Q DisSys School, 2016 · IMSc, Chennai

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Lecture 1 : Many-Body Localization & Thermalization in Isolated Systems

References :
- R. Nandkishore & D. Huse, Ann. Rev. of Cond Mat. Phys
- B. Bauer & C. Nayak, J. Stat. Mech.
- S.P., 2016 Cargèse lectures

+ references therein.

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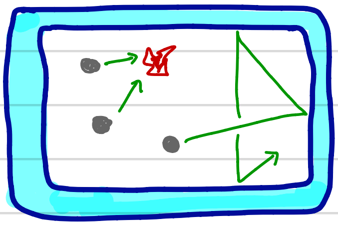
I. Introduction : Eigenstate Thermalization

Take an isolated classical system ("microcanonical"; microstates $\{\vec{p}_n, \vec{q}_n\}$)

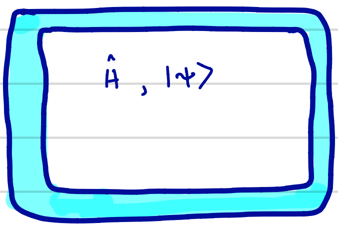
Fundamental Assumption of Stat. Mech: all microstates are equiprobable
(as long as they are consistent w/ conservation laws)

- Implicitly assumes ergodic dynamics - no "memory" of initial conditions
  (Can derive in some cases e.g. kinetic theory of gases
  - Newtonian dynamics + statistics + "molecular chaos")
  - NEEDS INTERACTIONS (even "ideal" gas is not really free!)



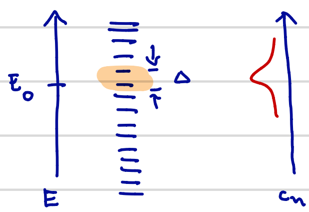
- Exceptions rare ~ "integrable systems" w/ extensive # of conservation laws.

What about isolated quantum systems?
(microstates ≡ many-body eigenstates)



Consider some initial state (e.g., built from states in window ~ Δ around energy E_0):

$$|\psi\rangle = \sum_n c_n |\psi_n\rangle \qquad (\text{at } t=0)$$



(e.g. $|c_n|^2 \sim e^{-(E_n-E_0)^2/\Delta^2}$)

Q. How does this evolve in time?

Schrödinger : $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$

$$\Rightarrow \quad |\psi(t)\rangle = \sum_n c_n e^{-iE_nt} |\psi_n\rangle$$

For stat. mech. ~ convenient to use density matrix formulation

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| \qquad \text{(note: \underline{pure} state!)}$$

$$= \sum_{m,n} c_m^* c_n e^{i(E_m-E_n)t} |\psi_m\rangle\langle\psi_n|$$

Consider making a measurement "averaged" over time T large compared to all $E_m - E_n$ (assume no degeneracies ~ "generic" system): this corresponds to performing averages using the appropriately time-averaged density matrix

$$\overline{\rho(t)} \equiv \frac{1}{T}\int_t^{t+T} dt\, \rho(t') \simeq \boxed{\sum_m |c_m|^2 |\psi_m\rangle\langle\psi_m|}$$

off-diag. pieces oscillate & cancel

THE "DIAGONAL ENSEMBLE".

Time-averaged density matrix "knows" the probability that system was initially in state n!

How do we define ergodicity for isolated quantum systems?

- Quantum evolution is <u>unitary</u> $\Rightarrow$ can't lose any quantum information
- Information must be "hidden" somehow (DECOHERENCE)

Consider a local observable: Hermitian operator $\hat{O}$ (acts only inside a subregion, A)

$$\langle \hat{O}(t)\rangle = \mathrm{tr}\, \overline{\rho}(t)\, \hat{O} = \sum_m |c_m|^2 \langle m|\hat{O}|m\rangle$$

We can relax our demand for ergodicity and simply ask that no local measurement can remember the initial conditions

i.e. we can ask that $\langle \mathcal{O}(t) \rangle \simeq \mathcal{O}(E_0)$ + small corrections

$$\Rightarrow \quad \sum_m |c_m|^2 \langle m|\mathcal{O}|m\rangle \simeq \mathcal{O}(E_0)$$

Suppose $\langle m|\mathcal{O}|m\rangle \simeq \mathcal{O}(E_m)$ + small corrections ($\mathcal{O}(E_m) \sim$ smooth fn.)

