs which hold asymptotically, and also tight bounds for $a'_r(G)$ for bounded degree graphs. Other types of generalisation should also be possible.

1.3.1 Bounds on acyclic chromatic index

The following theorem (which apears in [AMR91]) is an existential proof that $a'(G)$ is linearly upper bounded by Δ. It gives an asymptotically tight value.

Theorem 1.3.1. *For any graph G, $a'(G) = O(\Delta)$.*

The next is a result confirming Conjecture 1.2.1 for graphs with high girth. The result appears in [ASZ01], and the proof is by the probabilistic method.

Theorem 1.3.2. *There is a constant $c > 0$, such that, if G is a graph with $g(G) \geq c\Delta(G) \log \Delta(G)$, then $a'(G) \leq \Delta(G) + 2$.*

A weaker bound for a larger class of graphs (also appearing in [ASZ01]) is given below.

Theorem 1.3.3. *There is a constant $c > 0$, such that, if G is a graph with $g(G) \geq c \log \Delta$, then $a'(G) \leq 2\Delta + 2$.*

The next result, due to Nesetril and Wormald (see [NW05]), shows the conjecture is true for random regular graphs.

Theorem 1.3.4. *If G is a random d-regular graph on n vertices, then $a'(G) = d+1$ with very high probability. Here, it is assumed that d is fixed but arbitrary while n grows arbitrarily large.*

It is an improvement over an earlier result, where the upper bound was greater by 1, for graphs on an odd number of vertices (see [ASZ01]).

1.3.2 Known algorithmic aspects of acyclic edge colouring

Very few results have been obtained in this area. This is probably because, it is hard to prove bounds which are nearly tight even for highly structured classes of graphs. If we do not impose a structure on the graph, then again we do not expect to get very tight bounds. The result mentioned below is due to Skulrattankulchai (see [Sku04]).

Theorem 1.3.5. *If $\Delta(G) \leq 3$, then $a'(G) \leq 5$. There is a linear time algorithm which produces such a colouring.*

The following proof due to Subramanian, is the best bound obtained, without using the probabilistic method, on $a'(G)$ for all graphs (see [Sub06]). It is a constructive proof which produces such a colouring in polynomial time.

Theorem 1.3.6. *For any graph G, $a'(G) = O(\Delta \log \Delta)$. Also, such a colouring can be obtained in polynomial time. The colouring procedure is based on a greedy heuristic.*

The next result indicates why it might be hard to obtain optimal or almost optimal colourings even for very special classes of graphs (see [AZ02]).

Theorem 1.3.7. *It is NP-hard to determine $a'(G)$, where G is a 2-degenerate subcubic graph.*

This, of course, implies that the problem is NP-hard for general graphs also.

1.4 Main results obtained here

We obtain good estimates on the acyclic chromatic index of various classes of graphs and often efficient algorithms which either produce colourings with the estimated number of colours or a close approximation. The results can be broadly classified into two parts. First we mention bounds which we obtained using random colourings and probabilistic analysis. These results use no structural properties of the graph, except assumptions on its girth. We believe that the bounds obtained are unlikely to be tight to within an additive constant factor. The second set of results give bounds which are very close to the optimum value and apply to structured families of graphs. The proofs rely heavily on structural properties of the graph and also yield efficient algorithms to produce colourings.

1.4.1 Improved upper bounds on $a'(G)$

The following theorem is a result obtained by us, which gives an upper bound on the acyclic chromatic index of graphs whose girths are lower bounded by a specified constant.

Theorem. *For any graph G with girth, $g(G) \geq 9$, the acyclic chromatic index $a'(G) \leq 5.91\Delta(G)$.*

The next theorem gets a better upper bound but the class of graphs to which it is applicable is a strict subset of the class considered above. It applies only to graphs with higher girth. The proofs of these results are quite similar.

Theorem. *For any graph G with girth, $g(G) \geq 220$, the acyclic chromatic index $a'(G) \leq 4.52\Delta(G)$.*

The next result gives a general upper bound on the acyclic chromatic index of a graph as a function of its maximum degree and its girth. As we discussed earlier, the maximum degree of a graph is a very natural parameter, in terms of which its acyclic chromatic index can be bounded. Also, the girth of a graph is a measure of how easy it is to colour it acyclically with a fixed set of colours. The higher its girth the easier it seems to colour the graph, especially when we adopt a random colouring procedure. This result is interesting because it gets an upper bound in terms of these two crucial parameters.

Theorem. *There are absolute constants c_1, $c_2 > 0$ such that, for any G with $g \geq c_1 \log \Delta$ we have,*

$$a'(G) \leq \Delta + 1 + \left\lceil c_2 \left(\frac{\Delta \log \Delta}{g} \right) \right\rceil$$

All these results appear in [MNS05],[MNS07b].

1.4.2 Algorithmic bounds

Here, we state the results we obtained using constructive techniques.

Theorem. *Let G be a simple graph with $a'(G) = \eta$. Then,*

1. $a'(G \square P_2) \leq \eta + 1$, *if* $\eta \geq 2$.
2. $a'(G \square P_l) \leq \eta + 2$, *if* $\eta \geq 2$ *and* $l \geq 3$.
3. $a'(G \square C_l) \leq \eta + 2$, *if* $\eta > 2$ *and* $l \geq 3$.

The next two theorems follow as corollaries and deal with more concretely defined classes of graphs.

Theorem. *The following is true for each* $d \geq 1$.

- $a'(G) = \Delta(G) + 1 = d + 1$, *if G is a hypercube of dimension* $d \geq 2$; $a'(P_2) = 1$.
- $a'(G) = \Delta(G) = 2d$, *if G is a grid of dimension d.*
- $a'(G) = \Delta(G) + 1 = 2d + 1$, *if G is a torus of dimension d.*

Theorem. *When G has factors from at least 2 of the classes* CYCLES, PATHS *and* EDGES, $a'(G) \in \{\Delta(G), \Delta(G) + 1\}$, *and its exact value depends on the specific combination of factors.*

The results of the preceeding two theorems are constructive leading to efficient algorithms for obtaining almost optimal acyclic edge colourings. These theorems are stated more precisely in Chapter 3, where they are considered in greater detail. The preceeding three results appear in [MNS06].

The next theorem (see [MS07]) gives a slightly weaker bound on $a'(G)$ than the previous theorems, but it covers a far wider class of graphs. It relates $a'(G)$ and the cartesian product operation in a general setting.

Theorem. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two connected non-trivial graphs such that* $max\{a'(G), a'(H)\} > 1$. *Then,*

$$a'(G \square H) \leq a'(G) + a'(H)$$

The following result (see [MNS07a]) on acyclic edge colouring was also obtained by us, but forms a part of the Ph.D. thesis of my collaborator N. Narayanan.

Theorem 1.4.1. *If G is an outerplanar graph then $a'(G) \leq \Delta(G) + 1$. A colouring using this many colours can be obtained in $O(n \log \Delta)$ time.*

The next result states good bounds obtained by us on $a'(G)$ for the class of partial 2-trees (see [MNS08]). In this case also, the results translate directly into efficient algorithms which produce acyclic edge colourings using the claimed number of colours.

Theorem. *If G is a partial 2-tree then $a'(G) \leq \Delta(G) + 1$.*

1.5 Thesis outline

In Chapter 2, we present the upper bounds on $a'(G)$ for graphs with high girth. We also present therein, a result connecting acyclic chromatic index to girth and maximum degree. Chapter 3 gives an exposition of acyclic colouring of partial tori. The acyclic colouring of the cartesian product of arbitrary graphs is presented in Chapter 4. In Chapter 5, we study acyclic colouring of partial 2-trees. Chapter 6 contains some concluding remarks and outlines possible future directions for research.

Chapter 2

Acyclic edge colouring of high girth graphs

In this chapter we present an upper bound on the acyclic chromatic index, $a'(G)$, for all graphs with girth, $g(G)$, greater than a fixed constant. The proof is by probabilistic arguments and improves the estimates on this parameter for a fairly large class of graphs. We also obtain a general relationship between the girth of a graph and an upper bound on its acyclic chromatic index. Some of the ideas in the proof are similar to those used in earlier work by others, while other ideas are new.

After introducing the idea of attacking the acyclic edge colouring problem in Section 2.1 we introduce the necessary probabilistic tools in Section 2.2. A brief survey of other work on acyclic colouring using the probabilistic method is presented in Section 2.3. Section 2.4 contains our results and the proofs. We indicate open problems and present some conclusions in Section 2.5.

2.1 Introduction

The probabilistic method is a powerful tool to prove the existence of combinatorial structures having some desired property. It was pioneered by Erdös who applied it to various combinatorial problems (see [AS00]). It provides a proof of the existence of such structures using probabilistic arguments. Also, it is sometimes possible to translate the existence proof into an efficient algorithm which is significantly faster than the brute-force approach of enumerating all possibilities.

In the context of acyclic edge colouring, it has been shown, originally by Alon, McDiarmid and Reed, that the number of colours sufficient is $O(\Delta)$ using the probabilistic method (see [AMR91],[MR98]), but to design an efficient algorithm which provably produces a colouring using $o(\Delta^2)$ colours is non-trivial. Note that any proper edge colouring of G which requires that each path P on three edges should use three colours is also an acyclic edge colouring. It is easy to see that such a colouring can be obtained using $2\Delta^2$ colours. This can be seen by noting that each such colouring is a proper vertex colouring of the square of its line graph $L(G)$, namely $L(G)^2$, and that $\Delta(L(G)^2) \leq 2\Delta(G)^2$. This colouring is also efficiently constructible. The best, and so far the only constructive bound which is an improvement over this trivial bound is an algorithm, due to Subramanian, which produces a colouring using $O(\Delta \log \Delta)$ colours in polynomial time (see [Sub06]).

Thus, there is a large gap between what can be shown existentially and what can be constructed systematically. Even if we ignore the issue of algorithms, the scenario is bleak. There is no combinatorial or deterministic proof that $a'(G) = O(\Delta)$, to match the bound proved using recourse to randomness. Most constructive results on acyclic edge colouring have been obtained for highly structured classes of graphs where the extra structural information and properties have been used as a handle to obtain the solution.

2.2 Tools and methods

For an introduction to probability theory and tools of the probabilistic method we refer the reader to [Fel66],[AS00],[MR02].

The basic approach to proving bounds on the acyclic chromatic index, using the probabilistic method, is to randomly colour the edges of the graph with the specified number of colours and show that the colouring thus obtained is an acyclic colouring with positive probability. Modification of this basic approach adds sophistication and, sometimes, produces improvements. One possiblity is to combine the probabilistic method with a deterministic method. In this approach one might relax the requirements of the colouring produced by the random colouring and handle the defect by a deterministic method. On the other hand one may produce a colouring with relaxed constraints using a deterministic method and rectify it using a random procedure. It is also possible to interleave these methods an arbitrary number of times. Iterative random experiments have also provided breakthroughs when a one pass method has failed in some problems.

Let us consider the basic approach mentioned above and colour each edge uniformly at random and independently from a fixed set of colours. If we require that the random colouring be proper, we observe that long cycles are less likely to receive exactly two colours than their shorter counterparts. In fact, this statement holds even if we allow the random colouring to be improper. It follows that any attempt to prove a bound by this method is more likely to succeed if the class of graphs considered are assumed to have high girth. This is reflected by the bounds obtained on $a'(G)$ using the probabilistic method.

Below we state some of the tools we use in proving the results in this chapter.

Lemma 2.2.1 (The Probabilistic Method)**.** Let $(\Omega, p : \Omega \to [0,1])$ be a finite probability space. Let $\omega \in \Omega$ be a random point chosen from this space. Let $\mathcal{P} \subseteq \Omega$ be an arbitrary property. Then if $Prob(\omega \in \mathcal{P}) > 0$, then there exists a point $\omega \in \Omega$ such that $\omega \in \mathcal{P}$.

The next lemma is a specialised tool which is a powerful weapon in the arsenal of the probabilistic method. It helps in showing that with positive probability, none of a collection of bad events occur. Suppose $A_1, \ldots, A_n$ is a collection of bad events which we want to avoid and which are such that each A_i fails to occur with positive probability. If these are mutually independent, then it follows that with positive probability none of them occurs. The following tool is very powerful because it is applicable even in scenarios when there is limited interdependence between the events.

Lemma 2.2.2 (Lovaśz Local Lemma)**.** Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a set of events defined over a probability space. For each i, let $N_i \subseteq \mathcal{A}$ be such that A_i is mutually independent of all events in $\mathcal{A} \setminus N_i$. Suppose that for each event A_i, there exists a real $0 < x_i < 1$, such that $Pr(A_i) \leq x_i \prod_{j:A_j \in N_i}(1 - x_j)$. Then, $Pr(\bigwedge_{i=1}^{n} \overline{A_i}) > 0$.

Given below is a simpler form of the local lemma, applicable under some assumptions.

Lemma 2.2.3 (Symmetric form of the local lemma)**.** Let $A_1, \ldots, A_n$ be a set of events defined over a probability space. Suppose for some $p \in (0,1)$ and some integer $d \geq 0$, that for each i, $Pr(A_i) \leq p$, and also A_i is mutually independent of all but at most d other events A_j. If $ep(d+1) < 1$, then $Pr(\bigwedge_{i=1}^{n} \overline{A_i}) > 0$.

Typically, in proofs using various forms of the local lemma, we define a set of bad events over a suitable probability space, such that their non-occurrence guarantees the property we are trying to prove. We then use the local lemma to show that with positive probability none of these events occur. It follows that the property we are trying to prove holds with positive probability. The probability of a sample point meeting the specifications, however, might be very small so that an efficient randomised algorithm does not normally manifest itself.

2.3 Related results

In this section, we outline related results obtained by other researchers. That $a'(G) = O(\Delta^2)$ can be seen by observing that any colouring in which incident edges as well as edges at distance two get distinct colours is an acyclic colouring. There are $O(\Delta^2)$ edges within distance at most two from any given edge. Thus a simple greedy procedure produces a colouring with $O(\Delta^2)$ colours. A linear $O(\Delta)$ upper bound for all graphs was proved using a probabilistic method. Randomness has also been successfully utilised to prove a bound tight within an additive factor of 2, for all graphs with suitably high girth in terms of the maximum degree.

Theorem 2.3.1. *For any graph G, $a'(G) \leq 64\Delta$.*

This result (appearing in [AMR91]) was later improved to 16Δ, by Molloy and Reed (see [MR98]), using essentially the same experiment and analysis, but making more careful calculations. It can be shown that even with arbitrarily precise calculation, this experiment cannot yield a bound significantly better than 12Δ. The problem is that, using the probabilistic arguments of [AMR91] and [MR98], one requires almost 12Δ colours even for proper colouring alone without acyclicity. Consider the random experiment, used there, where each edge chooses a colour uniformly at random and independently from a set of $C = a\Delta$ colours for some $a > 1$. Even if we ignore acyclicity and forbid only the events corresponding to pairs of incident edges receiving the same colour, we can easily verify that any proof (based on Lovász Local Lemma with the same constant for all events) ensuring properness of the random colouring with positive probability, requires that $a \geq 4e$.

We define a bad event as a pair of incident edges receiving the same colour. The probability of this event is $\frac{1}{C} = \frac{1}{a\Delta}$. The number of other events on which any of these events depend is at most $2 \times 2(\Delta - 1) \leq 4\Delta$. If we take the associated constant of the local lemma (note that we follow their specification, that the same constant is used for all events) to be $\frac{1}{\alpha\Delta}$, the inequality which we get is

$$\frac{1}{a\Delta} \leq \frac{1}{\alpha\Delta}\left(1 - \frac{1}{\alpha\Delta}\right)^{4\Delta}$$

A solution to this in the valid range of α does not yield a value of $a < 4e$. The above arguments are based on the local lemma in its most general form. Thus, the proof of $a'(G) \leq 9\Delta$ given in [MR02], based on applying a specialized version of Lovász Local Lemma, cannot be correct. For further details on these arguments, see Appendix A of our paper [MNS07b].

However, if we bring in girth assumptions, we get better bounds.

