Aspects of Noncommutativity in Field and String theory and Closed String Tachyon Condensation

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DECLARATION

I declare that the thesis entitled " Aspects of Noncommutativity in Field and String theory and Closed String Tachyon Condensation", submitted by me for the Degree of Doctor of Philosophy is the record of work carried out by me during the period from April 2002 to August 2005 under the guidance of Prof. Balachandran Sathiapalan and has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

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CERTIFICATE

I certify that the Ph.D. thesis titled " Aspects of Noncommutativity in Field and String theory and Closed String Tachyon Condensation" submitted for the Degree of Doctor of Philosophy by Mr. Swarnendu Sarkar is the record of bona fide research work carried out by him during the period from April 2002 to August 2005 under my supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning. It is further certified that the thesis represents independent work by the candidate and collaboration was necessitated by the nature and scope of the problems dealt with.

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Abstract

This thesis is devoted to the study of two issues. One is the mixing of the ultraviolet and the infrared sectors in noncommutative field theories and their interpretation in terms of open-closed string duality in string theory. The other is the analysis of localised closed string tachyon condensation. A brief abstract of the studies is written separately below.

- **Noncommutativity in field and string theory**

In the first part we study some aspects of noncommutativity in field and string theory. Specifically we study the problem of UV/IR mixing that is one of the most important and generic features of noncommutative field theories. As a consequence of this coupling of the UV and IR sectors, the configuration of fields at the zero momentum limit in these theories is a very singular configuration. We show that the renormalisation conditions set at a particular momentum configuration with a fixed number of zero momenta, renormalises the Green's functions for any general momenta only when this configuration has same set of zero momenta. Therefore only when renormalisation conditions are set at a point where all the external momenta are nonzero, the quantum theory is renormalisable for all values of nonzero momentum. This arises as a result of different scaling behaviours of Green's functions with respect to the UV cutoff (Λ) for configurations containing different set of zero momenta. We study this in the noncommutative ϕ^4 theory and analyse similar results for the Gross-Neveu model at one loop level. We next show this general feature using Wilsonian Renormalisation Group equations of Polchinski in the globally O(N) symmetric scalar theory and prove the renormalisability of the

theory to all orders with an infrared cutoff. In the context of spontaneous symmetry breaking in noncommutative scalar theory, it is essential to note the different scaling behaviours of Green's functions with respect to Λ for different set of zero momenta configurations. We show that in the broken phase of the theory the Ward identities are satisfied to all orders only when one keeps an infrared regulator by shifting to a nonconstant vacuum.

The mixing of UV and IR sectors has a natural interpretation in string theory from the point of view of open-closed string duality. With this motivation we study closed string exchanges in background B-field. By analysing the two point one loop amplitude in bosonic string theory, we show that tree-level exchange of lowest lying, tachyonic and massless closed string modes, have IR singularities similar to those of the nonplanar sector in noncommutative gauge theories. We further isolate the contributions from each of the massless modes. We interpret these results as the manifestation of open-closed string duality, where the IR behaviour of the boundary noncommutative gauge theory is reconstructed from the bulk theory of closed strings.

Next using the same setup we study the phenomenon for noncommutative $\mathcal{N} = 2$ gauge theory realised on a D_3 fractional brane localised at the fixed point of C^2/Z_2. The IR singularities from the massless closed string exchanges are exactly equal to those coming from one-loop gauge theory. This is as a result of cancellation of all contributions from the massive modes.

- **Localised closed string tachyon condensation**

In the second part we study localised closed string tachyon condensation. We analyse the condensation of closed string tachyons on the C/Z_N orbifold. We construct the potential for the tachyons upto the quartic interaction term in the large N limit. In this limit there are near marginal tachyons. The quartic coupling for these tachyons is calculated by subtracting from the string theory amplitude for the tachyons, the contributions from the massless exchanges, computed from the effective field theory. We argue that higher point interaction terms are are also of the same order in 1/N as the quartic term and are necessary for existence of the minimum of the tachyon potential that is consistent with earlier analysis.

Publications

This Thesis is based on the following publications :

1. *On the UV renormalizability of noncommutative field theories*
   Swarnendu Sarkar, JHEP **0206** (2002) 003, [arXiv:hep-th/0202171].

2. *Closed string tachyons on C/Z(N)*
   Swarnendu Sarkar, B. Sathiapalan, Int. J. Mod. Phys. A **19** (2004) 5625, [arXiv:hep-th/0309029].

3. *Aspects of open-closed duality in a background B-field*
   Swarnendu Sarkar, B. Sathiapalan, JHEP **0505** (2005) 062, [arXiv:hep-th/0503009].

4. *Aspects of open-closed duality in a background B-field II*
   Swarnendu Sarkar, B. Sathiapalan, [arXiv:hep-th/0508004]. *Submitted to* JHEP.

5. *UV/IR mixing in noncommutative field theories and open-closed string duality*
   Swarnendu Sarkar, B. Sathiapalan, *Invited review submitted to* Int. J. Mod. Phys. A.

Other publications :

1. Swarnendu Sarkar and B. Sathiapalan, "Comments on the renormalizability of the broken symmetry phase in noncommutative scalar field theory," JHEP **0105** (2001) 049 [arXiv:hep-th/0104106]

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Chapter 1

Introduction

Unification of all the four known forces has been the central theme of research in theoretical physics in the past century. *String theory*, that was developed in order to explain phenomena including strong interactions soon turned out to be one of the most interesting candidates for this goal [1, 2]. In its initial formulation, it is a theory of first quantised relativistic one dimensional strings whose length is of the order of Plank length, 10^{-33} cm. Depending on whether the end points are identified or not, we get closed or open strings respectively. In this picture the elementary states are the various vibrational modes of the string. We will discuss some of the important features of string theory below.

1. *Gravity* : All closed string theories contain gravity, a spin two massless field. This has thus become a promising area for the quest for a quantum formulation of gravity that had turned out to be conceptually as well as technically difficult to formulate.

2. *Gauge Groups* : One of the most important lessons learnt from the original days of Quantum Field Theories is the notion of scales at which observations are made. An effective model may work perfectly well about given scale though it may lack the essential features of being a fundamental theory of nature. Deviations in experimental results from the theoretical predications of these effective theories point towards the existence of a theory encompassing a larger

domain of scales. The existing model for elementary particles, the *Standard Model* includes the electromagnetic, strong and weak interactions in an unified *gauge theory* with gauge group $SU(3) \times SU(2) \times U(1)$ with couplings to *chiral* fermions. This model has been extremely successful and predicts results with high degree of accuracy upto 100 Gev (10^{-16} cm). This is however low energy as compared to the Plank scale where one must include gravitational interactions and is thus believed to be an effective theory that may be embedded in a larger theory. String theory has surprised us many times with its riches of physical and mathematical structures and has offered much more than one expects. Gauge groups containing $SU(3) \times SU(2) \times U(1)$ as a subgroup have been found in various setups with chiral gauge couplings. Recently the effort is on to pin down the *Standard Model* like (realistic) gauge groups in the low energy limits of this theory that are ordinary Quantum Field Theories.

3. *Extra Dimensions* : A consistent string theory can only be realised in a space-time of dimension equal to ten. This is however not a problem as six of the extra dimensions can be compactified so that at low energies or at large length scales the space-time looks effectively four dimensional. The main point is that there exists a large number of possible compactifications and for each of these choices we get a different effective theory in four dimensions. The existing physical theory, the Standard Model at low energies serves as a guideline for selecting these compactifications.

4. *Supersymmetry* : All consistent string theories require supersymmetry. It is believed that at lower scales supersymmetry will be broken, though the exact mechanism is not known yet. In the effort to make contact with realistic theories, the primary aim is to embed the Standard Model into a Minimal Supersymmetric Model i.e $\mathcal{N} = 1$ in four dimensions. The earliest model that gave $\mathcal{N} = 1$ supersymmetry is the Heterotic string compactified on a Calabi-Yau manifold.

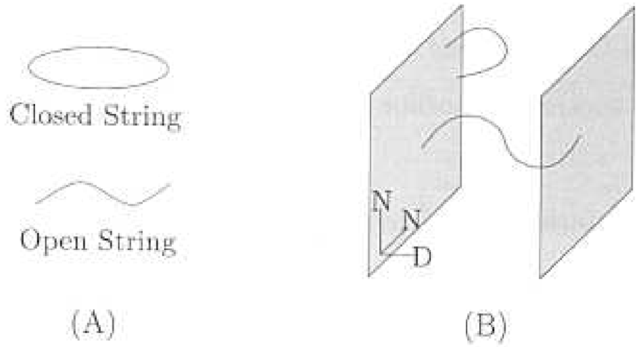

Figure 1.1: (A) Closed and open strings (B) Open strings ending on a D_2-brane, N and D are the *Neumann* and *Dirichlet* directions respectively.

5. *Finiteness* : One of the generic features that plagued Quantum field theories is the appearance of ultraviolet divergences when integrating over very high momentum modes in the loops. The theories are classified as *renormalisable* or *non-renormalisable* depending on whether they have finite or infinite number of diverging couplings. For the former case there exists a well defined renormalisation procedure that lays down the rules as to how to remove the infinities in a systematic way. However no such procedure exists for the non-renormalisable theories and are thus only well defined upto some finite momentum. Unfortunately perturbative gravity was soon discovered to be non-renormalisable. String theory on the other hand gives finite result. This is primarily due to the fact that the point interactions in quantum field theory is replaced by interactions smeared over a region because of the extended nature of the string.

The discovery of other extended hypersurfaces, *Dirichlet branes* or *D-branes* has accelerated progress [3] in various directions. At the end point of open strings one can impose either *Neumann* or *Dirichlet* boundary conditions. *D-branes* may be thought of as extended hypersurfaces on which the open strings end corresponding to Dirichlet conditions along the directions transverse to the brane and Neumann conditions along the brane (Figure 1.1). The low energy dynamics of

D-branes can be described by the low energy dynamics of open strings that is a gauge theory that lives on the world-volume of the brane. D-branes have another description from the closed strings. They are solitonic solutions of low energy closed string theory that is supergravity.

The merit of D-branes was uncovered in the study of *dualities* in string theory [4]. Perturbatively, string theory was known to have five consistent formulations, namely Type IIA, Type IIA, Type I $SO(32)$, Heterotic $SO(32)$, Heterotic $E_8 \times E_8$. In one of the revolutionary steps in String theory, it was discovered that all these theories are in fact related to each other and are part of a bigger theory. In this picture, each of these five theories is defined as perturbative expansions about five different points. See Figure 1.2. These dualities (except T-duality) are not visible in the perturbative sector of these theories, however non-perturbative techniques using D-branes have played a pivotal role in discovering these dualities. A more complicated web of dualities is also conjectured to exist for the compactified theories that give $\mathcal{N} = 1$ space-time supersymmetry on non-compact four dimensional space-time. We will now briefly state the various components of the Figure 1.2 which contains the minimum set of dualities connecting all the string theories.

- *T Duality* : Target space duality relates a theory compactified on a circle (S^1) of radius R to another theory compactified on a circle of radius $1/R$. This relates the Type IIA and IIB theories and also the two Heterotic theories.

- *S Duality* : This relates a strongly coupled theory to a weakly coupled theory [5]. This duality maps the Type IIA theory to the 11-dimensional M-theory compactified on a circle (S^1) and also the Heterotic $E_8 \times E_8$ theory to M-theory on an interval (I). The exact formulation of M-theory is not yet known, however its low-energy limit is known to be given by supergravity in 11 dimensions [6]. Other examples are the self-duality of the Type IIB theory and the Type I and Heterotic $SO(32)$ duality.

Recently a new form of duality that goes by the name of *Gauge/Gravity* correspondence has triggered a lot of interest. This was originally conjectured by

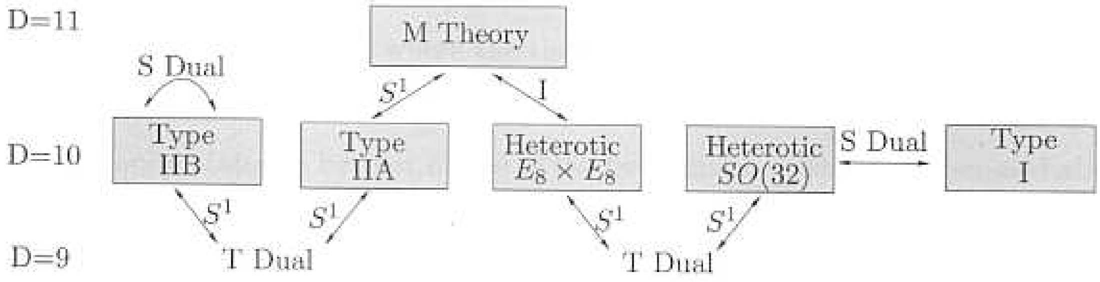

Figure 1.2: Dualities connecting the five consistent string theories

Maldacena [13] and relates a closed string theory to Gauge theory that is a field theory. This duality is the manifestation of world-sheet *open-closed string duality* in some special background. In the later chapters of this thesis we will study this duality in a background constant B-field.

Some of the important characteristics of all the above string theories are recollected in the following table.

	String Type	Supersymmetry	Gauge Group	D_p-branes
Type IIA	Closed Oriented	$\mathcal{N}=2$ Non-chiral	$U(1)$	$p=$ $0,2,4,8$
Type IIB	Closed Oriented	$\mathcal{N}=2$ Chiral	$U(1)$	$p=$ $(-1),1,3,5,7,9$
Type I	Open/Closed Unoriented	$\mathcal{N}=1$ Chiral	$SO(32)$	$p=$ $1,5,9$
Heterotic $SO(32)$	Closed Oriented	$\mathcal{N}=1$ Chiral	$SO(32)$	-
Heterotic $E_8 \times E_8$	Closed Oriented	$\mathcal{N}=1$ Chiral	$E_8 \times E_8$	-

Not included in the list of theories above is the 26-dimensional *Bosonic Theory* and a pair of 10-dimensional Type 0A and 0B theories. These theories are non-supersymmetric and do not contain fermions. There are tachyons in both the open and the closed string sectors. Tachyons are particles with negative $mass^2$

and their presence signals a wrong choice of ground state. It is believed that there exists stable ground states where the these theories will be free from tachyons and quite likely will eventually include fermions. Uniqueness of string theory requires these ground states to be that of one of the five consistent theories discussed above. However it has not been possible to show this rigorously yet. Tachyons also appear in various compactifications of the five theories listed in the table. We will study one of these models. A more detailed introduction to this problem is given in Section 1.2

Till now we have not mentioned anything about the backgrounds in which string propagation is possible. It is believed that string theory would ultimately be formulated in a background independent way and that background will be an emergent phenomenon. However severe constraints on the backgrounds can be readily seen even from perturbative string theory. Let us consider the action for the bosonic string in the presence of the background massless modes, the graviton ($G_{\mu\nu}$), dilaton (Φ) and the antisymmetric two form $B_{\mu\nu}$-field,

$$S = \frac{1}{4\pi\alpha'}\int_M g^{1/2}\left[\left(g^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu \partial_b X^\nu + \alpha' R\Phi(X)\right] \qquad (1.1)$$

From the two-dimensional world-sheet point of view the background fields are couplings when the fields are expanded about some constant values. For a consistent string theory the action (1.1) must be Weyl-invariant. This requires the beta functions corresponding to the background fields to vanish. This constraint gives the space-time equations of motion for fields. Any solution to these equations would define a consistent background for string dynamics. For example the the beta function for the metric gives Einstein's equation and any background metric must satisfy this equation. Thus the background space-time geometry must always be defined by *Riemannian geometry*. Various studies of string dynamics in nontrivial backgrounds have followed since the early days. Open string dynamics in a constant B-field background has recently been studied and it was found that in a certain limit one finds that the space-time on the world-volume of the D-brane is described by *Noncommutative geometry*. The list of such interesting observations is enormous each of which adds to our better understanding of string theory.

We will leave this very brief general introduction at this point and move on to the topics that would be studied in detail in the following chapters in this thesis.

1.1 Part I : Noncommutativity in field and string theory

When an antisymmetric two form constant field ($B_{\mu\nu}$) is turned on in the background, the world-volume theory on the brane is described by a noncommutative gauge theory [7, 8, 9, 10]. Quantum field theories on noncommutative space times have independently been studied for a long time with a hope to cure the ultraviolet divergence problem in QFT that arise in the continuum limit. The rationale for this being that discreteness of space time is inherent in any quantum formulation of gravity and that noncommutative space-time is one of the ways to achieve this. Studies have shown that QFTs on noncommutative backgrounds often lead to nonlocality and violate the conventional notions of local quantum field theories. Various aspects of these theories have been studied extensively over the past few years [11]. One of the well known generic features of these theories is the *mixing* of the ultraviolet (UV) and the infrared (IR) sectors contrary to the ordinary QFTs where they decouple [12]. Within the domain of QFT it is thus important to see how the usual notions of *Wilsonian Renormalisation Group* fits into these models. A thorough analysis shows that an IR cutoff is necessary for the Wilsonian RG to make sense here, and with the IR cutoff usual renormalisation can be done. However the embedding of these theories in string theories have lead to better understanding of the nonlocal properties. In string theory, the ultraviolet region of open strings can be mapped to the infrared region of closed strings by *open-closed string duality*.

As discussed dualities have played a vital role in the understanding of string theory. One of this is the open-closed string duality. Geometrically an open string one loop diagram, that is a cylinder, is a closed string tree-level diagram. The cylinder diagram can thus be interpreted as open string modes propagating in the loop or tree-level propagation of closed string modes. This observation is as old as the initial formulation of string theory. The map is nontrivial in the sense that we have an infinite number of modes on either side. The low energy limit of the open string that is governed by gauge theory, corresponds to the high energy limit of closed strings where all the massive closed string states are excited. This duality is also reflected in the description of D-branes as solitonic solutions of supergravity theory. The quantum dynamics of D-branes can be described by gauge theories or supergravity solutions. These descriptions are valid for the low energy regime of the open or the closed string respectively. There are certain backgrounds where the low energy description is valid simultaneously at both the ends. This is the AdS/CFT correspondence conjectured by Maldacena. The duality in its weakest form is between $D = 4$, $\mathcal{N} = 4$ $SU(N)$ superconformal (finite) Yang Mill's theory with sixteen supercharges and Type IIB supergravity on $AdS_5 \times S^5$ [13]. Efforts have been made to extend this duality to more realistic gauge theories with less supersymmetries and that are nonconformal. Since these theories are not expected to be finite, from the point of view of noncommutative gauge theories, it is interesting to study UV/IR mixing here. The UV behaviour of the world-volume gauge theory on the brane can then be mapped as infrared effect due to tree-level exchanges of closed string massless modes. We have analysed this aspect for the bosonic theory first and then for the type IIB theory with the gauge theory on a *fractional brane* localised on C^2/Z_2 *orbifold*. In the second case, the orbifold breaks half of the supersymmetries and we have $\mathcal{N} = 2$ gauge theory on the brane. For a *fractional brane* this theory is nonconformal. The bosonic model requires an infinite number of closed string modes for the dual description of the ultraviolet limit of the gauge theory. However for Type II strings on the C^2/Z_2 it is known that the one loop open string amplitude with only the massless modes propagating in the loop, can be reproduced by the tree-level exchange of massless closed string modes [14, 15]. This is due to the cancellation of all contributions from the massive modes, that is very specific to this model. Thus in the noncommutative theory one has a natural interpretation of the IR divergences that arise out of integrating high momentum modes as IR effect due to closed string modes. These aspects have been explored in Chapter 3.

1.2 Part II : Closed string tachyon condensation

The presence of tachyons in the spectrum signals instability of the vacuum. This instability is a sign of wrong choice of the ground state. It is therefore natural to ask what the stable ground state is or in other words what is the end point of the tachyon condensation process. In the bosonic theory there are tachyons in both the open and closed string sectors. In the supersymmetric Type II theory, the tachyon in the NS sector is removed by GSO projection, that is needed to make the partition function modular invariant. However open string tachyons occur in brane-antibrane systems and on non-BPS D-branes in Type II theories. For the open strings the condensation process is now very well understood mainly due to the work of Sen. See [16] for review and references therein. From the point of view of quantum field theory the object one needs to prove the existence of a stable ground state is the potential $V(T)$, where T is the tachyon field. The problem then boils down to the construction of $V(T)$ and finding its minimum. However the computation of the potential from string theory is difficult since conformal invariance constrains the amplitudes to have external particles on-shell. The zero momentum continuation of amplitudes with external massive and tachyonic particles turns out to be ambiguous. This is the reason, for which an off-shell formulation is a necessity. The following are the conjectures due to Sen regarding the tachyon potential and the endpoint of the condensation process [17, 18].

1. There exists minima for the tachyon potential $V(T)$. The shape of the potential is a double well with minima at $T = \pm T_0$ for non-BPS branes, where the tachyon field is real. For the brane-antibrane system one has complex tachyon and hence the minima are parametrised by, α, so that $T_{min} = T_0 e^{i\alpha}$.

2. The height of the potential is equal to the tension of the D-brane system.

3. The end point of open string tachyon condensation is a closed string vacuum with no open strings in its excited spectrum and no D-branes.

These conjectures have been verified to a high accuracy using various techniques including the off-shell formulation, open string field theory. The situation is however much more different in the case for the closed string tachyons. While the open string tachyon lives on the world volume of the D-brane and affects the D-branes as the condensation progresses, the closed string tachyon on the other hand couples to the graviton and the dilaton in the bulk. The condensation of the closed string tachyon is thus believed to be accompanied by drastic modifications of background space-time. Moreover in the absence of an off-shell formulation for closed string theory, the construction of of tachyon potential has also remained a difficult problem. However in the recent years some progress has been made in the study of *localised* tachyon condensation. The bosonic theory has a closed string tachyon in its spectrum, but it resides in the bulk of the 26 dimensional space time and this makes the study of this condensation even more intractable. One can follow the condensation process in a controlled way if the effects of condensation are localised.

Type II theories on noncompact orbifolds, C^p/Z_N usually contains tachyons in its spectrum. The presence of tachyons indicate that supersymmetry is completely broken. The localised tachyons lie in the twisted sectors of the closed string spectrum and reside at the fixed point on the orbifold plane. Condensation of these localised tachyons has been first studied in [19]. In this paper the authors, Adams, Polchinski and Silverstein have collected evidences using D-brane probe techniques in support of their following conjecture.

- The condensation of localised closed string tachyons removes the singularity at the orbifold fixed point and is replaced by a smooth surface. The endpoint of localised closed string tachyon condensation is supersymmetric Type II theory on flat space.

Various works have followed using other methods in the analysis of this problem. Some of these works and techniques are reviewed in [21, 22]. The simplest of these orbifold models is the C/Z_N. In this case the mass of the lowest lying tachyons in the k-th twisted sector is given by, $M^2 = -\frac{2}{\alpha'}\left(1-\frac{k}{N}\right)$. It was further conjectured by Dabholkar [20],

- The height of the potential for the C/Z_N orbifold, where the minimum of the potential is flat space, is proportional to the deficit angle of the orbifold that is given by,

$$\Delta \sim 2\pi\left(1-\frac{1}{N}\right) \tag{1.2}$$

One can notice that in the $(N-k)$-th twisted sector the mass vanishes as $N \to \infty$. In this limit one can therefore construct a potential $V(|\phi|^2)$ for the tachyon and analyse the problem in the light of the above conjectures. We have studied this in chapter 4.

1.3 Organisation of this thesis

This thesis is divided into two parts. Part I of this thesis includes studies of noncommutativity in field and string theory. This is addressed in Chapters 2 and 3. In Chapter 2 we study the renormalisation of noncommutative field theories and extend it to all loops for the scalar $\lambda\phi^4$ theory with global $O(N)$ symmetry using the Wilsonian Renormalisation Group equations as recast by Polchinski. In Chapter 3 we study the problem of UV/IR mixing in noncommutative field theories from the point of view of string theory. In Part II of this thesis we study condensation of localised closed string tachyons. Specifically we study closed string tachyons on C/Z_N orbifold in the $N \to \infty$ limit in Chapter 4. We summarise the results of this thesis in Chapter 5.

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Part I

Noncommutativity in Field and String Theory

Chapter 2

Renormalisation in Noncommutative field theory

Noncommutative space-times and field theories defined on them have been studied extensively for the past few years mainly motivated from string theory [1, 2, 3]. Apart from this fact that these theories arise as low energy limits of string theory in a constant $B_{\mu\nu}$ background, their study as field theories in their own right is quite fascinating . For reviews on the subject see [4, 5, 6]. In this chapter we will consider noncommutative R^n as the background space defined by,

$$[X^i, X^j] = i\theta^{ij} \tag{2.1}$$

Where, θ^{ij} is a constant antisymmetric matrix and $\theta = 1/B$. B is the antisymmetric two form closed string field that is turned on in the background in which the open strings propagate. Let us now see how to construct quantum field theories on these (2.1) noncommutative spaces. The objects one uses to study quantum field theories are Green's functions. In noncommutative quantum field theory these functions are calculated by the use of operator symbols. An operator symbol as it is defined in ordinary Quantum Mechanics, is a function on the phase space which is constructed with a definite rule from a given operator. Different rules for construction produce different symbols. Sets of such functions with a product defined on them called the $*$-product, form algebras which are isomorphic to the initial operator algebra. Let us see how this $*$-product is to be defined.

We are interested in the transition,

$$\phi(\hat{X}) \rightarrow \phi_W(x) \tag{2.2}$$

where $\phi_W(x)$ is the operator (Weyl) symbol. The field theories would be constructed using these. This transition is defined as

$$\phi(\hat{X}) = \frac{1}{(2\pi)^{n/2}} \int d^n k e^{ik_i \hat{X}^i} \widetilde{\phi}_W(k) \tag{2.3}$$

where,

$$\widetilde{\phi}_W(k) = \frac{1}{(2\pi)^{n/2}} \int d^n x e^{ik_i x^i} \phi_W(x) \tag{2.4}$$

Equations (2.3,2.4) define the map from functions $\phi(\hat{X})$ in the operator algebra, to those of classical functions $\phi_W(x)$. The product of two operators is,

$$\begin{aligned}
\phi_1(\hat{X})\phi_2(\hat{X}) &= \frac{1}{(2\pi)^n} \int d^n k d^n p e^{ik_i \hat{X}^i} e^{ip_j \hat{X}^j} (\widetilde{\phi}_W)_1(k)(\widetilde{\phi}_W)_2(p) && (2.5)\\
&= \frac{1}{(2\pi)^n} \int d^n k d^n p e^{i(k_i+p_i)\hat{X}^i + \frac{1}{2}k_i\theta^{ij}p_j} (\widetilde{\phi}_W)_1(k)(\widetilde{\phi}_W)_2(p) \\
&= e^{\frac{i}{2}\theta_{ij}\partial^i_y \partial^j_z} (\phi_W)_1(y)(\phi_W)_2(z) \mid y = z = x && (2.6)\\
&= (\phi_W)_1(x) * (\phi_W)_2(x) && (2.7)
\end{aligned}$$

Where we have used, $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}-\frac{1}{2}[\hat{A},\hat{B}]}$. A *Lagrangian* defined on noncommutative space-time can be mapped to that on commutative space-time with all products replaced by $*$-products. Thus,

$$\mathcal{L}[\phi(\hat{X})] \rightarrow \mathcal{L}[\phi_W(x), *] \tag{2.8}$$

The partition functional for the noncommutative field theory is then defined by,

$$Z[J] = \int \mathcal{D}\phi_W \exp\left(-S[\phi_W, J, *]\right) \tag{2.9}$$

Quantum field theories on noncommutative space-times can thus be constructed by replacing the usual product of fields by the $*$-product. The algebra of functions on noncommutative R^n can be viewed as the algebra of ordinary functions on the usual R^n with a deformed $*$-product. The noncommutativity of space-times is thus transfered to the noncommutativity of the product of fields and then we can apply the usual perturbative expansions of the correlation functions. Note that the non-polynomial nature of the interactions make the theories highly nonlocal. Such deformation occur in string theory when describing the world-volume theory on a D-brane in background B-field. We will review this in Section 2.1.

Various interesting aspects of these theories have been studied rigorously recently [7]-[32]. The most important of these being the intriguing mixing of UV and IR divergences which is a direct consequence of the noncommutativity of the background space-time. Perturbative studies of these field theories carried out extensively revealed various nontrivial aspects arising from this transmutation of UV into IR divergences. One of the most important being the alteration of the conventional Wilsonian picture of Renormalisation group flows in the very low momentum domain. The UV renormalisability of these theories has been argued [8] and in some cases explicitly shown upto two loops [20]. The $\lambda\phi^4$ theory has been shown to be renormalisable as long as the external momenta, p for the n-point functions are such that $\Lambda^2 pop > 1$ [18], where Λ is the UV cutoff and pop is defined in (2.29). However given that we are only interested in the continuum limit, $\Lambda \to \infty$, the inequality is not satisfied only when p is restricted to the zero momentum value.

The possibility that these theories would ultimately be defined with an infrared cutoff still exists. It was shown [17] that phase transitions if possible can only occur at a finite momentum leading to a non-homogeneous phase. This could be a cure for the infrared problem.

Normally in commutative field theories the Renormalisation conditions required to absorb the infinities in the Green's functions of a particular configuration of fields, with momenta around some scale, leads to the infinities being absorbed from the Green's functions at all scales i.e. the functional dependence of the divergent n-

point functions on the UV cutoff Λ_0 is same at all values of external momenta. On the other hand, in noncommutative theories, due to coupling of the UV and IR sectors, the zero momentum limit of a particular configuration of fields is singular. We show that Green's functions scale with different coefficients of Λ for configurations which differ by the number of zero external momenta [16]. If the Renormalisation conditions are set at a point where all the external momenta are nonzero, then the bare couplings defined through this would have a different Λ_0 dependence from the case when the Renormalisation point consists of a number of zero momenta. This shows that with the former Renormalisation condition, the theory for all values of external momenta, $p \neq 0$ is renormalisable, while the latter leads to a non-renormalisable theory. We discuss this issue in the noncommutative ϕ^4 theory as well as in the Gross-Neveu model as shown in [11].

We next study the renormalisability of the globally $O(N)$ symmetric noncommutative scalar theory in its symmetric and its broken phases to all orders. First we review the same for the commutative case in the Renormalisation group approach [33][34]. The noncommutative theory is then proved to be renormalisable to all orders with an infrared cutoff. Keeping in mind the observations stated in the previous paragraph, we separate the sector with external momenta such that $\Lambda^2 pop > 1$ from that of $\Lambda^2 pop < 1$. In the latter case, the continuum limit restricts p only to zero. We will concentrate on the first sector and work with an Infrared Cutoff. Because of the presence of the infrared cutoff for the external momenta, we shall also formally introduce an IR cutoff for the internal loop momenta (Λ_{IR}). However we shall see that the in the loop computations the IR cutoff for the internal momenta is not necessary as there are no IR divergences in the loop integrals in this RG approach. IR divergences appear in the continuum limit as the zero momentum field configuration is approached from a nonzero momentum field configuration. Of course with an IR cutoff in the external momenta, p such that, $\Lambda^2 pop > 1$, there are no IR divergences in the theory. However IR divergences do appear in perturbation theory. This is illustrated by an example of a 2-point diagram computation.

It may seem that (Λ_{IR}) may be needed, so that the the canonical scaling of the relevant and irrelevant operators are not affected by the UV/IR transmutation. However this is not the case. This point will be clarified in Section 2.4.

We then demonstrate how the different scaling behaviours of the Green's functions for configurations having different set of zero momenta plays a crucial role when proving the renormalisability of the spontaneously broken phase. It was shown in [15] that the renormalisability of the broken symmetric phase to one loop could be proved by shifting to a phase where the vacuum was a non constant background field. This non constant vacuum acts as an infrared regulator.

This chapter is organised as follows. In Section 2.1, we review the bosonic string dynamics in background B-field and the appearance of noncommutative gauge theory as the low energy description of D-branes. In Section 2.2 we study the noncommutative ϕ^4 theory and show that the UV cutoff (Λ) dependence of the bare couplings are different depending on whether we set the Renormalisation conditions at $p = 0$ or $p \neq 0$. In Section 2.3 we analyse some more results for the noncommutative Gross-Neveu model along the same line as Section 2.2. In Section 2.4 we study Spontaneous Symmetry Breaking in noncommutative scalar theory with global $O(N)$ symmetry in its symmetric phase. We first review the one loop results from [12, 15]. Next we prove the renormalisability of the symmetric phase of the globally $O(N)$ symmetric noncommutative scalar theory to all orders, after reviewing the same for the commutative case. For the broken phase we demonstrate that, by going to a phase which is translationally non invariant i.e. by keeping the shift v as a non constant background field one is able to work with a infrared regulator so that the problem of different scaling behaviours of Green's functions for nonzero external momenta from that of the zero momentum case does not arise. With this infrared regulator we prove the renormalisability of the broken phase of the theory to all orders. We give our conclusions in Section 2.5.