Then,

$$\sum_m |c_m|^2 \langle m|\mathcal{O}|m\rangle \simeq \sum_m |c_m|^2 \, \mathcal{O}(E_m) + \text{small corrections}$$

$\nwarrow$ sharply peaked around E_0

$$\simeq \mathcal{O}(E_0)$$

(Obvios generalizations to other conserved quantities, e.g. $\hat{N}$, $\hat{S}_z$, ...)

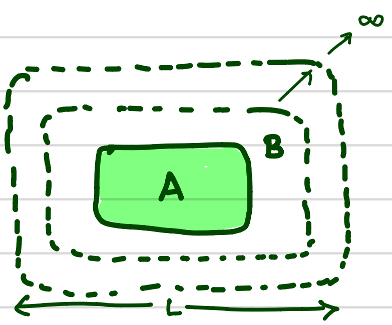
This is the idea of "typicality": states that are "close" in terms of conserved quantities have similar values of local observables.

AN ENTANGLEMENT PERSPECTIVE

We can formalize this notion by taking the above ideas to the logical conclusion: consider a single eigenstate $|\psi_m\rangle$. How do we recover stat. mech. ?

$$\hat{\rho}^{(m)} = |\psi_m\rangle\langle\psi_m| \qquad \text{(pure state)}$$

Let's consider a local observable, made inside some region A and take the thermodynamic limit while keeping A fixed: i.e. we add degrees of freedom only to $B = \bar{A}$, and hence grow the "environment" of A:



It's convenient to write

$$\langle \hat{O} \rangle = \mathrm{Tr}_{A\cup B}\, \hat{\rho}^{(m)} \hat{O} = \mathrm{Tr}_A \left\{ (\mathrm{Tr}_B \hat{\rho}^{(m)})\, \hat{O} \right\} \quad \text{(since } \hat{O} \text{ lives within A)}$$

$$= \mathrm{Tr}_A \left\{ \hat{\rho}^{(m)}_A\, \hat{O} \right\}$$

where $\hat{\rho}^{(m)}_A \equiv \mathrm{Tr}_B\, \hat{\rho}^{(m)}$ is the <u>reduced</u> <u>density</u> <u>matrix</u> of A in state m.
(can define even for non-eigenstate)

So, as $L \to \infty$, $\hat{\rho}_{mA}$ is changing. This reflects the fact that as we add degrees of freedom to B, they can get entangled with those in A. Note that even though $|\psi_m\rangle$ is a pure state - even an eigenstate! - $\hat{\rho}_{mA}$, the reduced density matrix of a subregion, can look like a mixed-state density matrix - reflecting the fact that entanglement of degrees of freedom inside A with those outside of it can generate [entanglement] entropy. [Q. What would it mean if $\hat{\rho}^{(m)}_A$ were a pure-state density matrix?]

We now require that this entanglement generates thermal equilibrium behavior within A - i.e., that B acts as an "environment" for A. To quantify when $\hat{\rho}_{mA}$ is "thermal", we recall that thermal equilibrium expectation values of local observables are given by

$$\langle \hat{O} \rangle_{th} = \mathrm{Tr}_{A\cup B}\, \frac{e^{-\mathcal{H}/T}}{Z(T)}\, \hat{O} = \mathrm{Tr}_A \left\{ \mathrm{Tr}_B \frac{e^{-H/T}}{Z(T)}\, \hat{O} \right\} \equiv \mathrm{Tr}\left\{ \hat{\rho}^{eq}_A(T)\, \hat{O} \right\}$$

where $Z(T) = \mathrm{Tr}\, e^{-\mathcal{H}/T}$ and $\hat{\rho}^{eq}_A(T) \equiv \mathrm{Tr}_B \dfrac{e^{-H/T}}{Z(T)}$ is the thermal equilibrium density matrix of A.

We are now ready to state the Eigenstate Thermalization Hypothesis (ETH):
(Deutsch, Srednicki)

A system satisfies ETH if in the thermodynamic limit where we hold the size of a subregion, A, fixed,

i) given eigenstate $|\psi_m\rangle$ ($\hat{H}|\psi_m\rangle = E_m|\psi_m\rangle$), $\hat{\rho}_A^m = \hat{\rho}_A^{eq}(T_m)$ where T_m is the temperature at which the thermal equilibrium energy is E_m; in other words, T_m is implicitly defined by $E_n = \mathrm{Tr}_A\, \hat{H}\, \hat{\rho}_A^{eq}(T_m)$.

ii) matrix elements $\langle m|\hat{O}|n\rangle$ vanish "fast enough" in the th. limit that temporal fluctuations of the $\rho_A(t)$ vanish.