Theorem 2.3.2. $\exists c > 0$*, such that* $\forall G$ *with girth,* $g(G) \geq c\Delta \log \Delta$*,* $a'(G) \leq \Delta + 2$*.*

This result, proved in [ASZ01] using the probabilistic method, reflects the fact that short cycles give rise to difficulties when getting good bounds on $a'(G)$. The next result, also appearing in [ASZ01], is similar.

Theorem 2.3.3. $\exists c > 0$*, such that* $\forall G$ *with girth,* $g(G) \geq c \log \Delta$*,* $a'(G) \leq 2\Delta + 2$*.*

As for constructive bounds, the following result was obtained in [Sub06].

Theorem 2.3.4. *For any graph* G*,* $a'(G) = O(\Delta \log \Delta)$*. There is a deterministic algorithm running in time* $O(mn\Delta^2(\log \Delta)^2)$ *that produces such a colouring with these many colours.*

This result is the first, and so far the only, $o(\Delta^2)$ upper bound for all graphs, proved by a constructive method. Also, tight bounds have been obtained for random regular graphs by Alon et. al. in [ASZ01], and Nesetril and Wormald in [NW05]. The following result is from [NW05].

Theorem 2.3.5. *If* G *is a random* d*-regular graph on* n *vertices (*d *is fixed,* $n \to \infty$*), then* $Pr\left(a'(G) = (\Delta + 1)\right) \to 1$*.*

There is no explicit assumption here on the girth of the graphs. In fact, in the case of random d-regular graphs, there are short cycles but there is a guarantee that any pair of short cycles are separted by a long path.

2.4 Our results

In this section we present the results we obtained using the probabilistic method. Like the earlier results, some assumptions are made about the girth of the graphs. There follows an elaborate description of the proofs of the results.

Theorem 2.4.1. *There are absolute constants $c_1, c_2 > 0$ such that, for any graph G with $g \geq c_1 \log \Delta$ we have,*

$$a'(G) \leq \Delta + 1 + \left\lceil c_2 \left(\frac{\Delta \log \Delta}{g} \right) \right\rceil$$

The above result generalises Theorems 2.3.2 and 2.3.3. The following results give bounds on $a'(G)$ for graphs with some assumptions on their girth.

Theorem 2.4.2. *For any graph G with girth, $g(G) \geq 9$, the acyclic chromatic index $a'(G) \leq 5.91\Delta(G)$.*

Theorem 2.4.3. *For any graph G with girth, $g(G) \geq 220$, the acyclic chromatic index $a'(G) \leq 4.52\Delta(G)$.*

First, we present the proofs of Theorems 2.4.2 and 2.4.3 in Section 2.4.1. After this, we present the proof of Theorem 2.4.1 in Section 2.4.2.

2.4.1 Proofs of improved upper bounds

We give a combined proof of Theorems 2.4.2 and 2.4.3 here. The colourings, that constitute the proofs, are obtained by producing improper random colourings satisfying certain constraints, which are then rectified. We define a measure of the improperness, and the two proofs differ only in the value of this measure and are thus similar. In applying Lovàsz Local Lemma we have not optimised the constants. With a more meticulous calculation it might be possible to improve the bound further.

Proof. It is known that, if $\Delta \leq 3$, then $a'(G) \leq \Delta + 2$ (see [Bur79], [Sku04]). Hence we may assume that $\Delta \geq 4$ in our arguments. Our proof consists of two stages. In the first stage, we show, by probabilistic arguments, the existence of a colouring $\mathcal{C}$, using a set C of $c\Delta$ colours (where $c > 1$ is a constant to be fixed later), such that $\mathcal{C}$ satisfies the following properties for some positive integer $\eta \leq 4$.

(i) every vertex has at most η incident edges of any single colour,

(ii) there are no properly two-coloured cycles, and

(iii) there are no monochromatically coloured cycles.

$\eta = 1$ corresponds to the standard proper edge colouring. The earlier results (Theorem 2.3.1 and its improved form in [MR98]), was obtained without recourse to this parameter η. Thus, this is the main new idea which allows us to get a significantly better bound, with a restriction to graphs of girth greater than a fixed constant. In applying the local lemma, the properness condition doesn't allow a bound significantly better than 12Δ, as stated earlier. Thus allowing limited improperness (every colour class is a bounded degree forest, instead of just a matching), and then partitioning the edge set of each forest, we obtain an acyclic colouring with a small multiplicative overhead in the number of colours. Even with the overhead it is much smaller than 12Δ.

Note that in $\mathcal{C}$, each *colour class* (set of edges receiving the same colour) is a forest of maximum degree at most η. In the second stage, we split each colour class into η parts by recolouring the edges of each colour c_i with the colours $c_i^1, \ldots, c_i^\eta$ to get a colouring $\mathcal{C}'$. We claim that $\mathcal{C}'$ is proper and acyclic. Since every forest of maximum degree at most d is properly edge colourable using d colours, it is easy to see that properness holds. Any bichromatic cycle in the colouring $\mathcal{C}'$ should either come from an existing two-coloured cycle in $\mathcal{C}$, or from a monochromatic even length cycle in $\mathcal{C}$ being split into two. Both of these possibilities are forbidden by properties (ii) and (iii) of the colouring, respectievely. It follows that the colouring $\mathcal{C}'$ is proper, acyclic and uses at most $c\eta\Delta$ colours.

To complete the proof, it is now sufficient to show that such a colouring $\mathcal{C}$, described above, exists. We do this probabilistically, using Lovàsz Local Lemma. For this, we do the following random experiment. Each edge chooses a colour uniformly at random and independently, from the set C. For the resulting random colouring to satisfy (i)-(iii) above, define the following three types of *bad* events. As explained below, in the absence of these events, the colouring obtained satisfies the above properties.

1. For a set of $\eta + 1$ edges $\{e_1, \ldots, e_{\eta+1}\}$ incident on a vertex u, let $E_{e_1,\ldots,e_{\eta+1}}$ be the event that all of them receive the same colour. We call this an event of Type I.

2. Let $E_{C,2k}$ denote the event that an even cycle C of length $2k$ is properly coloured with 2 colours. We call this an event of Type II.

3. Let $E_{C,\ell}$ denote the event that a cycle C of length ℓ is coloured monochromatically. We call this an event of Type III.

Suppose $\mathcal{C}$ be such that none of the above events hold. We claim that properties (i)-(iii) above are satisfied. It is easy to see that the absence of events of type I implies that (i) holds. Similarly, absence of type II, III events, respectively imply (ii) and (iii).

In order to apply the local lemma, we need estimates for the probabilities of each event, and also for the number of other events of each type which can possibly influence any given event. For the above random experiment, an event $\mathcal{E}$ is mutually independent of a set $\mathcal{B}$ of other events if the set of edges on which $\mathcal{E}$ depends is disjoint from the set of edges on which the events in $\mathcal{B}$ depend. Hence, we calculate the the number of events of each type that depend on a given edge, and multiply by the number of edges defining the event $\mathcal{E}$ to get an upper bound on the number of events influencing $\mathcal{E}$. The following two lemmas present the estimated bounds.

The proof of Lemma 2.4.1 is straightforward. It is based on the elementary fact that the probability of an event in a probability space is the ratio of the number of favourable cases to the total number of cases. This, combined with a bit of simplification of the expressions obtained give us the claimed values. Lemma 2.4.2 is proved by computing the number of subgraphs of a fixed type to which any given edge in the input graph belongs. As we stated earlier, we use the number of events of any type influenced by an edge to estimate the interdependence. These values are estimated using nothing more than the fact that the graph has maximum degree Δ and therefore the upper bounds obtained apply universally to all the graphs.

Lemma 2.4.1. The probabilities of events are as follows:

1. For each event $E_{e_1,\ldots,e_{\eta+1}}$ of type I, $Pr(E_{e_1,\ldots,e_{\eta+1}}) = \frac{1}{|C|^{\eta}}$.

2. For each event $E_{C,2k}$ of type II, where the length of C is $2k$, $Pr(E_{C,2k}) \leq \frac{1}{|C|^{2k-2}}$

3. For each event $E_{C,\ell}$ of type III, where C is of length ℓ, $Pr(E_{C,\ell}) = \frac{1}{|C|^{\ell-1}}$.

Lemma 2.4.2. The following is true for any given edge e:

1. Less than $\frac{2\Delta^{\eta}}{\eta!}$ events of type I depend on e.

2. Less than Δ^{2k-2} events of type II depend on e.

3. Less than $\Delta^{\ell-2}$ events of type III depend on e.

In order to apply Lovász Local Lemma, let $x_0 = 1/(\alpha\Delta)^\eta$, $x_k = 1/(\beta\Delta)^{2k-2}$ and $y_\ell = 1/(\gamma\Delta)^{\ell-1}$, be the values associated with events of Types I, II and III respectively, where α,β,$\gamma > 1$ are constants to be determined by calculation. Recall that we use g to denote girth. We conclude that, with positive probability none of the above events occur, provided $\forall k \geq \lceil \frac{g}{2} \rceil, \ell \geq g$

$$\frac{1}{(c\Delta)^\eta} \leq x_0 \left(1 - x_0\right)^{(\eta+1)\frac{2\Delta^\eta}{\eta!}} \prod_{\theta \geq \lceil \frac{g}{2} \rceil} \left(1 - x_\theta\right)^{(\eta+1)\Delta^{2\theta-2}} \prod_{\lambda \geq g} \left(1 - y_\lambda\right)^{(\eta+1)\Delta^{\lambda-2}}$$

$$\frac{1}{(c\Delta)^{2k-2}} \leq x_k \left(1 - x_0\right)^{2k\frac{2\Delta^\eta}{\eta!}} \prod_{\theta \geq \lceil \frac{g}{2} \rceil} \left(1 - x_\theta\right)^{2k\Delta^{2\theta-2}} \prod_{\lambda \geq g} \left(1 - y_\lambda\right)^{2k\Delta^{\lambda-2}}$$

$$\frac{1}{(c\Delta)^{\ell-1}} \leq y_\ell \left(1 - x_0\right)^{\ell\frac{2\Delta^\eta}{\eta!}} \prod_{\theta \geq \lceil \frac{g}{2} \rceil} \left(1 - x_\theta\right)^{\ell\Delta^{2\theta-2}} \prod_{\lambda \geq g} \left(1 - y_\lambda\right)^{\ell\Delta^{\lambda-2}}$$

Let $f(z) = (1 - \frac{1}{z})^z$. It is well-known that $(1 - \frac{1}{z})^z \uparrow \frac{1}{e}$. Defining

$$\Lambda = \min\left\{ f({x_0}^{-1}),\ \min_{\theta \geq \lceil \frac{g}{2} \rceil} f({x_\theta}^{-1}),\ \min_{\lambda \geq g} f({y_\lambda}^{-1}) \right\},\ \text{it follows that}$$

$$\left(1 - x_0\right)^{\frac{2\Delta^\eta}{\eta!}} = \left(1 - \frac{1}{(\alpha\Delta)^\eta}\right)^{\frac{2\Delta^\eta}{\eta!}} = \left(\left(1 - \frac{1}{(\alpha\Delta)^\eta}\right)^{(\alpha\Delta)^\eta}\right)^{\frac{2}{\eta!\alpha^\eta}} \geq \Lambda^{\frac{2}{\eta!\alpha^\eta}}.$$

Similarly,

$$\prod_{\theta \geq \lceil \frac{g}{2} \rceil} \left(1 - x_\theta\right)^{\Delta^{2\theta-2}} = \prod_{\theta \geq \lceil \frac{g}{2} \rceil} \left(1 - \frac{1}{(\beta\Delta)^{2\theta-2}}\right)^{\Delta^{2\theta-2}} \geq \prod_{\theta \geq \lceil \frac{g}{2} \rceil} \Lambda^{\beta^{-(2\theta-2)}} \geq \Lambda^{S_1}$$

where

$$S_1 = \sum_{\theta \geq \lceil \frac{g}{2} \rceil} \frac{1}{\beta^{2\theta-2}} \leq \frac{1}{(\beta^2 - 1)\beta^{2\lceil \frac{g}{2} \rceil - 4}}, \quad \text{and}$$

$$\prod_{\lambda \geq g} \left(1 - y_\lambda\right)^{\Delta^{\lambda-2}} = \prod_{\lambda \geq g} \left(1 - \frac{1}{(\gamma\Delta)^{\lambda-1}}\right)^{\Delta^{\lambda-2}} \geq \prod_{\lambda \geq g} \Lambda^{\gamma^{-(\lambda-1)}/\Delta} \geq \Lambda^{S_2}$$

where

$$S_2 = \sum_{\lambda \geq g} \frac{1}{\Delta \gamma^{\lambda-1}} \leq \frac{1}{\Delta \gamma^{g-2}(\gamma-1)}.$$

Thus, taking roots on both sides and simplifying, the three inequalities required by local lemma are satisfied $\forall k \geq \lceil \frac{g}{2} \rceil$, $\ell \geq g$, provided

$$\frac{1}{c} \leq \frac{1}{\alpha} \Lambda^{\frac{\eta+1}{\eta}\Upsilon}, \qquad \frac{1}{c} \leq \frac{1}{\beta} \Lambda^{\frac{2k}{2k-2}\Upsilon} \quad \text{and} \quad \frac{1}{c} \leq \frac{1}{\gamma} \Lambda^{\frac{\ell}{\ell-1}\Upsilon} \tag{2.1}$$

where

$$\Upsilon = \frac{2}{\eta!\alpha^\eta} + \frac{1}{(\beta^2-1)\beta^{2\lceil \frac{g}{2} \rceil - 4}} + \frac{1}{\Delta \gamma^{g-2}(\gamma-1)}.$$

Now we have to set specific values of α, β, γ and η. First we set $\eta = 2$ and $\alpha = \beta = \gamma = 2$. Using $g \geq 9$ and $\Delta \geq 4$, we have $\Lambda \geq (1 - \frac{1}{64})^{64} \geq 0.3649$. It can easily be verified that that the above inequalities (2.1) are satisfied by setting $c = 2.951$. It follows that $a'(G) \leq 5.91 < 6\Delta$ for all graphs G with girth $g \geq 9$. This proves Theorem 2.4.2.

Secondly, we set $\eta = 4$, $\alpha = 1.02$, $\beta = 1.04$ and $\gamma = 1.04$. Using $g \geq 220$ and $\Delta \geq 4$, we have $\Lambda \geq (1 - \frac{1}{256})^{256} \geq 0.3671$. It follows that by setting $c = 1.13$, $a'(G) \leq 4 \times 1.13\Delta = 4.52\Delta$ when girth $g \geq 220$. Hence Theorem 2.4.3. $\square$

Further improvements on $a'(G)$, which can be obtained (with this experiment) by strengthening the girth requirement are only marginal as long as we focus on constant lower bounds on girth.

2.4.2 Girth and acyclic chromatic index

An even cycle is called *half-monochromatic* with respect to a colouring if one of its *halves* (a set of alternate edges) is monochromatic. Notice that, this definition includes bichromatic cycles also.

Proof. For the sake of simplicity in the analysis, we write g in the form $c_1 \Delta^\varepsilon \log \Delta$, where $\varepsilon \geq 0$ and where c_1 is mentioned in Theorem 2.4.1. We can, assume without loss of generality, that $\varepsilon \leq 1$, because when ε exceeds 1, by choosing a large value of c_1, $a'(G) \leq \Delta + 2$ as in Theorem 2.3.2. As before, we assume $\Delta \geq 4$.

The proof consists of an initial deterministic phase followed by a random phase. We begin by obtaining a proper edge colouring of G using $\Delta + 1$ colours applying Vizing's method. We, then randomly recolour some of the edges with a new set of $o(\Delta)$ colours, and show that with positive probability, the colouring obtained is proper and acyclic. This random experiment is a slight modification of the ones used in the proofs of Theorems 2.3.3 and 2.3.2.

The random colouring is obtained as follows:

1. Obtain a proper colouring $\mathcal{C} : E \rightarrow S_1 = \{1, \ldots, \Delta + 1\}$.

2. In the second phase we do the following:

   - Activate each edge with independent probability $p = \frac{1}{\Delta^{\varepsilon}}$.
   - Each activated edge chooses a new colour uniformly at random and independently, from the set $S_2 = \{1', \ldots, (a\Delta^{1-\varepsilon})'\}$, where $a > 1$ is a constant to be determined later.