2.1 Strings in Background B-Field and Noncommutative Field Theory

In this section we give a short review of open string dynamics in the presence of constant background B-field leading to noncommutative field theory on the world volume of a D-brane [3]. In the presence of a constant background B-field, the world sheet action is given by,

$$S_b = \frac{1}{4\pi\alpha'}\int_{\Sigma}[g_{MN}\partial_a X^M \partial^a X^N - 2\pi i\alpha' B_{MN}\epsilon^{ab}\partial_a X^M \partial_b X^N] \tag{2.10}$$

Consider a D_p brane extending in the directions 1 to p, such that, $B_{MN} \neq 0$ only for $M, N \leq p+1$ and $B_{MN} = 0$ for $M \leq p+1, N > p$. The equation of motion gives the following boundary condition,

$$g_{MN}\partial_n X^N + 2\pi i\alpha' B_{MN}\partial_t X^N|_{\partial\Sigma} = 0 \tag{2.11}$$

The world sheet propagator on the boundary of a disc satisfying this boundary condition is given by,

$$\mathcal{G}(y, y') = -\alpha' G^{MN}\ln(y-y')^2 + \frac{i}{2}\theta^{MN}\epsilon(y-y') \tag{2.12}$$

where, $\epsilon(\Delta y)$ is 1 for $\Delta y > 0$ and -1 for $\Delta y < 0$. G_{MN}, θ_{MN} are given by,

$$\begin{aligned} G^{MN} &= \left(\frac{1}{g+2\pi\alpha' B} g \frac{1}{g-2\pi\alpha' B}\right)^{MN} \\ G_{MN} &= g_{MN} - (2\pi\alpha')^2 (Bg^{-1}B)_{MN} \\ \theta^{MN} &= -(2\pi\alpha')^2 \left(\frac{1}{g+2\pi\alpha' B} B \frac{1}{g-2\pi\alpha' B}\right)^{MN} \end{aligned} \tag{2.13}$$

The relations above define the open string metric G in terms of the closed string metric g and B. This difference in the two metrics as seen by the open strings on the brane and the closed strings in the bulk plays an important role in the discussions in the next chapter. We next turn to to the low energy limit, $\alpha' \to 0$. A nontrivial low energy theory results from the following scaling.

$$\alpha' \sim \epsilon^{1/2} \to 0 \; ; \; g_{ij} \sim \epsilon \to 0 \tag{2.14}$$

where, i, j are the directions along the brane. This is the Seiberg-Witten (SW) limit that gives rise to noncommutative field theory on the brane. The relations in eqn(2.13), to the leading orders, in this limit reduce to,

$$\begin{aligned} G^{ij} &= -\frac{1}{(2\pi\alpha')^2}(\theta g\theta)^{ij} \; ; \; G_{ij} = -(2\pi\alpha')^2(Bg^{-1}B)_{ij} \\ \theta^{ij} &= \left(\frac{1}{B}\right)^{ij} \end{aligned} \tag{2.15}$$

for directions along the D_p brane. $G_{MN} = g_{MN}$ and $\theta = 0$ otherwise. It was shown that the tree-level action for the low energy effective field theory on the brane has the following form,

$$S_{YM} = -\frac{1}{g_{YM}^2}\int \sqrt{G}G^{kk'}G^{ll'}\,\mathrm{Tr}(\hat{F}_{kl} * \hat{F}_{k'l'}) \tag{2.16}$$

where the $*$-product is defined by,

$$f * g(x) = e^{\frac{i}{2}\theta^{ij}\partial_i^y\partial_j^z} f(y)g(z)\mid_{y=z=x} \tag{2.17}$$

and $\hat{F}_{kl}$ is the noncommutative field strength, which is related to the ordinary field strength, F_{kl} by the Seiberg-Witten map,

$$\hat{F}_{kl} = F_{kl} + \theta^{ij}(F_{ki}F_{lj} - A_i\partial_j F_{kl}) + \mathcal{O}(F^3) \tag{2.18}$$

and,

$$\hat{F}_{kl} = \partial_k\hat{A}_l - \partial_l\hat{A}_k - i\hat{A}_k * \hat{A}_l + i\hat{A}_l * \hat{A}_k \tag{2.19}$$

This form of the tree-level action is derived from the n-point tree-level open string correlators with gauge field vertices and then keeping the surviving terms in the low energy limit (2.14). The vertex is given by (3.5) and the boundary propagator is (2.12).

The coupling of the UV and the IR regimes, manifested in the nonplanar sector, is a very important and generic feature of these theories [8]. To see this, let us consider a noncommutative scalar ($\lambda\phi^4$) theory in four dimensions. The noncommutative theory is written with all the products of fields replaced by $*$-products. The nonplanar one-loop two-point amplitude has the following form,

$$\Gamma^2_{NP}(p) \sim \Lambda^2_{eff} - m^2 \ln\left(\frac{\Lambda^2_{eff}}{m^2}\right) \tag{2.20}$$

where, Λ is the UV cutoff and Λ^2_{eff} is defined in (2.29) (see eqn 2.28). The amplitude is finite in the UV but is IR divergent, though we had a massive theory to start with. Note that $\tilde{p}^2$ plays the role of $1/\Lambda^2$ in the continuum limit. It was suggested [8] that these IR divergent terms could arise by integrating out massless modes at high energies. The effective action containing the two point function can be written as,

$$S' = S(\Lambda) + \int d^4x \left[\frac{1}{2}\partial\chi o\partial\chi + \frac{1}{2}\Lambda^2(\partial o\partial\chi)^2 + i\frac{1}{4\pi}\lambda\chi\phi\right] \tag{2.21}$$

$S(\Lambda)$ is the effective action for the cutoff field theory and χ is a massless field. Integrating out χ gives the quadratic piece in the effective action of the original theory in the continuum limit. It was further noted that both the quadratic and the log terms of eqn(2.20) can be recovered through massless tree-level exchanges if these modes are allowed to propagate in 0 and 2 extra dimensions transverse to the brane respectively [8]. This is quite like the open string one loop divergence which is reinterpreted as IR divergence coming from massless closed string exchange.

A similar structure arises for the nonplanar two point function for the gauge boson in noncommutative gauge theories,

$$\Pi^{ij}(p) \sim N_1[G^{ij}G^{kl} - G^{ik}G^{jl}]p_k p_l \ln(p^2\tilde{p}^2) + N_2\frac{\tilde{p}^i\tilde{p}^j}{\tilde{p}^4} \tag{2.22}$$

N_1 and N_2 depends on the matter content of the theory. For some early works on noncommutative gauge theories see [12]. The effective action with the two point function (2.22) is not gauge invariant. To write down a gauge invariant effective action one needs to introduce open Wilson lines [13]

$$W_C(p) = \int d^4xP * \exp\left(ig\int_C d\sigma\partial_\sigma y^i A_i(x+y(\sigma)\right) * e^{ipx} \tag{2.23}$$

The curve C is parametrised by $y^i(\sigma)$, where $0 \leq \sigma \leq 1$ such that, $y^i(1) - y^i(0) = \tilde{p}^i$. Correlators of Wilson lines in noncommutative gauge theories have been studied by various authors [14]. The terms in (2.22) are the leading terms in the expansion of the two point function for the open Wilson line. A crucial point to be noted is that for supersymmetric theories, N_2, the coefficient of the second term, which is allowed by the noncommutative gauge invariance vanishes [15]. Also see [16] for an elaborate discussion. An observation on the arising of tachyon in the closed string theory in the bulk and the non vanishing of N_2 with a negative sign was made in [20]. Thus when the closed string theory is unstable due to the presence of tachyons, the two point function in noncommutative gauge theory also diverges with a negative sign for low momenta. Various attempts have been made, along the lines as discussed above, to recover the nonplanar IR divergent terms from tree-level closed string exchanges. We shall address and try to resolve this issue in the next chapter.

2.2 UV/IR mixing and UV renormalisability

After the brief review of noncommutative field theory from the point of view of string theory we now return to the issue of renormalisability of these theories purely from the point of view of field theory. In this section we show that the one loop Green's functions defines two different Renormalisation conditions depending on whether we set $p = 0$ right in the beginning or approach this limit with a nonsingular, $p \neq 0$ configuration. We study the noncommutative ϕ^4 theory to show this. For an introduction to noncommutative scalar theory see [8].

The Lagrangian for the theory is,

$$\mathcal{L}_E = -[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi * \phi * \phi * \phi] \tag{2.24}$$

where,

$$\phi_1 * \phi_2 = e^{\frac{i}{2}\partial_\mu^y \theta^{\mu\nu} \partial_\nu^z} \phi_1(y)\phi_2(z)\big|_{y=z=x} \tag{2.25}$$

and θ is an antisymmetric matrix. The propagator for the theory, is same as that of the commutative theory. Only the interaction term has a nontrivial momentum dependence. The interaction vertex is given by,

$$\begin{aligned} -2\lambda V(\mathrm{p}) = -2\lambda[\cos(p_1 \wedge p_2)\cos(p_3 \wedge p_4) \; &+ \; \cos(p_1 \wedge p_3) cos(p_2 \wedge p_4) \\ &+ \; \cos(p_1 \wedge p_4)\cos(p_2 \wedge p_3)] \end{aligned} \tag{2.26}$$

The only divergent functions for the theory are the two point and the four point functions. The diagrams corresponding to the one loop contributions to these functions are shown in Figures 2.1 and 2.2.

Figure 2.1: One loop contribution to the two point function

$$\Gamma^2 = -[p^2 + m^2] - \lambda \int \frac{d^4k}{(2\pi)^4} \frac{2 + \cos(p \wedge k)}{k^2 + m^2} \tag{2.27}$$

The cos term inside the integral regulates the second part of the integral and is finite for $p \neq 0$.

$$\begin{aligned} \Gamma^2 = -[p^2 + m^2] &- \frac{\lambda}{8\pi^2}[\Lambda^2 - m^2 ln(\frac{\Lambda^2}{m^2}) + O(1)] \\ &- \frac{\lambda}{16\pi^2}[\Lambda^2_{eff}(p) - m^2 ln(\frac{\Lambda^2_{eff}(p)}{m^2}) + O(1)] \end{aligned} \tag{2.28}$$

where,

$$\Lambda^2_{eff}(p) = \frac{1}{\frac{1}{\Lambda^2} + pop}$$
$$pop = -\frac{p^\mu \theta^2_{\mu\nu} p^\nu}{4} \tag{2.29}$$

Λ is the UV cutoff. We shall call terms containing $\Lambda^2_{eff}(p)$ as nonplanar terms.

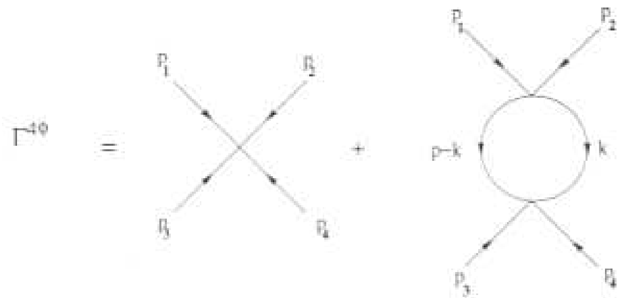

Figure 2.2: One loop contribution to the four point function

$$\Gamma^4 = -2\lambda V(\mathbf{p}) + 2\lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{F(p_1, p_2, p_3, p_4, p, k)}{(k^2+m^2)[(p-k)^2+m^2]} + \text{ s and u channels} \tag{2.30}$$

where $F(p_1, p_2, p_3, p_4, p, k)$ is a function of terms containing "cos" of the external momenta.

$$\Gamma^4 = -2\lambda V(\mathbf{p}) + 4\lambda^2 V(\mathbf{p}) \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+m^2)[(p-k)^2+m^2]} + NP \tag{2.31}$$

where NP are the Nonplanar terms. These terms would give rise to IR divergences as the external momenta goes to zero. The UV divergent piece for this four point amplitude is given by,

$$\Gamma^4 \sim \frac{\lambda^2}{4\pi^2} V(\mathbf{p}) ln(\frac{\Lambda^2}{m^2}) \tag{2.32}$$

The two and the four point contributions to the effective action to one loop is,

$$
\begin{aligned}
-S_{eff} &= \frac{1}{2!}\int dp\phi(p)\phi(-p)[p^2+m^2+\frac{\lambda}{8\pi^2}\Lambda^2-\frac{\lambda}{8\pi^2}m^2 ln(\frac{\Lambda^2}{m^2})+NP] \\
&+ \frac{1}{4!}\int dp_1dp_2dp_3dp_4[P+NP]\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\delta(\sum p_i) \quad (2.33)
\end{aligned}
$$

where, P is the planar term from the four point amplitude given by,

$$P = V(\mathbf{p})\{2\lambda - \frac{\lambda^2}{4\pi^2}ln(\frac{\Lambda^2}{m^2})\} \quad (2.34)$$

The renormalised parameters may now be defined as,

$$
\begin{aligned}
m_R^2 &= m^2+\frac{\lambda}{8\pi^2}[\Lambda^2-m^2 ln(\frac{\Lambda^2}{m^2})] \\
\lambda_R &= \lambda-\frac{\lambda^2}{8\pi^2}ln(\frac{\Lambda^2}{m^2}) \quad (2.35)
\end{aligned}
$$

As the zero momentum limit is approached, the NP terms in equation (2.30) give rise to IR divergences, however the Renormalisation conditions in equation (2.35) lead to a UV renormalisable quantum theory at one loop.

Now let us consider the case where the effective action at the zero momentum field configuration is defined by [35],

$$V_{eff} = \sum_{n=1}^{\infty}\frac{1}{n!}\Gamma^n(0,0,...,0)\phi^n \quad (2.36)$$

We use this to define $\lambda(\Lambda)$. In this case at one loop level there are no nonplanar diagrams and the potential to one loop is exactly equal to the commutative theory. The external momenta are all put to zero before all loop calculations. To one loop the effective potential is given by,

$$
\begin{aligned}
V_{eff} &= \frac{1}{2}m^2\phi^2+\frac{\lambda}{4}\phi^4+\frac{1}{2}\int^{\Lambda}\frac{d^4k}{(2\pi)^4}ln(1+\frac{3\lambda\phi^2}{k^2+m^2}) \quad (2.37) \\
&= \frac{1}{2}m^2\phi^2+\frac{\lambda}{4}\phi^4+\frac{3\lambda\phi^2}{32\pi^2}[\Lambda^2-m^2 ln(\frac{\Lambda^2}{m^2})]+\frac{9\lambda^2}{64\pi^2}\phi^4 ln(\frac{3\lambda\phi^2}{\Lambda^2}) \quad (2.38)
\end{aligned}
$$

where in the final expression we have dropped terms which have negative powers or are independent of Λ.

The renormalised quantities would now be defined by,

$$\frac{d^2V}{d\phi^2}|_{\phi=0} = m_R^2 \tag{2.39}$$

$$\frac{d^4V}{d\phi^4}|_{\phi=\phi_0} = 6\lambda_R \tag{2.40}$$

These lead to,

$$\begin{aligned} m_R^2 &= m^2 + \frac{3\lambda}{16\pi^2}[\Lambda^2 - m^2 ln(\frac{\Lambda^2}{m^2})] \\ \lambda_R &= \lambda - \frac{9\lambda^2}{16\pi^2} ln(\frac{\Lambda^2}{3\phi_0^2}) \end{aligned} \tag{2.41}$$

From equations (2.35) and (2.41) it is clear that the Renormalisation conditions defined by equation (2.41) would not lead to a renormalisable noncommutative theory for non zero external momenta. As noted before, the configuration of fields with $p = 0$ is singular and the two different Renormalisation conditions (2.35), (2.41) occur as a consequence of setting $p = 0$ right in the beginning or of approaching this configuration as a limit $p \to 0$. This is a generic feature of noncommutative theories. The origin of this is the transmutation of UV divergences into the IR divergences.

In terms of the Renormalisation group flows, equations (2.35), (2.41) states that at the one loop level the relevant (relevant plus marginal) coupling λ scales with respect to Λ with different coefficients. This means that the functional dependence of the bare coupling on the UV cutoff Λ are different in the two cases. We shall see this in Section 2.4, where the relevant couplings would scale with different coefficients of Λ depending on whether or not the external momenta are such that $\Lambda^2 p\circ p << 1$.

2.3 The noncommutative Gross-Neveu model

In this section we review some of the results of the noncommutative Gross-Neveu model [11] which are along the same line as those of the previous section.

The Lagrangian for the noncommutative Gross-Neveu model is,

$$\mathcal{L}_E = -[\frac{1}{2}\bar{\psi}^i\gamma^\mu\partial_\mu\psi^i + \frac{\lambda}{8N}\bar{\psi}^i * \psi^i * \bar{\psi}^j * \psi^j] \tag{2.42}$$

Where ψ is a 2-component spinor and γ^μ are 2×2 Dirac matrices. To evaluate the large N limit of the effective action it is helpful to introduce an auxiliary field σ, so that,

$$\mathcal{L}_E = -[\frac{1}{2}\bar{\psi}^i\gamma^\mu\partial_\mu\psi^i - \frac{8N}{\lambda}\sigma^2 - 2\sigma * \bar{\psi}^i * \psi^i] \tag{2.43}$$

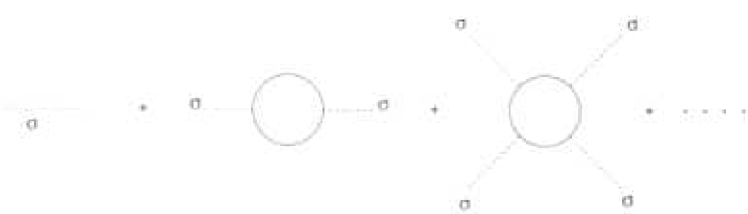

Figure 2.3: Diagrams contributing to lowest order in $1/N$

We now evaluate the effective action at the zero momentum field configuration as defined by equation (2.36). The processes with only σ field on the external legs would contribute to the lowest order in $\frac{1}{N}$ [36]. These diagrams contributing to this lowest order are shown in Figure 2.3.

$$\begin{aligned} V_{eff} &= \frac{8N}{\lambda}\sigma^2 - N\sum_{n=1}^{\infty}\int\frac{d^2k}{(2\pi)^2}\frac{1}{2n}(\frac{4\sigma^2}{k^2})^n & (2.44)\\ &= \frac{8N}{\lambda}\sigma^2 - \frac{N\sigma^2}{2\pi}[ln(\frac{\Lambda^2}{4\sigma^2}) + O(1)] & (2.45)\end{aligned}$$

The Renormalisation condition,

$$\frac{\partial^2 V_{eff}}{\partial\sigma^2}|_{\sigma^2=\sigma_0^2} = 1 \tag{2.46}$$

defines the renormalised coupling λ_R as,

$$\frac{1}{\lambda_R} = \frac{1}{\lambda} - \frac{1}{16\pi} ln(\frac{\Lambda^2}{4\sigma_0^2}) \tag{2.47}$$

We now write down the effective action for nonzero external momenta in the large N limit. Only the two point function is divergent. To extract the Renormalisation condition it is sufficient to evaluate the effective action with contributions upto the two point function only.

$$S_{eff} = \frac{1}{2}\int d^2p\sigma(p)\Gamma^{\sigma\sigma}\sigma(-p) \tag{2.48}$$

where,

$$\Gamma^{\sigma\sigma} = \frac{8N}{\lambda} - N\int \frac{d^2k}{(2\pi)^2}\frac{k.(p+k)}{k^2(k+p)^2} + NP \tag{2.49}$$

NP is the nonplanar term. The renormalised coupling will now be defined as,

$$\frac{1}{\lambda_R} = \frac{1}{\lambda} - \frac{1}{32\pi} ln(\frac{\Lambda^2}{p^2}) \tag{2.50}$$

The Renormalisation condition, equation (2.50) gives a UV renormalisable quantum theory for any nonzero value of external momenta, similar to the results obtained in the previous section. If one defines the renormalised coupling as equation (2.47), the quantum theory at nonzero external momenta is non-renormalisable. These two Renormalisation conditions correspond to two different bare theories as argued at the end of Section 2.2.

The ground state of the theory will be defined by a particular configuration of fields. If it is defined with zero external momenta, we have noted that this is a very singular field configuration and Green's functions in this configuration has a different Λ dependence from the nonzero momenta field configurations. If the Renormalisation conditions are set at zero momentum the theory is non-renormalisable. In the next section we shall see how this plays a crucial role in deciding the renormalisability of the Broken Phase in a spontaneously broken global symmetric theory.

2.4 Spontaneous Symmetry Breaking in Non-commutative scalar theory

In the previous two sections we have seen that due to the singular behaviour of the IR limit in noncommutative field theories, the Green's functions have different Λ dependences for nonzero and zero momentum field configurations. In this section we show how this plays a crucial role when proving renormalisability of the broken phase of a spontaneously broken globally $O(N)$ symmetric theory. We first review our one loop results [15] and then in the later part of this section we shall prove following [34], the renormalisability of the symmetric phase as long as the external momenta $p \neq 0$ such that $\Lambda^2 pop > 1$. Formally with an IR cutoff in the external momenta, one should also introduce an IR cutoff for the internal loop momenta (Λ_{IR}). The presence of this IR cutoff in the loop computations would be implicitly assumed. This point is elaborated in Section 2.4.3. The relevant couplings for the zero momentum configuration of fields scale with different coefficients of the UV cutoff from the $p \neq 0$ configurations. If the Renormalisation conditions are set in these two momentum regimes, the bare couplings will have different Λ dependences. At the end of the section we shall show the renormalisability of the broken phase of the theory to all orders in terms of the Renormalisation group.

2.4.1 One loop analysis

We consider a scalar field theory with global $O(2)$ symmetry in its symmetric phase in 4 dimensions. We refer the reader to [15] for the details of calculations.

The Lagrangian for the theory is[1],

$$\begin{aligned} L_S &= -[\frac{1}{2}(\partial_\mu\phi^i)^2 - \frac{1}{2}\mu^2(\phi^i)^2 + \frac{\lambda}{4}\phi^i * \phi^i * \phi^j * \phi^j] \\ i,j &= 1,2 \end{aligned} \tag{2.51}$$

Here $\mu^2 > 0$, so that at the tree-level the theory undergoes SSB. The global O(2) symmetry of the quantum theory implies that the green's functions satisfy the following set of Ward identities in the symmetric phase,

$$\begin{aligned} \frac{\delta^2\Gamma}{\delta\phi_1^2}\Big|_{\phi_1=\phi_2=0} &= \frac{\delta^2\Gamma}{\delta\phi_2^2}\Big|_{\phi_1=\phi_2=0} \\ \frac{\delta^4\Gamma}{\delta\phi_1^4}\Big|_{\phi_1=\phi_2=0} &= 3\frac{\delta^4\Gamma}{\delta\phi_1^2\delta\phi_2^2}\Big|_{\phi_1=\phi_2=0} \end{aligned} \tag{2.52}$$

and in the broken phase which is defined by shifting the fields, $\phi_1 = \sigma + v$ and $\phi_2 = \pi$,

$$v\frac{\delta^2\Gamma}{\delta\pi^2}\Big|_{\sigma=\pi=0} = \frac{\delta\Gamma}{\delta\sigma}\Big|_{\sigma=\pi=0} \tag{2.53}$$

where the ϕ_1 field has been shifted by a constant amount v which fixes the vacuum for the broken phase.

The set of Ward identities, equation (2.52), can be verified to one loop [15], showing that the quantum theory is symmetric. However care must be taken in defining the broken phase so that equation (2.53) holds.

---

[1]Note that in the noncommutative theory, there are various possible inequivalent orderings of the fields for the quartic term. One of these terms has been chosen as an example. The proof of the renormalisability of the globally $O(N)$ symmetric theory, in its symmetric as well as in its broken phases, to all orders, outlined in the latter part of this section remains unaltered for any such $O(N)$ symmetric quartic term.

If one naively shifts to a translationally invariant vacuum, the Ward identity, equation (2.53) only holds when the order of the continuum and the IR limits is such that the IR divergences on the RHS of (2.53) are transmuted to UV divergences i.e. the vacuum is defined with $p = 0$ before all loop calculations. It is important to note here that shifting fields by a constant amount v at the tree-level itself causes the σ-tadpole amplitude to scale with respect to Λ in the same way as the second case in Section 2.2, where as the $\Gamma^{\pi\pi}$ amplitude has the same Λ dependence when the external momenta are put to zero before all loop computations. This makes the broken phase non-renormalisable with the same number of counter terms as the symmetric phase. This point will be elaborated at the end of this section. However in order to define a UV renormalisable theory for all values of p we must work with an infrared cutoff and remove the cutoff only after all loop computations as the former case in Section 2.2. To do so one has to shift the fields by v which is not a constant. v may be set to a constant after all loop calculations. This would mean that the singular field configuration, defined as the vacuum, is approached as a limit. One can also leave v as a non-constant leading to a translationally non-invariant vacuum. This later case was studied in [17] where phase transitions when a finite number of momentum modes condense is studied. In the present chapter a non-constant v acts as an infrared regulator and help us to avoid problems as noted earlier.

The Ward identity in the case where v is not a constant will now be written as,

$$\int \frac{d^4p}{(2\pi)^4} v(-p) \frac{\delta^2\Gamma}{\delta\pi(p_1)\delta\pi(p)}|_{\sigma=\pi=0} = \frac{\delta\Gamma}{\delta\sigma(p_1)}|_{\sigma=\pi=0} \tag{2.54}$$

Explicit computations in this case shows that,

$$\begin{aligned}\Gamma^\sigma = v_0\mu^2 - \lambda v_0^3 \;&+\; v_0[-3\lambda(\Lambda^2 + \mu^2 ln(\frac{\Lambda^2}{-\mu^2})) + IR + F] + \\ &+\; v_0^3[\frac{5}{2}ln(\frac{\Lambda^2}{-\mu^2}) + IR + F]\end{aligned} \tag{2.55}$$

$$\Gamma^{\pi\pi} = -p^2 + \mu^2 - \lambda v_0^2 \;+\; [-3\lambda(\Lambda^2 + \mu^2 ln(\frac{\Lambda^2}{-\mu^2})) + IR + F] +$$
$$+\; v_0^2[\frac{5}{2}ln(\frac{\Lambda^2}{-\mu^2}) + IR + F] \tag{2.56}$$

Where IR are the infrared divergent terms and F are the finite terms. It can be seen that the UV divergence structure of the two amplitudes are exactly as the Ward identity (2.53). The IR divergences appear when we go to a translationally invariant vacuum i.e. v is set to a constant v_0 after all loop calculations.

2.4.2 Renormalisability to all orders : Review of The commutative case

We now consider the commutative globally symmetric $O(N)$ scalar theory in its symmetric phase and review its renormalisability to all orders following Polchinski [34]. We discuss here the set up of the RG equations and merely state the results, which will be necessary for the later parts of this section. The reader may refer to [34] for more elaborate details and proofs.

$$S(\phi) = \int \frac{d^4p}{(2\pi)^4}[-\frac{1}{2}\phi^a(p)\phi^a(-p)(p^2-\mu^2)K^{-1}(\frac{p^2}{\Lambda^2})] + L_{int}(\phi)$$
$$L_{int}(\phi) = \int d^4x[-\frac{1}{2}\rho_1^0(\phi^a(x))^2 - \frac{1}{2}\rho_2^0(\partial_\mu\phi^a(x))^2 - \frac{1}{4}\rho_3^0(\phi(x)^a\phi(x)^a)^2] \tag{2.57}$$

where, $K(\frac{p^2}{\Lambda^2})$ has a value of 1 for $p^2 < \Lambda^2$ and vanishes rapidly at infinity. ρ_a^0 are the bare couplings defined at an UV cutoff scale Λ_0.

The generating functional for the theory with the cutoff, Λ may be written as

$$Z[J,\Lambda] = \int D\phi \exp[\int \frac{d^4p}{(2\pi)^4}[\;-\;\frac{1}{2}\phi^a(p)\phi^a(-p)(p^2-\mu^2)K^{-1}(\frac{p^2}{\Lambda^2})$$
$$+\; J^a(p)\phi^a(-p)] + L_{int}(\phi)] \tag{2.58}$$

$$\Lambda\frac{dZ[J,\Lambda]}{d\Lambda} = \int \mathcal{D}\phi[\int \frac{d^4p}{(2\pi)^4}[\; - \; \frac{1}{2}\phi^a(p)\phi^a(-p)(p^2-\mu^2)\Lambda\frac{\partial K^{-1}}{\partial\Lambda}(\frac{p^2}{\Lambda^2}) + \; J^a(p)\phi^a(-p)] + \Lambda\frac{\partial L_{int}(\phi)}{\partial\Lambda}]\exp(S(\phi)) \quad (2.59)$$

The RHS of equation (2.59) vanishes if L varies as,

$$\Lambda\frac{\partial L}{\partial\Lambda} = - \frac{1}{2}\int d^4p(2\pi)^4(p^2-\mu^2)^{-1}\Lambda\frac{\partial K}{\partial\Lambda} [\; \frac{\partial L}{\partial\phi^a(-p)}\frac{\partial L}{\partial\phi^a(p)} + \frac{\partial^2 L}{\partial\phi^a(-p)\partial\phi^a(p)}] \quad (2.60)$$

L can now be expanded in terms of its Fourier modes. The global $O(N)$ symmetry of the quantum theory implies that we can arrange the expansion as follows.

$$L = \sum_{n=1}^{\infty}\frac{1}{2n!}\int\prod_{i=1}^{n}\prod_{j=n+1}^{2n} d^4p_i d^4p_j\phi^a(p_i)\phi^a(p_j)L_{2n}(p_1...p_{2n},\Lambda)\delta^4(\sum_{i,j}p_i+p_j) \quad (2.61)$$

Define the relevant operators as,

$$\rho_1(\Lambda) = -L_2(p,-p,\Lambda)\big|_{p^2=p_0^2} \quad (2.62)$$

$$\rho_2(\Lambda) = -\frac{\partial^2}{\partial p^2}L_2(p,-p,\Lambda)\big|_{p^2=p_0^2} \quad (2.63)$$

$$\rho_3(\Lambda) = -L_4(p_1,p_2,p_3,p_4,\Lambda)\big|_{p=\bar{p}} \quad (2.64)$$

Now, construct $V(\Lambda)$ such that,

$$V(\Lambda) = \Lambda_0\frac{\partial L(\Lambda)}{\partial\Lambda_0} - \sum_{a,b}\frac{\partial L(\Lambda)}{\partial\rho_a^0}\frac{\partial\rho_a^0}{\partial\rho_b(\Lambda)}\Lambda_0\frac{\partial\rho_b(\Lambda)}{\partial\Lambda_0} \quad (2.65)$$

where, a, b runs from 1 to 3. To prove that the theory is renormalisable or in other words to show that, for the low energy theory to be finite one has to tune a finite number of parameters, $V(\Lambda)$ must be shown to vanish at the low energy limit, $\Lambda/\Lambda_0 \to 0$.

Expanding the following quantities similar to equation (2.61),

$$\begin{aligned}\frac{\partial L(\Lambda)}{\partial \rho_b(\Lambda)} &= \sum_a \frac{\partial L(\Lambda)}{\partial \rho_a^0}\frac{\partial \rho_a^0}{\partial \rho_b(\Lambda)} \\ &= \sum_{n=1}^{\infty} \frac{\Lambda^{4-2n-2\delta_{b1}}}{2n!} \int \frac{\prod_{i=1}^{n}\prod_{j=n+1}^{2n} d^4p_i d^4p_j}{(2\pi)^{8n-4}} \phi^{\alpha}(p_i)\phi^{\alpha}(p_j) \times \\ &\times \; B_{b,2n}(p_1...p_{2n},\Lambda)\delta^4(\sum_{i,j} p_i + p_j) \end{aligned} \tag{2.66}$$

$$\begin{aligned} V(\Lambda) &= \sum_{n=1}^{\infty} \frac{\Lambda^{4-2n}}{2n!} \int \frac{\prod_{i=1}^{n}\prod_{j=n+1}^{2n} d^4p_i d^4p_j}{(2\pi)^{8n-4}} \phi^{\alpha}(p_i)\phi^{\alpha}(p_j) \times \\ &\times \; V_{2n}(p_1...p_{2n},\Lambda)\delta^4(\sum_{i,j} p_i + p_j) \end{aligned} \tag{2.67}$$

Now defining,

$$Q(p,\Lambda,\mu^2) = \frac{1}{(p^2-\mu^2)}\Lambda^3 \frac{\partial K(\frac{p^2}{\Lambda^2})}{\partial \Lambda} \tag{2.68}$$

one arrives at the RG equations for L, B, V, shown in the appendix, equations (A2.1),(A2.3),(A2.5) where,

$$\parallel f(p_1,...,p_{2n},\Lambda) \parallel = \max_{p_i^2 < c\Lambda^2} |f(p_1,...,p_{2n},\Lambda)| \tag{2.69}$$

so that,

$$\parallel \int \frac{d^4p}{(2\pi)^4} Q(p,\Lambda) L_{2n}(p_1,...,p_{2n-2},p,-p,\Lambda) \parallel < C\Lambda^4 \parallel L_{2n}(\Lambda) \parallel \tag{2.70}$$

where, c and C are constants independent of Λ. Now define the following initial conditions for the couplings,

$$\begin{aligned} \rho_1(\Lambda_R,\Lambda_0,\rho^0) &= 0 \\ \rho_2(\Lambda_R,\Lambda_0,\rho^0) &= 0 \\ \rho_3(\Lambda_R,\Lambda_0,\rho^0) &= 6\lambda_R \end{aligned} \tag{2.71}$$

Note that because of the $O(N)$ symmetry of the quantum theory it is sufficient to set boundary conditions as above. The irrelevant couplings are set to vanish at Λ_0. Perturbative renormalisability now means that order by order in λ_R the following limit exists,

$$\lim_{\Lambda_0\to\infty} \bar{L}(\phi,\Lambda_R,\lambda_R,\Lambda_0) = \bar{L}(\phi,\Lambda_R,\lambda_R,\infty) \tag{2.72}$$

where,

$$\bar{L}(\phi,\Lambda_R,\lambda_R,\Lambda_0) = L(\phi,\Lambda_R,\Lambda_0,\rho^0(\Lambda_R,\lambda_R,\Lambda_0)) \tag{2.73}$$

and specifically to rth order in λ_R,

$$\begin{aligned} \| \bar{L}^{(r)}_{2n}(\Lambda_R,\Lambda_0) - \bar{L}^{(r)}_{2n}(\Lambda_R \quad , \quad \infty) \| & \\ \leq & \ \Lambda_R^{4-2n}(\frac{\Lambda_R}{\Lambda_0})^2 P^{2r-n} ln(\frac{\Lambda_R}{\Lambda_0}), r+1-n \geq 0 \\ = & \ 0, r+1-n < 0 \end{aligned} \tag{2.74}$$

where, P^{2r-n} is a polynomial of degree $2r-n$. We do not include the proof here, for which the reader may refer to the original paper [34], but merely state that it follows by induction from the following assertions.

