Introduction

novel (topological) quantum matter

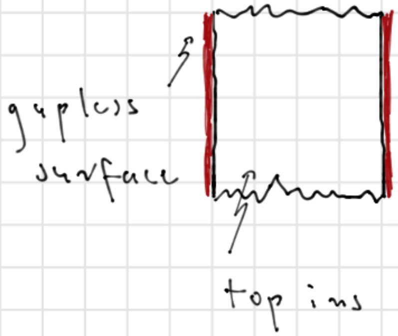
long range entangled

- defined by strong correlations
- often bosonic, fractional, or non-abelian excitations
- e.g. spin liquids, quantum codes, ...

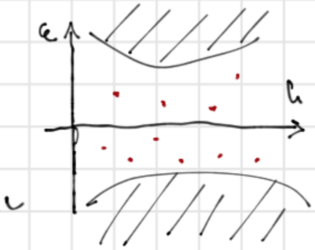
short range entangled

- ∃ quasiparticle pictures
- Often fermionic (metals, semiconductors, superconductors, (topological) insulators, semi-metals, ...)
- Often protected by symmetries
- interactions can be strong (but often manageable)
- disorder almost always important

Example for relevancy of disorder



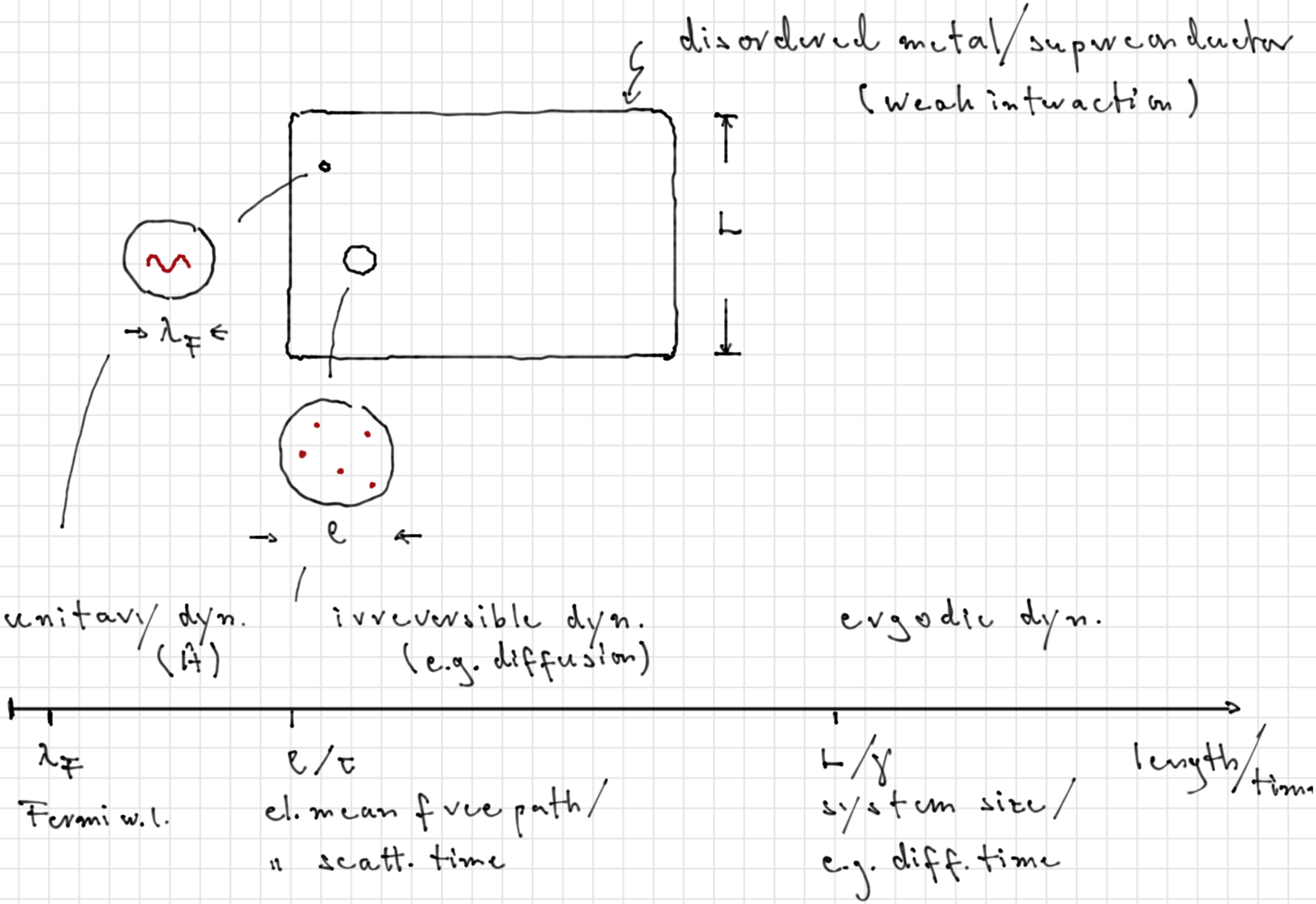
- protected by sym
- clean system:
  - bulk band gap
  - topological inv. from Brillouin zone

