ASYMPTOTIC BEHAVIOUR OF SOME OPTIMAL CONTROL PROBLEMS

Thesis submitted in partial fulfilment of the degree of Doctor of Philosophy (Ph.D.)
by

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May 2006

DECLARATION

I declare that the thesis entitled *Asymptotic Behaviour of some Optimal Control Problems* submitted by me for the degree of Doctor of Philosophy is the record of work carried out by me during the period from *April 2002* to *January 2006* under the guidance of *Prof. S. Kesavan* and has not formed the basis for the award of any degree, diploma, associateship, fellowship, titles in this or any other University or other similar institution of Higher Learning.

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CERTIFICATE

I certify that the thesis entitled *Asymptotic Behaviour of some Optimal Control Problems* submitted for the degree of Doctor of Philosophy by Mr. T. Muthukumar is the record of research work carried out by him during the period from April 2002 to January 2006 under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning. I further certify that the new results presented in this thesis represent his independent work in a very substantial measure.

S. Kesavan

S. Kesavan

Acknowledgements

I wish to record my sincere thanks to:

Prof. S. Kesavan, for accepting to be my doctoral supervisor and introducing me to the subject of Homogenization; for his efforts to improve my teaching and writing skills; for his patience in dealing with me; for letting me grow naturally at my own pace, but always ready to help when necessary.

The Institute of Mathematical Sciences (IMSc.) for its excellent facilities and supportive staffs.

M. Rajesh for willingly discussing with me when I started working on my thesis problems and to N. Sabu for various important exchanges and interesting stories.

My sister for lending her IBM Thinkpad when I was typing this thesis.

My friends at IMSc. for many pleasant, interesting and useful academic and non-academic conversations.

"... if you don't see that what you are working on is almost obvious, then you are not ready to work on that yet,... Prepare the way. ... everything should be so natural that it just seems completely straightforward."

— ARTHUR OGUS, On Grothendieck: Notices of the AMS (November 2004)

Contents

Notations

Symbols

$\mathbb{R}$ denotes the real line

$\mathbb{R}^n$ denotes the n-dimensional Euclidean space over $\mathbb{R}$. $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$

$\overline{\mathbb{R}}$ $\mathbb{R} \cup \{-\infty, +\infty\}$

$|E|$ is the Lebesgue measure of $E \subset \mathbb{R}^n$

$\overline{E}$ denotes the closure of $E \subset \mathbb{R}^n$ in the usual topology

Ω denotes an open bounded subset of $\mathbb{R}^n$

$\partial\Omega$ denotes the boundary of Ω

$\Omega \subset\subset \Omega'$ denotes a bounded open subset Ω of Ω' such that $\overline{\Omega} \subset \Omega'$

$\mathcal{M}(a, b, \Omega)$ denotes, for $0 < a < b$, the class of all $n \times n$ matrices, $A = A(x)$, with $L^\infty(\Omega)$ entries such that,

$$a|\xi|^2 \leq A(x)\xi.\xi \leq b|\xi|^2 \quad a.e.\ x \quad \forall \xi \in \mathbb{R}^n$$

I denotes the identity matrix

${}^tA$ denotes the transpose of a matrix A

Function Spaces

$\mathcal{D}(\Omega)$ is the class of all infinitely differentiable functions on Ω with compact support

$\mathcal{D}'(\Omega)$ is the topological dual of $\mathcal{D}(\Omega)$, the space of all distributions

$L^\infty(\Omega)$ is the space of all essentially bounded measurable functions and its norm is denoted by $\|\cdot\|_{\infty,\Omega}$

$L^p(\Omega)$ is the space of all p-summable measurable functions and its norm is denoted by $\|\cdot\|_{p,\Omega}$ $(1 \leq p < \infty)$

$W^{m,p}(\Omega)$ is the collection of all $L^p(\Omega)$ functions such that all distributional derivatives upto order m are also in $L^p(\Omega)$ and its norm is denoted by $\|\cdot\|_{W^{m,p}(\Omega)}$

$W_0^{m,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$

$W^{-m,q}(\Omega)$ denotes the dual of $W_0^{m,p}(\Omega)$ where p is such that $\frac{1}{p} + \frac{1}{q} = 1$

$H_0^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,2}(\Omega) = (H^1(\Omega))$ and its norm is denoted by $\|\cdot\|_{H_0^1(\Omega)}$

$H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$

General Conventions

V' denotes the topological dual (space of continuous linear functionals) of the space V

$\langle \cdot, \cdot \rangle$ denotes the inner product in the ambient Hilbert space

$\langle \cdot, \cdot \rangle_{V',V}$ denotes the duality pairing between V' and V

$\rightarrow$ will denote the convergence in the strong topology of the space

$\rightharpoonup$ will denote the convergence in the weak topology of the space

$B(x, r)$ denotes an open ball of radius r centred at x in any normed linear space

C_0 is a generic positive constant independent of the parameters w.r.t which a limit is taken; will be different in different inequalities

$\sim$ denotes the extension of a function by zero on the holes of Ω , see page 11

Chapter 1

General Introduction

1.1 Homogenization

The theory of *homogenization* of partial differential equations is a concept that deals with the study of the macroscopic behaviour of a composite medium through its microscopic properties. The origin of the word is related to the question of replacing a heterogeneous medium by a fictitious homogeneous one (the 'homogenized' material). The known and unknown quantities in the study of physical or mechanical processes in a medium with microstructure depend on a small parameter $\varepsilon = \frac{l}{L}$, where L is the macroscopic scale length of the dimension of a specimen of the medium and l is the characteristic length of the medium configuration. The study of the limit, as $\varepsilon \to 0$, is the aim of the mathematical theory of homogenization. Though the case $\varepsilon \to 0$ has no real physical meaning, it is important as a tool for numerical computations.

We shall now illustrate this notion with an example.

Consider a beam made of a homogeneous material with uniform cross section occupying Ω with $\gamma > 0$, a constant, representing the elastic property of the material. Then, to study its torsional rigidity we need to solve the following homogeneous Dirichlet problem:

$$\left\{ \begin{array}{rcl} -\mathrm{div}(\gamma \nabla u(x)) &=& 2 \text{ in } \Omega \\ u &=& 0 \text{ on } \partial\Omega. \end{array} \right. \tag{1.1.1}$$

Since γ is constant, the above equation can be rewritten as

$$\left\{ \begin{array}{rcl} -\gamma \Delta u &=& 2 \text{ in } \Omega \\ u &=& 0 \text{ on } \partial\Omega. \end{array} \right. \tag{1.1.2}$$

This is a classical second order elliptic boundary value problem and admits a unique solution.

If we consider the situation where a large number of fibres of different materials with reduced thickness are introduced along the length of the beam, then γ takes different values in each component of the composite, i.e., γ is a function which is discontinuous in Ω. To simplify, suppose we consider a beam made of two fibres of two different materials, the cross-section of one occupying the subdomain Ω_1 and the other Ω_2, with $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega = \Omega_1 \cup \Omega_2 \cup (\partial\Omega_1 \cap \partial\Omega_2)$ then problem (1.1.2) is replaced by

$$\left\{ \begin{array}{rcl} -\mathrm{div}(A\nabla u) & = & 2 \text{ in } \Omega \\ u & = & 0 \text{ on } \partial\Omega. \end{array} \right. \tag{1.1.3}$$

where $A(x) = \gamma(x)I$ with

$$\gamma(x) = \begin{cases} \gamma_1 & \text{if } x \in \Omega_1 \\ \gamma_2 & \text{if } x \in \Omega_2, \end{cases} \tag{1.1.4}$$

where γ_i is the elastic property of the material occupying Ω_i, for $i = 1, 2$. Suppose we now progressively increase the number of fibres while reducing their thickness then the coefficients of the matrix A oscillate rapidly. Thus, when we try to solve the problem numerically, we need to use a very fine grid or mesh to get a good approximation of the solution, and this is very expensive. The mathematical theory of homogenization 'averages out' the heterogeneities and studies an 'equivalent' homogeneous fictitious material whose behaviour reflects that of the original material, when the number of fibres is very large.

Homogenization, as a mathematical discipline, took shape only in the last three decades but the physical ideas of homogenization date back at least to [Poi22, Mos50, Max73, Cla79, Ray92]. A very good historical record of works related to homogenization until 1975 can be found in [Bab76] and the references therein.

An abstract theory of homogenization was introduced by S. Spagnolo in a paper of 1967 (cf. [Spa67]) under the name of G-convergence[1] (also cf. [Spa68, GS73, Spa76]) and further generalised as H-convergence by L. Tartar in [Tar77] and developed by F. Murat and L. Tartar (cf. [Mur78b, MT97]). There is also a variational theory of homogenization, known as Γ-convergence,

[1] The terminology denoting the convergence of Green's operators for boundary problems

proposed by Ennio De Giorgi in a sequence of papers (cf. [GS73, Gio75, GF75]). For a thorough introduction to this theory we refer to [Gio84, Att84, DM93, BD98]. The wide spread application and theory of homogenization can also be found in [BLP78, JKO94, Hor97, CD99, CP99].

1.2 Periodic Homogenization

We shall, in this section, illustrate homogenization in a periodic framework. The periodic framework models the case where the heterogeneities are very small with respect to the size of Ω and are evenly distributed. This is a realistic assumption for large class of applications. The periodicity can be represented by a small parameter ε. A very nice exposition on periodic homogenization is the book [BLP78]. Also the recent book by Cioranescu and Donato ([CD99]) is dedicated to the study of asymptotic analysis for periodic structures.

We shall now introduce the geometric model of a periodic mixture. Let us assume that $Y(=[0,1]^n$, for example) is a reference cell (or period) in $\mathbb{R}^n$. Let $A=(a_{ij}) \in \mathcal{M}(a,b,Y)$ be a $n \times n$ matrix that has Y-periodic entries, *i.e.*, a_{ij} take equal value on opposite faces of Y. If we now partition $\mathbb{R}^n$ into cells of size ε by translating the cell εY then we have a partition of Ω into ε-cells (cf. Fig. 1.1). The function a_{ij} now gives the function $a^{\varepsilon}_{ij}(x) = a_{ij}(\frac{x}{\varepsilon})$ on the cell εY and can be translated to each of the other cells. Thus we get a periodically oscillating function, with period ε, on $\mathbb{R}^n$. Define $A_\varepsilon = (a^{\varepsilon}_{ij})$, which is in $\mathcal{M}(a,b,\Omega)$.

We now introduce the auxiliary periodic function defined on the reference cell Y which is useful in identifying the 'limit' homogenized matrix A_0. For $i=1,\dots,n$, let w_i be the unique solution of the following problem:

$$\begin{cases} -\operatorname{div}(A(y)\nabla w_i) &= 0 \quad \text{in } Y \\ \dfrac{1}{|Y|}\displaystyle\int_Y (w_i(y) - y_i)\, dy &= 0 \\ w_i - y_i & \quad \text{is } Y\text{-periodic.} \end{cases} \tag{1.2.1}$$

The following theorem describes the asymptotic behaviour of a second order elliptic system with periodic coefficients. The proof of this theorem can be found in [CD99].

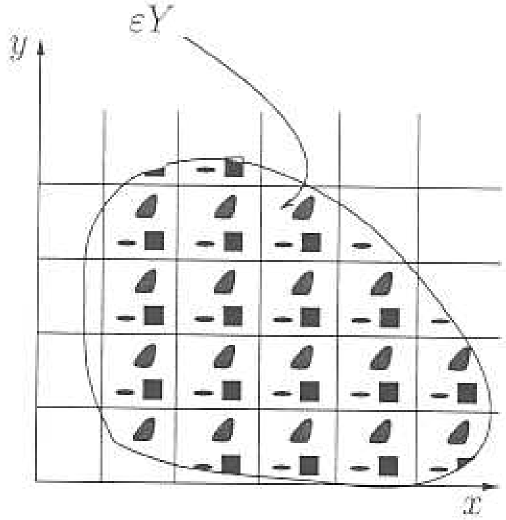

Figure 1.1: partition of Ω into ε-cells

Theorem 1.2.1. *Let $f \in H^{-1}(\Omega)$ and u_ε be the solution of*

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}^{\varepsilon}\frac{\partial u_\varepsilon}{\partial x_j}) &= f \quad in\ \Omega \\ u_\varepsilon &= 0 \quad on\ \partial\Omega. \end{cases}$$

Then

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H_0^1(\Omega) \quad and$$
$$A_\varepsilon \nabla u_\varepsilon \rightharpoonup A_0 \nabla u_0 \text{ weakly in } (L^2(\Omega))^n,$$

where u_0 is the unique solution of

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}^{0}\frac{\partial u_0}{\partial x_j}) &= f \quad in\ \Omega \\ u_0 &= 0 \quad on\ \partial\Omega \end{cases}$$

and $A_0 = (a_{ij}^0)$ is constant, elliptic and given by

$$a_{ij}^0 = \frac{1}{|Y|}\sum_{k,l=1}^{n}\int_Y a_{kl}\frac{\partial w_j}{\partial y_l}\frac{\partial w_i}{\partial y_k}\,dy \quad \forall i,j = 1,\ldots,n. \tag{1.2.2}$$

□

The homogenized coefficients a_{ij}^0 depend only on the matrix A, and not on the other data f and Ω.

Remark 1.2.1. In the one dimensional (for example, say, $Y = [0, 1]$) periodic case, A_0 is given by

$$\frac{1}{A_0} = \int_0^1 \frac{1}{A(y)}\, dy.$$

In fact, in the one dimensional case, the periodicity of $A(y)$ does not play a fundamental role in the above result. Thus, for the non periodic one dimensional case, one can deduce that the limiting coefficient A_0 is $\frac{1}{A^*}$ where A^* is the weak* limit of $\frac{1}{A_\varepsilon}$ in $L^\infty(\Omega)$. □

In the rest of the chapter we shall very briefly recall the basic notions of H-convergence, H_0-convergence and Γ-convergence. Most of the results included in this chapter form a base to the thesis. Those of which are already available in the literature have been stated without proof.

1.3 H-Convergence

Let $A_\varepsilon \in \mathcal{M}(a, b, \Omega)$ for some small parameter ε and let $f \in H^{-1}(\Omega)$. Then, the second order elliptic problem

$$\begin{cases} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) &= f \quad \text{in } \Omega \\ \qquad\qquad u_\varepsilon &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.3.1}$$

has a *unique* solution satisfying the estimate

$$||u_\varepsilon||_{H_0^1(\Omega)} \leq \frac{1}{a}\|f\|_{H^{-1}(\Omega)}. \tag{1.3.2}$$

Hence there exists a subsequence such that

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H_0^1(\Omega). \tag{1.3.3}$$

The uniqueness of u_ε follows from (1.3.2). The bounded elliptic operator $\mathcal{A}_\varepsilon = -\mathrm{div}(A_\varepsilon \nabla)$ from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ is an isomorphism and the norm of $(\mathcal{A}_\varepsilon)^{-1}$ is not larger than a^{-1} (cf. (1.3.2)).

Remark 1.3.1. The unique weak solution u_ε of (1.3.1) can also be characterized as the minimiser of

$$J_\varepsilon(v) = \frac{1}{2}\int_\Omega A_\varepsilon \nabla v.\nabla v\,dx - (f,v)_{H^{-1}(\Omega),H_0^1(\Omega)}$$

in $H_0^1(\Omega)$. □

Definition 1.3.1. *A sequence $\{A_\varepsilon\}$ of elements of $\mathcal{M}(a,b,\Omega)$ H-***converges** *to an element A_0 of $\mathcal{M}(a',b',\Omega)$ (denoted as $A_\varepsilon \xrightarrow{H} A_0$) iff for any $f \in H^{-1}(\Omega)$, the solution u_ε of (1.3.1) is such that*

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H_0^1(\Omega) \quad \text{and} \tag{1.3.4a}$$

$$A_\varepsilon \nabla u_\varepsilon \rightharpoonup A_0 \nabla u_0 \text{ weakly in } (L^2(\Omega))^n, \tag{1.3.4b}$$

where u_0 is the unique solution of

$$\begin{cases} -\operatorname{div}(A_0\nabla u_0) &= f \quad \text{in } \Omega \\ u_0 &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.3.5}$$

□

The matrix A_0 is called the H-limit of the sequence $\{A_\varepsilon\}$. The notion of H-convergence can also be interpreted as a statement about the convergence of the operators $(\mathcal{A}_\varepsilon)^{-1}$ when both the spaces $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ are endowed with the weak topologies. In other words, $A_\varepsilon \xrightarrow{H} A_0$ is equivalent to the convergence of the inverse operators in the following sense:

$$\langle \mathcal{A}_\varepsilon^{-1} f, g\rangle \to \langle \mathcal{A}_0^{-1} f, g\rangle \quad \forall f,g \in H^{-1}(\Omega)$$

where the operator $\mathcal{A}_0 = -\operatorname{div}(A_0\nabla)$.

The following theorem briefly lists some of the principal properties of H-convergence. For a proof of this we refer to [CD99, MT97].

Theorem 1.3.1. (Uniqueness) *The H-limit of a H-converging sequence $\{A_\varepsilon\} \subset \mathcal{M}(a,b,\Omega)$ is unique.*

(Transpose) *If $A_\varepsilon \xrightarrow{H} A_0$ then ${}^tA_\varepsilon \xrightarrow{H} {}^tA_0$.*

(Compactness) *For any given sequence A_ε in $\mathcal{M}(a,b,\Omega)$, there exists a subsequence $\{A_{\varepsilon'}\}$ and $A_0 \in \mathcal{M}(a,\frac{b^2}{a},\Omega)$ such that $A_{\varepsilon'} \xrightarrow{H} A_0$.*

(Energy convergence) *If* $A_\varepsilon \xrightarrow{H} A_0$ *then*

$$\int_\Omega A_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx \to \int_\Omega A_0 \nabla u_0 . \nabla u_0 \, dx \tag{1.3.6}$$

where u_ε *and* u_0 *are, respectively, the unique solution of* (1.3.1) *and* (1.3.5). □

Remark 1.3.2. The energy convergence stated in the above theorem is also valid when, in (1.3.1), instead of a fixed f, one has $f_\varepsilon \in H^{-1}(\Omega)$ such that $f_\varepsilon \to f$ strongly in $H^{-1}(\Omega)$. □

The energy convergence also amounts to saying that the quadratic forms associated with the operators converge, i.e., $\langle \mathcal{A}_\varepsilon u_\varepsilon, u_\varepsilon \rangle \to \langle \mathcal{A}_0 u_0, u_0 \rangle$. In section §1.5 (cf. Lemma 1.5.1), we will observe that this is actually subject to a special type of convergence called the Γ-convergence.

The energy functional (cf. (1.3.6)) involves a product of two weakly converging sequences and we have claimed that the limit of the product is equal to the product of the limit. This property does not hold in general. One of the main tools for getting across such difficulties is the theory of compensated compactness due to F. Murat and L. Tartar (cf. [Mur78a, Mur79, Tar79]). The following result is one of the first results of this theory and is very useful.

Theorem 1.3.2 (div-curl lemma). *Let* u_ε *and* v_ε *be two sequences in* $(L^2(\Omega))^n$ *such that*

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly in } (L^2(\Omega))^n$$

$$v_\varepsilon \rightharpoonup v_0 \text{ weakly in } (L^2(\Omega))^n.$$

If $\{\operatorname{div} u_\varepsilon\}$ *is compact in* $H^{-1}(\Omega)$ *and* $\{\operatorname{curl} v_\varepsilon\}$ *is bounded in* $(L^2(\Omega))^{n\times n}$, *then*

$$u_\varepsilon v_\varepsilon \to u_0 v_0 \text{ weak* in } \mathcal{D}'(\Omega).$$

□

We have from (1.3.4a) that

$$\nabla u_\varepsilon \rightharpoonup \nabla u_0 \text{ weakly in } (L^2(\Omega))^n.$$

In general, the above convergence is not strong. However, by adjusting the term ∇u_0, we get a strong convergence (cf. Theorem 1.3.3). This adjustment is done by introducing the corrector matrix.

The corrector matrices are obtained by looking for functions $\chi_\varepsilon^i \in H^1(\Omega)$, for $1 \leq i \leq n$, with the following properties:

$$\begin{cases} \chi_\varepsilon^i \rightharpoonup x_i \text{ weakly in } H^1(\Omega), \\ A_\varepsilon \nabla \chi_\varepsilon^i \rightharpoonup A_0 e_i \text{ weakly in } (L^2(\Omega))^n, \\ \operatorname{div}(A_\varepsilon \nabla \chi_\varepsilon^i) \text{ converges strongly in } H^{-1}(\Omega). \end{cases} \tag{1.3.7}$$

One procedure to build a function with above properties is by defining $\chi_\varepsilon^i \in H^1(\Omega)$, for $1 \leq i \leq n$, as a solution of

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla \chi_\varepsilon^i) &= -\operatorname{div}(A_0 e_i) \quad \text{in } \Omega \\ \chi_\varepsilon^i &= x_i \quad \text{on } \partial\Omega. \end{cases} \tag{1.3.8}$$

Then the corrector matrix $D_\varepsilon \in (L^2(\Omega))^{n\times n}$ is defined as $D_\varepsilon e_i = \nabla \chi_\varepsilon^i$ for $1 \leq i \leq n$. Some interesting properties of the corrector functions are given by the following proposition, the proof of which can be found in [CD99, MT97].

Proposition 1.3.1. *Let $A_\varepsilon \in \mathcal{M}(a, b, \Omega)$, χ_ε^i be a function with properties (1.3.7) and $D_\varepsilon e_i = \nabla \chi_\varepsilon^i$. Also, let A_ε H-converge to A_0, then the following are true:*

(a) $D_\varepsilon \rightharpoonup I$ weakly in $(L^2(\Omega))^{n\times n}$.

(b) $A_\varepsilon D_\varepsilon \rightharpoonup A_0$ weakly in $(L^2(\Omega))^{n\times n}$.

(c) ${}^tD_\varepsilon A_\varepsilon D_\varepsilon \rightharpoonup A_0$ weak in $[\mathcal{D}'(\Omega)]^{n\times n}$.* □

The interest of the corrector matrix D_ε is the following theorem:

Theorem 1.3.3 (cf. [CD99]). *If $A_\varepsilon \xrightarrow{H} A_0$, then*

$$\nabla u_\varepsilon - D_\varepsilon \nabla u_0 \to 0 \text{ strongly in } (L^1(\Omega))^n.$$

Moreover, if $D_\varepsilon \in (L^r(\Omega))^{n\times n}$, $\|D_\varepsilon\|_{(L^r(\Omega))^n} \leq C_0$ for $2 \leq r \leq +\infty$ and $\nabla u_0 \in (L^s(\Omega))^n$, $2 \leq s < +\infty$, then

$$\nabla u_\varepsilon - D_\varepsilon \nabla u_0 \to 0 \text{ strongly in } (L^t(\Omega))^n,$$

where $t = \min\left\{2, \frac{rs}{r+s}\right\}$. □

A question of similar interest is to know the limit of $\|\nabla u_\varepsilon\|^2_{2,\Omega}$. One knows that this quantity is uniformly bounded and hence, at least for a subsequence, converges. We know that the limit is *not* $\|\nabla u_0\|^2_{2,\Omega}$, since we know from the above theorem that u_ε does not converge to u_0 strongly in $H^1_0(\Omega)$. We would like to know the limit and whether it can be expressed in terms of the function u_0. More generally, the problem can be framed as identifying the limit of $\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx$ where B_ε is a family of matrices in $\mathcal{M}(c, d, \Omega)$. More precisely, does there exist a matrix $B' \in \mathcal{M}(c', d', \Omega)$ such that, at least for a subsequence, we have

$$\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx \to \int_\Omega B' \nabla u_0 . \nabla u_0 \, dx?$$

The convergence question posed above is answered when $B_\varepsilon = A_\varepsilon$ (cf. (1.3.6)), in which case, it has been observed that $B^\sharp = A_0$, the H-limit of A_ε. The general problem was studied by Kesavan and Rajesh in [KR02].

Proposition 1.3.2. *Let $A_\varepsilon \in \mathcal{M}(a, b, \Omega)$, $B_\varepsilon \in \mathcal{M}(c, d, \Omega)$ and χ^i_ε be a function with properties (1.3.7) and $D_\varepsilon e_i = \nabla \chi^i_\varepsilon$. Also, let A_ε H-converge to A_0, then the following are true:*

(a) There exists a $B^\sharp$ (depending only on $\{A_\varepsilon\}$ and $\{B_\varepsilon\}$) such that

$$^tD_\varepsilon B_\varepsilon D_\varepsilon \rightharpoonup B^\sharp \text{ weak* in } (\mathcal{D}'(\Omega))^{n\times n}. \tag{1.3.9}$$

(b) If $B_\varepsilon = A_\varepsilon$ for all ε, then $B^\sharp = A_0$.

(c) If B_ε's are symmetric, then $B^\sharp$ is symmetric.

(d) $B^\sharp \in \mathcal{M}\left(c, d(\frac{b}{a})^2, \Omega\right)$. □

The existence of the matix $B^\sharp$, mentioned in the above proposition, was shown by Kesavan and Vanninathan, for the periodic case (cf. [KV77]), and by Kesavan and Saint Jean Paulin in the general case (cf. [KP97]), in the process of homogenizing an optimal control problem.

It was observed that the required B' is actually the $B^\sharp$ obtained in Proposition 1.3.2 and thus

$$\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx \to \int_\Omega B^\sharp \nabla u_0 . \nabla u_0 \, dx. \tag{1.3.10}$$

Therefore, if C is the positive square root of the matrix $B^\sharp$ when $B_\varepsilon = I$, for all $\varepsilon > 0$, then

$$\|\nabla u_\varepsilon\|_{2,\Omega}^2 \to \|C\nabla u_0\|_{2,\Omega}^2.$$

An explicit formula for the matrix $B^\sharp$ can be found in §3.4.

In this section, we introduced the notion of H-convergence and the results on H-convergence are valid when the data converges *strongly* in $H^{-1}(\Omega)$. We will see in §3.3 that the notion of H-convergence can be applied to the case when data is from the positive cone of $H^{-1}(\Omega)$ and converges *weakly* in $H^{-1}(\Omega)$, however, the energy convergence fails, in general (cf. Remark 3.3.1).

1.4 H_0-Convergence

In this section, we shall introduce the theory of homogenization developed for perforated domains. Perforated domains are domains with holes. The common feature of problems posed over perforated domains is that the functions are defined over different domains Ω_ε (*viz.* Ω minus the perforations). Mathematically speaking, we consider a family of closed subsets $S_\varepsilon \subset \Omega$ and set $\Omega_\varepsilon = \Omega \setminus S_\varepsilon$, which we call the perforated domain. A detailed exposition of homogenization on perforated media can be found in [CP99].

There are two kinds of boundary conditions one could consider on the boundaries of the holes. The first is the Dirichlet boundary condition on the boundaries of the holes. The other case is to consider a Neumann type boundary condition on the boundaries of the holes and a Dirichlet (or any other) condition on the global boundary $\partial\Omega$. Both these cases were studied in the periodic case (cf. [CP79, CM97]). It was soon realised that to study the convergence of the solutions in these cases, one has to first extend these functions suitably on Ω.

In the case of homogeneous Dirichlet boundary conditions we have the trivial extension by zero over the holes. However, there are other kinds of difficulties in this case (cf. [CM97, DMM04]). In the case of Neumann type boundary condition the existence of a uniformly bounded prolongation operator is presumed from $H^1(\Omega_\varepsilon)$ to $H_0^1(\Omega)$. For this case, the notion of H-convergence was extended to the perforated domains, called H_0-convergence, by Briane, Damlamian, Donato in [BDD96].

We shall now introduce some machinery required for H_0-convergence. Let

χ_ε denote the characteristic function of the set Ω_ε in Ω,

$$\chi_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega_\varepsilon \\ 0 & \text{if } x \in S_\varepsilon. \end{cases}$$

Observe that (for a subsequence) $\chi_\varepsilon \rightharpoonup \chi_0$ weak* in $L^\infty(\Omega)$. Some properties specific to χ_ε and χ_0 are proved in Lemma 4.1.1 and Lemma 5.2.1. If f_ε is a function defined on Ω_ε, we denote by $\tilde{f}_\varepsilon$ its extension by zero across the holes, to all of Ω.

The difficulty specific to the perforated case, in contrast to the case where impurities exist instead of holes, is the inability to obtain estimates on the whole of Ω. A natural way to get around this difficulty is to assume the existence of extension operators which extend the solutions to Ω.

H 1. *There exists, for each $\varepsilon > 0$, an extension operator*

$$P_\varepsilon : V_\varepsilon \to H_0^1(\Omega)$$

where $V_\varepsilon = \{u \in H^1(\Omega_\varepsilon) \mid u = 0 \text{ on } \partial\Omega\}$, such that, for every $u \in V_\varepsilon$,

$$P_\varepsilon u|_{\Omega_\varepsilon} = u \text{ and } \|\nabla P_\varepsilon u\|_{2,\Omega} \leq C_0 \|\nabla u\|_{2,\Omega_\varepsilon}$$

where the constant C_0 is independent of ε.

The above hypothesis represents a condition on the regularity of the holes and the way they approach the boundary $\partial\Omega$. Such extension operators can be explicitly constructed in the case of periodic distribution of holes (cf. [CP79, CP99]). The space V_ε is the solution space of the system (1.4.1) and, given (**H**1), we can define the norm on V_ε as, $\|u\|_{V_\varepsilon} = \|\nabla u\|_{2,\Omega_\varepsilon}$. The independence of H_0-convergence from the extension operator is taken care of by the following hypothesis (cf. [BDD96, Lemma 2.1]):

H 2. *Every weak* limit point in $L^\infty(\Omega)$ of $\{\chi_\varepsilon\}$ is positive a.e. in Ω.*

We say that the family of holes $\{S_\varepsilon\}$ is an *admissible* family of holes in Ω, if the conditions (**H**1) and (**H**2) are satisfied. Throughout this thesis S_ε will denote an admissible family of holes in Ω.

Definition 1.4.1. *A sequence $\{A_\varepsilon\}$ of elements of $\mathcal{M}(a, b, \Omega)$ H_0-**converges** to an element A_0 of $\mathcal{M}(a', b', \Omega)$ iff for any $f \in H^{-1}(\Omega)$, the solution u_ε of*

$$\begin{cases} -\text{div}(A_\varepsilon \nabla u_\varepsilon) &= P_\varepsilon^* f \quad \text{in } \Omega_\varepsilon \\ A_\varepsilon \nabla u_\varepsilon . n_\varepsilon &= 0 \quad \text{on } \partial S_\varepsilon \\ u_\varepsilon &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.4.1}$$

(where n_ε is the unit outward normal on ∂S_ε and $P_\varepsilon^* : H^{-1}(\Omega) \to V_\varepsilon'$ denotes the adjoint of P_ε), is such that

$$P_\varepsilon u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H_0^1(\Omega) \quad \text{and} \tag{1.4.2a}$$

$$\widetilde{A_\varepsilon \nabla u_\varepsilon} \rightharpoonup A_0 \nabla u_0 \text{ weakly in } (L^2(\Omega))^n, \tag{1.4.2b}$$

where u_0 is the unique solution of (1.3.5). □

The matrix A_0 is, then, said to be the H_0-limit of $\{A_\varepsilon\}$. Analogous to the theory of H-convergence, if A_ε H_0-converges to A_0 then the energies converge, *i.e.*,

$$\int_{\Omega_\varepsilon} A_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx \to \int_\Omega A_0 \nabla u_0 . \nabla u_0 \, dx. \tag{1.4.3}$$

Moreover, $A_0 \in \mathcal{M}(\frac{a}{C_0^2}, \frac{b^2}{a}, \Omega)$ and the local property of H_0-limit is given by the following proposition.

Proposition 1.4.1 (Local Property). *Let A_ε and B_ε be two sequences in $\mathcal{M}(a, b, \Omega)$ that satisfy $A_\varepsilon \overset{H_0}{\rightharpoonup} A_0$ and $B_\varepsilon \overset{H_0}{\rightharpoonup} B_0$, and are such that $A_\varepsilon = B_\varepsilon$ on $\omega \setminus S_\varepsilon$, where ω is an open set contained in Ω. Then $A_0 = B_0$ on ω.*

We now clarify the case of varying right side in the definition of H_0-convergence.

Theorem 1.4.1. *Let $f_\varepsilon \rightharpoonup f$ weakly in $L^2(\Omega)$ and let $u_\varepsilon \in V_\varepsilon$ be the solution of*

$$\left\{ \begin{array}{rll} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) & = f_\varepsilon & \text{in } \Omega_\varepsilon \\ A_\varepsilon \nabla u_\varepsilon . n_\varepsilon & = 0 & \text{on } \partial S_\varepsilon \\ u_\varepsilon & = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1.4.4}$$

then $P_\varepsilon u_\varepsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$ where u_0 is the solution of (1.3.5) *and A_0 is the H_0-limit of $\{A_\varepsilon\}$.* □

If the right-hand side of (1.4.4) involves fixed $f \in L^2(\Omega)$, then we must take $f_\varepsilon = \chi_\varepsilon f$ which will converge weakly in $L^2(\Omega)$ to $\chi_0 f$. The proof of this result can be found in [BDD96] and the above theorem can be proved by rephrasing this proof suitably.

The notion of *correctors* was also developed for the theory of H_0 convergence (cf. [BDD96]) The corrector matrices are obtained by looking for functions $\mu_\varepsilon^i \in H^1(\Omega)$, for $1 \leq i \leq n$, with the following properties:

$$\begin{cases} \mu_\varepsilon^i \rightharpoonup x_i \text{ weakly in } H^1(\Omega), \\ A_\varepsilon \nabla \mu_\varepsilon^i \rightharpoonup A_0 e_i \text{ weakly in } (L^2(\Omega))^n, \\ \operatorname{div}(\chi_\varepsilon A_\varepsilon \nabla \mu_\varepsilon^i) \text{ converges strongly in } H^{-1}(\Omega). \end{cases} \tag{1.4.5}$$

We shall now detail one procedure to build a function with above properties. Let Ω' be a bounded open subset of $\mathbb{R}^n$ such that $\Omega \subset\subset \Omega'$. Let $\Omega'_\varepsilon = \Omega' \setminus S_\varepsilon$. Then the family of holes, S_ε, is also admissible for Ω'. This can be seen by extending P_ε, obtained in (**H1**), by zero in $\Omega' \setminus \Omega$. We denote this extension operator on Ω' by P_ε itself. The matrix A_ε, as a function, can be extended to Ω' by defining it to be aI in $\Omega' \setminus \Omega$, and denote the extension by A_ε itself. Clearly, $A_\varepsilon \in \mathcal{M}(a, b, \Omega')$ and the H_0 limit of A_ε in Ω' is denoted by A'. Then, by Proposition 1.4.1, A' restricted to Ω is A_0, the H_0-limit in Ω. Let $\phi \in \mathcal{D}(\Omega')$ with $\phi \equiv 1$ in Ω. Then, we define $\mu_\varepsilon^i \in H^1(\Omega'_\varepsilon)$, for $1 \leq i \leq n$, as a solution of

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla \mu_\varepsilon^i) &= P_\varepsilon^*(-\operatorname{div}(A' \nabla(\phi x_i))) & \text{in } \Omega'_\varepsilon \\ A_\varepsilon \nabla \mu_\varepsilon^i . n_\varepsilon &= 0 & \text{on } \partial S_\varepsilon \\ \mu_\varepsilon^i &= 0 & \text{on } \partial \Omega'. \end{cases} \tag{1.4.6}$$

Then, by H_0-convergence, we have $P_\varepsilon \mu_\varepsilon^i$ weakly converging to ϕx_i in $H_0^1(\Omega')$ and hence to x_i when restricted to Ω. We now define the corrector matrix[2] $D_\varepsilon \in (L^2(\Omega))^{n \times n}$ is defined as $D_\varepsilon e_i = \nabla(P_\varepsilon \mu_\varepsilon^i)$ for $1 \leq i \leq n$. Some properties of the corrector functions are given by the following proposition, the proof of which can be found in [BDD96].