(i) At order r in Λ_R,

$$\begin{aligned} \| \partial^{\mu_1}_{i_1 j_1} ... \partial^{\mu_p}_{i_p j_p} A^{(r)}_{2n}(p_i,...,p_{2n},\Lambda) \| & \leq \Lambda^{-p} P^{2r-n} ln(\frac{\Lambda_0}{\Lambda_R}), r+1-n \geq 0 \\ & = 0, r+1-n < 0 \end{aligned} \tag{2.75}$$

where, $A_{2n} = \Lambda^{2n-4} L_{2n}$ and,

$$\left(\frac{\partial}{\partial p^\mu_i} - \frac{\partial}{\partial p^\mu_j}\right) A_{2n} = \partial^\mu_{i,j} A_{2n} \tag{2.76}$$

(ii) At order r in Λ_R,

$$\begin{aligned} \| \partial^{\mu_1}_{i_1 j_1} ... \partial^{\mu_p}_{i_p j_p} B^{(r)}_{b,2n}(p_i,...,p_{2n},\Lambda) \| & \leq \Lambda^{-p} P^{2r-n+1+\delta_{b3}} ln(\frac{\Lambda_0}{\Lambda_R}), r+2-n \geq 0 \\ & = 0, r+2-n < 0 \end{aligned} \tag{2.77}$$

(iii) At order r in Λ_R,

$$\begin{aligned} \| \partial^{\mu_1}_{i_1,j_1}...\partial^{\mu_p}_{i_p,j_p} V^{(r)}_{2n}(p_i,...,p_{2n},\Lambda) \| &\leq \Lambda^{-p}(\frac{\Lambda}{\Lambda_0})^2 P^{2r-n} ln(\frac{\Lambda_0}{\Lambda_R}), r+1-n \geq 0 \\ &= 0, r+1-n < 0 \end{aligned} \quad (2.78)$$

We end this part of the section by writing down the tree-level and the one loop forms for the relevant and irrelevant parts of L. We use (A2.1) to obtain the perturbative expansion for L_{2n} order by order in λ_R.

Expanding the two and four point parts of L,

$$\begin{aligned} L_2(p,-p,\Lambda) &= L_2(p,-p,\Lambda)\big|_{p^2=p_0^2} + (p^2-p_0^2)\frac{\partial}{\partial p^2}L_2(p,-p,\Lambda)\big|_{p^2=p_0^2} + \Delta L_2 \\ &= -\rho_1(\Lambda) - (p^2-p_0^2)\rho_2(\Lambda) + \Delta L_2 \end{aligned} \quad (2.79)$$

$$\begin{aligned} L_4(p_i,\Lambda) &= L_4(p_i,\Lambda)\big|_{p_i=\bar{p}_i} + \Delta L_4 \\ &= -\rho_3(\Lambda) + \Delta L_4 \end{aligned} \quad (2.80)$$

Solutions for equation (A2.1) with boundary conditions (2.71) for zero and one loop gives,

$$\rho_1^{(0)} = 0, \rho_2^{(0)} = 0, \rho_3^{(0)} = 6\lambda_R \quad (2.81)$$

$$\Delta L_2^{(0)} = 0, \Delta L_4^{(0)} = 0 \quad (2.82)$$

$$(2.83)$$

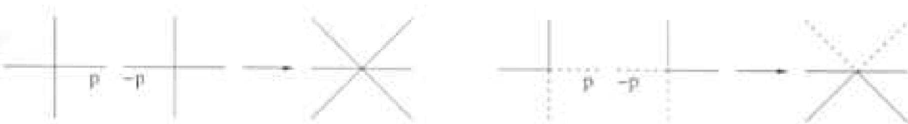

Figure 2.4: L_6 vertex from L_4 vertices

The $L_6^{(0)}$ vertex is obtained from two $L_4^{(0)}$ vertices as shown in Figure 2.4.

$$
\begin{aligned}
L_6^{(0)} = \int_{\Lambda}^{\Lambda_0} \frac{d\Lambda'}{\Lambda'^3} Q & (\Lambda', P) L_4^{(0)}(p_1, p_2, p_3, P, \Lambda') L_4^{(0)}(p_4, p_5, p_6, -P, \Lambda') \\
& + \frac{1}{2}\begin{pmatrix} 6 \\ 3 \end{pmatrix} - 1 \text{ permutations for all same external fields or,} \\
& + \frac{1}{2}\begin{pmatrix} 4 \\ 2 \end{pmatrix} \text{ permutations for 2 different external fields}
\end{aligned} \quad (2.84)
$$

where, $P = p_1 + p_2 + p_3$

Contracting two of the legs of the $L_4^{(0)}$ vertices we get the one loop L_2 amplitude, Figure 2.5. The weights of N coming from the global $O(N)$ symmetry are indicated in brackets in the figure.

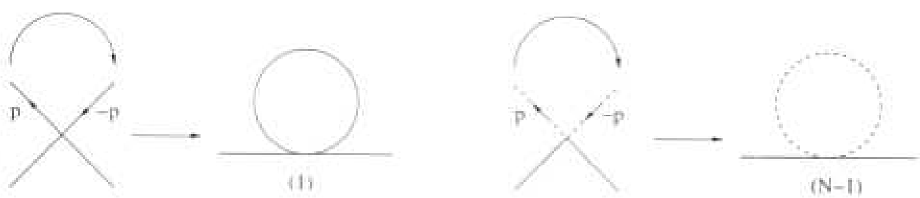

Figure 2.5: One loop L_2 from L_4 vertex

$$
\begin{aligned}
\rho_1^{(1)} &= \frac{1}{2}\int \frac{d^4p}{(2\pi)^4} \int_{\Lambda_R}^{\Lambda} \frac{d\Lambda'}{\Lambda'^3} Q(\Lambda', p) L_4^{(0)}(p, -p, q, -q, \Lambda')\Big|_{q^2=p_0^2} \\
&= (N+2)\frac{\lambda_R}{16\pi^2}\Lambda^2
\end{aligned} \quad (2.85)
$$

where we have only kept the Λ dependent term.

$$\rho_2^{(1)} = 0, \Delta L_2^{(1)} = 0 \quad (2.86)$$

(2.87)

Contracting two legs of the $L_6^{(0)}$ vertex, Figure 2.6, we obtain the one loop L_4 amplitude. The Λ dependent part of $\rho_3^{(1)}$ comes from this term only.

$$\rho_3^{(1)} \sim -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4} \int_{\Lambda_R}^{\Lambda} \frac{d\Lambda'}{\Lambda'^3} Q(p, \Lambda') L_6^{(0)}(p_1, p_2, p_3, p_4, p, -p, \Lambda')\Big|_{p_i=\bar{p}_i} \quad (2.88)$$

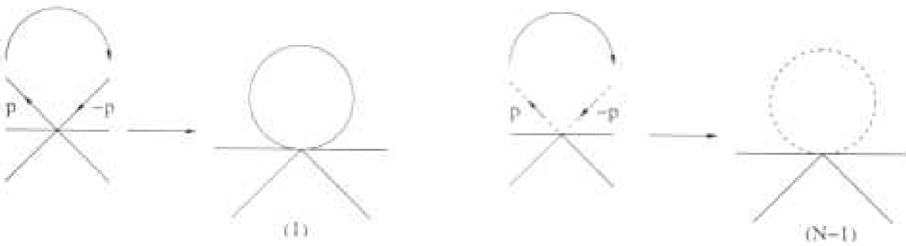

Figure 2.6: One loop L_4 from L_6 vertex

In the large Λ limit,

$$\rho_3^{(1)} \sim -(N+8)\frac{3\lambda_R^2}{8\pi^2} ln(\frac{\Lambda^2}{-\mu^2}) \tag{2.89}$$

$$\Delta L_4^{(1)} \sim \frac{f(\bar{p}_i)}{\Lambda} \tag{2.90}$$

$f(\bar{p}_i)$ is a function of external momenta. The relevant and the irrelevant operators scale at zero and one loop in the way asserted by equation (2.75) which is the primary step towards proving the renormalisability of the theory order by order in the loops by induction.

2.4.3 The noncommutative case

We have seen that the $O(N)$ symmetry of the quantum theory enables us to use the same set of RG equations for any N as the $N=1$ case. We now discuss the renormalisability of the noncommutative theory to all orders in its symmetric as well as in its broken phases. Apart from the UV cutoff Λ, we shall also introduce an IR cutoff, Λ_{IR} as mentioned earlier.

First note that the components of L, $L_{2n}(p_1, ...p_{2n}, \Lambda)$ contain a phase factor which accounts for the noncommutativity of the theory. This factor is of the form $e^{-\frac{i}{2}\sum_{i<j} p_i \wedge p_j}$ with all possible permutations of the external momenta. Now consider equation (2.70),

$$\| \int \frac{d^4p}{(2\pi)^4} Q(p,\Lambda) L_{2n} \quad (\quad p_1, \ldots p_{2n}, p, -p, \Lambda) \| + \text{all possible permutations of } p_i \tag{2.91}$$

$$= \| \int \frac{d^4p}{(2\pi)^4} Q(p,\Lambda) e^{-\frac{i}{2}\sum_{i<j} p_i \wedge p_j} \| \,.\, \| \tilde{L}_{2n}(p_1, \ldots p_{2n-2}, p, -p, \Lambda) \|$$
$$+ \quad \text{all possible permutations of } p_i \tag{2.92}$$

where $\tilde{L}_{2n}(p_1, \ldots p_{2n-2}, p, -p, \Lambda)$ is the part of L_{2n} not containing the phase factor. Equation (2.91) corresponds to the one loop vertex L_{2n-2} evaluated from the L_{2n} vertex.

Evaluating (2.91) for one particular permutation of the external momenta gives,

$$\int \frac{d^4p}{(2\pi)^4} Q(p,\Lambda) e^{-\frac{i}{2}\sum_{i<j} p_i \wedge p_j} e^{i(p \wedge \sum_l p_l)} \sim e^{-\frac{i}{2}\sum_{i<j} p_i \wedge p_j} \frac{\Lambda^4}{1 + \Lambda^2 (\sum_l p_l) o (\sum_l p_l)} \tag{2.93}$$

The term on the RHS of equation (2.93) is the nonplanar term. Depending upon whether the two external momenta being contracted are consecutive to each other or not we get the usual commutative planar term or nonplanar terms respectively. See Figure 2.7.

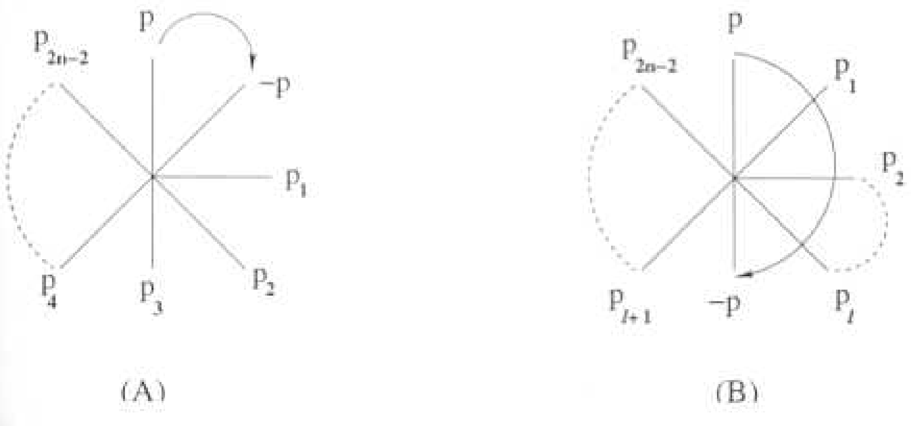

Figure 2.7: Contractions leading to planar and nonplanar terms are shown in (A) and (B) respectively

The exact weights of the planar and the nonplanar graphs would not be necessary for our following discussions. We just label the weight of the planar graph by N_p. For all permutations of the external momenta, the scaling behaviour of (2.93) with respect to Λ is,

$$N_p\Lambda^4 + \frac{\Lambda^4}{1+\Lambda^2(\sum_l p_l)o(\sum_l p_l)} + \text{...other nonplanar parts} \tag{2.94}$$

At this point it may be noted that, with every external momentum put to zero the weight of the planar term changes due to the transmutation of the nonplanar term into the planar term.

We now turn to the question of UV renormalisability of the theory by discussing the assertions as stated in (2.75), (2.77) and (2.78). First, let us consider the case where all the external momenta are such that $\Lambda^2 pop >> 1$, p being some combination of external momenta. In this limit the nonplanar term in (2.94) scales as Λ^2. The scaling behaviour of (2.94) is thus dominated by the planar term for large Λ. Therefore in this case the proof of assertion (i) follows exactly as the commutative case. Assuming that the assertion holds for $r = s-1$, from the RG equation (A2.1), we have,

$$\begin{aligned} &\| (\Lambda\frac{\partial}{\partial\Lambda}\partial^{\mu_1}_{i_1,j_1}...\partial^{\mu_p}_{i_p,j_p} L^{(s)}_{2n} \quad (\quad p_1,...,p_{2n},\Lambda) \| \\ &\qquad \leq \quad \Lambda^{4-2n-p}[N_p + \frac{1}{1+\Lambda^2(\sum_l p_l)o(\sum_l p_l)} + ...]P^{2s-n-1}ln(\frac{\Lambda_0}{\Lambda_R}) \end{aligned} \tag{2.95}$$

For the case considered, $\Lambda^2 pop >> 1$, we have,

$$\frac{1}{1+\Lambda^2(\sum_l p_l)o(\sum_l p_l)} \sim \frac{1}{\Lambda^2(\sum_l p_l)o(\sum_l p_l)} \tag{2.96}$$

The two powers of Λ in the denominator, decreases the overall power of Λ for L_{2n} in the nonplanar terms so that the overall scaling of L_{2n} with respect to Λ for large Λ is dominated by the planar term.

For $n \geq 3$ and $n = 2, p \geq 1$,the boundary values are set to zero at $\Lambda = \Lambda_0$. Therefore,

$$\begin{aligned}\| \partial^{\mu_1}_{i_1,j_1}...\partial^{\mu_p}_{i_p,j_p} L^{(s)}_{2n} & (p_1, ..., p_{2n}, \Lambda) \| \qquad (2.97)\\ & \leq \int_{\Lambda}^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} (\Lambda')^{4-2n-p} [N_p + \frac{1}{\Lambda'^2 (\sum_i p_i) o (\sum_i p_i)} + ...] P^{2s-n-1} ln(\frac{\Lambda_0}{\Lambda_R}) \\ & \leq \Lambda^{4-2n-p} P^{2s-n-1} ln(\frac{\Lambda_0}{\Lambda_R})\end{aligned}$$

In general L^{s-1}_{2n+2} would have nonplanar terms as shown above. When evaluating the L^s_{2n} vertex from L^{s-1}_{2n+2}, one has to integrate over all momenta thus including the small momentum modes. For these soft modes, k one can expand the nonplanar term as,

$$\frac{\Lambda_{eff}(k)}{\Lambda^2} = \frac{1}{1+\Lambda^2 kok} = [1 - \Lambda^2 kok + (\Lambda^2 kok)^2 ...] \qquad (2.98)$$

such that $\Lambda^2 kok \sim O(1)$. However in this region of internal momenta, $k \sim O(1/\theta\Lambda)$, the integral (2.93) without further exponential suppression is,

$$e^{-\frac{i}{2}\sum_{i<j} p_i \wedge p_j} \int_0^{\frac{1}{\theta\Lambda}} \frac{d^4 k}{(2\pi)^4} Q(k, \Lambda) [1 - \Lambda^2 kok + (\Lambda^2 kok)^2 ...] \qquad (2.99)$$

The contribution from this momentum shell is suppressed by $1/\theta\Lambda$. For larger values of k the integral is again suppressed by powers of Λ in the denominator. Therefore even though the very small momentum modes in the internal lines are included, it does not affect the canonical scaling of the irrelevant operators. We restrict ourselves to configurations of external momenta, p such that $\Lambda^2 pop > 1$ because relevant operators scale with different coefficients from the configurations with $\Lambda^2 pop < 1$ and in the continuum limit IR divergences appear when the external momenta are put to zero. It is thus clear that there is really no need for an IR cutoff, Λ_{IR} in the loop integrals. The integrals are IR finite since we are always working with a finite UV cutoff Λ. However restriction to configurations with external momenta such that $\Lambda^2 pop > 1$ implies the presence of an IR cutoff for the external momenta. Due to this reason the IR cutoff for the internal loop momenta would be formally assumed. The IR divergences in loop integrals however appear in perturbation theory. This is illustrated by an example (eqn. 2.111) at the end of this proof of UV renormalisability.

For $n = 2, p = 0$, the boundary values are set at $p_i = \bar{p}_i$ and $\Lambda = \Lambda_R$. So from (2.95),

$$\| \Lambda \frac{\partial}{\partial \Lambda} L_4^{(s)'}(\bar{p}_i, \Lambda) \| \;\leq\; P^{2s-3} ln(\frac{\Lambda_0}{\Lambda_R}) \tag{2.100}$$

which gives,

$$\begin{aligned} \| L_4^{(s)'}(\bar{p}_i, \Lambda) \| &\leq L_4^{(s)'}(\bar{p}_i, \Lambda_R) + P^{2s-2} ln(\frac{\Lambda_0}{\Lambda_R}) \\ &\leq P^{2s-2} ln(\frac{\Lambda_0}{\Lambda_R}) \end{aligned} \tag{2.101}$$

where $L_4^{(s)'}(\bar{p}_i, \Lambda_R)$ is a constant independent of Λ. $L_4^{(s)'}(p_i, \Lambda)$ can now be constructed for a general momentum configuration from (2.101) using the Taylor's expansion,

$$L_4^{(s)'}(p_i, \Lambda) = L_4^{(s)'}(\bar{p}_i, \Lambda) + \sum_{i,j=1}^{3} p_i^\mu p_j^\nu \int_0^1 d\lambda (1-\lambda) \partial^\mu_{i,4} \partial^\nu_{j,4} L_4^{(s)'}(p_i', \Lambda)|_{p_i'=\lambda p} \tag{2.102}$$

The terms on the RHS of (2.102) being bounded by (2.97) and (2.101), $L_4^{(s)'}(p_i, \Lambda)$ is also bounded. The bound for the components of L for $n = 1$ and $p = 0, 2$ follows along the same line as above. This proves the assertion (i).

Let us now see the scaling behaviours of the Green's functions with respect to Λ for a momentum configuration where some or all the external momenta are zero. As noted earlier, with each external momentum put to zero, the weights of the planar terms increases due to the transmutation of the nonplanar terms to the planar terms. Therefore although the components of L scale with same power of Λ as for the configuration with all momenta nonzero, they scale with different coefficients for the two configurations. Specifically let us take the example of L_4 where $p_3 = p_4 = 0$. The configuration of fields for which not all external momenta are nonzero is singular and so the scaling behaviour of functions for this configuration cannot be obtained from (2.102). Instead one has to define the Renormalisation conditions at a point where there are same number of zero external momenta so that, from (2.100) we have

$$\begin{aligned} \| L_4^{(s)}(\bar{p}_1, \bar{p}_2, 0, 0, \Lambda) \| &\leq L_4^{(s)}(\bar{p}_1, \bar{p}_2, 0, 0, \Lambda_R) + P^{2s-2} ln(\frac{\Lambda_0}{\Lambda_R}) \\ &\leq P^{2s-2} ln(\frac{\Lambda_0}{\Lambda_R}) \end{aligned} \tag{2.103}$$

The constant $L_4^{(s)}(\bar{p}_1, \bar{p}_2, 0, 0, \Lambda_R)$ differs from that of (2.101), but does not affect the scaling behaviour with respect to Λ. $L_4(p_1, p_2, 0, 0, \Lambda)$ can now be obtained form the Taylor's expansion about $(\bar{p}_1, \bar{p}_2, 0, 0)$. However due to (2.94) with $p_3 = 0$, $p_4 = 0$ $L_4^{(s)}(p_1, p_2, 0, 0, \Lambda)$ scales with a different weight of $ln(\frac{\Lambda}{\Lambda_R})$ from that of (2.102).

It is clear form these discussions that with the Renormalisation conditions set at nonzero external momenta the Green's functions for configurations of fields with some or all momenta zero cannot be renormalised. This was the issue in the initial sections of this chapter. The bare couplings defined through the Renormalisation conditions for the different configurations will have different Λ_0 dependences. Again the different scaling of the greens functions with respect to Λ for these configurations plays a crucial role in studying the renormalisability of the broken phase of a spontaneously broken $O(N)$ theory. We shall discuss this at the end of this section.

The proof of assertions (ii) and (iii) follow along the same line as the commutative case and the theory is renormalisable as long as we keep away from the zero momentum limit. This concludes the proof of the renormalisability of the noncommutative $O(N)$ symmetric theory.

We now give some computations of zero and one loop contributions of the relevant and irrelevant parts of L with the following boundary conditions.

$$\begin{aligned}\rho_1(\Lambda_R, \Lambda_0, \rho^0) &= 0 \\ \rho_2(\Lambda_R, \Lambda_0, \rho^0) &= 0 \\ \rho_3(\Lambda_R, \Lambda_0, \rho^0) &= 2V(\bar{\mathbf{p}})\lambda_R \end{aligned} \tag{2.104}$$

where $V(\bar{\mathbf{p}})$ is given by (2.26). The boundary values for the irrelevant couplings are set to zero at Λ_0 like the commutative case. As mentioned earlier the momentum integrals will now also be regulated in the IR by Λ_{IR}. However in the expressions for the one loop functions, to show the UV behaviour, we drop all the Λ_{IR} dependent terms and retain only the UV cutoff dependent pieces.

$$\rho_1^{(0)} = 0, \rho_2^{(0)} = 0, \rho_3^{(0)} = 2V(\bar{\mathbf{p}})\lambda_R \tag{2.105}$$

$$\Delta L_2^{(0)} = 0, \Delta L_4^{(0)} = 0 \tag{2.106}$$

$$\begin{aligned}\rho_1^{(1)} &= -\lambda_R \int \frac{d^4p}{(2\pi)^4} \int_{\Lambda_R}^{\Lambda} \frac{d\Lambda'}{\Lambda'^3} Q(\Lambda', p)[N+1+cos(p\wedge q)]\big|_{q^2=p_0^2} \\ &\sim -(N+1)\frac{\lambda_R}{16\pi^2}[\Lambda^2 + \mu^2 ln(\frac{\Lambda^2}{-\mu^2})] \\ &- \frac{\lambda_R}{16\pi^2}[\Lambda^2_{eff}(p_0) + \mu^2 ln(\frac{\Lambda^2_{eff}(p_0)}{-\mu^2})]\end{aligned} \tag{2.107}$$

$$\begin{aligned}\rho_2^{(1)} &= -\frac{\partial}{\partial p^2} L_2^{(1)}(p, -p, \Lambda)\big|_{p^2=p_0^2} \\ &\sim -\frac{\lambda_R}{16\pi^2} tr(\theta^2)[\Lambda^4_{eff}(p_0) + \mu^2 \Lambda^2_{eff}(p_0)]\end{aligned} \tag{2.108}$$

$$\begin{aligned}\Delta L_2^{(1)} &= \frac{\lambda_R}{16\pi^2}[\Lambda^2_{eff}(p) + \mu^2 ln(\frac{\Lambda^2_{eff}(p)}{-\mu^2})]\big|_{\Lambda}^{\Lambda_0} \\ &- \frac{\lambda_R}{16\pi^2}[\Lambda^2_{eff}(p_0) + \mu^2 ln(\frac{\Lambda^2_{eff}(p_0)}{-\mu^2})]\big|_{\Lambda}^{\Lambda_0} \\ &+ \frac{\lambda_R}{16\pi^2}(p^2 - p_0^2) tr(\theta^2)[\Lambda^4_{eff}(p_0) + \mu^2 \Lambda^2_{eff}(p_0)]\big|_{\Lambda}^{\Lambda_0}\end{aligned} \tag{2.109}$$

$$\rho_3^{(1)} \sim (N+3)V(\bar{\mathbf{p}})\frac{\lambda_R^2}{4\pi^2}ln(\frac{\Lambda^2}{-\mu^2}) + G(\bar{p}_i) \tag{2.110}$$

$G(\bar{p}_i)$ contains the nonplanar terms which in the continuum limit is only a function of the external momenta $\bar{p}_i$ and is divergent at low momenta.

Equations (2.105-2.110) show the scaling of the relevant and irrelevant operators with respect to Λ at zero and one loop. As long as the external momenta are such that $\Lambda_0^2 pop > 1$, these parts scale as asserted in (2.75). However we stress again that, ultimately one is interested in the $\Lambda_0 \to \infty$ limit, so that these scaling behaviours persist as long as we keep away from the $p = 0$ limit.

There are two points that may be noted here. For values of external momenta, p such that $\Lambda_0^2 pop << 1$,

(i) One can expand $\Lambda_{0eff}^2(p)$ in powers of $\Lambda_0^2 pop$, as in (2.98). The irrelevant coupling (2.109) now is dependent on the UV cutoff Λ_0, spoiling the usual Wilsonian picture as also noted in [18]. Thus a renormalisable noncommutative quantum theory has to be defined with an IR cutoff.

(ii) The relevant couplings scale with different coefficients from the noncommutative case where all external momenta are nonzero. They in fact scale like the corresponding one loop commutative theory couplings. In the limit $\Lambda^2 \bar{p}_i o \bar{p}_j << 1$, the would be IR divergences terms in $G(\bar{p}_i)$, equation (2.110) are transmuted into UV divergences and the noncommutative 4-point function scales with respect to $ln(\Lambda^2)$ with the same coefficient as the commutative 4-point function (2.88).

Before discussing the renormalisability of the broken phase of the spontaneously broken theory we now take a look at an example of the two point function in perturbation theory and see the appearance of IR divergences.[8]

The contribution from the diagram, shown in Figure 2.8, is given by,

$$\Gamma^2 \sim \int d^4k \frac{I(k)}{(k^2+m^2)^2} \tag{2.111}$$

where the nonplanar part of $I(k)$ with n tadpole insertions is given by,

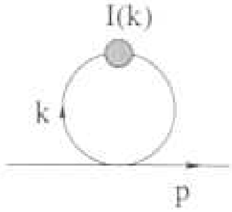

Figure 2.8: A higher order two-point diagram in perturbation theory. $I(k)$ contains a number of tadpole insertions

$$I(k) = \frac{1}{(\frac{1}{\Lambda^2} + kok)^n} \frac{1}{(k^2 + m^2)^{n-1}} + \text{less singular terms} \tag{2.112}$$

The effect of high momenta in the tadpole insertions is encoded in Λ. As all the high momenta modes are included, the singularities in $I(k)$ appears in the form of $1/(kok)^n$. The loop integral 2.111 is thus IR divergent. Note that this IR divergence in the loop integrals never comes up in the RG approach, as the loop integrals are always performed at a finite Λ.

We now turn to the issue of renormalisability of the broken phase in the spontaneously broken theory. The broken phase of the theory is defined by shifting the fields ϕ^α such that,

$$\begin{aligned} \phi^\alpha &\rightarrow \sigma + v, \alpha = 1 \\ &\rightarrow \pi^\alpha, \alpha = 2, N \end{aligned} \tag{2.113}$$

The two and the four point contributions to the effective Lagrangian of the broken phase is now given by,

$$\begin{aligned} \frac{1}{2}\int d^4p\phi^\alpha(p)\phi^\alpha(-p)L_2(p, &- p, \Lambda) \rightarrow \\ \frac{1}{2}\int d^4p[\sigma(p)\sigma(-p) &+ 2\sigma(p)v(-p) + \pi^\alpha(p)\pi^\alpha(-p)]L_2(p, -p, \Lambda) \end{aligned} \tag{2.114}$$

$$\frac{1}{4!}\int d^4p_1 d^4p_2 d^4p_3 d^4p_4 [\phi^a(p_1)\phi^a(p_2)][\phi^a(p_3)\phi^a(p_4) \quad] \quad L_4(p_1,p_2,p_3,p_4,\Lambda)\times$$
$$\times \quad \delta^4(\sum p_i) \rightarrow$$

$$\frac{1}{4!}\int d^4p_1 d^4p_2 d^4p_3 d^4p_4 \quad [\quad \sigma(p_1)v(p_2)v(p_3)v(p_4) + permutations + \pi^a(p_1)\pi^a(p_2)v(p_3)v(p_4)$$
$$+ \quad (p_1 \rightarrow p_3, p_2 \rightarrow p_4) + ...]L_4(p_1,p_2,p_3,p_4,\Lambda)\delta^4(\sum p_i) \qquad (2.115)$$

We have shown only these two contributions because these are the only divergent functions. The Ward identity (2.53), which is the consequence of the $O(N)$ symmetry of the symmetric phase tells that v times the two point π-π amplitude equals the σ-tadpole amplitude. This is manifest from the above two expansions. However, from the foregoing analysis we have seen that the relevant couplings scale with different coefficients whether or not the external momentum is such that $\Lambda_0^2 pop << 1$. In the $\Lambda_0 \rightarrow \infty$ limit this inequality is satisfied for $p \rightarrow 0$. The Renormalisation conditions at this limit defines different Λ_0 dependences of the bare couplings from the Renormalisation conditions at $p \neq 0$.

In particular, the contributions to the π-π amplitude are from,

$$L_2\pi^a(p)\pi^a(-p) + L_4\pi^a(p_1)\pi^a(p_2)v(p_3)v(p_4) + (p_1 \rightarrow p_3, p_2 \rightarrow p_4) \qquad (2.116)$$

and for the σ-tadpole amplitude from,

$$2L_2\sigma(p)v(-p) + L_4\sigma(p_1)v(p_2)v(p_3)v(p_4) + \text{permutations} \qquad (2.117)$$

We have seen that the behaviour of Green's functions with respect to Λ changes once we put one or more external momenta to zero. The same is true for the expressions (2.116) and (2.117). Once v is set to a constant at the tree level the momenta associated with v are set to zero. It is clear that in this case the L_2 and L_4 functions would have different Λ dependences from the case where v is not a constant, for reasons outlined earlier. Now, since an unequal number of v's multiply (2.116) and (2.117), the weights of the UV divergent, Λ dependent terms obtained due to transmutation of the nonplanar terms to the planar terms by setting v to a constant, are different in (2.116) and (2.117). This leads to a different Λ dependence in the σ-tadpole amplitude and the π-π amplitude. Consequently these two amplitudes cannot be renormalised by the same counter-term, thus violating the Ward identity (2.53). However the origin of the problem lies in the nonplanarity of the diagrams induced by external momenta in the loop diagrams, the resolution lies in keeping v as a non-constant background field, so that the components of (2.116) and (2.117) scale in the same way and that the Ward identity (2.53) holds. The symmetric phase of the theory has already been proved to be renormalisable as long as we have an infrared regulator. Spontaneous symmetry breaking does not affect the renormalisability and the broken phase is also renormalisable as long as v is kept as a non-constant background field.

2.5 Conclusion

We have studied the UV renormalisability of noncommutative field theories. In our discussions we have investigated this issue in the context of the $\lambda\phi^4$ theory, the Gross-Neveu model and the globally $O(N)$ symmetric ϕ^4 model in its symmetric as well as its spontaneously broken phases. The renormalisability of the globally $O(N)$ symmetric ϕ^4 theory is proved to all orders for both the symmetric as well as the broken phases with an IR cutoff.

The zero momentum configuration for these theories is singular. Through our discussions of one loop results followed by a general analysis in the language of the Wilsonian Renormalisation group, the following general features of noncommutative field theories evolved.

(i) We have seen that with the Renormalisation conditions set at a momentum configuration where all the external momenta are nonzero, a general Green's function with some or all the external momenta zero, cannot be renormalised. With each external momentum set to zero, the weights of the UV divergent planar graphs increases due to UV/IR mixing. This leads to a scaling behaviour of the greens functions with respect to the cutoff (Λ), with a different weight than a nonzero momenta configuration. This implies that for the relevant operators, the bare couplings will have different UV cutoff dependences for these different configurations.

(ii) The different scaling of greens functions for these separate configurations with respect to Λ has crucial implications on the renormalisability of the spontaneously broken phase of the $O(N)$ symmetric theory. In general the renormalisability of the broken phase of the theory is unaffected by spontaneous symmetry breaking. The underlying $O(N)$ symmetry makes the broken phase renormalisable with the same number of counter-terms as the symmetric phase. However in the case of noncommutative theory the one loop results indicate that this only happens when we break the symmetry by going to a vacuum which is translationally non-invariant. This can easily be understood in the language of the Wilsonian Renormalisation group. One of the consequences of the global $O(N)$ symmetry is the broken phase Ward identity (2.53). It can be seen from (2.116) and (2.117), keeping in mind the scaling behaviours in the foregoing discussions, that the σ-tadpole amplitude and the π-π amplitude would scale differently with respect to Λ when v is set to a constant. However when the constant v configuration is approached as a limit of the non-constant v configuration, the Ward identity (2.53) is still preserved. We have proved this to all orders following the proof of the symmetric phase of the $O(N)$ symmetric theory.