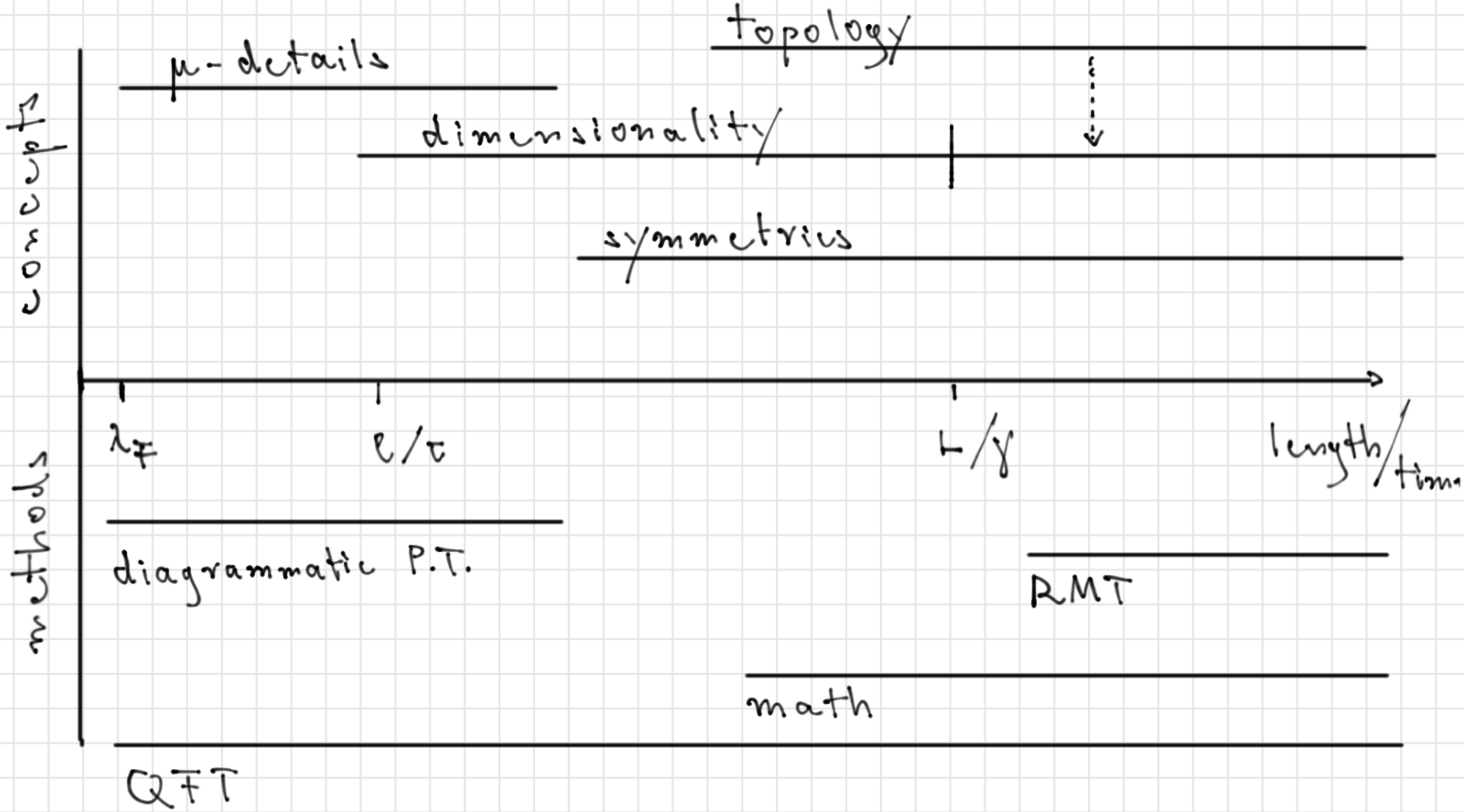


- disordered system:
  - symmetries?
  - Brillouin zone: destroyed
  - band gaps: "
  - high level of universality

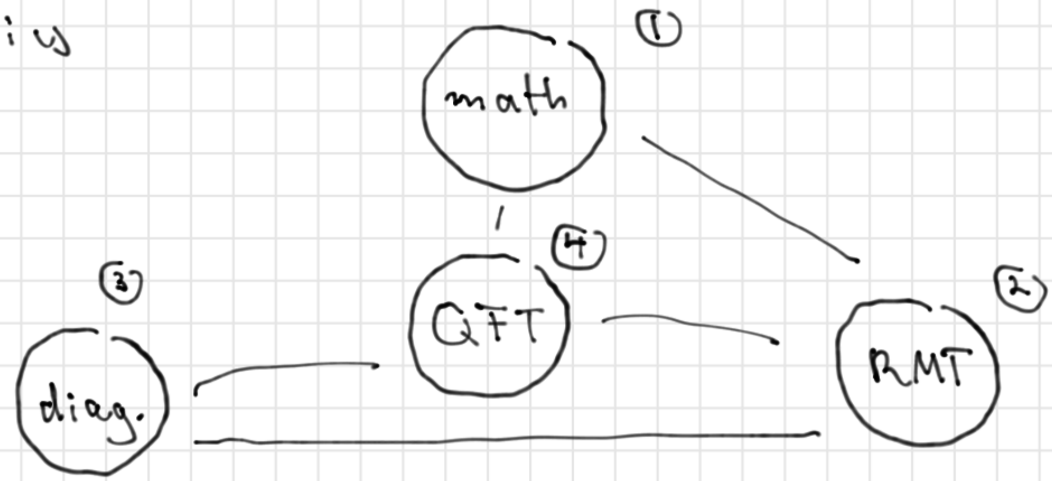
Effective field theories of disordered fermion systems

Intro





Building theories



Symmetries in QM

Setup: Hilbert space $\mathcal{H}$ Symmetry group: $G \ni g$

Hamiltonian $\hat{H}$ $g: \mathcal{H} \to \mathcal{H}$

$|\psi\rangle \mapsto g|\psi\rangle$

Unitary symmetries: G represented through unitary trafos $g = \mathcal{U}$

exmpl.: rotation, translation, space inversion, ...

generally fragile & compromised by other unitary operators

Anti-unitary symmetries: G represented through anti-unitary tr. $g = \Theta$

reminder: $\langle \Theta\psi, \Theta\psi' \rangle = \overline{\langle \psi, \psi' \rangle} = \langle \psi', \psi \rangle$

$$\Theta(z\psi) = \bar{z}\,\Theta\psi \qquad z \in \mathbb{C}$$

$\exists$ unitary $\mathcal{U}$: $\Theta = \mathcal{U}K$ K = complex conj.

physics: $\Theta^2 = \pm\, \mathrm{id}.$ e.g. $K^2 = \mathrm{id}$, $\left(\begin{pmatrix} & -i \\ i & \end{pmatrix} K\right)^2 = -\mathrm{id}$

exampl.: time reversal, particle-hole sym., charge conjugation sym.

more robust; not compromised by unit. op.

10/02/16

10 symmetry classes (shortcut intro.)

