1. In this problem we look at a few examples of double counting.

(a) (Handshaking Lemma) Prove that at any party, the number of people who have shaken hands with an odd number of people is always even.

(b) Let \( B \) be a set of subsets of the set \( \{1, \ldots, v\} \), containing exactly \( b \) sets. Suppose that

- every set in \( B \) contains exactly \( k \) elements;
- for \( i = 1, \ldots, v \), the element \( i \) is contained in exactly \( r \) members of \( B \)

Prove that \( bk = vr \).

(c) Prove the following binomial identity.

\[
\sum_{i=o}^{n} \binom{n}{i}^2 = \binom{2n}{n}
\]

(d) Fermat’s little theorem states that for any natural number \( n \) and a prime \( p \), \( p | (n^p - n) \). Can you come up with a proof for this statement using double counting.

2. Let \( A_1, \ldots, A_n \) be distinct subsets of \( [n] \). Show that there exists some \( x \in [n] \) such that \( A_1 \setminus \{x\}, \ldots, A_n \setminus \{x\} \) remain distinct.

3. Prove that the set of all finite subsets of \( \mathbb{N} \) is countable but the set of all subsets of \( \mathbb{N} \) is not.

4. Consider a \( 2^k \times 2^k \) chessboard whose one of the corner squares is removed. Prove that for every \( k \geq 1 \) such a chessboard can always be totally covered with L shaped dominos made with 3 squares.

5. Suppose that each edge of a \( K_6 \) (the complete graph on 6 vertices) is colored with one of two colors, say red or blue. Show that there exists a \( K_3 \) which has all its edges colored with the same color. Is the previous statement true if we replace \( K_6 \) by \( K_5 \)?

6. In this problem, we will look at some identities concerning Fibonacci numbers. We will denote the \( n^{th} \) Fibonacci number by \( F_n \).

(a) Fibonacci numbers are given by the recurrence \( F_n = F_{n-1} + F_{n-2} \). Give a closed form expression for \( F_n \).

(b) Show that the number of ways in which a non-negative integer \( n \) can be written as an ordered sum of ones and twos is \( F_n \).
(c) Show that the number of subsets of \([n]\) which do not contain consecutive numbers is \(F_n\).

(d) Show that
\[
\sum_{i=0}^{\lfloor(n-1)/2\rfloor} F_{n-2i} = F_{n+1} - 1
\]
for \(n \geq 1\).

(e) Prove that every non-negative integer \(x\) less that \(F_{n+1}\) can be expressed in a unique way in the form
\[F_{i_1} + F_{i_2} + \cdots + F_{i_r}\]
where \(i_1, \ldots, i_r \in [n]\) such that \(i_1, \ldots, i_r\) are all distinct and no two are consecutive.

(Hint: By the previous problem, the largest expression of the form \(F_{i_1} + F_{i_2} + \cdots + F_{i_r}\) for distinct, non-consecutive \(i_1, \ldots, i_r\) that can be made by Fibonacci numbers below \(F_n\) is \(F_{n-1}\)(why?) So, if \(F_n \leq x < F_{n+1}\), then \(F_n\) must be in the sum; and \(x - F_n < F_{n-1}\), so \(F_{n-1}\) cannot be included.)

7. Let \(C_n\) denote the number of ways in which a sum of \(n\) terms can be bracketed so that it can be calculated by adding two terms at a time. Show that \(C_n\) satisfies the recurrence
\[
C_n = \sum_{i=1}^{n-1} C_i C_{n-i}.
\]
Show that
\[
C_n = \frac{1}{n} \left( \frac{2n - 2}{n - 1} \right).
\]

8. A clown stands on the edge of a swimming pool, holding a bag containing \(n\) red and \(n\) blue balls. He draws the balls one at a time and discards them. If he draws a blue ball, he takes one step back; if a red ball, one step forward. Show that the probability that the clown remains dry is \(1/(n + 1)\).

9. Let
\[
\prod_{n \geq 1} (1 + t^n) = \sum_{n \geq 0} a_n t^n
\]
Prove that \(a_n\) is the number of ways of writing \(n\) as the sum of distinct positive integers.