Proposition 1.4.2. *Let $A_\varepsilon \in \mathcal{M}(a, b, \Omega)$, μ_ε^i be a function with properties (1.4.5) and D_ε is the corrector matrix as defined above. Also, let A_ε H_0-converge to A_0, then the following are true:*

(a) $D_\varepsilon \rightharpoonup I$ *weakly in* $(L^2(\Omega))^{n \times n}$.

(b) $\chi_\varepsilon A_\varepsilon D_\varepsilon \rightharpoonup A_0$ *weakly in* $(L^2(\Omega))^{n \times n}$.

(c) $\chi_\varepsilon {}^t D_\varepsilon A_\varepsilon D_\varepsilon \rightharpoonup A_0$ *weak* in* $[\mathcal{D}'(\Omega)]^{n \times n}$. □

[2] Caution: Same notation D_ε for correctors is being employed in both H and H_0 convergence

Questions similar to those posed in the previous section regarding the convergence of ∇u_ε can be posed here too. The existence of the matix[3] $B^\sharp$ for the perforated case was shown by Kesavan and Saint Jean Paulin (cf. [KP99]), in the process of homogenizing an optimal control problem.

Proposition 1.4.3. *Let $A_\varepsilon \in \mathcal{M}(a, b, \Omega)$, $B_\varepsilon \in \mathcal{M}(c, d, \Omega)$, μ_ε^i be a function with properties (1.4.5) and D_ε is the corrector matrix as defined above. Also, let A_ε H_0-converge to A_0, then the following are true:*

(a) There exists a matrix $B^\sharp$ (depending only on $\{A_\varepsilon\}$ and $\{B_\varepsilon\}$) such that

$$\chi_\varepsilon {}^t D_\varepsilon B_\varepsilon D_\varepsilon \rightharpoonup B^\sharp \text{ weak* in } (\mathcal{D}'(\Omega))^{n\times n}. \tag{1.4.7}$$

(b) If $B_\varepsilon = A_\varepsilon$ for all ε, then $B^\sharp = A_0$.

(c) If B_ε's are symmetric, then $B^\sharp$ is symmetric.

(d) $B^\sharp \in \mathcal{M}\left(\frac{c}{C_0^2}, d(\frac{b}{a})^2, \Omega\right)$. □

Moreover, the energy functional converges, *i.e.*,

$$\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx \to \int_\Omega B^\sharp \nabla u_0 . \nabla u_0 \, dx, \tag{1.4.8}$$

where u_0 is the solution of (1.3.5). In particular, if C denotes the positive square root of the matrix $B^\sharp$ when $B_\varepsilon = I$, for all $\varepsilon > 0$, then we have that

$$\|\nabla u_\varepsilon\|^2_{2,\Omega_\varepsilon} \to \|C \nabla u_0\|^2_{2,\Omega}.$$

1.5 Γ-Convergence

The notion of Γ-convergence was introduced by Ennio De Giorgi in a sequence of papers (cf. [GS73, Gio75, GF75]). An excellent account of this concept is the book of Dal Maso [DM93]. In this section we will introduce the sequential notion of Γ-limit and K-limit (Kuratowski) in a topological space for completeness sake. We shall also give simple proofs of some important results. Let us point out that all the Γ-limits and K-limits used in this thesis are of sequential kind and the space X, in this section, denotes a topological vector space.

[3] Caution: Same notation $B^\sharp$ is being employed in both perforated and non-perforated case

Definition 1.5.1. *A function* $F : X \to \overline{\mathbb{R}}$ *is* **sequentially lower semicontinuous (lsc)** *at a point* $x \in X$ *if*

$$F(x) \leq \liminf_{n\to\infty} F(x_n)$$

for every sequence $\{x_n\}$ *converging to* $x \in X$.

F *is* **sequentially lower semicontinuous on** X *if* F *is sequentially lower semicontinuous at each point* $x \in X$. □

Definition 1.5.2. *A set* E *of* X *is* **sequentially compact** *if every sequence in* E *has a subsequence which converges to a point of* E. □

Definition 1.5.3. *A function* $F : X \to \overline{\mathbb{R}}$ *is* **sequentially coercive** *on* X *if the closure of the set* $\{x \in X : F(x) \leq t\}$ *is sequentially compact in* X *for every* $t \in \mathbb{R}$. □

Remark 1.5.1. If F is sequentially coercive on X, then every sequence $\{x_n\}$ in X with $\limsup_{n\to\infty} F(x_n) < +\infty$ has a convergent subsequence in X. □

Remark 1.5.2. Let X be a reflexive Banach space. A function $F : X \to \overline{\mathbb{R}}$ is sequentially coercive in the weak topology of X if and only if $F(x)$ tends to $+\infty$ as $\|x\|$ tends to $+\infty$. □

Definition 1.5.4. *A* **minimizing sequence** *for* F *in* X *is a sequence* $\{x_n\}$ *in* X *such that*

$$\inf_{y\in X} F(y) = \lim_{n\to\infty} F(x_n)$$

□

Theorem 1.5.1. *Assume that the function* $F : X \to \overline{\mathbb{R}}$ *is coercive and lower semicontinuous. Then* F *attains a minimum in* X. □

Definition 1.5.5. *We say that a function* $F : X \to \overline{\mathbb{R}}$ *is* **convex** *if*

$$F(tx + (1-t)y) \leq tF(x) + (1-t)F(y)$$

for every $t \in (0,1)$ *and for every* $x, y \in X$ *such that* $F(x) < +\infty$ *and* $F(y) < +\infty$. □

Definition 1.5.6. *We say that a function $F : X \to \overline{\mathbb{R}}$ is* **strictly convex** *if F is not identically $+\infty$ and*

$$F(tx + (1-t)y) < tF(x) + (1-t)F(y)$$

for every $t \in (0,1)$ and for every $x, y \in X$ such that $x \neq y$, $F(x) < +\infty$ and $F(y) < +\infty$. □

Proposition 1.5.1. *Let $F : X \to \overline{\mathbb{R}}$ be a strictly convex function. Then F has at most one minimum point in X.* □

Let $\{F_n\}$ be a sequence of functions from X in to $\overline{\mathbb{R}}$ and let $\{E_n\}$ be a sequence of subsets of X.

Definition 1.5.7. *A function F is said to be the* **sequential Γ-limit** *of F_n (denoted as $F_n \overset{\Gamma_{seq}}{\to} F$) w.r.t the topology of X, if the following two conditions are satisfied:*

(i) For every $x \in X$ and for every sequence $\{x_n\}$ converging to x in X, we have

$$\liminf_{n\to\infty} F_n(x_n) \geq F(x).$$

(ii) For every $x \in X$, there exists a sequence $\{x_n\}$ converging to x in X (called the Γ-realising sequence) such that

$$\lim_{n\to\infty} F_n(x_n) = F(x).$$

□

Lemma 1.5.1. *If $A_\varepsilon \overset{H}{\rightharpoonup} A_0$ then $J_\varepsilon \overset{\Gamma_{seq}}{\rightharpoonup} J$ in the weak topology of $H_0^1(\Omega)$ where*

$$J_\varepsilon(u) = \int_\Omega A_\varepsilon \nabla u . \nabla u \, dx$$

and

$$J(u) = \int_\Omega A_0 \nabla u . \nabla u \, dx.$$

Proof. Let $u \in H_0^1(\Omega)$ and let $w_\varepsilon \in H_0^1(\Omega)$ for all ε be a sequence such that $w_\varepsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Let $u_\varepsilon \in H_0^1(\Omega)$ be the solution of

$$-\text{div}(A_\varepsilon \nabla u_\varepsilon) = -\text{div}(A_0 \nabla u). \tag{1.5.1}$$

Then, it follows from H-convergence that $u_\varepsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $\int_\Omega A_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx \to \int_\Omega A_0 \nabla u . \nabla u \, dx$. Thus, we have shown that there exists a $\{u_\varepsilon\}$ in $H_0^1(\Omega)$ converging weakly to u in $H_0^1(\Omega)$ such that

$$\lim_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon) = J(u).$$

Also, it follows from Remark 1.3.1 that

$$\begin{aligned} \frac{1}{2}\int_\Omega A_\varepsilon \nabla w_\varepsilon . \nabla w_\varepsilon \, dx - \int_\Omega A_0 \nabla u . \nabla w_\varepsilon \, dx \;\; &\geq \;\; \frac{1}{2}\int_\Omega A_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx \\ &\quad - \int_\Omega A_0 \nabla u . \nabla u_\varepsilon \, dx \end{aligned}$$

and taking liminf on both sides of above inequality we have

$$\liminf_{\varepsilon \to 0} J_\varepsilon(w_\varepsilon) \geq J(u).$$

Hence $J_\varepsilon \stackrel{\Gamma_{seq}}{\to} J$ in the weak topology of $H_0^1(\Omega)$. □

Definition 1.5.8. *A point $x \in X$ is said to be in the* **sequential K- lower limit**, *E', of E_n (denoted by K-$\liminf_{n\to\infty} E_n$) w.r.t the topology in X, if and only if there exists a $k \in \mathbb{N}$ and a sequence $\{x_n\}$ converging to x in X such that $x_n \in E_n$, for all $n \geq k$.*

Definition 1.5.9. *A point $x \in X$ is said to be in the* **sequential K- upper limit**, *E'', of E_n (denoted by K-$\limsup_{n\to\infty} E_n$) w.r.t the topology in X, if and only if there exists a subsequence $\{E_{n_k}\}$ of $\{E_n\}$ and a sequence $\{x_k\}$ converging to x in X such that $x_k \in E_{n_k}$, for all $k \in \mathbb{N}$.*

Definition 1.5.10. *A set E is said to be the* **sequential K-limit** *of E_n (denoted as $E_n \stackrel{K_{seq}}{\to} E$) w.r.t the topology in X, if the following two conditions are satisfied:*

(i) For every $x \in E$ there exists a $k \in \mathbb{N}$ and a sequence $\{x_n\}$ converging to x in X such that $x_n \in E_n$, for all $n \geq k$.

(ii) If $\{E_{n_k}\}$ is a subsequence of $\{E_n\}$ and $\{x_k\}$ is a sequence converging to x in X such that $x_k \in E_{n_k}$, for all $k \in \mathbb{N}$, then $x \in E$. □

Let $J_n : E_n \to \mathbb{R}$ be a sequence of functionals on $E_n \subset X$ having a minimiser $x_n^* \in E_n$. Let $F_n : X \to \overline{\mathbb{R}}$ be defined as,

$$F_n(x) = \begin{cases} J_n(x) & \text{if } x \in E_n \\ +\infty & \text{if } x \in X \setminus E_n. \end{cases} \tag{1.5.2}$$

The following lemma is, essentially, Corollary 7.20 of [DM93] and generalises Lemma 2.1.1 and Proposition 2.1.1 of [Raj00]. We state and prove it here, under the sequential characterization of Γ-limits, in a form suitable for the kind of functionals as given in (1.5.2).

Lemma 1.5.2. *Let $F_n \overset{\Gamma_{seq}}{\to} F$ and $x_n^* \to x^*$ in X. Define the set $\mathcal{E}$ as follows, $\mathcal{E} = \{x \in X \mid F(x) < +\infty\}$. If $\mathcal{E}$ is non-empty, then $x^* \in \mathcal{E}$ and is a minimiser of F on $\mathcal{E}$. Also, $F_n(x_n^*) \to F(x^*)$.*

Proof. $\mathcal{E}$ being non-empty, we can choose a $x \in \mathcal{E}$. Now, since $F_n \overset{\Gamma_{seq}}{\to} F$ in X, there exists a sequence $x_n \to x$ in X such that

$$\lim_{n\to\infty} F_n(x_n) = F(x) < +\infty.$$

Hence there exists a $n_0 \in \mathbb{N}$ such that $x_n \in E_n$, for all $n \geq n_0$. Also, since $x_n^* \to x^*$ in X, we have

$$\liminf_{n\to\infty} F_n(x_n^*) \geq F(x^*). \tag{1.5.3}$$

Since x_n^* is the minimiser of F_n, we have $F_n(x_n^*) \leq F_n(x_n)$ for all n. Therefore

$$F(x^*) \leq \liminf_{n\to\infty} F_n(x_n^*) \leq \liminf_{n\to\infty} F_n(x_n) = \lim_{n\to\infty} F_n(x_n) = F(x) < +\infty.$$

Hence $x^* \in \mathcal{E}$ and, since $x \in \mathcal{E}$ was arbitrary, we have shown that x^* minimizes F in $\mathcal{E}$.

Again by the hypothesis, since $x^* \in \mathcal{E}$, there exists a sequence $y_n \to x^*$ in X such that

$$\lim_{n\to\infty} F_n(y_n) = F(x^*)$$

and $y_n \in E_n$ for all $n \geq k$ for some $k \in \mathbb{N}$. Taking limit supremum on both sides of the inequality, $F_n(x_n^*) \leq F_n(y_n)$, we have,

$$\limsup_{n\to\infty} F_n(x_n^*) \leq \limsup_{n\to\infty} F_n(y_n) = \lim_{n\to\infty} F_n(y_n) = F(x^*).$$

Now combining with the inequality (1.5.3) we have, $\lim_{n\to\infty} F_n(x_n^*) = F(x^*)$.

□

Remark 1.5.3. We observe that the set $\mathcal{E}$ defined in the above lemma satisfies the property of Definition 1.5.8, *i.e.*, we have $\mathcal{E} \subseteq E' = K\text{-}\liminf_{n\to\infty} E_n$. In particular, if $E_n \overset{K_{seq}}{\to} E$ we have $\mathcal{E} \subseteq E$ (cf. Open Problem 1 in page 95). □

Lemma 1.5.3. *Let $E_n \overset{K_{seq}}{\to} E$ and $x \in X \setminus E$. Then,*

(a) there exists n_0 such that $F_n(x) = +\infty$, for all $n \geq n_0$.

(b) If $x_n \to x$ in X, we have $\liminf F_n(x_n) = +\infty$.

Proof. Suppose for all $n \in \mathbb{N}$, there exists $m_n \geq n$ such that $F_{m_n}(x) < +\infty$, then we have a subsequence $\{E_{m_n}\}$ such that $x \in E_{m_n}$ and hence $x \in E$, a contradiction. This proves (a).

Suppose $\liminf F_n(x_n) < +\infty$, then we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $F_{n_k}(x_{n_k}) < +\infty$, for all k. Thus $x_{n_k} \in E_{n_k}$ and $x_{n_k} \to x$. Then, by definition, we have $x \in E$, a contradiction. This proves (b). □

1.6 Summary of the Thesis

The aim of this thesis is to analyse the asymptotic behaviour of various classes of optimal control problems. In Chapter 1, some preliminary concepts necessary for the thesis have been introduced. Most of the results in this chapter have been stated without proof, but with good references. In Chapter 2, the notion of control problems is introduced and a brief review of the existing literature on the homogenization of optimal control problems is given. It surveys some of the existing results in the literature and highlights the difficulties presented by some of these problems, which are taken up in later chapters. In Chapter 3, the asymptotic behaviour of a class of optimal control problems with varying control sets and the corresponding low cost control problems is studied. This chapter attempts to answer the open problems posed in [KP02]. In Chapter 4, the asymptotic behaviour of low cost control problems on perforated domains is studied. In Chapter 3 and Chapter 4, the results detailed in Chapter 2 are extended and improved. In Chapter 5, an optimal control problem with constraints on state is studied.

Chapter 2

Introducing Control Problems

2.1 Introducing control problems

The theory of optimal control is a part of optimization theory. Our interest will be in the asymptotic behaviour of optimal control problems governed by elliptic partial differential equations with appropriate boundary conditions. The term *control* was coined by Pontryagin and his collaborators (cf. [PBGM62]) in the context of problems defined by ordinary differential equations.

The general form of the optimal control problem that would be of interest to us is the following. Let $\mathcal{H}$ be a Hilbert space denoting the *state space*, $\mathcal{U}$ another Hilbert space denoting the *control space* and U, the set of *admissible controls*, be a closed convex subset of $\mathcal{U}$. For $\theta \in U$, the cost functional is given as

$$J(\theta) = T(u(\theta)) + NS(\theta)$$

- where $u \mapsto T(u)$ is a functional on the state space, $\mathcal{H}$, with values in $\mathbb{R}$,
- $\theta \mapsto S(\theta)$ is a functional on the control space, $\mathcal{U}$, with values in $\mathbb{R}$,
- $N > 0$ is the cost of the control and
- $\theta \mapsto u(\theta)$ is given by the state equation $\mathcal{A}u = f + \mathcal{B}\theta$ where $f \in \mathcal{H}'$ (the dual of $\mathcal{H}$), $\mathcal{A} : \mathcal{H} \to \mathcal{H}'$ and $\mathcal{B} : \mathcal{U} \to \mathcal{H}'$.

The optimal control problem is to minimise the cost functional J over the control set U. The case where $U = \mathcal{U}$ is referred to as the unconstrained case. It is a well known fact from the calculus of variations that for a coercive, lower semicontinuous and strictly convex function $J : U \to \overline{\mathbb{R}}$ there exists a unique $\theta^* \in U$ that minimises J over U (cf. Theorem 1.5.1 and Proposition 1.5.1). Such an element $\theta^* \in U$ is called the *optimal control*. For a detailed study of optimal control problems governed by partial differential equations we refer to the books [Lio71, Lio81] of Lions. In this thesis, we consider optimal control problems governed by second order elliptic partial differential equations, which we will now describe.

Let $A \in \mathcal{M}(a, b, \Omega)$ and $B \in \mathcal{M}(c, d, \Omega)$, and assume that B is symmetric. Let $U \subset L^2(\Omega)$ be a closed convex subset. Let $f \in L^2(\Omega)$ be a given function and $N > 0$ be a given constant. The basic optimal control problem is the following: Find $\theta^* \in U$ such that

$$J(\theta^*) = \min_{\theta \in U} J(\theta),$$

where the cost functional, $J(\theta)$, is defined by

$$J(\theta) = \frac{1}{2} \int_{\Omega} B\nabla u.\nabla u \, dx + \frac{N}{2} \|\theta\|^2_{2,\Omega},$$

and the state $u = u(\theta)$ is the weak solution in $H^1_0(\Omega)$ of the boundary value problem

$$\begin{cases} -\mathrm{div}(A\nabla u) &= f + \theta \quad \text{in } \Omega \\ \qquad\quad u &= 0 \qquad\quad \text{on } \partial\Omega. \end{cases} \tag{2.1.1}$$

It follows from Theorem 1.5.1 and Proposition 1.5.1 that for the problem posed above there exists a unique optimal control. Our interest will be in the asymptotic analysis of the above problem when $A = A_\varepsilon$ and $B = B_\varepsilon$, *i.e.*, when the coefficients vary rapidly with a parameter, $\varepsilon > 0$, which tends to zero. We shall consider the situations where Ω is fixed (non-perforated material) as well as the case where Ω varies with ε (perforated material).

2.2 Fixed Cost of the Control

In this section, we give a survey of results on the asymptotic behaviour of optimal control problem of the above form in both perforated and non-perforated domains.

2.2.1 Non-perforated Domains

Let $A_\varepsilon \in \mathcal{M}(a,b,\Omega)$ and $B_\varepsilon \in \mathcal{M}(c,d,\Omega)$ be two sequences of matrices, where the B_ε's are assumed to be symmetric. For each ε, there exists a unique optimal control θ_ε^* such that

$$J_\varepsilon(\theta_\varepsilon^*) = \min_{\theta\in U} J_\varepsilon(\theta),$$

where the cost functional, $J_\varepsilon(\theta)$, is defined by

$$J_\varepsilon(\theta) = \frac{1}{2}\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{N}{2}\|\theta\|_{2,\Omega}^2 \tag{2.2.1}$$

and the state $u_\varepsilon = u_\varepsilon(\theta)$ is the weak solution in $H_0^1(\Omega)$ of the boundary value problem

$$\begin{cases} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) &= f + \theta \quad \text{in } \Omega \\ \qquad\qquad u_\varepsilon &= 0 \qquad\quad \text{on } \partial\Omega \end{cases} \tag{2.2.2}$$

for $\theta \in U$.

Let us now introduce the adjoint system and the optimality condition related to (2.2.1)–(2.2.2) which plays an important role in the identification of the limit problem. The minimiser θ_ε^* is characterised by the optimality condition

$$\int_\Omega (B_\varepsilon \nabla u_\varepsilon^* . \nabla (u_\varepsilon - u_\varepsilon^*) + N\theta_\varepsilon^*(\theta - \theta_\varepsilon^*))\, dx \geq 0, \quad \forall \theta \in U$$

where $u_\varepsilon^* \in H_0^1(\Omega)$, called the *optimal state*, is the state corresponding to θ_ε^*. The above condition can be rewritten as

$$\int_\Omega (p_\varepsilon^* + N\theta_\varepsilon^*)(\theta - \theta_\varepsilon^*)\, dx \geq 0 \quad \forall \theta \in U, \tag{2.2.3}$$

using the adjoint optimal state $p_\varepsilon^* \in H_0^1(\Omega)$ given as the weak solution of

$$\begin{cases} -\mathrm{div}({}^tA_\varepsilon \nabla p_\varepsilon^*) &= -\mathrm{div}(B_\varepsilon \nabla u_\varepsilon^*) \quad \text{in } \Omega \\ \qquad\qquad p_\varepsilon^* &= 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega. \end{cases} \tag{2.2.4}$$

The optimality condition (2.2.3) actually implies that θ_ε^* is the projection in $L^2(\Omega)$ of $\frac{-p_\varepsilon^*}{N}$ on U.

The natural questions that arise in the study of the asymptotic behaviour of the above system are:

(a) Does the optimal control θ^*_ε converge, in a suitable topology, to some limit function θ^*?

(b) In case of (a) being true, is θ^* the optimal control of some limit problem of the same (or similar) type as (2.2.1)–(2.2.2)?

(c) Can the convergence of θ^*_ε to θ^* be suitably strengthened?

We have already described the concept of periodic homogenization in §1.2. The questions that were posed above on the asymptotic behaviour of optimal control problems were answered for the system (2.2.1)–(2.2.2) in the periodic case by Kesavan and Vanninathan (cf. [KV77]). Let $A_\varepsilon = A(\frac{x}{\varepsilon})$ and $B_\varepsilon = B(\frac{x}{\varepsilon})$ where $A = (a_{ij})$ and $B = (b_{ij})$ are matrices with Y-periodic coefficients, defined on a reference cell $Y \subset \mathbb{R}^n$ with B symmetric. For $1 \leq i \leq n$, define v_i analogous to w_i by replacing A with B in (1.2.1). Let A_0 be the homogenized limit matrix as given by (1.2.2) and let $B_0 = (b^0_{ij})$ is the homogenized limit matrix corresponding to B_ε (replace a by b and w_i by v_i in (1.2.2)). Define the matrix $B^\sharp = (b^\sharp_{ij})$ by

$$b^\sharp_{ij} = b^0_{ij} + \int_Y b_{kl} \frac{\partial(w_j - v_j)}{\partial y_l} \frac{\partial(w_i - v_i)}{\partial y_k} \, dy.$$

It was shown in [KV77] that θ^* is the optimal control of the problem:

$$\min_{\theta \in U} J(\theta)$$

where

$$J(\theta) = \frac{1}{2} \int_\Omega B^\sharp \nabla u_0 . \nabla u_0 \, dx + \frac{N}{2} \|\theta\|^2_{2,\Omega} \tag{2.2.5}$$

where $u_0 = u_0(\theta)$ solves

$$\begin{cases} -\text{div}(A_0 \nabla u_0) &= f + \theta \quad \text{in } \Omega \\ \quad\quad\quad\quad u_0 &= 0 \quad\quad\;\; \text{on } \partial\Omega. \end{cases} \tag{2.2.6}$$

The optimal control problem described in (2.2.1)–(2.2.2) was studied by Kesavan and Saint Jean Paulin for the general non-perforated case (cf. [KP97]). They showed the existence of a matrix $B^\sharp$ such that the limit of the optimal controls is the optimal control of the system (2.2.5)–(2.2.6).

In the process they had actually homogenized the adjoint equation (2.2.4), *in spite of the right hand side being bounded only in* $H^{-1}(\Omega)$. It was also observed that the weak convergence of the optimal controls θ^*_ε can be upgraded to strong convergence in $L^2(\Omega)$. The result of [KP97] is summarised in the following theorem.

Theorem 2.2.1 (cf. [KP97]). *(i) For given $f \in L^2(\Omega)$ and $\theta \in L^2(\Omega)$, let $(u_\varepsilon, p_\varepsilon) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be the weak solution of the system*

$$\begin{cases} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) &= f + \theta \quad \text{in } \Omega \\ -\mathrm{div}({}^tA_\varepsilon \nabla p_\varepsilon - B_\varepsilon \nabla u_\varepsilon) &= 0 \quad \text{in } \Omega \\ u_\varepsilon = p_\varepsilon &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.2.7}$$

and let $u_\varepsilon \rightharpoonup u_0$ and $p_\varepsilon \rightharpoonup p_0$ weakly in $H_0^1(\Omega)$. Then there exists a matrix $B^\sharp$ (depending only on the A_ε's and B_ε's) such that

$${}^tA_\varepsilon \nabla p_\varepsilon - B_\varepsilon \nabla u_\varepsilon \rightharpoonup {}^tA_0 \nabla p_0 - B^\sharp \nabla u_0$$

weakly in $(L^2(\Omega))^n$ and the pair $(u_0, p_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is the solution of

$$\begin{cases} -\mathrm{div}(A_0 \nabla u_0) &= f + \theta \quad \text{in } \Omega \\ -\mathrm{div}({}^tA_0 \nabla p_0 - B^\sharp \nabla u_0) &= 0 \quad \text{in } \Omega \\ u_0 = p_0 &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.2.8}$$

where A_0 is the H-limit of A_ε.

(ii) Let $\theta_\varepsilon^* \rightharpoonup \theta^*$ weakly in $L^2(\Omega)$ then $\theta^* \in U$ is the optimal control of the system (2.2.5)–(2.2.6) where A_0 is now the H-limit of A_ε. Further, $\theta_\varepsilon^* \to \theta^*$ strongly in $L^2(\Omega)$ and $J_\varepsilon(\theta_\varepsilon^*) \to J(\theta^*)$. Also, θ^* verifies the limit optimality condition,

$$\int_\Omega (p^* + N\theta^*)(\theta - \theta^*)\,dx \geq 0 \quad \forall \theta \in U \tag{2.2.9}$$

where p^* is the weak limit of p_ε^* in $H_0^1(\Omega)$ that solves the equation

$$\begin{cases} -\mathrm{div}({}^tA_0 \nabla p^*) &= -\mathrm{div}(B^\sharp \nabla u^*) \quad \text{in } \Omega \\ p^* &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.2.10}$$

□

Remark 2.2.1. The result (i) in the above theorem does not require the hypothesis that B_ε are symmetric. The weak convergence (for a subsequence) hypothesis of the optimal controls in (ii) can be established from the ellipticity and boundedness assumptions on A_ε and B_ε. In the case of low cost control, we are unable to deduce the weak convergence of the optimal controls. □

Let us point out that in the problems described above, the admissible control set U is independent of ε. One can then pose the following question:

P 1. What happens when the admissible set U is varying with ε?

In the case of (**P1**), since the control set is varying, the usual argument for existence of bound for θ_ε^* in $L^2(\Omega)$ fails and one also has to identify the control set of the limit problem. Also, if possible, one has to improve the convergence of the optimal controls. We shall address this question in §3.1 of the next chapter.

2.2.2 Perforated domains

Let $A_\varepsilon \in \mathcal{M}(a,b,\Omega)$ and $B_\varepsilon \in \mathcal{M}(c,d,\Omega)$ be two sequences of matrices, where the B_ε's are assumed to be symmetric. Let Ω_ε be as defined in §1.4. Let $U_\varepsilon \subset L^2(\Omega_\varepsilon)$ be a closed convex subset. Let $f \in L^2(\Omega)$ be a given function and $N > 0$ be a given constant. For each ε, the optimal control problem

$$\min_{\theta_\varepsilon \in U_\varepsilon} J_\varepsilon(\theta_\varepsilon),$$

where the cost functional, $J_\varepsilon(\theta_\varepsilon)$, is defined by

$$J_\varepsilon(\theta_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{N}{2}\|\theta_\varepsilon\|^2_{2,\Omega_\varepsilon} \tag{2.2.11}$$

and the state $u_\varepsilon = u_\varepsilon(\theta_\varepsilon)$ is the weak solution in V_ε of the boundary value problem

$$\left\{\begin{array}{rll} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) & = f + \theta_\varepsilon & \text{in } \Omega_\varepsilon \\ A_\varepsilon \nabla u_\varepsilon . n_\varepsilon & = 0 & \text{on } \partial S_\varepsilon \\ u_\varepsilon & = 0 & \text{on } \partial\Omega, \end{array}\right. \tag{2.2.12}$$

(where n_ε is the unit outward normal on ∂S_ε) has a unique optimal control θ_ε^* in U_ε. The adjoint system of (2.2.11)-(2.2.12) is given as

$$\left\{\begin{array}{rll} -\operatorname{div}({}^tA_\varepsilon \nabla p_\varepsilon) & = -\operatorname{div}(B_\varepsilon \nabla u_\varepsilon) & \text{in } \Omega_\varepsilon \\ ({}^tA_\varepsilon \nabla p_\varepsilon - B_\varepsilon \nabla u_\varepsilon).n_\varepsilon & = 0 & \text{on } \partial S_\varepsilon \\ p_\varepsilon & = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{2.2.13}$$

The optimal control problem described in (2.2.11)-(2.2.12) was studied by Kesavan and Saint Jean Paulin in [KP99]. They showed the existence of a matrix $B^\sharp$ such that the limit of the optimal controls is the optimal control of a limit system.

It can be shown that $\tilde{\theta}_\varepsilon^*$, the extension of θ_ε^* by zero on the holes of Ω, is a bounded sequence in $L^2(\Omega)$ and hence, for a subsequence, $\tilde{\theta}_\varepsilon^* \rightharpoonup \theta^*$ weakly in $L^2(\Omega)$. Let A_0 be the H_0-limit of A_ε, and let χ_0 be the weak* limit of χ_ε in $L^\infty(\Omega)$ (cf. §1.4). Then the following theorem of [KP99] summarises the asymptotic behaviour of the state-adjoint system.

Theorem 2.2.2 (cf. [KP99]). *Let $f \in L^2(\Omega)$ and $\theta_\varepsilon \in L^2(\Omega_\varepsilon)$ be such that $\{\tilde{\theta}_\varepsilon\}$ is bounded in $L^2(\Omega)$ and let $(u_\varepsilon, p_\varepsilon) \in V_\varepsilon \times V_\varepsilon$ be the weak solution of the system (2.2.12)–(2.2.13) then there exists a matrix $B^\sharp$ and functions $\theta \in L^2(\Omega)$ and $u_0, p_0 \in H_0^1(\Omega)$ such that (for a subsequence)*

$$
\begin{aligned}
P_\varepsilon u_\varepsilon &\rightharpoonup u_0 \text{ weakly in } H_0^1(\Omega)\\
P_\varepsilon p_\varepsilon &\rightharpoonup p_0 \text{ weakly in } H_0^1(\Omega)\\
\tilde{\theta}_\varepsilon &\rightharpoonup \theta \text{ weakly in } L^2(\Omega)
\end{aligned}
$$

and the pair $(u_0, p_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is the solution of

$$
\left\{
\begin{aligned}
-\mathrm{div}(A_0 \nabla u_0) &= \chi_0 f + \theta && \text{in } \Omega\\
-\mathrm{div}({}^t A_0 \nabla p_0 - B^\sharp \nabla u_0) &= 0 && \text{in } \Omega\\
u_0 = p_0 &= 0 && \text{on } \partial\Omega.
\end{aligned}
\right. \tag{2.2.14}
$$

□

In [KP99], Kesavan and Saint Jean Paulin considered the problem (2.2.11) solving (2.2.12) with the admissible control set U of obstacle type. In particular, when the control set U_ε is the positive cone of $L^2(\Omega_\varepsilon)$ they proved the following result.

Theorem 2.2.3. *In addition to (H1)–(H2) assume the following:*

H 3. *If $\chi_\varepsilon \rightharpoonup \chi_0$ in $L^\infty(\Omega)$ weak*, then $\chi_0^{-1} \in L^\infty(\Omega)$.*

Given $\tilde{\theta}_\varepsilon^* \rightharpoonup \theta^*$ weakly in $L^2(\Omega)$ then θ^* is in the limit admissible set U, the positive cone of $L^2(\Omega)$, and is the optimal control of the limit system

$$
J(\theta) = \frac{1}{2}\int_\Omega B^\sharp \nabla u_0 . \nabla u_0 \, dx + \frac{N}{2}\int_\Omega \frac{\theta^2}{\chi_0}\, dx \tag{2.2.15}
$$

where $u_0 = u_0(\theta)$ *solves*

$$\begin{cases} -div(A_0 \nabla u_0) & = \chi_0 f + \theta \quad in\ \Omega \\ u_0 & = 0 \qquad on\ \partial\Omega. \end{cases} \tag{2.2.16}$$

in U. *Further,* $\tilde{\theta}^*_\varepsilon - \frac{\chi_\varepsilon}{\chi_0}\theta^* \to 0$ *strongly in* $L^2(\Omega)$ *and* $J_\varepsilon(\theta^*_\varepsilon) \to J(\theta^*)$. □

In both the subsections of this section, we dealt with problems where the cost of the control N (fixed cost) was independent of ε. A question that can be asked at this juncture is:

P 2. What happens when the cost of the control, N, is dependent on the parameter ε, say, $N = \varepsilon$? Study these problems in both perforated and non-perforated settings.