[various modifications of this are possible, won't discuss these much ...]

ETH formalizes what it means for a system to "act as its own heat bath".

Are there systems that don't satisfy ETH?

- One example is furnished by quantum integrable systems - due to the extensive number of conserved quantities, they thermalize to an equilibrium density matrix with an extensive number of Lagrange multipliers - this is called the Generalized Gibbs Ensemble (GGE). However, these are very fragile - generic perturbations destroy integrability & restore ergodicity. So, they don't provide a route to an ETH-violating phase.

- A more interesting class of ETH-violating systems is the set of Anderson-localized systems. These are also non-generic: they are non-interacting. However, seminal work by Basko, Aleiner, Altshuler (2006) (notable prior work by Fleischman & Anderson ('80), Altshuler-Gefen-Kamenev-Levitov (1997) and Gornyi, Mirlin, & Polyakov (2005)) showed that Anderson localization is perturbatively stable against interactions — so we term these more generic localized systems "many-body localized". We will examine their phenomenology and new physics afforded by them in the remainder.

II. ENTANGLEMENT, LOCALIZATION & ETH VIOLATION

To see why localized systems violate ETH, let's go back to the statement of ETH. This says that the reduced density matrix of a state is equivalent (in the thermodynamic limit) to the thermal density matrix at a temperature corresponding to the energy of the state.

Consider the entanglement entropy of subregion A in state m:

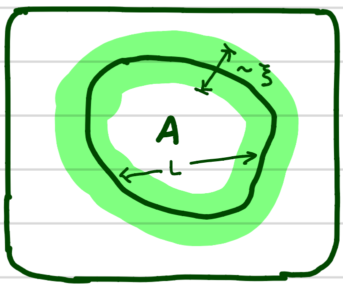
$$S_{EE}^{(m)}(A) \equiv -\mathrm{Tr}\, \hat{\rho}_A^{(m)} \ln \hat{\rho}_A^{(m)}$$

Now, consider a state at finite energy density, corresponding to high temperature. If ETH holds, then $\hat{\rho}_A^{(m)} = \hat{\rho}_A^{eq}(T_m)$; in that case, the entanglement entropy must coincide with the thermal entropy. Crucially, thermal entropy is extensive. Therefore, we have

$$\begin{aligned} S_{EE}^{(m)}(A) &= -\mathrm{Tr}\, \hat{\rho}_A^{eq}(T_m) \ln \hat{\rho}_A^{eq}(T_m) \\ &= S^{th}(A, T_m) \\ &= s(T_m) \times \mathrm{vol}(A) \end{aligned}$$

Now, consider an Anderson-localized system. Since it's a free-fermion problem, every many-body state is a Slater determinant of single-particle states, each of which is exponentially localized, with some finite localization length, ξ.

Then, we can compute the entanglement entropy by noting that degrees of freedom located $\gg \xi$ from the boundary of A don't contribute to the entanglement of A with the rest of the system.



$$S_A \sim c\, \xi\, L^{d-1}$$ (true for any state in the spectrum)

[Exercise: argue this more formally by actually computing the reduced density matrix. Hint: you can use "Peschel's trick."]

In other words, ETH $\Rightarrow$ $S_{EE}(A) \sim \text{vol}(A)$; but this is not true in the Anderson-localized phase $\Rightarrow$ ETH is violated.

This is stable to adding interactions $\Rightarrow$ MBL systems violate ETH and are "generic"!

[Detailed Numerical Study: Pal & Huse, Bauer & Nayak + many others...
Putative Proof of MBL: J.Z. Imbrie, arXiv 2014 - so far unpublished...]

III. PHENOMENOLOGY OF THE MBL PHASE

We now discuss the known phenomenology of the MBL phase. Much of this has been inferred from numerics on modest system-sizes; (Nandkishore & Huse have the appropriate references; the pioneering numerics were done by Arijeet Pal and David Huse c. 2008-10.)

It is useful to set up our discussion by first discussing the non-interacting ("Anderson") localized phase. This will be brief - the other lecturers have already done a wonderful job of detailing the intricacies of Anderson localization. For simplicity, henceforth I will focus on $d=1$.