Denote the resulting random colouring by $\mathcal{C}'$. With respect to $\mathcal{C}'$, we define the following *bad* events.

1. For a pair of incident edges e and f, let $E_{e,f}$ denote the event that they are both recoloured with the same new colour. We call this an event of type I.

2. Let $E_{C,2k}$ denote the event that a bichromatic cycle C of length $2k$ in $\mathcal{C}$ is undisturbed in the recolouring process. Call this a type II event.

3. Let $E_{C,2\ell}$ denote the event that a half-monochromatic cycle C of length 2ℓ in $\mathcal{C}$ becomes bichromatic by retaining the same colour on a half and receiving a common new colour on the other half, a type III event.

4. Let $E_{C,2m}$ denote the type IV event where an even length cycle C of length $2m$ becomes properly bichromatic with 2 of the new colours.

We claim that the absence of type I-IV events imply that the colouring $\mathcal{C}'$ is proper and acyclic. Since $\mathcal{C}$ is proper, the absence of events of type I ensures that $\mathcal{C}'$ is also proper. The absence of events of type II, III and IV ensure respectively, (i) the absence of bichromatic cycles using both colours from S_1, (ii) one colour from each of S_1 and S_2 and (iii) both colours from S_2. It is therefore sufficient to show the absence of the above four types of events which we do by using Lovász Local Lemma.

To apply the local lemma we need estimates for the probabilities of each event, and for the number of events of each type possibly influencing a given event. As before, we calculate the number of events of each type that depend on a single edge and multiply by the number of edges in any event to get an upper bound on the total dependence. The following two lemmas present the estimated bounds.

Lemma 2.4.3. The probabilities of events are as follows: For each

1. event $E_{f,g}$ of type I, $Pr(E_{f,g}) = \frac{p^2}{a\Delta^{1-\varepsilon}} = \frac{1}{a\Delta^{1+\varepsilon}}$.
2. event $E_{C,2k}$ of type II, $Pr(E_{C,2k}) = (1-p)^{2k} \leq e^{\frac{-2k}{\Delta^{\varepsilon}}}$.
3. event $E_{C,2\ell}$ of type III, $Pr(E_{C,2\ell}) \leq \frac{2p^{\ell}(1-p)^{\ell}}{(a\Delta^{1-\varepsilon})^{\ell-1}} < \frac{2a\Delta^{1-\varepsilon}}{(a\Delta)^{\ell}}$.
4. event $E_{C,2m}$ of type IV, $Pr(E_{C,2m}) = p^{2m}\binom{a\Delta^{1-\varepsilon}}{2}\frac{2}{(a\Delta^{1-\varepsilon})^{2m}} < \frac{(a\Delta^{1-\varepsilon})^2}{(a\Delta)^{2m}}$.

Lemma 2.4.4. The following is true for any given edge e:

1. Less than 2Δ events of type I depend on e.
2. Less than Δ events of type II depend on e.
3. Less than $2\Delta^{\ell-1}$ events of type III depend on e, for each $\ell \geq 2$.
4. Less than Δ^{2m-2} events of type IV depend on e, for each $m \geq 2$.

To apply Lovász Local Lemma, let $x_0 = 1/(\alpha\Delta^{1+\varepsilon})$, $x_1 = 1/(\beta\Delta^{1+2\varepsilon})$, $y_\ell = (2a\Delta^{1-\varepsilon})/(\gamma\Delta)^{\ell}$ and $z_m = (a\Delta^{1-\varepsilon})^2/((\delta\Delta)^{2m})$ be the values associated with events of type I, II, III and IV, where the lengths of the cycles in Type III and IV events are 2ℓ and $2m$, respectively. Here $\alpha, \beta, \gamma, \delta > 1$ are real values to be determined by calculation. We conclude that with positive probability none of the above events occur, provided $\forall k, \ell, m \geq \lceil \frac{g}{2} \rceil$,

$$\frac{1}{a\Delta^{1+\varepsilon}} \leq x_0(1-x_0)^{4\Delta}(1-x_1)^{2\Delta} \prod_{\theta \geq \lceil \frac{g}{2} \rceil} (1-y_\theta)^{4\Delta^{\theta-1}} \prod_{\lambda \geq \lceil \frac{g}{2} \rceil} (1-z_\lambda)^{2\Delta^{2\lambda-2}}$$

$$e^{\frac{-2k}{\Delta^{\varepsilon}}} \leq x_1(1-x_0)^{4k\Delta}(1-x_1)^{2k\Delta} \prod_{\theta \geq \lceil \frac{g}{2} \rceil} (1-y_\theta)^{4k\Delta^{\theta-1}} \prod_{\lambda \geq \lceil \frac{g}{2} \rceil} (1-z_\lambda)^{2k\Delta^{2\lambda-2}}$$

$$\frac{2a\Delta^{1-\varepsilon}}{(a\Delta)^{\ell}} \leq y_\ell(1-x_0)^{4\ell\Delta}(1-x_1)^{2\ell\Delta} \prod_{\theta \geq \lceil \frac{g}{2} \rceil} (1-y_\theta)^{4\ell\Delta^{\theta-1}} \prod_{\lambda \geq \lceil \frac{g}{2} \rceil} (1-z_\lambda)^{2\ell\Delta^{2\lambda-2}}$$

$$\frac{(a\Delta^{1-\varepsilon})^2}{(a\Delta)^{2m}} \leq z_m(1-x_0)^{4m\Delta}(1-x_1)^{2m\Delta}\prod_{\theta\geq\lceil\frac{g}{2}\rceil}(1-y_\theta)^{4m\Delta^{\theta-1}}\prod_{\lambda\geq\lceil\frac{g}{2}\rceil}(1-z_\lambda)^{2m\Delta^{2\lambda-2}}$$

Setting $\alpha = \beta = \gamma = \delta = 1000$ and $a = 4000$ and using the fact that $(1-\frac{1}{z})^z \geq \frac{1}{4}$ $\forall z \geq 2$ we have,

$$(1-x_0)^{2\Delta} \geq \left(\frac{1}{4}\right)^{2\Delta x_0} = \left(\frac{1}{4}\right)^{\frac{2}{\alpha\Delta^\varepsilon}}$$

$$(1-x_1)^{\Delta} \geq \left(\frac{1}{4}\right)^{\Delta x_1} = \left(\frac{1}{4}\right)^{\frac{1}{\beta\Delta^{2\varepsilon}}}$$

$$\prod_{\theta\geq\lceil\frac{g}{2}\rceil}(1-y_\theta)^{2\Delta^{\theta-1}} \geq \left(\frac{1}{4}\right)^{\mathcal{S}_1}$$

$$\prod_{\lambda\geq\lceil\frac{g}{2}\rceil}(1-z_\lambda)^{\Delta^{2\lambda-2}} \geq \left(\frac{1}{4}\right)^{\mathcal{S}_2}$$

where,

$$\mathcal{S}_1 = \sum_{\theta\geq\lceil\frac{g}{2}\rceil} 2y_\theta\Delta^{\theta-1} = \frac{4a}{\Delta^\varepsilon}\sum_{\theta\geq\lceil\frac{g}{2}\rceil}\frac{1}{\gamma^\theta} \leq \frac{4a}{\Delta^\varepsilon\gamma^{\lceil\frac{g}{2}\rceil-1}(\gamma-1)}$$

and

$$\mathcal{S}_2 = \sum_{\lambda\geq\lceil\frac{g}{2}\rceil} z_\lambda\Delta^{2\lambda-2} = \frac{a^2}{\Delta^{2\varepsilon}}\sum_{\lambda\geq\lceil\frac{g}{2}\rceil}\frac{1}{(\delta)^{2\lambda}} \leq \frac{a^2}{\Delta^{2\varepsilon}\delta^{2\lceil\frac{g}{2}\rceil-2}(\delta^2-1)}.$$

Let $\mathcal{P}_i$, $\mathcal{N}_i$ and x_i denote, respectively, the probabilities, number of edges and local lemma constants associcated with events of type i. We can see that, as in the previous proof, the inequalities required by local lemma are satisfied provided

$$\mathcal{P}_i \leq x_i\left(\frac{1}{4}\right)^{\mathcal{N}_i\Upsilon}, \quad \forall i \tag{2.2}$$

where

$$\Upsilon = \frac{2}{\alpha\Delta^\varepsilon} + \frac{1}{\beta\Delta^{2\varepsilon}} + \frac{4a}{\Delta^\varepsilon\gamma^{\lceil\frac{g}{2}\rceil-1}(\gamma-1)} + \frac{a^2}{\Delta^{2\varepsilon}\delta^{\lceil\frac{g}{2}\rceil-2}(\delta^2-1)}$$

By choosing c_1 suitably large, we can verify that $\Delta^\varepsilon\Upsilon \leq \frac{1}{125}$ and each of the inequalities (2.2) are satisfied. As a result, the inequalities corresponding to local lemma are also satisfied. Finally fixing $c_2 = a \cdot c_1$, the theorem is proved.

□

2.5 Conclusions

The work presented in this chapter has been published by us. The references are [MNS07b], [MNS05].

We obtained a reasonably significant improvement over 16Δ, by a new idea which initially allows limited improperness in the colouring. To do this, we only require to assume a constant lower bound on girth, which is not a very severe constraint. We tried to get better bounds by modifying the ideas presented in this chapter. Some of the ideas tried include *non-uniform* assignment of colours randomly, and also allowing different values of the parameter η for different colours. The optimum values occurred when everything was identical. It would be interesting to find something radically different which lowers the bound closer to its conjectured value of $(\Delta + 2)$. A probabilistic argument might yield an improvement but it is naturally unlikely to give a bound which is very tight.

J. Beck has designed a method (see [Bec91]) for translating existential proofs using Lovász Local Lemma into efficient randomised algorithms for constructing an object guaranteed by the proof. It would be interesting to investigate the applicability of Beck's method in the context of our proof.

The result we obtained in Theorem 2.4.1, unifies the two results of Theorems 2.3.3 and 2.3.2 in a generalised framework. Theorem 2.3.2 gives a very tight bound but the class of graphs it applies to is quite restrictive. For example, Theorem 2.3.2, when applied to graphs G with minimum degree, $\delta(G) \geq 3$, requires that $\Delta = O(\log n)$. This is because it is well-known that graphs having $\delta \geq 3$ always have a cycle of length $O(\log n)$. Theorem 2.3.3 gives a weaker bound applicable for a wider class of graphs. Our result, Theorem 2.4.1, shows these to be special extreme cases of a trend.

Theorem 2.3.4 represents the first step towards getting an asymptotic improvement over the trivial $O(\Delta^2)$ bound towards the known value of $O(\Delta)$ by constructive means. It would be interesting to improve this constructive bound further, possibly to $O(\Delta)$ itself.

Chapter 3

Acyclic edge colouring of partial tori

In this chapter we consider the problem of acyclic colouring of partial tori (*a.k.a. grid-like graphs*). We obtain tight bounds on the number of colours required and also design efficient algorithms to produce such colourings. We mention classes of graphs for which good algorithms have been obtained for the acyclic colouring problem in Section 3.1. We define partial tori in Section 3.2 and also introduce the notation we use. Section 3.3 states our results and the proofs follow in Section 3.4. An algorithm to acyclically colour partial tori is presented in Section 3.5. These algorithms are a direct result of the preceeding proofs. Section 3.6 makes some concluding remarks.

3.1 Introduction

Determining $a'(G)$ to great accuracy is a very difficult problem. Even for the highly structured and simple class of complete graphs, the value of $a'(G)$ is not yet determined. First, we recall the following known facts about acyclic edge colouring. It is NP-Hard to determine $a'(G)$. Also, $a'(G) \geq \chi'(G) \geq \Delta$ for any graph G. Recall, that $\chi'(G)$ denotes the chromatic index of G.

Often, the gap between the trivial lower bound of Δ and the demonstrated upper bound is at least a linear function of Δ. This is the case for the family of *odd graphs* which are a generalisation of the Petersen graph. The odd graph O_k has as vertex set the $\binom{2k+1}{k}$, k-element subsets of $[1, \ldots, 2k+1]$, and an edge between two vertices precisely when the corresponding sets are disjoint. It can be shown quite easily that $\Delta(G) + 1 \leq$

$a'(G) \leq 2\Delta(G)-1$, for the class of odd graphs. While we believe the actual value is closer to the lower bound, the problem is still open. Thus, results which are off by an additive constant are good, and all the more so if the constant is 1 or 2.

Most of the results obtained in Chapter 2 and also those mentioned therein, are existential in nature and are not constructive. In those cases, there is no known effecient way of obtaining such colourings which is better than looking through all possible colourings until one is found. In this chapter we present results obtained by us, which provide good estimates on the acyclic chromatic index of partial tori. In most of the cases the values we provide are exact, and in the few remaining cases the value is off by an additive factor of at most 1. Our main result, which is on the acyclic chromatic index of partial tori, is obtained by an application of a more general result, which we prove first.

These graphs can actually be coloured in polynomial time with these many colours. The only other examples of graph families where such tight bounds have been proved and are constructible efficiently are graphs with $\Delta(G) \leq 3$, due to Skulrattankulchai (see [Sku04]), our results on outer planar graphs (see [MNS07a]) and partial 2-trees (see [MNS08]) and a result on 2-degenerate graphs ([CM07]) due to Manu and Chandran. The methods used in our results here can be extended to give bounds on any graph expressed as the cartesian product of other graphs (see [MS07]) . The bounds are in terms of corresponding values of the constituents of the product, and are not as tight as these bounds. These results are presented in Chapter 4.

3.2 Definitions and notations

Some of the notations and definitions we use in this chapter have already been given in Chapter 1, but we provide them again for ready reference. We use P_k to denote a simple path on k vertices. Without loss of generality, we assume that $V(P_k) = \{0, \ldots, k-1\}$ and $E(P_k) = \{(i,j) : |i-j| = 1\}$. Similarly, we use C_k to denote a cycle $(0, \ldots, k-1, 0)$ on k vertices. We use PATHS to denote the set $\{P_3, P_4, \ldots\}$ of all paths on 3 or more vertices. Similarly, we use CYCLES to denote the set $\{C_3, C_4, \ldots\}$ of all cycles. We sometimes use EDGE to denote P_2. The standard notation $[n]$ is used to denote the set $\{1, 2, \ldots, n\}$.

Our definition of the class of partial tori is based on the so-called *cartesian product* of graphs defined below.

Definition 3.2.1. Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *cartesian product* of G_1 and G_2, denoted by $G_1 \square G_2$, is defined to be the graph $G = (V, E)$ where $V = V_1 \times V_2$ and E contains the edge joining (u_1, u_2) and (v_1, v_2) if and only if *either* $u_1 = v_1$ and $(u_2, v_2) \in E_2$ *or* $u_2 = v_2$ and $(u_1, v_1) \in E_1$.

Note that $\square$ is a binary operation on graphs which is commutative in the sense that $G_1 \square G_2$ and $G_2 \square G_1$ are isomorphic. Similarly, it is also associative. Hence, the graph $G_0 \square G_1 \square \cdots \square G_d$ is unambiguously defined for any d. We use G^d to denote the d-fold Cartesian product of G with itself. It was shown independently by Sabidussi, and Vizing, and others (see [AHI92],[IK00]) that any connected graph G can be expressed as a product $G \cong G_1 \square \cdots \square G_k$ of prime factors G_i. Here, a graph is said to be *prime* with respect to the $\square$ operation if it has at least two vertices and if it is not isomorphic to the product of two non-trivial graphs (those having at least two vertices). Also, this factorisation (or decomposition) is unique except for a re-ordering of the factors and will be referred to as the *Unique Prime Factorisation (UPF)* of the graph. Since $a'(G)$ is a graph invariant, we assume, without loss of generality, that any G_i from EDGE $\cup$ PATHS $\cup$ CYCLES has as its vertex set $\{0, 1\}$, $\{0, \ldots, k-1\}$, $\{0, \ldots, k-1\}$ respectively, and the adjacency as described above. This allows us to explicitly handle concrete example graphs of these isomorphism classes.