We call a Ham. Op. $\hat{H}$ **time reversal symmetric** if $\exists\, \Theta_T$ with $\Theta_T^{-1} \hat{H} \Theta_T = +\hat{H}$ (T)

3 options:

T	Θ_T^2	
✓	id	+1
✓	-id	-1
↯		0

We call a Ham. Op. $\hat{H}$ symmetric under **charge conjugation** if $\exists\, \Theta_C$ with $\Theta_C^{-1} \hat{H} \Theta_C = -\hat{H}$ (C). 3 options:

C	Θ_C^2	
✓	id	+1
✓	-id	-1
↯		0

We call a Ham. op. **particle-hole symmetric** if it is symmetric under $T \circ C$ (and therefore also under $C \circ T = T \circ \underbrace{T^{-1} C T}_{C'}$) $\rightsquigarrow$

$\underbrace{\Theta_T^{-1} \cdot \Theta_C^{-1}}_{U^{-1}} \hat{H}\, \Theta_C \cdot \Theta_T = -\hat{H}$ (S, sometimes: **chiral sym.**)

options:

T	C	S
±1	±1	1
0	±1	0
±1	0	0
0	0	0
0	0	1

(first three rows:) S fixed by C, T

Counting:

T, C $\rightarrow$ (T, C)

T, C: 3 options; (T, C): 9 options

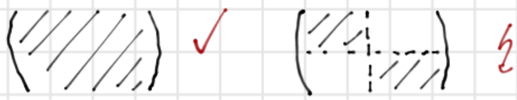
$9 = 8 + 1 \rightarrow$ 10 options

$1 \rightsquigarrow S = (0, 1)$

Classifying symmetries

- Consider long time limit of QM time evolution 

- Hamiltonian fully described by triplet (T, C, S). More formally: Unitary (sub) group of QM time evolution does not have irreducible representation space in $\mathcal{H}$ other than $\mathcal{H}$ itself

matrix $e^{-i\hat{H}t}$ $t \to \infty$ 

What can we say about symmetry classification of time evolution?

Exmpl.: • $(T, C, S) = (0, 0, 0)$. $\hat{H} = \hat{H}^\dagger$. $\hat{\mathcal{U}} = \exp(-i\hat{H}t) \in \mathcal{U}(N)$

• $(T, C, S) = (+1, 0, 0)$. $\hat{H} = \hat{H}^\dagger$

$$\hat{H} = \Theta_T^{-1} \hat{H} \Theta_T \qquad \Theta_T^2 = \mathrm{id.} \rightsquigarrow \Theta_T \underset{\substack{\uparrow \\ \text{unitary equiv.}}}{\sim} K \rightsquigarrow \hat{H} = \hat{H}^T$$

What subspace of $\mathcal{U}(N)$ does time ev. generate?

$$\hat{X} \equiv -i\hat{H}t \qquad \hat{X}^\dagger = -\hat{X} \in u(N) \quad (\text{Lie algebra})$$

$$\hat{X}^T = -\hat{X} \in o(N)$$

$$\hat{X}^T = +\hat{X} \in u(N)/o(N)$$

$$\rightsquigarrow \exp(-i\hat{H}t) = \exp(\hat{X}) \in \mathcal{U}(N)/O(N)$$

Exercise: Show that for $(T, C, S) = (-1, 0, 0)$ and $\Theta_T \sim \begin{pmatrix} & i \\ -i & \end{pmatrix} \otimes \mathbb{1}_{N/2} \cdot K$

$$\mathcal{U} \in U(N)/Sp(N)$$

Note: $U(N)$, $U(N)/O(N)$, $U(N)/Sp(N)$ are compact symmetric spaces

Symmetric space • heuristic: a compact Riemannian manifold of globally uniform geometry

looks the same everywhere

S^2 is a sym. space.

• formal: check out math textbooks

• Cartan ~ 100 yrs. ago: there exist 10 families of sym. spaces (just as many as symmetry classes.)