The question posed in (**P**2) addresses a class of problems called *low cost control* problems. In this case too, similar to the varying control set case, the argument for existence of bound for θ^*_ε in $L^2(\Omega)$ fails (cf. Remark 2.2.1). Thus, the ideas of [KP97] do not carry over to this case. Moreover, the optimal controls θ^*_ε are seen to converge only in the weak topology of the dual of some higher order Sobolev space and it has not been possible to improve this convergence. Due to this, tackling the problem for an arbitrary control set U is difficult. In §2.3, we shall introduce the low cost control problem and give an overview of the existing literature. We shall address these problems for the non-perforated case in §3.2 of next chapter and for the perforated case in Chapter 4.

2.3 Low Cost Controls

The notion of *low cost control* was introduced by J. L. Lions in [Lio73]. Lions had originally called it *cheap control* and the current terminology was used by Kesavan and Saint Jean Paulin in [KP02]. The terminology is due to the fact that the cost of the control (cf. N in (2.2.1)) is of the order of ε that tends to zero.

2.3.1 Control and State on Domain

The cost functional J_ε is defined as follows:

$$J_\varepsilon(\theta) = \frac{1}{2}\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{\varepsilon}{2}\|\theta\|^2_{2,\Omega} \tag{2.3.1}$$

and the state $u_\varepsilon = u_\varepsilon(\theta)$ is the weak solution in $H_0^1(\Omega)$ of the boundary value problem (2.2.2). In other words, we have set $N = \varepsilon$ in the system (2.2.1)–(2.2.2). Of course, the two parameters involved in the coefficients and the cost of the control could be of different orders, but we shall not consider such situations in this thesis.

The low cost control problems were addressed by Kesavan and Saint Jean Paulin in [KP02]. They had considered the case where the admissible control set U is the positive cone in $L^2(\Omega)$ (the set of non-negative functions in $L^2(\Omega)$), i.e., $U = \{\theta \in L^2(\Omega) \mid \theta \geq 0 \text{ a.e. in } \Omega\}$ and the term $\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx$ in the cost functional replaced by $\|u_\varepsilon\|_{2,\Omega}^2$. Thus, the cost functional is given by

$$J_\varepsilon(\theta) = \frac{1}{2}\|u_\varepsilon\|_{2,\Omega}^2 + \frac{\varepsilon}{2}\|\theta\|_{2,\Omega}^2, \tag{2.3.2}$$

where the state $u_\varepsilon \in H_0^1(\Omega)$ is the weak solution of (2.2.2). The result of [KP02] is summarised in the following theorem.

Theorem 2.3.1 (cf. [KP02, Theorem 2.1]). *If*

$$U = \{\theta \in L^2(\Omega) \mid \theta \geq 0 \text{ a.e. in } \Omega\}$$

is the admissible control set for the system (2.3.2) solving (2.2.2), then there exists u^* and θ^* such that

$$u_\varepsilon^* \rightharpoonup u^* \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega), \tag{2.3.3}$$

$$\varepsilon^{\frac{1}{2}}\theta_\varepsilon^* \to 0 \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega), \tag{2.3.4}$$

$$J_\varepsilon(\theta_\varepsilon^*) \to \frac{1}{2}\|u^*\|_{2,\Omega}^2 \tag{2.3.5}$$

$$\text{and for a subsequence, } \theta_\varepsilon^* \rightharpoonup \theta^* \text{ weakly in } H^{-1}(\Omega). \tag{2.3.6}$$

Further, u^* is the projection of 0 on to $\overline{K}$ in $L^2(\Omega)$, i. e., $u^* \in \overline{K}$ and

$$\int_\Omega u^*(v - u^*)\, dx \geq 0 \quad \forall v \in \overline{K}$$

where

$$K = \left\{ v \in H_0^1(\Omega) \mid \begin{array}{c} \exists \text{ a sequence } v_\varepsilon \in H_0^1(\Omega) \text{ s.t. } v_\varepsilon \rightharpoonup v \text{ in } H_0^1(\Omega), \\ -\mathrm{div}(A_\varepsilon \nabla v_\varepsilon) \in L^2(\Omega) \text{ and is } \geq f \text{ a.e. in } \Omega \end{array} \right\}$$

and $\overline{K}$ is the closure of K in $L^2(\Omega)$. □

In the above theorem though the limit optimal state, u^*, was shown to satisfy a kind of variational inequality, no relation was noted between u^* and θ^* and the description of the set K is somewhat complicated. Also, the limit control, θ^* was not given as an optimal control of a homogenized problem. One also observes that, in contrast to the limit cost functional (2.2.5), the possible limit cost functional of the system (2.3.2) solving (2.2.2)

$$J(\theta) = \frac{1}{2}\|u(\theta)\|^2_{2,\Omega} \tag{2.3.7}$$

may not be coercive in the weak topology of $L^2(\Omega)$ (cf. Example 2.3.1). Thus J may not have a minimiser in U. In spite of these difficulties, the problem (2.3.2) solving (2.2.2) is settled in §3.5. The problem (2.3.1) solving (2.2.2) is taken up in §3.4.

Example 2.3.1. The cost functional J as defined in (2.3.7) is not coercive, in general, in the weak topology of $L^2(\Omega)$. We give a one-dimensional example to observe this fact. Let $\Omega = (-1, 1)$. Let ρ_ε denote the sequence of **mollifiers** defined as,

$$\rho_\varepsilon(x) = \begin{cases} k\varepsilon^{-1}\exp\left(\frac{-\varepsilon^2}{\varepsilon^2-|x|^2}\right), & |x| < \varepsilon \\ 0, & |x| \geq \varepsilon \end{cases} \tag{2.3.8}$$

where $k^{-1} = \int_{|x|\leq 1} \exp\left(\frac{-1}{1-|x|^2}\right) dx$, so that $\int_{-1}^{1} \rho_\varepsilon(x)\, dx = 1$. We now observe that $\|\rho_\varepsilon\|^2_{2,(-1,1)} \to +\infty$ as $\varepsilon \to 0$.

$$\begin{aligned} \int_{-1}^{1} \rho_\varepsilon^2(x)\, dx &= \frac{k^2}{\varepsilon^2}\int_{-\varepsilon}^{\varepsilon} \exp\left(\frac{-2\varepsilon^2}{\varepsilon^2 - |x|^2}\right) dx \\ &= \frac{k^2}{\varepsilon^2}\int_{-\varepsilon}^{\varepsilon} \exp\left(\frac{-2}{1-\frac{|x|^2}{\varepsilon^2}}\right) dx \end{aligned}$$

Putting $y = \frac{x}{\varepsilon}$, we have

$$= \frac{k^2}{\varepsilon}\int_{-1}^{1} \exp\left(\frac{-2}{1-|y|^2}\right) dy \to +\infty \text{ as } \varepsilon \to 0.$$

Using the mollifiers as controls we define u_ε as the solution of

$$-\frac{d^2u_\varepsilon}{dx^2} = \rho_\varepsilon \text{ in } \Omega = (-1, 1)$$

such that $u_\varepsilon(-1) = u_\varepsilon(1) = 0$. Hence

$$-u_\varepsilon'(x) = \int_{-1}^{x} \rho_\varepsilon(y)\,dy - u_\varepsilon'(-1)$$

and $|u_\varepsilon'(-1)| \leq 1 + |u_\varepsilon'(x)|$. Integrating both sides over $(-1,1)$, we have

$$2|u_\varepsilon'(-1)| \leq 2 + \int_{-1}^{1} |u_\varepsilon'(x)|\,dx \leq 2 + \left(\int_{-1}^{1} |u_\varepsilon'|^2\right)^{\frac{1}{2}} \sqrt{2}.$$

By the variational formulation of the equation, we have

$$\int_{-1}^{1} |u_\varepsilon'(x)|^2\,dx = \int_{-1}^{1} \rho_\varepsilon u_\varepsilon\,dx \leq \|u_\varepsilon\|_{\infty,(-1,1)}$$

and hence

$$\begin{aligned} \|u_\varepsilon'\|_{\infty,(-1,1)} &\leq 1 + |u_\varepsilon'(-1)| \\ &\leq 1 + 1 + \frac{1}{\sqrt{2}}\left(\int_{-1}^{1} |u_\varepsilon'|^2\right)^{\frac{1}{2}} \\ &\leq 2 + \frac{1}{\sqrt{2}}\|u_\varepsilon\|_{\infty,(-1,1)}^{\frac{1}{2}}. \end{aligned}$$

Now, since $u_\varepsilon(x) = \int_{-1}^{x} u_\varepsilon'(y)\,dy$, we have

$$|u_\varepsilon(x)| \leq \|u_\varepsilon'\|_{\infty,(-1,1)}|x+1| \leq 2\|u_\varepsilon'\|_{\infty,(-1,1)}.$$

Hence, $\|u_\varepsilon\|_{\infty,(-1,1)} \leq 4 + \sqrt{2}\|u_\varepsilon\|_{\infty,(-1,1)}^{\frac{1}{2}}$. The (positive) root of the quadratic equation $\alpha^2 - \sqrt{2}\alpha - 4 = 0$ is $2\sqrt{2}$ and so $\|u_\varepsilon\|_{\infty,(-1,1)} \leq 8$ and hence $\|u_\varepsilon\|_{2,(-1,1)} = \left(\int_{-1}^{1} u_\varepsilon^2\,dx\right)^{\frac{1}{2}} \leq 8\sqrt{2}$. Thus, $\|u_\varepsilon\|_{2,(-1,1)}$ is bounded while $\|\rho_\varepsilon\|_{2,(-1,1)}^2 \to \infty$. Thus, J as defined in (2.3.7) is not coercive in the weak topology of $L^2(-1,1)$ (cf. Remark 1.5.2). □

2.3.2 Control and State on Boundary

A variant of the system (2.3.2) and (2.2.2) was considered by Kesavan and Saint Jean Paulin in [KP02] with Neumann boundary condition in the state equation and the admissible control set being a subset of $L^2(\partial\Omega)$. In this case, the control θ appears in the boundary condition of the state equation and thus the cost functional is given as

$$J_\varepsilon(\theta) = \frac{1}{2}\|u_\varepsilon\|^2_{2,\partial\Omega} + \frac{\varepsilon}{2}\|\theta\|^2_{2,\partial\Omega}. \tag{2.3.9}$$

where the state $u_\varepsilon \in H^1(\Omega)$ is the weak solution of

$$\begin{cases} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) + u_\varepsilon &= 0 \quad \text{in } \Omega \\ A_\varepsilon \nabla u_\varepsilon.\nu &= f + \theta \quad \text{on } \partial\Omega \end{cases} \tag{2.3.10}$$

where ν is the unit outward normal on $\partial\Omega$.

The asymptotic behaviour of the system (2.3.9)–(2.3.10) was studied in [KP02] when the admissible set U is the positive cone of $L^2(\partial\Omega)$. Let θ^*_ε, as usual, denote the optimal control of the system then the result of [KP02] is summarised in the following theorem.

Theorem 2.3.2 (cf. [KP02, Theorem 3.1]). *If*

$$U = \{\theta \in L^2(\partial\Omega) \mid \theta \geq 0 \ a.e. \ on \ \partial\Omega\}$$

is the admissible control set for the system (2.3.9)–(2.3.10), then there exists u^* and $\theta^* \in U$ satisfying the homogenized problem:

$$\begin{cases} -\mathrm{div}(A_0 \nabla u^*) + u^* &= 0 \quad \text{in } \Omega \\ A_0 \nabla u^*.\nu &= f + \theta^* \quad \text{on } \partial\Omega \end{cases} \tag{2.3.11}$$

such that

$$u^*_\varepsilon \rightharpoonup u^* \text{ weakly in } H^{1/2}(\partial\Omega) \text{ and strongly in } L^2(\partial\Omega), \tag{2.3.12}$$

$$\varepsilon^{\frac{1}{2}}\theta^*_\varepsilon \rightharpoonup 0 \text{ weakly in } H^{1/2}(\partial\Omega) \text{ and strongly in } L^2(\partial\Omega), \tag{2.3.13}$$

$$J_\varepsilon(\theta^*_\varepsilon) \to \frac{1}{2}\|u^*\|^2_{2,\partial\Omega} \tag{2.3.14}$$

$$\text{and } \theta^*_\varepsilon \rightharpoonup \theta^* \text{ weakly in } H^{-1/2}(\partial\Omega) \tag{2.3.15}$$

(Recall that $H^{\frac{1}{2}}(\partial\Omega)$ is the range of the trace map $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$ is its dual). Further, $u^* \in \overline{K}$ and

$$\int_{\partial\Omega} u^*(v - u^*)\, d\sigma \geq 0 \quad \forall v \in \overline{K}$$

where K is the set of all traces of elements of

$$\left\{ v \in H^1(\Omega) \mid \begin{array}{rll} -\mathrm{div}(A_0 \nabla v) + v & = 0 & in\ \Omega \\ A_0 \nabla v.\nu & = f + \theta & on\ \partial\Omega \end{array} \quad for\ \theta \in U \right\}$$

and $\overline{K}$ is the closure of K in $L^2(\partial\Omega)$. □

In contrast to the control on domain case, in boundary control it was possible to express the limit optimal state and limit optimal control in terms of the homogenized operator.

2.4 Summary

In this chapter a class of control problems has been introduced and a brief survey has been done on the homogenization results available in the literature. The difficulties involved in its study have been indicated and the open problems have been listed. In §2.2, we introduced control problems with fixed cost of control. We stated the results available for these problems in both perforated and non-perforated domains. We concluded the section by highlighting some difficulties involved and gave references to later chapters where the problems are addressed afresh. In §2.3, we introduced low cost control problems and stated results available in the literature. The results of low cost control for the non-perforated case will be improved in the next chapter. In Chapter 4, the low cost control problems for perforated domains are treated.

Chapter 3

Control Problems on Non-Perforated Domains

In this chapter we attempt to answer the problems (**P1**) and (**P2**) posed in the previous chapter (page 25 and 27 respectively). In §3.1, we study the problem (**P1**), *i.e.*, we study the homogenization of system (2.2.1)–(2.2.2) when the admissible control set U is dependent on ε. In the rest of the chapter we study the low cost control problems on the non-perforated domains (part of the question posed in (**P2**)).

In this chapter, $A_\varepsilon \in \mathcal{M}(a, b, \Omega)$, $B_\varepsilon \in \mathcal{M}(c, d, \Omega)$ be two sequences of matrices and $B^\sharp$ is as defined in (1.3.9).

3.1 Varying Control Set

Let the matrices B_ε's be symmetric. Let the admissible set of controls denoted as U_ε, be closed convex subsets of $L^2(\Omega)$ and the cost functional be given as,

$$J_\varepsilon(\theta) = \frac{1}{2}\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{N}{2}\|\theta\|^2_{2,\Omega}, \text{ for } \theta \in U_\varepsilon, \tag{3.1.1}$$

where the state $u_\varepsilon = u_\varepsilon(\theta)$ is the weak solution in $H^1_0(\Omega)$ of the boundary value problem

$$\begin{cases} -\text{div}(A_\varepsilon \nabla u_\varepsilon) &= f + \theta \quad \text{in } \Omega \\ \qquad\qquad u_\varepsilon &= 0 \qquad\quad \text{on } \partial\Omega. \end{cases} \tag{3.1.2}$$

Let θ_ε^* be the minimiser of J_ε on U_ε, *i.e.*, $\theta_\varepsilon^* \in U_\varepsilon$ is the solution of the optimal control problem

$$J_\varepsilon(\theta_\varepsilon^*) = \min_{\theta \in U_\varepsilon} J_\varepsilon(\theta).$$

Even to begin addressing this problem in a fashion similar to that of the fixed control set case, we need to have an extra hypothesis that there exists a sequence $\theta_\varepsilon \in U_\varepsilon$ such that $\{\theta_\varepsilon\}$ is bounded in $L^2(\Omega)$. Given this extra hypothesis, we can show the weak convergence of the optimal controls (for a subsequence) as follows: Since θ_ε^* is the optimal control, we have

$$\frac{N}{2}\|\theta_\varepsilon^*\|_{2,\Omega}^2 \leq J_\varepsilon(\theta_\varepsilon^*) \leq J_\varepsilon(\theta),\ \forall \theta \in U_\varepsilon$$

and hence, in particular

$$\frac{N}{2}\|\theta_\varepsilon^*\|_{2,\Omega}^2 \leq J_\varepsilon(\theta_\varepsilon)$$

for the θ_ε whose existence has been assumed. Therefore

$$\begin{aligned}
\frac{N}{2}\|\theta_\varepsilon^*\|_{2,\Omega}^2 &\leq \frac{1}{2}\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{N}{2}\|\theta_\varepsilon\|_{2,\Omega}^2 \\
&\leq \frac{d}{2}\|u_\varepsilon\|_{H_0^1(\Omega)}^2 + \frac{N}{2}\|\theta_\varepsilon\|_{2,\Omega}^2 \\
&\leq \frac{d}{2a}\|f + \theta_\varepsilon\|_{2,\Omega}^2 + \frac{N}{2}\|\theta_\varepsilon\|_{2,\Omega}^2.
\end{aligned}$$

Thus, θ_ε^* is bounded in $L^2(\Omega)$ and hence, for a subsequence (still denoted by ε), $\theta_\varepsilon^* \rightharpoonup \theta^*$ in $L^2(\Omega)$ for some θ^*.

Remark 3.1.1. Let $A_\varepsilon \xrightarrow{H} A_0$. If $\theta_\varepsilon \rightharpoonup \theta$ weakly in $L^2(\Omega)$ then u_ε, the solution of (3.1.2), converges (weakly in $H_0^1(\Omega)$) to $u_0 = u_0(\theta)$, the unique solution of

$$\left\{ \begin{array}{rll} -\text{div}(A_0 \nabla u_0) & = f + \theta & \text{in } \Omega \\ u_0 & = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{3.1.3}$$

Also,

$$\frac{1}{2}\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx \to \frac{1}{2}\int_\Omega B^\sharp \nabla u_0 . \nabla u_0 \, dx. \tag{3.1.4}$$

This result is a consequence of Theorem 2.2.1(i) (cf. also (1.3.10)). □

If one can establish a stronger convergence result of the optimal controls[1], then the theorem below states that the results of [KP97] are also valid when the admissible set U depends on the parameter ε.

Let F_ε denote the extension of J_ε to the extended real line, *i.e.*,

$$F_\varepsilon(\theta) = \begin{cases} J_\varepsilon(\theta) & \text{if } \theta \in U_\varepsilon \\ +\infty & \text{if } \theta \in L^2(\Omega) \setminus U_\varepsilon. \end{cases} \tag{3.1.5}$$

Theorem 3.1.1. *Assume that U, the sequential K-limit of $\{U_\varepsilon\}$ in the strong topology of $L^2(\Omega)$, exists and that $\theta_\varepsilon^* \to \theta^*$ strongly in $L^2(\Omega)$. Let J be defined on U as,*

$$J(\theta) = \frac{1}{2}\int_\Omega B^\sharp \nabla u_0 . \nabla u_0 \, dx + \frac{N}{2}\|\theta\|_{2,\Omega}^2, \text{ for } \theta \in U \tag{3.1.6}$$

where $u_0 = u_0(\theta)$ solves (3.1.3). Let

$$F(\theta) = \begin{cases} J(\theta) & \text{if } \theta \in U \\ +\infty & \text{if } \theta \in L^2(\Omega) \setminus U. \end{cases} \tag{3.1.7}$$

Then $F_\varepsilon \overset{\Gamma_{seq}}{\to} F$ in the strong topology of $L^2(\Omega)$. Also θ^* is the unique minimiser of J in U.

Proof. We begin by showing the Γ-convergence of the extended functionals in $L^2(\Omega)$.

Step 1: Let $\{\theta_\varepsilon\}$ be a sequence in $L^2(\Omega)$ such that $\theta_\varepsilon \to \theta$ strongly in $L^2(\Omega)$. If $\theta \notin U$, then by Lemma 1.5.3(b), we have $\liminf F_\varepsilon(\theta_\varepsilon) = +\infty = F(\theta)$.

If $\theta \in U$ and $\theta_\varepsilon \notin U_\varepsilon$ for small ε, then again $\liminf F_\varepsilon(\theta_\varepsilon) = +\infty > F(\theta)$. But if $\theta \in U$ and there exists a subsequence $\{\theta_{\varepsilon_n}\}$ such that $\theta_{\varepsilon_n} \in U_{\varepsilon_n}$ then

$$\liminf_{\varepsilon\to 0} F_\varepsilon(\theta_\varepsilon) = \liminf_{\varepsilon_n \to 0} F_{\varepsilon_n}(\theta_{\varepsilon_n}) = \lim_{\varepsilon_n\to 0} J_{\varepsilon_n}(\theta_{\varepsilon_n}).$$

Thus, by Remark 3.1.1 and the strong convergence of θ_ε, we have

$$\lim_{\varepsilon_n \to 0} J_{\varepsilon_n}(\theta_{\varepsilon_n}) = J(\theta) = F(\theta).$$

Step 2: Let $\theta \notin U$. Hence $F(\theta) = +\infty$. Then we choose $\theta_\varepsilon = \theta$ for all ε and, by Lemma 1.5.3(a), there exists a $\delta > 0$ such that $F_\varepsilon(\theta_\varepsilon) = +\infty$ for all $\varepsilon < \delta$.

[1] cf. Open Problem 2 in page 95

Now, let $\theta \in U$. Since U is the strong K-limit of U_ε in $L^2(\Omega)$, there exists a $\delta > 0$ and a sequence θ_ε such that $\theta_\varepsilon \to \theta$ strongly in $L^2(\Omega)$ and $\theta_\varepsilon \in U_\varepsilon$, $\forall \varepsilon < \delta$. Therefore, Remark 3.1.1 and the strong convergence of θ_ε together imply that

$$\lim_{\varepsilon \to 0} F_\varepsilon(\theta_\varepsilon) = \lim_{\varepsilon \to 0} J_\varepsilon(\theta_\varepsilon) = J(\theta) = F(\theta).$$

Thus, we have shown that $F_\varepsilon \overset{\Gamma_{seq}}{\to} F$ strongly in $L^2(\Omega)$.
Step 3: We deduce from the hypothesis on U that $\theta^* \in U$. It now follows from Lemma 1.5.2 that $\theta^* \in U$ is the minimiser of J over U and $J(\theta^*_\varepsilon) \to J(\theta^*)$. The uniqueness of θ^* follows from the fact that J is strictly convex. □

One observes from the above theorem that for the results of [KP97] to be valid for the system (3.1.1)–(3.1.2) with varying admissible set, one needs to improve the convergence of the optimal controls[2] and identify the strong K-limit of U_ε, if it exists.

We now prove some results, under the weak convergence hypothesis of the optimal controls, which are useful when the convergence of the optimal controls cannot be improved. Let U' be the (possibly empty) strong K-lower limit of U_ε in $L^2(\Omega)$, *i.e.*, $U' = K\text{-}\liminf_{\varepsilon \to 0} U_\varepsilon$ in the strong topology of $L^2(\Omega)$. For the next theorem, let J be defined on U' by the equation (3.1.6).

Theorem 3.1.2. *Let the minimisers $\theta^*_\varepsilon \rightharpoonup \theta^*$ weakly in $L^2(\Omega)$ for a subsequence, then the following are equivalent:*

(a) $\theta^* \in U'$.

(b) $J_\varepsilon(\theta^*_\varepsilon) \to J(\theta^*)$.

(c) $\theta^*_\varepsilon \to \theta^*$ strongly in $L^2(\Omega)$.

In the case of any one of the above being true, θ^* is the unique minimiser of J on U'.

Proof. (a) $\Longrightarrow$ (b): Let $\theta^* \in U'$. Then it follows from the definition of U' that there exists a $\delta > 0$ and a sequence $\{\theta_\varepsilon\}$ such that $\theta_\varepsilon \to \theta^*$ strongly in $L^2(\Omega)$ and $\theta_\varepsilon \in U_\varepsilon$, $\forall \varepsilon < \delta$. Since,

$$J_\varepsilon(\theta^*_\varepsilon) \le J_\varepsilon(\theta), \quad \forall \theta \in U_\varepsilon,$$

[2] cf. Open Problem 2 in page 95

we have in particular,

$$J_\varepsilon(\theta_\varepsilon^*) \leq J_\varepsilon(\theta_\varepsilon).$$

Taking limsup both sides, we obtain (cf. Remark 3.1.1)

$$\limsup_{\varepsilon\to 0} J_\varepsilon(\theta_\varepsilon^*) \leq J(\theta^*)$$

and by the weak lower semi-continuity of the L^2-norm we deduce

$$J(\theta^*) \leq \liminf_{\varepsilon\to 0} J_\varepsilon(\theta_\varepsilon^*) \leq \limsup_{\varepsilon\to 0} J_\varepsilon(\theta_\varepsilon^*) \leq J(\theta^*).$$

Thus, $J_\varepsilon(\theta_\varepsilon^*) \to J(\theta^*)$ and (b) holds.
(b) $\Longrightarrow$ (c): Let $J_\varepsilon(\theta_\varepsilon^*) \to J(\theta^*)$. We need to show that $\theta_\varepsilon^* \to \theta^*$ strongly in $L^2(\Omega)$. Remark 3.1.1 implies that the first term of $J_\varepsilon(\theta_\varepsilon^*)$ converges to the first term of $J(\theta^*)$, thus we deduce that $\|\theta_\varepsilon^*\|_{2,\Omega}^2 \to \|\theta^*\|_{2,\Omega}^2$, which combined with the weak convergence of θ_ε^*, implies that $\theta_\varepsilon^* \to \theta^*$ strongly in $L^2(\Omega)$. Thus (c) holds.
(c) $\Longrightarrow$ (a): Given (c), (a) follows from the definition of U'.

We now show that if $\theta^* \in U'$, it is the minimiser of J over U'. Let $\theta \in U'$ be an arbitrary control, then there exists a $\delta > 0$ and a sequence θ_ε such that $\theta_\varepsilon \to \theta$ strongly in $L^2(\Omega)$ and $\theta_\varepsilon \in U_\varepsilon$, $\forall \varepsilon < \delta$. We know that,

$$J_\varepsilon(\theta_\varepsilon^*) \leq J_\varepsilon(\theta), \quad \forall \theta \in U_\varepsilon$$

and hence, in particular,

$$J_\varepsilon(\theta_\varepsilon^*) \leq J_\varepsilon(\theta_\varepsilon).$$

Now, taking limit both sides implies that

$$J(\theta^*) \leq J(\theta), \quad \forall \theta \in U'.$$

Thus, θ^* is the minimiser of J over U' and the uniqueness follows from the strict convexity of J. □

In the corollary below, we show that one can actually improve the weak convergence of the optimal controls if $\theta^* \in U$. The proof of the corollary is very similar to the proof of Theorem 3.1.2, except that U' is now replaced with U. For the corollary below, let J on U be defined as in (3.1.6).

Corollary 3.1.1. *Assume that U, the strong K-limit of U_ε in $L^2(\Omega)$, exists and let the minimisers $\theta_\varepsilon^* \to \theta^*$ weakly in $L^2(\Omega)$ for a subsequence, then $\theta^* \in U$ if and only if $\theta_\varepsilon^* \to \theta^*$ strongly in $L^2(\Omega)$ and in this case θ^* is the unique minimiser of J in U.*

3.2 Low Cost Control Problems

Let the admissible set U be a closed convex subset of $L^2(\Omega)$. Consider the low cost control problem

$$J_\varepsilon(\theta) = \frac{1}{2}\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{\varepsilon}{2}\|\theta\|^2_{2,\Omega} \tag{3.2.1}$$

where the state $u_\varepsilon = u_\varepsilon(\theta)$ is the weak solution in $H^1_0(\Omega)$ of the boundary value problem (3.1.2).

Theorem 3.2.1. *If the minimisers θ^*_ε of the system (3.2.1) solving (3.1.2) is bounded in $L^2(\Omega)$, then $\theta^*_\varepsilon \rightharpoonup \theta^*$ weakly in $L^2(\Omega)$, where θ^* is the unique minimiser of J on U where J is given by*

$$J(\theta) = \frac{1}{2}\int_\Omega B^\sharp \nabla u . \nabla u \, dx \tag{3.2.2}$$

and $u \in H^1_0(\Omega)$ is the solution of (3.1.3).

Proof. Since $\{\theta^*_\varepsilon\}$ is bounded in $L^2(\Omega)$, $\|\varepsilon^{1/2}\theta^*_\varepsilon\|^2_{2,\Omega} \to 0$. Let, for a subsequence, $\theta^*_\varepsilon \rightharpoonup \theta^*$. Thus, passing to the limit in

$$J_\varepsilon(\theta^*_\varepsilon) \leq J_\varepsilon(\theta), \quad \forall \theta \in U$$

we deduce using Remark 3.1.1,

$$J(\theta^*) \leq J(\theta), \quad \forall \theta \in U.$$

Therefore, θ^* is a minimiser of J over U and the uniqueness follows from the strict convexity of J. It follows from the uniqueness of θ^* that $\theta^*_\varepsilon \rightharpoonup \theta^*$ for the entire sequence. □

Remark 3.2.1. In contrast to the fixed cost case, the limit cost functional J as defined in (3.2.2) is not coercive, in general (cf. Remark 2.3.1), and hence may not possess a minimiser in U. But if J has a minimiser, say θ^*, then it is unique and satisfies the optimality condition

$$\int_\Omega p^*(\theta - \theta^*)\, dx \geq 0, \quad \forall \theta \in U \tag{3.2.3}$$

where p^* is the solution of

$$\begin{cases} -\text{div}({}^tA_0 \nabla p^*) &= -\text{div}(B^\sharp \nabla u^*) \quad \text{in } \Omega \\ p^* &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.2.4}$$

and u^* is the state corresponding to θ^*. □

As was noted in the last chapter, the main difficulty in the study of low cost control problems is the presence of a small order parameter in the cost functional. Moreover, one is unable to show that the minimisers $\{\theta_\varepsilon^*\}$ are bounded in $L^2(\Omega)$ (the hypothesis of Theorem 3.2.1), even by other means. Thus, carrying over the ideas of 'fixed cost of control' is out of question. Also, for the same reason, tackling the low cost control problem in an arbitrary closed convex subset U of $L^2(\Omega)$ becomes difficult[3]. We, however, observe that the problem turns out to be quite trivial for two cases of U.

Case 1. THE CASE WHERE $-f \in U$ FOR THE GIVEN FUNCTION $f \in L^2(\Omega)$. If $-f \in U$ then its corresponding state is zero. Thus,

$$\frac{c}{2}\|\nabla u_\varepsilon^*\|_{2,\Omega}^2 \leq J_\varepsilon(\theta_\varepsilon^*) \leq J_\varepsilon(-f) = \frac{\varepsilon}{2}\|f\|_{2,\Omega}^2$$
$$\text{and } \frac{\varepsilon}{2}\|\theta_\varepsilon^*\|_{2,\Omega}^2 \leq J_\varepsilon(\theta_\varepsilon^*) \leq J_\varepsilon(-f) = \frac{\varepsilon}{2}\|f\|_{2,\Omega}^2.$$

Therefore, we deduce that $u_\varepsilon^* \to 0$ strongly in $H_0^1(\Omega)$ and that the sequence $\{\theta_\varepsilon^*\}$ is bounded in $L^2(\Omega)$. Hence, by Theorem 3.2.1, $\theta_\varepsilon^* \rightharpoonup \theta^*$ weakly in $L^2(\Omega)$. Now, by H-convergence, we have $u^* = 0$ and $\theta^* = -f$. Also the convergence is valid for the whole sequence. We now note that the convergence is, in fact, strong by observing that,

$$\limsup_{\varepsilon\to 0} \|\theta_\varepsilon^*\|_{2,\Omega}^2 \leq \|f\|_{2,\Omega}^2 \leq \liminf \|\theta_\varepsilon^*\|_{2,\Omega}^2.$$

Hence, $\|\theta_\varepsilon^*\|_{2,\Omega}^2 \to \|f\|_{2,\Omega}^2$. Thus, $\theta_\varepsilon^* \to -f$ strongly in $L^2(\Omega)$. Also note that $J_\varepsilon(\theta_\varepsilon^*) \to 0$, which is the minimum of J over U.

Case 2. THE CASE WHEN U IS BOUNDED IN $L^2(\Omega)$. If U is bounded in $L^2(\Omega)$ then the optimal controls θ_ε^* are bounded in $L^2(\Omega)$ and thus the results of Theorem 3.2.1 hold.

Though we are unable to prove the strong convergence of the optimal control, in general, we can do so when the bounded admissible set U is a ball of radius R in $L^2(\Omega)$. Such an improvement of convergence of the optimal control was proved for the unit ball in [KP02] for a different low cost control problem. We show the same for the system (3.2.1) solving (3.1.2) using a different argument, which also throws some light on much more general bounded control sets U.

[3]cf. Open Problem 3 in page 95

Theorem 3.2.2. *Let U be a bounded admissible set in $L^2(\Omega)$ such that $-f \notin U$, then the optimal controls $\theta^*_\varepsilon \in \partial U$ for ε small enough and its weak limit $\theta^* \in \partial U$.*

Proof. Suppose $\theta^*_\varepsilon \notin \partial U$, then for some $r > 0$ there exists a ball $B(\theta^*_\varepsilon, r) \subset U$ and thus,

$$\theta^*_\varepsilon + t\eta \in U \quad \forall \eta \in B(0,1) \text{ and } |t| < r.$$

Using this in the optimality condition

$$\int_\Omega (p^*_\varepsilon + \varepsilon\theta^*_\varepsilon)(\theta - \theta^*_\varepsilon)\,dx \geq 0 \quad \forall \theta \in U, \tag{3.2.5}$$

we have, for $0 < t < r$

$$t\int_\Omega (p^*_\varepsilon + \varepsilon\theta^*_\varepsilon)\eta\,dx \geq 0 \quad \forall \eta \in B(0,1).$$

Changing η to $-\eta$, we deduce easily that $\theta^*_\varepsilon = \frac{-p^*_\varepsilon}{\varepsilon}$ and hence $p^*_\varepsilon \to 0$ strongly in $L^2(\Omega)$ which implies that $u^* = 0$ and $\theta^*_\varepsilon \rightharpoonup \theta^* = -f$. This contradicts the fact that $-f \notin U$.