Definition 3.2.2. A *d-dimensional partial torus* is a connected graph G whose unique prime factorisation is of the form $G \cong G_1 \square \cdots \square G_d$ where $G_i \in$ EDGE $\cup$ PATHS $\cup$ CYCLES for each $i \leq d$. We denote the class of such graphs by $\mathcal{P}_d$.

Definition 3.2.3. If each prime factor of a graph $G \in \mathcal{P}_d$ is a P_2, then G is called the *d-dimensional hypercube*. This graph is denoted by P_2^d.

Definition 3.2.4. If each prime factor of a graph $G \in \mathcal{P}_d$ is from PATHS, then G is called a *d-dimensional grid* or *mesh*. The class of all such graphs is denoted by $\mathcal{M}_d$.

Definition 3.2.5. If each prime factor of a graph $G \in \mathcal{P}_d$ is from CYCLES, then G is called a *d-dimentional torus*. The class of all such graphs is denoted by $\mathcal{T}_d$.

3.3 Results

The results we have obtained for grid-like graphs is stated in Theorem 3.3.2 below. The proofs (given in Section 3.4.1) are based on the following useful theorem whose proof is given later in Section 3.4.2.

Theorem 3.3.1. *Let G be a simple graph with $a'(G) = \eta$. Then,*

1. $a'(G \square P_2) \leq \eta + 1$, *if* $\eta \geq 2$.
2. $a'(G \square P_l) \leq \eta + 2$, *if* $\eta \geq 2$ *and* $l \geq 3$.
3. $a'(G \square C_l) \leq \eta + 2$, *if* $\eta > 2$ *and* $l \geq 3$.

As a corollary, we obtain the following results.

Theorem 3.3.2. *The following is true for each $d \geq 1$.*

- $a'(P_2^d) = \Delta(P_2^d) + 1 = d + 1$ *if* $d \geq 2$; $a'(P_2) = 1$.
- $a'(G) = \Delta(G) = 2d$ *for each* $G \in \mathcal{M}_d$.
- $a'(G) = \Delta(G) + 1 = 2d + 1$ *for each* $G \in \mathcal{T}_d$.
- *Let $G \in \mathcal{P}_d$ be any graph. Let e (respectively p and c) denote the number of prime factors of G which are from* EDGE *(respectively from* PATHS *and* CYCLES*). Then,*
  - $a'(G) = \Delta(G) + 1 = e + 2c + 1$ *if* $p = 0$.
  - $a'(G) = \Delta(G) = e + 2p + 2c$ *if* either $p \geq 2$, or $p = 1$ *and* $e \geq 1$.
  - $a'(G) = \Delta(G) = 2 + 2c$ *if* $p = 1$, $e = 0$ *and if at least one prime factor of G is an even cycle.*
  - $a'(G) \in \{\Delta = 2 + 2c, \Delta + 1 = 2 + 2c + 1\}$ *if* $p = 1$, $e = 0$ *and if all prime factors of G (except the one path) are odd cycles. There are examples for both values of $a'(G)$.*

3.4 Proofs

We repeat here a fact about acyclic edge colouring which we have stated before. It can be easily verified and would be used often in our proofs. We follow the statement of the fact with a brief explanation as to why it is correct.

Fact 3.4.1. *If a graph G is regular with $\Delta(G) \geq 2$, then $a'(G) \geq \Delta(G) + 1$.*

This is because in any proper edge-colouring of G with $\Delta(G)$ colours, each colour is used on some edge incident incident at any vertex. Hence, for each pair of distinct colours a and b and for each vertex u, there is a unique cycle in G going through u and which is coloured with a and b.

We first present the proof of Theorem 3.3.2. In this proof, we assume the truth of Theorem 3.3.1, and apply its various cases to complete the proof of Theorem 3.3.2.

3.4.1 Proof of Theorem 3.3.2

Case 1 (G is the d-dimensional hypercube P_2^d)
Clearly, $a'(P_2) = 1$ and $a'(P_2^2) = a'(C_4) = 3$. For $d > 2$, we start with $G = P_2^2$ and repeatedly and inductively apply Statement (1) of Theorem 3.3.1 to deduce that $a'(P_2^d) \leq d + 1$. Combining this with Fact 3.4.1, we get $a'(P_2^d) = d + 1$ for $d \geq 2$.
Case 2 (Case G is a d-dimensional mesh $\mathcal{M}_d$)
Again, we prove by induction on d. If $d = 1$, then $G \in$ PATHS and hence $a'(G) = 2 = \Delta(G)$. For $d > 1$, repeatedly and inductively apply Statement (2) of Theorem 3.3.1 to deduce that $a'(G) \leq 2(d-1)+2 = 2d$. Combining this with the trivial lower bound $a'(G) \geq \Delta(G)$, we get $a'(G) = 2d$ for each $G \in \mathcal{M}_d$ and each $d \geq 1$.
Case 3 (Case G is a d-dimensional torus $\mathcal{T}_d$)
We prove by induction on d. If $d = 1$, then $G \in$ CYCLES and hence $a'(G) = 3 = \Delta(G)+1$. For $d > 1$, repeatedly and inductively apply Statement (3) of Theorem 3.3.1 to deduce that $a'(G) \leq 2(d-1)+1+2 = 2d+1$. Combining this with Fact 3.4.1, we get $a'(G) = 2d+1$ for each $G \in \mathcal{T}_d$ and each $d \geq 1$.
Case 4 (Case G is a d-dimensional partial torus $\mathcal{P}_d$)
Let e, p and c be as defined in the statement of the theorem. If $p = 0$, then G is the product of edges and cycles and hence G is regular and $a'(G) \geq \Delta(G) + 1$ by Fact 3.4.1.

Also, we can assume that $c > 0$. Otherwise, $G = P_2^d$ and this case has already been established. Again, without loss of generality, we can assume that the first factor G_1 of G is from CYCLES and $a'(G_1) = 3$. Now, as in the previous cases, we apply induction on d and also repeatedly apply one of the Statements (1) and (3) of Theorem 3.3.1 to deduce that $a'(G) \leq \Delta(G) + 1$. This settles the case $p = 0$.

Now, suppose *either* $p \geq 2$, *or* $p = 1$ and $e \geq 1$. Order the d prime factors of G so that $G \cong G_1 \Box \cdots \Box G_d$ and the first p factors are from PATHS and the next e factors are copies of P_2. By the previously established cases and from Theorem 3.3.1, it follows that

$$a'(G_1 \Box \cdots \Box G_{p+e}) = \Delta(G_1 \Box \cdots \Box G_{p+e}) = 2p + e \ \geq \ 3.$$

As before, applying (3) of Theorem 3.3.1 inductively, it follows that

$$a'(G) = a'(G_1 \Box \cdots \Box G_{p+e+c}) \leq \Delta(G) = 2p + e + 2c.$$

Combining this with the trivial lower bound establishes this case also.

Suppose $p = 1$, $e = 0$ and at least one prime factor of G is an even cycle. Let $G_1 = P_k$ for some $k \geq 3$ and $G_2 = C_{2l}$ for some $l \geq 2$. We note that it is enough to show that $G' = G_2 \Box G_1$ is acyclically colourable with $\Delta(G') = 4$ colours. Extending this colouring to an optimal colouring of G can be achieved by repeated applications of Statement (3) of Theorem 3.3.1 as before. Hence we focus on showing $a'(G') = 4$.

Firstly, colour the cycle $G_2 = C_{2l} = \langle 0, 1, \ldots, 2l-1, 0 \rangle$ acyclically as follows. For each $i, 0 \leq i \leq 2l-2$, colour the edge $(i, i+1)$ with 1 if i is even and with 2 if i is odd. Colour the edge $(2l-1, 0)$ with 3. Now, use the same colouring on each of the k isomorphic copies (numbered with $0, \ldots k-1$) of G_2. For each $j, 0 \leq j < k-1$, the j^{th} and $(j+1)^{th}$ copies of G_2 are joined by cross-edges which constitute a perfect matching between similar vertices in the two copies. These cross-edges are coloured as follows. For every i and j, the cross edge joining (i, j) and $(i, j+1)$ is coloured with 4 *if* $(i + j)$ is even *and* is coloured with the unique colour from $\{1, 2, 3\}$ which is missing at this vertex i in both copies *if* $(i + j)$ is odd. See Figure 3.1 for an illustration.

The colouring is such that in each perfect matching joining two adjacent copies of G_2, the cross edges which are part of this matching are alternately coloured with 4 and a colour from $\{1, 2, 3\}$. Note that there can be no bichromatic cycle within each copy of

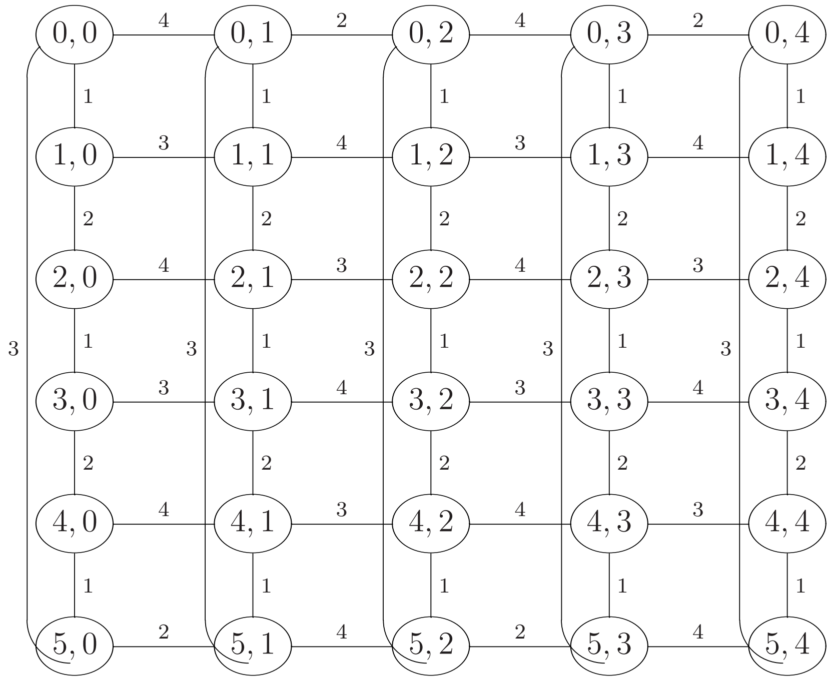

Figure 3.1: colouring of $C_6 \Box P_5$

G_2. Hence, any bichromatic cycle (if it exists) should use cross edges.

First, we claim that there can be no $(4, c)$-coloured cycle for any $c \in \{1, 2, 3\}$. To see this, note that no two successive edges of any such cycle can be from the same copy of G_2 since there is no edge coloured 4 in any copy of G_2. In addition, to complete a cycle it is necessary that there must be two adjacent copies, say the j^{th} and the $(j+1)^{th}$, such that the cycle passes from the j^{th} to the $(j+1)^{th}$ and back to j^{th} copy using exactly 3 edges. This contradicts the fact that the cross edges between adjacent copies are alternately coloured with 4 and a colour from $\{1, 2, 3\}$.

In addition, there can be no (c, c')-coloured cycle for any $c, c' \in \{1, 2, 3\}$. To see this, we first note that any maximal (c, c')-coloured path in the j^{th} (for any j) copy of G_2 is of odd length (counted as the number of edges) and hence the first and last edge of such a path are coloured the same, say with c. This means the c'-coloured edges incident at the two end points u and v connect them to the different, namely the $(j-1)^{th}$ and $(j+1)^{th}$, copies (because of the way cross edges are coloured). Extending this further, we see that

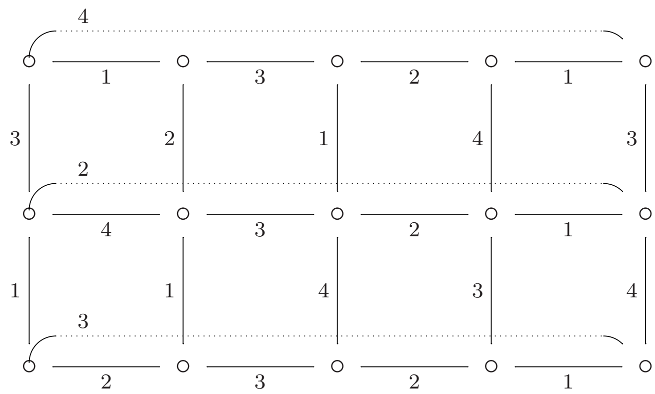

Figure 3.2: colouring of $P_3 \square C_5$

any (c, c')-coloured maximal path starts at either $(u, 0)$ or $(v, 0)$ and ends at $(u, k-1)$ or $(v, k-1)$ and does not complete to a cycle. This shows that $a'(G') = 4$ as desired.

Suppose $p = 1$, $e = 0$ and all prime factors of G (except the one path) are odd cycles. In this case, $a'(G)$ can take both values as the following examples show. If $G = P_3 \square C_3$, then it can be easily verified that $a'(G) = 5 = \Delta + 1$. Also, if $G = P_3 \square C_5$, then $a'(G) = 4 = \Delta$ as shown by the colouring in Figure 3.2.

3.4.2 proof of Theorem 3.3.1

We now present the proof of Theorem 3.3.1.

A restricted class of bijections (defined below) would play an important role in this proof.

Definition 3.4.1. A bijection σ from a set $\mathcal{A}$ to an equivalent set $\mathcal{B}$ is a *non-fixing* bijection if $\sigma(i) \neq i$ for each i.

Since $a'(G) = \eta$, we can edge-colour G acyclically using colours from $[\eta]$. Fix one such colouring $\mathcal{C}_0 : E(G) \to [\eta]$.

Define $\mathcal{C}_1$ to be the colouring defined by $\mathcal{C}_1(e) = \sigma(\mathcal{C}_0(e))$ where $\sigma : [\eta] \to [\eta]$ is any bijection which is *non-fixing*. For concreteness, define $\sigma(i) = (i \bmod \eta) + 1$.

Case 1 ($a'(G \Box P_2)$)

Let G_0, G_1 be the two isomorphic copies of G induced respectively by the sets $\{(u,0) : u \in V(G)\}$ and $\{(u,1) : u \in V(G)\}$. Let G_0 and G_1 be edge coloured respectively by $\mathcal{C}_0$ and $\mathcal{C}_1$. For each of the remaining edges (termed *cross-edges* and which constitute a perfect matching between G_0 and G_1) of the form $((u,0),(u,1))$, give a new colour α. Denote by $\mathcal{C}$, the resultant colouring of $G \Box P_2$. We claim that $\mathcal{C}$ is proper and acyclic.

It is easy to see that $\mathcal{C}$ is proper. Also note that any bichromatic cycle in $\mathcal{C}$ should necessarily use the colour α (since the colourings of G_0 and G_1 are acyclic).

Suppose that $G \Box P_2$ has a bichromatic cycle C using the colours α and some other colour, say i, from the set $[\eta]$. In $\mathcal{C}$, G_0 and G_1 are both coloured α-free and hence any proper α, i-coloured cycle should contain the α-coloured edges an even number of times. Hence we have $|C| \equiv 0 \bmod 4$. Fix a vertex $(u_1, 0)$ as the starting point of C. Then C looks like $C = \langle (u_1,0) \xrightarrow{\alpha} (u_1,1) \xrightarrow{i} (u_2,1) \xrightarrow{\alpha} (u_2,0) \cdots (u_k,0) \xrightarrow{i} (u_1,0) \rangle$.

Notice that k is of even parity (since $|C| \equiv 0 \bmod 4$). For each i-coloured edge $(u_{2\ell+1}, 1) \to (u_{2\ell+2}, 1)$ of G_1 in C, its isomorphic copy in G_0, namely, the edge $(u_{2\ell+1}, 0) \to (u_{2\ell+2}, 0)$ is coloured with a colour $j = \sigma^{-1}(i) \neq i$ (since σ is a non-fixing bijection of $[\eta]$). Now it can be seen that the cycle $\langle (u_1,0) \xrightarrow{j} (u_2,0) \xrightarrow{i} (u_3,0) \ldots \xrightarrow{j} (u_k,0) \xrightarrow{i} (u_1,0) \rangle$ is an i,j-coloured cycle in G_0. This is a contradiction to the fact that G_0 is acyclically coloured. Hence the colouring $\mathcal{C}$ is acyclic.