Anderson Localization

Let's take a simple tight-binding model,

$$H_0 = -J\sum_i c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + \sum_i h_i c_i^\dagger c_i \qquad \text{where } h_i \in \text{unif.}[-W,W].$$

As originally shown by Anderson (1958) for "sufficiently strong" disorder (as parametrized by the magnitude of W/J) in any dimension, all single-particle eigenstates of H_0 are localized, i.e. have the form

$$\psi_\alpha(\vec{r}) \simeq e^{-|\vec{r}-\vec{R}_\alpha|/\xi}$$

(any $\frac{W}{J} > 1$ is "sufficiently strong" for $d \leq 2$)

A simple change of basis allows us to recast H_0 in the localized single-particle eigenbasis:

$$H_0 = \sum_\alpha \epsilon_\alpha c_\alpha^\dagger c_\alpha$$

In $d=1$, we can map the fermion problem using a Jordan-Wigner transformation into a spin model

$$H_0 = J \sum_i (\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+) + \sum_i h_i \sigma_i^z .$$

"random-field XX chain"

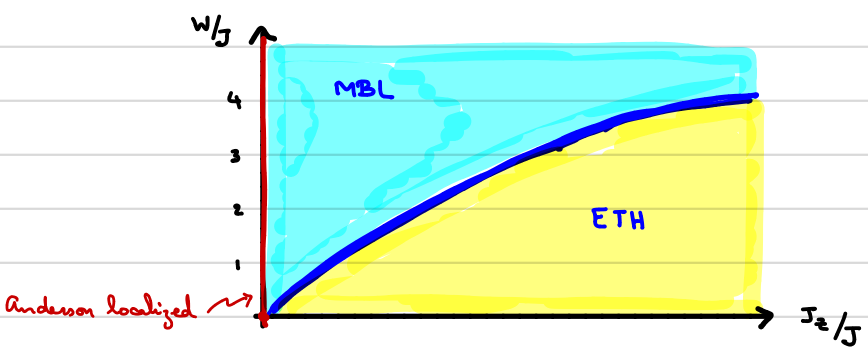
Adding Interactions

We can add interactions to the Anderson insulator: this takes the form of four-fermion terms:

$$H = H_0 + H_{int} \; ; \qquad H_{int} = \sum_i J_z \hat{n}_i \hat{n}_{i+1} \qquad (\hat{n}_i = c_i^\dagger c_i)$$

- In the eigenbasis, $\mathcal{H}_{int} = \sum_{\alpha\beta\gamma\delta} J_z^{eff} c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta$, with $J_z^{eff} \sim \mathcal{O}(J_z)$.
- In the spin language $H_{int} = J_z \sum_i \sigma_i^z \sigma_{i+1}^z$ (the XXZ chain w/ random fields).

With interactions, even in $d=1$, disorder needs to be "sufficiently strong" to induce localization $\Rightarrow$ there is a many-body localization phase transition driven by either disorder or by interactions. (more on this in Lecture 3, hopefully)



"Quench" Experiments

So far, we have focused mainly on the similarities of the Anderson and MBL insulators. However, there are key differences between the two, and numerical studies of "quench" dynamics are very useful in illuminating this. Two such experiments are

[1] Bardarson, Pollmann, Moore, Phys. Rev. Lett. 109, 017202 (2012)

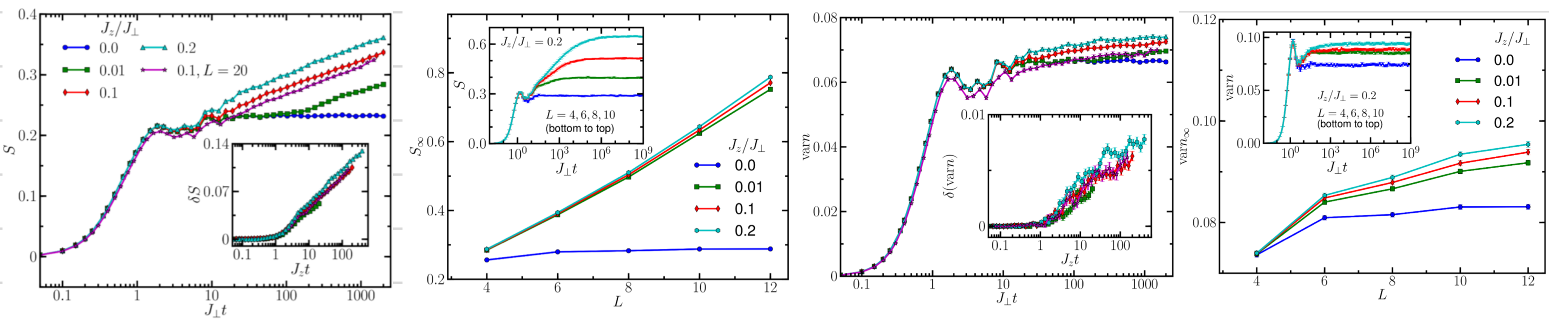
[2] Vasseur, Parameswaran, Moore, Phys. Rev. B. 91, 140202 (2015)