We now show that if $-f \notin U$, then $\theta^* \in \partial U$. Suppose $\theta^* \notin \partial U$, then for some $r > 0$ there exists a ball $B(\theta^*, r) \subset U$ and thus,

$$\theta^* + t\eta \in U \quad \forall \eta \in B(0,1) \text{ and } |t| < r.$$

Using this in (3.2.3) we have, for $0 < t < r$,

$$t\int_\Omega p^*\eta\,dx \geq 0 \quad \forall \eta \in B(0,1)$$

and again this yields $p^* = 0$ which in turn implies $u^* = 0$ and $\theta^* = -f \in U$ which contradicts our hypothesis. □

Corollary 3.2.1. *Let U be the ball in $L^2(\Omega)$ centred at 0 of radius R. If $f \notin U$, then $\|\theta^*_\varepsilon\|_{2,\Omega} = \|\theta^*\|_{2,\Omega} = R$ (for ε small enough) and thus $\theta^*_\varepsilon \to \theta^*$ strongly in $L^2(\Omega)$.*

We have so far, in this section, studied the most simple situations of the low cost control problem (3.2.1) solving (3.1.2) and observed in the process that one requires more information on optimal controls. The behaviour of the low cost control problems for an arbitrary admissible set U is unknown[4]. The case where U is the positive cone of $L^2(\Omega)$, *i.e.*, $U = \{\theta \in L^2(\Omega) \mid \theta \geq 0 \text{ a.e. in } \Omega\}$ was considered by Kesavan and Saint Jean Paulin (cf. [KP02]) for cost functionals different from (3.2.1). However, they were unable to identify the limit for those systems due to the very weak convergence of the optimal controls. Their results are described in the previous chapter (cf. §2.3). In the rest of the chapter we develop the necessary tools and study the asymptotic behaviour of some low cost control problems when U is the positive cone of $L^2(\Omega)$. In the next section, we present results very crucial for the homogenization of low cost control problems on the positive cone and present some elementary results on the closure of the positive cone in various spaces.

3.3 Data from the positive cone of H^{-1}

In this section, we prove some results that will be useful in the sequel. To begin, we shall state a result called Meyers' regularity result, whose proof can be found in [BLP78, Page 38] (or cf. [Mey63]).

Theorem 3.3.1. *Let $A \in \mathcal{M}(a, b, \Omega)$ and $u \in H_0^1(\Omega)$ be the solution of*

$$\begin{cases} -\operatorname{div}(A\nabla u) &= f \quad \text{in } \Omega \\ \qquad\quad u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.3.1}$$

where $f \in H^{-1}(\Omega)$. There exists a number $p > 2$ (which depends on a, b, Ω and on the dimension n) such that if $f \in W^{-1,p}(\Omega)$, then the solution u belongs to $W_0^{1,p}(\Omega)$ and satisfies

$$\|u\|_{W_0^{1,p}(\Omega)} \leq C_0 \|f\|_{W^{-1,p}(\Omega)} \tag{3.3.2}$$

(where C_0 depends on the same quantities as p does). □

The highlight of the above theorem, other than the regularity aspect, is that p and C_0 will be independent of ε, if the equation involves oscillating coefficients, say A_ε, and also that the p is same for ${}^tA$ instead of A in the state equation above.

We shall now state a result proved by F. Murat (cf. [Mur81]) which plays a crucial role in the results proved in §3.5 and §3.4.

[4] cf. Open Problem 3 in page 95

Theorem 3.3.2 (F. Murat, 1981). *Let Ω be an open subset of $\mathbb{R}^n$. Consider a sequence $\{g_\varepsilon\} \subset H^{-1}(\Omega)$ such that*

$$g_\varepsilon \rightharpoonup g \text{ weakly in } H^{-1}(\Omega)$$

and $g_\varepsilon \geq 0$ for all ε, then

$$g_\varepsilon \to g \text{ strongly in } W^{-1,q}_{loc}(\Omega), \quad \forall q < 2$$

i.e,

$$\phi g_\varepsilon \to \phi g \text{ strongly in } W^{-1,q}(\Omega), \quad \forall q < 2 \text{ and } \forall \phi \in \mathcal{D}(\Omega).$$

□

The following is a H-convergence result for weak data from the positive cone of $H^{-1}(\Omega)$. We now prove the theorem in a particular case. The theorem in its full generality is stated and proved in [DMM04, Theorem 3.1].

Theorem 3.3.3. [5] *Let $\{A_\varepsilon\}$ be a sequence of matrices in $\mathcal{M}(a,b,\Omega)$ which H-converges to a matrix A_0 and let $f \in H^{-1}(\Omega)$. If $u_\varepsilon \in H^1_0(\Omega)$ is the weak solution of*

$$\left\{ \begin{array}{rll} -\text{div}(A_\varepsilon \nabla u_\varepsilon) & = f + g_\varepsilon & \text{in } \Omega \\ u_\varepsilon & = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{3.3.3}$$

with $g_\varepsilon \rightharpoonup g$ weakly in $H^{-1}(\Omega)$ and g_ε's belong to the positive cone of $H^{-1}(\Omega)$. Then,

$$\left. \begin{array}{r} u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H^1_0(\Omega) \\ A_\varepsilon \nabla u_\varepsilon \rightharpoonup A_0 \nabla u_0 \text{ weakly in } (L^2(\Omega))^n, \end{array} \right\} \tag{3.3.4}$$

where $u_0 \in H^1_0(\Omega)$ is the unique solution of

$$\left\{ \begin{array}{rll} -\text{div}(A_0 \nabla u_0) & = f + g & \text{in } \Omega \\ u_0 & = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{3.3.5}$$

Proof. Observe that there exists a subsequence (still denoted by ε) such that

$$\begin{array}{c} u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H^1_0(\Omega) \\ A_\varepsilon \nabla u_\varepsilon \rightharpoonup \sigma \text{ weakly in } (L^2(\Omega))^n. \end{array}$$

Let, now, $v \in \mathcal{D}(\Omega)$ and let $v_\varepsilon \in H^1_0(\Omega)$ be the solution of

$$\left\{ \begin{array}{rll} -\text{div}({}^t A_\varepsilon \nabla v_\varepsilon) & = -\text{div}({}^t A_0 \nabla v) & \text{in } \Omega \\ v_\varepsilon & = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{3.3.6}$$

[5] cf. Open Problem 4 in page 96

Since the sequence $\{{}^tA_\varepsilon\}$ H-converges to ${}^tA_0$, we have

$$v_\varepsilon \rightharpoonup v \text{ weakly in } H_0^1(\Omega) \tag{3.3.7}$$
$${}^tA_\varepsilon\nabla v_\varepsilon \rightharpoonup {}^tA_0\nabla v \text{ weakly in } (L^2(\Omega))^n \tag{3.3.8}$$

Also, by Theorem 3.3.1 (since $v \in \mathcal{D}(\Omega)$ and the constant C_0 obtained there is independent of ε), we have

$$v_\varepsilon \rightharpoonup v \text{ weakly in } W_0^{1,p}(\Omega) \text{ for some } p > 2. \tag{3.3.9}$$

Let $\phi \in \mathcal{D}(\Omega)$. Using $v_\varepsilon\phi$ as test function in (3.3.3), and $u_\varepsilon\phi$ as test function in (3.3.6), we have

$$\begin{aligned}\langle f + g_\varepsilon, v_\varepsilon\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} &= \langle -\mathrm{div}({}^tA_0\nabla v), u_\varepsilon\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &\quad - \int_\Omega ({}^tA_\varepsilon\nabla v_\varepsilon.\nabla\phi)u_\varepsilon\,dx \\ &\quad + \int_\Omega (A_\varepsilon\nabla u_\varepsilon.\nabla\phi)v_\varepsilon\,dx.\end{aligned} \tag{3.3.10}$$

Since $g_\varepsilon \rightharpoonup g$ in $H^{-1}(\Omega)$ and $g_\varepsilon \geq 0$ a.e., by Theorem 3.3.2, we have

$$\psi g_\varepsilon \to \psi g \text{ strongly in } W^{-1,q}(\Omega) \text{ for every } q < 2 \text{ and for every } \psi \in \mathcal{D}(\Omega).$$

Let us choose ψ in $\mathcal{D}(\Omega)$ which is equal to 1 in a neighbourhood of $\mathrm{supp}(\phi)$ and q such that $1/p + 1/q = 1$ for the p obtained in (3.3.9). Then passing to the limit in (3.3.10), the left hand side becomes,

$$\begin{aligned}\lim_{\varepsilon\to 0}\langle f + g_\varepsilon, v_\varepsilon\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} &= \lim_{\varepsilon\to 0}\langle \psi(f + g_\varepsilon), v_\varepsilon\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &= \langle \psi(f+g), v\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &= \langle f+g, v\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.\end{aligned}$$

Thus passing to the limit in (3.3.10) gives

$$\begin{aligned}\langle f+g, v\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} &= \langle -\mathrm{div}({}^tA_0\nabla v), u_0\phi\rangle - \int_\Omega ({}^tA_0\nabla v.\nabla\phi)u_0\,dx \\ &\quad + \int_\Omega (\sigma.\nabla\phi)v\,dx \\ &= \int_\Omega ({}^tA_0\nabla v.\nabla u_0)\phi\,dx + \int_\Omega (\sigma.\nabla\phi)v\,dx \\ &= \int_\Omega (A_0\nabla u_0.\nabla v)\phi\,dx + \int_\Omega (\sigma.\nabla\phi)v\,dx.\end{aligned}$$

Since $-\text{div}(\sigma) = f + g$ in $\mathcal{D}'(\Omega)$, we deduce that

$$\int_\Omega (\sigma.\nabla v)\phi\, dx = \int_\Omega (A_0 \nabla u_0.\nabla v)\phi\, dx,$$

for every ϕ and v in $\mathcal{D}(\Omega)$. Since, for every $x \in \Omega$, v can be chosen such that $\nabla v(x)$ coincides with any vector of $\mathbb{R}^n$, we have $\sigma = A_0 \nabla u_0$ a.e. in Ω. The uniqueness of the limits in (3.3.4) implies that the convergences hold for the entire sequence. □

Remark 3.3.1. We note that, in general, the energy functional does not converge for weakly converging data (from the positive cone) in $H^{-1}(\Omega)$, even if the coefficients are fixed, as the following example shows. Let $\Omega = (-1,1) \subset \mathbb{R}$. Define $u_\varepsilon : \Omega \to \mathbb{R}$ as

$$u_\varepsilon(x) = \begin{cases} \frac{1+x}{\varepsilon} & \text{if } x \in (-1, -1+\varepsilon^2), \\ \varepsilon & \text{if } x \in [-1+\varepsilon^2, 1-\varepsilon^2], \\ \frac{1-x}{\varepsilon} & \text{if } x \in (1-\varepsilon^2, 1) \end{cases}$$

then its first derivative u'_ε is given as

$$u'_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon} & \text{if } x \in (-1, -1+\varepsilon^2), \\ 0 & \text{if } x \in [-1+\varepsilon^2, 1-\varepsilon^2], \\ \frac{-1}{\varepsilon} & \text{if } x \in (1-\varepsilon^2, 1). \end{cases}$$

Observe that the distribution $-u''_\varepsilon = \frac{1}{\varepsilon}(\delta_{-1+\varepsilon^2} + \delta_{1-\varepsilon^2}) \geq 0$ is in the positive cone of $H^{-1}(\Omega)$ and converges weakly to 0. Also $u_\varepsilon \in H_0^1(\Omega)$ and $u_\varepsilon \to 0$ weakly in $H_0^1(\Omega)$, while the associated energy functional $\|u'_\varepsilon\|_2^2 = 2$ is a constant independent of ε. □

We now prove some results which seem to be intuitively obvious but do not appear to have been proved anywhere in the available literature.

A distribution is said to be *non-negative* if it takes non-negative values for all non-negative test functions. Now, if $f, g \in L^2(\Omega)$ are non-negative then, clearly, $\int_\Omega fg\, dx \geq 0$. At this juncture one is interested to know if a similar statement is also valid in the dual of $H_0^1(\Omega)$, *i.e.*, if $w \geq 0$ in $H^{-1}(\Omega)$ and $v \geq 0$ in $H_0^1(\Omega)$ then is it true that $\langle w, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \geq 0$? The answer is trivial to observe in the case when $\Omega = \mathbb{R}^n$ than in the case of a bounded open set in $\mathbb{R}^n$.

The basic idea for the $\Omega = \mathbb{R}^n$ case is that for any $v \in H_0^1(\mathbb{R}^n)$ such that $v \geq 0$ there exists a sequence of positive test functions converging strongly to v in $H_0^1(\mathbb{R}^n)$. These positive test functions are obtained by the convolution of v with the mollifiers (cf. (2.3.8)) and then using the cut-off function technique to make the support compact, *i.e.*, define $v_k(x) = \zeta_k(x)(\rho_{\varepsilon_k} * v)(x)$ where the cut-off function $\zeta_k(x) = \zeta(x/k)$ for a function $\zeta \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \leq \zeta \leq 1, \zeta \equiv 1$ on $B(0,1)$ and Supp(ζ)$\subset B(0,2)$. This a standard technique in the theory of Sobolev spaces to prove results on the entire space $\mathbb{R}^n$. But these techniques break down when Ω is a bounded open subset of $\mathbb{R}^n$. This difficulty is overcome by Proposition 3.3.1.

One knows that $H_0^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$. In the following proposition we prove that for a given *positive* H_0^1 function we can extract a sequence of *positive* H_0^1 functions with compact support in Ω which converges to the given function in H_0^1.

Proposition 3.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $v \in H_0^1(\Omega)$ and $v \geq 0$ then there exists a sequence $\{\psi_n\} \subset H_0^1(\Omega)$ such that $\psi_n \to v$ in $H_0^1(\Omega)$, $\psi_n \geq 0$ for all n and ψ_n has compact support in Ω.*

Proof. Since $v \in H_0^1(\Omega)$, there exists a sequence $\{\phi_n\} \subset \mathcal{D}(\Omega)$ such that $\phi_n \to v$ in $H_0^1(\Omega)$. In particular, $\phi_n \to v$ in $L^2(\Omega)$ and hence $|\phi_n| \to |v| = v$ in $L^2(\Omega)$. Thus, for a subsequence,

$$\phi_n^+ + \phi_n^- \to v \text{ a.e. in } \Omega,$$
$$\phi_n^+ - \phi_n^- \to v \text{ a.e. in } \Omega.$$

Therefore, $\phi_n^+ \to v$ a.e. in Ω and $\phi_n^- \to 0$ a.e in Ω.

Observe that $|\phi_n^-|^2 \leq |\phi_n|^2$ and $|\phi_n|^2 \to v^2$ a.e. in Ω. Now, since $|\phi_n^-|^2 \to 0$ a.e. in Ω and $\int_\Omega |\phi_n|^2\,dx \to \int_\Omega v^2\,dx < +\infty$, it follows from the generalised Lebesgue convergence theorem that $\|\phi_n^-\|_{2,\Omega}^2 \to 0$. Thus, $\phi_n^+ \to v$ strongly in $L^2(\Omega)$.

Since, $\|\nabla\phi_n\|_{2,\Omega}^2 = \|\nabla\phi_n^+\|_{2,\Omega}^2 + \|\nabla\phi_n^-\|_{2,\Omega}^2 \to \|\nabla v\|_{2,\Omega}^2$, we deduce that $\{\phi_n^-\}$ is bounded in $H_0^1(\Omega)$. Thus, for a subsequence, $\phi_n^- \rightharpoonup \phi$ in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. Then from the previous paragraph we conclude $\phi = 0$.

Consider,

$$\begin{aligned}\|\nabla(v - \phi_n^+)\|_{2,\Omega}^2 &= \|\nabla(v - \phi_n) - \nabla\phi_n^-\|_{2,\Omega}^2 \\ &= \|\nabla(v - \phi_n)\|_{2,\Omega}^2 + \|\nabla\phi_n^-\|_{2,\Omega}^2 - 2\int_\Omega \nabla(v - \phi_n).\nabla\phi_n^-\,dx.\end{aligned}$$

Since the first and last term on the right hand side goes to zero as n tends to infinity, we have

$$\lim_{n\to\infty} \|\nabla(v-\phi_n^+)\|_{2,\Omega}^2 = \lim_{n\to\infty} \|\nabla\phi_n^-\|_{2,\Omega}^2 = l, \text{ say.} \tag{3.3.11}$$

Then by passing to the limit in both sides of

$$\begin{aligned}\|\nabla(v-\phi_n)\|_{2,\Omega}^2 &= \|\nabla(v-\phi_n^+)+\nabla\phi_n^-\|_{2,\Omega}^2 \\ &= \|\nabla(v-\phi_n^+)\|_{2,\Omega}^2 + \|\nabla\phi_n^-\|_{2,\Omega}^2 + 2\int_\Omega \nabla(v-\phi_n^+).\nabla\phi_n^-\,dx,\end{aligned}$$

we have,

$$0 = 2l + 2\lim_{n\to\infty}\int_\Omega \nabla v.\nabla\phi_n^-\,dx.$$

Since $\phi_n^- \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, we have deduced that $l=0$. Thus, proving that (cf. (3.3.11))

$$\phi_n^+ \to v \text{ strongly in } H_0^1(\Omega).$$

Since $\mathrm{Supp}(\phi_n^+) \subset \mathrm{Supp}(\phi_n)$, by choosing $\psi_n = \phi_n^+$, we have proved our result. □

Remark 3.3.2. In the result proved above if we choose $\psi_n = v-(v-\phi_n^+)^+$ then, in addition to the properties proved above, we also have that $0 \le \psi_n \le v$ for all n. Since, $v-\phi_n^+ \to 0$ strongly in $H_0^1(\Omega)$ we have $\|\nabla(v-\psi_n)\|_{2,\Omega}^2 \to 0$ because

$$\|\nabla(v-\phi_n^+)^+\|_{2,\Omega}^2 \le \|\nabla(v-\phi_n^+)\|_{2,\Omega}^2 \to 0.$$

Hence $\psi_n \to v$ strongly in $H_0^1(\Omega)$ and by definition $0 \le \psi_n \le v$. □

Proposition 3.3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $g \in H^{-1}(\Omega)$ be such that $g \ge 0$ and let $u \in H_0^1(\Omega)$ be such that $u \ge 0$ a.e. in Ω then $\langle g,u\rangle \ge 0$, where $\langle\cdot,\cdot\rangle$ denotes the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.*

Proof. Let $v \in H_0^1(\Omega)$ be such that $v \ge 0$ a.e. in Ω and $\mathrm{Supp}(v)$ is compact in Ω. If ρ_ε denotes the mollifiers, then $\mathrm{Supp}(\rho_\varepsilon * v) \subset B(0,\varepsilon)+\mathrm{Supp}(v) \subset \Omega$ for small ε and is compact. Now, $\rho_\varepsilon * v \to v$ strongly in $H_0^1(\Omega)$. Moreover, $\rho_\varepsilon * v \in \mathcal{D}(\Omega)$ and $\rho_\varepsilon * v \ge 0$ in Ω. Since $\langle g, \rho_\varepsilon * v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \ge 0$, passing to the limit we have $\langle g,v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \ge 0$.

Given $u \in H_0^1(\Omega)$ such that $u \ge 0$ a.e. in Ω, by Proposition 3.3.1, there exists a sequence $\{v_n\} \subset H_0^1(\Omega)$ with compact support in Ω such that $v_n \ge 0$ for all n and $v_n \to u$ strongly in $H_0^1(\Omega)$. The argument in the above paragraph shows that $\langle g,v_n\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \ge 0$ for all n and hence $\langle g,u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \ge 0$. □

We shall now prove a result which shows the equivalence of the above result to a statement on the closure of the positive cone.

Proposition 3.3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. The following statements are true and are equivalent:*

(i) The closure of the positive cone of $L^2(\Omega)$ in $H^{-1}(\Omega)$ is the positive cone of $H^{-1}(\Omega)$.

(ii) The closure of the positive cone of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$ is the positive cone of $H_0^1(\Omega)$.

(iii) If $g \in H^{-1}(\Omega)$ is such that $g \geq 0$ and $u \in H_0^1(\Omega)$ is such that $u \geq 0$ a.e. in Ω then $\langle g, u \rangle \geq 0$.

Proof. (i) $\Longrightarrow$ (iii): Let $g \in H^{-1}(\Omega)$ be such that $g \geq 0$ and $u \in H_0^1(\Omega)$ is such that $u \geq 0$ a.e. in Ω. By (i), there exists a sequence $\{g_n\} \subset L^2(\Omega)$ such that $g_n \geq 0$ a.e. in Ω for all n and $g_n \to g$ strongly in $H^{-1}(\Omega)$. Therefore, $\int_\Omega g_n u\, dx \geq 0$ and *i.e.*, $\langle g_n, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \geq 0$ and by passing to the limit we have $\langle g, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \geq 0$.

(iii) $\Longrightarrow$ (i): Suppose (i) is false. Then there exists a $g \in H^{-1}(\Omega)$ such that $g \geq 0$ a.e. and g not in the H^{-1}-closure of the positive cone of $L^2(\Omega)$. Thus, by Hahn-Banach separation theorem, there exists a F in the dual of $H^{-1}(\Omega)$ and a $\alpha \in \mathbb{R}$ such that

$$F(g) < \alpha < F(f), \quad \forall f \in L^2(\Omega) \text{ and } f \geq 0 \text{ a.e. in } \Omega.$$

Since $H^{-1}(\Omega)$ is a reflexive space, there exists $u \in H_0^1(\Omega)$ such that

$$\langle g, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} < \alpha < \langle f, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega f u\, dx, \forall f \in L^2(\Omega)$$
$$\text{s.t. } f \geq 0 \text{ a.e. in } \Omega.$$

On replacing f by nf, we have $\langle f, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} > \frac{\alpha}{n} \quad \forall n \in \mathbb{N}$ and hence

$$\langle f, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega f u\, dx \geq 0 \quad \forall f \in L^2(\Omega) \text{ and } f \geq 0 \text{ a.e. in } \Omega.$$

Therefore, $u \geq 0$ a.e. in Ω. Now, by choosing $f = 0$, we deduce that $\alpha < 0$ and hence $\langle g, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} < 0$ which contradicts (iii).

(ii) $\Longrightarrow$ (iii): Let g and u satisfy the hypotheses of (iii). By (ii), there exists a sequence $\{u_n\} \subset \mathcal{D}(\Omega)$ such that $u_n \geq 0$ a.e. for all n and $u_n \to u$ strongly in $H_0^1(\Omega)$. Since $\langle g, u_n\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \geq 0$ for all n, we have on the limit $\langle g, u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \geq 0$, thus proving (iii).

(iii) $\Longrightarrow$ (ii): Suppose (ii) is false. Then there exists a $u \in H_0^1(\Omega)$ such that $u \geq 0$ a.e. and u not in the H_0^1-closure of the positive cone of $\mathcal{D}(\Omega)$. Thus, by Hahn-Banach separation theorem, there exists a $g \in H^{-1}(\Omega)$ and a $\alpha \in \mathbb{R}$ such that

$$\langle g, u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} < \alpha < \langle g, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall v \in \mathcal{D}(\Omega) \text{ and } v \geq 0 \text{ a.e. in } \Omega.$$

On replacing v by nv, we have $\langle g, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} > \frac{\alpha}{n}$ $\forall n \in \mathbb{N}$ and hence

$$\langle g, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \geq 0 \quad \forall v \in \mathcal{D}(\Omega) \text{ and } v \geq 0 \text{ a.e. in } \Omega.$$

Therefore, $g \geq 0$ a.e. in Ω. Now, by choosing $v = 0$, we deduce that $\alpha < 0$ and hence $\langle g, u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} < 0$ which contradicts (iii).

Observe that Proposition 3.3.2 proves (iii) of Proposition 3.3.3 and hence all the three equivalent statements are true. □

3.4 Dirichlet type integral in cost

After the necessary tools developed in the previous section, we are now in a position to address the problem (3.2.1) solving (3.1.2) defined in §3.2. To refresh the memory of the readers, we recall that our aim was to study the asymptotic behaviour of the system (3.2.1) solving (3.1.2) for an arbitrary admissible set U in $L^2(\Omega)$. We had, however, remarked that the problem is open[6] for an arbitrary control set in $L^2(\Omega)$. The approach used by [KP97] in the fixed cost case was to homogenize the optimality system (consisting of the state and adjoint state equations) and then showing the convergence of the energy. However, in the case of weakly converging data (even from the positive cone), we cannot expect, in general, the convergence of the energy (cf. Remark 3.3.1). Thus we are still not in a position to establish a limit optimal control problem for the positive cone case[7] in spite of the machinery developed in the previous section. However, in this section, we prove a result analogous to Theorem 2.2.1(i) when the data is from the positive cone

[6] cf. Open Problem 3 in page 95
[7] cf. Open Problem 3 in page 95

in $H^{-1}(\Omega)$, wherein we homogenize the optimality system which has weak converging data in $H^{-1}(\Omega)$.

To begin, we recall the corrector functions χ_ε^i defined in (1.3.8) satisfying the properties (1.3.7). We now define a set of test functions $\psi_\varepsilon^i \in H_0^1(\Omega)$, for $1 \le i \le n$, which solves

$$\left\{ \begin{array}{rll} -\mathrm{div}({}^tA_\varepsilon \nabla \psi_\varepsilon^i) & = \mathrm{div}({}^tB_\varepsilon \nabla \chi_\varepsilon^i) & \text{in } \Omega \\ \psi_\varepsilon^i & = 0 & \text{on } \partial\Omega. \end{array} \right. \quad (3.4.1)$$

Observe that $\{\psi_\varepsilon^i\}$ converges weakly in $H_0^1(\Omega)$, up to a subsequence, say to a function ψ_0^i and $\{{}^tA_\varepsilon \nabla \psi_\varepsilon^i + {}^tB_\varepsilon \nabla \chi_\varepsilon^i\}$ converges weakly, say to τ_0^i, in $(L^2(\Omega))^n$. Note that $\mathrm{div}\tau_0^i = 0$, for $1 \le i \le n$. It was shown in [KP97] that ${}^tB^\#e_i = \tau_0^i - {}^tA_0 \nabla \psi_0^i$. The following lemma sheds some light on the regularity of the functions χ_ε^i and ψ_ε^i.

Lemma 3.4.1. *There exists a $p > 2$ for which the corrector functions $\{\chi_\varepsilon^i\}$ defined in* (1.3.8) *are bounded in $W^{1,p}(\Omega)$ and the functions $\{\psi_\varepsilon^i\}$ are bounded in $W_0^{1,p}(\Omega)$.*

Proof. Let $\chi_\varepsilon^i = -\omega_\varepsilon^i + x_i$. Now, since $-\mathrm{div}(A_0 e_i) + \mathrm{div}(A_\varepsilon e_i)$ is bounded in $W^{-1,\widehat{p}}(\Omega)$ for all $1 \le \widehat{p} \le \infty$, it follows from Theorem 3.3.1 (Meyers' regularity result), that $\{\omega_\varepsilon^i\}$ is bounded in $W_0^{1,p}(\Omega)$ for some $p > 2$. Thus χ_ε^i is bounded in $W^{1,p}(\Omega)$ for the same $p > 2$. Note that p is independent of the parameter ε, since $A_\varepsilon \in \mathcal{M}(a, b, \Omega)$.

Now, since $\{\chi_\varepsilon^i\}$ is bounded in $W^{1,p}(\Omega)$, we have $\mathrm{div}({}^tB_\varepsilon \nabla \chi_\varepsilon^i)$ bounded in $W^{-1,p}(\Omega)$. Thus, again by Theorem 3.3.1, ψ_ε^i is bounded in $W_0^{1,p}(\Omega)$. Note that the p obtained in the Meyers' result is same for both A_ε and ${}^tA_\varepsilon$. □

We now prove the most important theorem of this section. We recall that Theorem 3.3.3 is actually the H-convergence result for the weak data from the positive cone of $H^{-1}(\Omega)$. The following theorem is the H-convergence result for a system of equations involving two set of matrix coefficients with weak data from positive cone.

Theorem 3.4.1. *Let $A_\varepsilon \in \mathcal{M}(a, b, \Omega)$ and $B_\varepsilon \in \mathcal{M}(c, d, \Omega)$. For a given sequence θ_ε from the positive cone of $H^{-1}(\Omega)$ converging weakly to θ in $H^{-1}(\Omega)$, let $(u_\varepsilon, p_\varepsilon)$ be the solution of*

$$\left\{ \begin{array}{rll} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) & = f + \theta_\varepsilon & \text{in } \Omega \\ -\mathrm{div}({}^tA_\varepsilon \nabla p_\varepsilon - B_\varepsilon \nabla u_\varepsilon) & = 0 & \text{in } \Omega \\ u_\varepsilon = p_\varepsilon & = 0 & \text{on } \partial\Omega \end{array} \right. \quad (3.4.2)$$

then

$$z_0 = {}^tA_0\nabla p_0 - B^\sharp\nabla u_0$$

where z_0, p_0 *and* u_0 *are the weak limits of* $z_\varepsilon \equiv {}^tA_\varepsilon\nabla p_\varepsilon - B_\varepsilon\nabla u_\varepsilon$, p_ε *and* u_ε *in* $(L^2(\Omega))^n$, $H_0^1(\Omega)$ *and* $H_0^1(\Omega)$ *respectively. Thus* $(u_0, p_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *solves*

$$\left\{\begin{array}{rll} -\mathrm{div}(A_0\nabla u_0) & = f + \theta & \text{in } \Omega \\ -\mathrm{div}({}^tA_0\nabla p_0 - B^\sharp\nabla u_0) & = 0 & \text{in } \Omega \\ u_0 = p_0 & = 0 & \text{on } \partial\Omega \end{array}\right. \tag{3.4.3}$$

where A_0 *is the* H*-limit of* A_ε *and* $B^\sharp$ *is same as the one obtained in* (1.3.9).

Proof. It is obvious that $\{u_\varepsilon\}$ and $\{p_\varepsilon\}$ are bounded sequences in $H_0^1(\Omega)$ and $\{z_\varepsilon\}$ bounded in $(L^2(\Omega))^n$. Thus, for a subsequence,

$$\begin{aligned} u_\varepsilon &\rightharpoonup u_0 \text{ weakly in } H_0^1(\Omega) \\ p_\varepsilon &\rightharpoonup p_0 \text{ weakly in } H_0^1(\Omega) \\ z_\varepsilon &\rightharpoonup z_0 \text{ weakly in } (L^2(\Omega))^n. \end{aligned}$$

Also $\xi_\varepsilon \equiv A_\varepsilon\nabla u_\varepsilon$ is bounded in $(L^2(\Omega))^n$. Hence $\xi_\varepsilon \rightharpoonup \xi_0$ weakly in $(L^2(\Omega))^n$. Under the given hypotheses, it now follows from Theorem 3.3.3 that

$$\begin{aligned} \xi_0 &= A_0\nabla u_0 \text{ and} \\ -\mathrm{div}(A_0\nabla u_0) &= f + \theta \end{aligned}$$

where A_0 is the H-limit of $\{A_\varepsilon\}$.

Let $\phi \in \mathcal{D}(\Omega)$ be an arbitrary function. Using $\chi_\varepsilon^i\phi$ as a test function in the second equation of (3.4.2), we get

$$\int_\Omega {}^tA_\varepsilon\nabla p_\varepsilon.\nabla(\chi_\varepsilon^i\phi)\,dx = \int_\Omega B_\varepsilon\nabla u_\varepsilon.\nabla(\chi_\varepsilon^i\phi)\,dx,$$

which yields

$$\begin{aligned} 0 &= \int_\Omega ({}^tA_\varepsilon\nabla p_\varepsilon - B_\varepsilon\nabla u_\varepsilon).(\nabla\phi)\chi_\varepsilon^i\,dx + \int_\Omega {}^tA_\varepsilon\nabla p_\varepsilon.(\nabla\chi_\varepsilon^i)\phi\,dx \\ &\quad - \int_\Omega B_\varepsilon\nabla u_\varepsilon.(\nabla\chi_\varepsilon^i)\phi\,dx \\ &= \int_\Omega z_\varepsilon.(\nabla\phi)\chi_\varepsilon^i\,dx + \int_\Omega A_\varepsilon\nabla\chi_\varepsilon^i.(\nabla p_\varepsilon)\phi\,dx - \int_\Omega {}^tB_\varepsilon\nabla\chi_\varepsilon^i.(\nabla u_\varepsilon)\phi\,dx \\ 0 &= \int_\Omega z_\varepsilon.(\nabla\phi)\chi_\varepsilon^i\,dx - \int_\Omega \mathrm{div}(A_\varepsilon\nabla\chi_\varepsilon^i)p_\varepsilon\phi\,dx - \int_\Omega A_\varepsilon\nabla\chi_\varepsilon^i.(\nabla\phi)p_\varepsilon\,dx \\ &\quad - \int_\Omega {}^tB_\varepsilon\nabla\chi_\varepsilon^i.(\nabla u_\varepsilon)\phi\,dx. \end{aligned} \tag{3.4.4}$$

Now using $\psi_\varepsilon^i\phi$ as a test function in the state equation corresponding to u_ε (cf. (3.4.2)), we have

$$\begin{aligned}
\int_\Omega (f+\theta_\varepsilon)\psi_\varepsilon^i\phi\,dx &= \int_\Omega A_\varepsilon\nabla u_\varepsilon.(\nabla\phi)\psi_\varepsilon^i\,dx + \int_\Omega A_\varepsilon\nabla u_\varepsilon.(\nabla\psi_\varepsilon^i)\phi\,dx \\
&= \int_\Omega \xi_\varepsilon.\nabla\phi\psi_\varepsilon^i\,dx + \int_\Omega {}^tA_\varepsilon\nabla\psi_\varepsilon^i.\nabla u_\varepsilon\phi\,dx \\
&= \int_\Omega \xi_\varepsilon.\nabla\phi\psi_\varepsilon^i\,dx - \int_\Omega \mathrm{div}({}^tA_\varepsilon\nabla\psi_\varepsilon^i)u_\varepsilon\phi\,dx \\
&\quad - \int_\Omega {}^tA_\varepsilon\nabla\psi_\varepsilon^i.(\nabla\phi)u_\varepsilon\,dx \\
&= \int_\Omega \xi_\varepsilon.\nabla\phi\psi_\varepsilon^i\,dx + \int_\Omega \mathrm{div}({}^tB_\varepsilon\nabla\chi_\varepsilon^i)u_\varepsilon\phi\,dx \\
&\quad - \int_\Omega {}^tA_\varepsilon\nabla\psi_\varepsilon^i.(\nabla\phi)u_\varepsilon\,dx \\
\int_\Omega (f+\theta_\varepsilon)\psi_\varepsilon^i\phi\,dx &= \int_\Omega \xi_\varepsilon.\nabla\phi\psi_\varepsilon^i\,dx - \int_\Omega {}^tB_\varepsilon\nabla\chi_\varepsilon^i.(\nabla u_\varepsilon)\phi\,dx \qquad (3.4.5) \\
&\quad - \int_\Omega {}^tB_\varepsilon\nabla\chi_\varepsilon^i.(\nabla\phi)u_\varepsilon\,dx - \int_\Omega {}^tA_\varepsilon\nabla\psi_\varepsilon^i.(\nabla\phi)u_\varepsilon\,dx.
\end{aligned}$$

Subtracting (3.4.4) from (3.4.5), we get

$$\begin{aligned}
\int_\Omega (f+\theta_\varepsilon)\psi_\varepsilon^i\phi\,dx &= \int_\Omega \xi_\varepsilon.\nabla\phi\psi_\varepsilon^i\,dx - \int_\Omega z_\varepsilon.(\nabla\phi)\chi_\varepsilon^i\,dx \\
&\quad + \int_\Omega \mathrm{div}(A_\varepsilon\nabla\chi_\varepsilon^i)p_\varepsilon\phi\,dx + \int_\Omega A_\varepsilon\nabla\chi_\varepsilon^i.(\nabla\phi)p_\varepsilon\,dx \\
&\quad - \int_\Omega \left[{}^tA_\varepsilon\nabla\psi_\varepsilon^i + {}^tB_\varepsilon\nabla\chi_\varepsilon^i\right].(\nabla\phi)u_\varepsilon\,dx. \qquad (3.4.6)
\end{aligned}$$

It was observed in Lemma 3.4.1 that the corrector functions $\{\psi_\varepsilon^i\}$ is bounded in $W_0^{1,p}(\Omega)$ for some $p > 2$. Since $\theta_\varepsilon \rightharpoonup \theta$ in $H^{-1}(\Omega)$ and $\theta_\varepsilon \geq 0$ a.e. in Ω, by Theorem 3.3.2, we have

$$\psi\theta_\varepsilon \to \psi\theta \text{ strongly in } W^{-1,q}(\Omega) \text{ for every } q < 2 \text{ and for every } \psi \in \mathcal{D}(\Omega).$$

Let us choose ψ in $\mathcal{D}(\Omega)$ which is equal to 1 in a neighbourhood of supp(ϕ) and q such that $1/p + 1/q = 1$ for the p obtained in Lemma 3.4.1. Then while passing to the limit in (3.4.6), the left hand side becomes,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega (f+\theta_\varepsilon)\psi_\varepsilon^i\phi\,dx &= \lim_{\varepsilon\to 0}\int_\Omega f\psi_\varepsilon^i\phi\,dx + \lim_{\varepsilon\to 0}\langle\psi\theta_\varepsilon,\psi_\varepsilon^i\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\\ &= \int_\Omega f\psi_0^i\phi\,dx + \lim_{\varepsilon\to 0}\langle\psi\theta_\varepsilon,\psi_\varepsilon^i\phi\rangle_{W^{-1,q}(\Omega),W_0^{1,p}(\Omega)}\\ &= \langle f+\theta,\psi_0^i\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.\end{aligned}$$

Thus passing to the limit in (3.4.6), we get

$$\begin{aligned}\langle f+\theta,\psi_0^i\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} &= \int_\Omega \xi_0.\nabla\phi\psi_0^i\,dx - \int_\Omega z_0.\nabla\phi x_i\,dx\\ &\quad+\int_\Omega \mathrm{div}(A_0e_i)p_0\phi\,dx + \int_\Omega A_0e_i.\nabla\phi p_0\,dx\\ &\quad-\int_\Omega \tau_0^i.\nabla\phi u_0\,dx.\end{aligned}$$

Using integration by parts and the fact that

$$\mathrm{div} z_0 = 0 = \mathrm{div}\tau_0^i \text{ and } -\mathrm{div}\xi_0 = -\mathrm{div}(A_0\nabla u_0) = f+\theta,$$

we derive

$$z_0.e_i = {}^tA_0\nabla p_0.e_i + {}^tA_0\nabla\psi_0^i.\nabla u_0 - \tau_0^i.\nabla u_0.$$

Now, from the definition of $B^\sharp$, we can write

$$z_0 = {}^tA_0\nabla p_0 - B^\sharp\nabla u_0.$$

and thus $(u_0,p_0)\in H_0^1(\Omega)\times H_0^1(\Omega)$ solves (3.4.3). □

The result proved above is an example of the homogenization of a system where the data converges only weakly in $H^{-1}(\Omega)$. So far, in this section, the hypothesis that B_ε is symmetric was not required.