Case 2 ($a'(G \Box P_k)$)

Let the k isomorphic copies of G in $G \Box P_k$ be $G_0, G_1, \ldots, G_{k-1}$ where G_i is induced by the set $\{(u, i) : u \in V(G)\}$. Let α_0 and α_1 be two new colours which are not in $[\eta]$. Our colouring is as follows.

For each i, colour the edges of copy G_i with $\mathcal{C}_{i \bmod 2}$. Also, for each i, colour the edges of the form $((u, i), (u, i + 1))$ with the new colour $\alpha_{i \bmod 2}$. Denote by $\mathcal{C}$ the resultant colouring of $G \Box P_k$. It is easy to see that $\mathcal{C}$ is proper. We claim that $\mathcal{C}$ is also acyclic.

For each i $(0 \leq i < k - 1)$ and for each edge e of G, notice that G_i and G_{i+1} have different colours on their respective copies of e (since the colourings $\mathcal{C}_0$ and $\mathcal{C}_1$ are based on mutually non-fixing bijections over $[\eta]$). Hence bichromatic cycles between two consecutive copies of G are ruled out by Case 1. So any bichromatic cycle C should pass through at least three consecutive copies of G, thus fixing the colours of C to be α_0 and α_1. Since all the copies of G are free of both α_0 and α_1, and the edges joining (u, i) and $(u, i + 1)$ between successive copies of G alone do not form a cycle, the colouring $\mathcal{C}$ is acyclic.

Case 3 $(a'(G \Box C_k))$
In this case, we have k isomorphic copies of G numbered $G_0, G_1, \ldots, G_{k-2}, G_{k-1}$ such that there is a perfect matching between successive copies G_i and $G_{(i+1) \bmod k}$ (see Figure 3.3). Our colouring is as follows.

For each i, $1 \leq i \leq k - 2$, colour the edges of G_i with $\mathcal{C}_{(i+1) \bmod 2}$.

As before, let α_0, α_1 be two new colours which are not in $[\eta]$. Let $\mathcal{D}_0$ be a colouring of G_0 defined by $\mathcal{D}_0(e) = \tau(\mathcal{C}_0(e))$ where $\tau(i) = i + 1, i < \eta$, $\tau(\eta) = \alpha_1$.

In order to colour G_{k-1}, define a colouring $\mathcal{D}_1(e) = \mu(\mathcal{C}_0(e))$ where $\mu(i) = i + 2$, $i < \eta - 1$ and $\mu(\eta - 1) = \alpha_{(k+1) \bmod 2}$, $\mu(\eta) = 2$.

Now, colour any edge of the form $((u, i), (u, i + 1))$, $0 \leq i < k - 1$ with the new colour $\alpha_{i \bmod 2}$. Colour the edges of the form $((u, k - 1), (u, 0))$ with the colour 1. Denote this colouring of $G \Box C_k$ by $\mathcal{C}$.

We claim that $\mathcal{C}$ is proper and acyclic. For each i, the colouring $\mathcal{C}$ restricted to G_i is proper and acyclic by definition. Also note that, each cross-edge $((u, i), (u, (i+1) \bmod k))$ is coloured with a colour γ (say) which is not used in either of the copies G_i and G_{i+1}. Hence $\mathcal{C}$ is proper.

Also, in $\mathcal{C}$, any edge $e \in G_i$ and its isomorphic copy $e' \in G_{(i+1) \bmod k}$ receive different colours (since the colourings on successive copies of G are based on mutually non-fixing bijections). Hence, as shown for the Case $G \Box P_2$, there can be no bichromatic cycle in $\mathcal{C}$

restricted to two successive copies of G. Hence any such bichromatic cycle C should pass through at least 3 consecutive copies of G, again fixing the two colours of C to be those used on two incident cross edges. Also, it is easy to see that there can be no bichromatic cycle involving *only* cross-edges since any such cycle uses the three colours $\{\alpha_0, \alpha_1, 1\}$.

Note that each of $G_1, \ldots, G_{k-2}$ are coloured free of both α_0 and α_1. Hence any α_0, α_1-bichromatic cycle C should start from some vertex $(u_1, 0)$ in G_0, then reach $(u_1, k-1)$ using only cross edges, then go to some vertex $(u_2, k-1)$ using an edge of G_{k-1}, then reach $(u_2, 0)$ using only cross edges and then some vertex $(u_3, 0)$ using a α_1-coloured edge of G_0 and continue this (possibly) again and again and finally reach a vertex $(u_k, 0)$ (where k is an even number) and then go to $(u_1, 0)$ using a α_1-coloured edge of G_0. Here the only non-cross edges used in C are either from G_0 (and coloured with α_1) or from G_{k-1} (and coloured with either α_0 or α_1 depending on the parity of k). From the definitions of $\mathcal{D}_0$ and $\mathcal{D}_1$, it follows that for each edge $(u_{2l+1}, k-1) \rightarrow (u_{2l+2}, k-1)$ from G_{k-1} used in C, its isomorphic copy in G_0, namely $(u_{2l+1}, 0) \rightarrow (u_{2l+2}, 0)$, is coloured with η. This implies the existence of a α_1, η-coloured bichromatic cycle in G_0 and this is a contradiction.

Similarly, any $\alpha_0, 1$-coloured bichromatic cycle should only visit vertices in the copies $G_1, G_0, G_{k-1}, G_{k-2}$ (or G_1, G_0, G_{k-1}) depending on whether k is even (or odd). As argued before, this would imply the existence of a $(1, \eta)$-coloured cycle in G_0 (or a $(1, (\eta - 1))$-coloured cycle in G_0) contradicting our definition of $\mathcal{C}$.

Also, if k is even, then any $(1, \alpha_1)$-coloured cycle should only visit vertices in G_0 and G_{k-1} (which are consecutive) and hence cannot exist. If k is odd, then such a cycle can only visit vertices in G_0, G_{k-1} and G_{k-2} and its existence would imply the existence of a $(2, \alpha_1)$-coloured cycle in G_0 which is again a contradiction. This shows that $\mathcal{C}$ is acyclic.

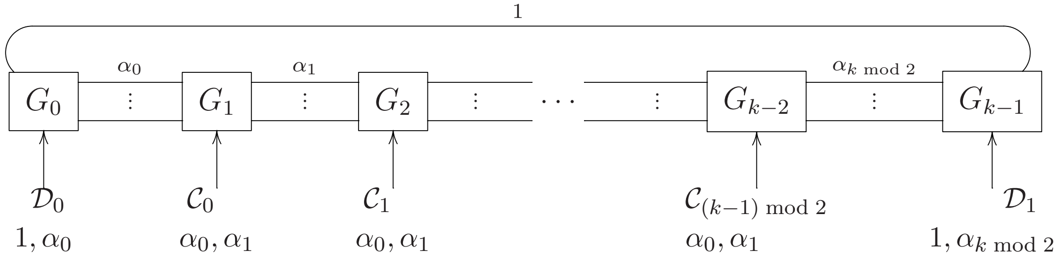

Figure 3.3: colouring of $G \square C_k$

3.5 Algorithmic aspects

There has been very little study of algorithmic aspects of acyclic edge colouring. In [AZ02], Alon and Zaks prove that it is NP-complete to determine if $a'(G) \leq 3$ for an arbitrary graph G. They also describe a deterministic polynomial time algorithm which obtains an acyclic $(\Delta + 2)$-edge-colouring for any graph G whose girth g is at least $c\Delta^3$ for some large absolute constant c. Skulrattanakulchai [Sku04] presents a linear time algorithm to acyclically edge colour any graph with $\Delta \leq 3$ using at most 5 colours. Apart from this, and a few other results mentioned elsewhere in this thesis, no significant progress has been made on the algorithmic aspects of acyclic edge colouring.

All of our proofs given in the previous section are constructive and readily translate to efficient algorithms which find optimal (or almost optimal) acyclic edge colourings of the partial tori. Formally,

Theorem 3.5.1. *Let $G \in \mathcal{P}_d$ be a graph (on n vertices and m edges) specified by its Unique Prime Factorisation. Then, an acyclic edge colouring of G using Δ or $\Delta + 1$ colours can be obtained in $O(n + m)$ time. Also, the colouring is optimal except when G is a product of a path and a number of odd cycles.*

For the sake of completeness, we now present a brief and formal description of these algorithms. Before we finish, we need to say a few words about how the input is presented to the algorithm. It is known from the work of Aurenhammer, Hagauer and Imrich [AHI92] that the UPF of a connected graph G (on n vertices and m edges) can be obtained in $O(m \log n)$ time. Hence we assume that our connected input $G \in \mathcal{P}_d$ is given by the list of its prime factors $G_1, \ldots, G_d$. Also, without loss of generality, we assume that the list is such that

(i) $G_i \in$ PATHS for $i = 1, \ldots, p$;

(ii) $G_i = P_2$ for $i = p + 1, \ldots, p + e$;

(iii) $G_i \in$ CYCLES for $i = p + e + 1, \ldots, d = p + c + e$ and all even cycles appear before all odd cycles in the order.

Here p, e, c denote respectively the number of prime factors which are from PATHS, EDGES and CYCLES.

Algorithmic Version of Theorem 3.3.2

Algorithm 1 AcycColPCGrid($G_1, \ldots, G_d$)

if $d = 1$, **then** output an optimal acyclic edge-colouring of G_1 using 2 (1 or 3) colours depending on whether $G_1 \in$ PATHS ($G_1 = P_2$ or $G_1 \in$ CYCLES) and exit.

if $d = 2$ **then**

  if both $G_1 = G_2 = P_2$, **then** output an optimal colouring of $G_1 \square G_2$ using 3 colours and exit.

  if *either* $G_1 = P_2$ and $G_2 \in$ CYCLES *or* $G_1 \in$ PATHS and G_2 is an even cycle, **then** interchange G_1 and G_2; Otherwise, let G_1 and G_2 remain the same.

  Let $\mathcal{C}_0$ be an optimal acyclic colouring of G_1 (on l vertices) defined as follows : For each i, $0 \leq i < l - 1$, colour the edge $(i, i+1)$ with $i \bmod 2$. Colour the edge $(l-1, 0)$ (if it exists) with 3.

  Output the optimal acyclic edge colouring obtained by applying Acycol2fac($G_2, G_1, \mathcal{C}_0$) and exit.

end if

if $d > 2$ **then**

  Obtain an optimal colouring $\mathcal{C}_0$ of $G = G_1 \square \cdots \square G_{d-1}$ by applying AcycColPCGrid($G_1, \ldots, G_{(d-1)}$).

  Obtain an optimal colouring of $G \square G_d$ by applying Acycol2fac($G, G_d, \mathcal{C}_0$).

  Output the optimal colouring of $G_1 \square \cdots \square G_d$ thus obtained and exit.

end if

Algorithmic Version of Theorem 3.3.1

Algorithm 2 Acycol2fac$(G, H, \mathcal{C}_0)$

Let H be a path or a cycle on $k \geq 2$ vertices $\{0, \ldots, k-1\}$. Let $G_0, \ldots, G_{k-1}$ be the k isomorphic copies of G induced respectively by the sets $\{(u, i) : u \in V(G)\}$ for each i.

if G is an even cycle C_{2l} and $H = P_k$, **then** colour each of the k isomorphic copies of G by the same colouring $\mathcal{C}_0$. For every j $(0 \leq j < k-1)$ and i $(0 \leq i \leq 2l-1)$, colour the edge joining (i, j) and $(i, j+1)$ with 4 *if* $i + j$ is even and colour it with the unique colour from $\{1, 2, 3\}$ which is missing at both copies of i *if* $i + j$ is odd and exit.

Otherwise, suppose $\mathcal{C}_0$ uses colours from $[\eta] = \{1, \ldots, \eta\}$ for some $\eta > 0$. Let σ, τ, μ be three permutations over $[\eta + 2] = \{1, \ldots, \eta + 2\}$ defined by

$\sigma(i) = (i \bmod \eta) + 1$ for $i \in [\eta]$ and $\sigma(i) = i$ for $i > \eta$.

$\tau(i) = i + 1$ for $i < \eta$, $\tau(\eta) = \eta + 1$, $\tau(\eta + 1) = 1$ and $\tau(\eta + 2) = \eta + 2$.

$\mu(i) = i + 2$ for $i < \eta - 1$, $\mu(\eta - 1) = \eta + 1 + ((k+1) \bmod 2)$, $\mu(\eta) = 2$, $\mu(\eta + 1 + ((k+1) \bmod 2)) = 1$ and $\mu(\eta + 1 + (k \bmod 2)) = \eta + 1 + (k \bmod 2)$.

Let $\mathcal{C}_1, \mathcal{D}_0, \mathcal{D}_1$ be three new colourings of G obtained respectively by colouring each edge e of G by the colour $\sigma(\mathcal{C}_0(e))$, $\tau(\mathcal{C}_0(e))$, $\mu(\mathcal{C}_0(e))$.

if $H = P_k$, **then** colour each copy G_i by the colouring $\mathcal{C}_{i \bmod 2}$. Also, for each $i < k-1$, colour the cross-edges between G_i and G_{i+1} with the common colour missing from both of them. This missing colour is $\eta + 1 + (i \bmod 2)$.

if $H = C_k$, **then**, for each $i, 0 < i < k-1$, colour G_i by the colouring $\mathcal{C}_{(i+1) \bmod 2}$. Also, colour G_0 by $\mathcal{D}_0$ and colour G_{k-1} by $\mathcal{D}_1$. Also, for each $0 \leq i < k-1$, colour the cross-edges between G_i and G_{i+1} with the common colour, namely $\eta + 1 + (i \bmod 2)$, missing from both of them. In addition, the cross-edges between G_0 and G_{k-1} are coloured with 1.

3.6 Remarks

The work presented in this chapter has been published by us. The reference is [MNS06].

If G is isomorphic to the product of a path and a number of odd cycles, it can take either of the values in $\{\Delta, \Delta+1\}$. It would be interesting to see if we can classify such graphs for which $a'(G) = \Delta$. It would also be nice to construct an optimal colouring efficiently. Another direction is to extend this result and prove $a'(G) \leq \Delta(G) + 1$, where G is any subgraph of a hypercube. If such a result is obtained, it can be used to get results for more complicated kinds of products.

A standard kind of product called *strong product* is another operation whose effect on the acyclic chromatic index would be interesting to study and we are looking at this. The strong product of two graphs $G = (V, E)$ and $H = (V', E')$ has as vertex set $V \times V'$ and edge set $((u_1, v_1), (u_2, v_2))$ if $(u_1 = u_2 \textit{ or } (u_1, u_2) \in E) \textit{ and } (v_1 = v_2 \textit{ or } (v_1, v_2) \in E')$.

It is clear that the graph resulting from applying the strong product operator to a pair of graphs is a supergraph, on the same vertex set, of the graph obtained by applying the cartesian product operator to the same pair of graphs.

Chapter 4

Cartesian product and acyclic edge colouring

In this chapter, we extend and generalise the results obtained in Chapter 3 to the cartesian product of any two graphs. These graphs are more general than those considered in Chapter 3, since those graphs are all obtained by applying the cartesian product operator to a collection of only edges, paths and cycles.

In Section 4.1 we introduce the idea of acyclically colouring the cartesian product of graphs in a more general setting than grid-like graphs. We state useful properties of the cartesian product operator in Section 4.2. Section 4.3 contains our result and also a detailed proof. A few immediate consequences of this result are given as corollaries in Section 4.4. Section 4.5 contains some concluding remarks.