Symmetry class ↔ symmetric space correspondence culminates in a large table ...

no.	T	C	S	Cartan	space	physics
1	0	0	0	A	$U(N)$	no symmetries "unitary class"
2	1	0	0	AI	$U(N)/O(N)$	T-inv. integer spin "orthogonal class"
3	-1	0	0	AII	$U(N)/Sp(N)$	" " half " " "symplectic class"[1]
4	0	1	0	D	$SO(N)$	superconductor w.o. sym. ("Majorana wire")
5	0	-1	0	C	$Sp(N)$	spin singlet s.c. w.o. sym - plain vanilla s.c.
6	1	1	1	BDI	$O(N+M)/O(N)\times O(M)$	T-inv. spin triplet s.c. ("Kitaev chain")[q]
7	1	-1	1	CI	$Sp(N)/U(N)$	T-inv. spin singlet s.c. (rare, e.g. d-wave s.c.)
8	-1	1	1	DIII	$SO(N)/U(N)$	... even rarer
9	-1	-1	1	CII	$Sp(N+M)/Sp(N)\times Sp(M)$	"chiral symplectic"[q]
10	0	0	1	AIII	$U(N+M)/U(N)\times U(M)$	chiral, no further sym (SSH model)[q]

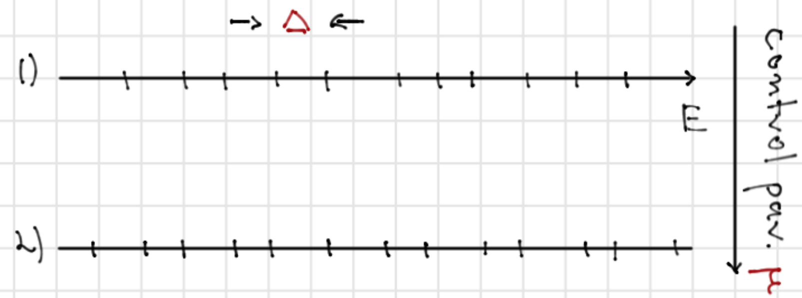
q) first appeared in lattice QCD

1) $\exists$ AII systems without spin (random mass 2d Dirac, ...)

Symmetries and ergodic regime

Working hypothesis: physical properties of fermionic quantum systems in ergodic regimes, $t > \gamma^{-1}$, solely determined by symmetries

⟶ $\sim$-like $\hat{H}$ ⟶



Phys. inf. stored in wave functions and spectrum

statistical approach! (Wigner 1950s)

- Characterize spectra in terms of correlation functions

$$\langle \rho(E) \rangle_\mu = -\frac{1}{\pi} \langle \mathrm{Im}\, \mathrm{tr}\, (G^+(E)) \rangle_\mu \qquad \Delta \equiv 1/\langle \rho(0) \rangle \qquad G^\pm(E) = (E \pm i0 - \hat{H})^{-1}$$

$$R_2(E) = \langle \rho(E)\, \rho(E+\omega) \rangle_{c,\mu}$$

Empirical observation: correlation functions depend on

i) parameter $s \equiv \pi\epsilon/\Delta$

ii) symmetry class

iii) nothing else

Analytic approach: Random Matrix Theory

Idea: Define ensemble of model Hamiltonians $\hat{H}$, subject to conditions:

i) fix $N = \dim(\mathcal{H})$ iii) fix symmetry class

ii) fix Δ (average) iv) determine $P(\hat{H})\,d\hat{H}$ that 'minimizes information' mod i)-iii)

consequence of i)-iii): matrix elements $H_{\mu\nu}$ Gaussian i.i.d. variables constrained only by sym., e.g AI: $H_{\mu\nu} = H_{\nu\mu}$ etc.

Example: Wigner-Dyson classes

A	AI	AII	
2	1	4	β

$$P(\hat{H})\,d\hat{H} = e^{-\frac{\beta N}{\lambda^2}\operatorname{tr}(\hat{H}^2)} \prod_{\mu\nu} dH_{\mu\nu}$$

↳ unconstrained matrix el.