In the rest of this section, we shall highlight the difficulty involved in the study of the asymptotic behaviour of the optimal control problem (3.2.1) solving (3.1.2), when the admissible control set U is the positive cone in $L^2(\Omega)$. It is known that for this optimal control problem there exists a unique optimal control $\theta_\varepsilon^*\in U$ such that

$$J_\varepsilon(\theta_\varepsilon^*) = \min_{\theta\in U} J_\varepsilon(\theta)$$

and let u_ε^* be state corresponding to θ_ε^*. Due to the small order of the cost of control, one is unable to check whether θ_ε^* is bounded in $L^2(\Omega)$ or not. Thus, one is forced to look for other means of homogenizing the optimality system. The following theorem is a step towards such an approach when the positive cone of $L^2(\Omega)$ is the control set of the optimal problem.

We now introduce the adjoint state $p_\varepsilon \in H_0^1(\Omega)$ as the weak solution of

$$\left\{\begin{array}{lll} -\mathrm{div}({}^tA_\varepsilon\nabla p_\varepsilon) & = -\mathrm{div}(B_\varepsilon\nabla u_\varepsilon) & \text{in } \Omega \\ p_\varepsilon & = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{3.4.7}$$

Observe that p_ε^*, the adjoint state corresponding to u_ε^*, is bounded in $H_0^1(\Omega)$ and thus, for a subsequence, converges to some p^*.

Theorem 3.4.2. *If $U = \{\theta \in L^2(\Omega) \mid \theta \geq 0$ a.e. in $\Omega\}$ is the admissible control set for the system (3.2.1) solving (3.1.2) and V is the positive cone of $H^{-1}(\Omega)$, then there exists u^* and θ^* such that, for a subsequence,*

$$u_\varepsilon^* \rightharpoonup u^* \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega), \tag{3.4.8}$$

$$p_\varepsilon^* \rightharpoonup p^* \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega), \tag{3.4.9}$$

$$\varepsilon^{\frac{1}{2}}\theta_\varepsilon^* \rightharpoonup 0 \text{ weakly in } L^2(\Omega), \tag{3.4.10}$$

$$\theta_\varepsilon^* \rightharpoonup \theta^* \in V \text{ weakly in } H^{-1}(\Omega), \tag{3.4.11}$$

$$J_\varepsilon(\theta_\varepsilon^*) \to \frac{1}{2}\int_\Omega B^\sharp\nabla u^*.\nabla u^*\,dx - \frac{1}{2}\langle\theta^*, p^*\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \tag{3.4.12}$$

Further, $p^* \in U$, $\int_\Omega p_\varepsilon^*\theta_\varepsilon^*\,dx \leq 0$ and the pair (u^*, p^*) solves the homogenized system

$$\left\{\begin{array}{rll} -\mathrm{div}(A_0\nabla u^*) & = f + \theta^* & \text{in } \Omega \\ -\mathrm{div}({}^tA_0\nabla p^* - B^\sharp\nabla u^*) & = 0 & \text{in } \Omega \\ u^* = p^* & = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{3.4.13}$$

Proof. It is easy to observe that, for a fixed $\theta \in U$, $u_\varepsilon(\theta)$ is bounded uniformly in $H_0^1(\Omega)$. Therefore, u_ε^* is bounded in $H_0^1(\Omega)$ and $\varepsilon^{1/2}\theta_\varepsilon^*$ is bounded in $L^2(\Omega)$. Moreover, for $v \in H_0^1(\Omega)$, we have

$$\begin{aligned} \int_\Omega \theta_\varepsilon^* v\,dx &= \int_\Omega A_\varepsilon.\nabla u_\varepsilon^*.\nabla v\,dx - \int_\Omega fv\,dx \\ &\leq (b\|u_\varepsilon^*\|_{H_0^1(\Omega)} + \|f\|_{2,\Omega})\|v\|_{H_0^1(\Omega)}. \end{aligned}$$

Thus, θ_ε^* is bounded in $H^{-1}(\Omega)$. Thus, there exists u^*, θ' and θ^* such that, for a subsequence,

$$\begin{aligned} u_\varepsilon^* &\rightharpoonup u^* \text{ weakly in } H_0^1(\Omega),\\ \varepsilon^{\frac{1}{2}}\theta_\varepsilon^* &\rightharpoonup \theta' \text{ weakly in } L^2(\Omega),\\ \theta_\varepsilon^* &\rightharpoonup \theta^* \text{ weakly in } H^{-1}(\Omega).\end{aligned}$$

Since θ_ε^* is bounded in $H^{-1}(\Omega)$, we in fact have $\theta' = 0$ and thus

$$\varepsilon^{\frac{1}{2}}\theta_\varepsilon^* \rightharpoonup 0 \text{ weakly in } L^2(\Omega).$$

It then follows from Theorem 3.4.1 that (u^*, p^*) solves (3.4.13). Now, consider the optimality condition associated with the optimal problem, given as

$$\int_\Omega (p_\varepsilon^* + \varepsilon\theta_\varepsilon^*)(\theta - \theta_\varepsilon^*)\, dx \geq 0, \quad \forall \theta \in U. \tag{3.4.14}$$

It follows from the above inequality that $\varepsilon\theta_\varepsilon^*$ is the projection of $-p_\varepsilon^*$ on U in $L^2(\Omega)$, i.e., $\theta_\varepsilon^* = \varepsilon^{-1}(p_\varepsilon^*)^-$. Thus we can rewrite (3.4.14), as

$$\int_\Omega p_\varepsilon^*\theta\, dx + \int_\Omega (\varepsilon^{1/2}\theta_\varepsilon^*)(\varepsilon^{1/2}\theta)\, dx \geq 0, \quad \forall \theta \in U,$$

since $\int_\Omega (p_\varepsilon^* + \varepsilon\theta_\varepsilon^*)\theta_\varepsilon^*\, dx = 0$. Passing to the limit, we deduce that

$$\int_\Omega p^*\theta\, dx \geq 0, \quad \forall \theta \in U.$$

Thus, proving that $p^* \geq 0$, *i.e.*, $p^* \in U$. Also the fact that

$$\int_\Omega p_\varepsilon^*\theta_\varepsilon^*\, dx = -\varepsilon\|\theta_\varepsilon^*\|_{2,\Omega}^2 \leq 0.$$

implies that $\int_\Omega p_\varepsilon^*\theta_\varepsilon^*\, dx \leq 0$.

Consider

$$\begin{aligned} J_\varepsilon(\theta_\varepsilon^*) &= \frac{1}{2}\int_\Omega B_\varepsilon \nabla u_\varepsilon^* . \nabla u_\varepsilon^*\, dx + \frac{\varepsilon}{2}\|\theta_\varepsilon^*\|_{2,\Omega}^2\\ &= \frac{1}{2}\int_\Omega (f + \theta_\varepsilon^*)p_\varepsilon^*\, dx + \frac{\varepsilon}{2}\|\theta_\varepsilon^*\|_{2,\Omega}^2\\ &= \frac{1}{2}\int_\Omega f p_\varepsilon^*\, dx\\ &\to \frac{1}{2}\int_\Omega f p^*\, dx\\ &= \frac{1}{2}\int_\Omega B^\sharp \nabla u^* . \nabla u^*\, dx - \frac{1}{2}\langle \theta^*, p^*\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.\end{aligned}$$

Hence, (3.4.12) holds. □

From the fact that both p^* and θ^* are non-negative, we know that

$$\langle \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \geq 0.$$

Also, as observed in the above theorem, $\langle \theta_\varepsilon^*, p_\varepsilon^* \rangle_{L^2(\Omega), L^2(\Omega)} \leq 0$. Given these, one would expect $\langle \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0$ (the optimality condition for the limit system) but we have no means of arriving at this result, which keeps the problem open[8]. If $\langle \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0$ then θ^* is the unique optimal control of the problem

$$J(\theta) = \frac{1}{2} \int_\Omega B^\sharp \nabla u_0 . \nabla u_0 \, dx \tag{3.4.15}$$

over the set V, the positive cone of $H^{-1}(\Omega)$, where $u_0 \in H_0^1(\Omega)$ solves

$$\begin{cases} -\mathrm{div}(A_0 \nabla u_0) &= f + \theta \quad \text{in } \Omega \\ u_0 &= 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{3.4.16}$$

Further, $J_\varepsilon(\theta_\varepsilon^*) \to J(\theta^*)$ and the convergences (3.4.8), (3.4.9), (3.4.10) and (3.4.11) holds for the entire sequence.

To see this, if $\langle \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0$ then, by (3.4.12), $J_\varepsilon(\theta_\varepsilon^*) \to J(\theta^*)$. Since θ_ε^* is the minimizer of J_ε over U, by passing to the limit in

$$J_\varepsilon(\theta_\varepsilon^*) \leq J_\varepsilon(\theta), \quad \forall \theta \in U$$

we deduce using Remark 3.1.1,

$$J(\theta^*) \leq J(\theta), \quad \forall \theta \in U.$$

Now, since V is the strong closure of U in $H^{-1}(\Omega)$, we actually have,

$$J(\theta^*) \leq J(\theta), \quad \forall \theta \in V.$$

Thus θ^* minimises J over V. The strict convexity of J implies the uniqueness of θ^* and thus the convergences hold for the entire sequence.

[8] cf. Open Problem 3 in page 95

3.5 L^2-norm of state in cost

So far, our interest was in the study of a system involving Dirichlet-type integral in the cost functional. Though some interesting results are proved, one is unable to completely settle the problem. However, we now change the cost functional (cf. (3.5.1)) and note that one can improve upon the results described in Theorem 2.3.1 using the machinery developed in §3.3, which we proceed to do in this section.

Consider the system

$$J_\varepsilon(\theta) = \frac{1}{2}\|u_\varepsilon\|_{2,\Omega}^2 + \frac{\varepsilon}{2}\|\theta\|_{2,\Omega}^2, \tag{3.5.1}$$

where the state $u_\varepsilon(\theta) \in H_0^1(\Omega)$ is the weak solution of (3.1.2). Studying the limit system of the problem defined above is still open[9] for an arbitrary admissible set U in $L^2(\Omega)$. However, we settle the problem for the case of the positive cone.

Let the admissible control set U be the positive cone in $L^2(\Omega)$, i e.,

$$U = \left\{\theta \in L^2(\Omega) \mid \theta \geq 0 \text{ a.e. in } \Omega\right\}.$$

We shall now introduce the adjoint problem and the optimality condition associated with the above described system.

The minimizer θ_ε^* is characterised by the optimality condition

$$\int_\Omega (u_\varepsilon^*(u_\varepsilon - u_\varepsilon^*) + \varepsilon\theta_\varepsilon^*(\theta - \theta_\varepsilon^*))\, dx \geq 0, \quad \forall \theta \in U. \tag{3.5.2}$$

where u_ε is the state corresponding to θ. We can rewrite the optimality condition as

$$\int_\Omega (p_\varepsilon^* + \varepsilon\theta_\varepsilon^*)(\theta - \theta_\varepsilon^*)\, dx \geq 0 \quad \forall \theta \in U,$$

using the adjoint optimal state $p_\varepsilon^* \in H_0^1(\Omega)$ given as the weak solution of

$$\begin{cases} -\mathrm{div}({}^tA_\varepsilon \nabla p_\varepsilon^*) &= u_\varepsilon^* \quad \text{in } \Omega \\ \qquad\qquad p_\varepsilon^* &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.5.3}$$

Now,

$$\|u_\varepsilon^*\|_{2,\Omega}^2 \leq J_\varepsilon(\theta_\varepsilon^*) \leq J_\varepsilon(\theta), \forall \theta \in U.$$

[9] cf. Open Problem 5 in page 96

Therefore

$$\begin{aligned}
\|u_\varepsilon^*\|_{2,\Omega}^2 &\leq \frac{1}{2}\|u_\varepsilon\|_{2,\Omega}^2 + \frac{\varepsilon}{2}\|\theta\|_{2,\Omega}^2 \\
&\leq \frac{1}{2}\|u_\varepsilon\|_{H_0^1(\Omega)}^2 + \frac{1}{2}\|\theta\|_{2,\Omega}^2 \\
&\leq \frac{1}{2a}\|f+\theta\|_{2,\Omega}^2 + \frac{1}{2}\|\theta\|_{2,\Omega}^2.
\end{aligned}$$

Thus, since $\{u_\varepsilon^*\}$ is bounded in $L^2(\Omega)$, by H-convergence, there exists a matrix A_0 (called the H-limit of $\{A_\varepsilon\}$) such that

$$\begin{cases} -\mathrm{div}({}^tA_0\nabla p^*) &= u^* \quad \text{in } \Omega \\ \qquad p^* &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.5.4}$$

and $p_\varepsilon^* \rightharpoonup p^*$ weakly in $H_0^1(\Omega)$.

The problem (3.5.1) solving (3.1.2) was studied in [KP02] when the set U is the positive cone of $L^2(\Omega)$ and their results are recalled in §2.3. In the following theorem, we establish a relation between u^* and θ^* and show θ^* as an optimal control of a homogenized problem.

Theorem 3.5.1. *If $U = \{\theta \in L^2(\Omega) \mid \theta \geq 0$ a.e. in $\Omega\}$ is the admissible control set for the system (3.5.1) solving (3.1.2), then there exist u^* and θ^* such that*

(a)

$$u_\varepsilon^* \rightharpoonup u^* \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega), \tag{3.5.5}$$

$$\varepsilon^{\frac{1}{2}}\theta_\varepsilon^* \rightharpoonup 0 \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega), \tag{3.5.6}$$

$$J_\varepsilon(\theta_\varepsilon^*) \to \frac{1}{2}\|u^*\|_{2,\Omega}^2 \tag{3.5.7}$$

(b) $\theta_\varepsilon^* \rightharpoonup \theta^*$ weakly in $H^{-1}(\Omega)$ for the entire sequence.

(c) u^* solves

$$\begin{cases} -\mathrm{div}(A_0\nabla u^*) &= f + \theta^* \quad \text{in } \Omega \\ \qquad u^* &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.5.8}$$

where, now, $\theta^* \in H^{-1}(\Omega)$.

(d) θ^* is the unique minimizer of $J(\theta) = \frac{1}{2}\|u(\theta)\|_{2,\Omega}^2$ over V, the positive cone of $H^{-1}(\Omega)$.

(e) u^* *is the projection of* 0 *on to* $\overline{K'}$ *in* $L^2(\Omega)$, *i. e.,* $u^* \in \overline{K'}$ *and*

$$\int_\Omega u^*(v - u^*)\, dx \geq 0 \quad \forall v \in \overline{K'}$$

where

$$K' = \left\{ v \in H_0^1(\Omega) \mid -\text{div}(A_0 \nabla v) - f \in V \right\}.$$

Proof. (a) follows from Theorem 2.3.1. Also, (b) holds for a subsequence (cf. (2.3.6)) and by Theorem 3.3.3 we have that u^* is the solution of (3.5.8), thus proving (c).

It follows from Proposition 3.3.3 that V is the strong closure of U in $H^{-1}(\Omega)$. Observe that V is a closed convex subset of $H^{-1}(\Omega)$. Thus, V is also the weak closure of U in $H^{-1}(\Omega)$ and hence $\theta^* \in V$. We know that,

$$J_\varepsilon(\theta_\varepsilon^*) \leq J_\varepsilon(\theta), \quad \forall \theta \in U. \tag{3.5.9}$$

Therefore, passing to the limit as ε goes to 0 we have

$$J(\theta^*) \leq J(\theta), \quad \forall \theta \in U$$

and hence

$$J(\theta^*) \leq J(\theta), \quad \forall \theta \in V. \tag{3.5.10}$$

By the strict convexity of J, θ^* is the unique minimizer of J over V, thus proving (d). The uniqueness of θ^* implies (b).

Let $\overline{K'}$ denote the closure of K' in $L^2(\Omega)$. This is then a closed convex subset of $L^2(\Omega)$. Observe that $u^* \in K' \subset \overline{K'}$, since $\theta^* \in V$. Let $\theta \in U$ and $v(\theta)$ be the solution of

$$\begin{cases} -\text{div}(A_0 \nabla v) &= f + \theta \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.5.11}$$

Then passing to the limit in the optimality condition (3.5.2) and noting that $u_\varepsilon \rightharpoonup v(\theta)$ in $H_0^1(\Omega)$, we have

$$\int_\Omega u^*(v(\theta) - u^*)\, dx \geq 0 \quad \forall \theta \in U.$$

Let $v \in K'$ and let $\theta = -\text{div}(A_0 \nabla v) - f$. Then there exists a sequence $\{\theta_n\} \subset U$ such that $\theta_n \to \theta$ strongly in $H^{-1}(\Omega)$. Let $v_n \in K'$ be the states corresponding to θ_n for which the above inequality holds. Thus,

$$\int_\Omega u^*(v - u^*)\, dx \geq 0 \quad \forall v \in K'$$

and a simple density argument proves (e). □

Remark 3.5.1. Since θ^* is a unique minimizer of J over V, it is characterised by the condition

$$\langle \theta - \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \geq 0 \quad \forall \theta \in V.$$

Now, by choosing $\theta = 0$ and $\theta = 2\theta^*$, we deduce $\langle \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0$. Also, by choosing $\theta = \theta^* + \eta$, for arbitrary $\eta \in V$, we get $\langle \eta, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \geq 0$ implying that $p^* \geq 0$ a.e. in Ω. □

Remark 3.5.2. We now observe that the K' we defined in the above theorem is same as the K defined in Theorem 2.3.1, i. e., $K' = K$. Let $v \in K$ then there exists a sequence $\{v_\varepsilon\} \subset H_0^1(\Omega)$ such that $v_\varepsilon \rightharpoonup v$ weakly in $H_0^1(\Omega)$ and $\theta_\varepsilon = -\text{div}(A_\varepsilon \nabla v_\varepsilon) - f \in U$. Then, by Theorem 3.3.3, it follows that $v \in K'$ for some $\theta \in V$ which comes as the weak limit of θ_ε in $H^{-1}(\Omega)$. Thus, $K \subset K'$. Now, let $v \in K'$ and $\theta \in V$. Then there exists a sequence $\{\theta_\varepsilon\} \subset U$ such that $\theta_\varepsilon \to \theta$ strongly in $H^{-1}(\Omega)$. Set v_ε to be the solution of

$$\begin{cases} -\text{div}(A_\varepsilon \nabla v_\varepsilon) &= f + \theta_\varepsilon \quad \text{in } \Omega \\ v_\varepsilon &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.5.12}$$

and thus $v_\varepsilon \rightharpoonup v$ weakly in $H_0^1(\Omega)$. Hence, we have shown $v \in K$ and therefore $K' \subset K$. □

Remark 3.5.3. The highlight of Theorem 3.5.1 is the result (d). We conclude that the optimal controls θ_ε^* converge weakly in $H^{-1}(\Omega)$ to θ^* which is a unique optimal control for the problem of minimising

$$J(\theta) = \frac{1}{2} \|u_0(\theta)\|_{2,\Omega}^2$$

over the set V, the positive cone of $H^{-1}(\Omega)$, where $u_0 \in H_0^1(\Omega)$ solves (3.4.16). Further, $J_\varepsilon(\theta_\varepsilon^*) \to J(\theta^*)$. This was a problem open in [KP02] (cf. Theorem 2.3.1). They were also unable to estabilsh the relation between u^* and θ^*. Also, the description of the set K' was bit quite complicated.

3.6 Summary

In this chapter, the questions (**P1**) and (**P2**) posed in the previous chapter are answered for some particular cases or under certain assumptions. In §3.1, the problem (**P1**) is answered under certain assumptions. In the rest of the chapter the machinery required to address problem (**P2**) was developed and the case where the cost functional involves the L^2-norm of the state variable, a problem left open in [KP02], is settled for the positive cone case (cf. §3.5). The problem is still open for an arbitrary admissible set[10]. Also, the case with Dirichlet-type integral in the cost functional is still unsettled[11], in spite of relaxing the control set, even for the positive cone case. It was shown in [KR02, Theorem 2.1] that when the optimal controls are bounded in $L^2(\Omega)$, the homogenization of the state-adjoint system (3.4.2) implies the convergence of energy. However, we have shown in this chapter that this result is no longer valid when the optimal controls are bounded only in $H^{-1}(\Omega)$ (cf. Remark 3.3.1).

[10]cf. Open Problem 5 in page 96

[11]cf. Open Problem 3 in page 95

Chapter 4

Low Cost Controls on Perforated Domains

In this chapter, we study the asymptotic behaviour of low cost control problems on perforated domains. The fixed cost of the control case for the perforated domain was studied in [KP99] which is described in §2.2.2. A general introduction on the perforated domains can be found in §1.4.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $S_\varepsilon \subset \Omega$ be a family of closed subsets (called the 'holes'). Let $\Omega_\varepsilon = \Omega \setminus S_\varepsilon$ represent the perforated domain.

Let $U_\varepsilon \subset L^2(\Omega_\varepsilon)$, the set of admissible controls, be a closed convex set and let $f \in L^2(\Omega)$ be given. We note that the homogenization of the system

$$J_\varepsilon(\theta_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{\varepsilon}{2}\|\theta_\varepsilon\|^2_{2,\Omega_\varepsilon}, \quad \forall \theta_\varepsilon \in U_\varepsilon, \tag{4.0.1}$$

where the state $u_\varepsilon = u_\varepsilon(\theta_\varepsilon) \in V_\varepsilon$ is the weak solution of

$$\left\{ \begin{array}{rll} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) & = f + \theta_\varepsilon & \text{in } \Omega_\varepsilon \\ A_\varepsilon \nabla u_\varepsilon . n_\varepsilon & = 0 & \text{on } \partial S_\varepsilon \\ u_\varepsilon & = 0 & \text{on } \partial\Omega \end{array} \right. \tag{4.0.2}$$

is still open[1] and we will not study this system in this chapter. However, we will consider the system with the cost functional involving L^2-norm of the state and see if this can be homogenized as has been done for the non-perforated case (cf. Chapter3). In this chapter, we shall consider the cases when both the control and state are given in the domain (cf. §4.1) and are on the boundary (cf. §4.2).

[1] cf. Open Problem 6 in page 96

4.1 Control and State on the domain

In this section, we consider the perforated version of the system (2.3.2) solving (2.2.2). Before we describe the problem, we recall some notations introduced in §1.4. Let χ_ε denote the characteristic function of the set Ω_ε in Ω,

$$\chi_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega_\varepsilon \\ 0 & \text{if } x \in S_\varepsilon \end{cases}$$

and let χ_0 be a weak* limit of χ_ε in $L^\infty(\Omega)$. The extension of a function on Ω_ε by zero on the holes of Ω is denoted with a $\tilde{}$ in the superscript. We shall now prove a result which will be useful in the sequel.

It is easy to observe that when a sequence $f_\varepsilon \to f$ strongly in $L^2(\Omega)$ then we have $\int_\Omega \chi_\varepsilon f_\varepsilon\, dx \to \int_\Omega \chi_0 f\, dx$. We shall now prove a lemma that discusses about the L^2-norm convergence of $\chi_\varepsilon f_\varepsilon$.

Lemma 4.1.1. *If $f_\varepsilon \to f$ strongly in $L^2(\Omega)$ then $\|\chi_\varepsilon f_\varepsilon\|^2_{2,\Omega} \to \int_\Omega \chi_0 f^2\, dx$.*

Proof. Since $f_\varepsilon \to f$ in $L^2(\Omega)$, we have $\|f_\varepsilon\|_{2,\Omega} \to \|f\|_{2,\Omega}$ and (for a subsequence) $f_\varepsilon(x) \to f(x)$ pointwise a.e. (since the limit is independent of the subsequence, the convergence occurs for the entire sequence). Equivalently, we have $\|f_\varepsilon^2\|_{1,\Omega} \to \|f^2\|_{1,\Omega}$. Now, it can be shown as a consequence of Egoroff's theorem and Fatou's lemma (cf. [Rud87, Exercise 17(b), page 73]) that $f_\varepsilon^2 \to f^2$ strongly in $L^1(\Omega)$. Thus, we have (recall that $\chi_\varepsilon^2 = \chi_\varepsilon$),

$$\|\chi_\varepsilon f_\varepsilon\|^2_{2,\Omega} = \int_\Omega \chi_\varepsilon f_\varepsilon^2\, dx \to \int_\Omega \chi_0 f^2\, dx$$

using the $L^\infty(\Omega)$ weak* convergence of $\{\chi_\varepsilon\}$. □

We now state the problem we are interested in: For a given $\theta_\varepsilon \in U_\varepsilon$, the cost functional is given by

$$J_\varepsilon(\theta_\varepsilon) = \frac{1}{2}\|u_\varepsilon\|^2_{2,\Omega_\varepsilon} + \frac{\varepsilon}{2}\|\theta_\varepsilon\|^2_{2,\Omega_\varepsilon} \tag{4.1.1}$$

where the state $u_\varepsilon = u_\varepsilon(\theta_\varepsilon) \in V_\varepsilon$ is the weak solution of (4.0.2). Recall from §1.4 that $V_\varepsilon = \{u \in H^1(\Omega_\varepsilon) \mid u = 0 \text{ on } \partial\Omega\}$. For $u \in V_\varepsilon$, we define the norm on V_ε as, $\|u\|_{V_\varepsilon} = \|\nabla u\|_{2,\Omega_\varepsilon}$. Let P_ε be the extension operator as assumed in page 11, then the following lemma shows that it is bounded in $H_0^1(\Omega)$.

Lemma 4.1.2. *If there exists, for each ε, $\theta_\varepsilon \in U_\varepsilon$ such that $\{\tilde{\theta}_\varepsilon\}$ is bounded in $L^2(\Omega)$ then $\{P_\varepsilon u_\varepsilon\}$ is bounded in $H_0^1(\Omega)$.*

Proof. By the ellipticity of A_ε,

$$\begin{aligned}
a\|u_\varepsilon\|_{V_\varepsilon}^2 &\leq \int_{\Omega_\varepsilon} A_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx \\
&= \int_{\Omega_\varepsilon} (f + \theta_\varepsilon) u_\varepsilon \, dx \\
&= \int_{\Omega} (\chi_\varepsilon f + \tilde{\theta}_\varepsilon) P_\varepsilon u_\varepsilon \, dx \\
&\leq \|\chi_\varepsilon f + \tilde{\theta}_\varepsilon\|_{2,\Omega} \|P_\varepsilon u_\varepsilon\|_{2,\Omega} \\
&\leq C_0 \|\chi_\varepsilon f + \tilde{\theta}_\varepsilon\|_{2,\Omega} \|\nabla P_\varepsilon u_\varepsilon\|_{2,\Omega} \\
\|u_\varepsilon\|_{V_\varepsilon}^2 &\leq \frac{C_0}{a} \|\chi_\varepsilon f + \tilde{\theta}_\varepsilon\|_{2,\Omega} \|\nabla P_\varepsilon u_\varepsilon\|_{2,\Omega}.
\end{aligned}$$

Therefore, by (**H1**) (in page 11), we have

$$\begin{aligned}
\|\nabla P_\varepsilon u_\varepsilon\|_{2,\Omega}^2 &\leq C_0 \|u_\varepsilon\|_{V_\varepsilon}^2 \\
&\leq \frac{C_0}{a} \|\chi_\varepsilon f + \tilde{\theta}_\varepsilon\|_{2,\Omega} \|\nabla P_\varepsilon u_\varepsilon\|_{2,\Omega}
\end{aligned}$$

and thus,

$$\|\nabla P_\varepsilon u_\varepsilon\|_{2,\Omega} \leq \frac{C_0}{a} \|\chi_\varepsilon f + \tilde{\theta}_\varepsilon\|_{2,\Omega}$$

showing that $\{P_\varepsilon u_\varepsilon\}$ is bounded in $H_0^1(\Omega)$. Note that the constant C_0 is generic and is not fixed in the above inequalities. □

The problem (4.1.1) solving (4.0.2) admits a unique optimal solution, which minimizes J_ε in U_ε and is denoted by θ_ε^*. The corresponding optimal states is denoted by u_ε^*.

We now introduce the adjoint optimal state $p_\varepsilon^* \in V_\varepsilon$ as the weak solution of the problem

$$\left\{ \begin{aligned} -\mathrm{div}({}^t A_\varepsilon \nabla p_\varepsilon^*) &= u_\varepsilon^* \quad \text{in } \Omega_\varepsilon \\ {}^t A_\varepsilon \nabla p_\varepsilon^* . n_\varepsilon &= 0 \quad \text{on } \partial S_\varepsilon \\ p_\varepsilon^* &= 0 \quad \text{on } \partial\Omega. \end{aligned} \right. \tag{4.1.2}$$

Then the optimality condition

$$\int_{\Omega_\varepsilon} [u_\varepsilon^*(u_\varepsilon - u_\varepsilon^*) + \varepsilon\theta_\varepsilon^*(\theta_\varepsilon - \theta_\varepsilon^*)] \, dx \geq 0 \quad \forall \theta_\varepsilon \in U_\varepsilon \tag{4.1.3}$$

can be rewritten as

$$\int_{\Omega_\varepsilon} (p_\varepsilon^* + \varepsilon\theta_\varepsilon^*)(\theta_\varepsilon - \theta_\varepsilon^*)\,dx \geq 0 \quad \forall \theta_\varepsilon \in U_\varepsilon.$$

We observe that θ_ε^* is the projection in $L^2(\Omega_\varepsilon)$ of $\frac{-p_\varepsilon^*}{\varepsilon}$ onto U_ε.