4.1 Introduction

The results in Chapter 3 involove graphs, all of whose factors, under the unique prime factorisation with respect to the cartesian product, are from the family EDGES $\cup$ PATHS $\cup$ CYCLES. The strongest requirement of the graph G in any of the cases of Theorem 3.3.1 is that $a'(G) > 2$. Theorem 3.3.2 follows as a consequence by iterative applications of various cases of Theorem 3.3.1. Here, we look at graphs which are the cartesian product of any two graphs.

The factors of the graph being more general, the bounds we get on $a'(G)$ are not as tight as for grid-like graphs. In the case of grid-like graphs, for each additional prime factor incorporated we extend the colouring by introducing only as many new colours as the increase in maximum degree. Here, we show that we can incorporate each prime factor, say $\mathcal{F}$, using an additional $a'(\mathcal{F})$ colours. Even though these bounds may not be tight, they can be used to obtain bounds for any graph if we know the $a'(H)$ values for each of its prime factors H.

4.2 Cartesian products

We have already defined the cartesian product formally (Definition 3.2.1). As was stated in Chapter 3, any non-prime graph with respect to this operator has a unique factorisation, upto a reordering of the factors. Note that $G \square H$ can be thought of as being obtained as follows. Take $|V(H)|$ isomorphic copies of G and label them with vertices from $V(H)$. For each edge (u, v) in $E(H)$, introduce a perfect matching between G_u and G_v which joins each vertex in $V(G_u)$ with its isomorphic image in $V(G_v)$. Equivalently, one can also think of this as obtained by taking $|V(G)|$ isomorphic copies of H and introducing a perfect matching between corresponding copies of H for each edge in $E(G)$.

The following facts are consequences of the definition of the cartesian product .

Fact 4.2.1. *The cartesian product $G_1 \square G_2$ is commutative in the sense that $G_1 \square G_2$ is isomorphic to $G_2 \square G_1$. Similarly, this operation is also associative. Hence the product $G_1 \square G_2 \square \ldots \square G_k$ is well-defined for each k. For each G and $k \geq 1$, we define G^k as follows : $G^1 \cong G$ and $G^k \cong G^{k-1} \square G$ for $k > 1$.*

Fact 4.2.2. *If $G = G_1 \square G_2 \square \ldots \square G_k$, then $G = (V, E)$ where V is the set of all k-tuples of the form $(u_1, \ldots, u_k)$ with each $u_i \in V(G_i)$ and the edge joining $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_k)$ is in E if and only if for some i, $1 \leq i \leq k$, (i) $u_j = v_j$ for all $j \neq i$ and (ii) the edge (u_i, v_i) is in $E(G_i)$.*

Fact 4.2.3. *$G_1 \square G_2$ is connected if and only if both G_1 and G_2 are connected.*

4.3 Our result and proof

In view of Fact 4.2.3, it is sufficient to consider only connected graphs. Also, if H is trivial (that is, H is a graph on just one vertex), then $G \Box H$ is isomorphic to G for any G. Hence, we focus only on connected non-trivial graphs.

We obtain the following general statement relating $a'(G)$ and the cartesian product operator.

Theorem 4.3.1. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two connected non-trivial graphs such that $\max\{a'(G), a'(H)\} > 1$. Then,*

$$a'(G \Box H) \leq a'(G) + a'(H).$$

Note : If G and H are both connected and non-trivial with $a'(G) = a'(H) = 1$, then each of G and H is a P_2. In that case, $G \Box H \cong C_4$ where C_4 is a cycle on 4 vertices. Only in this case, we have $a'(G \Box H) = 3$ whereas $a'(G) + a'(H) = 2$.

The proof is presented in two stages. We first describe a colouring in Section 4.3.1. In Section 4.3.2, we prove that the colouring we obtain is proper and acyclic, thus completing the proof.

4.3.1 The colouring

Let $a'(G) = \eta$ and $a'(H) = \beta$. Without loss of generality, assume that $\eta \geq \beta$. Let Δ denote the maximum degree of H. Set d to be $\Delta + 1$ *if* H is *either* a complete graph *or* an odd cycle and to be Δ *otherwise*. In either case, by Brooks' Theorem, H can be properly *vertex* coloured using colours from the set $[d] = \{0, \ldots d-1\}$.

We know that $\beta = a'(H) \geq \Delta$ always. If $H = K_{\Delta+1}$ or C_{2k+1}, then (since H is Δ-regular) $a'(H) \geq \Delta + 1$ (except when $H = K_2$). In both cases, $\eta \geq \beta \geq d$. If $H = K_2$, then $d = \Delta + 1 = 2$ and $\eta \geq 2$ by assumption. In any case, we have $\eta \geq d$.

Let $X_G : E_G \rightarrow [\eta] = \{0, \ldots, \eta - 1\}$ and $X_H : E_H \rightarrow [\beta'] = \{0', \ldots, (\beta-1)'\}$ be acyclic edge colourings of G and H respectively, using disjoint sets of colours.

Each edge in $G \Box H$ is either (i) an edge joining (u_1, v) and (u_2, v) for some $e = \{u_1, u_2\} \in E_G$ and $v \in V_H$ or (ii) an edge joining (u, v_1) and (u, v_2) for some $f = \{v_1, v_2\} \in E_H$ and $u \in V_G$. We denote the former edges by e_v (where $e \in E_G, v \in V_H$) and the latter edges by f_u (where $f \in E_H, u \in V_G$). Note that each edge of $G \square H$ lies either in some isomorphic copy H_u of H or in some isomorphic copy G_v of G.

For each $i \in \{0, \ldots, d-1\}$, let $\sigma_i : [\eta] \to [\eta]$ be a bijection defined by

$$\sigma_i(j) = (j+i) \bmod \eta, \quad \forall j \in [\eta].$$

Since $\eta \geq \beta \geq d$, we notice that the bijections $\sigma_i (i \in [d])$ are mutually non-fixing, that is, for all $i, k \in \{0, \ldots, d-1\}$ such that $i \neq k$, and for each $j \in [\eta]$, $\sigma_i(j) \neq \sigma_k(j)$.

Let $Y_H : V_H \to \{0, \ldots, d-1\}$ be a proper vertex colouring of V_H. We define a colouring of the edges of $G \square H$ based on the colourings X_G, X_H and Y_H as follows.

For each edge in E of the form f_u, where $f \in E_H$ and $u \in V_G$, we colour f_u using the colour $X_H(f)$. Now consider any arbitrary edge of the form e_v where $e \in E_G$ and $v \in V_H$. Let $i = Y_H(v)$ be the colour used by Y_H on v. Colour e_v using the colour $\sigma_i(X_G(e))$.

In other words, edges f_u in each isomorphic copy H_u is coloured the same way as f in H is coloured by X_H. But edges e_v in each isomorphic copy G_v is coloured essentially (ignoring the labels of colours) the same way as G is coloured but the colour labels are rotated by mutually non-fixing permutations. The permutation that is used for a G_v is decided by the vertex colour assigned to v by Y_H. As a result, for each edge $f = (v_1, v_2) \in E_H$ and for each edge $e = (u_1, u_2) \in E_G$, e_{v_1} and e_{v_2} get different colours but always from $[\eta]$.

4.3.2 Correctness of the colouring

Let $X : E(G \square H) \to \{0, \ldots, \eta - 1\} \cup \{0', \ldots, (\beta - 1)'\}$ be the colouring defined in the previous section. We will show that X is proper and acyclic.

Claim. *X is proper.*

Proof. Consider any vertex (u, v). The set of edges in $G \square H$ which are incident on (u, v) can be partitioned into two subsets $A_u = \{f_u : v \in f \in E_H\}$ and $A_v = \{e_v : u \in e \in E_G\}$. Since edges in these two sets are coloured using colours from disjoint sets, namely from $[\eta]$ and $[\beta']$, there is no conflict between these two sets. Now, let us focus on edges in A_u. Since f_u's are coloured in the same way as f's are coloured in H, there is no conflict among edges in A_u. Similarly, the edges e_v's in A_v are coloured with distinct colours, there is no conflict among members of A_v also. Hence X is proper. $\square$

It is only left to prove the acyclicity of X. We prove by contradiction. Suppose there is a bichromatic cycle C in G, with respect to the colouring X. First, we note that

Claim. *C cannot lie entirely within any isomorphic copy G_v or H_u of G or H respectively.*

Proof. Note that X restricted to H_u (or G_v) is basically either X_H (or X_G except for renaming of the colours). Hence if C lies within such an isomorphic copy, it implies that either X_H or X_G has a bichromatic cycle, which is a contradiction. □

By the above claim, it follows that C should visit vertices in at least two different copies G_v and $G_{v'}$. But different copies are only joined by edges of type f_u for some $f \in E_H$ and $u \in V_G$. Thus, it follows that C has at least one edge each of the two types e_v $(e \in E_G,\ v \in V_H)$ and f_u $(f \in E_H,\ u \in V_G)$ which are coloured with respectively, say, $a \in [\eta]$ and $b \in [\beta']$.

Claim. *Let (u_1, v_1) be some arbitrary vertex in C. Let (u_1, v_2) for some $v_2 \in V_H$ be the other end point of the unique b-coloured edge in C incident at (u_1, v_1). C lies entirely within G_{v_1} and G_{v_2}.*

Proof. The proof is by induction on the distance l in C from (u_1, v_1) *along the direction* specified by the edge $\{v_1, v_2\}_{u_1}$. For $l = 0$, it is clearly true. Suppose it is true for vertices whose above-defined distance is at most l'. Let $(u_{l'}, v_{l'})$ be the vertex at distance l'. By inductive hypothesis, $v_{l'}$ is either v_1 or v_2. Let $c \in \{a, b\}$ be the colour of the edge joining $(u_{l'}, v_{l'})$ and $(u_{l'+1}, v_{l'+1})$. If $c = a$, then $v_{l'+1} = v_{l'}$ and hence the hypothesis is clearly true for $l = l' + 1$. If $c = b$ (hence $u_{l'+1} = u_{l'}$) and if $v_{l'} = v_1$, then $v_{l'+1} = v_2$. This follows from (i) the b-coloured edge incident at the copy of u_1 in G_{v_1} joins it to the copy of u_1 in G_{v_2} and hence (ii) all edges of the perfect matching joining isomorphic copies of vertices in G_{v_1} and G_{v_2} are coloured with b. In particular, the b-coloured edge incident at $(u_{l'}, v_1)$ joins it to $(u_{l'}, v_2)$. Similarly, one can argue that if $c = b$ and if $v_{l'} = v_2$, then $v_{l'+1} = v_1$. In any case, $v_{l'+1} \in \{v_1, v_2\}$, there by proving that C lies entirely within G_{v_1} and G_{v_2}. □

Since the edges in G_{v_1} and G_{v_2} are coloured without using colour b and since every alternate edge of C is coloured with b, we see that b is used an even number of times in C. This implies $|C| = 0 \pmod 4$. Thus, C looks like

$$C = \langle\ (u_1, v_1), (u_1, v_2), (u_2, v_2), (u_2, v_1), \ldots, (u_{2k-1}, v_2), (u_{2k}, v_2), (u_{2k}, v_1), (u_1, v_1)\ \rangle.$$

For each of the a-coloured edges in G_{v_2} joining (u_{2l-1}, v_2) and (u_{2l}, v_2), its isomorphic copy in G_{v_1} joins (u_{2l-1}, v_1) and (u_{2l}, v_1) and is coloured with the colour $c = \sigma_i(\sigma_j^{-1}(a)) \neq a$ where $i = Y_H(v_1)$ and $j = Y_H(v_2)$. These isomorphic copies in G_{v_1} of a-coloured edges of C in G_{v_2} together a-coloured edges of C in G_{v_1} constitute the following bichromatic cycle

$$D = \langle\ (u_1, v_1), (u_2, v_1), (u_3, v_1), \ldots, (u_{2k}, v_1), (u_1, v_1)\ \rangle.$$

This is a contradiction to the fact that X restricted to G_{v_1} is acyclic. This shows that X admits no bichromatic cycle and hence X is proper and acyclic. Since X uses only colours from $[\eta] \cup [\beta']$, we get $a'(G \Box H) \leq a'(G) + a'(H)$.

4.4 Consequences

The following results for certain special families of graphs, are immediate consequences of the result of the previous section.

Corollary 4.4.1. *Let $G_1, \ldots, G_k$ be k connected non-trivial graph such that for each i, $1 \leq i \leq k$, $a'(G_i) = \Delta(G_i)$ and $\max\{a'(G_1), \ldots, a'(G_k)\} > 1$. Then,*

$$a'(G_1 \Box \ldots \Box G_k) = \Delta(G_1 \Box \ldots \Box G_k).$$

Proof. Follows from
(i) $a'(G) \geq \Delta(G)$ for any G,
(ii) $\Delta(G_1 \Box \ldots \Box G_k) = \Delta(G_1) + \ldots + \Delta(G_k)$,
(iii) Theorem 4.3.1. □

Corollary 4.4.2. *Let G be a connected non-trivial graphs such that $a'(G) = \Delta(G) > 1$. Then, for each $d \geq 1$,*

$$a'(G^d) = d\Delta(G).$$

Even though the following corollary has already been presented in Chapter 3, we present it here for the sake of completeness.

Corollary 4.4.3. *Let $G = P_2^d = P_2 \Box \cdots \Box P_2$ be the d-dimensional hypercube for some $d \geq 1$. Then,*

$$a'(P_2) = 1 \quad \textit{and} \quad a'(P_2^d) = d + 1 \quad \textit{for} \quad d > 1.$$

Proof. Suppose $d > 1$. Since $G = P_2^d$ is d-regular, we need at least $d + 1$ colours in any acyclic edge colouring of P_2^d and hence $a'(G) \geq d+1$. Also, $a'(P_2^2) = a'(C_4) = 3$. Starting with $G = P_2^2$ and applying Theorem 4.3.1 repeatedly by setting $H = P_2$ each time, we get $a'(P_2^d) \leq a'(P_2^2) + (d-2) \leq d+1$. Combining both the lower and upper bounds, we get the result. □

4.5 Conclusions

The work presented in this chapter has been submitted to a journal. The reference is [MS07].

It is quite possible that Conjecture 1.2.1 is true. Under this assumption, we know that the gap between the maximum degree of a graph and its acyclic chromatic index is at most 2. Note that for any Δ-regular graph ($\Delta > 1$), the gap is at least 1. Thus, by applying Theorem 4.3.1 repeatedly on such graphs, the difference between the bound obtained and the maximum degree increases for each additional factor added. Thus if the conjecture is true, the bounds obtained by applying our result cannot be optimal. Nevertheless, it is not possible to make an unqualified claim of the type $a'(G \square H) \leq a'(G) + \Delta(H)$. It is well-known that $\Delta(G \square H) = \Delta(G) + \Delta(H)$. Therefore, if the gap is 0 for G but positive for $G \square H$, then this will violate a claim of the form made previously. We conclude that, if the conjecture is true, then a statement of the form $a'(G \square H) \leq a'(G) + \Delta(H)$ can be made, only if the gap is 2 for G.

From an algorithmic point of view, our proof immediately yields an efficient algorithm to obtain a colouring of $G \square H$ using $a'(G) + a'(H)$ colours, provided colourings for G and H are available using disjoint sets of $a'(G)$ and $a'(H)$ colours, respectively. Its strength as an algorithmic result, however, is undermined by the fact that optimal colourings of the prime factors themselves are not always easily computable.

Chapter 5

Acyclic edge colouring of partial 2-trees

In this chapter we consider the problem of acyclically edge colouring the class of *partial* 2-*trees*. These are precisely the graphs of treewidth at most 2. More generally, for $k \geq 0$, the partial k-trees are precisely the class of all graphs with treewidth at most k. The partial k-trees are a strict subclass of the class of all k-degenerate graphs. We define these closely related classes of graphs and describe a hierarchy among them, in Section 5.2. We elaborate some properties of these graphs in Section 5.1, which facilitate the acyclic edge colouring problem on partial 2-trees. Section 5.4 states the results we have obtained and the proofs follow in subsequent sections. Closely related results are mentioned in Section 5.3. A brief description of the algorithm corresponding to the proof is given in Section 5.7. Section 5.8 incorporates a few concluding remarks and outlines possible future directions for research.