(iii) There are no IR divergences in the loop integrals of the Renormalisation group equations. This is as a consequence of always working with a finite UV cutoff Λ. IR divergences only show up when we approach a singular, zero external momentum configuration after taking the continuum limit in the solutions for the RG equations. If one keeps away from these singular field configurations, for generic external momenta, p, the theory is free from infrared divergences. It is because of this reason and the scaling behaviours of Green's functions in the two different momentum domains as discussed in (i), that an IR cutoff for the external momenta is necessary. IR divergences appearing in equation (2.111) are thus artifacts of perturbation theory.

(iv) Finally (2.109) indicates that the irrelevant operators are badly behaved at the zero momentum configuration.

In the light of all these remarks it may be concluded that the zero momentum configuration in noncommutative seems to make sense only when it is approached as a limit from a nonzero momentum configuration, or in other words, the noncommutative theories are renormalisable as long as one works with an infrared cutoff.

Having studied renormalisability and UV/IR mixing purely in the framework of quantum field theory, it is now natural to ask what new insights about the problems can be gained form string theory, since these theories occur in the low energy limit of open strings in background B-field. We will pursue this in the next chapter.

A2 Appendix : RG Equations

RG equation for L :

$$\Lambda\frac{\partial}{\partial\Lambda}L_{2n} = -\sum_{l=1}^{n}[\frac{Q(P,\Lambda)}{\Lambda^2}L_{2l}(p_1,...,p_{2l-1},P,\Lambda\quad)\quad L_{2n+2-2l}(p_{2l},...,p_{2n},-P,\Lambda) + \\ + \frac{1}{2}\begin{pmatrix} 2n \\ 2l-1 \end{pmatrix} - 1 \text{ permutations}]$$

$$-\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\frac{Q(p,\Lambda)}{\Lambda^2}L_{2n+2}(p_1,...,p_{2n},p,-p,\Lambda) \tag{A2.1}$$

where, $P = \sum_{i=1}^{2l-1} p_i$, such that,

$$\| (\Lambda^3\frac{\partial}{\partial\Lambda}L_{2n}) \| \leq \sum_{l=1}^{n}[\frac{1}{2}D_0 \| L_{2l}(\Lambda) \| \quad . \quad \| L_{2n+2-2l}(\Lambda) \|] \\ + \frac{1}{2}C\Lambda^4 \| L_{2n+2}(\Lambda) \| \tag{A2.2}$$

where, D, C are constants independent of Λ.

RG equation for B :

$$\begin{aligned}(\Lambda\frac{\partial}{\partial\Lambda} &+ 4-2n-2\delta_{b1})B_{b,2n}(p_1,...,p_{2n}) = \\ &- \sum_{l=1}^{n}[Q(P,\Lambda)A_{2l}(p_1,...,P,\Lambda)B_{b,2n+2-2l}(p_{2l},...,p_{2n},\Lambda) + \begin{pmatrix} 2n \\ 2l-1 \end{pmatrix} - 1 \text{ permutations}] \\ &- \frac{1}{2}\int\frac{d^4p}{(2\pi\Lambda)^4}B_{b,2n+2}(p_1,...,p_{2n},p,-p,\Lambda)Q(p,\Lambda) \\ &+ B_{1,2n}(p_1,...,p_{2n},\Lambda)\frac{1}{2}\int\frac{d^4q}{(2\pi\Lambda)^4}B_{b,4}(p,-p,q,-q)Q(q,\Lambda)\big|_{p^2=p_0^2} \\ &+ B_{2,2n}(p_1,...,p_{2n},\Lambda)\frac{\Lambda^2}{2}\int\frac{d^4q}{(2\pi\Lambda)^4}\frac{\partial}{\partial p^2}B_{b,4}(p,-p,q,-q)Q(q,\Lambda)\big|_{p^2=p_0^2} \\ &+ B_{3,2n}(p_1,...,p_{2n},\Lambda)\frac{1}{2}\int\frac{d^4q}{(2\pi\Lambda)^4}B_{b,6}(p_i,q,-q)Q(q,\Lambda)\big|_{p_i=\bar{p}_i} \qquad \text{(A2.3)}\end{aligned}$$

$$\begin{aligned}\parallel(\Lambda\frac{\partial}{\partial\Lambda} &+ 4-2n-2\delta_{b1})B_{b,2n}(p_1,....,p_{2n})\parallel \\ &\leq \sum_{l=1}^{n}[D_0 \parallel A_{2l}(\Lambda)\parallel . \parallel B_{b,2n+2-2l}(\Lambda)\parallel] \\ &+ \frac{1}{2}C \parallel B_{b,2n+2}(\Lambda)\parallel + \frac{1}{2}C\parallel B_{1,2n}(\Lambda)\parallel . \parallel B_{b,4}(\Lambda)\parallel \\ &+ \frac{1}{2}C\Lambda^2 \parallel B_{2,2n}(\Lambda)\parallel . \parallel \partial^{\mu}_{1,2}\partial^{\mu}_{1,2}B_{b,4}(\Lambda)\parallel \\ &+ \frac{1}{2}C\parallel B_{3,2n}(\Lambda)\parallel . \parallel B_{b,6}(\Lambda)\parallel \qquad \text{(A2.4)}\end{aligned}$$

RG equation for V :

$$\begin{aligned}(\Lambda\frac{\partial}{\partial\Lambda} &+ 4-2n)V_{2n}(p_1,....,p_{2n}) = \\ &- \sum_{l=1}^{n}[Q(P,\Lambda)A_{2l}(p_1,...,P,\Lambda)V_{2n+2-2l}(p_{2l},...,p_{2n},\Lambda) + \begin{pmatrix} 2n \\ 2l-1 \end{pmatrix} - 1 \text{ permutations}] \\ &- \frac{1}{2}\int\frac{d^4p}{(2\pi\Lambda)^4}V_{2n+2}(p_1,...,p_{2n},p,-p,\Lambda)Q(p,\Lambda) \\ &+ B_{1,2n}(p_1,...,p_{2n},\Lambda)\frac{1}{2}\int\frac{d^4q}{(2\pi\Lambda)^4}V_4(p,-p,q,-q)Q(q,\Lambda)\big|_{p^2=p_0^2} \\ &+ B_{2,2n}(p_1,...,p_{2n},\Lambda)\frac{\Lambda^2}{2}\int\frac{d^4q}{(2\pi\Lambda)^4}\frac{\partial}{\partial p^2}V_4(p,-p,q,-q)Q(q,\Lambda)\big|_{p^2=p_0^2} \\ &+ B_{3,2n}(p_1,...,p_{2n},\Lambda)\frac{1}{2}\int\frac{d^4q}{(2\pi\Lambda)^4}V_6(p_i,q,-q)Q(q,\Lambda)\big|_{p_i=\bar{p}_i} \qquad \text{(A2.5)}\end{aligned}$$

$$\begin{aligned}
\| (\Lambda\frac{\partial}{\partial\Lambda} &+ 4-2n)V_{2n}(p_1,...,p_{2n}) \| \\
&\leq \sum_{l=1}^{n}[D_0 \| A_{2l}(\Lambda) \| . \| V_{2n+2-2l}(\Lambda) \|] \\
&+ \frac{1}{2}C \| V_{2n+2}(\Lambda) \| + \frac{1}{2}C \| B_{1,2n}(\Lambda) \| . \| V_4(\Lambda) \| \\
&+ \frac{1}{2}C\Lambda^2 \| B_{2,2n}(\Lambda) \| . \| \partial^{\mu}_{1,2}\partial^{\mu}_{1,2}V_4(\Lambda) \| \\
&+ \frac{1}{2}C \| B_{3,2n}(\Lambda) \| . \| V_6(\Lambda) \| \qquad \text{(A2.6)}
\end{aligned}$$

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Chapter 3

Open-Closed String Duality in background B-Field

World sheet duality is a very fundamental feature of string theories and one expects that because of this there is a duality between open string theories and closed string theories. AdS/CFT [1] can be viewed as a particular realization of this where to leading order we see a relation between the massless sectors of the two theories. It is useful to study such dualities in other backgrounds to further elucidate the key ingredients. One such background is the constant B-field. As will be shown below because of the regulatory nature of the B-field duality statements in some cases can be made more sharply.

Open string dynamics in constant background B-field have been studied for the past few years with renewed interest for other reasons also. This is mainly due to the fact that the low energy dynamics of open strings in background constant B-field can be studied as a gauge theory on noncommutative space-times [2, 3, 4, 5]. Embedding these theories in string theory helps us to analyse them from a wider perspective.

In the recent years there has been an extensive study of noncommutative field theories from two directions [6]. One by starting from noncommutativity at the field theory level and exploring various phenomena that arise which may be absent in commutative models. The other direction is by studying them from the point of view of string theory. Most often the latter perspective has led to better understanding of the various unusual features in the noncommutative models.

As we have seen in the previous chapter, a generic feature of noncommutative field theories is the appearance of infrared singularities by integrating out high momentum modes propagating in the loop, popularly known as UV/IR mixing. Various attempts have been made to interpret these results in the usual Wilsonian renormalisation group picture. In the previous chapter we have seen that this phenomenon in noncommutative field theories fits into the usual notion of Wilsonian renormalisation group if we include an infrared cutoff [9]. A different approach to cure the problem of UV/IR mixing has also been pursued [10, 11]. It is however, not clear whether this is to be viewed as an inequivalent quantisation and therefore a different theory or a different cure to the infrared divergences.

The coupling of the ultraviolet to the infrared is inherent in string theory and manifests itself as a consequence of open-closed string duality. With this hindsight it was proposed that the new IR singularities should appear at the field theory level by integrating out additional massless closed string modes that couple to the gauge theory [7, 8]. This was studied for the one loop N-point tachyonic amplitude in bosonic theory in [19]. Interesting connections between the closed string tachyons and noncommutative divergences was shown in [20]. Usually the ultraviolet divergences of the open string modes can be interpreted as infrared divergences from massless closed string exchanges. In the presence of the background B-field these divergences are regulated and thus a quantitative analysis can be made. The one-loop two point diagram for open strings is a cylinder with a modular parameter t and vertex operator insertions at the boundaries. The two point one-loop noncommutative field theory diagram results in the Seiberg-Witten limit by keeping surviving terms in the integrand for the integral over t for $t \to \infty$. This limit suppresses all contributions from massive modes in the loop. The resulting diagram is that of the gauge theory with massless propagating modes. This amplitude is usually divergent in the ultraviolet when integrated over t. The source of ultraviolet divergence is the same as that of those in string theory i.e. $t \to 0$. It is therefore natural to analyse the amplitude directly in this limit when only the low lying closed string exchanges contribute.

In the bosonic string theory setting, we first analyse the two-point one loop amplitude for gauge bosons on the brane, in the closed string channel [21]. We argue that the region of the modulus giving rise to divergences (that are regulated in the nonplanar amplitudes) in noncommutative field theories can be identified as the region where the lightest closed string modes dominate in the dual picture. The full two point open string amplitude also contains finite contributions which would require the entire tower of closed string states for its dual description. However, the singular IR behaviour of the nonplanar amplitudes, in the boundary noncommutative gauge theory can be seen from the exchange of closed strings in the bulk. Though there are additional tachyonic divergences, we are able to show that the form of IR divergences with appropriate tensor structures can be extracted by considering only lowest lying modes (tachyonic and massless). We further analyse the two point amplitude by studying massless closed string exchanges in background constant B-field. From this analysis we are able to isolate the individual contributions from the massless closed string exchanges. We further argue that the exact correspondence can occur in some special supersymmetric models. Open strings on fractional branes localised at the fixed point of C^2/Z_2 naturally satisfy these conditions. In the next part we study this issue in this orbifold background [22].

The fact that the ultraviolet behaviour of the one loop gauge theory is same as that of the massless closed string tree-level exchanges in this model with $\mathcal{N} = 2$ supersymmetry have been pointed out in [23]. Further studies as the consequence of this duality have been done by various authors [24, 25, 26, 27, 28]. Also see [29] for a recent review and references therein. We show that the UV/IR mixing phenomenon of $\mathcal{N} = 2$ gauge theory can be naturally interpreted as a consequence of open-closed string duality in the presence of background B-field. The effective action for the full two point function from gauge theory differs from that with closed string exchanges only by finite derivative corrections. However as far as the divergent UV/IR mixing term is concerned, it is exactly equal to the infrared contribution from the massless closed string exchanges. Using world-sheet duality these modes can be identified as coming from the twisted NS-NS and R-R sectors. This model was also studied in [18] in the context of closed string realization of the IR singular terms in gauge theory. Here we show how only the twisted NS-NS and R-R sectors closed strings couplings to the gauge theory survive and that the closed string interpretation of the UV/IR-terms naturally follows as a consequence of open-closed duality. The crucial feature that plays a role here is that the contributions from the massive modes cancel in this model.

This chapter is organised as follows. In Section 3.1, we study the one loop open string amplitude in the UV limit and write down the contribution from the lowest states. In Section 3.2, we analyse massless closed string exchange in background B-field and reconstruct the massless contribution computed in Section 3.1. In Section 3.3 we study superstrings in B-field background and give a short review of strings on C^2/Z_2 orbifold and the massless spectrum of open strings ending on fractional D_3-brane localised at the fixed point and closed strings. In section 3.4 we compute the two point function for one loop open strings in this orbifold background with the B-field turned on, and analyse it in the open and closed string channels. By taking the field theory limit, we show using open-closed string duality that the new IR divergent term from the nonplanar amplitude is exactly equal to the IR divergent contributions from massless closed string exchanges. We conclude this chapter with discussions.

Conventions: We will use capital letters $(M, N, ...)$ to denote general spacetime indices and small letters $(i, j, ...)$ for coordinates along the D-brane. Small Greek letters $(\alpha, \beta...)$ will be used to denote indices for directions transverse to the brane.

3.1 Open string one loop amplitude

In the previous chapter we have outlined how new IR divergent terms appear in the nonplanar loop amplitudes of noncommutative field theories. To interpret these in terms of closed string exchanges we now embed the problem in string theory. The main idea is summarised in Figure 3.6. The various steps involved in the problem will be clarified as we proceed. In this section we compute the open string one loop amplitude with insertion of two gauge field vertices. We will compute the two point amplitude in the closed string channel keeping only the contributions from the tachyon and the massless modes. One loop amplitudes for open strings with two vertex insertions in the presence of a constant background B-field have been computed by various authors, and field theory amplitudes were obtained in the $\alpha' \to 0$ limit [17].

Firstly, the one loop partition function is written as [32, 31]

$$Z(t) = \det(g + 2\pi\alpha' B)V_{p+1}(8\pi^2\alpha' t)^{-\frac{p+1}{2}} Z_0(t) \tag{3.1}$$

with,

$$Z_0(t) = \mathrm{Tr}[\exp(-2\pi t L_0')] \tag{3.2}$$

$\det(g+2\pi\alpha' B)$ comes from the trace over the zero modes of the world-sheet bosons. See Appendix A3 eqn(A3.14). t is the modulus of the cylinder and L_0' contains the oscillators. This gives,

$$Z(t) = \det(g + 2\pi\alpha' B)V_{p+1}(8\pi^2\alpha' t)^{-\frac{p+1}{2}} \eta(it)^{-(D-2)} \tag{3.3}$$

V_{p+1} is the volume of the D_p brane. We are interested in the two point one loop amplitude. Specifically we write down here the nonplanar amplitude for reasons mentioned earlier. The two point one loop amplitude has the form,

$$A(p_1, p_2) = \int_0^\infty \frac{dt}{2t} Z(t) \int_0^{2\pi t} dy \int_0^{2\pi t} dy' < V(p_1, x, y) V(p_2, x', y') > \tag{3.4}$$

where $Z(t)$ is as defined in eqn(3.3). The required vertex operator is given by,

$$V(p, y) = -i\frac{g_o}{(2\alpha')^{1/2}} \epsilon_j \partial_y X^j e^{ip.X}(y) \tag{3.5}$$

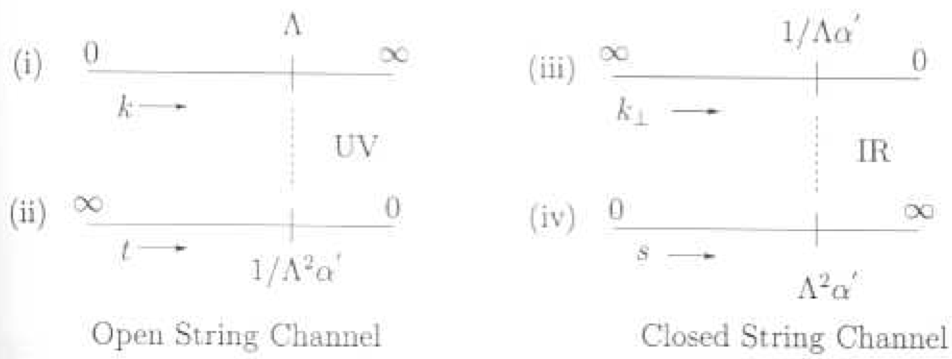

Figure 3.1: UV and IR regions in the open and closed string channels. (i) k is the momentum in the gauge theory one loop diagram. (ii) t is the modulus of the cylinder in the open string one loop diagram (iii) $k_\perp$ is the transverse momentum of the closed string modes emitted from the brane (iv) $s = 1/t$

The noncommutative field theory results are recovered from region of the modulus where $t \to \infty$ in the SW limit. As mentioned, the nonplanar diagrams in the noncommutative field theory give rise to terms which manifest coupling of the UV to the IR sector of the field theory.

The $t \to 0$ limit, picks out the contributions only from the tree-level massless closed string exchange. This is the UV limit of the open string. The amplitude is usually divergent. However, in the usual case, these divergences are reinterpreted as IR divergences due to the massless closed string modes. What is the role played by the B-field? In the presence of the background B-field, the integral over the modulus is regulated. On the closed string side, this would mean that the propagator for the massless modes are modified so as to remove the IR divergences. We would now like to investigate this end of the modulus.

Before going into the actual form let us see heuristically what we can expect to compare on both ends of the modulus. First consider the one loop amplitude,

$$\mathcal{Z} \sim \int \frac{dt}{t}(\alpha' t)^{-\frac{p+1}{2}}\eta(it)^{-(D-2)}\exp(-C/\alpha' t) \qquad (3.6)$$

where C is some constant which in our case is dependent on the B-field. In the $t \to \infty$ limit,

$$\mathcal{Z}_{op} \sim \int \frac{dt}{t}(\alpha' t)^{-\frac{p+1}{2}} \left[e^{2\pi t} + (D-2) + O(e^{-2\pi t})\right] \exp(-C/\alpha' t) \qquad (3.7)$$

If we throw out the tachyon, and restrict ourselves only to the $O(1)$ term in the expansion of the η-function, α' and t occur in pairs. This means that in the $\alpha' \to 0$ limit the finite contributions to the field theory come from the region where t is large. We can break the integral over t into two parts, $1/\Lambda^2\alpha' < t < \infty$ and $0 < t < 1/\Lambda^2\alpha'$, where Λ translates into the UV cutoff for the field theory on the brane. The second interval is the source of divergences in the field theory that is regulated by C. This is the region of the modulus dominated by massless exchanges in the closed string channel. See Figure 3.1. For the closed string channel, we have

$$\mathcal{Z}_{cl} \sim \int ds(\alpha')^{-\frac{p+1}{2}} s^{-l/2} \left[e^{2\pi s} + (D-2) + O(e^{-2\pi s})\right] \exp(-Cs/\alpha') \qquad (3.8)$$

where $l = D-(p+1)$, is the number of dimensions transverse to the D_p brane. The would be divergences as $C \to 0$ manifest themselves as $1/C$ or $\ln(C)$, depending on l [8]. The full open string channel result will always require all the closed string modes for its dual description. As far as the divergent (UV/IR mixing) terms are concerned, we can hope to realise them through some field theory of the massless closed string modes. However, the exact correspondence between the divergences in both the channels, is destroyed by the presence of the tachyons. Also note that, at the $t \to 0$ end of the open string one loop amplitude, the divergence is contributed by the full tower of open string modes. However, in the cases where the one loop open string amplitude restricted to only the massless exchanges can be rewritten as massless closed string exchanges, the integrand as a function of t in one loop amplitude should have the same asymptotic form as $t \to 0$ and $t \to \infty$ so that eqn(3.8) is exactly the same as that of eqn(3.7) integrated between $[0, 1/\Lambda^2\alpha']$. There are examples of supersymmetric configurations where the one loop open string amplitude restricted to the massless sector can be rewritten exactly as tree-level massless closed string exchanges. It was shown that in these situations the potential between two branes with separation r is the same at both the $r \to 0$, and $r \to \infty$ corresponding to $t \to \infty$ and $t \to 0$ ends respectively [23]. This has lead to further interesting studies on the gauge/gravity correspondence. For a review see [29] and references therein. We can expect that in these cases the IR singularities of the noncommutative gauge theory match with those computed from the closed string massless exchanges. In the bosonic case, this is true for $p = 13$, if we remove the tachyons. However, we are concerned with reproducing UV/IR effects of four dimensional gauge theory for which we need to set $l = 2$. The broader purpose of the exercise that follows is to outline a construction that can be set up for supersymmetric case, that is to follow in the latter sections.

We now return to the original computation of the amplitude in the closed string channel. The nonplanar world sheet propagator obtained by restricting to the positions at the two boundaries is [31, 17],

$$\mathcal{G}^{ij}(y, y^{'}) = -\alpha^{'} G^{ij} \ln \left| e^{-\frac{\pi}{4t}} \frac{\vartheta_4\left(\frac{\Delta y}{2\pi t}, \frac{i}{t}\right)}{t^{-1}\eta(i/t)^3} \right|^2 - i\frac{\theta^{ij}\Delta y}{2\pi t} - \alpha^{'} g^{ij}\frac{\pi}{2t} \tag{3.9}$$

where, $\Delta y = y - y^{'}$. In the limit $t \to 0$ the propagator has the following structure,

$$\mathcal{G}^{ij} = -4\alpha^{'} G^{ij} \left[cos(\Delta y/t) e^{-\frac{\pi}{t}} - e^{-\frac{2\pi}{t}} \right] - i\frac{\theta^{ij}\Delta y}{2\pi t} - \alpha^{'} g^{ij}\frac{\pi}{2t} \tag{3.10}$$

Inserting this into the correlator for two gauge bosons and keeping only terms that would contribute to the tachyonic and massless closed string exchanges, we get,

$$< ... >= \left[p_k p_l \frac{(8\pi\alpha^{'})^2}{(2\pi t)^2} (G^{ij}G^{kl} - G^{ik}G^{jl}) sin^2(\Delta y/t) e^{-\frac{2\pi}{t}} + \frac{\tilde{p}^i \tilde{p}^j}{(2\pi t)^2} \right] e^{p_i \mathcal{G}^{ij} p_j} \tag{3.11}$$

expanding $\eta(it)$ in this limit,

$$\eta(it)^{-(D-2)} = t^{\frac{D-2}{2}} \eta(i/t)^{-(D-2)} \sim t^{\frac{D-2}{2}} [e^{\frac{2\pi}{t}} + (D-2) + O(e^{-\frac{2\pi}{t}})] \tag{3.12}$$

The two point amplitude with only the tachyonic and the massless closed string exchange can now be written down,

$$A_2(p,-p) = -i\det(g+2\pi\alpha' B)\mathcal{V}_{p+1}(\frac{g_o^2}{2\alpha'})(8\pi^2\alpha')^{-\frac{p+1}{2}}\epsilon_i\epsilon_j I(p) \tag{3.13}$$

with $I(p) = I_T(p) + I_\chi(p)$ and,

$$\begin{aligned} I_T(p) &= \tilde{p}^i\tilde{p}^j \int ds s^{-\frac{l}{2}} \exp\left\{-(\frac{\alpha'\pi}{2}p_i g^{ij} p_j - 2\pi)s\right\} \\ &= 4\pi(2\pi^2\alpha')^{\frac{l}{2}-1}\tilde{p}^i\tilde{p}^j \int \frac{d^l k_\perp}{(2\pi)^l} \frac{1}{k_\perp^2 + p_i g^{ij} p_j - 4/\alpha'} \end{aligned} \tag{3.14}$$

We have written the integral over t in terms of $s = 1/t$ in (3.14) and further in the last expression we have replaced the integral over s with that of $k_\perp$. The dimension of the $k_\perp$ integral is the number of directions transverse to the brane and is thus the momentum of the closed string along these directions. Note that the s integral has to be cutoff at the lower end at some value $\Lambda^2\alpha'$. This corresponds to the UV transverse momentum cutoff for the closed strings, that allows us to extract the contribution from the IR region (see Figure 3.1).

$$I(p;\Lambda) \sim \int_{\Lambda^2\alpha'}^{\infty} \frac{ds}{s} e^{-p^2\alpha' s} \sim \int_0^\infty d^2k_\perp \frac{e^{-(k_\perp^2+p^2)\Lambda^2\alpha'^2}}{(k_\perp^2+p^2)\alpha'} \tag{3.15}$$

The integral over $k_\perp$, eqn(3.15) receives contribution upto $k_\perp \sim O(1/\Lambda\alpha')$. The included region of the $k_\perp$ integral is the required IR sector for the transverse closed string modes or the UV for the open string channel. With this observation, for the tachyon with $l = 2$, we get

$$I_T(p,\Lambda) = 4\pi^2(2\pi^2\alpha')^{\frac{l}{2}-1}\tilde{p}^i\tilde{p}^j \ln\left(\frac{p_i g^{ij} p_j - 4/\alpha' + \frac{1}{(\Lambda\alpha')^2}}{p_i g^{ij} p_j - 4/\alpha'}\right) \tag{3.16}$$

For the noncommutative limit (2.14), we can expand the answer (3.16) in powers of $1/(\alpha' p g^{-1} p)$,

$$\ln\left(pg^{-1}p - 4/\alpha'\right) \sim \ln\left(pg^{-1}p\right) - \frac{4}{\alpha' pg^{-1}p} - \frac{1}{2}\left(\frac{4}{\alpha' pg^{-1}p}\right)^2 - \dots \tag{3.17}$$

The $(1/\alpha' pg^{-1}p)^2$ term in the expansion (3.17) above corresponds to the IR singular term which appears in the noncommutative gauge theory. To compare with the second term of (2.22), we should set $G = \eta$, so that $g^{-1} \sim -\theta^2/\alpha'^2$. Here we have got this from one of the terms in the expansion of the amplitude with tachyon exchange. However one can easily see that any massive spin zero closed string exchange would produce such a term. As far as the exact coefficient is concerned, the full tower of massive states would contribute. The absence of this term in the supersymmetric theories can only be due to exact cancellations between the bosonic and fermionic sector contributions [20].

As for the tachyon, similarly we now write down the contribution from the massless exchanges,

$$\begin{aligned} I_\chi(p,\Lambda) &= 4\pi(2\pi^2\alpha')^{\frac{l}{2}-1}\left[(D-2)\tilde{p}^i\tilde{p}^j + 8(2\pi\alpha')^2 p_k p_l (G^{ij}G^{kl} - G^{ik}G^{jl})\right] \times \\ &\times \int \frac{d^l k_\perp}{(2\pi)^l} \frac{1}{k_\perp^2 + p_i g^{ij} p_j} \end{aligned} \tag{3.18}$$

One can observe that the terms occurring with $\alpha'^2(\sim \epsilon)$ as the coefficient, relative to the other terms in (3.17) and (3.18), appear in the gauge theory result in eqn(2.22). In the closed string channel we have got this for the number of transverse dimensions, $l = 2$. This means that $p + 1 = D - 2 = 24$ is the dimension of the gauge theory on the string side. However the result of eqn(2.22) is valid for the NC gauge theory defined in 4-dimensions. To understand why it is these terms that occur in the four dimensional gauge theory, we must have a string setting where $l = 2$ and $p = 3$. However, at this point, as discussed earlier, it is only necessary that $l = 2$ so that the lowest lying closed string exchanges reproduce the correct form of the IR singularities as that of the gauge theory in eqn(2.22).

We mention again that the exact correspondence between the UV behavior of the noncommutative gauge theory and closed string exchanges would require the full tower of closed string states. The contribution from the massive closed string states are likely to be suppressed only in some supersymmetric configurations [23]-[29]. We will see how this works out in these setups in Sections 3.3 and 3.4, but before that let us study the massless closed string exchanges in the presence of background B-field.

3.2 Closed string exchange

In this section we reconstruct the two point function of two gauge fields eqn(3.18) with massless closed string exchanges. The aim here is to write the amplitude as sum of massless closed string exchanges in the presence of constant background B-field. To proceed, by considering the effective field theory of massless closed strings, we construct the propagators for these modes (graviton, dilaton and B-field) with a constant background B-field. As a next step we compute the couplings of the gauge field on the brane with the massless closed strings from the DBI action. Finally we combine these results to construct the two point function. We will consider three separate cases when computing the two point amplitude in this section.

1. In this case the background B field is assumed to be small and the closed string metric, $g = \eta$.
2. The Seiberg Witten limit when $g = \epsilon\eta$.
3. The case when the open string metric on the brane, $G = \eta$.

The amplitude eqn(3.18) in the closed string channel is the closed form result of the massless exchanges. In each of the above cases, we will compare this amplitude to respective orders with the ones we compute here in this section. Let us first begin by considering the field theory of the massless modes of the closed string string propagating in the bulk. The spacetime action for closed string fields is written as,

$$S = \frac{1}{2\kappa^2}\int d^D X\sqrt{-g}[R - \frac{1}{12}e^{-\frac{8\phi}{D-2}}H_{LMN}H^{LMN} - \frac{4}{D-2}g^{MN}\partial_M\phi\partial_N\phi] \quad (3.19)$$

where, D is the number of dimensions in which the closed string propagates. The indices are raised and lowered by g. We will now construct the tree-level propagators that will be necessary in the next section to compute two point amplitudes. For each of the cases as defined above, the propagator will take different forms. Let us first consider the dilaton. For $g = \eta$ the propagator is the usual one,

$$< \phi\phi > \;=\; -\frac{(D-2)i\kappa^2}{4}\frac{1}{k_\perp^2 + k_\parallel^2} \tag{3.20}$$

The next limit for the metric is $g = \epsilon\eta$ along the world volume directions of the brane. In this limit, the dilaton part of the action can be written as,

$$S_\phi = -\frac{4}{\kappa^2(D-2)}\int d^DX\frac{1}{2}[\partial_a\phi\partial^a\phi + \epsilon^{-1}\partial_i\phi\partial^i\phi] \tag{3.21}$$

This gives the propagator,

$$< \phi\phi > \;=\; -\frac{(D-2)i\kappa^2}{4}\frac{1}{k_\perp^2 + \epsilon^{-1}k_\parallel^2} \tag{3.22}$$

Finally, when the open string metric is set to, $G = \eta$, the lowest order solution for g along the brane directions is,

$$g = -(2\pi\alpha')^2B^2 + \mathcal{O}(\alpha'^4) \tag{3.23}$$

which gives,

$$< \phi\phi > \;=\; -\frac{(D-2)i\kappa^2}{4}\frac{1}{k_\perp^2 + \bar{k}_\parallel^2/(2\pi\alpha')^2} \tag{3.24}$$

where,

$$\bar{k}_\parallel^2 = -k_{\parallel i}\left(\frac{1}{B^2}\right)^{ij}k_{\parallel j} \tag{3.25}$$

Let us now turn to the free part for the antisymmetric tensor field,

$$S_b = -\frac{1}{24\kappa^2}\int d^DXH_{LMN}H^{LMN} \tag{3.26}$$

where,

$$H_{LMN} = \partial_L b_{MN} + \partial_M b_{NL} + \partial_N b_{LM} \tag{3.27}$$

Using the following gauge fixing condition,

$$g^{MN}\partial_M b_{NL} = 0 \tag{3.28}$$

The action reduces to,

$$S_b = -\frac{(2\pi\alpha^{'})^2}{8\kappa^2}\int d^D X \left[g^{\alpha\beta}\partial_\alpha b_{IJ}\partial_\beta b_{KL} + g^{ij}\partial_i b_{IJ}\partial_j b_{KL}\right] g^{IK} g^{JL} \tag{3.29}$$

The factor of $(2\pi\alpha^{'})^2$ in the b-field action has been included because the sigma model is defined with $(2\pi\alpha^{'})B$ coupling. The propagator then is,

$$< b_{IJ} b_{I'J'} >= -\frac{2i\kappa^2}{(2\pi\alpha^{'})^2}\frac{g_{I[J'}g_{I']J}}{k_\perp^2 + g^{ij}k_{\|i}k_{\|j}} \tag{3.30}$$

Finally, the gravitational part of the action. As will turn out in the next section that we will only have to consider graviton exchanges for the case $g = \eta$. The propagator for the graviton here is the usual propagator from the action,

$$S_h = \frac{1}{2\kappa^2}\int d^D X \sqrt{-g} R \tag{3.31}$$

By considering fluctuations about η, and in the gauge (3.33),

$$g_{MN} = \eta_{MN} + h_{MN} \tag{3.32}$$

$$g^{MN}\Gamma^L_{MN} = 0 \tag{3.33}$$

the graviton propagator is,

$$< h_{IJ} h_{I'J'} >= -2i\kappa^2 \frac{[\eta_{I\{J'}\eta_{I'\}J} - 2/(D-2)\eta_{IJ}\eta_{I'J'}]}{k_\perp^2 + k_\parallel^2} \tag{3.34}$$

After writing down the required propagators, we now turn to the computation of the vertices. As mentioned in the beginning of this section, we will consider each of the three cases separately. To begin, we first write down the DBI action for a D_p brane,

$$S_p = -T_p \int d^{p+1}\xi e^{-\Phi} \sqrt{g' + 2\pi\alpha'(B+b)} \tag{3.35}$$

Where, g' is the closed string metric in the string frame, B is the constant two form background field and b is the fluctuation of the two form field. The b-field on the brane is interpreted as the two form field strength for the $U(1)$ gauge field and in the bulk it is the usual two form potential. Going to the Einstein frame by defining,

$$g = g' e^{2\omega}; \quad \omega = \frac{2(\phi_0 - \Phi)}{D-2}; \quad \Phi = \phi + \phi_0; \quad \omega = \frac{-2\phi}{D-2} \tag{3.36}$$

the action can be rewritten as,

$$\begin{aligned} S_p = & - \tau_p \int d^{p+1}\xi \mathcal{L}(\phi, , h, b) \\ & - \tau_p \int d^{p+1}\xi e^{-\phi(1-\frac{2(p+1)}{D-2})} \sqrt{g + 2\pi\alpha'(B+b)e^{-\frac{4\phi}{D-2}}} \end{aligned} \tag{3.37}$$

where, $\tau_p = T_p e^{-\phi_0}$ and ϕ is the propagating dilaton field. We will now consider each of the three cases separately and compute the two point function upto the respective orders.