Lemma 4.1.3. *If there exists, for each $\varepsilon > 0$, $\theta_\varepsilon \in U_\varepsilon$ such that $\{\tilde{\theta_\varepsilon}\}$ is bounded in $L^2(\Omega)$, then we have both $\{\chi_\varepsilon P_\varepsilon u_\varepsilon^*\}$, $\{\varepsilon^{1/2}\tilde{\theta_\varepsilon^*}\}$ bounded in $L^2(\Omega)$ and $\{P_\varepsilon p_\varepsilon^*\}$ bounded in $H_0^1(\Omega)$.*

Proof. It follows from (4.1.1) that

$$\frac{1}{2}\|u_\varepsilon^*\|_{2,\Omega_\varepsilon}^2 \leq J_\varepsilon(\theta_\varepsilon^*) \leq J_\varepsilon(\eta) \quad \forall \eta \in U_\varepsilon.$$

In particular, for θ_ε from the hypothesis,

$$\begin{aligned} \frac{1}{2}\|u_\varepsilon^*\|_{2,\Omega_\varepsilon}^2 &\leq J_\varepsilon(\theta_\varepsilon) \\ &= \frac{1}{2}\|u_\varepsilon\|_{2,\Omega_\varepsilon}^2 + \frac{1}{2}\|\theta_\varepsilon\|_{2,\Omega_\varepsilon}^2 \\ &= \frac{1}{2}\|\chi_\varepsilon P_\varepsilon u_\varepsilon\|_{2,\Omega}^2 + \frac{1}{2}\|\tilde{\theta_\varepsilon}\|_{2,\Omega}^2. \end{aligned}$$

Since the RHS of the above inequality is bounded (from Lemma 4.1.2), we have $\|\chi_\varepsilon P_\varepsilon u_\varepsilon^*\|_{2,\Omega}^2$ $(= \|u_\varepsilon^*\|_{2,\Omega_\varepsilon}^2)$ is bounded. Similarly, we also have

$$\frac{\varepsilon}{2}\|\theta_\varepsilon^*\|_{2,\Omega_\varepsilon}^2 \leq J_\varepsilon(\theta_\varepsilon^*) \leq J_\varepsilon(\eta) \quad \forall \eta \in U_\varepsilon$$

and arguing as above, we have

$$\|\varepsilon^{\frac{1}{2}}\tilde{\theta_\varepsilon^*}\|_{2,\Omega}^2 = \|\varepsilon^{\frac{1}{2}}\theta_\varepsilon^*\|_{2,\Omega_\varepsilon}^2 \leq \|\chi_\varepsilon P_\varepsilon u_\varepsilon\|_{2,\Omega}^2 + \|\tilde{\theta_\varepsilon}\|_{2,\Omega}^2$$

is bounded. Now by ellipticity of A_ε we have,

$$\begin{aligned} a\|p_\varepsilon^*\|_{V_\varepsilon}^2 &\leq \int_{\Omega_\varepsilon} A_\varepsilon \nabla p_\varepsilon^* . \nabla p_\varepsilon^*\,dx \\ &= \int_{\Omega_\varepsilon} u_\varepsilon^* p_\varepsilon^*\,dx \\ &= \int_{\Omega} (\chi_\varepsilon P_\varepsilon u_\varepsilon^*)(P_\varepsilon p_\varepsilon^*)\,dx \\ &\leq \|\chi_\varepsilon P_\varepsilon u_\varepsilon^*\|_{2,\Omega}\|P_\varepsilon p_\varepsilon^*\|_{2,\Omega} \\ &\leq C_0\|\chi_\varepsilon P_\varepsilon u_\varepsilon^*\|_{2,\Omega}\|\nabla P_\varepsilon p_\varepsilon^*\|_{2,\Omega} \\ \|p_\varepsilon^*\|_{V_\varepsilon} &\leq \frac{C_0}{a}\|\chi_\varepsilon P_\varepsilon u_\varepsilon^*\|_{2,\Omega}\|\nabla P_\varepsilon p_\varepsilon^*\|_{2,\Omega}. \end{aligned}$$

Therefore, by (H1) (in page 11), we have

$$\begin{aligned}\|\nabla P_\varepsilon p_\varepsilon^*\|_{2,\Omega}^2 &\leq C_0\|p_\varepsilon^*\|_{V_\varepsilon}^2\\ &\leq \frac{C_0}{a}\|\chi_\varepsilon P_\varepsilon u_\varepsilon^*\|_{2,\Omega}\|\nabla P_\varepsilon p_\varepsilon^*\|_{2,\Omega}\end{aligned}$$

and thus,

$$\|\nabla P_\varepsilon p_\varepsilon^*\|_{2,\Omega} \leq \frac{C_0}{a}\|\chi_\varepsilon P_\varepsilon u_\varepsilon^*\|_{2,\Omega}$$

showing that $\{P_\varepsilon p_\varepsilon^*\}$ is bounded in $H_0^1(\Omega)$. Note that, as usual, the constant C_0 is generic and varies in the above inequalities. □

It now follows from Lemma 4.1.3 that, up to a subsequence,

$$\varepsilon^{1/2}\tilde{\theta}_\varepsilon^* \rightharpoonup \theta' \text{ weakly in } L^2(\Omega) \tag{4.1.4}$$

$$\chi_\varepsilon P_\varepsilon u_\varepsilon^* \rightharpoonup u' \text{ weakly in } L^2(\Omega) \tag{4.1.5}$$

$$P_\varepsilon p_\varepsilon^* \rightharpoonup p^* \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega). \tag{4.1.6}$$

We observe that the adjoint equation (4.1.2) can be rewritten in the following way:

$$\left\{\begin{aligned} -\operatorname{div}({}^tA_\varepsilon\nabla p_\varepsilon^*) &= \chi_\varepsilon P_\varepsilon u_\varepsilon^* && \text{in } \Omega_\varepsilon\\ {}^tA_\varepsilon\nabla p_\varepsilon^*.n_\varepsilon &= 0 && \text{on } \partial S_\varepsilon\\ p_\varepsilon^* &= 0 && \text{on } \partial\Omega.\end{aligned}\right. \tag{4.1.7}$$

Thus, under the hypothesis of Lemma 4.1.3, we can homogenize the adjoint equation (4.1.2) (cf. [KP99, Proposition 2.1]). In other words, by the theory of H_0-convergence, there exists a matrix A_0 such that (up to a subsequence) A_ε H_0-converges to A_0 and p^* is the solution of,

$$\left\{\begin{aligned} -\operatorname{div}({}^tA_0\nabla p^*) &= u' && \text{in } \Omega\\ p^* &= 0 && \text{on } \partial\Omega.\end{aligned}\right. \tag{4.1.8}$$

Let us now extend the admissible set to the space $L^2(\Omega)$ in the following way:

$$\widetilde{U_\varepsilon} = \{\tilde{\theta}_\varepsilon \in L^2(\Omega) \mid \theta_\varepsilon \in U_\varepsilon\} \subset L^2(\Omega).$$

Theorem 4.1.1. *Let A_0 be the H_0-limit of $\{A_\varepsilon\}$ and let the sequential K-limit of $\{\widetilde{U_\varepsilon}\}$ in the weak topology of $L^2(\Omega)$ exist, denoted by U. Also let* *the optimal controls $\tilde{\theta}_\varepsilon^*$ converge to θ^* weakly in $L^2(\Omega)$, then θ^* is the unique minimizer of*

$$J(\theta) = \frac{1}{2}\int_\Omega \chi_0|u|^2\,dx$$

in U, where $u = u(\theta) \in H_0^1(\Omega)$ is the weak solution of,

$$\begin{cases} -\mathrm{div}(A_0\nabla u) & = \chi_0 f + \theta \quad \text{in } \Omega \\ \qquad\quad u & = 0 \qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{4.1.9}$$

Further

$$P_\varepsilon u_\varepsilon^* \rightharpoonup u^* \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega),$$
$$J_\varepsilon(\theta_\varepsilon^*) \to J(\theta^*),$$

$u' = \chi_0 u^*$ and $\theta' = 0$.

Proof. The fact that $\theta' = 0$ follows from the weak convergence hypothesis of the optimal controls θ_ε^*. Now, since U is the sequential K-limit of $\{\widetilde{U_\varepsilon}\}$, we have $\theta^* \in U$. Also, for any given $\theta \in U$, there exists a $\delta > 0$ and a sequence $\{\theta_\varepsilon\}$ such that $\theta_\varepsilon \rightharpoonup \theta$ weakly in $L^2(\Omega)$ and $\theta_\varepsilon \in \widetilde{U_\varepsilon}$, $\forall \varepsilon < \delta$. Now, since θ_ε^* is the minimizer of J_ε in U_ε, we have, for $\varepsilon < \delta$,

$$J_\varepsilon(\theta_\varepsilon^*) \le J_\varepsilon(\theta_\varepsilon)$$

(we denote the restriction of θ_ε to Ω_ε by θ_ε itself). Taking limit on both sides of the above inequality, we have

$$\lim_{\varepsilon\to 0}\frac{1}{2}\left[\|\chi_\varepsilon P_\varepsilon u_\varepsilon^*\|_{2,\Omega}^2 + \varepsilon\|\tilde{\theta}_\varepsilon^*\|_{2,\Omega}^2\right] \le \lim_{\varepsilon\to 0}\frac{1}{2}\left[\|\chi_\varepsilon P_\varepsilon u_\varepsilon\|_{2,\Omega}^2 + \varepsilon\|\theta_\varepsilon\|_{2,\Omega}^2\right].$$

It now follows from the theory of H_0-convergence (cf. [KP99, Proposition 2.1]) that $P_\varepsilon u_\varepsilon^* \rightharpoonup u^*$ and $P_\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$ where the u^* and u are the solutions of the homogenized problem (4.1.9) corresponding to θ^* and θ, respectively. Thus, $u' = \chi_0 u^*$. Hence, it now follows from Lemma 4.1.1 that

$$\frac{1}{2}\int_\Omega \chi_0|u^*|^2\,dx \le \frac{1}{2}\int_\Omega \chi_0|u|^2\,dx,$$

i.e. $J(\theta^*) \le J(\theta)$. Since $\theta \in U$ was arbitrary, we have shown that θ^* is the minimiser of J over U. The uniqueness of θ^* is proved by passing to the limit in (4.1.3). Observe that (4.1.3) can be rewritten in the following way:

$$\int_\Omega [\chi_\varepsilon P_\varepsilon u_\varepsilon^*(P_\varepsilon u_\varepsilon - P_\varepsilon u_\varepsilon^*) + \varepsilon\tilde{\theta}_\varepsilon^*(\theta_\varepsilon - \tilde{\theta}_\varepsilon^*)]\,dx \ge 0 \quad \forall\theta_\varepsilon \in U_\varepsilon$$

where θ_ε is as chosen above that converges to θ weakly in $L^2(\Omega)$. Now passing to the limit in the above inequality, we have

$$\int_\Omega \chi_0 u^*(u - u^*)\, dx \geq 0 \quad \forall u \in G(U)$$

where G is the map $\theta \mapsto u$, where u is the solution of (4.1.9). Note that, since U is closed and convex, $G(U)$ is a closed convex subset of $L^2(\Omega)$ and thus we have u^* as a projection of 0 onto $G(U)$ in $L^2_\mu(\Omega)$ where $d\mu = \chi_0 dx$. Thus, from the uniqueness of u^* follows the uniqueness of θ^*. □

Remark 4.1.1. We observe that the optimality condition involving the adjoint state

$$\int_{\Omega_\varepsilon} (p^*_\varepsilon + \varepsilon\theta^*_\varepsilon)(\theta_\varepsilon - \theta^*_\varepsilon)\, dx \geq 0 \quad \forall \theta_\varepsilon \in U_\varepsilon$$

can be rewritten in the following way:

$$\int_\Omega (P_\varepsilon p^*_\varepsilon + \varepsilon\tilde{\theta}^*_\varepsilon)(\theta_\varepsilon - \tilde{\theta}^*_\varepsilon)\, dx \geq 0 \quad \forall \theta_\varepsilon \in U_\varepsilon$$

and by passing to the limit, we obtain the optimality condition for the limit system

$$\int_\Omega p^*(\theta - \theta^*)\, dx \geq 0, \quad \forall \theta \in U$$

where p^* is the solution of (4.1.8) with $u' = \chi_0 u^*$.

We observe that one is, in general, unable to verify the weak convergence hypothesis of the optimal controls as in Theorem 4.1.1 for the system (4.1.1) solving (4.0.2). However, we shall observe some trivial cases of the above mentioned system. Observe that, under the hypothesis of Theorem 4.1.1, if $-\chi_0 f \in U$ then by uniqueness of θ^*, we have $\theta^* = -\chi_0 f$ and $u^* = 0$.

Corollary 4.1.1. *Under the hypothesis of Theorem 4.1.1, if $-\chi_0 f \notin U$ then $\theta^* \in \partial U$.*

Proof. Suppose $\theta^* \notin \partial U$, then for some $r > 0$ there exists a ball $B(\theta^*, r) \subset U$. Thus,

$$\theta^* + t\eta \in U \quad \forall \eta \in B(0,1) \text{ and } t < r$$

. Using this in the optimality condition of the limit system,

$$\int_\Omega p^*(\theta - \theta^*)\, dx \geq 0, \quad \forall \theta \in U$$

we have, $\forall \eta \in B(0,1)$

$$t\int_\Omega p^*\eta \geq 0.$$

Hence, $p^* = 0$ which in turn implies $u^* = 0$ and thus $\theta^* = -\chi_0 f \in U$, a contradiction. Thus, $\theta^* \in \partial U$. □

Proposition 4.1.1. *If there exists a $\delta > 0$ such that $-f \in U_\varepsilon$, $\forall \varepsilon < \delta$, then*

$$P_\varepsilon u_\varepsilon^* \rightharpoonup 0 \text{ weakly in } H_0^1(\Omega)$$
$$\tilde{\theta_\varepsilon^*} \rightharpoonup \theta^* = -\chi_0 f \text{ weakly in } L^2(\Omega)$$
$$J_\varepsilon(\theta_\varepsilon^*) \to 0.$$

Proof. It follows from the hypothesis that $J_\varepsilon(\theta_\varepsilon^*) \leq J_\varepsilon(-f)$, $\forall \varepsilon < \delta$. Thus,

$$\frac{1}{2}\|\chi_\varepsilon P_\varepsilon u_\varepsilon^*\|_{2,\Omega}^2 + \frac{\varepsilon}{2}\|\tilde{\theta_\varepsilon^*}\|_{2,\Omega}^2 \leq \frac{\varepsilon}{2}\|\chi_\varepsilon f\|_{2,\Omega}^2.$$

Hence, we deduce that $\chi_\varepsilon P_\varepsilon u_\varepsilon^* \to 0$ strongly in $L^2(\Omega)$ and $\tilde{\theta_\varepsilon^*} \rightharpoonup \theta^*$ weakly(for a subsequence) in $L^2(\Omega)$. Also, we have, $J_\varepsilon(\theta_\varepsilon^*) \to 0$. It now follows from the theory of H_0-convergence that $P_\varepsilon u_\varepsilon^* \rightharpoonup u^*$ weakly in $H_0^1(\Omega)$ and hence we observe that $u^* = 0$ and $\theta^* = -\chi_0 f$, also the convergence of the optimal states holds for the entire sequence. □

As we observe from the results developed so far that one lacks information on the optimal controls when the admissible sets are arbitrary. We now consider the case of the positive cone as the admissible set and hope to establish stronger convergence results for u_ε^* and θ_ε^* without any hypothesis on the optimal controls.

Theorem 4.1.2. *Let $U_\varepsilon = \{\theta \in L^2(\Omega_\varepsilon) \mid \tilde{\theta} \geq 0\, a.e. \text{ in } \Omega\}$. Then $\{P_\varepsilon u_\varepsilon^*\}$ bounded in $H_0^1(\Omega)$ and hence we have (for a subsequence),*

$$P_\varepsilon u_\varepsilon^* \rightharpoonup u^* \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega) \tag{4.1.10}$$
$$\tilde{\theta_\varepsilon^*} \rightharpoonup \theta^* \text{ weakly in } H^{-1}(\Omega) \tag{4.1.11}$$
$$J_\varepsilon(\theta_\varepsilon^*) \to \frac{1}{2}\int_\Omega \chi_0 |u^*|^2\,dx. \tag{4.1.12}$$

Further $u' = \chi_0 u^*$, $\theta' = 0$ and $p^* \geq 0$.

Proof. Since U_ε is the positive cone, we have $\varepsilon\theta_\varepsilon^* = (p_\varepsilon^*)^-$ in Ω_ε. Observe that $\varepsilon\tilde{\theta}_\varepsilon^* = \chi_\varepsilon P_\varepsilon(p_\varepsilon^*)^- = \chi_\varepsilon(P_\varepsilon p_\varepsilon^*)^-$ in Ω. The hypothesis of Lemma 4.1.3 is satisfied by U_ε (since $0 \in U_\varepsilon$, for all ε). Hence the convergences in (4.1.4), (4.1.5) and (4.1.6) are valid.

Using u_ε^* as a test function in the weak form of the state equation satisfied by u_ε^*, we have

$$\begin{aligned}\int_{\Omega_\varepsilon} A_\varepsilon \nabla u_\varepsilon^* . \nabla u_\varepsilon^* \, dx &= \int_{\Omega_\varepsilon} (f + \theta_\varepsilon^*) u_\varepsilon^* \, dx \\ &= \int_\Omega \chi_\varepsilon f P_\varepsilon u_\varepsilon^* \, dx + \varepsilon^{-1} \int_{\Omega_\varepsilon} (p_\varepsilon^*)^- u_\varepsilon^* \, dx.\end{aligned}$$

Now using $(p_\varepsilon^*)^-$ as a test function in the weak form of the adjoint equation (4.1.8), we have

$$\int_{\Omega_\varepsilon} (p_\varepsilon^*)^- u_\varepsilon^* \, dx = \int_{\Omega_\varepsilon} A_\varepsilon \nabla (p_\varepsilon^*)^- . \nabla p_\varepsilon^* \, dx = - \int_{\Omega_\varepsilon} A_\varepsilon \nabla (p_\varepsilon^*)^- . \nabla (p_\varepsilon^*)^- \, dx$$

and hence we derive the equality,

$$\int_{\Omega_\varepsilon} A_\varepsilon \nabla u_\varepsilon^* . \nabla u_\varepsilon^* \, dx + \varepsilon^{-1} \int_{\Omega_\varepsilon} A_\varepsilon \nabla (p_\varepsilon^*)^- . \nabla (p_\varepsilon^*)^- \, dx = \int_\Omega \chi_\varepsilon f P_\varepsilon u_\varepsilon^* \, dx. \quad (4.1.13)$$

Since (by Lemma 4.1.3) $\{\chi_\varepsilon P_\varepsilon u_\varepsilon^*\}$ is bounded in $L^2(\Omega)$, we deduce from (4.1.13) that $\{P_\varepsilon u_\varepsilon^*\}$ and $\{\varepsilon^{-1/2} P_\varepsilon (p_\varepsilon^*)^-\}$ are bounded in $H_0^1(\Omega)$. Therefore, for a subsequence, (4.1.10) holds and

$$\varepsilon^{-1/2} P_\varepsilon (p_\varepsilon^*)^- \rightharpoonup q \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega). \quad (4.1.14)$$

Hence

$$\chi_\varepsilon P_\varepsilon u_\varepsilon^* \rightharpoonup \chi_0 u^* \text{ weakly in } L^2(\Omega)$$

and by (4.1.5) it follows that $u' = \chi_0 u^*$. Also

$$\varepsilon^{-1/2} \chi_\varepsilon P_\varepsilon (p_\varepsilon^*)^- \rightharpoonup \chi_0 q \text{ weakly in } L^2(\Omega)$$

i.e.

$$\varepsilon^{1/2} \tilde{\theta}_\varepsilon^* \rightharpoonup \chi_0 q \text{ weakly in } L^2(\Omega).$$

Therefore, by (4.1.4), we have $\theta' = \chi_0 q$.

For $v \in H_0^1(\Omega)$, consider

$$\begin{aligned}\left|\int_\Omega \tilde{\theta_\varepsilon^*} v\,dx\right| &= \left|\int_{\Omega_\varepsilon} \theta_\varepsilon^* v\,dx\right| \\ &= \left|\int_{\Omega_\varepsilon} A_\varepsilon \nabla u_\varepsilon^*.\nabla v\,dx - \int_\Omega \chi_\varepsilon f v\,dx\right| \\ &\leq (b\|u_\varepsilon^*\|_{V_\varepsilon} + C_0\|\chi_\varepsilon f\|_{2,\Omega})\,\|v\|_{H_0^1(\Omega)}.\end{aligned}$$

Hence, it follows that $\{\tilde{\theta_\varepsilon^*}\}$ is bounded in $H^{-1}(\Omega)$ and thus there exists a $\theta^* \in H^{-1}(\Omega)$ such that (4.1.11) holds. Consequently,

$$\varepsilon^{1/2}\tilde{\theta_\varepsilon^*} \to 0 \text{ strongly in } H^{-1}(\Omega)$$

and thus $\theta' = \chi_0 q = 0$. Now, since $\varepsilon\tilde{\theta_\varepsilon^*} = \chi_\varepsilon(P_\varepsilon p_\varepsilon^*)^-$ in Ω we have, using (4.1.6)

$$\varepsilon\tilde{\theta_\varepsilon^*} \rightharpoonup \chi_0(p^*)^- \text{ weakly in } L^2(\Omega).$$

Therefore, $\chi_0(p^*)^- = 0$ which implies $(p^*)^- = 0$ and hence $p^* \geq 0$.

It now follows from (4.1.10) and Lemma 4.1.1 that

$$\|u_\varepsilon^*\|_{2,\Omega_\varepsilon}^2 = \|\chi_\varepsilon P_\varepsilon u_\varepsilon^*\|_{2,\Omega}^2 \to \int_\Omega \chi_0|u^*|^2\,dx$$

and from (4.1.14) and Lemma 4.1.1 that

$$\|\varepsilon^{1/2}\tilde{\theta_\varepsilon^*}\|_{2,\Omega}^2 = \|\varepsilon^{-1/2}\chi_\varepsilon P_\varepsilon(p_\varepsilon^*)^-\|_{2,\Omega}^2 \to \int_\Omega \chi_0 q^2\,dx = 0.$$

Since $J_\varepsilon(\theta_\varepsilon^*) = \frac{1}{2}\left(\|u_\varepsilon^*\|_{2,\Omega_\varepsilon}^2 + \|\varepsilon^{1/2}\tilde{\theta_\varepsilon^*}\|_{2,\Omega}^2\right)$, (4.1.12) holds. □

Remark 4.1.2. The penultimate line in the above proof shows that, in fact, $\varepsilon^{1/2}\tilde{\theta_\varepsilon^*} \to 0$ strongly in $L^2(\Omega)$. Also, since θ^* and p^* are positive, we have $(\theta^*, p^*)_{H^{-1}(\Omega),H_0^1(\Omega)} \geq 0$. On the other hand, observe that $\int_{\Omega_\varepsilon}(p_\varepsilon^* + \varepsilon\theta_\varepsilon^*)\theta_\varepsilon^*\,dx = 0$ and hence $\int_{\Omega_\varepsilon} p_\varepsilon^*\theta_\varepsilon^*\,dx = -\varepsilon\|\theta_\varepsilon^*\|_{2,\Omega_\varepsilon}^2 \leq 0$. Thus $\int_{\Omega_\varepsilon} p_\varepsilon^*\theta_\varepsilon^*\,dx \leq 0$. But we are unable to conclude that $(\theta^*, p^*)_{H^{-1}(\Omega),H_0^1(\Omega)} \leq 0$, owing to the weak convergences of p_ε^* in $H_0^1(\Omega)$ and θ_ε^* in $H^{-1}(\Omega)$. □

Remark 4.1.3. Using p_ε^* as a test function in the state equation (4.0.2) corresponding to θ_ε^* and u_ε^* as a test function in the adjoint-state equation

(4.1.2), for the case U_ε as in Theorem 4.1.2, we have

$$\begin{aligned}\int_\Omega \chi_\varepsilon (P_\varepsilon u_\varepsilon^*)^2\,dx = \int_{\Omega_\varepsilon} (u_\varepsilon^*)^2\,dx &= \int_{\Omega_\varepsilon} A_\varepsilon \nabla u_\varepsilon^* . \nabla p_\varepsilon^*\,dx \\ &= \int_{\Omega_\varepsilon} (f + \theta_\varepsilon^*) p_\varepsilon^*\,dx \\ &= \int_\Omega \chi_\varepsilon f P_\varepsilon p_\varepsilon^*\,dx - \varepsilon \int_{\Omega_\varepsilon} (\theta_\varepsilon^*)^2\,dx.\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$, it follows that

$$\int_\Omega \chi_0 |u^*|^2\,dx = \int_\Omega \chi_0 f p^*\,dx.$$

This result is crucial in the sense that it hints to the fact that one can have $\langle \theta^*, p^* \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0$, if one could homogenize the state equation (4.0.2) with the controls θ_ε^*. □

The absence of the result equivalent to Theorem 3.3.3 for the Neumann boundary condition problem hinders one from writing down the limit control problem for (4.1.1) solving (4.0.2) as was done for the non-perforated case in §3.5, which keeps the problem still open.

Due to the nature of the problem we do not have the uniqueness characterization of θ^*, in general. We compensate this lack by proving a uniqueness characterization of u^*.

Let us define the set,

$$E = \left\{ v \in H_0^1(\Omega) \mid \begin{array}{c} \exists v_\varepsilon \in V_\varepsilon \text{ s.t. } P_\varepsilon v_\varepsilon \rightharpoonup v \text{ in } H_0^1(\Omega), \\ -\operatorname{div}(A_\varepsilon \nabla v_\varepsilon) \in L^2(\Omega_\varepsilon) \text{ and is } \geq f \text{ a.e. in } \Omega_\varepsilon \end{array} \right\}$$

and let $\bar{E}$, a closed convex set in $L^2(\Omega)$, denote the norm-closure of E in $L^2(\Omega)$. It follows from (4.1.10) that $u^* \in E \subset \bar{E}$ and hence E is non-empty. Let $G_\varepsilon : L^2(\Omega_\varepsilon) \to V_\varepsilon$ be the map $\theta_\varepsilon \mapsto u_\varepsilon$ where u_ε is the solution of (4.0.2).

Proposition 4.1.2. *Let U_ε be as given in Theorem 4.1.2. Then E is the K-limit of the sets $E_\varepsilon = P_\varepsilon G_\varepsilon(U_\varepsilon)$ in the weak topology of $H_0^1(\Omega)$.*

Proof. (a) Let $v \in E$. We need to find a $\eta > 0$ and a sequence $v_\varepsilon \rightharpoonup v$ in $H_0^1(\Omega)$ such that $v_\varepsilon \in E_\varepsilon$, $\forall \varepsilon \leq \eta$.

Given $v \in E$, by definition of E, there exists $w_\varepsilon \in V_\varepsilon$ s.t. $P_\varepsilon w_\varepsilon \rightharpoonup v$ in $H_0^1(\Omega)$. Set $\theta_\varepsilon = -\operatorname{div}(A_\varepsilon \nabla w_\varepsilon) - f$. Hence, by definition of E, $\theta_\varepsilon \in U_\varepsilon$, $\forall \varepsilon$. Therefore $w_\varepsilon = G_\varepsilon(\theta_\varepsilon)$. Now, choose $v_\varepsilon = P_\varepsilon w_\varepsilon$, $\forall \varepsilon$. Hence our claim.

(b) Suppose $v_\varepsilon \in E_\varepsilon$ and $v_\varepsilon \rightharpoonup v$ in $H_0^1(\Omega)$, then we need to show that $v \in E$.

Let $v_\varepsilon = P_\varepsilon w_\varepsilon$ where $w_\varepsilon \in G_\varepsilon(U_\varepsilon) \subset V_\varepsilon$. Note that, in fact, w_ε is v_ε restricted to Ω_ε . Also, $\theta_\varepsilon = -\text{div}(A_\varepsilon \nabla w_\varepsilon) - f$ is in U_ε and hence $-\text{div}(A_\varepsilon \nabla w_\varepsilon) \in L^2(\Omega_\varepsilon)$. Hence our claim.

Thus, we have shown that $E_\varepsilon \xrightarrow{K} E$ in the weak topology of $H_0^1(\Omega)$. □

Remark 4.1.4. In the non-perforated case the above proposition reduces to saying that $G_\varepsilon(U) \xrightarrow{K} E$ in the weak topology of $H_0^1(\Omega)$ where,

$$U = \{\theta \in L^2(\Omega) \mid \theta \geq 0\, a.e. \text{ in } \Omega\},$$

$$E = \left\{ v \in H_0^1(\Omega) \mid \begin{array}{c} \exists v_\varepsilon \in H_0^1(\Omega) \text{ s.t. } v_\varepsilon \rightharpoonup v \text{ in } H_0^1(\Omega), \\ -\text{div}(A_\varepsilon \nabla v_\varepsilon) \in L^2(\Omega) \text{ and is } \geq f\, a.e. \text{ in } \Omega \end{array} \right\}$$

and $G_\varepsilon : L^2(\Omega) \to H_0^1(\Omega)$ is the map $\theta_\varepsilon \mapsto u_\varepsilon$ where u_ε is the solution of the counterpart of (4.0.2) in the non-perforated case. □

Theorem 4.1.3. *If U_ε is as in Theorem 4.1.2, then u^* is the projection of 0 onto $\bar{E}$ in $L^2_\mu(\Omega)$ where $d\mu = \chi_0 dx$. In other words,*

$$\int_\Omega \chi_0 u^*(v - u^*)\, dx \geq 0 \quad \forall v \in \bar{E}.$$

Proof. Let $v \in E$ and set $\theta_\varepsilon = -\text{div}(A_\varepsilon \nabla v_\varepsilon) - f$. Then we have $\theta_\varepsilon \in U_\varepsilon$ and arguing as in Theorem 4.1.2 we prove $\tilde{\theta}_\varepsilon$ is bounded in $H^{-1}(\Omega)$. Using this θ_ε in (4.1.3) we have,

$$\begin{aligned} \int_{\Omega_\varepsilon} [u_\varepsilon^*(v_\varepsilon - u_\varepsilon^*) + \varepsilon\theta_\varepsilon^*(\theta_\varepsilon - \theta_\varepsilon^*)]\, dx &\geq 0 \\ \text{i.e. } \int_{\Omega_\varepsilon} u_\varepsilon^* v_\varepsilon\, dx + \varepsilon \int_{\Omega_\varepsilon} \theta_\varepsilon^* \theta_\varepsilon\, dx &\geq \int_{\Omega_\varepsilon} (u_\varepsilon^*)^2\, dx + \varepsilon \int_{\Omega_\varepsilon} (\theta_\varepsilon^*)^2\, dx \\ \text{i.e. } \int_\Omega \chi_\varepsilon P_\varepsilon u_\varepsilon^* P_\varepsilon v_\varepsilon\, dx + \varepsilon \int_\Omega \tilde{\theta}_\varepsilon^* \tilde{\theta}_\varepsilon\, dx &\geq \int_\Omega \chi_\varepsilon (P_\varepsilon u_\varepsilon^*)^2\, dx + \varepsilon \int_\Omega (\tilde{\theta}_\varepsilon^*)^2\, dx. \end{aligned}$$

whence, on passing to the limit

$$\int_\Omega \chi_0 u^* v\, dx \geq \int_\Omega \chi_0 (u^*)^2\, dx.$$

Since $v \in E$ was arbitrary we have,

$$\int_\Omega \chi_0 u^*(v - u^*)\, dx \geq 0 \quad \forall v \in E$$

and by simple density argument we have the inequality for all $v \in \bar{E}$. □

Remark 4.1.5. By the uniqueness of u^*, the convergence in (4.1.4) and (4.1.10) holds for the entire sequence and not just for a subsequence. □

Let us now consider the cases where f has a sign. If $f \leq 0$ a.e. in Ω. Then $-f \in U_\varepsilon$ (as defined in Theorem 4.1.2) and hence the result of proposition 4.1.1 holds. Moreover, from (4.1.13), we have $P_\varepsilon u_\varepsilon^* \to 0$ strongly in $H_0^1(\Omega)$.

Observe that the weak maximum principle remains valid for the state equation (4.0.2) due to the homogeneous Dirichlet boundary condition on $\partial\Omega$ and the homogeneous Neumann boundary condition on the holes. If $f \geq 0$ a.e. in Ω and since $\theta_\varepsilon^* \geq 0$ a.e. in Ω_ε, it follows from the weak maximum principle that $u_\varepsilon^* \geq 0$ a.e. in Ω_ε. Thus by using the weak maximum principle for the adjoint equation 4.1.2, we have $p_\varepsilon^* \geq 0$ a.e. in Ω_ε and hence $\theta_\varepsilon^* = 0$ in Ω_ε. Thus, $\theta^* = 0$ and the state equation becomes

$$\left\{ \begin{aligned} -\text{div}(A_\varepsilon \nabla u_\varepsilon^*) &= f \quad \text{in } \Omega_\varepsilon \\ A_\varepsilon \nabla u_\varepsilon^* . n_\varepsilon &= 0 \quad \text{on } \partial S_\varepsilon \\ u_\varepsilon^* &= 0 \quad \text{on } \partial\Omega. \end{aligned} \right. \tag{4.1.15}$$

Then, by H_0 convergence, it follows that u^* is the solution of the homogenized problem

$$\left\{ \begin{aligned} -\text{div}(A_0 \nabla u^*) &= \chi_0 f \quad \text{in } \Omega \\ u^* &= 0 \quad \text{on } \partial\Omega. \end{aligned} \right. \tag{4.1.16}$$

Theorem 4.1.4. *Let $U_\varepsilon = L^2(\Omega_\varepsilon)$ then we have, $u' = \theta' = p^* = 0$ and*

$$\begin{aligned} P_\varepsilon u_\varepsilon^* &\to 0 \text{ strongly in } H_0^1(\Omega) \\ P_\varepsilon p_\varepsilon^* &\to 0 \text{ strongly in } H_0^1(\Omega) \\ \varepsilon^{1/2}\theta_\varepsilon^* &\to 0 \text{ strongly in } L^2(\Omega) \\ \theta_\varepsilon^* \rightharpoonup \theta^* \text{ weakly in } L^2(\Omega) &\text{ and } \theta^* = -\chi_0 f \\ J_\varepsilon(\theta_\varepsilon^*) &\to 0 \end{aligned}$$

Proof. Since $-f$ restricted to Ω_ε is in $U_\varepsilon = L^2(\Omega_\varepsilon)$, the results of proposition 4.1.1 stays valid. Also, the convergences (4.1.4), (4.1.5) and (4.1.6) remain valid. It follows from the strong convergence of $P_\varepsilon u_\varepsilon^*$ that $u' = 0$ and hence $p^* = 0$. Now, since $\{\theta_\varepsilon^*\}$ is bounded in $L^2(\Omega)$, we have $\varepsilon^{1/2}\theta_\varepsilon^* \to 0$ strongly in $L^2(\Omega)$ and thus $\theta' = 0$.

Also, from the optimality condition, we have $\varepsilon\theta_\varepsilon^* = -p_\varepsilon^*$ in Ω_ε and hence $\varepsilon\theta_\varepsilon^* = -\chi_\varepsilon P_\varepsilon p_\varepsilon^*$ in Ω. An argument similar to the one in theorem 4.1.2 gives the equality corresponding to (4.1.13), i.e.,

$$\int_{\Omega_\varepsilon} A_\varepsilon \nabla u_\varepsilon^*.\nabla u_\varepsilon^*\, dx + \varepsilon^{-1}\int_{\Omega_\varepsilon} A_\varepsilon \nabla p_\varepsilon^*.\nabla p_\varepsilon^*\, dx = \int_\Omega \chi_\varepsilon f P_\varepsilon u_\varepsilon^*\, dx.$$

We deduce from the above equality that $P_\varepsilon p_\varepsilon^* \to 0$ and $P_\varepsilon u_\varepsilon^* \to 0$ strongly in $H_0^1(\Omega)$. □

4.2 Control and State on Boundary

In this section, we consider the case of perforated domain for the boundary control problem described in §2.3. To begin we need to reformulate the notion of *admissible* family of holes. For this section, the family of holes, $\{S_\varepsilon\}$, is said to be admissible in Ω if, along with (**H2**) (in page 11), the following is satisfied:

H 4. *There exists, for each $\varepsilon > 0$, an extension operator*

$$Q_\varepsilon : H^1(\Omega_\varepsilon) \to H^1(\Omega)$$

such that, for every $u \in H^1(\Omega_\varepsilon)$,

$$Q_\varepsilon u|_{\Omega_\varepsilon} = u \text{ and } \|Q_\varepsilon u\|_{H^1(\Omega)} \leq C_0 \|u\|_{H^1(\Omega_\varepsilon)}$$

where C_0 is independent of ε.