⇝ Compute expectation values of operators depending on $\hat{H}$ by averaging over RMT distribution.

$$\langle X(\hat{H})\rangle = \int d\hat{H}\, P(\hat{H})\, X(\hat{H})$$

$$X(\hat{H}) = \rho(E),\quad \rho(E)\,\rho(E+\omega),\ \dots$$

If X depends only on EV of $\hat{H}$: $\hat{H} = \hat{g}\hat{D}\hat{g}^{-1}$ $\operatorname{tr}(\hat{H}^2) = \operatorname{tr}(\hat{D}^2)$

$$\langle X(\hat{H})\rangle = \int d\hat{D}\, \mu(\hat{D})\, P(\hat{D})\, X(\hat{D})$$

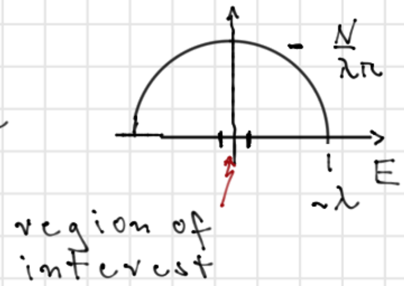
μ: Jacobian

} can compute a lot!

RMT Results

average density of states (DOS)

- Wigner Dyson classes (A, AI, AII) : $\rho(E) = \frac{N}{\pi\lambda}\left(1-\left(\frac{E}{2\lambda}\right)^2\right)^{1/2}$



define $s = \pi E/\Delta = \pi E \rho(0) = \frac{EN}{\lambda} \sim O(1)$

- Non WD classes: exmpl.: AIII (0, 0, 1)

$[\hat{H}, \hat{D}]_+ = 0 \qquad \hat{D}: \mathrm{diag}(\underbrace{1, \dots, 1}_{r}, \underbrace{-1, \dots, -1}_{s}) \qquad r+s=N.$

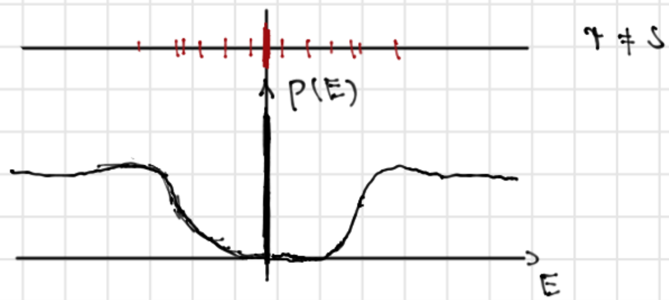
Linear algebra:

- spectrum symmetric around $E=0$ ('particle-hole')

  $\hat{H}|\chi\rangle = \varepsilon|\chi\rangle \Rightarrow \hat{H}\hat{D}|\chi\rangle = -\varepsilon D|\chi\rangle$

- For $r \neq s$ $\exists$ $r-s$ $\varepsilon=0$ states (a shade of topology..)

Expect:



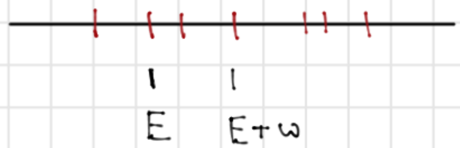
RMT : $\rho(s) = \frac{\pi s}{2\Delta}\left(J_0^2(s) + J_1^2(s)\right)$

Bessel fct.

$\rho(s) = \frac{\pi N}{\Delta}\delta(s) + \frac{\pi s}{2\Delta}\left(J_n^2(s) - J_{n-1}(s)\, J_{n+1}(s)\right)$

RMT results cont'd

$$R_2(\varepsilon) = \Delta^2 \langle \rho(E)\, \rho(E+\omega) \rangle_c = \Delta^2 \langle \rho(E)\, \rho(E+\omega) \rangle - \overset{\text{disconnected}}{1} \quad \sim \text{level repulsion}$$



A: $R_2(s) = - \dfrac{\sin^2(s)}{s^2} \xrightarrow{s \to 0} -1$

AI: a more complicated expression

AII, ... still more complicated

but qual. similar

RMT vs. diagrammatic perturbation theory

Goal: Understand RMT results from perturbative perspective. Why?!