3.2.1 Expansion for small B

In this part we compute the couplings of the gauge field on the brane to the massless closed strings in the bulk. We will assume the background constant B-field to be small and compute the lowest order contribution to the two point function considered as an expansion in B. The first thing to note is that, since B is antisymmetric, there cannot be a non vanishing amplitude with a single B in one vertex only. We need at least two powers of B. One on each vertex or both on one. The graviton and the dilaton need one on each vertex. The b-field can couple to the gauge field without a B. So for the b-field we need to consider couplings upto $\mathcal{O}(B^2)$. The closed string tree-level diagrams contributing to the three massless modes are shown in Figure 3.2.

$$\mathcal{L} = \sqrt{e^{-P\phi}\left[g + (2\pi\alpha')e^{-Q\phi}(B+b)\right]} \tag{3.38}$$

where

$$P = \frac{2}{p+1} - \frac{4}{(D-2)} \qquad Q = \frac{4}{(D-2)} \tag{3.39}$$

We now expand of $\mathcal{L}$ for small B, with $g = \eta + h$,

$$\mathcal{L} = \sqrt{e^{-P\phi}[g + (2\pi\alpha')e^{-Q\phi}b]}\left[1 + \frac{(2\pi\alpha')e^{-Q\phi}}{g + (2\pi\alpha')e^{-Q\phi}b}B\right]^{1/2} \tag{3.40}$$

To the linear order in B,

$$\begin{aligned}\mathcal{L} &= \sqrt{e^{-P\phi}[g + (2\pi\alpha')e^{-Q\phi}b]}\left[1 + \frac{(2\pi\alpha')}{2}e^{-Q\phi}\mathrm{Tr}\frac{1}{g + (2\pi\alpha')e^{-Q\phi}b}B\right] \\ &= \sqrt{e^{-P\phi}[g + (2\pi\alpha')e^{-Q\phi}b]}\left[1 - \frac{1}{2}(2\pi\alpha')^2 e^{-2Q\phi}\mathrm{Tr}\left\{g^{-2}bB\right\}\right]\end{aligned} \tag{3.41}$$

In the last line the trace was expanded in powers of α'. The first term is zero because it is trace over an antisymmetric matrix. Let us define the term under the square-root in the last line as Y and the second term as X,

$$Y = \sqrt{e^{-P\phi}[\eta + h + (2\pi\alpha')e^{-Q\phi}b]} \tag{3.42}$$

$$X = \left[1 - \frac{1}{2}(2\pi\alpha')^2 e^{-2Q\phi}\mathrm{Tr}\left\{(\eta+h)^{-2}bB\right\}\right] \tag{3.43}$$

To get the vertices, we need to find,

$$\frac{\delta^2\mathcal{L}}{\delta b\delta\chi} = \frac{\delta^2(XY)}{\delta b\delta\chi} \tag{3.44}$$

$$= \left[X\frac{\delta^2 Y}{\delta b\delta\chi} + \frac{\delta Y}{\delta\chi}\frac{\delta X}{\delta b} + \frac{\delta X}{\delta\chi}\frac{\delta Y}{\delta b} + Y\frac{\delta^2 X}{\delta b\delta\chi}\right] \tag{3.45}$$

where, $\chi \equiv \phi, b, h$

Now, listing the required derivatives at $\phi, b, h = 0$.

$$\frac{\delta X}{\delta h_{ij}} = 0 \;\; ; \;\; \frac{\delta X}{\delta b_{kl}} = -\frac{1}{2}(2\pi\alpha^{'})^2 B^{lk} \;\; ; \;\; \frac{\delta X}{\delta\phi} = 0 \tag{3.46}$$

$$\frac{\delta^2 X}{\delta b_{kl}\delta h_{ij}} = (2\pi\alpha^{'})^2\eta^{jk}B^{li} \; ; \; \frac{\delta^2 X}{\delta b_{kl}\delta\phi} = (2\pi\alpha^{'})^2 Q B^{lk} \; ; \; \frac{\delta^2 X}{\delta b_{kl}\delta b_{ij}} = 0 \tag{3.47}$$

$$\frac{\delta Y}{\delta h_{ij}} = \frac{1}{2}\eta^{ij} \;\; ; \;\; \frac{\delta Y}{\delta b_{kl}} = 0 \;\; ; \;\; \frac{\delta Y}{\delta\phi} = -\frac{p+1}{2}P \tag{3.48}$$

$$\frac{\delta^2 Y}{\delta b_{kl}\delta h_{ij}} = 0 \; ; \; \frac{\delta^2 Y}{\delta b_{kl}\delta\phi} = 0 \; ; \; \frac{\delta^2 Y}{\delta b_{kl}\delta b_{ij}} = (2\pi\alpha^{'})\eta^{li}\eta^{jk} \tag{3.49}$$

Using these derivatives, the vertices for the graviton and dilaton are,

$$V_h^{ij} = -\tau_p(2\pi\alpha^{'})^2\left[-\frac{1}{4}B^{lk}\eta^{ij} + \eta^{jk}B^{li}\right] \tag{3.50}$$

$$V_\phi = -\tau_p(2\pi\alpha^{'})^2\left[\frac{1}{4}(p+1)P + Q\right]B^{lk} \tag{3.51}$$

For the b-field we need to consider couplings upto $\mathcal{O}(B^2)$, the next order term in B in the expansion of eqn(3.40). Since we are not interested in the graviton and dilaton exchange at this order, so putting them to zero,

$$\begin{aligned}
\mathcal{O}(B^2) &= \sqrt{\eta + (2\pi\alpha^{'})b}\left[-\frac{1}{4}\mathrm{Tr}\left(\frac{(2\pi\alpha^{'})}{\eta + (2\pi\alpha^{'})b}B\right)^2 + \frac{1}{8}\left(\mathrm{Tr}\frac{(2\pi\alpha^{'})}{\eta + (2\pi\alpha^{'})b}B\right)^2\right] \\
&= (2\pi\alpha^{'})^2\left(1 - \frac{(2\pi\alpha^{'})^2}{4}\mathrm{Tr}(b^2)\right) \times \\
&\times \left[-\frac{1}{4}\mathrm{Tr}\left(B^2 + (2\pi\alpha^{'})^2(bBbB + 2b^2B^2)\right) + \frac{(2\pi\alpha^{'})^2}{8}\mathrm{Tr}(bB)\mathrm{Tr}(bB)\right]
\end{aligned} \tag{3.52}$$

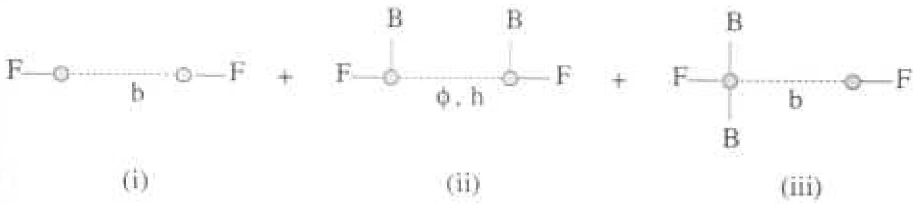

Figure 3.2: Two point amplitude upto quadratic order in B. (i) and (iii) are due to only b-field exchange, (ii) is due to graviton and dilaton exchange.

This along with the $\mathcal{O}(1)$ term gives the following vertex for the b-field.

$$
\begin{aligned}
V_b^{ij} &= \tau_p \frac{(2\pi\alpha')^2}{2} \eta^{li}\eta^{jk} \left(1 - (2\pi\alpha')^2 \frac{1}{4} \mathrm{Tr}(B^2)\right) \\
&- \tau_p (2\pi\alpha')^4 \left[\frac{1}{4} B^{kl} B^{ij} - \frac{1}{2} B^{li} B^{jk} - (B^2)^{li} \eta^{jk}\right]
\end{aligned}
\tag{3.53}
$$

The propagators are the usual ones, rewriting them from eqns(3.20, 3.30,3.34),

$$
< h_{ij} h_{i'j'} > = -2i\kappa^2 \frac{[\eta_{ii'}\eta_{jj'} + \eta_{ij'}\eta_{i'j} - 2/(D-2)\eta_{ij}\eta_{i'j'}]}{k_\perp^2 + k_\parallel^2} \tag{3.54}
$$

$$
< \phi\phi > = -\frac{(D-2)i\kappa^2}{4} \frac{1}{k_\perp^2 + k_\parallel^2} \tag{3.55}
$$

$$
< b_{ij} b_{i'j'} > = -\frac{2i\kappa^2}{(2\pi\alpha')^2} \frac{[\eta_{ii'}\eta_{jj'} - \eta_{ji'}\eta_{ij'}]}{k_\perp^2 + k_\parallel^2} \tag{3.56}
$$

With these, the contributions from the three modes to the two point function can be worked out. We are interested in the correction to the quadratic term in the effective action for the gauge field on the brane. This can be constructed with the vertices computed above and the propagators for the intermediate massless closed string states. This correction for the nonplanar diagram can be written as,

$$
A_2(bb) = \int d^{p+1}\xi \int d^{p+1}\xi' b(\xi) b(\xi') V < \chi(\xi)\chi(\xi') > V \tag{3.57}
$$

where,

$$< \chi(\xi)\chi(\xi') >= \int \frac{d^D k}{(2\pi)^D} < \chi(k_\perp, k_\parallel)\chi(-k_\perp, -k_\parallel) > e^{-ik_\parallel(\xi-\xi')} \tag{3.58}$$

We can rewrite eqn(3.57) in momentum space coordinates as,

$$\begin{aligned} A_2(bb) &= \mathcal{V}_{p+1} \int \frac{d^{p+1}p}{(2\pi)^{p+1}} b(p)b(-p) \int \frac{d^l k_\perp}{(2\pi)^l} V < \chi(k_\perp, -p)\chi(-k_\perp, p) > V \\ &= \mathcal{V}_{p+1} \int \frac{d^{p+1}p}{(2\pi)^{p+1}} b(p)b(-p) L_2(p, -p) \end{aligned} \tag{3.59}$$

In the planar two point function, both the vertices are on the same end of the cylinder in the world-sheet computation. In the field theory this corresponds to putting both the vertices at the same position on the D-brane. In other words, in the expansion of the DBI action, we should be looking for $b^2\chi$ vertices on one end and a χ tadpole on the other. In this case, from the above calculation, $k_\parallel = 0$. So the closed string propagator is just $1/k_\perp^2$, i.e. the propagator is not modified by the momentum of the gauge field on the brane. This is what we expect, as in the field theory on the brane, the loop integrals are not modified for the planar diagrams. Here we will only concentrate on the nonplanar sector.

As mentioned earlier, on the brane we will identify,

$$b_{kl}(p) \equiv \frac{g_0}{\sqrt{2\alpha'}} F_{kl}(p) = \frac{g_0}{\sqrt{2\alpha'}} p_{[k} A_{l]}(p) \tag{3.60}$$

For the graviton we have,

$$L_2(bhb) = \int \frac{d^l k_\perp}{(2\pi)^l} V_h^{ij,kl} < h_{ij} h_{i'j'} > V_h^{i'j',k'l'} \tag{3.61}$$

$$= -i\kappa^2 \tau_p^2 (2\pi\alpha')^4 \int \frac{d^l k}{(2\pi)^l} \frac{2}{k_\perp^2 + p^2} \times \tag{3.62}$$

$$\times \left[-B^{2ll'}\eta^{kk'} + B^{lk'}B^{l'k} + \left(\frac{p+1}{8} + \frac{p-1}{D-2} - \frac{(p+1)^2}{8(D-2)} - 1 \right) B^{lk}B^{l'k'} \right]$$

For the dilaton,

$$
\begin{aligned}
L_2(b\phi b) &= \int \frac{d^l k_\perp}{(2\pi)^l} V_\phi^{kl} < \phi\phi > V_\phi^{k'l'} \qquad (3.63)\\
&= -i\kappa^2\tau_p^2(2\pi\alpha')^4 \int \frac{d^l k}{(2\pi)^l}\frac{1}{k_\perp^2+p^2}\frac{(D-2)}{4}\left(\frac{1}{2}-\frac{p+1}{D-2}+\frac{4}{D-2}\right)^2 B^{lk}B^{l'k'}
\end{aligned}
$$

Adding the contributions from the graviton and the dilaton,

$$
\begin{aligned}
L_2(bhb+b\phi b) &= -i\kappa^2\tau_p^2(2\pi\alpha')^4 \int \frac{d^l k_\perp}{(2\pi)^l}\frac{1}{k_\perp^2+p^2}\times \qquad (3.64)\\
&\times \left[-2B^{2ll'}\eta^{kk'} + 2B^{lk'}B^{l'k} + B^{lk}B^{l'k'}\left(\frac{D-2}{16}-1\right)\right]
\end{aligned}
$$

Similarly for the b-field we have,

$$
\begin{aligned}
L_2(bbb) &= \int \frac{d^l k_\perp}{(2\pi)^l} V_b^{ij,kl} < b_{ij}b_{i'j'} > V_b^{i'j',k'l'} \qquad (3.65)\\
&= -i\kappa^2\tau_p^2 \int \frac{d^l k}{(2\pi)^l}\frac{1}{k_\perp^2+p^2}\times\\
&\times \left[\frac{(2\pi\alpha')^2}{4}\{1-\frac{(2\pi\alpha')^2}{2}\mathrm{Tr}(B^2)\}(\eta^{ll'}\eta^{kk'}-\eta^{lk'}\eta^{kl'})\right.\\
&+ (2\pi\alpha')^4\{\frac{1}{2}B^{lk}B^{l'k'} + ((B^2)^{ll'}\eta^{kk'} - (B^2)^{lk'}\eta^{kl'})\\
&- \left.\frac{1}{2}(B^{lk'}B^{l'k} - B^{ll'}B^{k'k}) + (lk)\leftrightarrow(l'k')\}\right]
\end{aligned}
$$

For the full two point function, there are cancellations between the eqn(3.64) and eqn(3.65). The final answer is,

$$
\begin{aligned}
L_2 &= -i\kappa^2\tau_p^2 \int \frac{d^l k_\perp}{(2\pi)^l}\frac{1}{k_\perp^2+p^2}\times \qquad (3.66)\\
&\times \left[(2\pi\alpha')^4 B^{lk}B^{l'k'}\frac{D-2}{32} + \frac{(2\pi\alpha')^2}{4}\{1-\frac{(2\pi\alpha')^2}{2}\mathrm{Tr}(B^2)\}(\eta^{ll'}\eta^{kk'}-\eta^{lk'}\eta^{kl'})\right.\\
&+ \left.\frac{(2\pi\alpha')^4}{2}\{(B^2)^{ll'}\eta^{kk'} - (B^2)^{lk'}\eta^{kl'}\} + (lk)\leftrightarrow(l'k')\right]
\end{aligned}
$$

The full two point effective action, can now be constructed by putting back L_2 in eqn(3.59) along with the identification eqn(3.132). To compare this with the closed string channel result with only massless exchanges, eqn(3.18) we must note the expansions of the following quantities to appropriate powers of B.

$$G^{ij} \sim \eta^{ij} + (2\pi\alpha')^2(B^2)^{ij} + \mathcal{O}(B^4) \tag{3.67}$$

$$\theta^{ij} \sim -(2\pi\alpha')^2 B^{ij} + \mathcal{O}(B^3) \tag{3.68}$$

$$\sqrt{\eta + (2\pi\alpha')B} \sim \left[1 - \frac{(2\pi\alpha')^2}{4}\mathrm{Tr}(B^2) + \mathcal{O}(B^4)\right] \tag{3.69}$$

With these expansions, we can see that eqn(3.18) equals the sum of massless contributions, in eqn(3.66).

3.2.2 Noncommutative case ($g = \epsilon\eta$)

We now turn to the Seiberg Witten limit, (2.14) which gives rise to noncommutative field theory on the brane. Here again we will be interested in writing out the two point function eqn(3.18) in the closed string channel as a sum of the massless closed string modes. Due to the scaling of the closed string metric, unlike the earlier case, we will now expand all results in powers of the scale parameter for closed string metric, ϵ. We begin by expanding the DBI action,

$$\mathcal{L} = \sqrt{(2\pi\alpha')e^{-(P+Q)\phi}(B+b)}\left[1 + \frac{1}{(2\pi\alpha')e^{-Q\phi}(B+b)}\epsilon(\eta+h)\right]^{1/2} \tag{3.70}$$

For a matrix M, we have that following expansion,

$$\sqrt{1+M} = \exp\left[\frac{1}{2}\mathrm{Trlog}(1+M)\right] \tag{3.71}$$

$$= 1 + \frac{1}{2}\mathrm{Tr}(M - \frac{M^2}{2} + ...) + \frac{1}{8}\left[\mathrm{Tr}(M - \frac{M^2}{2} + ...)\right]^2 + ... \tag{3.72}$$

For M antisymmetric, terms containing $\mathrm{Tr}(M)$ vanishes, hence to order ϵ^2, we have,

$$\begin{aligned}
\mathcal{L} &= \sqrt{(2\pi\alpha')e^{-(P+Q)\phi}(B+b)}\left[1 - \frac{\epsilon^2}{4}\mathrm{Tr}\left(\frac{1}{(2\pi\alpha')e^{-Q\phi}(B+b)}(\eta+h)\right)^2\right] \\
&= \sqrt{(2\pi\alpha')e^{-(P+Q)\phi}(B+b)} \times \\
&\times \left[1 - \frac{\epsilon^2 e^{2Q\phi}}{4(2\pi\alpha')^2}\mathrm{Tr}\left[\frac{1}{B^2}\left(1 - \frac{2}{B}b + 3\frac{1}{B}b\frac{1}{B}b - ...\right)(\eta+h)^2\right]\right]
\end{aligned} \tag{3.73}$$

Let us now first consider the $\mathcal{O}(1)$ term in ϵ,

$$\mathcal{L}\mid_{\mathcal{O}(1)}=\sqrt{(2\pi\alpha')e^{-(P+Q)\phi}(B+b)} \tag{3.74}$$

There is no graviton coupling at this order. The ϕ and b-field vertices from this are,

$$V_\phi^1 = -\frac{1}{2}\sqrt{(2\pi\alpha')B}\left(\frac{1}{B}\right)^{lk} \tag{3.75}$$

$$V_b^1 = \sqrt{(2\pi\alpha')B}\left[\frac{1}{4}\left(\frac{1}{B}\right)^{lk}\left(\frac{1}{B}\right)^{ji}-\frac{1}{2}\left(\frac{1}{B}\right)^{jk}\left(\frac{1}{B}\right)^{li}\right] \tag{3.76}$$

Now, let us consider the ϵ^2 term. As in the earlier case let us define,

$$Y = \sqrt{(2\pi\alpha')e^{-(P+Q)\phi}(B+b)} \tag{3.77}$$

$$X = -\frac{\epsilon^2 e^{2Q\phi}}{4(2\pi\alpha')^2}\mathrm{Tr}\left[\frac{1}{B^2}\left(1-\frac{2}{B}b+3\frac{1}{B}b\frac{1}{B}b-...\right)(\eta+h)^2\right] \tag{3.78}$$

We are interested in the two point function only upto $\mathcal{O}(\epsilon^2)$, hence we need not consider the graviton vertex. Also the b-field propagator has a ϵ^2 factor (3.30). So, it is only necessary to compute the dilaton vertex at this order. Listing the required derivatives,

$$\frac{\delta Y}{\delta\phi}=-\sqrt{(2\pi\alpha')B} \quad \frac{\delta Y}{\delta b_{kl}}=\frac{1}{2}\sqrt{(2\pi\alpha')B}\left(\frac{1}{B}\right)^{lk} \tag{3.79}$$

$$\frac{\delta^2 Y}{\delta b_{kl}\delta\phi}=V_\phi^1 \quad \frac{\delta^2 X}{\delta b_{kl}\delta\phi}=\frac{\epsilon^2 4Q}{4(2\pi\alpha')^2}\left(\frac{1}{B^3}\right)^{lk} \tag{3.80}$$

$$\frac{\delta X}{\delta\phi}=-\frac{\epsilon^2 2Q}{4(2\pi\alpha')^2}\mathrm{Tr}\frac{1}{B^2} \quad \frac{\delta X}{\delta b_{kl}}=\frac{\epsilon^2 2}{4(2\pi\alpha')^2}\left(\frac{1}{B^3}\right)^{lk} \tag{3.81}$$

After putting in all the appropriate derivatives, the vertices for the dilaton and the b-field upto $\mathcal{O}(\epsilon^2)$ is given by,

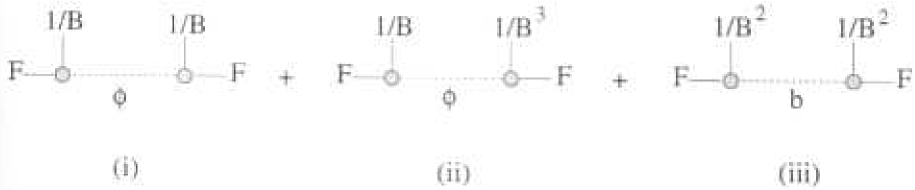

Figure 3.3: Two point amplitude upto $\mathcal{O}(\epsilon^2)$. (i) and (ii) are due to dilaton exchange, (iii) is due to b-field exchange.

$$
\begin{aligned}
V_\phi &= \sqrt{(2\pi\alpha')B}\left[-\frac{1}{2}\left(\frac{1}{B}\right)^{lk} + \frac{\epsilon^2(4Q-2)}{4(2\pi\alpha')^2}\left(\left(\frac{1}{B^3}\right)^{lk} - \frac{1}{4}\mathrm{Tr}\left(\frac{1}{B^2}\right)\left(\frac{1}{B}\right)^{lk}\right)\right] \\
V_b &= V_b^1
\end{aligned}
\tag{3.82}
$$

The situation in this case is similar to that of the earlier small B expansion and is shown in Figure 3.3. The propagators in this limit, eqns(3.22,3.30),

$$
< \phi\phi > \; = \; -\frac{(D-2)i\kappa^2}{4}\frac{1}{k_\perp^2 + \epsilon^{-1}k_\parallel^2} \tag{3.83}
$$

$$
< b_{ij}b_{i'j'} > \; = \; -\frac{2i\kappa^2\epsilon^2}{(2\pi\alpha')^2}\frac{[\eta_{ii'}\eta_{jj'} - \eta_{ji'}\eta_{ij'}]}{k_\perp^2 + \epsilon^{-1}k_\parallel^2} \tag{3.84}
$$

With the vertices computed above and the propagator in this limit, the two point function for the dilaton is,

$$
\begin{aligned}
I_1(b\phi b) &= -i\det(2\pi\alpha' B)\kappa^2\tau_p^2\frac{(D-2)}{4}\int\frac{d^lk_\perp}{(2\pi)^l}\frac{1}{k_\perp^2+\epsilon^{-1}p^2} \times \\
&\times \left[\frac{1}{8}\left(\frac{1}{B}\right)^{lk}\left(\frac{1}{B}\right)^{l'k'} - \frac{\epsilon^2(4Q-2)}{8(2\pi\alpha')^2}\left(\left(\frac{1}{B^3}\right)^{lk} - \frac{1}{4}\mathrm{Tr}\left(\frac{1}{B^2}\right)\left(\frac{1}{B}\right)^{lk}\right)\left(\frac{1}{B}\right)^{l'k'}\right] \\
&+ (lk) \leftrightarrow (l'k')
\end{aligned}
\tag{3.85}
$$

For the b-field,

$$
\begin{aligned}
L_2(bbb) &= -i\det(2\pi\alpha' B)\kappa^2\tau_p^2\frac{\epsilon^2}{(2\pi\alpha')^2}\int\frac{d^l k_\perp}{(2\pi)^l}\frac{2}{k_\perp^2+\epsilon^{-1}p^2}\times \\
&\times\ [\left(\frac{1}{4}\left(\frac{1}{B^3}\right)^{lk}-\frac{1}{16}\mathrm{Tr}\left(\frac{1}{B^2}\right)\left(\frac{1}{B}\right)^{lk}\right)\left(\frac{1}{B}\right)^{l'k'} \\
&+\ \frac{1}{8}\left(\frac{1}{B^2}\right)^{kk'}\left(\frac{1}{B^2}\right)^{ll'}-\frac{1}{8}\left(\frac{1}{B^2}\right)^{k'l}\left(\frac{1}{B^2}\right)^{lk'} \\
&+\ (lk)\leftrightarrow(l'k')]
\end{aligned}
\tag{3.86}
$$

The first two terms cancel with the Q-dependent terms of the dilaton, the resulting amplitude can now be written as,

$$
L_2 = -i\det(2\pi\alpha' B)\kappa^2\tau_p^2\int\frac{d^l k_\perp}{(2\pi)^l}\frac{1}{k_\perp^2+\epsilon^{-1}p^2}[\mathcal{O}(1)+\mathcal{O}(\epsilon^2)] \tag{3.87}
$$

where,

$$
\mathcal{O}(1)=\left[\frac{(D-2)}{32}\left(\frac{1}{B}\right)^{lk}\left(\frac{1}{B}\right)^{l'k'}+(lk)\leftrightarrow(l'k')\right] \tag{3.88}
$$

$$
\begin{aligned}
\mathcal{O}(\epsilon^2) &= \frac{\epsilon^2}{(2\pi\alpha')^2}\frac{(D-2)}{16}\left[\left[\left(\frac{1}{B^3}\right)^{lk}-\frac{1}{4}\mathrm{Tr}\left(\frac{1}{B^2}\right)\left(\frac{1}{B}\right)^{lk}\right]\left(\frac{1}{B}\right)^{l'k'}\right] \\
&+\ \frac{\epsilon^2}{(2\pi\alpha')^2}\left[\frac{1}{4}\left(\frac{1}{B^2}\right)^{kk'}\left(\frac{1}{B^2}\right)^{ll'}-\frac{1}{4}\left(\frac{1}{B^2}\right)^{k'l}\left(\frac{1}{B^2}\right)^{lk'}\right] \\
&+\ (lk)\leftrightarrow(l'k')
\end{aligned}
\tag{3.89}
$$

We can now reconstruct the quadratic term in effective action, (3.59) following the earlier case. With the following expansions, it is easy to check that the sum of the massless contributions adds upto eqn(3.18).

$$
G^{ij} \sim -\frac{\epsilon}{(2\pi\alpha')^2}\left(\frac{1}{B^2}\right)^{ij}+\mathcal{O}(\epsilon^3) \tag{3.90}
$$

$$
\theta^{ij} \sim \left(\frac{1}{B}\right)^{ij}+\frac{\epsilon^2}{(2\pi\alpha')^2}\left(\frac{1}{B^3}\right)^{ij} \tag{3.91}
$$

$$
\sqrt{\epsilon\eta+(2\pi\alpha')B} \sim \sqrt{(2\pi\alpha')B}\left[1-\frac{\epsilon^2}{4(2\pi\alpha')^2}\mathrm{Tr}\left(\frac{1}{B^2}\right)\right] \tag{3.92}
$$

Note that, at the tree-level, to the linear order, $\hat{F} = F$, (2.18). At this quadratic order in the effective action there is no need for redefinition of F to equate the result here with that of string theory result in eqn(3.18).

3.2.3 Noncommutative case ($G = \eta$)

In this part we finally consider the restriction of the open string metric, $G = \eta$. The lowest order solution for the closed string metric, g in α' in this limit is,

$$g = -(2\pi\alpha')^2 B^2 + \mathcal{O}(\alpha'^4) \tag{3.93}$$

We will now consider expansions of the two point functions in powers of α'. We begin again with the following DBI Lagrangian,

$$\mathcal{L} = \sqrt{(2\pi\alpha')e^{-(P+Q)\phi}(B+b)}\left[1 - \frac{1}{e^{-Q\phi}(B+b)}(2\pi\alpha')B^2(\eta+h)^2\right]^{1/2} \tag{3.94}$$

The calculation for the vertices is same as before, there is no graviton vertex to the leading orders. The dilaton and the b-field vertices are,

$$\begin{aligned} V_\phi &= \sqrt{(2\pi\alpha')B}\left[-\frac{1}{2}\left(\frac{1}{B}\right)^{lk} + \frac{(2\pi\alpha')^2(4Q-2)}{4}\left(B^{lk} - \frac{1}{4}\mathrm{Tr}(B^2)\left(\frac{1}{B}\right)^{lk}\right)\right] \\ V_b &= \sqrt{(2\pi\alpha')B}\left[\frac{1}{4}\left(\frac{1}{B}\right)^{lk}\left(\frac{1}{B}\right)^{ji} - \frac{1}{2}\left(\frac{1}{B}\right)^{jk}\left(\frac{1}{B}\right)^{li}\right] \end{aligned} \tag{3.95}$$

The propagators for the dilaton and the b-field are modified as,

$$< \phi\phi > = -\frac{(D-2)i\kappa^2}{4}\frac{1}{k_\perp^2 + \bar{k_\parallel}^2/(2\pi\alpha')^2} \tag{3.96}$$

$$< b_{ij}b_{i'j'} > = -2i\kappa^2(2\pi\alpha')^2\frac{[B^2_{ii'}B^2_{jj'} - B^2_{ji'}B^2_{ij'}]}{k_\perp^2 + \bar{k_\parallel}^2/(2\pi\alpha')^2} \tag{3.97}$$

With these vertices (shown in Figure 3.4) and the propagators from eqns(3.24,3.30), the two point functions are now given by,

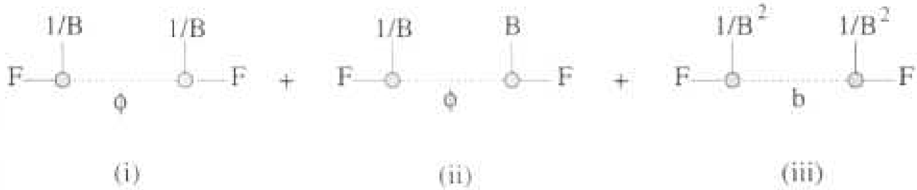

Figure 3.4: Two point amplitude upto $\mathcal{O}(\alpha'^2)$. (i) and (ii) are due to dilaton exchange, (iii) is due to b-field exchange.

$$\begin{aligned} L_2(b\phi b) &= -i\det(2\pi\alpha' B)\kappa^2\tau_p^2\frac{(D-2)}{4}\int\frac{d^l k_\perp}{(2\pi)^l}\frac{1}{k_\perp^2+\vec{p}^2/(2\pi\alpha')^2}\times \\ &\times\left[\frac{1}{8}\left(\frac{1}{B}\right)^{lk}\left(\frac{1}{B}\right)^{l'k'}-\frac{(2\pi\alpha')^2(4Q-2)}{8}\left(B^{lk}-\frac{1}{4}\mathrm{Tr}(B^2)\left(\frac{1}{B}\right)^{lk}\right)\left(\frac{1}{B}\right)^{l'k'}\right] \\ &+\ (lk)\leftrightarrow(l'k') \end{aligned} \tag{3.98}$$

$$\begin{aligned} L_2(bbb) &= -i\det(2\pi\alpha' B)\kappa^2\tau_p^2(2\pi\alpha')^2\int\frac{d^l k_\perp}{(2\pi)^l}\frac{2}{k_\perp^2+\vec{p}^2/(2\pi\alpha')^2}\times \\ &\times\left[\left(\frac{1}{4}B^{lk}-\frac{1}{16}\mathrm{Tr}(B^2)\left(\frac{1}{B}\right)^{lk}\right)\left(\frac{1}{B}\right)^{l'k'}+\frac{1}{8}(\eta^{ll'}\eta^{kk'}-\eta^{kl'}\eta^{lk'})\right] \\ &+\ (lk)\leftrightarrow(l'k') \end{aligned} \tag{3.99}$$

As before, the first term of the b exchange cancels with the Q dependent term of the dilaton exchange. The full two point answer is

$$L_2 = -i\det(2\pi\alpha' B)\kappa^2\tau_p^2\int\frac{d^l k_\perp}{(2\pi)^l}\frac{1}{k_\perp^2+\vec{p}^2(2\pi\alpha')^2}[\mathcal{O}(1)+\mathcal{O}(\alpha'^2)] \tag{3.100}$$

$$\mathcal{O}(1) = \left[\frac{(D-2)}{32}\left(\frac{1}{B}\right)^{lk}\left(\frac{1}{B}\right)^{l'k'}+(lk)\leftrightarrow(l'k')\right] \tag{3.101}$$

$$
\begin{aligned}
\mathcal{O}(\alpha'^2) &= (2\pi\alpha')^2\frac{(D-2)}{16}\left[\left[B^{lk}-\frac{1}{4}\mathrm{Tr}(B^2)\left(\frac{1}{B}\right)^{lk}\right]\left(\frac{1}{B}\right)^{l'k'}\right] \\
&+ (2\pi\alpha')^2\left[\frac{1}{4}\left(\eta^{ll'}\eta^{kk'}-\eta^{kl'}\eta^{lk'}\right)\right] \\
&+ (lk)\leftrightarrow(l'k')
\end{aligned}
\tag{3.102}
$$

We will need the following expansions in this limit, to expand the closed string channel result upto this order. We have already set,

$$
G^{ij} = \eta^{ij} \tag{3.103}
$$

and with the solution for g, eqn(3.93) to the lowest order in α',

$$
\theta^{ij} \sim \left(\frac{1}{B}\right)^{ij} + (2\pi\alpha')^2 B^{ij} \tag{3.104}
$$

$$
\sqrt{g+(2\pi\alpha')B} \sim \sqrt{(2\pi\alpha')B}\left[1-\frac{(2\pi\alpha')^2}{4}\mathrm{Tr}(B^2)\right] \tag{3.105}
$$

As in the earlier cases, the massless contributions computed here, eqn(3.100) adds upto eqn(3.18). Note that the situation here is similar to that of the earlier case in Section 3.2.2. As $\alpha' \sim \sqrt{\epsilon}$, the closed string metric in both the cases goes to zero as $g \sim \epsilon$. However the difference being that the two point amplitude differ by powers of B in both the cases, due to the relative power of B^2 in g in this case. Here too, the SW map between the usual and the noncommutative field strength eqn(2.18), remains the same. The differences in the powers of B in the two point amplitudes, eqn(3.87) and eqn(3.100) are absorbed in G, θ and $\sqrt{g+(2\pi\alpha')B}$ in the two cases. We can work with any of the forms of the closed string metric g, the important point being that g should go to zero as ϵ which gives the noncommutative gauge theory on the brane.