Such family of admissible holes has been considered by Hruslov in [Hru79]. We note that the holes allowed by (**H2**) and (**H4**) is not very different from those allowed by (**H2**) and (**H1**) (in page 11). We can, in fact, construct Q_ε from the extension operator P_ε obtained in (**H1**), provided we have the following:

H 5. *There exists a positive constant C_0 independent of ε such that for every $u \in H^1(\Omega_\varepsilon)$,*

$$\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C_0 \|u\|_{H^1(\Omega_\varepsilon)}$$

(Recall that $H^{\frac{1}{2}}(\partial\Omega)$ is the range of the trace map $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$).

To see this, assume (**H5**). Let $u \in H^1(\Omega_\varepsilon)$. Since u restricted to $\partial\Omega$ is in $H^{\frac{1}{2}}(\partial\Omega)$, there exists a $v \in H^1(\Omega)$ such that

$$\|v\|_{H^1(\Omega)} \leq C_0\|u\|_{H^{\frac{1}{2}}(\partial\Omega)}. \tag{4.2.1}$$

Thus, $u - v \in V_\varepsilon$. Then, by (**H1**), $P_\varepsilon(u-v) \in H_0^1(\Omega)$. Define $Q_\varepsilon u = P_\varepsilon(u-v)+v$. Then v restricted to $\partial\Omega$ is same as $Q_\varepsilon u$ restricted to $\partial\Omega$, which is u restricted to $\partial\Omega$. Now, consider

$$\begin{aligned}
\|Q_\varepsilon u\|_{H^1(\Omega)} &= \|P_\varepsilon(u-v)+v\|_{H^1(\Omega)} \\
&\leq \|P_\varepsilon(u-v)\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)} \\
&= \|P_\varepsilon(u-v)\|_{H_0^1(\Omega)} + \|v\|_{H^1(\Omega)} \\
&\leq C_0\|u-v\|_{V_\varepsilon} + \|v\|_{H^1(\Omega)} \\
&\leq C_0\left(\|u\|_{V_\varepsilon} + \|v\|_{V_\varepsilon}\right) + \|v\|_{H^1(\Omega)} \\
&\leq C_0\left(\|u\|_{V_\varepsilon} + \|\nabla v\|_{2,\Omega}\right) + \|v\|_{H^1(\Omega)} \\
&\leq C_0\|u\|_{V_\varepsilon} + C_1\|v\|_{H^1(\Omega)}.
\end{aligned}$$

Therefore, by (4.2.1), we have

$$\|Q_\varepsilon u\|_{H^1(\Omega)} \leq C_0\|u\|_{V_\varepsilon} + C_1\|u\|_{H^{\frac{1}{2}}(\partial\Omega)}$$

and then by, (**H5**),

$$\begin{aligned}
\|Q_\varepsilon u\|_{H^1(\Omega)} &\leq C_0\|u\|_{V_\varepsilon} + C_1\|u\|_{H^1(\Omega_\varepsilon)} \\
&\leq C_2\|u\|_{H^1(\Omega_\varepsilon)}.
\end{aligned}$$

Thus, we have constructed a Q_ε such that (**H4**) is valid.

Conversely, if (**H4**) is valid then we always have (**H5**). To see this, note that for $u \in H^1(\Omega_\varepsilon)$, u restricted to $\partial\Omega$ is same as $Q_\varepsilon u$ restricted to $\partial\Omega$. Now, it follows from trace theory that, for $Q_\varepsilon u \in H^1(\Omega)$,

$$\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C_0\|Q_\varepsilon u\|_{H^1(\Omega)}$$

and from (**H4**), it follows that

$$\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C_0\|u\|_{H^1(\Omega_\varepsilon)}.$$

In short, for state equations with Neumann (or more general) condition on the boundary $\partial\Omega$ in perforated domains, the discussion above suggests that the admissible family of holes are required to satisfy either (H2) and (H4) or, equivalently, (H1), (H2) and (H5). To maintain consistency throughout the section, we shall work with the hypotheses (H2) and (H4).

We now state the optimal control problem to be studied in this section. Let $U_\varepsilon \subset L^2(\partial\Omega)$ and $f \in L^2(\partial\Omega)$ be given. For $\theta_\varepsilon \in U_\varepsilon$, the cost functional is given by,

$$J_\varepsilon(\theta_\varepsilon) = \frac{1}{2}\|u_\varepsilon\|^2_{2,\partial\Omega} + \frac{\varepsilon}{2}\|\theta_\varepsilon\|^2_{2,\partial\Omega} \tag{4.2.2}$$

where the state $u_\varepsilon = u_\varepsilon(\theta_\varepsilon)$ in $H^1(\Omega_\varepsilon)$ is the unique solution of

$$\left\{\begin{array}{rll} -\mathrm{div}(A_\varepsilon\nabla u_\varepsilon) + u_\varepsilon & = 0 & \text{in } \Omega_\varepsilon \\ A_\varepsilon\nabla u_\varepsilon.n_\varepsilon & = 0 & \text{on } \partial S_\varepsilon \\ A_\varepsilon\nabla u_\varepsilon.\nu & = f + \theta_\varepsilon & \text{on } \partial\Omega \end{array}\right. \tag{4.2.3}$$

n_ε and ν are the unit outward normal on ∂S_ε and $\partial\Omega$, respectively.

We now prove a result analogous to Lemma 4.1.2. Assume that there exists a sequence $\theta_\varepsilon \in U_\varepsilon$ such that $\{\theta_\varepsilon\}$ is bounded in $L^2(\partial\Omega)$. We then show that $\{u_\varepsilon(\theta_\varepsilon)\}$ is bounded in $H^1(\Omega_\varepsilon)$. To see this, observe that by the ellipticity of A_ε,

$$\begin{aligned} a\|u_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} &\leq \int_{\Omega_\varepsilon} A_\varepsilon\nabla u_\varepsilon.\nabla u_\varepsilon\, dx + \int_{\Omega_\varepsilon} |u_\varepsilon|^2\, dx \\ &= \int_{\partial\Omega} (f+\theta_\varepsilon)u_\varepsilon\, d\sigma \\ &\leq \|f+\theta_\varepsilon\|_{2,\partial\Omega}\|u_\varepsilon\|_{2,\partial\Omega} \\ &= \|f+\theta_\varepsilon\|_{2,\partial\Omega}\|Q_\varepsilon u_\varepsilon\|_{H^{1/2}(\partial\Omega)} \\ \|u_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} &\leq \frac{C_0}{a}\|f+\theta_\varepsilon\|_{2,\partial\Omega}\|Q_\varepsilon u_\varepsilon\|_{H^1(\Omega)}. \end{aligned}$$

Therefore, by (H4), we have

$$\begin{aligned} \|Q_\varepsilon u_\varepsilon\|^2_{H^1(\Omega)} &\leq C_0\|u_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} \\ &\leq \frac{C_0}{a}\|f+\theta_\varepsilon\|_{2,\partial\Omega}\|Q_\varepsilon u_\varepsilon\|_{H^1(\Omega)} \end{aligned}$$

and thus,

$$\|Q_\varepsilon u_\varepsilon\|_{H^1(\Omega)} \leq \frac{C_0}{a}\|f+\theta_\varepsilon\|_{2,\partial\Omega}$$

showing that $\{Q_\varepsilon u_\varepsilon(\theta_\varepsilon)\}$ is bounded in $H^1(\Omega)$ and hence its trace is bounded in $H^{1/2}(\partial\Omega)$ and $L^2(\partial\Omega)$. Note that the constant C_0 is generic and is not fixed in the above inequalities.

As usual, the problem (4.2.2)–(4.2.3) admits a unique optimal solution, which minimizes J_ε in U_ε and is denoted by θ_ε^*. The corresponding optimal states is denoted by u_ε^*. Also, the adjoint optimal state $p_\varepsilon^* \in H^1(\Omega_\varepsilon)$ is given as the weak solution of the problem

$$\left\{\begin{aligned} -\mathrm{div}({}^tA_\varepsilon\nabla p_\varepsilon^*) + p_\varepsilon^* &= 0 \quad \text{in } \Omega_\varepsilon \\ {}^tA_\varepsilon\nabla p_\varepsilon^*.n_\varepsilon &= 0 \quad \text{on } \partial S_\varepsilon \\ {}^tA_\varepsilon\nabla p_\varepsilon^*.\nu &= u_\varepsilon^* \quad \text{on } \partial\Omega \end{aligned}\right. \tag{4.2.4}$$

Then the optimality condition

$$\int_{\partial\Omega} [u_\varepsilon^*(u_\varepsilon - u_\varepsilon^*) + \varepsilon\theta_\varepsilon^*(\theta_\varepsilon - \theta_\varepsilon^*)]\,d\sigma \geq 0 \quad \forall\theta_\varepsilon \in U_\varepsilon \tag{4.2.5}$$

can be rewritten as

$$\int_{\partial\Omega} (p_\varepsilon^* + \varepsilon\theta_\varepsilon^*)(\theta_\varepsilon - \theta_\varepsilon^*)\,d\sigma \geq 0 \quad \forall\theta_\varepsilon \in U_\varepsilon$$

and hence $\varepsilon\theta_\varepsilon^*$ is the projection in $L^2(\partial\Omega)$ of $-p_\varepsilon^*$ onto U_ε.

Also, a proof similar to the one of Lemma 4.1.3, with obvious changes, will prove the following:

Lemma 4.2.1. *If there exists, for each $\varepsilon > 0$, $\theta_\varepsilon \in U_\varepsilon$ such that $\{\theta_\varepsilon\}$ is bounded in $L^2(\partial\Omega)$ then we have both $\{u_\varepsilon^*\}$, $\{\varepsilon^{1/2}\theta_\varepsilon^*\}$ bounded in $L^2(\partial\Omega)$ and $\{Q_\varepsilon p_\varepsilon^*\}$ bounded in $H^1(\Omega)$.*

It now follows from Lemma 4.2.1 that, up to a subsequence,

$$\varepsilon^{1/2}\theta_\varepsilon^* \rightharpoonup \theta' \text{ weakly in } L^2(\partial\Omega) \tag{4.2.6}$$

$$u_\varepsilon^* \rightharpoonup u' \text{ weakly in } L^2(\partial\Omega) \tag{4.2.7}$$

$$\begin{aligned} &Q_\varepsilon p_\varepsilon^* \rightharpoonup p^* \text{ weakly in } H^1(\Omega) \text{ and hence we have} \\ &p_\varepsilon^* \rightharpoonup p^* \text{ weakly in } H^{1/2}(\partial\Omega) \text{ and strongly in } L^2(\partial\Omega). \end{aligned} \tag{4.2.8}$$

Under the hypothesis of Lemma 4.2.1, we can homogenize the adjoint-state equation (4.2.4) (cf. [KP99, Proposition 2.1]). By the theory of H_0 convergence there exists a matrix A_0 such that A_ε H_0-converges to A_0 and p^* is the solution of,

$$\left\{\begin{aligned} -\mathrm{div}({}^tA_0\nabla p^*) + \chi_0 p^* &= 0 \quad \text{in } \Omega \\ {}^tA_0\nabla p^*.\nu &= u' \quad \text{on } \partial\Omega. \end{aligned}\right. \tag{4.2.9}$$

An argument analogous to the one in the proof of Theorem 4.1.1 will prove the following theorem.

Theorem 4.2.1. *Let A_0 be the H_0-limit of $\{A_\varepsilon\}$ and let the sequential K-limit of $\{U_\varepsilon\}$ in the weak topology of $L^2(\partial\Omega)$ exist, denoted by U. Also let the optimal controls θ_ε^* converge to θ^* weakly in $L^2(\partial\Omega)$, then θ^* is the unique minimizer of*

$$J(\theta) = \frac{1}{2}\int_{\partial\Omega} u^2\, d\sigma$$

in U, where $u = u(\theta) \in H^1(\Omega)$ is the weak solution of,

$$\left\{\begin{array}{rll} -\mathrm{div}(A_0\nabla u) + \chi_0 u &= 0 & \text{in } \Omega \\ A_0\nabla u.\nu &= f + \theta & \text{on } \partial\Omega. \end{array}\right. \tag{4.2.10}$$

Further $u' = u^*$ and $\theta' = 0$.

We now establish stronger convergence results for u_ε^* and θ_ε^* and homogenize the system when the admissible control set is the positive cone of $L^2(\partial\Omega)$.

Theorem 4.2.2. *Let $U = \{\theta \in L^2(\partial\Omega) \mid \theta \geq 0 \,a.e. \text{ on } \partial\Omega\}$, for all $\varepsilon > 0$. Then $Q_\varepsilon u_\varepsilon^* \rightharpoonup u^*$ weakly in $H^1(\Omega)$ and hence,*

$$u_\varepsilon^* \rightharpoonup u^* = u' \text{ weakly in } H^{1/2}(\partial\Omega) \text{ and strongly in } L^2(\partial\Omega) \tag{4.2.11}$$

$$\theta_\varepsilon^* \rightharpoonup \theta^* \text{ weakly in } H^{-1/2}(\partial\Omega) \tag{4.2.12}$$

$$\varepsilon^{1/2}\theta_\varepsilon^* \rightharpoonup \theta' = 0 \text{ weakly in } H^{1/2}(\partial\Omega) \text{ strongly in } L^2(\partial\Omega) \tag{4.2.13}$$

$$J_\varepsilon(\theta_\varepsilon^*) \to \frac{1}{2}\int_{\partial\Omega} |u^*|^2\, d\sigma \tag{4.2.14}$$

Further, u^* and θ^* satisfy the homogenized problem as in (4.2.10).

Proof. Since U is the positive cone, we have $\varepsilon\theta_\varepsilon^* = (p_\varepsilon^*)^-$ a.e. in $\partial\Omega$. The hypothesis of Lemma 4.2.1 is satisfied by U (since $0 \in U$). Hence the convergences in (4.2.6), (4.2.7) and (4.2.8) are valid.

Now computing, as done in the proof of Theorem 4.1.2, we derive the equality

$$a_\varepsilon(u_\varepsilon^*, u_\varepsilon^*) + \varepsilon^{-1}a_\varepsilon((p_\varepsilon^*)^-, (p_\varepsilon^*)^-) = \int_{\partial\Omega} f u_\varepsilon^*\, d\sigma \tag{4.2.15}$$

where

$$a_\varepsilon(v, w) = \int_{\Omega_\varepsilon} A_\varepsilon\nabla v.\nabla w\, dx + \int_{\Omega_\varepsilon} vw\, dx$$

is the bilinear form on $H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon)$.

Since (cf. Lemma 4.2.1) $\{u_\varepsilon^*\}$ is bounded in $L^2(\partial\Omega)$, we deduce from (4.2.15) that $\{Q_\varepsilon u_\varepsilon^*\}$ and $\{\varepsilon^{-1/2}Q_\varepsilon(p_\varepsilon^*)^-\}$ are bounded in $H^1(\Omega)$. Therefore, for a subsequence, (4.2.11) holds and (4.2.6) holds weakly in $H^{1/2}(\partial\Omega)$ and strongly in $L^2(\partial\Omega)$. To show $\theta' = 0$, we shall show that θ_ε^* is bounded in $H^{-1/2}(\partial\Omega)$. We have for $v \in H^1(\Omega)$,

$$\int_{\partial\Omega} \theta_\varepsilon^* v \, d\sigma = a_\varepsilon(u_\varepsilon^*, v) - \int_{\partial\Omega} f v \, d\sigma.$$

Therefore, $(\theta_\varepsilon^*, \psi)_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)}$ is bounded uniformly w.r.t. ε for each ψ, since any $\psi \in H^{1/2}(\partial\Omega)$ can be continuously lifted to a $v \in H^1(\Omega)$. Hence θ_ε^* is bounded in $H^{-1/2}(\partial\Omega)$, thus, (4.2.12) holds for some $\theta^* \in H^{-1/2}(\partial\Omega)$ and also $\theta' = 0$. Thus we have shown (4.2.13), and (4.2.14) follows from (4.2.11) and (4.2.13). Moreover, since $\theta_\varepsilon^* \geq 0$, we have that $\theta^* \geq 0$ in the sense of $H^{-1/2}(\partial\Omega)$.

It follows from the H_0-convergence that

$$(\widetilde{A_\varepsilon \nabla u_\varepsilon^*}) \rightharpoonup A_0 \nabla u^* \text{ weakly in } (L^2(\Omega))^n.$$

Let $v \in H^1(\Omega)$. Then, by passing to the limit in

$$\begin{aligned}
\int_{\partial\Omega} \theta_\varepsilon^* v \, d\sigma &= \int_{\Omega_\varepsilon} A_\varepsilon \nabla u_\varepsilon^* . \nabla v \, dx + \int_{\Omega_\varepsilon} u_\varepsilon^* v \, dx - \int_{\partial\Omega} f v \, d\sigma \\
&= \int_{\Omega} (\widetilde{A_\varepsilon \nabla u_\varepsilon^*}) . \nabla v \, dx + \int_{\Omega} Q_\varepsilon u_\varepsilon^* \chi_\varepsilon v \, dx - \int_{\partial\Omega} f v \, d\sigma
\end{aligned}$$

we have,

$$\begin{aligned}
\langle \theta^*, v \rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} &= \int_{\Omega} A_0 \nabla u^* . \nabla v \, dx + \int_{\Omega} \chi_0 u^* v \, dx - \int_{\partial\Omega} f v \, d\sigma \\
&= \int_{\Omega} -\text{div}(A_0 \nabla u^*) . \nabla v \, dx + \int_{\Omega} \chi_0 u^* v \, dx \\
&\quad + \int_{\partial\Omega} A_0 \nabla u^* . \nu \, v \, d\sigma - \int_{\partial\Omega} f v \, d\sigma
\end{aligned}$$

and hence for all $v \in H^1(\Omega)$,

$$\int_{\partial\Omega} A_0 \nabla u^* . \nu \, v \, d\sigma = \langle \theta^*, v \rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} + \int_{\partial\Omega} f v \, d\sigma.$$

Thus, θ^* and u^* satisfy the homogenized problem as in (4.2.10). □

Remark 4.2.1. Using p_ε^* as a test function in the state equation for u_ε^* and u_ε^* as a test function in the adjoint-state equation, for U as in Theorem 4.2.2, we have

$$\begin{aligned}\int_{\partial\Omega}(u_\varepsilon^*)^2\,d\sigma = a_\varepsilon(u_\varepsilon^*,p_\varepsilon^*) &= \int_{\partial\Omega}(f+\theta_\varepsilon^*)p_\varepsilon^*\,d\sigma\\ &= \int_{\partial\Omega} fp_\varepsilon^*\,d\sigma - \varepsilon\int_{\partial\Omega}(\theta_\varepsilon^*)^2\,d\sigma.\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$, it follows that

$$\int_{\partial\Omega}(u^*)^2\,d\sigma = \int_{\partial\Omega} fp^*\,dx. \tag{4.2.16}$$

Since, we could homogenize the state equation, it follows that

$$\begin{aligned}\int_{\partial\Omega} fp^*\,d\sigma + \langle\theta^*,p^*\rangle_{H^{-\frac12}(\partial\Omega),H^{\frac12}(\partial\Omega)} &= \int_\Omega A_0\nabla u^*.\nabla p^*\,dx + \int_\Omega \chi_0 u^* p^*\,dx\\ &= \int_{\partial\Omega} {}^tA_0\nabla p^*.\nu\, u^*\,d\sigma\\ &= \int_{\partial\Omega}(u^*)^2\,d\sigma.\end{aligned}$$

Hence, using (4.2.16), we deduce that $\langle\theta^*,p^*\rangle_{H^{-\frac12}(\partial\Omega),H^{\frac12}(\partial\Omega)} = 0.$ □

We now study the unconstrained control set case.

Theorem 4.2.3. *Let $U = L^2(\partial\Omega)$ then we have, $u' = \theta' = p^* = 0$ and*

$$\begin{aligned} Q_\varepsilon u_\varepsilon^* \to u^* = 0 \ &\textit{strongly in } H^1(\Omega)\\ \theta^* &= -f\\ J_\varepsilon(\theta_\varepsilon^*) &\to 0.\end{aligned}$$

Proof. Since $0 \in U$, we have from Lemma 4.2.1 that the convergences in (4.2.6), (4.2.7) and (4.2.8) are valid. Also, by the optimality condition, we have $\varepsilon\theta_\varepsilon^* = p_\varepsilon^*$ a.e. in $\partial\Omega$.

The analogous equality of (4.2.15) will be,

$$a_\varepsilon(u_\varepsilon^*,u_\varepsilon^*) + \varepsilon^{-1}a_\varepsilon(p_\varepsilon^*,p_\varepsilon^*) = \int_{\partial\Omega} fu_\varepsilon^*\,d\sigma.$$

It follows from the above equality that $p^* = 0$ and from the homogenized adjoint equation, it follows that $u' = 0$. Hence, by above equality and (4.2.7), we have $Q_\varepsilon u_\varepsilon^* \to u^* = 0$ strongly in $H^1(\Omega)$. Also, we have $\varepsilon^{-1/2} Q_\varepsilon p_\varepsilon^* \to 0$ strongly in $H^1(\Omega)$ and hence, by (4.2.6), $\theta' = 0$. Thus $\theta^* = -f$ and $J_\varepsilon(\theta_\varepsilon^*) \to 0$. □

4.3 Summary

In this chapter, low cost control problems on perforated domains are considered (**P2**). Two types of problems are addressed: The case where the state and control are given on the domain (cf. §4.1) and the case where the state and control are given on the boundary of the domain (cf. §4.2). The asymptotic behaviour is studied when the admissible control set is the positive cone. Due to the absence of the result equivalent to Theorem 3.3.3 for the Neumann boundary condition problem, one is unable to write down the limit system for these problems as was done for the non-perforated case in §3.5, which keeps the problem still open.

It would be interesting to study the fixed cost case and low cost control case for perforated domains with Dirichlet boundary conditions on the holes[2]. The periodic case of this problem with fixed coefficients has been studied in [Raj00].

[2]cf. Open Problem 7 in page 96

Chapter 5

Control Problems with State Constraints

So far, in this thesis, we have been studying optimal control problems with constraints only on the control. In this chapter, we shall study the asymptotic behaviour of optimal control problems with constraints on the state. We shall study the problem with fixed cost and low cost control in both perforated and non-perforated case.

5.1 Non-Perforated Case

We consider a state-constraint optimal control problem in non-perforated domains, where the admissible set varies with the parameter. We shall consider the two cases where the cost of the control is, respectively, dependent and independent of ε.

Let $f \in L^2(\Omega)$ and $U_\varepsilon = \{\theta \in L^2(\Omega) \mid \|\nabla u_\varepsilon(\theta)\|_{2,\Omega} \leq 1\}$, where the state $u_\varepsilon(\theta) = u_\varepsilon$ is the weak solution in $H_0^1(\Omega)$ of the equation

$$\left\{ \begin{array}{rll} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) & = f + \theta & \text{in } \Omega \\ u_\varepsilon & = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{5.1.1}$$

Observe that U_ε is a closed convex subset of $L^2(\Omega)$. The admissible set U_ε is non-empty, since $-f \in U_\varepsilon$, for all ε. We shall now prove a proposition which will identify the limit set for the problem to be considered in §5.1.1 and §5.1.2. Let C be the positive square root of the matrix $B^\sharp$ (cf. (1.3.10)) when $B_\varepsilon = I$, the identity matrix, for all $\varepsilon > 0$. Equivalently, by (1.3.9), C

is the positive square root of the matrix obtained as a distribution limit of $\{^tD_\varepsilon D_\varepsilon\}$. Let us now define

$$U = \{\theta \in L^2(\Omega) \mid \|C\nabla u\|_{2,\Omega} \leq 1\}$$

where the state $u = u(\theta)$ is the weak solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\text{div}(A_0\nabla u) &= f + \theta \quad \text{in } \Omega \\ \qquad\quad u &= 0 \qquad\quad \text{on } \partial\Omega \end{cases} \tag{5.1.2}$$

and A_0 is the H-limit of $\{A_\varepsilon\}$.

Proposition 5.1.1. *U is the K-limit of the sets U_ε in the strong topology of $L^2(\Omega)$.*

Proof. Given $\theta \in U$, we need to find a $\eta > 0$ and a sequence $\theta_\varepsilon \to \theta$ strongly in $L^2(\Omega)$ such that $\theta_\varepsilon \in U_\varepsilon$, for all $\varepsilon \leq \eta$.

Case 1. Let $\theta \in U$ be such that $\|C\nabla u\|_{2,\Omega} < 1$. Then by (1.3.10), there exists δ_θ such that

$$\|\nabla u_\varepsilon(\theta)\|_{2,\Omega} \leq 1, \quad \forall \varepsilon \leq \delta_\theta$$

and hence $\theta \in U_\varepsilon$, for all $\varepsilon \leq \delta_\theta$. Therefore, we set $\theta_\varepsilon = \theta$, for all $\varepsilon \leq \eta = \delta_\theta$ and hence our claim.

Case 2. Let $\theta \in U$ be such that $\|C\nabla u\|_{2,\Omega} = 1$. Choose a sequence of positive real numbers, $\{\alpha_n\}$, such that $0 < \alpha_n < 1$, for all n and $\alpha_n \to 1$. Set $\theta_n = \alpha_n(\theta + f) - f$. Then, its corresponding state, $u_n = \alpha_n u$ is such that $\|C\nabla u_n\|_{2,\Omega} = |\alpha_n|\,\|C\nabla u\|_{2,\Omega} = \alpha_n < 1$. Also, $\theta_n \to \theta$ in $L^2(\Omega)$ because

$$\begin{aligned} \|\theta_n - \theta\|_{2,\Omega} &= \|\alpha_n\theta + (\alpha_n - 1)f - \theta\|_{2,\Omega} \\ &= \|(\alpha_n - 1)\theta + (\alpha_n - 1)f\|_{2,\Omega} \\ &= |\alpha_n - 1|\,\|\theta + f\|_{2,\Omega} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Now, by the previous case, there exists $\delta_n = \delta(\theta_n)$ such that $\theta_n \in U_\varepsilon$, for all $\varepsilon \leq \delta_n$. If $\inf\{\delta_n\} > 0$, then we choose $\eta = \inf\{\delta_n\}$ and set

$$\theta_\varepsilon = \theta_n, \text{ when } \frac{\eta}{n+1} \leq \varepsilon < \frac{\eta}{n}.$$

But if $\inf\{\delta_n\} = 0$, we choose the subsequence (again labelled as δ_n) such that

$$\delta_1 > \delta_2 > \delta_3 > \ldots > \delta_n > \ldots > 0 \text{ and } \delta_n \to 0.$$

We now define, $\theta_\varepsilon = \theta_i$ for $\delta_{i+1} < \varepsilon \leq \delta_i$. This sequence $\{\theta_\varepsilon\}$ satisfies our requirement with $\eta = \delta_1$.

It now remains to be shown that, given $\theta_\varepsilon \in U_\varepsilon$ and $\theta_\varepsilon \to \theta$ strongly in $L^2(\Omega)$, $\theta \in U$. Since $\theta_\varepsilon \in U_\varepsilon$, $\|\nabla u_\varepsilon\|_{2,\Omega} \leq 1$, and by (1.3.10) we have $\|C\nabla u\|_{2,\Omega} \leq 1$. Therefore $\theta \in U$. Thus, we have shown that $U_\varepsilon \overset{K_{seq}}{\to} U$ in $L^2(\Omega)$ strong-topology. □

Example 5.1.1. We describe the situation in the one-dimensional periodic case which gives a nice formula for C. Let λ be a periodic function on $(0,1)$ such that $0 < a \leq \lambda(y) \leq b$ and let $\lambda_\varepsilon(x) = \lambda(\frac{x}{\varepsilon})$. Given $f = 0$ and

$$\begin{cases} -\frac{d}{dx}(\lambda_\varepsilon \frac{du_\varepsilon}{dx}) & = \theta \qquad \text{in } (0,1) \\ u_\varepsilon(0) & = u_\varepsilon(1) \quad = 0, \end{cases}$$

it was shown (cf. [KP97]) that for any function μ_ε such that $0 < c \leq \mu_\varepsilon \leq d$,

$$\int_0^1 \mu_\varepsilon \frac{du_\varepsilon}{dx}\frac{du_\varepsilon}{dx}\,dx \to \int_0^1 \mu^* \frac{du}{dx}\frac{du}{dx}\,dx$$

with $\mu^* = \frac{\lambda_0^2}{\nu_0}$ where $\frac{1}{\lambda_\varepsilon} \rightharpoonup \frac{1}{\lambda_0} = \|1/\lambda\|_{1,\Omega}$ and $\frac{\mu_\varepsilon}{\lambda_\varepsilon^2} \rightharpoonup \frac{1}{\nu_0}$ in $L^\infty(0,1)$ weak*. When $\mu_\varepsilon \equiv 1$, we have $\nu_0 = \left(\|1/\lambda\|_{2,\Omega}^2\right)^{-1}$ and hence $\mu^* = \left(\frac{\|1/\lambda\|_{2,\Omega}}{\|1/\lambda\|_{1,\Omega}}\right)^2$. Then, $C = \frac{\|1/\lambda\|_{2,\Omega}}{\|1/\lambda\|_{1,\Omega}}$.

Thus the strong K-limit of the set

$$U_\varepsilon = \left\{\theta \in L^2(0,1) \mid \left\|\frac{du_\varepsilon}{dx}\right\|_{2,\Omega} \leq 1\right\}$$

is given as

$$U = \left\{\theta \in L^2(0,1) \mid \left\|\frac{du}{dx}\right\|_{2,\Omega} \leq \left\|\frac{1}{\lambda}\right\|_{1,\Omega}\left\|\frac{1}{\lambda}\right\|_{2,\Omega}^{-1}\right\}.$$

where $u = u(\theta)$ solves,

$$\begin{cases} -\frac{d}{dx}(\lambda_0 \frac{du}{dx}) & = \theta \qquad \text{in } (0,1) \\ u(0) & = u(1) \quad = 0. \end{cases}$$

□

In this section, for U_ε as defined earlier and for given $\theta \in U_\varepsilon$, we study the limiting behaviour of

$$J_\varepsilon(\theta) = \frac{1}{2}\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{N}{2}\|\theta\|^2_{2,\Omega} \tag{5.1.3}$$

where u_ε is the solution of (5.1.1). Let θ^*_ε be the unique minimizer of J_ε in U_ε. We now remark that for the admissible set considered in this chapter, whatever may be N (dependent or independent of ε), θ^*_ε is bounded in $L^2(\Omega)$. To see this note that, since $-f \in U_\varepsilon$, the corresponding state $u_\varepsilon(-f) = 0$, for all ε. Thus, we have

$$\|\theta^*_\varepsilon\|^2_{2,\Omega} \leq \|f\|^2_{2,\Omega}, \forall \varepsilon. \tag{5.1.4}$$

Hence θ^*_ε is bounded in $L^2(\Omega)$. Thus, it admits a subsequence weakly converging, say to θ^*, in $L^2(\Omega)$.

5.1.1 N independent of ε

For the case when N is fixed and independent of ε, we have by the theory of H-convergence that the state equation (5.1.1) can be homogenized, and (1.3.10) holds for the optimal states, where $B^\sharp$ is as defined in (1.3.9).

We wish to compute the limit of $J_\varepsilon(\theta^*_\varepsilon)$ and identify θ^* as the optimal control for the limit functional on U. Let us extend the cost functional given in (5.1.3) to all of $L^2(\Omega)$ as follows:

$$F_\varepsilon(\theta) = \begin{cases} J_\varepsilon(\theta) & \text{if } \theta \in U_\varepsilon \\ +\infty & \text{if } \theta \in L^2(\Omega) \setminus U_\varepsilon. \end{cases}$$

Now for U, as obtained in Propositon 5.1.1, we define $J : U \to \mathbb{R}$ as,

$$J(\theta) = \frac{1}{2}\int_\Omega B^\sharp \nabla u \nabla u \, dx + \frac{N}{2}\|\theta\|^2_{2,\Omega} \tag{5.1.5}$$

where the state $u = u(\theta)$ is the weak solution in $H^1_0(\Omega)$ of (5.1.2) and call its extension to $L^2(\Omega)$ as $F : L^2(\Omega) \to \overline{\mathbb{R}}$, defined by

$$F(\theta) = \begin{cases} J(\theta) & \text{if } \theta \in U \\ +\infty & \text{if } \theta \in L^2(\Omega) \setminus U. \end{cases} \tag{5.1.6}$$

We shall now verify the hypotheses of Lemma 1.5.2 in the following theorem.