- connect from spectra to physics
- connect to other approaches

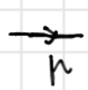
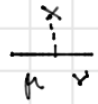
$\sim$ aim to compute $\quad \mathrm{tr}\, G^{\pm}(E) \leftrightarrow \rho(E) \quad$ for classes A, AI by

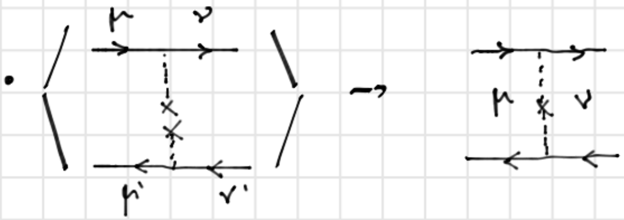
$\quad \mathrm{tr}\,(G^{+}(E+\omega)\, G^{-}(E)) \leftrightarrow R_2(\omega)$

formal expansion in $H_{\mu\nu}$ around E.

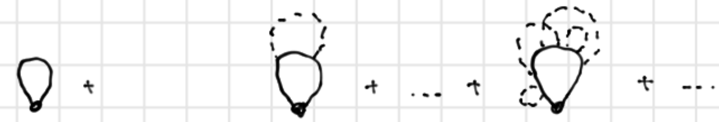
- Def.: $G_0^{\pm} = \{ (E^{\pm})^{-1} \delta_{\mu\nu} \}$
- $\langle H_{\mu\nu} H_{\nu'\mu'} \rangle = \dfrac{\lambda^2}{N} \begin{cases} \delta_{\mu\mu'}\delta_{\nu\nu'} & \text{A} \\ \delta_{\mu\mu'}\delta_{\nu\nu'} + \delta_{\mu\nu'}\delta_{\nu\mu'} & \text{AI} \end{cases}$

RMT diagrammatic code

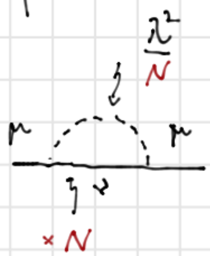
• $G_0^\pm$  • $H_{\mu\nu}$ 

• 

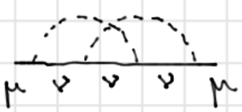
• Average Green function: $\mathrm{tr}\left(G^+(E)\right) =$ 

$\left\langle \mathrm{tr}\left(G^+(E)\right)\right\rangle =$ 

Expansion parameter: $N^{-1} \ll 1$

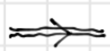


$\sim O(N^0)$ ✓  $\sim O(N^0)$ ✓

 $\sim O(N^{-1})$ ↯

discard diagrams with crossed lines!

Self consistent Born approx. for GF (SCBA)

$\langle G^+(E)\rangle =$ 

 $=$ $\frac{1}{E^+ - \Sigma^+}\,\delta_{\mu\nu}$

$$\Sigma^+ = \frac{\lambda^2}{N}\, G^+ = \frac{\lambda^2}{N}\cdot\frac{1}{E^+ - \Sigma^+}\cdot N \;\Rightarrow\; \Sigma^+ = \frac{E}{2} \pm i\left(\lambda^2 - \frac{E^2}{4}\right)^{1/2}$$

RMT diagrammatics cont'd

$$\sim \quad \rho(E) = -\frac{1}{\pi}\,\mathrm{Im}\,G^+(E) = \frac{N}{\pi\lambda}\left(1-\frac{E^2}{4}\right)^{1/2}$$

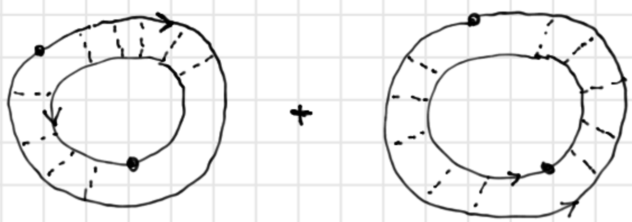
Notice: • infinitesimal $i\delta$ determines sign of $\mathrm{Im}\,\Sigma$ ~ 'spontaneous sym. breaking'

• average GF 'decays' on 'shortest' scales $\langle G_{\mu\nu}\rangle \sim \delta_{\mu\nu}$

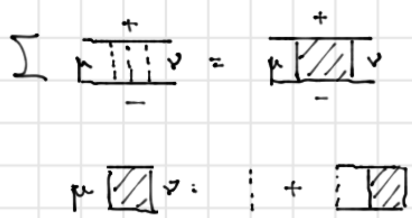
Two point function

$\left\langle \mathrm{tr}\left(G^+(E+\omega/2)\right)\,\mathrm{tr}\left(G^-(E-\omega/2)\right)\right\rangle$ = write $\Longrightarrow = \longrightarrow$ for simplicity

$\simeq$



A, AI AI

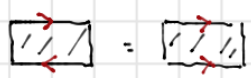


$= i\frac{\lambda^2}{\pi\omega} = P_{\mu\nu}$

homework (or ask me)

$P_{\mu\nu}$: 'particle/hole mode', propagator, ...