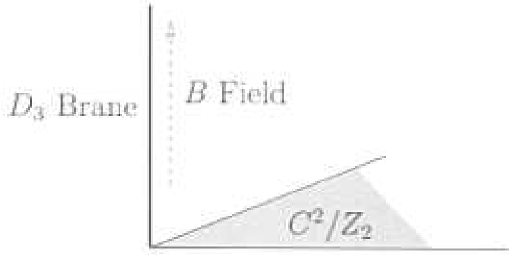

Figure 3.5: The background for string propagation. D_3 brane world volume directions are 0,1,2,3. Orbifolded directions 6,7,8,9.

3.3 Open Superstring in Background B-field

After studying the bosonic case, let us now turn to the supersymmetric example. With the observations made in Section 3.1 and following [23], we will consider $\mathcal{N}=2$ gauge theory that can be realised on a *fractional* D_3 brane located at the fixed point of C^2/Z_2 orbifold. The setup is shown in Figure 3.5. With the B-field turned on along the world-volume directions, the low-energy effective theory on the brane is described by a noncommutative $\mathcal{N}=2$ gauge theory. The world-sheet action for the fermions coupled to B-field is given by,

$$S_F = \frac{i}{4\pi\alpha'}\int_\Sigma g_{MN}\bar{\psi}^M\rho^\alpha\partial_\alpha\psi^N - \frac{i}{4}\int_{\partial\Sigma} B_{MN}\bar{\psi}^N\rho^0\psi^M \tag{3.106}$$

The full action including the bosons (2.10) and the fermions (3.106) with the bulk and the boundary terms are invariant under the following supersymetry transformations,

$$\begin{aligned}\delta X^M &= \bar{\epsilon}\psi^M \\ \delta\psi^M &= -i\rho^\alpha\partial_\alpha X^M\epsilon\end{aligned} \tag{3.107}$$

We now write down the boundary equations by varying (3.106) with the following constraints,

$$\delta\psi_L^M = \delta\psi_R^M \mid_{\sigma=\pi} \quad \text{and} \quad \delta\psi_L^M = -(-1)^a \delta\psi_R^M \mid_{\sigma=0} \tag{3.108}$$

where, $a = 0, 1$ gives the NS and the R sectors respectively This gives the following boundary equations,

$$g_{MN}(\psi_L^N - \psi_R^N) + 2\pi\alpha' B_{MN}(\psi_L^N + \psi_R^N) = 0 \mid_{\sigma=\pi} \tag{3.109}$$

$$g_{MN}(\psi_L^M + (-1)^a \psi_R^M) + 2\pi\alpha' B_{MN}(\psi_L^N - (-1)^a \psi_R^N) \mid_{\sigma=0} \tag{3.110}$$

To write down the correlator for the fermions, first define,

$$\begin{aligned} \psi^M &= \psi_L^M(\sigma, \tau) \qquad 0 \leq \sigma \leq \pi \\ &= \left(\frac{g - 2\pi\alpha' B}{g + 2\pi\alpha' B}\right)^M_N \psi_R^N(2\pi - \sigma, \tau) \qquad \pi \leq \sigma \leq 2\pi \end{aligned} \tag{3.111}$$

This is the usual doubling trick that ensures the boundary conditions (3.109). In the following section we would compute the two point function for the gauge field on the brane by inserting two vertex operators at the boundaries of the cylinder. Restricting ourselves to the directions along the brane, this vertex operator for the gauge field in the zero picture is given by,

$$V(p, x, y) = \frac{g_o}{(2\alpha')^{1/2}} \epsilon_j \left(i\partial_y X^j + 4p.\Psi\Psi^j\right) e^{ip.X}(x, y) \tag{3.112}$$

where Ψ^i is given by,

$$\begin{aligned} \Psi^i(0, \tau) &= \frac{1}{2}\left(\psi_L^i(0, \tau) + (-1)^{a+1}\psi_R^i(0, \tau)\right) = \left(\frac{1}{g - 2\pi\alpha' B} g\right)^i_j \psi_L^j(0, \tau) \\ \Psi^i(\pi, \tau) &= \frac{1}{2}\left(\psi_L^i(\pi, \tau) + \psi_R^i(\pi, \tau)\right) = \left(\frac{1}{g - 2\pi\alpha' B} g\right)^i_j \psi_L^j(\pi, \tau) \end{aligned} \tag{3.113}$$

Using (3.113), the correlation function for Ψ is given by,

$$\left\langle \Psi^i(w)\Psi^j(w')\right\rangle = G^{ij}\mathcal{G}\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](w - w') \tag{3.114}$$

where G^{ij} is the open string metric defined in (2.13) and, $\mathcal{G}\left[\substack{\alpha\\\beta}\right](w-w')$ is given by [30],

$$\mathcal{G}\left[\substack{\alpha\\\beta}\right](w-w') = \frac{\alpha'}{4\pi}\frac{\vartheta\left[\substack{\alpha\\\beta}\right]\left(\frac{w-w'}{2\pi}, it\right)\vartheta'\left[\substack{1/2\\1/2}\right](0, it)}{\vartheta\left[\substack{1/2\\1/2}\right]\left(\frac{w-w'}{2\pi}, it\right)\vartheta\left[\substack{\alpha\\\beta}\right](0, it)} \tag{3.115}$$

α, β denotes the spin structures. $\alpha = (0, 1/2)$ are the NS and the R sectors and $\beta = (0, 1/2)$ stands for the absence or the presence of the world-sheet fermion number $(-1)^F$ with ψ being antiperiodic or periodic along the τ direction on the world-sheet. w, w' are located at the boundaries of the world-sheet for the open string which is a cylinder, i.e. at $\sigma = 0, \pi$.

3.3.1 Strings on C^2/Z_2 orbifold

An efficient and simple way to break $\mathcal{N} = 4$ supersymmetry and obtain gauge theories with less supersymmetries is by orbifolding the background space. Specifically strings on C^2/Z_2 gives rise to $\mathcal{N} = 2$ supersymmetric gauge theory on D3-branes with world volume directions transverse to the orbifilded planes. The open and the closed string spectrum on this orbifold have been nicely worked out in [27]. We include a brief analysis here that will be relevant for the later discussions. We will take the orbifolded directions to be 6, 7, 8, 9 with $Z_2 = \{g_i \mid e, g\}$ such that $g^2 = e$ The action of g on these coordinates is given by,

$$gX^I = -X^I \qquad \text{for} \qquad I = 6, 7, 8, 9 \tag{3.116}$$

In order to preserve world sheet supersymmetry we must also consider the action of Z_2 on the fermionic partners, ψ^I.

3.3.1.1 Open String Spectrum and Fractional Branes

On a particular state of the open string the orbifold action is on the oscillators, ψ^I_{-r} along with the Chan-Paton indices associated with it. Let us consider the massless bosonic states from the NS sector.

$$g|i,j,\psi^I_{-1/2}> = \gamma_{ii'}|i',j',\hat{g}\psi^I_{-1/2}> \gamma^{-1}_{j'j} \tag{3.117}$$

where γ is a representation of Z_2 The spectrum is obtained by keeping the states that are invariant under the above action. To derive this it is easier to work in the basis where γ is diagonal.

$$\gamma = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.118}$$

The action on the Chan-Paton indices can be thought of as,

$$\gamma \begin{pmatrix} 11 & 12 \\ 21 & 22 \end{pmatrix} \gamma^{-1} = \begin{pmatrix} 11 & -12 \\ -21 & 22 \end{pmatrix} \tag{3.119}$$

Thus the diagonal ones survive for the Z_2 action on the oscillators is $\hat{g}|\psi^I_{-1/2}> = |\psi^I_{-1/2}>$, i.e. for $I = 2,3,4,5$ and the off-diagonal ones are preserved for oscillators that are odd under the Z_2 action, $\hat{g}|\psi^I_{-1/2}> = -|\psi^I_{-1/2}>$, i.e. for $I = 6,7,8,9$. The spectrum can thus be summarised as,

$$A^I \rightarrow \begin{bmatrix} A^I_1 & A^I_2 & \text{2 gauge fields} & I = 2,3 \\ \phi^I_1 & \phi^I_2 & \text{4 real scalars} & I = 4,5 \\ \Phi^I_1 & \Phi^I_2 & \text{8 real scalars} & I = 6,7,8,9 \end{bmatrix} \tag{3.120}$$

The orbifold action on the space-time spinors is given by,

$$\chi_{ij} \rightarrow \gamma_{ii'} e^{i\pi(s_3+s_4)} \chi_{i'j'} \gamma^{-1}_{j'j} \tag{3.121}$$

where $\chi_{i'j'} = |s_1, s_2, s_3, s_4\rangle_{i'j'}$ are the sixteen spinors of $SO(8)$, with $s_i = \pm 1/2$. The spinors in (3.121) are left invariant for, $s_3 + s_4 = 0$ and $s_3 + s_4 = \pm 1$. The first one leaves χ_{11} and χ_{22} invariant and the second one leaves χ_{12} and χ_{21}. Projection onto one of the chiralities leaves four copies of each of the spinors.

The above fields can be grouped into two vector multiplets and two hypermultiplets of $\mathcal{N} = 2$ with gauge group $U(1) \times U(1)$. The beta function for the gauge couplings for this theory vanishes and the theory is conformally invariant. Now consider an irreducible representation $\gamma = \pm 1$. This acts trivially on the Chan-Paton indices. These branes are known as *fractional branes*. From the geometric point of view there is no image for the $D3$ brane. The brane is localised at the fixed point on the orbifold plane but is free to move in the non-orbifolded (4,5) directions.

Following the above analysis, the spectrum consists of a single gauge field and two scalars completing the vector multiplet of $\mathcal{N} = 2$ with gauge group $U(1)$. The beta function for this theory is nonzero. With a constant B-field turned on along the world volume directions of the D_3-brane,$(0, 1, 2, 3)$, the low energy dynamics on the brane will be described be noncommutative gauge theory in the Seiberg-Witten limit. In the following section we will study the ultraviolet behavior of this theory and see how the UV divergences have a natural interpretation in terms of IR divergences due to massless closed string modes as a result of open-closed string duality.

3.3.1.2 Closed String Spectrum

The closed string theory consists of additional twisted sectors apart from the untwisted sectors. The orbifold action an the space-time implies the following boundary conditions on the world-sheet bosons and fermions,

$$\begin{aligned} X^I(\sigma + 2\pi, \tau) &= \pm X^I(\sigma, \tau) \\ \psi^I(\sigma + 2\pi, \tau) &= \pm \psi^I(\sigma, \tau) \quad I = 6, 7, 8, 9 \end{aligned} \tag{3.122}$$

For the world sheet fermions, the $(+)$-sign stands for the NS-sector and the $(-)$-sign for the R-sector. For the other directions the boundary conditions on the world-sheet fields are as usual. We will first list the fields in the untwisted sectors. In the NS-NS sector the massless states invariant under the orbifold projection are,

$$\psi^I_{-1/2}\tilde{\psi}^J_{-1/2}\,|0,k\rangle \tag{3.123}$$

where, $I,J=\{2,3,4,5\}$ or $I,J=\{6,7,8,9\}$. The first set of oscillators give the graviton, antisymmetric 2-form field, and the dilaton. The second set gives sixteen scalars.

The orbifold action on the spinor of $SO(8)$ is given by,

$$|s_1,s_2,s_3,s_4\rangle \rightarrow e^{i\pi(s_3+s_4)}\,|s_1,s_2,s_3,s_4\rangle \tag{3.124}$$

The Z_2 invariant R-R state is formed by taking both the left the right states to be either even or odd under Z_2 projection corresponding to $s_3+s_4=0$ or $s_3+s_4=\pm 1$ respectively. GSO projection, restricting to both the left and right states to be of the same chirality gives thirty two states. These states correspond to four 2-form fields and eight scalars.

Let us now turn to the twisted sectors. For the twisted sectors the ground state energy for both the NS and the R sectors vanish. In the NS sector the massless modes come from ψ^I_0, $I=6,7,8,9$ oscillators which form a spinor representation of $SO(4)$. With the GSO and the orbifold projections, the closed string spectrum is given by, $2\times 2=[0]+[2]$. The $[0]$ and the self-dual $[2]$ constitute the four massless scalars in the NS-NS sector. Similarly, in the R sector, the massless modes are given by ψ^I_0 for $I=2,3,4,5$. Thus giving a scalar and a two-form self-dual field in the closed string R-R sector. The couplings for the massless closed string states to the fractional D_3-brane have been worked out by various authors. See for example [27].

3.4 Two point one loop amplitude

In this section we compute the two point function for the gauge fields on the brane. The necessary ingredients are given in Section 3.3 and in the Appendix A3. The vacuum amplitude without any vertex operator insertion vanishes as a result of supersymmetry, i.e.

$$\det(g+2\pi\alpha' B)\int_0^\infty \frac{dt}{4t}(8\pi^2\alpha' t)^{-2}\sum_{(\alpha,\beta,g_i)} Z\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_{g_i}=0 \tag{3.125}$$

The factor of $\det(g+2\pi\alpha' B)$ comes from the trace over the world sheet bosonic zero modes. The sum is over the spin structures $(\alpha,\beta)=(0,1/2)$ corresponding to the $NS-R$ sectors and the GSO projection and the orbifold projection. The elements $Z_{g_i}\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ are computed in the Appendix A3. Let us now compute the two point function. This is given by,

$$\begin{aligned} A(p,-p) &= \det(g+2\pi\alpha' B)\int_0^\infty \frac{dt}{4t}(8\pi^2\alpha' t)^{-2}\times \\ &\times \sum_{(\alpha,\beta,g_i)} Z\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_{g_i}\int_0^{2\pi t}dy\int_0^{2\pi t}dy'\left\langle V(p,x,y)V(-p,x',y')\right\rangle_{(\alpha,\beta)} \end{aligned} \tag{3.126}$$

For the flat space, it is well known that amplitudes with less that four boson insertions vanish. However, in this model the two point amplitude survives. We will now compute this amplitude in the presence of background B-field. First note that the bosonic correlation function, $\langle :\partial_y X^i e^{ip.X} :: \partial'_y X^i e^{-ip.X}:\rangle$, does not contribute to the two point amplitude as it is independent of the spin structure. The two point function would involve the sum over the $Z_{g_i}\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ which makes this contribution zero due to (3.125). The nonzero part of the amplitude will be obtained from the fermionic part,

$$\begin{aligned} \epsilon_k\epsilon_l\left\langle : p.\Psi\Psi^k e^{ip.X} :: p.\Psi\Psi^l e^{-ip.X}:\right\rangle &= \epsilon_k\epsilon_l p_i p_j\left(G^{il}G^{jk}-G^{ij}G^{kl}\right)\times \\ &\times\ \mathcal{G}^2\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](w-w')\left\langle : e^{ip.X} :: e^{-ip.X}:\right\rangle \end{aligned} \tag{3.127}$$

For the planar two point amplitude, both the vertex operators would be inserted at the same end of the cylinder (i.e. at $w=0+iy$ or $\pi+iy$). In this case, the sum in the two point amplitude reduces to,

$$\sum_{(\alpha,\beta,g_i)} Z\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_{g_i} \mathcal{G}^2\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](i\Delta y/2\pi) = \sum_{(\alpha,\beta)} Z\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_e \mathcal{G}^2\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](i\Delta y/2\pi) + \sum_{(\alpha,\beta)} Z\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_g \mathcal{G}^2\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](i\Delta y/2\pi)$$

$$= \frac{4\pi^2}{\eta(it)^6\vartheta_1^2(i\Delta y/2\pi, it)} \sum_{(\alpha,\beta)} \vartheta^2(0,it)\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\vartheta^2\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](i\Delta y/2\pi, it) + \tag{3.128}$$
$$+ \frac{16\pi^2}{\vartheta_1^2(i\Delta y/2\pi, it)\vartheta_2^2(0,it)}\left[\vartheta_3^2(i\Delta y/2\pi, it)\vartheta_4^2(0,it) - \vartheta_4^2(i\Delta y/2\pi, it)\vartheta_3^2(0,it)\right]$$

where, $\Delta y = y - y'$. We have separated the total sum as the sum over the two Z_2 group actions. In writing this out we have used the following identity

$$\eta(it) = \left[\frac{\partial_\nu \vartheta_1(\nu, it)}{-2\pi}\right]^{1/3}_{\nu=0} \tag{3.129}$$

Now, the first term vanishes due to the following identity

$$\sum_{(\alpha,\beta)} \vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](u)\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](v)\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](w)\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](s) = 2\vartheta\begin{bmatrix}1/2\\ 1/2\end{bmatrix}(u_1)\vartheta\begin{bmatrix}1/2\\ 1/2\end{bmatrix}(v_1)\vartheta\begin{bmatrix}1/2\\ 1/2\end{bmatrix}(w_1)\vartheta\begin{bmatrix}1/2\\ 1/2\end{bmatrix}(s_1) \tag{3.130}$$

where,

$$\begin{aligned} u_1 &= \frac{1}{2}(u+v+w+s) \quad v_1 = \frac{1}{2}(u+v-w-s) \\ w_1 &= \frac{1}{2}(u-v+w-s) \quad s_1 = \frac{1}{2}(u-v-w+s) \end{aligned} \tag{3.131}$$

and noting that, $\vartheta\begin{bmatrix}1/2\\ 1/2\end{bmatrix}(0,it) = 0$, in the same way as the flat case that makes amplitudes with two vertex insertions vanish. The second term is a constant also due to,

$$\vartheta_4^2(z,it)\vartheta_3^2(0,it) - \vartheta_3^2(z,it)\vartheta_4^2(0,it) = \vartheta_1^2(z,it)\vartheta_2^2(0,it) \tag{3.132}$$

For the nonplanar amplitude, which we are ultimately interested in, we need to put the two vertices at the two ends of the cylinder such that, $w = \pi + iy$ and $w' = iy'$. It can be seen that the fermionic part of the correlator is constant and independent of t, same as the planar case following from the identity (3.132). The effect of nonplanarity and the regulation of the two point function due to the background B-field is encoded in the correlation functions for the exponentials. The two point function thus reduces to,

$$A(p,-p) \sim \epsilon_k \epsilon_l p_i p_j \left(G^{il}G^{jk} - G^{ij}G^{kl}\right) \int_0^\infty \frac{dt}{4t}(8\pi^2\alpha' t)^{-2} \int_0^{2\pi t} dy dy' \left\langle e^{ip.X} e^{-ip.X} \right\rangle \tag{3.133}$$

The noncommutative gauge theory two point function is obtained in the limit $t \to \infty$ and $\alpha' \to 0$. The correlation function in this limit can be computed from the bosonic correlation functions [17, 31]. We give below the function for the nonplanar case in this limit.

$$\begin{aligned} \left\langle e^{ip.X} e^{-ip.X} \right\rangle &= \exp\left\{ -p^2 t \Delta x(\Delta x - 1) - \frac{1}{4t} p_i (g^{-1} - G^{-1})^{ij} p_j \right\} \\ &= \exp\left\{ -p^2 t \Delta x(\Delta x - 1) - \frac{\tilde{p}^2}{4t} \right\} \end{aligned} \tag{3.134}$$

where, $\tilde{p} = (\theta p)$. We have redefined the world sheet coordinate as $\Delta x = \Delta y/(2\pi t)$ and have scaled $t \to t/(2\pi\alpha')$. We have also used the following relation in writing down the last expression.

$$g^{-1} = G^{-1} - \frac{(\theta G \theta)}{(2\pi\alpha')^2} \tag{3.135}$$

The first term in the exponential in (3.134) regulates the integral over t in the infrared, for $p \neq 0$ and the second term regulates it in the ultraviolet that is usually observed in noncommutative field theories. The $t \to \infty$ limit suppresses the contributions from all the open string massive modes. However as, discussed in Section 3.1, the field theory divergences still come from the $t \to 0$ region. We can thus break the integral over t into two intervals $1/\Lambda^2\alpha' < t < \infty$ and $0 < t < 1/\Lambda^2\alpha'$ (see Figure 3.1). The second interval which is the source of the UV divergence is also the regime dominated by massless closed string exchanges. We now evaluate the two point function in this limit. First, the correlation function for the exponential in the $t \to 0$ limit is given by

$$\langle e^{ip.X} e^{-ip.X} \rangle = \exp\left\{ -\frac{\alpha' \pi}{2t} p_i g^{ij} p_j \right\} \tag{3.136}$$

where g^{ij} is the closed string metric. Modular transformation , $(t \to 1/t)$ allows us to rewrite the one loop amplitude as the sum over closed string modes in a tree diagram. In the limit $t \to 0$, the amplitude will be dominated by massless closed string modes. In this model however, the effect of the massive modes in the loop cancel amongst themselves for any value of t. In the open string channel the $t \to 0$ limit would usually be contributed by the full tower of open string modes. However since we have seen that the effect of the massive string modes cancel anyhow for all values of t, the contribution to this limit from the open string modes comes only from the massless ones. The additional term in (3.134) as compared to (3.136) gives finite derivative corrections to the effective action. These would in general require the massive closed string states for its dual description. Without these derivative corrections, the contributions from the massless open string loop and the massless closed string tree are exactly equal. The divergent ultraviolet behavior of the massless open string modes can thus be captured by the the massless closed string modes that have momentum in the limit $[0, 1/\Lambda\alpha']$. The amplitude can now be written as,

$$A(p,-p) = V_4 \det(g + 2\pi\alpha' B) \left(\frac{g_o^2}{8\pi^2\alpha'} \right) \epsilon_k \epsilon_l p_i p_j \left(G^{il} G^{jk} - G^{ij} G^{kl} \right) I(p) \tag{3.137}$$

where,

$$
\begin{aligned}
I(p) &= \int ds s^{-1} \exp\left\{-\frac{\alpha' \pi s}{2} p_i g^{ij} p_j\right\} \\
&= 4\pi \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{k_\perp^2 + p_i g^{ij} p_j}
\end{aligned}
\tag{3.138}
$$

The integral is written in terms of $s = 1/t$ and in the last line we have rewritten it as an integral over $k_\perp$, the momentum in the directions transverse to the brane for closed strings. The nonzero contribution to the two point amplitude in (3.126) comes from the $\mathrm{Tr}_{NS}\left[gq^{L_0}\right]$ and $\mathrm{Tr}_{NS}\left[g(-1)^F q^{L_0}\right]$, that are evaluated in (A3.35). These correspond to antiperiodic (NS-NS) and periodic (R-R) closed strings in the twisted sectors respectively. The fractional D_3-brane is localised at the fixed point of C^2/Z_2. Thus the twisted sector closed string states that couple to it are twisted in all the directions of the orbifold. These modes are localised at the fixed point and are free to move in the six directions transverse to the orbifold. This is the origin of the momentum integral (3.138) in two directions transverse to the D-brane. These twisted states come from both the $NS-NS$ and the $R-R$ sectors are listed in Section 3.3.1.2. The couplings for the massless closed string states to the fractional D_3-brane have been worked out by various authors. See for example [27].

As we are interested in seeing the ultraviolet effect of the open string channel as an infrared effect in the closed string channel, like in the bosonic case (see eqn.(3.15)), we must cut off the s integral at the lower end at some value $\Lambda^2 \alpha'$ corresponding to the UV cutoff for the momentum of the massless closed strings in the directions transverse to the brane. With this, we have,

$$
I(p, \Lambda) = 4\pi^2 \ln\left(\frac{p_i g^{ij} p_j + 1/(\Lambda \alpha')^2}{p_i g^{ij} p_j}\right) \tag{3.139}
$$

This is the ultraviolet behavior of the two point function for two gauge fields in $\mathcal{N}=2$ theory. For the noncommutative theory it is regulated for $p \neq 0$. The fact that we are able to rewrite the gauge theory two point function as massless closed string tree-level exchanges is very specific to the $\mathcal{N}=2$ theory. The computations above show that the origin of this can be traced to open-closed string duality where

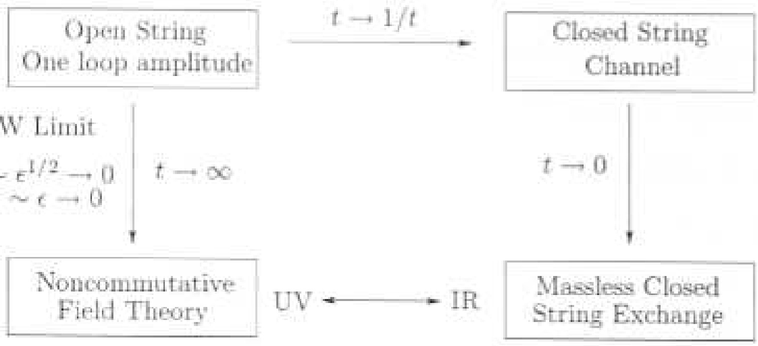

Figure 3.6: Noncommutative field theory and closed string channel limits

the orbifold background cancels all contributions from the massive states as far as the UV singular terms are concerned. The background B-field in the SW limit only acts as a physical regulator.

3.5 Discussions

In the previous sections we have addressed the issue of open closed duality in string theory in the presence of B-field. This duality lies at the heart of duality between gravity and gauge theory as exemplified by the AdS/CFT conjecture. In that situation the gauge theory has $\mathcal{N} = 4$ supersymmetry and is finite. The duality of the annulus diagram then reduces to a trivial identity namely, $0 = 0$. To get something non trivial one reduces the amount of supersymmetry by orbifolding, but taking care to preserve enough supersymmetry that there are no tachyons. In this case one loop amplitudes are divergent. One can compare divergences in the closed and open string channels and if one makes a suitable identification of the cutoffs one can show the equality of amplitudes. What we have done here is to consider bosonic and supersymmetric theories in the presence of B-field so that some of the amplitudes (non planar) are actually finite and can be compared in an exact way. The B-field plays the role of a regulator that preserves the duality.

Once we turn on a B-field we also make contact with another phenomenon: UV/IR mixing that is known to happen in non commutative field theories. This has a natural explanation when we consider this theory as the $\alpha' \to 0$ limit of a string theory a la Seiberg-Witten. The open string loop UV region is reproduced by closed string trees with small (i.e IR) momentum exchange. The B-field acts as a regulator for both amplitudes but the regulation goes away as the external momentum goes to zero. It is not surprising that the tree diagram diverges as the external momentum goes to zero, but by the duality map this must also be true for the UV divergence of the gauge theory.

Figure 3.6 sums up the various limits involved in the problem addressed in this chapter. Noncommutative field theory arises in the Seiberg Witten limit. In the open string one loop amplitude, the $t \to \infty$ region of the moduli space of the cylinder corresponds to the IR regime with only contributions to the amplitude coming from the massless open string modes propagating in the loop. As a result we get a one loop two point function in noncommutative field theory. However the UV divergences of the noncommutative field theory still come from the $t \to 0$ end. On the closed string side, the contributions in this region come from the massless modes. In the context of the bosonic theory, that we have discussed in Section 3.1, the divergences arising from the two ends are related to each other (upto some overall normalisation). This relation could not be made exact in the bosonic setup due to the presence of tachyons, which act as additional sources for divergences. We have shown that the tensor structure for the noncommutative field theory (2.22) two point amplitude can be recovered by considering massless and tachyonic exchanges of closed strings in the presence of background constant B-field. For the coefficients to match with the gauge theory result, in the bosonic string case, the full tower of the closed string states are required. We concluded there that an exact correspondence between the UV behavior of the noncommutative gauge theory and massless closed string exchanges may be made in some compactified superstring theory, where the gauge theory is four dimensional and the closed strings move in exactly two extra transverse directions. This would cure the problem of tachyons as well as lead to the desired forms of propagators in the closed string channel. We have also studied massless closed string exchanges in the background B-field. The full two point amplitude in the presence of background B-field must be of the form (3.18). We have reconstructed this from the sum of massless (graviton, dilaton and b-field) exchanges with the vertices computed from the DBI action, by considering expansions of the amplitude in three different cases. This exercise has helped in isolating the contributions from each of the massless closed string modes separately.

With these insights into the problem, in Sections 3.3 and 3.4 we have studied a noncommutative $\mathcal{N} = 2$ gauge theory realised on a fractional D_3-brane localised at the fixed point of C^2/Z_2 orbifold. The one loop two point open string amplitude gives the gauge theory two point amplitude in the $\alpha' \to 0$ SW limit. This assumes there is no tachyon as is the case in this model. As discussed above, we then see that the UV divergences of the gauge theory comes from the $t \to 0$ end that is dominated by massless closed strings. In general the massless closed string exchanges account for the UV contribution due to all the open string modes and similarly the dual description of the gauge theory would thus require the contributions from all the massive closed string states as well. This is the situation in the bosonic case. But in this supersymmetric case, the contributions from the massive modes cancel and hence the duality is between the finite number of massless states on both the open and the closed string sides. This is what is manifested through the equality of (3.139) to the gauge theory amplitude, both ends being regulated by the presence of the B-field.

We now discuss about the closed string couplings to the noncommutative gauge theory on the brane. To see the closed string coupling to the gauge theory, consider for the moment, eqn(3.7) in the bosonic theory. Let us set $\alpha' t = T$

$$\mathcal{Z}_{op} \sim \int \frac{dT}{T} (T)^{-\frac{p+1}{2}} \left[e^{2\pi \frac{T}{\alpha'}} + (D-2) + O(e^{-2\pi \frac{T}{\alpha'}}) \right] \exp(-C/T) \tag{3.140}$$

The $O(1)$ term in the expansion corresponds to the massless open string modes in the loop. If we take the $\alpha' \to 0$ limit the contribution of the massive modes drop out. If we ignore the tachyon we get the massless mode contribution. In the supersymmetric case there is no tachyon. However in the present case dropping the tachyon term makes an exact comparison of the massless sectors of the two cases meaningless because the powers of α' cannot match. Nevertheless the comparison is instructive.

The UV contribution of (3.7,3.140), as shown in Figure 3.1, comes from the region $0 < t < 1/\Lambda^2\alpha'$. The UV divergences coming from this region is regulated by C. In the closed string channel we have,

$$\begin{aligned} Z_{cl} &\sim \int ds(\alpha')^{-\frac{p+1}{2}} s^{-l/2} \left[e^{2\pi s} + (D-2) + O(e^{-2\pi s})\right] \exp(-Cs/\alpha') \\ &\sim (\alpha')^{-\frac{p+1}{2}} (\alpha')^{\frac{l}{2}-1} \int d^l k_\perp \frac{1}{k_\perp^2 + C/\alpha'^2} \end{aligned} \tag{3.141}$$

The $\alpha' \to 0$ limit does not pull out the massless sector (even if we ignore the tachyon) and this makes it clear that in general all the massive closed string modes are required to reproduce the massless open string contribution. But let us focus on the massless states of the closed string sector. In the second expression of (3.141), we have kept only the contribution from the massless closed string mode. This expression can be interpreted as the amplitude of emission and absorption of a closed string state from the D_p-brane with transverse momentum $k_\perp$, integrated over $0 < k_\perp < 1/\Lambda\alpha'$. The domain of the $k_\perp$ integral corresponds to the UV region in the open string channel. $l = D - (p+1)$, is the number of transverse directions in which the closed string propagates. For $l \neq 2$ there is an extra factor of $(\alpha')^{\frac{l}{2}-1}$ in (3.141) over (3.7), that makes the couplings of the individual closed string modes vanish when compared to the open string channel. It is only for the special case $l = 2$ that the powers match.