5.1 Introduction

As the reader is familiar, the acyclic edge colouring problem is about colouring the edges of a graph properly while simultaneously avoiding bichromatic cycles. It stands to reason that the more cyclic structure a graph has the more difficult it is to acyclically colour it. As always, the difficulty is present because the number of colours to be used is stringently restricted. In order to understand this better, a rigourous quantitative notion of the cyclic structure of a graph is needed.

The treewidth of a graph is indirectly a measure of its cyclic structure and indicates how close the graph is to a tree. The treewidth, defined formally below, is 1 for the class of trees while it is $n-1$ for the complete graph K_n. These are the ends of the spectrum and every conected graph has treewidth in this range. The acyclic chromatic index a' of a tree is its maximum degree Δ. Further, it is absolutely straightforward to obtain an acyclic edge colouring of trees using a' colours. In sharp contrast, it has proved extremely difficult to obtain tight estimates on a' for complete graphs, inspite of their simple and symmetric structure.

This suggests that it might be easier to get tighter bounds on the acyclic chromatic index when we focus on classes of graphs with small treewidth. Motivated by this, we study the acyclic edge colouring problem for such graphs.

5.2 Definitions

The treewidth of a graph is defined in terms of a notion called tree decomposition of the graph. We make these notions precise in the following definitions.

Definition 5.2.1. Given a graph $G = (V, E)$, a tree decomposition of G is any tree T whose nodes are labelled by subsets of V, such that:

- every vertex $v \in V$ appears in the label of at least one node of T
- for every edge $e = (u, v) \in E$, there is at least one node of T whose label contains both u and v
- for any vertex $v \in V$ the set of nodes whose labels contain v induces a connected subgraph of T

In a tree decomposition the number of vertices of the original graph used to label a node of the tree is called the label size of that node. The width of the tree decomposition is the largest label size of any node in that decomposition.

Definition 5.2.2. The treewidth of a graph G, denoted by $tw(G)$, is exactly one less than the minimum width of any tree decomposition of G.

Treewidth is a monotonic property, in the sense that the treewidth of a subgraph is at most the treewidth of the original graph.

We now define k-trees which are the basis of the definition of partial k-trees.

Definition 5.2.3. A k-tree is any graph obtained from the complete graph K_{k+1}, by a sequence of zero or more operations of adding a new vertex adjacent to the vertices of an existing k-clique in the graph.

Definition 5.2.4. A partial k-tree is any subgraph of a k-tree.

It can be seen that k-trees have treewidth exactly k. The monotonicity of the treewidth property implies that partial k-trees also have treewidth at most k. In fact, it is well-known that they are exactly the class of graphs of treewidth at most k. A closely related notion is the class of k-degenerate graphs defined below.

Definition 5.2.5. A k-degenerate graph is any graph obtained from the graph on a single vertex, K_1, by a sequence of zero or more operations of adding a new vertex adjacent to *at most* k existing vertices in the graph.

Note that every partial k-tree is also k-degenerate. These classes, though their definitions appear similar, differ strongly. It is known that all planar graphs are 5-degenerate, but planar graphs have unbounded treewidth. An infinite family of such examples is the family of square grids, $P_n \square P_n$. All these graphs are planar but have treewidth at least n.

A property of any k-degenerate G, is that its minimum degree $\delta(G) \leq k$. Additionally, for every subgraph, $H \subseteq G$, $\delta(H) \leq k$. This is an alternative, equivalent characterisation of k-degenerate graphs which often proves useful.

5.3 Related results

As we mentioned earlier, partial 2-trees are a subclass of 2-degenerate graphs. We state here some results on acyclic edge colouring of 2-degenerate graphs. The earliest result on acyclic edge colouring of 2-degenerate graphs was by Card and Roditty [CR94], where they proved that $a'(G) \leq \Delta + k - 1$, where k is the maximum edge-connectivity, defined as $k = \max_{u,v \in V(G)} \lambda(u,v)$, where $\lambda(u,v)$ is the edge-connectivity of the pair u, v. Note that, here, k can be as high as Δ.

Alon and Zaks show that it is NP-Hard to determine a' for subcubic graphs (see [AZ02]). It follows, from the reduction they use, that the problem is hard even for 2-degenerate graphs. It is possible that the problem is hard for many non-trivial subclasses of 2-degenerate graphs. In this context, it is significant that there have been a few results which give bounds on a' which differ from the optimum by at most one or two, and also provide efficient algorithms to produce such colourings. Recently, it has been shown by Muthu, Narayanan and Subramanian, that if G is an outerplanar graph then $a'(G) \leq \Delta+1$ [MNS07a]. Subsequently, a weaker bound of $a'(G) \leq \Delta + 2$ has been obtained by Manu and Chandran for the larger class of all 2-degenerate graphs [CM07]. More recent information indicates that Manu and Chandran have proved a better upper bound of $\Delta + 1$ for the class of 2-degenerate graphs also.

5.4 Our results

Here, we formally state the results we obtain. By setting $k = 2$ in Definitions 5.2.3 and 5.2.4 we obtain the following definitions.

Definition 5.4.1. A 2-*tree* is any graph obtained from the triangle K_3, by a sequence of zero or more operations of adding a new vertex adjacent to the endpoints of an existing edge in the graph.

Definition 5.4.2. A *partial* 2-*tree* is any subgraph of a 2-tree.

The main contribution of this chapter is the following result.

Theorem 5.4.1. *If G is a 2-tree or a partial 2-tree, then $a'(G) \leq \Delta + 1$.*

Since every series-parallel graph (defined below) is a partial 2-tree (see [Gra99]), we obtain the following corollary.

Corollary. *If G is a series-parallel graph, $a'(G) \leq \Delta + 1$.*

Definition 5.4.3. A *series-parallel graph* is any simple graph obtained starting from K_2 and performing any sequence of the following two operations:
(i) subdivide an edge
(ii) add edges parallel to existing edges.
Finally all multiple edges are eliminated to render the graph simple.

An independent proof of this result has been obtained in [CM07]. That proof uses ideas similar to the proof that $a'(G) \leq \Delta + 2$ for 2-degenerate graphs G, given in the same paper.

We first prove the bound for 2-trees in Section 5.5 and then extend the arguments to work for the more general partial 2-trees in Section 5.6.

5.5 Bound for 2-trees

First, we define the term 2-*ear*. The process of adding a new vertex w adjacent to two existing vertices u and v (not necessarily adjacent) in a graph G, is called adding a 2-*ear* to G. In the case of 2-trees, the vertices u and v must be adjacent. The path added through the new vertex w, described above is called a 2-*ear* of the graph. Note, that 2-trees can be constructed starting from a triangle and repeateding the procedure of adding 2-*ears* to the endpoints of an edge.

From Definition 5.4.1, it is easily seen that 2-trees are triangulated (chordal), planar and 2-degenerate. With respect to that definition of 2-trees, we introduce the following terms and notations. The triangle with which the construction is initiated is called the *initial triangle* or *base triangle*. For a given edge (u, v), the set of all *ears* added between its endpoints, is denoted by $ext(u, v)$. (u, v) is called the *base edge* for each of these *ears*.

We colour the graph incrementally by adding more edges to a partial colouring in batches. The incrementation used here is different from other incremental procedures where only a single edge is introduced at a time. Here, in each stage, we introduce a set of all 2-*ears* having the same base edge. The edge order is described in greater detail later.

5.5.1 Assumptions

At any intermediate stage the number of colours used is one more than the *current maximum degree*, Δ. Here, *a stage refers to the addition of all* 2-*ears having the same base edge*, and the *current maximum degree refers to the maximum degree after the introduction of this entire batch of* 2-*ears*.

We use $\mathcal{L}_v$ to denote the subset of colours not seen by the vertex v in the current partial colouring of G (*prior to colouring the current batch of edges*). Note that $|\mathcal{L}_v| \geq U_v + 1$

(since we use $\Delta + 1$ colours), where U_v is the number of uncoloured edges incident to v. $d(v)$ denotes the degree of the vertex v. The above described notation is used in the description of all our colouring procedures.

5.5.2 Acyclic colouring of $K_{2,t}$

Now, we describe how to acyclically edge colour any member of a special class of graphs. In the colouring of 2-trees, the graphs induced by the edges added in a batch as described above, all belong to this special family. This colouring procedure is, thus, a subroutine used in our colouring of 2-trees. The special class we refer to is the family of complete bipartite graphs ($K_{2,t}$), where one of the partite sets has exactly two vertices. We describe one *normal* acyclic edge colouring of this class of graphs and another acyclic edge colouring based on *lists* associated with each edge.

The following lemma describes an acyclic edge colouring for the complete bipartite graph, $K_{2,t}$, using t colours if $t \geq 3$, and using $t + 1$ colours otherwise.

Lemma 5.5.1. For the complete bipartite graph, $K_{2,t}$,

$$a'(K_{2,t}) = \begin{cases} t & \text{if } t \geq 3 \\ t+1 & \text{otherwise.} \end{cases}$$

Proof. Let $\mathcal{A} = \{u, v\}$ be the partite set of size two, and let $\mathcal{B} = \{w_1, \ldots, w_t\}$ be the partite set of size t. If $t = 1$ or 2, the colouring is straightforward. If $t \geq 3$, colour (u, w_i) with colour i, for $i \in \{1, \ldots, t\}$ and colour (v, w_i) with colour $i + 1$, for $i \in \{1, \ldots, t - 1\}$ and colour (v, w_t) with colour 1. It is easy to observe that the subgraph induced by any pair of colour classes is either a path on 4 edges, or a collection of two vertex disjoint paths on 2 edges each (see Figure 5.1). In each case, the subgraph is acyclic and hence the colouring is proper and acyclic. Further, if $t \geq 3$, the endpoints of these paths both lie in $\mathcal{B}$. Even if $t = 2$, one endpoint of any maximal bichromatic path lies in $\mathcal{B}$. We call this scheme *colouring by shifting* and use it as a subroutine in the colouring of partial 2-trees. □

We use the above-mentioned procedure in our colouring of 2-trees, when the two endpoints u and v of a base edge $e = (u, v)$, such that $|ext(e)| = k$ ($k \geq 3$), have at least k common free colours.

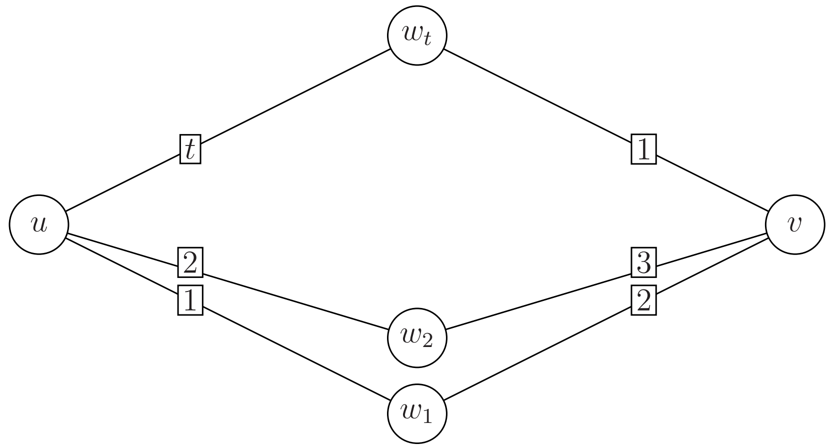

Figure 5.1: acyclic colouring of $K_{2,t}$

We now describe a generalised version of the previous lemma which describes an acyclic edge colouring of the same graph when the colours allowed for any edge is restricted to an associated list.

Lemma 5.5.2. Consider the complete bipartite graph $H = (\mathcal{A}, \mathcal{B}, F)$ with $\mathcal{A} = \{u, v\}$ and $\mathcal{B} = \{w_1, \ldots, w_t\}$. Let L_u denote a set of t colours which are permitted for edges incident at u. L_v is defined similarly. Then, there is an acyclic edge colouring of H using only colours from L_u and L_v for edges incident at u and v respectively.

Proof. Without loss of generality, assume that $I = L_u \cap L_v = \{1, \ldots, i\}$ is the set of $i \geq 0$ colours available for edges incident at both u and v and also that $L_u \setminus I = \{i+1, \ldots, t\}$ and also that $L_v \setminus I = \{t+1, \ldots 2t-i\}$. Then, colour the edges $(u, w_1), \ldots (u, w_t)$ with $1, \ldots, t$ respectively. Colour the edges $(v, w_1), \ldots, (v, w_{i-1})$ with $2, \ldots, i$ respectively. Colour the edges $(v, w_i) \ldots, (v, w_{t-1})$ with $t+1, \ldots, 2t-i$ respectively and (v, w_t) with colour 1 (see Figure 5.2). It can be seen that this colouring is proper and acyclic. We call this scheme also *colouring by shifting* and use it as a subroutine in the colouring of partial 2-trees. □

We use this procedure (described in the proof of Lemma 5.5.2) in our colouring of 2-trees, when the two endpoints u and v of a base edge $e = (u, v)$, such that $|ext(e)| = k$ ($k \geq 2$), do not *necessarily* have k common free colours. Note that the colouring which results from applying Lemma 5.5.2, when $t \leq 2$ and the lists L_u and L_v are identical is not acyclic. The colouring scheme described in the proof produces a proper and acyclic colouring only if *either* the lists are distinct *or* $t \geq 3$.

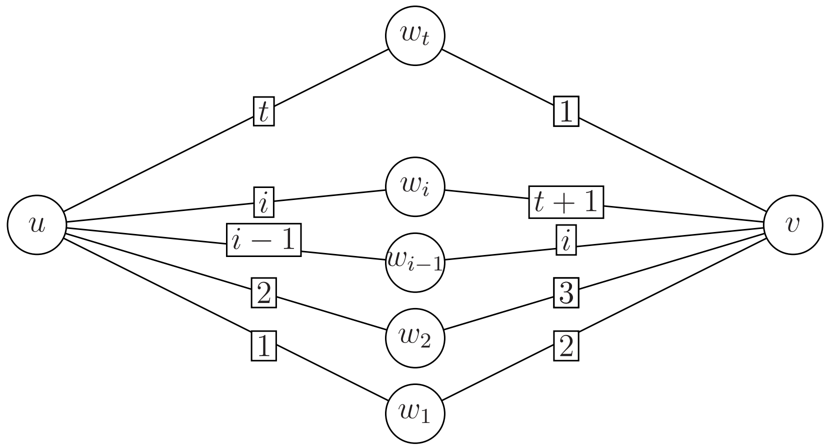

Figure 5.2: list acyclic colouring of $K_{2,t}$

5.5.3 The colouring procedure

We may assume that the given graph is biconnected, since an acyclic colouring of any graph can be obtained from the acyclic colourings of its biconnected components using $a'(G)$ colours. First we prove the result for the class $\mathcal{T}$ of 2-trees and then extend it, in the next section, to include all of $\mathcal{P}$, the partial 2-trees. We use the following easy to verify fact repeatedly in our proofs.

Observation 1. *If G is a 2-tree, one can construct G from any arbitrary triangle of G by repeatedly adding 2-ears according to Definition 5.4.1.*

We obtain a $(\Delta + 1)$-acyclic edge colouring of any 2-tree by an iterative colouring procedure which incorporates more edges at each stage into an existing partial colouring until the graph is fully coloured. There is in general more than one way in which a 2-tree can be constructed according to Definition 5.4.1. We fix one such construction, and with respect to it, define a notion of *level* to classify edges. The edges are introduced in increasing order of level number and coloured immediately. The procedure never alters the colour of an edge once it has been assigned.

We reconstruct the graph G by starting from any triangle $T = \{a, b, c\}$ as mentioned in Observation 1, and building the graph *ear* by *ear*. We call this the *base triangle* or *initial triangle*. Recall that $ext(u, v)$ denotes the set of all 2-$ears$ having (u, v) as its base edge.