Both study dynamics of the XXZ chain, focusing on the behavior as J_z is increased from 0 at strong disorder; in each case, the initial state is the "Néel state"

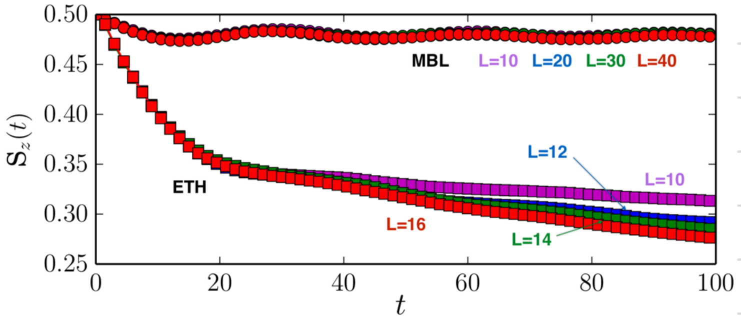
$$|\psi_0\rangle = |\uparrow\downarrow\uparrow\ldots\downarrow\uparrow\rangle \quad ; \quad |\psi(t)\rangle = e^{-iHt}|\psi_0\rangle .$$

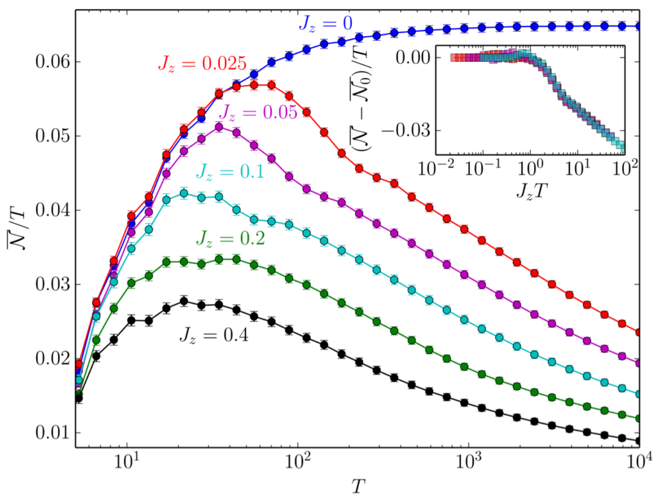
and time-evolution is performed using Time Evolving Block Decimation BD) or exact diagonalization.

[1] studied the behavior of entanglement entropy across a "cut in the middle of the system; they showed that in the AL phase ie entanglement grew a little after the quench, and it saturates. In contra in the MBL phase, S grows logarithmically, on a time-scale set by the interaction strength (J_z); while S saturates in finite s. ns, the saturation value increases with system size, showing that e entanglement grows without bound. They also showed that particle nu r fluctuations across the cut did not exhibit this growth. (This mc ed the phenomenological model we will discuss shortly; [2] served as onsistency check.)



[2] examined the post-quench dynamics of a single spin (for convenience, one with $J_z = h_i = 0$, but this is inessential), and measured the rate of "revivals" – how often the spin returned to its initial value of $+1/2$ per unit time. For the AL phase, this rate saturated to a constant value. For the MBL phase, the rate of revival decays on a time scale set by J_z. Meanwhile, the $t \to \infty$ expectation value of the "classical" component $\langle \sigma_z^{test} \rangle$ of the





test spin remains nonzero in both AL, MBL phases (w/ no finite-size scaling) whereas in the ETH phase it goes to zero as $t \to \infty$ (as determined by finite-size scaling).

Both results are consistent with a picture of the AL/MBL phase that is as follows

i) both are localized – exhibiting no transport, and retain memory of the initial conditions to late times $\Rightarrow$ no dissipation

ii) the interactions in the MBL phase induce dephasing, leading to the growth of entanglement & the decay of revivals – neither of which occurs in the AL phase.

We may now summarize the phenomenology of the ETH/AL/MBL phases.