In the supersymmetric case, for the C^2/Z_2 orbifold, we have seen that the closed strings that contribute to the dual description of the nonplanar divergences are from the twisted sectors. They are free to move in 6 directions transverse to the orbifold. For the D_3-brane that is localised at the fixed point with world volume directions perpendicular to the orbifold, these closed string twisted states propagate in exactly two directions transverse to the brane. Thus in this case $l = 2$ and from the above discussions this makes the power of α' in the coupling of closed string with the gauge field strength same as that of the open string channel.

Although in general, the closed string couplings to the gauge field when the closed string modes are restricted to the massless ones do not give the same normalisation as the gauge theory, the massive closed string modes are expected to contribute so that the normalisations at both the ends are equal. This is guaranteed by open-closed string duality. For the C^2/Z_2 orbifold, since the massive states cancel, the finite number of closed string modes must give the same normalisation as the gauge theory two point function. This is the reason why we are able to see the IR behavior of noncommutative $\mathcal{N} = 2$ theory in terms of only the massless closed string modes in the twisted sectors.

The role played by the B-field is essentially that of a regulator that preserves the open closed duality. The fact that we see the UV divergence at the field theory level as IR divergence depends on the the special nature of this regulator that is dependent on the B-field and external momenta, thus giving rise to UV/IR mixing in noncommutative gauge theories. However the B-field does not affect the correspondence between the modes on the open and closed string sides that arise as a result of the world-sheet duality. The only modification of the partition function due to the B-field is the inclusion of a constant determinant (see Appendix A3). In conclusion, the IR divergences in noncommutative gauge theories that arise by integrating over high momentum modes in the loops can be seen as IR divergences due to closed string exchanges, as a result of open-closed string duality. The question of whether a finite or infinite number of closed string modes are necessary for the dual description depends on the commutative theory without the B-field.

At higher orders one expects the duality to be true for the full string theory and not for the massless sectors. But in limits such as in the AdS/CFT case one can expect a duality for the massless sectors. It would be interesting to study the corresponding AdS/CFT - like limit here. This is presumably some orbifolded version of the AdS/CFT [33] with B-field [34].

Finally, it will be interesting to analyse the massless closed string exchanges in the background B-field for the C^2/Z_2 orbifold as we have done in Section 3.2 for the bosonic case [35]. We have seen from the string theory one loop computation that the surviving contributions here come from the twisted NS-NS and R-R states. A fractional D_3-brane at the fixed point of C^2/Z_2 can be seen as an ordinary D_5-brane wrapped on a vanishing 2-sphere. The twisted NS-NS and R-R couplings to the D-brane can be obtained from the dimensional reduction of the Born-Infeld (3.37) and the Chern-Simons [36, 37] couplings respectively.

A3 Appendix : Evaluation of Vacuum amplitude

In this appendix we calculate the vacuum amplitude for the open strings with end points on a D_3-brane that is located at the fixed point of C^2/Z_2 orbifold. Let us first start with the bosonic part of the world-sheet action,

$$S_B = -\frac{1}{4\pi\alpha'}\int_\Sigma g_{MN}\partial_a X^M \partial^a X^N + \frac{1}{2}\int_{\partial\Sigma} B_{MN} X^M \partial_\tau X^N \tag{A3.1}$$

(A3.2)

The boundary condition for the world-sheet bosons from the above action is,

$$g_{MN}\partial_\sigma X^N + 2\pi\alpha' B_{MN}\partial_\tau X^N = 0 \,|_{\sigma=0,\pi} \tag{A3.3}$$

In the Seiberg-Witten limit, $g_{ij} = \epsilon\eta_{ij}$ we choose the B field along the brane to be of the form,

$$B = \frac{\epsilon}{2\pi\alpha'}\begin{pmatrix} 0 & b_1 & 0 & 0 \\ -b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & -b_2 & 0 \end{pmatrix} \tag{A3.4}$$

With the above form for the B-field, and defining,

$$X^{\pm}_{(1)} = 2^{-1/2}(X^0 \pm X^1) \quad \text{and} \quad X^{\pm}_{(2)} = 2^{-1/2}(X^2 \pm iX^3) \tag{A3.5}$$

the boundary condition (A3.3) can be rewritten as,

$$\partial_\sigma X^{\pm}_{(1)} = \pm b_1 \partial_\tau X^{\pm}_{(1)} \mid_{\sigma=0,\pi} \quad \text{and} \quad \partial_\sigma X^{\pm}_{(2)} = \pm i b_2 \partial_\tau X^{\pm}_{(1)} \mid_{\sigma=0,\pi} \tag{A3.6}$$

The mode expansions for the open string satisfying the above boundary conditions are given by,

$$\begin{aligned} X^{\pm}_{(1)} &= x^{\pm}_{(1)} + \frac{2\alpha'}{1-b_1^2}(\tau \pm b_1\sigma)p^{\pm}_{(1)} + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{a^{\pm}_{(1)n}}{n}e^{-i(n\tau\pm\nu_1)}\cos(n\sigma \mp \nu_1) \\ X^{\pm}_{(2)} &= x^{\pm}_{(2)} + \frac{2\alpha'}{1+b_2^2}(\tau \pm ib_2\sigma)p^{\pm}_{(2)} + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{a^{\pm}_{(2)n}}{n}e^{-i(n\tau\pm\nu_2)}\cos(n\sigma \mp \nu_2) \end{aligned} \tag{A3.7}$$

where we have defined,

$$i\nu_1 = \frac{1}{2}\log\left(\frac{1+b_1}{1-b_1}\right) \qquad i\nu_2 = \frac{1}{2}\log\left(\frac{1+ib_2}{1-ib_2}\right) \tag{A3.8}$$

The coefficients of the mode expansions (A3.7) are fixed so as to satisfy,

$$\left[X^{+}_{(1)}(\tau,\sigma), P^{-}_{(1)}(\tau,\sigma')\right] = -2\pi\alpha'\delta(\sigma-\sigma') \tag{A3.9}$$

and that the zero modes and the other oscillators satisfy the usual commutation relations,

$$\left[a^{+}_{(1)m}, a^{-}_{(1)n}\right] = -m\delta_{m+n} \qquad \left[a^{+}_{(2)m}, a^{-}_{(2)n}\right] = m\delta_{m+n} \tag{A3.10}$$

$$\left[x^{+}_{(1)}, p^{-}_{(1)}\right] = -i \qquad \left[x^{+}_{(2)}, p^{-}_{(2)}\right] = i \tag{A3.11}$$

There is no shift in the moding of the oscillators, the zero point energy and the spectrum is the same as the $B=0$ case. The situation is the same as that of a neutral string in electromagnetic background [31]. Note that the commutator for $X^{\pm}$ now does not vanish at the boundary, for example,

$$
\begin{aligned}
\left[X^{+}_{(1)}(\tau,0), X^{-}_{(1)}(\tau,0)\right] &= -2\pi i\alpha' \frac{b_1}{1-b_1^2} \\
\left[X^{+}_{(1)}(\tau,\pi), X^{-}_{(1)}(\tau,\pi)\right] &= 2\pi i\alpha' \frac{b_1}{1-b_1^2}
\end{aligned} \tag{A3.12}
$$

The zero mode for the energy momentum tensor can now be worked out and is given by,

$$
L^{\parallel}_{(b)0} = \frac{2\alpha'}{b_1^2-1} p^{+}_{(1)} p^{-}_{(1)} + \frac{2\alpha'}{b_2^2+1} p^{+}_{(2)} p^{-}_{(2)} - \sum_{n\neq 0} \left[a^{+}_{(1)-n} a^{-}_{(1)} - a^{+}_{(2)-n} a^{-}_{(2)}\right] \tag{A3.13}
$$

Since the spectrum remains the same, the contribution to the vacuum amplitude from the bosonic modes is the same as the usual $B = 0$ case except that there is a factor of $\sqrt{(b_i^2 \pm 1)}$ which comes from the trace over the zero modes for each direction along the brane. From (A3.4) in the limit (2.14), $b_i \sim 1/\sqrt{\epsilon}$ for B to be finite. With this,

$$
\epsilon^2 \prod_i^2 (b_i^2 \pm 1) \rightarrow \det(g + 2\pi\alpha' B) \tag{A3.14}
$$

Including contributions from all the directions,

$$
L_{(b)0} = L^{\parallel}_{(b)0} + L^{\perp}_{(b)0} + L^{orb}_{(b)0} - \frac{5}{12} \tag{A3.15}
$$

$\perp$ denotes the $4,5$ directions and $6,7,8,9$ are the orbifolded directions. Let us now compute the contributions from the world sheet fermions. The action is given by,

$$
S_F = \frac{i}{4\pi\alpha'} \int_{\Sigma} g_{MN} \bar{\psi}^M \rho^{\alpha} \partial_{\alpha} \psi^N - \frac{i}{4} \int_{\partial\Sigma} B_{MN} \bar{\psi}^N \rho^0 \psi^M \tag{A3.16}
$$

We rewrite the boundary equations from (3.109),

$$
{}_{MN}(\psi_L^N - \psi_R^N) + 2\pi\alpha' B_{MN}(\psi_L^N + \psi_R^N) = 0 \,|_{\sigma=\pi} \tag{A3.17}
$$

$$
g_{MN}(\psi_L^M + (-1)^a \psi_R^M) + 2\pi\alpha' B_{MN}(\psi_L^N - (-1)^a \psi_R^N) \,|_{\sigma=0} \tag{A3.18}
$$

Now defining,

$$
\psi^{\pm}_{(1)R,L} = 2^{-1/2}(\psi^0_{R,L} \pm \psi^1_{R,L}) \quad \text{and} \quad \psi^{\pm}_{(2)R,L} = 2^{-1/2}(\psi^2_{R,L} \pm i\psi^3_{R,L}) \tag{A3.19}
$$

For the Ramond Sector ($a = 1$) with the constant B-field given by (A3.4),

$$
\psi^{\pm}_{(1)R}(1 \pm b_1) = \psi^{\pm}_{(1)L}(1 \mp b_1) \,|_{\sigma=0,\pi} \tag{A3.20}
$$

Mode expansion,

$$
\psi^{\pm}_{(1)L,R} = \sum_n d^{\pm}_{(1)n} \chi^{\pm}_{(1)L,R}(\sigma, \tau, n) \tag{A3.21}
$$

where,

$$
\chi^{\pm}_{(1)R} = \sqrt{2\alpha'} \exp\{-in(\tau - \sigma) \mp \nu_1\} \tag{A3.22}
$$

$$
\chi^{\pm}_{(1)L} = \sqrt{2\alpha'} \exp\{-in(\tau + \sigma) \pm \nu_1\} \tag{A3.23}
$$

and

$$
\nu_1 = \frac{1}{2} \log\left(\frac{1 + b_1}{1 - b_1}\right) = \tanh^{-1} b_1 \tag{A3.24}
$$

The boundary condition for the other two directions are,

$$
\psi^{\pm}_{(2)R}(1 \pm ib_2) = \psi^{\pm}_{(2)L}(1 \mp ib_2) \,|_{\sigma=0,\pi} \tag{A3.25}
$$

This gives the same mode expansion as (A3.21),

$$
\psi^{\pm}_{(2)L,R} = \sum_n d^{\pm}_{(2)n} \chi^{\pm}_{(1)L,R}(\sigma, \tau, n) \tag{A3.26}
$$

$$\chi^{\pm}_{(2)R} = \sqrt{2\alpha'}\exp\{-in(\tau-\sigma)\mp\nu_2\} \tag{A3.27}$$

$$\chi^{\pm}_{(2)L} = \sqrt{2\alpha'}\exp\{-in(\tau+\sigma)\pm\nu_2\} \tag{A3.28}$$

and

$$\nu_2 = \frac{1}{2}\log\left(\frac{1+ib_2}{1-ib_2}\right) = \tan^{-1} b_2 \tag{A3.29}$$

Like the bosonic partners there is no shift in the frequencies. The oscillators are integer moded as usual. For the Neveu-Schwarz sector, ($a = 0$), the relative sign between $\psi^{\pm}_{R}$ and $\psi^{\pm}_{L}$ at the $\sigma = \pi$ end in eqn(6) can be brought about by the usual restriction on n to only run over half integers in the mode expansions (A3.21,A3.26). The oscillators satisfy the standard anticommutation relations,

$$\{d^{+}_{(1)n}, d^{-}_{(1)m}\} = -\delta_{m+n} \quad ; \quad \{d^{+}_{(2)n}, d^{-}_{(2)m}\} = \delta_{m+n} \tag{A3.30}$$

The zero mode for the energy momentum tensor for the fermions along the brane can be written as,

$$L^{\|}_{(f)0} = \sum_{n} n\left[d^{-}_{(2)-n}d^{+}_{(2)n} - d^{-}_{(1)-n}d^{+}_{(1)n}\right] \tag{A3.31}$$

For all the fermions including the contributions from the other directions we have,

$$L_{(f)0} = L^{\|}_{(f)0} + L^{\perp}_{(f)0} + L^{orb}_{(f)0} + c_f(a) \tag{A3.32}$$

where $L^{\perp}_{(f)0}$ and $L^{orb}_{(f)0}$ have the usual representation in terms of oscillators.

$$c_f(1) = \frac{5}{12} \quad ; \quad c_f(0) = -\frac{5}{24} \tag{A3.33}$$

We now compute the vacuum amplitude including the contributions from the ghosts. This is given by,

$$Z_C = V_4 \det(g + 2\pi\alpha' B) \int_0^\infty \frac{dt}{t} (8\pi^2\alpha' t)^{-2} \mathrm{Tr}_{NS-R}\left[\left(\frac{1+g}{2}\right)\left(\frac{1+(-1)^F}{2}\right) q^{L_0}\right] \tag{A3.34}$$

The origin of the $\det(g + 2\pi\alpha' B)$ term is given in (A3.14) and V_4 is the volume of the D_3-brane and $q = e^{-2\pi t}$. The trace is summed over the spin structures with the orbifold projection. The required traces are listed below in terms of the *Theta Functions*, $\vartheta_i(\nu, it)$ (see for example [32]).

$$\begin{aligned} Z\left[{0 \atop 0}\right]_e(it) &= \mathrm{Tr}_{NS}\left[q^{L_0}\right] \\ &= \left[q^{-1/3}\prod_{m=1}^{\infty}(1-q^m)^{-8}\right]\left[q^{-1/6}\prod_{m=1}^{\infty}(1+q^{m-1/2})^{-8}\right] \\ &= \eta(it)^{-12}\vartheta_3^4(0, it) \end{aligned} \tag{A3.35}$$

$$\begin{aligned} Z\left[{0 \atop 1/2}\right]_e(it) &= \mathrm{Tr}_{NS}\left[(-1)^F q^{L_0}\right] \\ &= -\left[q^{-1/3}\prod_{m=1}^{\infty}(1-q^m)^{-8}\right]\left[q^{-1/6}\prod_{m=1}^{\infty}(1-q^{m-1/2})^{-8}\right] \\ &= -\eta(it)^{-12}\vartheta_4^4(0, it) \end{aligned} \tag{A3.36}$$

$$\begin{aligned} Z\left[{1/2 \atop 0}\right]_e(it) &= \mathrm{Tr}_{R}\left[q^{L_0}\right] \\ &= -\left[q^{-1/3}\prod_{m=1}^{\infty}(1-q^m)^{-8}\right]\left[q^{1/3}\prod_{m=1}^{\infty}(1+q^{m})^{8}\right] \\ &= -\eta(it)^{-12}\vartheta_2^4(0, it) \end{aligned} \tag{A3.37}$$

$$Z\left[{1/2 \atop 1/2}\right]_e(it) = \mathrm{Tr}_{R}\left[(-1)^F q^{L_0}\right] = 0 \tag{A3.38}$$

$$\begin{aligned} Z\left[{0 \atop 0}\right]_g(it) &= \mathrm{Tr}_{NS}\left[g q^{L_0}\right] \\ &= \left[q^{-1/3}\prod_{m=1}^{\infty}(1-q^m)^{-4(1+q^m)^{-4}}\right]\left[q^{1/3}\prod_{m=1}^{\infty}(1+q^{m-1/2})^{4}(1-q^{m-1/2})^{4}\right] \\ &= 4\eta(it)^{-6}\vartheta_3^2(0, it)\vartheta_4^2(0, it)\vartheta_2^{-2}(0, it) \end{aligned} \tag{A3.39}$$

$$\begin{aligned} Z\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]_g (it) &= \mathrm{Tr}_{NS}\left[g(-1)^F q^{L_0}\right] \\ &= -\left[q^{-1/3}\prod_{m=1}^{\infty}(1-q^m)^{-4(1+q^m)^{-4}}\right]\left[q^{1/3}\prod_{m=1}^{\infty}(1-q^{m-1/2})^4(1+q^{m-1/2})^4\right] \\ &= -4\eta(it)^{-6}\vartheta_3^2(0,it)\vartheta_4^2(0,it)\vartheta_2^{-2}(0,it) \end{aligned} \tag{A3.40}$$

$$Z\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]_g (it) = \mathrm{Tr}_R\left[g q^{L_0}\right] = 0 \tag{A3.41}$$

$$Z\left[\begin{matrix} 1/2 \\ 1/2 \end{matrix}\right]_g (it) = \mathrm{Tr}_R\left[g(-1)^F q^{L_0}\right] = 0 \tag{A3.42}$$

Recalling,

$$\vartheta_3^4(0,it) - \vartheta_4^4(0,it) - \vartheta_2^4(0,it) = 0 \tag{A3.43}$$

and noting that,

$$Z\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]_g (it) = -Z\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]_g (it) \tag{A3.44}$$

the vacuum amplitude vanishes. This is as a result of supersymmetry.

Theta Functions :

$$q = \exp(-2\pi t) \qquad z = \exp(2\pi i\nu) \tag{A3.45}$$

$$\vartheta_{00}(\nu, it) = \vartheta_3(\nu, it) = \prod_{m=1}^{\infty}(1-q^m)(1+zq^{m-1/2})(1+z^{-1}q^{m-1/2}) \tag{A3.46}$$

$$\vartheta_{01}(\nu, it) = \vartheta_4(\nu, it) = \prod_{m=1}^{\infty}(1-q^m)(1-zq^{m-1/2})(1-z^{-1}q^{m-1/2}) \qquad (A3.47)$$

$$\begin{aligned}\vartheta_{10}(\nu, it) &= \vartheta_2(\nu, it) = 2\exp(-\pi t/4)\cos(\pi\nu) \times \\ &\times \prod_{m=1}^{\infty}(1-q^m)(1+zq^{m-1/2})(1+z^{-1}q^{m-1/2})\end{aligned} \qquad (A3.48)$$

$$\begin{aligned}\vartheta_{11}(\nu, it) &= \vartheta_1(\nu, it) = -2\exp(-\pi t/4)\sin(\pi\nu) \times \\ &\times \prod_{m=1}^{\infty}(1-q^m)(1-zq^{m-1/2})(1-z^{-1}q^{m-1/2})\end{aligned} \qquad (A3.49)$$

$$\eta(it) = q^{1/24}\prod_{m=1}^{\infty}(1-q^m) \qquad (A3.50)$$

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Part II

Closed String Tachyon Condensation

Chapter 4

Localised Closed String Tachyon Condensation

Localised closed string tachyon condensation have been the topic of intensive study recently. This was pioneered by the work of Adams, Polchinski, and Silverstein (APS) [1]. Unlike the open string the closed string tachyon condensation is accompanied by changes in the background space time. This is due to the fact that the closed string tachyon couples to the graviton and the dilaton that are present in all closed string theories. Thus in general the condensation process is very hard to follow as it quickly leads to the region in the string coupling where the usual string perturbation theory is no longer reliable.

To know where the condensation process leads to, one must have a full knowledge of the tachyon potential, that gives information about the stable ground state. With the conventional conformal field theory techniques, one can only compute n-point amplitudes for the tachyons that are on-shell, the zero momentum continuation of which is ambiguous. Though progress have been made in the construction of an off-shell formalism for closed strings, construction of tachyon potential has still remained a difficult problem.

However there are models where the closed string tachyons are localised in some region of space. It may thus seem that the condensation process may be analysed in a more manageable way as the initial region of space affected is the region where the tachyons are localised. These tachyons often arise in the twisted sectors of orbifolded models. Recent studies include tachyons of C^p/Z_N.

Let us now concentrate on the simplest of these models which is the C/Z_N orbifold. This orbifold is a cone in two dimensions with a singularity at the tip where the twisted sector tachyons are localised. Various possibilities have been outlined in [1] that may follow as the tachyons condense. (a) The effect at the tip may start off with a hole that would in time propagate and eventually leave nothing behind. (b) An infinite throat may develop at the tip. (c) There may be no topology change but the singularity at the tip may be removed and replaced by a smooth cap. It was conjectured by APS [1] that this is what occurs. Evidences in support of this conjecture have been worked out using various techniques. These have been reviewed in [12]. We include here a short discussion on some of these approaches.

1. *D-brane probe* : The fact that D-branes can be used to probe distances shorter that the string scale was shown in [17]. This probe brane can thus be used to follow the condensation process in the *substring regime*. The world volume theory on the brane is a quiver gauge theory [18]. On the C/Z_N orbifold, though supersymmetry is broken, the D-term potential is known, as the scalar potential descends from the supersymmetric theory with orbifold projection. The moduli space of the scalars is that of the orbifold geometry of background space. The twisted tachyons of the closed string theory couple to the world volume scalars. On the world volume the potential will be modified by a mass term. The new moduli space can be verified to be that of a smooth space without the Z_N singularity at the tip. There are also specific deformations due to the mass terms that lead the condensation process to follow in a series of steps involving lower order orbifolds and ultimately to flat space.

2. *Supergravity* : This becomes a valid description at large length scales compared to the string length. It is expected that after the initial condensation process, when the excited massive closed strings have radiated all the energy

to the massless ones, we may study the condensation process by solving the supergravity equations of motion [1, 14].

3. *World sheet RG* : From the point of view of the two dimensional world sheet, strings on C/Z_N can be described by an exact conformal field theory. The tachyons in the spectrum correspond to relevant operators on the world sheet CFT. So the condensation process can be studied as a world sheet RG flow when the CFT is perturbed by these (tachyons) relevant operators. Generally without supersymmetry such flows are not easy to analyse. However for the C/Z_N orbifold, one can use the world sheet $\mathcal{N} = (2, 2)$ supersymmetry. The results here are in support of the APS conjecture. There are indeed specific perturbations which drive C/Z_N to C/Z_{N-k} or other lower nonsupersymmetric orbifolds [2]. Condensation of tachyons for a more general background namely the twisted circles confirms these observations [9].

Other studies on condensation of localised closed string tachyons include [3, 5, 6, 7, 13, 14, 15, 16]. In this chapter we consider the problem of tachyon condensation for Type II theory on the C/Z_N orbifold in the large N limit [10]. In this limit there are tachyons which become almost marginal and it makes sense to write an effective action involving the tachyons and the other massless particles (graviton, dilaton) while integrating out the massive string modes. The aim is to compute the effective tachyon potential for large N. With this as the guideline, we construct the tachyon potential upto the quartic interaction term. The procedure followed is along the lines of [19]. The four point amplitude for the twisted sector tachyons (of $m^2 = -1/N$) is first calculated following [20]. In the large N limit when the tachyons are nearly massless we take the zero momentum limit of this amplitude. The non-derivative quartic coupling for the tachyons is then obtained by subtracting the contribution of the massless exchanges from the string four point amplitude. The four point amplitude with massless exchanges, which in this case are the graviton and the dilaton is obtained from the low energy effective field theory of tachyons coupled to these massless fields.

The quartic coupling for the tachyon potential is found to be of the order $1/N$. We get the the height of the potential to the lowest order in $1/N$. We expect this minimum to correspond to the C/Z_{N-k} orbifold.

However there are various points which show that the higher point interaction couplings are also comparable to the quartic coupling. One being that, with a quartic potential having global $O(2)$ symmetry, we expect a particle of positive $(mass)^2 = -2m^2$, where m^2 is the mass of the tachyon, and the usual Goldstone boson. However these modes are not present in the spectrum of closed string on C/Z_{N-k}. Furthermore if we stick to the predicted height of the tachyon potential [4], we find that there is a mismatch by a factor of $1/N$ in the height of the potential, when the minimum is expected to correspond to the C/Z_{N-k} orbifold. However since the above modes are absent in the tree-level spectrum of closed string on C/Z_{N-k} orbifold we conclude that the higher point amplitudes are also of the order $1/N$. This includes the term ϕ^N allowed by the twist symmetry. These higher order terms modify the potential and the spectrum already to the lowest order in $1/N$.

Furthermore, the four point coupling is subject to field redefinitions. One can make the off-shell contact term as large or small as one wishes and can also change the sign. This makes the coupling non-universal and the existence of the minimum is not clear in this approach if one truncates the potential upto the quartic term. We elaborate on these points in Section 4.4.

This chapter is organised as follows. In Section 4.1, we compute the spectrum for closed strings on C/Z_N orbifold and show that there are tachyons. In Section 4.2, using the conformal field theory of C/Z_N orbifold we review the calculation of the four point amplitude of the tachyons in the large N limit where the tachyon is nearly marginal. In Section 4.2.1, we find the OPE of two tachyon vertices and show that the only intermediate massless exchanges are the graviton and dilaton. In Section 4.3, from the effective field theory of the tachyon coupled to the graviton and dilaton we compute the exact contribution of these massless exchanges to the four point tachyon amplitude. The quartic coupling for the tachyon is then obtained after subtraction of the massless exchanges from the string theory amplitude and we write down the potential upto the quartic interaction term in Section 4.4. We conclude this chapter with discussions in Section 4.5.

4.1 Closed string spectrum on C/Z_N

In this section we compute the spectrum of the closed string on the C/Z_N orbifold. We will be concerned with the NS-NS sector as it is in this sector that the twisted tachyons appear. In the RNS formulation of superstring, we consider x^μ and Ψ^μ as world sheet fields corresponding to the nonorbifolded directions. For the orbifolded directions we have the complex X,$\bar{X}$,ψ,$\bar{\psi}$ fields as defined below,

$$X = X^8 + iX^9, \quad \bar{X} = X^8 - iX^9; \quad \psi = \psi^8 + i\psi^9, \quad \bar{\psi} = \psi^8 - i\psi^9 \tag{4.1}$$

The Z_N group action on C defines the following boundary conditions on X,$\bar{X}$ and ψ, $\bar{\psi}$ for the closed string in the NS-NS sector,

$$\begin{aligned}
X(\sigma + 2\pi, \tau) &= e^{2\pi i \frac{k}{N}} X(\sigma, \tau) \\
\bar{X}(\sigma + 2\pi, \tau) &= e^{-2\pi i \frac{k}{N}} \bar{X}(\sigma, \tau) \\
\psi(\sigma + 2\pi, \tau) &= e^{2\pi i (\frac{k}{N} + \frac{1}{2})} \psi(\sigma, \tau) \\
\bar{\psi}(\sigma + 2\pi, \tau) &= e^{-2\pi i (\frac{k}{N} - \frac{1}{2})} \bar{\psi}(\sigma, \tau)
\end{aligned} \tag{4.2}$$

These boundary conditions give the following mode expansions for the world sheet scalars, [1],

$$\partial_z X(z) = -i \sum_{m=-\infty}^{\infty} \frac{\alpha_{m-\frac{k}{N}}}{z^{m+1-\frac{k}{N}}} \quad \partial_z \bar{X}(z) = -i \sum_{m=-\infty}^{\infty} \frac{\bar{\alpha}_{m+\frac{k}{N}}}{z^{m+1+\frac{k}{N}}} \tag{4.3}$$

$$\partial_{\bar{z}} X(\bar{z}) = -i \sum_{m=-\infty}^{\infty} \frac{\tilde{\alpha}_{m+\frac{k}{N}}}{\bar{z}^{m+1+\frac{k}{N}}} \quad \partial_{\bar{z}} \bar{X}(\bar{z}) = -i \sum_{m=-\infty}^{\infty} \frac{\bar{\tilde{\alpha}}_{m-\frac{k}{N}}}{\bar{z}^{m+1-\frac{k}{N}}} \tag{4.4}$$

[1] $\alpha' = 2$ in all calculations

and for the fermions,

$$\psi(z) = \sum_{r\in Z+\frac{1}{2}} \frac{\psi_{r-\frac{k}{N}}}{z^{r+\frac{1}{2}-\frac{k}{N}}} \quad \bar{\psi}(z) = \sum_{r\in Z+\frac{1}{2}} \frac{\bar{\psi}_{r+\frac{k}{N}}}{z^{r+\frac{1}{2}+\frac{k}{N}}} \tag{4.5}$$

$$\psi(\bar{z}) = \sum_{r\in Z+\frac{1}{2}} \frac{\tilde{\psi}_{r+\frac{k}{N}}}{\bar{z}^{r+\frac{1}{2}+\frac{k}{N}}} \quad \bar{\psi}(\bar{z}) = \sum_{r\in Z+\frac{1}{2}} \frac{\bar{\tilde{\psi}}_{r-\frac{k}{N}}}{\bar{z}^{r+\frac{1}{2}-\frac{k}{N}}} \tag{4.6}$$

From the usual OPEs of the world sheet fields and the mode expansions we get the following canonical commutation relations,

$$[\alpha_{m-\frac{k}{N}}, \bar{\alpha}_{n+\frac{k}{N}}] = (m-\frac{k}{N})\delta_{m+n,0} \quad [\alpha^{\mu}_{m}, \alpha^{\nu}_{n}] = m\eta^{\mu\nu}\delta_{m+n,0} \tag{4.7}$$

$$[\tilde{\alpha}_{m+\frac{k}{N}}, \bar{\tilde{\alpha}}_{n-\frac{k}{N}}] = (m+\frac{k}{N})\delta_{m+n,0} \quad [\tilde{\alpha}^{\mu}_{m}, \tilde{\alpha}^{\nu}_{n}] = m\eta^{\mu\nu}\delta_{m+n,0} \tag{4.8}$$

$$\{\psi_{r-\frac{k}{N}}, \bar{\psi}_{s+\frac{k}{N}}\} = \delta_{r+s,0} \quad \{\psi^{\mu}_{r}, \psi^{\nu}_{s}\} = \eta^{\mu\nu}\delta_{r+s,0} \tag{4.9}$$

$$\{\tilde{\psi}_{r+\frac{k}{N}}, \bar{\tilde{\psi}}_{s-\frac{k}{N}}\} = \delta_{r+s,0} \quad \{\tilde{\psi}^{\mu}_{r}, \tilde{\psi}^{\nu}_{s}\} = \eta^{\mu\nu}\delta_{r+s,0} \tag{4.10}$$

The holomorphic part of the energy momentum tensor for the world sheet fields corresponding to the orbifolded directions is given by,

$$T_{zz}(z) = -\partial X\partial\bar{X} - \frac{1}{2}\bar{\psi}\partial\psi - \frac{1}{2}\psi\partial\bar{\psi} \tag{4.11}$$

Using the mode expansions and the canonical commutation relations one gets the zero mode part of the energy momentum tensor,

$$\begin{aligned}
L_0 &= \sum_{n=0}^{\infty}\left[\alpha_{-n-\frac{k}{N}}\bar{\alpha}_{n+\frac{k}{N}}+\frac{1}{2}(n+\frac{k}{N})\right]+\sum_{n=1}^{\infty}\left[\bar{\alpha}_{-n+\frac{k}{N}}\alpha_{n-\frac{k}{N}}+\frac{1}{2}(n-\frac{k}{N})\right]\\
&+ \sum_{r\geq 1/2}\left[(r+\frac{k}{N})\psi_{-r-\frac{k}{N}}\bar{\psi}_{r+\frac{k}{N}}+(r-\frac{k}{N})\bar{\psi}_{-r+\frac{k}{N}}\psi_{r-\frac{k}{N}}-r\right] \qquad (4.12)
\end{aligned}$$

Where we have included the contributions from the normal ordering constant. The contribution to the zero point energy from the fields on the orbifolded complex plane is now given by,

$$\begin{aligned}
E_{orb} &= \frac{1}{2}\frac{k}{N}+\sum_{n=0}^{\infty}n-\sum_{r\geq 1/2}r\\
&= \frac{1}{2}\frac{k}{N}-\frac{1}{8} \qquad (4.13)
\end{aligned}$$

In NS-NS sector in the light cone gauge, we have six real periodic bosons and six antiperiodic fermions which contributes an amount of $-3/8$ to the zero point energy. Adding the zero point energies for all the directions, for the left moving part, we get,

$$E_L = -\frac{1}{2}(1-\frac{k}{N}) \qquad (4.14)$$

It may seem that for $1/2 < k/N < 1$, the $r = 1/2$ term from the second fermionic part in (4.12) contributes an additional $(1/2 - k/N)$ after normal ordering. Therefore, for this case we should have,

$$E_L = -\frac{k}{2N} \qquad (4.15)$$

However, since normal ordering is only a prescription for removing infinities from the zero point energy, we must choose one so that the zero point energy is consistent with that which is obtained from the world sheet conformal field theory. It will be seen in the next section, that a choice of the prescription, where we keep the $r = 1/2$ as it is, is consistent with the dimensions of the twist operators. The mass spectrum is thus given by,

$$M^2 = n + \tilde{n} - (1 - \frac{k}{N}) \text{ for } 0 < k/N < 1 \tag{4.16}$$

Where n and $\tilde{n}$ are level numbers which are no longer integers for the twisted sectors. The ground state of the NS-NS sector is thus tachyonic. For the $(N-k)th$ sector we have a tachyon of $(mass)^2 = -k/N$. Some of the excited states are also tachyonic their masses are given by,

$$\tilde{\psi}_{-\frac{1}{2}+\frac{k}{N}} |0\rangle \equiv -\frac{k}{N} : \text{ marginal for, } N \to \infty \tag{4.17}$$

$$\alpha_{-\frac{k}{N}} |0\rangle \equiv -(1 - \frac{3k}{N}) : \text{ tachyonic for, } 3k < N \tag{4.18}$$

From the moding of the oscillators it can be seen that there are no massless states in the twisted sector, as the oscillators are moded by k/N and the zero point energy is $-k/2N$. The massless states arise from the untwisted sector and these are the usual graviton, dilaton and the antisymmetric second rank tensor. In the twisted sectors, the GSO projection removes the ground state tachyon for k even and keeps it for k odd. The GSO action can be explicitly seen by considering the modular invariant partition function [12].