We also assign a nonnegative integer value to each edge e and call it the *level* of e and denote it by $level(e)$. Initially, the three edges of T are assigned level 0. The level number is defined inductively. If $e = (u, v)$ is any edge already added such that $level(e) = i$, then for each 2-ear $(u, w, v) \in ext(u, v)$, we assign $level(u, w) = level(w, v) = i + 1$ and add this 2-ear. In addition, we we follow the convention that:
(a) edges are added in increasing order of their level numbers and
(b) if (u, w, v) is a level i ear, then all $(i + 1)$-level $ears$ of $ext(u, w)$ appear contiguously (the same holds for $ext(w, v)$ also) with 2-$ears$ of one set appearing *immediately before or after* the 2-$ears$ of the other set.
This is the order in which the edges are introduced and coloured. Figure 5.3 indicates how the graph looks just prior to the addition of a set of $level(i + 1)$ edges.

In the following, we use Δ to denote, always, the maximum degree of the current graph (after adding the edges to be coloured at this step). The colouring procedure can be summarised as follows.

1. Colour the base triangle $T = \{ab, bc, ac\}$ with the colours $1, 2, 3$, respectively.

2. Colour the *level* 1 edges in the order $ext(a, b)$, followed by $ext(b, c)$, and finally $ext(a, c)$. We assume, without loss of generality, that $|ext(a, b)| \leq |ext(b, c)| \leq |ext(a, c)|$, where the notation $|ext(u, v)|$ represents the number of ears having the edge (u, v) as its base edge.

    - If $|ext(a, b)| = 1$, then colour the new edge incident to vertex a with colour 2 (which is free there) and the edge incident to b with the newly available colour (due to the increase in Δ). If $|ext(a, b)| = 2$ colour according to Lemma 5.5.2 where $L_a = \{2, 4\}$ and $L_b = \{3, 4\}$. If $|ext(a, b)| \geq 3$ colour according to Lemma 5.5.1 using colours from $\{4, \ldots, |ext(a, b)| + 3\}$.

- If $|ext(b,c)| = 1$, then colour the new edge incident to b with the newly available colour (again due to increase in Δ), and the edge incident to c with any missing colour. If $|ext(b,c)| = 2$, Δ increases by 2. Let the two new colours be n_1 and n_2. Let α be some original colour missing at b. Colour $ext(b,c)$ using Lemma 5.5.2, with $L_b = \{n_1, \alpha\}$ and $L_c = \{n_1, n_2\}$. If $|ext(b,c)| \geq 3$, then colour $ext(b,c)$ using Lemma 5.5.1, since there are $|ext(b,c)|$ common free colours (the set of new colours due to increase in Δ).

- If $|ext(a,c)| = 1$, then, we know that $|ext(a,b)| \leq 1$ and $|ext(b,c)| \leq 1$. The reader can verify that the colouring can be extended using the stipulated $\Delta + 1 \leq 5$ colours. If $|ext(a,c)| = 2$, then *either* $|ext(a,b)| = 2$ and $|ext(b,c)| = 2$ *or* Δ increases on account of adding $ext(a,c)$. In the former case, there exist non-identical lists of two colours each, missing at a and c. Colour $ext(a,c)$ using Lemma 5.5.2 with these lists. In the latter case, colour $ext(a,c)$ using Lemma 5.5.2 where $L_a = \{\alpha, n\}$ and $L_b = \{\beta, n\}$. Here, α and β are distinct colours missing at a and b respectively, while n is the new colour. If $|ext(a,c)| \geq 3$, colour $ext(a,c)$ using Lemma 5.5.1 provided there is a list of $|ext(a,c)|$ common free colours at a and c, and using Lemma 5.5.2 otherwise.

3. For $i \geq 1$, the procedure for colouring *level*-$(i+1)$ edges is as follows. Assume that all edges up to *level*-i have already been added and coloured acyclically (using $\Delta + 1$ colours). Assume that for some *level* i 2-*ear*, (u, w, v), we add the edges in $ext(u,w)$ followed by those in $ext(w,v)$ in incrementally building the graph. We refer to (u, w, v) as the *base ear*. Refer to Figure 5.3. Let the number of 2-*ears* in $ext(u,w)$ and $ext(w,v)$ be, respectively, k_1 and k_2. We assume, without loss of generality, that $k_1 \leq k_2$. Colour the new edges as described below, under *colour extension*.

5.5.3.1 Colour extension

We describe below how to extend the colouring $\mathcal{C}$ to the newly added *ears*. The procedure falls under a number of cases according to the values of k_1 and k_2. We colour the *ears* in $ext(u,w)$ first and then those in $ext(v,w)$. Let $\mathcal{C}(uw) = x$ and $\mathcal{C}(vw) = y$ and $\mathcal{C}(uv) = a$. Notice that since we use $\Delta + 1$ colours, we have $|\mathcal{L}_u| \geq k_1 + 1, |\mathcal{L}_v| \geq k_2 + 1$ and

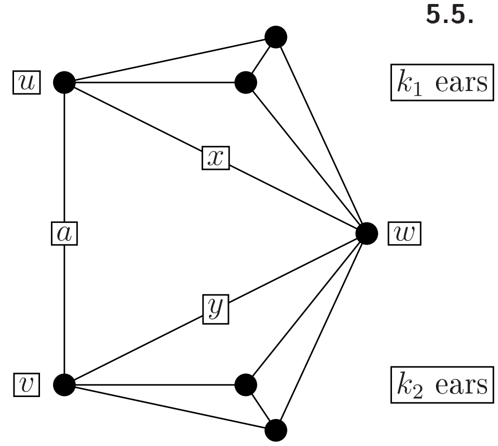

Figure 5.3: colour extension

$|\mathcal{L}_w| \geq k_1 + k_2 + 1$. One should also note that $|\mathcal{L}_u \cap \mathcal{L}_w| \geq k_1$ at the point of beginning the colouring of $ext(u, w)$ (because we colour $ext(u, w)$ before colouring $ext(v, w)$). Recall that $k_1 \leq k_2$ as assumed before.

In the colouring scheme below, we always colour the k_1 *ears* in $ext(u, w)$ by applying Lemma 5.5.1. It is worth recalling from the proof of Lemma 5.5.1 that any maximal bichromatic path in $ext(u, w)$ has at least one endpoint distinct from u and w. As a result, the subsequent colouring of ears in $ext(v, w)$ cannot create any bichromatic cycle passing through $ext(u, w)$.

- **Case** $k_1 = 0$**:**
  If $k_2 = 1$, colour the edge of the *ear* incident to w using $\mathcal{C}(uv) = a$ and other edge with any colour from $\mathcal{L}_v \setminus \{x\}$. Similarly if $k_2 = 2$, colour one *ear* with 2 colours from $\mathcal{L}_v \setminus \{x\}$ and the other *ear* with one of these colours and colour a. Otherwise, the k_2 *ears* based on edge (v, w) are coloured using k_2 colours from $\mathcal{L}_v \setminus \{x\}$, as given in Lemma 5.5.1.

- **Case** $k_1 = 1$**:**
  In this case, we color the single *ear* based on (u, w) with the colours a and some colour from $\mathcal{L}_u \setminus \{y\}$. If $k_2 = 1$, we colour the ear based on (v, w) properly, avoiding the colour x. If $k_2 = 2$, we colour the ears based on (v, w), using nonidentical list of colours, avoiding the colour x. If $k_2 \geq 3$, we pick a subset of k_2 colours from $\mathcal{L}_v \setminus \{x\}$ and a set of same size from $\mathcal{L}_w$. Note that these sets might be identical or distinct.

We accordingly use the colouring procedure in Lemma 5.5.1 or Lemma 5.5.2 to extend the partial colouring.

- **Case** $k_1 = 2$**:**
  In this case, one of the *ears* based on (u, w) is coloured using 2 colours from $\mathcal{L}_u \setminus \{y\}$ while the other is coloured using a and one of these colours. If $k_2 = 2$, we pick non-identical lists of colours and colour $ext(v, w)$ using these lists according to Lemma 5.5.2. If $k_2 \geq 3$, for the *ears* based on (v, w), we pick a subset of k_2 colours from $\mathcal{L}_v \setminus \{x\}$ and a set of the same size from $\mathcal{L}_w$. If these sets are identical we extend the colouring according to Lemma 5.5.1, and otherwise according to Lemma 5.5.2.

- **Case** $k_1 \geq 3$**:**
  In this case, the *ears* based on (u, w) are coloured using k_1 colours from $\mathcal{L}_u \setminus \{y\}$. Observe that here, $L_w \subset L_u \setminus \{y\}$, so the selected set of colours are free at both endpoints of the *ear* set, thus enabling the application of Lemma 5.5.1. For the *ears* based on (v, w), we pick a subset of colours from $\mathcal{L}_v \setminus \{x\}$ of cardinality k_2 and a set of same size from $\mathcal{L}_w$. We colour according to Lemma 5.5.1 or Lemma 5.5.2, depending on whether these lists are identical or distinct.

We now need to argue that the new partial colouring obtained as a result of the *colour extension* procedure is proper and acyclic, in each of the four cases. However, we argue in detail only for the case $k_1 \geq 3$. The arguments for the other cases are of a similar nature and are simpler.

Prior to colouring $ext(u, w)$ and $ext(v, w)$, the graph is assumed to be coloured properly and acyclically. Observe that the colouring procedure of Lemma 5.5.1, never creates bichromatic cycles in the graph induced by the edges of $ext(u, w)$. From that lemma, it is also clear that there is no maximal bichromatic path involving the edges of $ext(u, w)$ with endpoints as u and w. It follows that there is no bichromatic cycle using a combination of the old and new edges. Thus the colouring is proper and acyclic after the addition of $ext(u, w)$.

Now, $ext(v, w)$ is coloured according to Lemma 5.5.2, since the list of available colours available at v and w need not necessarily have k_2 common colours. From the lemma, we know that the edges of $ext(v, w)$ do not induce any bichromatic cycles. We do not use the colour x in $ext(v, w)$, so any bichromatic cycle using edges of $ext(v, w)$ must also use edges of $ext(u, w)$. However, that is not possible, since any bichromatic path starting at w and entering $ext(u, w)$ terminates at a vertex in $ext(u, w)$ distinct from u and w. Thus there is no bichromatic cycle using a combination of old and new edges.

5.6 Partial 2-trees

Here, we extend the proof given above to partial 2-trees.

Given any partial 2-tree T, we consider any 2-tree G which contains T as a subgraph. We mark all the edges of G which are not in T as *imaginary edges*. We use the imaginary edges only to classify the level of edges for the further addition of *ears*. They do not contribute to the degree of a vertex in G. They are never coloured. The important point to notice is that, again we need only $\Delta(T) + 1$ colours to extend the partial colouring at any stage. As before, $\Delta(T)$ refers to the maximum degree of the partial 2-*tree* at the current stage. An *ear* consisting of two real edges is called a *full ear*, while *ears* with one real edge and one imaginary edge are called *half ears*. Observe that *empty ears* (both edges are imaginary) are inconsequential, since we do not colour them at all, and only use their endpoints for the addition of higher level *ears*.

Suppose, at any point, we are to colour k_1, k_2 pairs of *ears* (some of them could be *half ears*). We notice that if there are k uncoloured *real* edges at an endpoint, then we have at least $k + 1$ available colours for the edges incident at the endpoint.

Here, the *ears* having the same base edge (real or imaginary) are ordered with all the *full ears* first followed by the *half ears* and finally by the *empty ears*. Colour the *full ears* as mentioned earlier for 2-trees and extend the colouring to *half ears* in a proper fashion. It follows that such a colouring is proper and acyclic. It is identical to the case of 2-trees, except for the *half ears*. However, *half ears* only give rise to pendant edges and cannot create bichromatic cycles, so any proper colouring is sufficient. This completes the proof of Thorem 5.4.1 for partial 2-trees.

5.7 Algorithmic aspects

Our proof that a partial 2-tree can be acyclically coloured using $\Delta + 1$ colours, can be made constructive yielding an efficient algorithm to produce such a colouring.

The proof consists of a colouring procedure which colours the set of edges considered in a specific order. This naturally divides the procedure into two phases. In the first phase, the order of the edges is computed. In the second phase the edges are coloured considering them in this order. Strictly speaking, the edge order is a partial order, and not a total order, since we introduce them in batches rather than one by one.

By observation 1, we can construct the graph starting from any triangle. Finding a triangle in a graph can be done using a standard graph searching algorithm like Breadth First Search (BFS). Subsequent computation of the edge order consists of finding the set of all common neighbours of the endpoints of each base edge considered in increasing order of levels. This can be accomplished by a modification of the basic BFS procedure.

Our colouring of 2-trees begins with the colouring of the base triangle. At each subsequent stage, a bipartite graph is coloured *either* in a very simple way *or* using one of Lemmas 5.5.1 and 5.5.2.

The colouring of the initial triangle takes constant time. The colouring is then extended to include at each stage the edges extending a fixed base edge. In order to perform this step, we need to compute the list of available colours at each of the endpoints of the base edge. Prior to colouring an extension, comparison between these lists needs to be made in order to determine the set of common colours and also the symmetric difference of these lists. After these lists are computed, we order the edges of the extension and assign colours to each of the edges. The extension procedure can be performed at a cost of $O(\Delta^2)$ to the running time.

A simple calculation then reveals that the entire graph can be coloured within $O(n^2)$ time. This is also an upper bound on the time taken to compute the edge order of the whole graph.

5.8 Future directions

The work presented in this chapter has been submitted to a conference. The reference is [MNS08]. It would be interesting to extend these ideas and see if similar or even weakened results can be obtained for the partial k-trees for higher values of k. The results obtained by similar methods is likely to yield bounds as a function of both Δ and k, rather than only Δ.

Chapter 6

Conclusions

6.1 Summary

In this thesis, we studied the problem of acyclic edge colouring of graphs. We introduced a new colouring idea and proved that $a'(G) \leq 4.52\Delta$ for all graphs of girth $g(G) \geq 220$. This improves the previous best bound of 16Δ for graphs with girth at least 220. This is a step towards obtaining tight bounds on $a'(G)$.

We also illustrated a general relationship between the girth of a graph and its acyclic chromatic index, which highlights the fact that acyclically colouring a graph seems harder for graphs with small cycles.

We obtained optimal or nearly optimal estimates on the acyclic chromatic index of some structured classes of graphs. For these classes, we also provided efficient algorithms to construct the corresponding optimal colourings. The classes of graphs for which we have obtained near optimal estimates on the acyclic chromatic index are grid-like graphs, partial 2-trees and outerplanar graphs (not in this thesis). This is interesting considering that for other structured clases like complete graphs no optimal or near optimal estimate is known at present.

We also correlate $a'(G)$ to the corresponding values of its prime factors under the cartesian product operation. Thus, tight bounds on the a' of the prime factors of a graph G, lead to reasonably tight bounds on $a'(G)$ as well. Also, if the actual colourings of the prime factors can be computed efficiently, then a colouring of the resultant graph can also be computed efficiently by our method of proof.

6.2 Future directions

It is easy to see that $a'(K_n) \leq p$, where p is the smallest prime greater than or equal to n. But the gap between p and n could be as large as $\sqrt{n}$ for certain values of n. So, it would be interesting to obtain a $\Delta + O(1)$ bound for K_n.

It would also be interesting to obtain improved bounds for general classes of graphs.

If G is a partial torus isomorphic to the product of a path and a number of odd cycles, $a'(G)$ can take either of the values in $\{\Delta, \Delta+1\}$. It would be interesting to see if we can classify such graphs for which $a'(G) = \Delta$ and also construct optimal colourings efficiently.

It is quite possible that Conjecture 1.2.1 is true. Under this assumption, we know that the gap between the maximum degree of a graph and its acyclic chromatic index is at most 2. Note that for any Δ-regular graph ($\Delta > 1$), the gap is at least 1. Thus, by applying Theorem 4.3.1 repeatedly on such graphs, the difference between the bound obtained and the maximum degree of the resultant graph increases for each additional factor.

Thus, assuming the truth of the conjecture, it is not possible to make a statement of the form $a'(G \Box H) \leq a'(G) + \Delta(H)$. It would be interesting to find conditions on G and H which would enable us to make such a statement.

It is a challenge to obtain constructive bounds better than the currently best known $O(\Delta \log \Delta)$, for all graphs. It would be nice to improve the best known bound of 16Δ for all graphs.

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