Problem Set 1
Automata Theory

Definition 1. A deterministic finite automaton (DFA) is a structure

\[ M = (Q, \Sigma, \delta, s, F) \],

where

- \( Q \) is a finite set; elements of \( Q \) are called states.
- \( \Sigma \) is a finite set, the input alphabet.
- \( \delta : Q \times \Sigma \to Q \) is the transition function.
- \( s \in Q \) is the start state
- \( F \subseteq Q \); elements of \( F \) are called accept states or final states.

Transition function \( \delta \) can be extended to another function \( \delta' : Q \times \Sigma^* \to Q \) by induction on the length of \( x \in \Sigma^* \) as:

\[
\delta'(q, \epsilon) = q \\
\delta'(q, xa) = \delta(\delta'(q, x))
\]

A word \( x \) is said to be accepted by the automaton \( M \) if \( \delta'(s, x) \in F \) and rejected by the automaton \( M \) if \( \delta'(s, x) \notin F \).

The set or language accepted by \( M \) is \( \text{Lang}(M) = \{ x \in \Sigma^* \mid \delta'(s, x) \in F \} \).

Question 1. Let \( \Sigma = \{a, b\} \). Construct a DFA for the languages

1. \( \{ x \in \Sigma^* \mid x \text{ starts with three consecutive } a's \} \).
2. \( \{ x \in \Sigma^* \mid x \text{ contains a substring of three consecutive } a's \} \).
3. \( \{ x \in \Sigma^* \mid \text{length of the string } x \text{ is even} \} \).
4. \( \{ x \in \Sigma^* \mid \text{length of the string } x \text{ is divisible by 3} \} \).
5. \( \{ x \in \Sigma^* \mid \text{length of the string } x \text{ is divisible by 2 or 3} \} \).

Definition 2. A nondeterministic finite automaton (NFA) is a structure \( M = (Q, \Sigma, \delta, S, F) \) where

- \( Q \) is a finite set; elements of \( Q \) are called states.
- \( \Sigma \) is a finite set, the input alphabet.
- \( \delta : Q \times \Sigma \to 2^Q \) is the transition function.
- \( S \subseteq Q \) is the set of start states
- \( F \subseteq Q \); elements of \( F \) are called accept states or final states.

Transition function \( \delta \) can be extended to another function \( \delta' : Q \times \Sigma^* \to 2^Q \) by induction on the length of \( x \in \Sigma^* \) as:

\[
\delta'(q, \epsilon) = \{ q \} \\
\delta'(q, xa) = \{ p \mid \text{for some state } r \in \delta'(q, x), p \in \delta(r, a) \}
\]

A word \( x \) is said to be accepted by the automaton \( M \) if \( \delta'(s, x) \) contains a state from \( F \).

The set or language accepted by \( M \) is \( \text{Lang}(M) = \{ x \in \Sigma^* \mid x \text{ is accepted by } M \} \).
Question 2. Let $\Sigma = \{a, b\}$. Construct NFA for the languages

1. $\{x \in \Sigma^* \mid x$ ends with three consecutive $a$’s$\}.
2. $\{x \in \Sigma^* \mid x$ ends with string $ab$ $\}.
3. $\{x \in \Sigma^* \mid 2$nd letter from the last is $a$ $\}.
4. $\{x \in \Sigma^* \mid x$ contains odd number of $a$’s$\}.
5. $\{x \in \Sigma^* \mid$ length of the string $x$ is divisible by 2 or 3$\}.$

Question 3.
Let $M_1$ and $M_2$ be two DFAs. Construct a DFA for accepting language of $\text{Lang}(M_1) \cap \text{Lang}(M_2)$.

Question 4. Can you construct an NFA or DFA for the languages given below:

1. $\{a^n b^n \mid 1 \leq n \leq 5\}.$
2. $\{a^n b^n \mid 1 \leq n \leq 100\}.$
3. $\{(ab)^n \mid 1 \leq n \leq 100\}.$
4. $\{a^n b^n \mid n \geq 1\}.$
5. $\{(ab)^n \mid n \geq 1\}.$

Question 5. Let $w \in \Sigma^*$ and $M = (Q, \Sigma, \delta, q_0, F)$ be a finite automaton.

1. If $M$ is a DFA, then give an algorithm to determine whether $w \in \text{Lang}(M)$.
   What is the complexity of your algorithm?
2. If $M$ is a NFA, then give an algorithm to determine whether $w \in \text{Lang}(M)$.
   What is the complexity of your algorithm?

Question 6. The shuffle of two words $f$ and $g$ from $\Sigma^*$, written as $f \parallel g$, is the subset of $\Sigma^*$ defined by:

$f \parallel g = \{f_1 g_1 f_2 g_2 \cdots f_n g_n \mid f_i \in \Sigma^*, g_j \in \Sigma^*, f_1 f_2 \cdots f_n = f$ and $g_1 g_2 \cdots g_n = g\}$.

Shuffle of two languages $L_1, L_2 \subseteq \Sigma^*$, denoted by $L_1 \parallel L_2$, is defined as:

$L_1 \parallel L_2 = \bigcup \{f \parallel g \mid f \in L_1, g \in L_2\}$.

Can you construct a finite automaton accepting $L_1 \parallel L_2$?