Apart from the GSO projection the spectrum is obtained by projecting into Z_N invariant states. The orbifold action on these space time spinors is given by,

$$R = e^{\frac{2\pi i}{N} J_{89}} \tag{4.19}$$

where, J_{89} is the angular momentum. This orbifold action projects out all spinors even in the untwisted sector (N-th twisted sector). However in order to preserve supersymmetry in the untwisted sector and avoid bulk closed string tachyons we choose the following action on the space time spinors,

$$R = (-1)^F e^{\frac{2\pi i}{N} J_{89}} \tag{4.20}$$

where F is the space time fermion number. This preserves all fermions in the untwisted sector for odd N [1]. Since the twisted fields are localised at the fixed point, for odd N, the theory away from the fixed point is type II. With these constraints on k and N in mind, in the following sections we will construct the tachyon potential in the large N limit when the tachyons become marginal.

4.2 Four point amplitude from CFT

In this section we review the computation of the four point amplitude for the tachyons that we found in the spectrum [20]. This gives the four point tachyon amplitude with all the massless and the massive exchanges. In the next section we will compute the exact contribution from the massless exchanges. Subtracting this from the amplitude computed here gives the effective four point coupling for the tachyon field.

The vacuum for the twisted sector that is labelled by k/N is created from the untwisted vacuum by the action of the bosonic twist fields, $\sigma_{\pm\frac{k}{N}}$ and the fermionic twist fields $s_{\pm\frac{k}{N}}$.

The OPEs of these twist fields with the world sheet fields, $X, \bar{X}, \psi, \bar{\psi}$ are given by [20],

$$\begin{aligned}
\partial_z X(z)\sigma_+(w,\bar{w}) &\sim (z-w)^{-(1-\frac{k}{N})}\tau_+(w,\bar{w}) \\
\partial_z \bar{X}(z)\sigma_+(w,\bar{w}) &\sim (z-w)^{-\frac{k}{N}}\tau_+^{'}(w,\bar{w}) \\
\partial_{\bar{z}} X(\bar{z})\sigma_+(w,\bar{w}) &\sim (\bar{z}-\bar{w})^{-\frac{k}{N}}\tilde{\tau}_+^{'}(w,\bar{w}) \\
\partial_{\bar{z}} \bar{X}(\bar{z})\sigma_+(w,\bar{w}) &\sim (\bar{z}-\bar{w})^{-(1-\frac{k}{N})}\tilde{\tau}_+(w,\bar{w})
\end{aligned} \tag{4.21}$$

where, τ_+, $\tau_+^{'}$, $\tilde{\tau}_+$, $\tilde{\tau}_+^{'}$ are excited twist fields. Using (4.21) and the OPE of the twist fields with the energy momentum tensor, the world sheet dimension of the bosonic twist fields are found to be,

$$h_\sigma = \bar{h}_\sigma = \frac{1}{2}\frac{k}{N}(1 - \frac{k}{N}) \tag{4.22}$$

Bosonising the world sheet fermions,

$$\psi(z) = -i\sqrt{2}e^{iH(z)} \tag{4.23}$$

$$\bar{\psi}(z) = -i\sqrt{2}e^{-iH(z)} \tag{4.24}$$

giving the fermionic twist fields as, $s_\pm = e^{\pm i\frac{k}{N}H(z)}$. From this we get the following OPEs of the fermionic fields with the twist fields.

$$\begin{aligned}\bar{\psi}(z)s_+(w) &= -i\sqrt{2}e^{-iH(z)}e^{i\frac{k}{N}H(w)} \\ &\sim -i\sqrt{2}(z-w)^{-\frac{k}{N}}e^{-i(1-\frac{k}{N})H(z)}[1 + (z-w)\partial H(z)]\end{aligned} \tag{4.25}$$

Similarly,

$$\begin{aligned}\psi(z)s_+(w) &= -i\sqrt{2}e^{iH(z)}e^{i\frac{k}{N}H(w)} \\ &\sim -i\sqrt{2}(z-w)^{\frac{k}{N}}e^{i(1+\frac{k}{N})H(z)}[1 + (z-w)\partial H(z)]\end{aligned} \tag{4.26}$$

For the fermionic string, vertices for the twist fields in the $(-1,-1)$ and $(0,0)$ pictures are given by,

$$V^+_{(-1,-1)}(z,\bar{z}) = e^{-\phi}e^{-\bar{\phi}}\bar{s}_+s_+\sigma_+e^{ik.x}(z,\bar{z}) \tag{4.27}$$

$$V^+_{(0,0)}(z,\bar{z}) = e^{\phi}T_F e^{\bar{\phi}}\bar{T}_F V^+_{(-1,-1)}(z,\bar{z}) \tag{4.28}$$

Where,

$$T_F = -\frac{1}{4}(\partial X\bar{\psi} + \partial\bar{X}\psi) - \frac{1}{2}\partial x.\Psi \tag{4.29}$$

Note that the dimension of the vertex gives the mass of the tachyon,

$$M^2 = -(1-\frac{k}{N}) \tag{4.30}$$

This corresponds to the ground state tachyons in the twisted sector. For the near marginal tachyons, in the large N limit, which are in the $(N-k)th$ sector, the vertex operator in the $(-1,-1)$ picture is,

$$V^{+}_{(-1,-1)}(z,\bar{z}) = e^{-\phi}e^{-\tilde{\phi}}e^{i(1-\frac{k}{N})H(z)}e^{-i(1-\frac{k}{N})\tilde{H}(\bar{z})}\sigma_{+}e^{ik.x}(z,\bar{z}) \tag{4.31}$$

The four point amplitude for these lowest lying tachyons can now be computed by taking two vertices in the $(0,0)$ picture and two in the $(-1,-1)$ picture.

$$C\int_C d^2z \left\langle V^{-}_{(-1,-1)}(z_\infty,\bar{z}_\infty)e^{\phi}T_F e^{\tilde{\phi}}\tilde{T}_F V^{+}_{(-1,-1)}(1)V^{-}_{(-1,-1)}(z,\bar{z})e^{\phi}T_F e^{\tilde{\phi}}\tilde{T}_F V^{+}_{(-1,-1)}(0)\right\rangle \tag{4.32}$$

The constant $C = g_c^4 C_s^2$. Where C_s^2 is related to g_c by,

$$C_s^2 = \frac{4\pi}{g_c^2} \tag{4.33}$$

This amplitude can now be computed and is given by [20],

$$I = C(k_1.k_3)^2 \int_C d^2z \frac{|z|^{-2-s}|1-z|^{-2-t}}{|F(z)|^2} \tag{4.34}$$

Where, $F(z)$ is the hypergeometric function,

$$F(z) \equiv F(\frac{k}{N},1-\frac{k}{N};1;z) = \frac{1}{\pi}\int_0^1 dyy^{-\frac{k}{N}}(1-y)^{-(1-\frac{k}{N})}(1-yz)^{-\frac{k}{N}} \tag{4.35}$$

and, $s=-(k_1+k_2)^2$, $t=-(k_2+k_3)^2$, $s=-(k_3+k_1)^2$.

In the large N approximation,

$$F(z) \sim 1 + \frac{k}{N}(z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + ...) + O((k/N)^2) \tag{4.36}$$

Note that the terms proportional to k/N in (4.36) shift the s-channel pole. There is an additional factor of $(k_1.k_2)^2$, due to which the contact term from any of the terms of (4.36) apart from 1, would at least be of $O((k/N)^2)$. With this observation, the integral can now be performed for $F(z) \to 1$.

$$\begin{aligned} I &= C2\pi(k_1.k_3)^2 \frac{\Gamma(-\frac{s}{2})\Gamma(-\frac{t}{2})\Gamma(1+\frac{s}{2}+\frac{t}{2})}{\Gamma(-\frac{s}{2}-\frac{t}{2})\Gamma(1+\frac{s}{2})\Gamma(1+\frac{t}{2})} \\ &= -(4\pi)^2 g_c^2 \times \frac{1}{4}(u-2m^2)^2(\frac{1}{s}+\frac{1}{t})\frac{\Gamma(1-\frac{s}{2})\Gamma(1-\frac{t}{2})\Gamma(1+\frac{s}{2}+\frac{t}{2})}{\Gamma(1-\frac{s}{2}-\frac{t}{2})\Gamma(1+\frac{s}{2})\Gamma(1+\frac{t}{2})} \end{aligned} \tag{4.37}$$

Now using $s + t + u = 4m^2$,

$$\begin{aligned} I = -4\pi^2 g_c^2 [\frac{(t-2m^2)^2}{s} + \frac{(s-2m^2)^2}{t} &+ 3(s+t) - 8m^2] \\ &\times \frac{\Gamma(1-\frac{s}{2})\Gamma(1-\frac{t}{2})\Gamma(1+\frac{s}{2}+\frac{t}{2})}{\Gamma(1-\frac{s}{2}-\frac{t}{2})\Gamma(1+\frac{s}{2})\Gamma(1+\frac{t}{2})} \end{aligned} \tag{4.38}$$

We have to expand the gamma functions. Now since we are interested in the order $O(\frac{1}{N})$ of the amplitude, any correction to the expansion of the gamma functions to that when the limit $s, t \to 0$ is taken, will be at least of order $O(\frac{1}{N})$. But the factor multiplying the gamma functions is already of order $O(\frac{1}{N})$ except for the pole terms. So we can take the contribution of the gamma functions to be 1. Thus we can write the string amplitude in the zero momentum limit as,

$$I \sim -4\pi^2 g_c^2 \left[\frac{(t-2m^2)^2}{s} + \frac{(s-2m^2)^2}{t} + 3(s+t) - 8m^2\right] \tag{4.39}$$

4.2.1 OPE of two tachyon vertices

In this section we compute the OPE of two tachyon vertices with one in the $(0,0)$ picture and another in the $(-1,-1)$ picture and find the couplings of tachyon to the massless particles.

The OPE we wish to find is,

$$\begin{aligned} V^{-}_{(0,0)}(z,\bar{z})V^{+}_{(-1,-1)}(w,\bar{w}) &= [\frac{1}{4}(\partial X\bar{\psi}+\partial\bar{X}\psi)+\frac{1}{2}\partial x.\Psi][\frac{1}{4}(\bar{\partial}X\bar{\tilde{\psi}}+\bar{\partial}\bar{X}\tilde{\psi})+\frac{1}{2}\bar{\partial}x.\tilde{\Psi}] \\ \times\tilde{s}_{-}s_{-}\sigma_{-}e^{ik.x}(z,\bar{z}) &\times\ e^{-\phi}e^{-\tilde{\phi}}\tilde{s}_{+}s_{+}\sigma_{+}e^{ip.x}(w,\bar{w}) \\ &= [\frac{1}{4}(\partial X\bar{\psi}+\partial\bar{X}\psi)+k.\Psi][\frac{1}{4}(\bar{\partial}X\bar{\tilde{\psi}}+\bar{\partial}\bar{X}\tilde{\psi})+k.\tilde{\Psi}] \\ \times\tilde{s}_{-}s_{-}\sigma_{-}e^{ik.x}(z,\bar{z}) &\times\ e^{-\phi}e^{-\tilde{\phi}}\tilde{s}_{+}s_{+}\sigma_{+}e^{ip.x}(w,\bar{w}) \end{aligned} \tag{4.40}$$

Now, the following OPEs are necessary to compute (4.40).

$$\begin{aligned} e^{ik.x}(z,\bar{z})e^{ip.x}(w,\bar{w}) &\sim |z-w|^{2k.p}e^{i(p+k).x}(w,\bar{w})[1+(z-w)(k-p)_{\mu}\partial x^{\mu} \\ &+ (\bar{z}-\bar{w})(k-p)_{\mu}\bar{\partial}x^{\mu}+|z-w|^{2}(k-p)_{\mu}(k-p)_{\nu}\partial x^{\mu}\bar{\partial}x^{\mu}] \end{aligned} \tag{4.41}$$

For the fermionic twist fields,

$$\begin{aligned} s_{-}(z)s_{+}(w) &= e^{-i\frac{k}{N}H(z)}e^{i\frac{k}{N}H(w)} \\ &\sim (z-w)^{-(\frac{k}{N})^{2}}[1-(z-w)\frac{2k}{N}\partial H(z)] \end{aligned} \tag{4.42}$$

$$\begin{aligned} \tilde{s}_{-}(z)\tilde{s}_{+}(w) &= e^{-i\frac{k}{N}H(\bar{z})}e^{i\frac{k}{N}H(\bar{w})} \\ &\sim (\bar{z}-\bar{w})^{-(\frac{k}{N})^{2}}[1-(\bar{z}-\bar{w})\frac{2k}{N}\bar{\partial}H(\bar{z})] \end{aligned} \tag{4.43}$$

For the bosonic twist fields,

$$\sigma_{-}(z)\sigma_{+}(w)\sim|z-w|^{-2\frac{k}{N}(1-\frac{k}{N})}[1+\cdots] \tag{4.44}$$

Using these, the OPE of the compact part of T_F with the twist fields is,

$$(\partial X\bar{\psi}+\partial\bar{X}\psi)s_{+}\sigma_{+}\sim(z-w)^{-1}\tau_{+}e^{-i(1-\frac{k}{N})H(z)}+\tau'_{+}e^{i(1+\frac{k}{N})H(z)} \tag{4.45}$$

This OPE includes higher twist operators and hence does not contain massless states which we are looking for. The massless state is obtained from the $(k.\Psi k.\bar{\Psi})$ term in the expansion (4.40) with the other twist fields contracted. The coupling for the term is,

$$V_{\mu\nu}(k) = k_\mu k_\nu \tag{4.46}$$

This term is completely symmetric in the indices. It thus corresponds to the massless vertex for the graviton and the dilaton in the $(-1,-1)$ picture, which is the symmetric part of

$$e^{-\phi}e^{-\bar{\phi}}\Psi_\mu\bar{\Psi}_\nu e^{i(k+p).x} \tag{4.47}$$

The four point tachyon scattering amplitude with a massless graviton and dilaton exchange is given by,

$$\begin{aligned} A_4 &= V_{\mu\nu}(p_1)\frac{1}{q^2}[\delta_{\mu\alpha}\delta_{\nu\beta}]V_{\alpha\beta}(p_3) \\ &= p_{1\mu}p_{1\nu}\frac{1}{q^2}[\delta_{\mu\alpha}\delta_{\nu\beta}]p_{3\alpha}p_{3\beta} \\ &= -\frac{(p_1.p_3)^2}{s} = -\frac{1}{4}\frac{(u-2m^2)^2}{s} \end{aligned} \tag{4.48}$$

where, $q^2 = -(p_1+p_2)^2 = s$ and $p_i^2 = -m^2$. We have chosen the configuration of the momenta for the tachyon fields such that it matches with the original configuration used in (4.32) for convenience. Namely p_1 and p_3 corresponds to the momenta of the external ϕ^* field corresponding to the V^- vertices. We still have to add the t-channel contribution to (4.48). Adding this we have,

$$A_4 = -\frac{1}{4}\left[\frac{(u-2m^2)^2}{s} + \frac{(u-2m^2)^2}{t}\right] \tag{4.49}$$

This reproduces the poles which we have found in (4.39) as expected apart from a factor of 2π which comes in (4.39) from the integral over the vertex position.

4.3 Amplitude from effective field theory

In the previous section we have seen that the massless exchanges in the four point amplitude of the twisted sector tachyons are the graviton and the dilaton. In this section we calculate the tachyon four point amplitude with these massless exchanges, namely the graviton and dilaton from the effective field theory.

The action for the complex tachyon coupled to graviton and dilaton is given by,

$$S = \frac{1}{\kappa^2}\int d^Dx\sqrt{(-g)}\left[-R + \frac{4}{D-2}\Phi\partial^2\Phi + \frac{1}{2}\phi^*(-\partial^2)\phi + \frac{1}{2}m^2e^{\frac{4}{D-2}\Phi}\phi^*\phi\right] \quad (4.50)$$

Expanding $g_{\mu\nu}$ about the flat metric,

$$g_{\mu\nu} = \delta_{\mu\nu} + \kappa h_{\mu\nu} \quad (4.51)$$

and rescaling the dilaton field by,

$$\Phi \to \sqrt{\frac{8}{D-2}}\Phi \quad (4.52)$$

We have,

$$S = \frac{1}{\kappa^2}\int d^Dx\left[\frac{1}{2}h_{\mu\nu}(-\partial^2+\cdots)h_{\mu\nu} - \frac{1}{2}\Phi\partial^2\Phi + \frac{1}{2}\phi^*(-\partial^2+m^2)\phi - h_{\mu\nu}T_{\mu\nu} + T\Phi\right] \quad (4.53)$$

The couplings of the graviton and dilaton to the complex scalar field are now,

$$-\kappa h_{\mu\nu}T_{\mu\nu} \qquad \text{and} \qquad \kappa T\Phi \quad (4.54)$$

Where,

$$\begin{aligned}
T_{\mu\nu} &= -\frac{1}{2}\left[\partial_\mu\phi^*\partial_\nu\phi+\partial_\nu\phi^*\partial_\mu\phi\right]+\frac{1}{2}\delta_{\mu\nu}\left[\partial_\alpha\phi^*\partial_\alpha\phi+m^2|\phi|^2\right]\\
T &= \sqrt{\frac{2}{D-2}}m^2\phi^*\phi
\end{aligned}\qquad(4.55)$$

The tachyon-graviton vertex and the graviton propagator in the harmonic gauge are given by,

$$V_{\mu\nu}(p,k)=i\kappa\left[-\frac{1}{2}(p_\mu k_\nu+p_\nu k_\mu)+\frac{1}{2}\delta_{\mu\nu}(k.p-m^2)\right]\qquad(4.56)$$

$$\Delta_{\mu\nu\alpha\beta}(q^2)=\frac{1}{q^2}\left[\delta_{\mu\alpha}\delta_{\nu\beta}+\delta_{\mu\beta}\delta_{\nu\alpha}-\frac{2}{D-2}\delta_{\mu\nu}\delta_{\alpha\beta}\right]\qquad(4.57)$$

The four point amplitude for four massless scalar scattering with a graviton exchange is,

$$\begin{aligned}
A_4^g &= V_{\mu\nu}(p_1,p_2)\Delta_{\mu\nu\alpha\beta}(q^2)V_{\alpha\beta}(p_3,p_4) \qquad(4.58)\\
&= \frac{\kappa^2}{q^2}[(p_1.p_3)(p_2.p_4)+(p_1.p_4)(p_2.p_3)-(p_1.p_2)(p_3.p_4)\\
&+ m^2(p_1.p_2+p_3.p_4)-\frac{D}{D-2}m^4]
\end{aligned}$$

Similarly, for the dilaton exchange we have,

$$A_4^d=\frac{\kappa^2}{q^2}\frac{2}{D-2}m^4\qquad(4.59)$$

Therefore, the four point tachyon amplitude with graviton and dilaton exchange is,

$$\begin{aligned}
A_4 &= A_4^g+A_4^d\\
&= \frac{\kappa^2}{q^2}\left[(p_1.p_3)(p_2.p_4)+(p_1.p_4)(p_2.p_3)-(p_1.p_2)(p_3.p_4)+m^2(p_1.p_2+p_3.p_4)-m^4\right]\\
&= -\frac{\kappa^2}{4s}\left[(u-2m^2)^2+(t-2m^2)^2-s^2\right] \qquad(4.60)\\
&= -\frac{\kappa^2}{2}\left[\frac{(t-2m^2)^2}{s}+t-2m^2\right] \qquad(4.61)
\end{aligned}$$

In writing (58) from the previous step, we have used $p_i^2 = -m^2$ and put in $q^2 = (p_1 + p_2)^2 = -s$. From (58) to (59) we have used $s + t + u = 4m^2$ which uses the mass shell conditions. Now including the t-channel process we get,

$$A_4 = -\frac{\kappa^2}{2}\left[\frac{(t-2m^2)^2}{s} + \frac{(s-2m^2)^2}{t} + (t+s) - 4m^2\right] \tag{4.62}$$

Comparing with the pole term of the string amplitude (4.39), we see that the pole is due to graviton exchange. This also relates κ to g_c which is found to be,

$$\frac{\kappa^2}{2} = 4\pi^2 g_c^2 \tag{4.63}$$

After subtraction, the non-derivative quartic term and the derivative terms left behind are,

$$-4\pi^2 g_c^2[-4m^2 + 2(s+t)] \tag{4.64}$$

4.4 The Potential

We can now write down the potential for the tachyon upto the quartic term. It may be noted that the sign of the quartic term is to be fixed relative to the sign of the pole terms which has to be positive. So we get the quartic coupling as,

$$\lambda = (4\pi^2 g_c^2) \times (-4m^2) \tag{4.65}$$

which is positive since $m^2 = -k/N$. The tachyon potential is now,

$$\begin{aligned} V(\phi^*\phi) &= \frac{1}{2}m^2(\phi^*\phi) + \frac{\lambda}{4}(\phi^*\phi)^2 \\ &= -\frac{k}{2N}(\phi^*\phi) + 4\pi^2 g_c^2 \frac{k}{N}(\phi^*\phi)^2 \end{aligned} \tag{4.66}$$

The potential has a minimum at $|\phi|^2 = 1/(16\pi^2 g_c^2)$ which is of $O(1)$. We expect the nearest minimum to correspond to the C/Z_{N-k} orbifold. In this case the height of the potential from C/Z_N to C/Z_{N-k} is given by,

$$\Delta = \frac{1}{64\pi^2 g_c^2}\frac{k}{N} \tag{4.67}$$

One may compare this, upto normalisations, with the conjectured height [4] which is,

$$\Delta = 4\pi(\frac{1}{N-k} - \frac{1}{N}) \sim 4\pi\frac{k}{N^2} \tag{4.68}$$

This shows that the perturbative result (4.67) is off by a factor of $1/N$. At this point we are not in a position to trust this perturbative result. Higher point amplitudes will most likely modify this.

There are various indications that higher point interaction terms in the amplitude are in fact important and are not of order less than $1/N$. We may note that with a potential upto the quartic coupling having global $O(2)$ symmetry, in the spontaneously broken theory there is a massless scalar corresponding to the Goldstone boson and a massive particle of mass, $-2m^2 = 2k/N$. This means that in the spectrum of C/Z_{N-k} to which the theory is supposed to flow, there must be a massless and a massive scalar of mass $2k/(N-k)(\sim 2k/N + \cdots;$ for large $N)$. However the spectrum does not contain these scalars. The absence of the massless Goldstone particle indicates that the O(2) symmetry of the tachyon potential has to be broken. This can only happen if the correlation functions of N twist operators are also of order $1/N$.

This fact may also seen as follows. The three point graviton vertex and the two tachyon and one graviton vertex, both have two powers of momentum (Figure 4.4). The four point tachyon amplitude with a graviton exchange has two positive powers of momentum.

With the addition of two more external tachyons, using (A), the positive power of momentum for the six point amplitude (C), remains two. Of course with

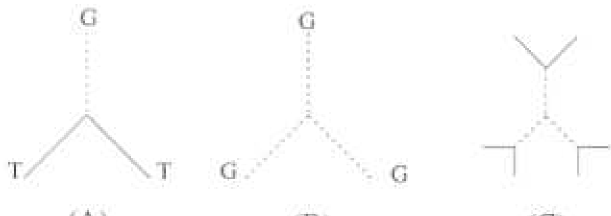

Figure 4.1: (A) Two tachyon one graviton vertex, (B) Three graviton vertex, (C) Six-point tachyon amplitude with graviton exchange.

a three graviton vertex insertion, we can introduce more negative powers, but there are tree diagrams where we can have two positive powers of momentum. The N point contact term is obtained by subtracting these graviton exchanges from the full N point tachyon amplitude from string theory. The graviton exchange diagrams as mentioned contains two positive powers of momentum, which in the on-shell limit give terms proportional to $m^2 \sim O(1/N)$. It is thus very likely that the subtracted term would give a $1/N$ dependence as the leading part.

We further see that the dilaton field redefinitions such as,

$$\Phi \to \Phi + c\phi^* \phi \tag{4.69}$$

where c is a constant, can change the value of the contact term and can even change the sign. The minimum thus depends crucially on the expectation value of the dilaton field which when becomes large, makes this perturbative analysis anyway unreliable. A similar observation was made in [16], in the context of closed string tachyon condensation with Rohm's Compactification. In general the existence of the minimum is independent of field redefinitions. The fact that we are able to change the nature of the potential by field redefinitions implies the potential upto the quartic term does not shed much light on the minimum of the theory. It is argued in various approaches that the Type II theory on the C/Z_N orbifold ultimately upon closed string tachyon condensation goes to the Type II theory on flat space. Our analysis does not give a proof of this observation. If we assume this to be true, that a stable minimum exists, then following the above arguments, we may conclude that the higher point terms are indeed necessary and are of the order $1/N$.

4.5 Conclusion

In this chapter we have studied the condensation of closed string tachyons for Type II strings on the C/Z_N orbifold. We constructed the potential for the tachyons upto the quartic term in the large N limit by subtracting the massless exchanges from the four point tachyon amplitude computed from string theory. We expect the minimum of the potential for the near marginal tachyons for the k-th twisted sector, in the large N limit to correspond to the C/Z_{N-k} orbifold. When compared to the conjectured value for the height of the potential for the C/Z_N orbifold, we find a mismatch by a factor of $1/N$. However, we have argued that the higher point amplitudes are indeed important and are of the same order in $1/N$ as the quartic term. A potential upto the quartic term after spontaneous symmetry breaking gives masses which are not there in the spectrum for closed string on C/Z_{N-k} to which the C/Z_N theory is expected to flow. This leads us to conclude that the higher point amplitudes including the global $O(2)$ breaking term, ϕ^N, must all be of order $1/N$ so that the potential gives a mass spectrum, consistent with that of the C/Z_{N-k} orbifold. We have also argued that field redefinitions can alter the contact term and can even change the sign. If the theory can be deformed so that the minimum can be reliably reached in perturbation theory, then, a direct approach such as the one discussed in this chapter can ascertain whether this minimum has the required properties.

This computation of the quartic coupling for the twisted sector tachyon is also done in [11]. It was pointed out by the authors that additional contributions to the four point contact term will also come from the massive untwisted states with momentum along the orbifold plane, C/Z_N. This modifies the $1/N$ dependence of the quartic term which we computed here to a more suppressed $1/N^3$ dependence. However with this modification, the expectation that the large N approximation may may be used to study the RG flow due to tachyon condensation from the C/Z_N orbifold to lower nonsupersymmetric orbifolds is even further weakened. The conclusions above thus remain unaltered.

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Chapter 5

Conclusion

In this thesis we have studied two problems. In the first part we have analysed some of the aspects of noncommutativity in field and string theory and in the second part we have studied localised closed string tachyon condensation. In this conclusion we highlight some of the important issues and summarise the results.

5.1 Noncommutativity in field and string theory

We have seen that when a constant antisymmetric two-form B-field is turned on along the world-volume directions of a D-brane, the low energy dynamics of the D-brane is described by a noncommutative gauge theory in the *Seiberg-Witten limit*. Noncommutative field theories are non-local in nature, however we have seen that they can be efficiently handled. They inherit many of the properties of the parent string theory. In this way noncommutative field theories differ from ordinary field theories where all the features of the one dimensional string theory are lost. One of the generic features in noncommutative field theories is the mixing of the ultra-violet and the infrared regimes of the theory that arise in the nonplanar sectors. This is very unusual in the Wilsonian sense of commutative quantum field theories. In the ordinary case the two sectors decouple, i.e. the low energy effective theory does not have any dependence on the high energy modes except through the renormalisation of a finite number of couplings. We have seen that the zero momentum limit in noncommutative field theories is a singular configuration. It would be thus natural to ask, in what sense the noncommutative field theories fit into the usual notion of Wilsonian renormalisation group. This question was addressed in Chapter 2. We have shown that when we restrict ourselves to configuration of fields that have non-zero external momentum, renormalisation can be proved to all orders. We have shown this to all loops using the Wilsonian renormalisation group equation of Polchinski for the scalar theory with global $O(N)$ symmetry. This means that at the level of quantum field theory we must define an infrared cutoff for the noncommutative field theory. We have also shown that the spontaneously broken phase of the noncommutative scalar theory is renormalisable to all orders with an IR cut-off. This means that the ground state of the broken phase will be translationally non-invariant.

The coupling of the ultraviolet and infrared sectors has a natural interpretation in terms of world-sheet open-closed string duality in string theory. We have thus looked into the problem of UV/IR mixing in noncommutative field theories from this point in Chapter 3 and have elucidated on the role played by the B-field . We have first studied the bosonic string model. In this model analysing the two point one loop open string amplitude with two gauge boson vertex insertions, we have shown that the infrared singularities that appear as a result of integrating over high momentum modes in loops in noncommutative field theories can be obtained by tree-level exchange of closed string modes. In general the the divergence of the gauge theory would require an infinite number of closed string modes for its dual description. Along with this observation and the presence of the tachyons in the bosonic theory the closed string interpretation of the gauge theory IR divergences is not exact in terms of the lowest modes of closed strings for the bosonic theory. To overcome these problems we have studied a supersymmetric model. Specifically we considered gauge theory on a fractional D_3-brane localised at the fixed point of C^2/Z_2 orbifold. We have shown that in this case, the infrared singularities in noncommutative gauge theory are exactly equal to the infrared singularities due to massless closed string modes. These modes come from the twisted NS-NS and R-R sectors. They are localised at the fixed point on the orbifold plane and are free to move in the six directions transverse to the orbifold. In this model the realisation of open-closed string duality is manifested only by the massless sectors on both the open and closed string sides. This is due to the fact that the contribution from the massive modes cancel. We have seen that the role played by the B-field is mostly that of the regulator that preserves this duality and thus helps in quantitatively analysing the UV/IR duality. We conclude that the one loop UV/IR mixing terms can always be obtained in terms of closed string tree-level amplitudes, however the question whether the duality is manifested by a finite set of modes on either side depends on the theory without the B-field. In these cases there is an exact correspondence between the gauge theory and supergravity. We have studied the duality between the open and the closed string channels only for the nonplanar sector, where in the presence of background B-field the amplitudes are regulated. It will be interesting to consider a limit where only the nonplanar sector of the gauge theory survives.

5.2 Localised closed string tachyon condensation

Localised tachyons on orbifolds have served as useful testing grounds for studying closed string tachyon condensation. The closed string tachyons couple to the graviton and the dilaton and hence their condensation is expected to be accompanied by large modifications of background space-time. Tachyons that arise in the twisted sectors of C^p/Z_N orbifolds, are localised at the fixed point and only freely propagate in directions transverse to the orbifold. It is thus expected the the initial condensation process will start on the orbifold plane only at the fixed point. Study of this tachyon condensation process for type II theory have led Adams, Polchinski and Silverstein to conjecture that the end point of condensation is a supersymmetric type II string theory on flat space. It was further conjectured by Dabholkar that the height of potential for the C/Z_N orbifold is proportional to the deficit angle of the orbifold. Specific perturbations by tachyons however lead to orbifolds of lower order.

Using these guidelines we have studied the condensation of twisted sector tachyons on the C/Z_N orbifold in the limit $N \to \infty$, in Chapter 4. On this orbifold, there are tachyons whose $(mass)^2$ goes as $1/N$. In the above limit these tachyons become nearly massless. On can now sensibly write down a potential for these tachyons and compute the height. Specifically perturbation by a tachyon in the $(N-k)$-th twisted sector should lead to the C/Z_{N-k} orbifold. We have computed the potential upto the quartic term for these tachyons in the large N approximation. However, we showed that if we stick to the conjectured height the answer is off by a factor of $1/N$. Moreover the quadratic and the quartic terms are of the same order in $1/N$, suggesting that higher point amplitudes may also be of the same order in $1/N$ as these terms. We have seen that a potential upto the quartic term after spontaneous symmetry breaking gives masses which are not there in the spectrum for closed string on C/Z_{N-k} to which the C/Z_N theory is expected to flow. This leads us to conclude that the higher point amplitudes including the global $O(2)$ breaking term, ϕ^N, must all be of order $1/N$ so that the potential gives a mass spectrum, consistent with that of the C/Z_{N-k} orbifold. We have also argued that field redefinitions can alter the contact term and can even change the sign. The problem of construction of tachyon potential for this model still remains open. If the theory can be deformed so that the minimum can be reliably reached in perturbation theory, then, a direct approach such as the one discussed here can ascertain whether this minimum has the required properties.