Theorem 5.1.1. *$F_\varepsilon \overset{\Gamma_{seq}}{\rightharpoonup} F$ in the weak (strong) topology of $L^2(\Omega)$. Consequently, θ^* is the unique minimizer of J over U. Also $J_\varepsilon(\theta^*_\varepsilon) \to J(\theta^*)$ and hence $\theta^*_\varepsilon \to \theta^*$ strongly in $L^2(\Omega)$.*

Proof. Given a sequence $\theta_\varepsilon \rightharpoonup \theta$ in $L^2(\Omega)$, we need to show

$$\liminf_{\varepsilon\to 0} F_\varepsilon(\theta_\varepsilon) \geq F(\theta).$$

If $\theta \notin U$, then the result holds trivially by Lemma 1.5.3(b). Now, let $\theta \in U$. Then $u_\varepsilon(\theta_\varepsilon) \rightharpoonup u$ in $H^1_0(\Omega)$ where u satisfies (5.1.2). By (1.3.10) and the weak convergence of $\{\theta_\varepsilon\}$, we have

$$\begin{aligned}
\liminf_{\varepsilon\to 0} F_\varepsilon(\theta_\varepsilon) &= \lim_{\varepsilon\to 0} \frac{1}{2}\int_\Omega B_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon \, dx + \liminf_{\varepsilon\to 0} \frac{N}{2}\|\theta_\varepsilon\|^2_{2,\Omega} \\
&= \frac{1}{2}\int_\Omega B^\sharp \nabla u \nabla u \, dx + \liminf_{\varepsilon\to 0} \frac{N}{2}\|\theta_\varepsilon\|^2_{2,\Omega} \\
&\geq \frac{1}{2}\int_\Omega B^\sharp \nabla u \nabla u \, dx + \frac{N}{2}\|\theta\|^2_{2,\Omega} = F(\theta).
\end{aligned}$$

It now remains, given $\theta \in L^2(\Omega)$, to find a sequence $\{\theta_\varepsilon\}$ such that $\theta_\varepsilon \rightharpoonup \theta$ in $L^2(\Omega)$ and $\lim_{\varepsilon\to 0} F_\varepsilon(\theta_\varepsilon) = F(\theta)$. If θ is such that $\theta \notin U$, then by Lemma 1.5.3(a) the result follows trivially by choosing $\theta_\varepsilon = \theta$, for all ε. Now, let $\theta \in U$. Then by Proposition 5.1.1 there exists $\delta > 0$ and $\theta_\varepsilon \to \theta$ (and hence $\theta_\varepsilon \rightharpoonup \theta$) such that $\theta_\varepsilon \in U_\varepsilon$, for all $\varepsilon \leq \delta$. For this sequence, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0} F_\varepsilon(\theta_\varepsilon) &= \lim_{\varepsilon\to 0} \left[\frac{1}{2}\int_\Omega B_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon \, dx + \frac{N}{2}\|\theta_\varepsilon\|^2_{2,\Omega}\right] \\
&= \frac{1}{2}\int_\Omega B^\sharp \nabla u \nabla u \, dx + \frac{N}{2}\|\theta\|^2_{2,\Omega} = F(\theta).
\end{aligned}$$

Thus, we have proved that $F_\varepsilon \overset{\Gamma_{seq}}{\rightharpoonup} F$ in $L^2(\Omega)$ weak (strong) topology. Now, since U is non-empty ($-f \in U$), by Lemma 1.5.2 and Proposition 5.1.1, we see that θ^* is a minimizer of J on U and $J_\varepsilon(\theta^*_\varepsilon) \to J(\theta^*)$ as $\varepsilon \to 0$.

θ^* is the unique minimizer, since J is strictly convex, and hence $\theta^*_\varepsilon \rightharpoonup \theta^*$ weakly in $L^2(\Omega)$ for the entire sequence and not just for a subsequence. The fact that $J_\varepsilon(\theta^*_\varepsilon) \to J(\theta^*)$ and (1.3.10) holds, together implies that

$$\|\theta^*_\varepsilon\|^2 \to \|\theta^*\|^2.$$

Hence we have that $\theta^*_\varepsilon \to \theta^*$ strongly in $L^2(\Omega)$. □

Remark 5.1.1. In the theorem above, we proved that the optimal controls, in fact, converge strongly in $L^2(\Omega)$. This result stays valid for the problem (5.1.1)–(5.1.3) on an arbitrary closed convex subset, $U_\varepsilon \subset L^2(\Omega)$, under the assumption that $U_\varepsilon \xrightarrow{K_{seq}} U$ strongly in $L^2(\Omega)$. Under this assumption, another proof, for the results proved in this section, is possible by passing to the limit in the optimality condition associated to the system, an idea used by Kesavan and Saint Jean Paulin (cf. [KP97]) for the situation $U_\varepsilon = U$, for all $\varepsilon > 0$. □

5.1.2 Low Cost Control ($N = \varepsilon$)

In this section, given $\theta \in U_\varepsilon$, we study the limiting behaviour of (cf. (2.3.1))

$$J_\varepsilon(\theta) = \frac{1}{2}\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{\varepsilon}{2}\|\theta\|^2_{2,\Omega} \tag{5.1.7}$$

where u_ε is the solution of (5.1.1). Let θ^*_ε be the unique minimizer of J_ε in U_ε. Therefore, by (5.1.4), θ^*_ε admits a subsequence converging weakly, say to θ^*, in $L^2(\Omega)$. By H-convergence, we can homogenize (5.1.1), and (1.3.10) is valid for the optimal states. Let us extend the cost functional given in (5.1.7) to all of $L^2(\Omega)$ as follows:

$$F_\varepsilon(\theta) = \begin{cases} J_\varepsilon(\theta) & \text{if } \theta \in U_\varepsilon \\ +\infty & \text{if } \theta \in L^2(\Omega) \setminus U_\varepsilon. \end{cases}$$

We shall now define $J : U \to \mathbb{R}$ as,

$$J(\theta) = \frac{1}{2}\int_\Omega B^\sharp \nabla u \nabla u \, dx \tag{5.1.8}$$

where $B^\sharp$ is as defined before and the state $u = u(\theta)$ is the weak solution in $H^1_0(\Omega)$ of (5.1.2). We extend it to all of $L^2(\Omega)$ by $F : L^2(\Omega) \to \overline{\mathbb{R}}$, defined as,

$$F(\theta) = \begin{cases} J(\theta) & \text{if } \theta \in U \\ +\infty & \text{if } \theta \in L^2(\Omega) \setminus U. \end{cases} \tag{5.1.9}$$

Theorem 5.1.2. *$F_\varepsilon \xrightarrow{\Gamma_{seq}} F$ in the weak (strong) topology of $L^2(\Omega)$. Consequently, $J_\varepsilon(\theta^*_\varepsilon) \to 0$ and $\theta^* = -f$ is the unique minimizer of J on U. Also, $\theta^*_\varepsilon \to \theta^* = -f$ strongly in $L^2(\Omega)$.*

Proof. The proof of $F_\varepsilon \xrightarrow{\Gamma_{seq}} F$ in the weak (strong) topology of $L^2(\Omega)$ is exactly along the lines of the proof in Theorem 5.1.1. In fact, whenever $\theta_\varepsilon \rightharpoonup \theta$ in $L^2(\Omega)$ and $F_\varepsilon(\theta_\varepsilon)$, $F(\theta)$ are all finite, the fact that $\{\theta_\varepsilon\}$ is bounded in $L^2(\Omega)$ and the result of (1.3.10) show that $F_\varepsilon(\theta_\varepsilon) \to F(\theta)$.

Hence by Lemma 1.5.2 and Proposition 5.1.1, θ^* is a minimizer of J on U and $J_\varepsilon(\theta_\varepsilon^*) \to J(\theta^*)$ as $\varepsilon \to 0$.

Let u_ε^* and u^* be the states corresponding to θ_ε^* and θ^*, respectively. Then, since $-f \in U_\varepsilon$, we have

$$\int_\Omega B_\varepsilon \nabla u_\varepsilon^*.\nabla u_\varepsilon^*\,dx \le \varepsilon\|f\|_{2,\Omega}^2,\ \varepsilon > 0.$$

Thus, passing to the limit we have $\int_\Omega B_\varepsilon \nabla u_\varepsilon^*.\nabla u_\varepsilon^*\,dx \to 0$ and therefore, by (1.3.10), $\int_\Omega B^\sharp \nabla u^*.\nabla u^*\,dx = 0$. Hence $J_\varepsilon(\theta_\varepsilon^*) \to J(\theta^*) = 0$. Also, by the ellipticity of the B_ε's, we have

$$c\|u_\varepsilon^*\|_{H_0^1(\Omega)}^2 \le \int_\Omega B_\varepsilon \nabla u_\varepsilon^*.\nabla u_\varepsilon^*\,dx$$

and so $u_\varepsilon^* \to 0 = u^*$ strongly in $H_0^1(\Omega)$ and hence $\theta^* = -f$ is the unique minimizer of J on U. Thus, we have $\theta_\varepsilon^* \rightharpoonup -f$ weakly in $L^2(\Omega)$ for the entire sequence and by (5.1.4) it follows that,

$$\limsup_{\varepsilon\to 0} \|\theta_\varepsilon^*\|_{2,\Omega}^2 \le \|f\|_{2,\Omega}^2 \le \liminf_{\varepsilon\to 0} \|\theta_\varepsilon^*\|_{2,\Omega}^2.$$

Hence, $\|\theta_\varepsilon^*\|_{2,\Omega}^2 \to \|f\|_{2,\Omega}^2$ implying the strong convergence of the optimal controls. □

5.2 Perforated Case

We have already described the perforated domain set-up in §1.4. We have from **(H2)** (in page 11) that $\chi_0 > 0$ a.e. in Ω. We now prove a lemma on its upper bound.

Lemma 5.2.1. *If $\chi_\varepsilon \rightharpoonup \chi_0$ weak* in $L^\infty(\Omega)$, then $\chi_0 \le 1$ a.e. in Ω.*

Proof. Suppose not, then the set $\mathfrak{E} = \{x \in \Omega \mid \chi_0(x) > 1\}$ has non-zero measure, i.e., $\mu(\mathfrak{E}) > 0$ where μ is the Lebegue measure in $\mathbb{R}^n$. Let $\chi_{\mathfrak{E}} \in L^1(\Omega)$ be defined as,

$$\chi_{\mathfrak{E}}(x) = \begin{cases} 1 & \text{if } x \in \mathfrak{E} \\ 0 & \text{if } x \in \Omega \setminus \mathfrak{E}. \end{cases}$$

Then $\int_\Omega \chi_\varepsilon \chi_{\mathfrak{E}}\,dx \to \int_\Omega \chi_0 \chi_{\mathfrak{E}}\,dx$. Equivalently, $\mu(\mathfrak{E}\cap\Omega_\varepsilon) \to \int_{\mathfrak{E}} \chi_0\,dx$. But, $\mu(\mathfrak{E}\cap\Omega_\varepsilon) \le \mu(\mathfrak{E})$, for all ε, and hence $\int_{\mathfrak{E}} \chi_0\,dx \le \mu(\mathfrak{E})$ which contradicts the inequality, $\int_{\mathfrak{E}} \chi_0\,dx > \mu(\mathfrak{E}) > 0$, that follows from our supposition. □

Henceforth, we will assume (by working with a suitable subsequence, if necessary) that $\chi_\varepsilon \rightharpoonup \chi_0$ weak* in $L^\infty(\Omega)$.

We call $\{S_\varepsilon\}$ to be an *admissible* family of holes in Ω, if (H2) and (H1). We also recall the norm on V_ε as, $\|u\|_{V_\varepsilon} = \|\nabla u\|_{2,\Omega_\varepsilon}$. Let $f \in L^2(\Omega)$ be given and $U_\varepsilon = \{\theta \in L^2(\Omega_\varepsilon) \mid \|\nabla u_\varepsilon(\theta)\|_{2,\Omega_\varepsilon} \le 1\}$, where the state $u_\varepsilon(\theta) = u_\varepsilon$ is the weak solution in V_ε of (4.0.2) and n_ε is the unit outward normal on ∂S_ε.

Observe that $U_\varepsilon = \{\theta \in L^2(\Omega_\varepsilon) \mid \|u_\varepsilon(\theta)\|_{V_\varepsilon} \le 1\}$. This is a closed convex subset of $L^2(\Omega_\varepsilon)$. The set is non-empty, since $-f$ restricted to Ω_ε is in U_ε. Let C be the positive square root of the matrix $B^\sharp$ when $B_\varepsilon = I$, the identity matrix, for all $\varepsilon > 0$. Equivalently, C is the positive square root of the matrix obtained as a distribution limit of $\{\chi_\varepsilon {}^tD_\varepsilon D_\varepsilon\}$ (cf. (1.4.7)).

Let us now define $U = \{\theta \in L^2(\Omega) \mid \|C\nabla u\|_{2,\Omega} \le 1\}$, where the state $u = u(\theta)$ is the weak solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\mathrm{div}(A_0\nabla u) &= \chi_0 f + \theta \quad \text{in } \Omega \\ \qquad\quad u &= 0 \qquad\qquad \text{on } \partial\Omega \end{cases} \tag{5.2.1}$$

and A_0 is the H_0-limit of $\{A_\varepsilon\}$. Using the extension by zero on the holes, we can consider U_ε as a subset of $L^2(\Omega)$. Similarly, $\theta \in L^2(\Omega)$ vanishing on the holes S_ε will be considered as an element of $L^2(\Omega_\varepsilon)$.

Proposition 5.2.1. *U is the K-limit of the sets U_ε in the weak topology of $L^2(\Omega)$.*

Proof. The arguments for the proof is similar to the one in Proposition 5.1.1. We note here the changes required to make the proof go through.

Given $\theta \in U$, we need to find a $\eta > 0$ and a sequence $\theta_\varepsilon \rightharpoonup \theta$ in $L^2(\Omega)$ such that $\theta_\varepsilon \in U_\varepsilon$, for all $\varepsilon \le \eta$. As done previously, we argue in two parts.

Case 1. Let $\theta \in U$ be such that $\|C\nabla u\|_{2,\Omega} < 1$ and choose $\theta_\varepsilon = (\chi_\varepsilon/\chi_0)\theta$. Then $(\chi_\varepsilon/\chi_0)\theta \rightharpoonup \theta$ weakly in $L^2(\Omega)$ and, by (1.4.8), there exists δ_θ such that $\theta_\varepsilon \in U_\varepsilon$, for all $\varepsilon \le \delta_\theta$.

Case 2. Now, suppose $\theta \in U$ is such that $\|C\nabla u\|_{2,\Omega} = 1$, we choose a sequence $\{\alpha_n\}$ as done in the proof of Proposition 5.1.1 and set

$$\theta_n = \alpha_n(\theta + \chi_0 f) - \chi_0 f.$$

Then the corresponding state $u_n = \alpha_n u$ is such that $\|C\nabla u_n\|_{2,\Omega} = \alpha_n < 1$. Also, $\|\theta_n - \theta\|_{2,\Omega} = |\alpha_n - 1| \, \|\theta + \chi_0 f\|_{2,\Omega} \to 0$ as $n \to \infty$. By the previous case, there exists $\delta_n = \delta(\theta_n)$ such that $(\chi_\varepsilon/\chi_0)\theta_n \in U_\varepsilon$, for all $\varepsilon \leq \delta_n$ and $(\chi_\varepsilon/\chi_0)\theta_n \rightharpoonup \theta$ in $L^2(\Omega)$. Depending on whether $\inf\{\delta_n\} > 0$ or $\inf\{\delta_n\} = 0$ we argue as in the proof Proposition 5.1.1 and choose the sequence accordingly.

If $\inf\{\delta_n\} > 0$, then by choosing $\eta = \inf\{\delta_n\}$ we have our claim with the required sequence being

$$\theta_\varepsilon = \frac{\chi_\varepsilon}{\chi_0}\theta_n, \text{ when } \frac{\eta}{n+1} \leq \varepsilon < \frac{\eta}{n}.$$

But if $\inf\{\delta_n\} = 0$, we choose the subsequence (again labelled as δ_n) such that

$$\delta_1 > \delta_2 > \delta_3 > \ldots > \delta_n > \ldots > 0 \text{ and } \delta_n \to 0.$$

We now define, $\theta_\varepsilon = \frac{\chi_\varepsilon}{\chi_0}\theta_i$ for $\delta_{i+1} < \varepsilon \leq \delta_i$. This sequence $\{\theta_\varepsilon\}$ satisfies our requirement with $\eta = \delta_1$.

Now, given $\theta_\varepsilon \in U_\varepsilon$ and $\theta_\varepsilon \rightharpoonup \theta$ in $L^2(\Omega)$, we need to show that $\theta \in U$. We argue as before and use (1.4.8) to get $\|C\nabla u\|_{2,\Omega} \leq 1$. Hence $\theta \in U$. Thus we have shown that $U_\varepsilon \overset{K_{seq}}{\rightharpoonup} U$ in $L^2(\Omega)$ weak-topology. □

Remark 5.2.1. In contrast to the situation in non-perforated case, here we do not have U as a strong K-limit of $\{U_\varepsilon\}$ in $L^2(\Omega)$. □

In this section, for U_ε as defined above and for given $\theta \in U_\varepsilon$, we study the limiting behaviour of

$$J_\varepsilon(\theta) = \frac{1}{2}\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{N}{2}\|\theta\|^2_{2,\Omega_\varepsilon} \tag{5.2.2}$$

where u_ε is the solution of (4.0.2). Let θ^*_ε be the unique minimizer of J_ε in U_ε. We now remark that for the admissible set considered in this section, whatever may be N (dependent or independent of ε), θ^*_ε is bounded in $L^2(\Omega)$. To see this note that, since $-f$ restricted to Ω_ε is in U_ε and $u_\varepsilon(-f) = 0$, for all ε, we have

$$\|\theta^*_\varepsilon\|^2_{2,\Omega_\varepsilon} \leq \|\chi_\varepsilon f\|^2_{2,\Omega} \leq \|f\|^2_{2,\Omega}, \text{ for all } \varepsilon \tag{5.2.3}$$

i.e. $\tilde{\theta^*_\varepsilon}$ is bounded in $L^2(\Omega)$. Thus, it admits a subsequence weakly converging, say to θ^*, in $L^2(\Omega)$.

5.2.1 N independent of ε

For the case when N is fixed and independent of ε, we have by the theory of H_0-convergence that the state equation (4.0.2) can be homogenized, and (1.4.8) is valid for the optimal states.

We wish to compute the limit of $J_\varepsilon(\theta_\varepsilon^*)$ and identify θ^* as the optimal control for the limit functional on U. Using the extension by zero on the holes, we can consider U_ε as a subset of $L^2(\Omega)$. Thus J_ε, as defined in (5.2.2), can be extended to all of $L^2(\Omega)$ as follows:

$$F_\varepsilon(\theta) = \begin{cases} J_\varepsilon(\theta) & \text{if } \theta \in \widetilde{U_\varepsilon} \\ +\infty & \text{if } \theta \in L^2(\Omega) \setminus \widetilde{U_\varepsilon} \end{cases}$$

where $\widetilde{U_\varepsilon}$ denotes the extension by zero on S_ε of the elements of U_ε. Now for U, as obtained in Propositon 5.2.1, we define $J : U \to \mathbb{R}$ as,

$$J(\theta) = \frac{1}{2}\int_\Omega B^\sharp \nabla u \nabla u \, dx + \frac{N}{2}\int_\Omega \frac{\theta^2}{\chi_0}\, dx \qquad (5.2.4)$$

where the state $u = u(\theta)$ is the weak solution in $H_0^1(\Omega)$ of (5.2.1). Extending it to all of $L^2(\Omega)$ as, $F : L^2(\Omega) \to \overline{\mathbb{R}}$ defined by,

$$F(\theta) = \begin{cases} J(\theta) & \text{if } \theta \in U \\ +\infty & \text{if } \theta \in L^2(\Omega) \setminus U. \end{cases} \qquad (5.2.5)$$

Theorem 5.2.1. *$F_\varepsilon \overset{\Gamma_{seq}}{\rightharpoonup} F$ in the weak topology of $L^2(\Omega)$. Consequently, θ^* is the unique minimizer of J over U. Also $J_\varepsilon(\theta_\varepsilon^*) \to J(\theta^*)$. Further, we have*

$$\tilde{\theta}_\varepsilon^* - \frac{\chi_\varepsilon}{\chi_0}\theta^* \to 0 \text{ strongly in } L^2(\Omega).$$

Proof. Let $\theta_\varepsilon \rightharpoonup \theta$ in $L^2(\Omega)$. It is enough to consider the case when $\theta \in U$ (cf. Lemma 1.5.3(b)) and $F_\varepsilon(\theta_\varepsilon)$ are all finite. By (1.4.8) and [KP99, Propositon 2.2], we have

$$\begin{aligned}
\liminf_{\varepsilon\to 0} F_\varepsilon(\theta_\varepsilon) &= \lim_{\varepsilon\to 0}\frac{1}{2}\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon \, dx + \liminf_{\varepsilon\to 0}\frac{N}{2}\|\theta_\varepsilon\|_{2,\Omega}^2 \\
&= \frac{1}{2}\int_\Omega B^\sharp \nabla u \nabla u \, dx + \liminf_{\varepsilon\to 0}\frac{N}{2}\|\theta_\varepsilon\|_{2,\Omega}^2 \\
&\geq \frac{1}{2}\int_\Omega B^\sharp \nabla u \nabla u \, dx + \frac{N}{2}\int_\Omega \frac{\theta^2}{\chi_0}\, dx = F(\theta).
\end{aligned}$$

It now remains, given $\theta \in L^2(\Omega)$, to find a sequence $\{\theta_\varepsilon\}$ such that $\theta_\varepsilon \rightharpoonup \theta$ in $L^2(\Omega)$ and $\lim_{\varepsilon\to 0} F_\varepsilon(\theta_\varepsilon) = F(\theta)$. By Lemma 1.5.3(a), it is enough to prove this when $\theta \in U$. Now, for $\theta \in U$, by the construction in the proof of Proposition 5.2.1, there exists $\delta > 0$ and $\tilde{\theta}_\varepsilon \rightharpoonup \theta$ such that $\theta_\varepsilon \in U_\varepsilon$, for all $\varepsilon \leq \delta$. Then, again by (1.4.8) and [KP99, Proposition 2.2] we have,

$$\begin{aligned}\lim_{\varepsilon\to 0} F_\varepsilon(\theta_\varepsilon) &= \lim_{\varepsilon\to 0}\left[\frac{1}{2}\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon \, dx + \frac{N}{2}\int_\Omega (\tilde{\theta}_\varepsilon)^2 \, dx\right] \\ &= \frac{1}{2}\int_\Omega B^\sharp \nabla u \nabla u \, dx + \frac{N}{2}\int_\Omega \frac{\theta^2}{\chi_0} \, dx = F(\theta).\end{aligned}$$

Therefore, we have proved $F_\varepsilon \overset{\Gamma_{seq}}{\rightharpoonup} F$ in $L^2(\Omega)$ weak topology. This with the results of Lemma 1.5.2 and Proposition 5.2.1 implies that θ^* is a minimizer of J on U and $J_\varepsilon(\theta_\varepsilon^*) \to J(\theta^*)$ as $\varepsilon \to 0$.

Since J is strictly convex, θ^* is the unique minimizer and hence $\tilde{\theta}_\varepsilon^* \rightharpoonup \theta^*$ weakly in $L^2(\Omega)$ for the entire sequence and not just for a subsequence. The fact that $J_\varepsilon(\theta_\varepsilon^*) \to J(\theta^*)$ and (1.4.8) holds, together implies that

$$\int_{\Omega_\varepsilon} (\theta_\varepsilon^*)^2 \, dx \longrightarrow \int_\Omega \frac{(\theta^*)^2}{\chi_0} \, dx.$$

Hence, by [KP99, Theorem 4.2], we have $\tilde{\theta}_\varepsilon^* - \frac{\chi_\varepsilon}{\chi_0}\theta^* \to 0$ strongly in $L^2(\Omega)$. □

Remark 5.2.2. In contrast to the case in non-perforated domains, here we do not have F as a strong Γ-limit of $\{F_\varepsilon\}$. □

Remark 5.2.3. Using the adjoint optimal state equation, one can deduce, as deduced by Kesavan and Saint Jean Paulin in [KP99], θ^* as a projection of $(\frac{-1}{N})\chi_0 p^*$ on to the convex set U in the weighted space $L^2_\mu(\Omega)$ where $d\mu = \frac{dx}{\chi_0}$ and p^* is the optimal adjoint state. □

5.2.2 Low Cost Control ($N = \varepsilon$)

In this section, given $\theta \in U_\varepsilon$, we study the limiting behaviour of

$$J_\varepsilon(\theta) = \frac{1}{2}\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{\varepsilon}{2}\|\theta\|^2_{2,\Omega_\varepsilon} \tag{5.2.6}$$

where u_ε is the solution of (4.0.2). Let θ_ε^* be the unique minimizer of J_ε in U_ε. Therefore, by (5.2.3), $\tilde{\theta}_\varepsilon^*$ admits a subsequence converging weakly, say to θ^*, in $L^2(\Omega)$. By H_0-convergence, we can homogenize (4.0.2), and (1.4.8) is valid for the optimal states.

Using the extension by zero on the holes, we can consider U_ε as a subset of $L^2(\Omega)$. Thus J_ε, as defined in (5.2.6), can be extended to all of $L^2(\Omega)$ as follows:

$$F_\varepsilon(\theta) = \begin{cases} J_\varepsilon(\theta) & \text{if } \theta \in \widetilde{U_\varepsilon} \\ +\infty & \text{if } \theta \in L^2(\Omega) \setminus \widetilde{U_\varepsilon} \end{cases}$$

where $\widetilde{U_\varepsilon}$ denotes the extension by zero on S_ε of the elements of U_ε. We shall now define $J : U \to \mathbb{R}$ as,

$$J(\theta) = \frac{1}{2}\int_\Omega B^\sharp \nabla u \nabla u \, dx \tag{5.2.7}$$

where the state $u = u(\theta)$ is the weak solution in $H_0^1(\Omega)$ of (5.2.1) and now extending it to all of $L^2(\Omega)$ as $F : L^2(\Omega) \to \overline{\mathbb{R}}$ defined by,

$$F(\theta) = \begin{cases} J(\theta) & \text{if } \theta \in U \\ +\infty & \text{if } \theta \in L^2(\Omega) \setminus U. \end{cases} \tag{5.2.8}$$

Theorem 5.2.2. *$F_\varepsilon \stackrel{\Gamma_{seq}}{\rightharpoonup} F$ in the weak topology of $L^2(\Omega)$. Consequently, $J_\varepsilon(\theta_\varepsilon^*) \to 0$ and $\theta^* = -f$ is the unique minimizer of J on U. Also,*

$$\tilde{\theta_\varepsilon^*} + \frac{\chi_\varepsilon}{\chi_0} f \to 0 \text{ strongly in } L^2(\Omega).$$

Proof. The proof of $F_\varepsilon \stackrel{\Gamma_{seq}}{\rightharpoonup} F$ in the weak topology of $L^2(\Omega)$ is along the lines of the proof in Theorem 5.2.1. Also, if $\theta_\varepsilon \rightharpoonup \theta$ in $L^2(\Omega)$ and $F_\varepsilon(\theta_\varepsilon)$, $F(\theta)$ are all finite, we have $F_\varepsilon(\theta_\varepsilon) \to F(\theta)$ by virtue of (1.4.8) and the fact that $\{\theta_\varepsilon\}$ is bounded in $L^2(\Omega)$.

Hence by Lemma 1.5.2 and Proposition 5.2.1, θ^* is a minimizer of J on U and $J_\varepsilon(\theta_\varepsilon^*) \to J(\theta^*)$ as $\varepsilon \to 0$.

Let u_ε^* and u^* be the states corresponding to θ_ε^* and θ^*, respectively. Then, since $-f \in U_\varepsilon$, we have

$$\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon^* . \nabla u_\varepsilon^* \, dx \le \varepsilon \|\chi_\varepsilon f\|_{2,\Omega}^2 \le \varepsilon \|f\|_{2,\Omega}^2, \ \varepsilon > 0.$$

Thus, $\int_\Omega B_\varepsilon \nabla u_\varepsilon^* . \nabla u_\varepsilon^* \, dx \to 0$ and hence $J_\varepsilon(\theta_\varepsilon^*) \to J(\theta^*) = 0$, since by (1.4.8) we have $\int_\Omega B^\sharp \nabla u^* . \nabla u^* \, dx = 0$. Also, by the ellipticity of B_ε, $\|u_\varepsilon^*\|_{V_\varepsilon} \to 0$ and hence $P_\varepsilon u_\varepsilon^* \to u^* = 0$ strongly in $H_0^1(\Omega)$ and hence $\theta^* = -f$ is the unique minimizer of J on U. Thus, we have $\tilde{\theta}_\varepsilon^* \rightharpoonup -f$ weakly in $L^2(\Omega)$ for the entire sequence and by [KP99, Proposition 2.2] it follows that,

$$\liminf_{\varepsilon \to 0} \|\tilde{\theta}_\varepsilon^*\|_{2,\Omega}^2 \geq \int_\Omega \frac{f^2}{\chi_0}\, dx. \tag{5.2.9}$$

Also from Lemma 5.2.1 we deduce that,

$$\|\tilde{\theta}_\varepsilon^*\|_{2,\Omega}^2 \leq \int_\Omega f^2\, dx \leq \int_\Omega \frac{f^2}{\chi_0}\, dx$$

and, now, taking limsup both sides, we have

$$\limsup_{\varepsilon \to 0} \|\theta_\varepsilon^*\|_{2,\Omega}^2 \leq \int_\Omega \frac{f^2}{\chi_0}\, dx$$

which combined with (5.2.9) gives, $\|\tilde{\theta}_\varepsilon^*\|_{2,\Omega}^2 \to \int_\Omega \frac{f^2}{\chi_0}\, dx$. Hence, by [KP99, Theorem 4.2], we deduce $\tilde{\theta}_\varepsilon^* + \frac{\chi_\varepsilon}{\chi_0} f \to 0$ strongly in $L^2(\Omega)$. □

5.3 Summary

In this chapter, we studied the asymptotic behaviour of an optimal control problem with constraints on the state. The admissible control set involved in the problem is defined through the state variable. The problem is settled for both the fixed cost of the control and low cost control cases in both perforated and non-perforated settings. A state constraint problem with a different control set is given as a open problem[1].

[1]cf. Open Problem 8 in page 97

Open Problems

Open Problem 1. It would be interesting to see whether the set $\mathcal{E}$ defined in Lemma 1.5.2 is actually all of E'. In particular, when $E_n \overset{K_{seq}}{\rightarrow} E$, then is $\mathcal{E} = E$?

Open Problem 2. Given $\{U_\varepsilon\}$, a class of closed convex subsets of $L^2(\Omega)$, the problem given by the cost functional

$$J_\varepsilon(\theta) = \frac{1}{2}\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{N}{2}\|\theta\|^2_{2,\Omega}, \text{ for } \theta \in U_\varepsilon, \tag{5.3.1}$$

where the state $u_\varepsilon = u_\varepsilon(\theta)$ is the weak solution in $H^1_0(\Omega)$ of the boundary value problem

$$\begin{cases} -\text{div}(A_\varepsilon \nabla u_\varepsilon) &= f + \theta \quad \text{in } \Omega \\ \qquad\qquad u_\varepsilon &= 0 \qquad\quad \text{on } \partial\Omega \end{cases} \tag{5.3.2}$$

has a unique minimiser denoted as θ^*_ε. It has been observed in §3.1 that $\theta^*_\varepsilon \rightharpoonup \theta^*$ weakly in $L^2(\Omega)$ for some θ^*. It would be interesting to see whether the convergence of the optimal controls can be improved, *i.e.*, $\theta^*_\varepsilon \to \theta^*$ strongly in $L^2(\Omega)$.

Open Problem 3. It would be interesting to study the asymptotic behaviour of the system where the cost functional is defined as,

$$J_\varepsilon(\theta) = \frac{1}{2}\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{\varepsilon}{2}\|\theta\|^2_{2,\Omega}, \quad \text{for } \theta \in U \tag{5.3.3}$$

where the state $u_\varepsilon = u_\varepsilon(\theta)$ is the weak solution in $H^1_0(\Omega)$ of the boundary value problem (5.3.2) and U is an arbitrary admissible control set in $L^2(\Omega)$. A study of the above system even for special cases of U is worth the time spent. In fact, even the case where U is the positive cone in $L^2(\Omega)$ is still open. Some trivial cases of U has been considered in §3.2.

Open Problem 4. Prove the equivalent of Theorem 3.3.3 when the Dirichlet boundary condition in (3.3.3) is replaced with Neumann condition.

Open Problem 5. The asymptotic behaviour of the system with cost functional given as,

$$J_\varepsilon(\theta) = \frac{1}{2}\|u_\varepsilon\|^2_{2,\Omega} + \frac{\varepsilon}{2}\|\theta\|^2_{2,\Omega}, \quad \text{for } \theta \in U \tag{5.3.4}$$

where the state $u_\varepsilon = u_\varepsilon(\theta)$ is the weak solution in $H^1_0(\Omega)$ of (5.3.2) and the admissible control set U is the positive cone in $L^2(\Omega)$ is settled in §3.5 (cf. Theorem 3.5.1). It would be interesting to study the system (5.3.2)–(5.3.4) for an arbitrary admissible control set.

Open Problem 6. In the case of perforated domains, it would be interesting to study the asymptotic behaviour of the system where the cost functional is defined as,

$$J_\varepsilon(\theta_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{\varepsilon}{2}\|\theta_\varepsilon\|^2_{2,\Omega_\varepsilon}, \quad \text{for } \theta_\varepsilon \in U_\varepsilon \subset L^2(\Omega_\varepsilon) \tag{5.3.5}$$

where the state $u_\varepsilon = u_\varepsilon(\theta_\varepsilon) \in V_\varepsilon$ is the weak solution of

$$\left\{ \begin{aligned} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) &= f + \theta_\varepsilon && \text{in } \Omega_\varepsilon \\ A_\varepsilon \nabla u_\varepsilon . n_\varepsilon &= 0 && \text{on } \partial S_\varepsilon \\ u_\varepsilon &= 0 && \text{on } \partial\Omega \end{aligned} \right. \tag{5.3.6}$$

(where n_ε is the unit outward normal on ∂S_ε). Even the system with other cost functionals as considered in Chapter 4 for the positive cone case are open.

Open Problem 7. The above problem has Neumann boundary condition on the body of the holes. It would be interesting to study the case of Dirichlet boundary conditions on the holes. Study the asymptotic behaviour of the system where the cost functional is defined as,

$$J_\varepsilon(\theta_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{N}{2}\|\theta_\varepsilon\|^2_{2,\Omega_\varepsilon}, \quad \text{for } \theta_\varepsilon \in U_\varepsilon \subset L^2(\Omega_\varepsilon) \tag{5.3.7}$$

where the state $u_\varepsilon = u_\varepsilon(\theta_\varepsilon) \in H^1_0(\Omega_\varepsilon)$ is the weak solution of

$$\left\{ \begin{aligned} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) &= f + \theta_\varepsilon && \text{in } \Omega_\varepsilon \\ u_\varepsilon &= 0 && \text{on } \partial\Omega_\varepsilon . \end{aligned} \right. \tag{5.3.8}$$

Note that the cost of the control here is fixed. The periodic case of this problem with fixed coefficients has been studied in [Raj00]. One is also interested in the low cost version of the above problem.

Open Problem 8. Let $U_\varepsilon = \{\theta \in L^2(\Omega) \mid |\nabla u_\varepsilon(\theta)| \leq 1 \text{ a.e. }\}$ be the admissible control set in $L^2(\Omega)$, where the state $u_\varepsilon(\theta) = u_\varepsilon$ is the weak solution in $H_0^1(\Omega)$ of the equation (5.3.2). It would be interesting to study the asymptotic behaviour of the state constraint problems (5.3.3) & (5.3.2) and the system

$$J_\varepsilon(\theta) = \frac{1}{2}\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{N}{2}\|\theta\|^2_{2,\Omega}, \quad \text{for } \theta \in U \tag{5.3.9}$$

and (5.3.2) for both perforated and non-perforated domains, as done in §5, with the above defined U_ε as the control set.

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