PROPERTY	ERGODIC SYSTEMS	ANDERSON - LOCALIZED	MBL
MEMORY OF INITIAL CONDITIONS	"hidden" from local probes as $t \to \infty$	Some memory of initial conditions preserved at long times, locally	
ETH	True	False	
$\sigma_{d.c.}$	can be $\neq 0$	vanishes	
LOCAL SPECTRUM	continuous	discrete	
EIGENSTATE ENTANGLEMENT	Volume - law	area - law	
ENTANGLEMENT GROWTH	$S_E(t) \underset{t\to\infty}{\sim} t^{\alpha}$	$S_E(t) \underset{t\to\infty}{\sim}$ const.	$S_E(t) \underset{t\to\infty}{\sim} \ln(J_z t)$
DEPHASING	Yes	No	Yes
DISSIPATION	Yes.	No	

Phenomenological Model

We now construct a phenomenological model for the MBL phase, valid at strong disorder when all (upto set of measure 0) eigenstates are localized (note that the existence of a many-body mobility edge is still rather controversial). It is useful to recast the AL problem in the spin language. Recall that we could rewrite the AL Hamiltonian by changing basis to that of localized states

$$H = \sum_{\alpha} E_{\alpha} c^{\dagger}_{\alpha} c_{\alpha} \,, \qquad \text{where} \quad c^{\dagger}_{\alpha} = \sum_j \psi^{*}_{\alpha}(r_j)\, c^{\dagger}_j$$

Note that this clearly identifies an extensive set of conserved quantities, corresponding to the occupancies of the modes: $[H, c^{\dagger}_{\alpha} c_{\alpha}] = 0$ for $\alpha = 1, \dots, L$

Since the AL problem maps to the XX chain, it follows that we can perform a unitary transformation on the bare spins or "physical bits" σ_i^μ, to a set of localized bits or "l-bits" τ_i^μ. For the AL, $\mathcal{H}$ written in terms of the l-bits only includes the τ_i^z (cf. only having $c_\alpha^\dagger c_\alpha$ above).

So, we have $\quad \tau_i^\mu = \sum f_{ij}^{\mu\nu} \sigma_j^\nu \quad$ with $\quad f_{ij}^{\mu\nu} \sim e^{-|i-j|/\xi}$

and $\quad H \simeq H_{\text{eff}}^{AL} = \text{const} + \sum_i E_i \tau_i^z \quad$ (phenomenological conjecture)

Note that we label the τ_i's with the same site index - in any localized phase we can find a 1-1 correspondence between the "bare" sites & the centers of the localized states. We also see clearly that the l-bit Hamiltonian has exactly enough states to capture the full spectrum of 2^L many-body states - yet, it only has L <u>independent</u> energies - one reason why free systems are special. Note also that $[H, \tau_i^z] = 0 \Rightarrow$ we have L conserved "integrals of motion".

What happens if we add interactions?

First, τ_i^μ will depend on higher-body combinations of σ_i^μ's; yet it all the weights will <u>still</u> decay exponentially away from some central site and the $(n \geq 2)$-body terms will be suppressed by powers of J_z.

Second, as a natural corollary of the above, the effective Hamiltonian now also has higher-body terms:

$$H_{\text{eff}}^{MBL} = \text{const.} + \sum_i E_i \tau_i^z + \sum_{ij} \overset{\sim O(J_z)}{J_{ij}} \tau_i^z \tau_j^z + \sum_{ij\{k\}} K_{ij}^{\{k\}} \tau_i^z \tau_{k_1}^z \cdots \tau_{k_n}^z \tau_j^z$$

vanish as $J_z \to 0$

decay exponentially for $|i-j| \gg \xi$

Crucially, $\mathcal{H}_{\text{eff}}$ only contains $\tau_i^z \Rightarrow [\mathcal{H}, \tau_i^z] = 0$ <u>still</u>.

So, we retain the localized conserved integrals of motion, but $\mathcal{H}$ is much more "generic" - e.g., all 2^L energy levels are independent.

The l-bit Hamiltonian has no dissipation (τ_i^z is conserved) and retains memory of initial conditions (e.g., $\sigma_i^\mu \to \tau_i^z$s within some ξ-region, and then the τ_i^zs are conserved). However, the energy of a state - and hence its dynamical phase e^{iEt} - is affected by the random Hartree energy shifts generated for $J_z \neq 0$, leading to dephasing.

It is a useful exercise to verify that H_{eff}^{AL}, H_{eff}^{MBL} have the right phenomenology as we detailed earlier.

x ——— x