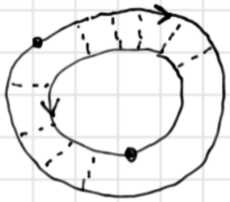
• independent of μ,ν : ergodicity

• 

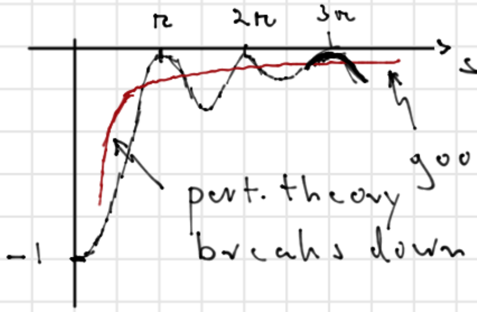
• $P_{\mu\nu}(\omega)$ = 'classical prob. to propagate $\mu \to \nu$ in time $\sim \omega^{-1}$'. RMT: $P_{\mu\nu}(t) = \Theta(t)$

• anticipate generalisation $P_{\mu\nu} \to P(\underline{x},\underline{x}')$ = 'diffusion prop.'

Two point function cont'd



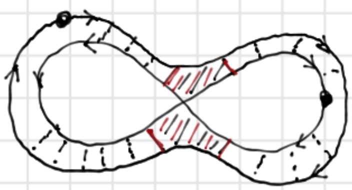
$\rightsquigarrow \quad R_2(s) = -\frac{1}{2s^2} \quad$ cf. exact res.: $-\frac{\sin^2(s)}{s^2}$



- Perturbative approach becomes problematic at time scales comparable to the Heisenberg time $t_h = \Delta^{-1}$.

- Identify diagrams of higher order in $s^{-1} \lesssim 1$ expansion

For example (AI):

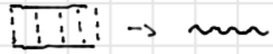


- interpretation: traversal of mutually time reversed paths
- contains 4 modes $P \sim s^{-4}$, however

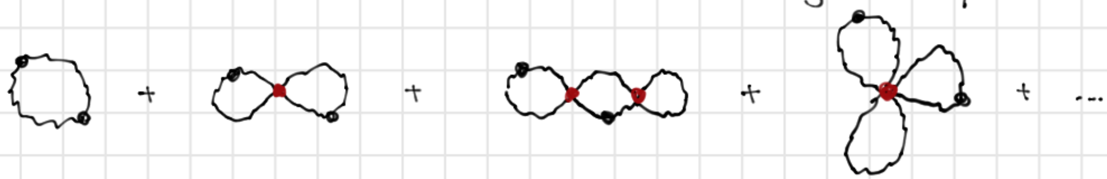
- correction $\mathcal{O}(s^{-3})$ to AI $R_2(s)$

-  $: \sim s$ (a complicated object, 'Hikami box', 'encounter region')
- 'topology' of diagram model independent ($\rightsquigarrow$ RMT, impurity diagrams, quantum chaos, ...)

Higher order diagrams cont'd

Compactified notation: 

Class A: $R_2(\omega) \to$  (sin(s) has essential singularity at $s \to \infty$)

AI: $R_2(\omega) \to$ 

s^{-2} s^{-3} s^{-4} s^{-4}

- Can obtain asymptotic series expansions of RMT corr. fcts.
- Diagrams determined by topology/combinatorics

  Note: Diagrammatic representations of pert. theory can be developed for disordered metals, chaotic cavities, etc. details differ but top./comb. identical. For $t < \gamma^{-1}$ quantitative equiv. to RMT diags.
- Could be that diag's encode some 'geometric' object?

$\rightsquigarrow$ $\boxed{\text{field theory}}$

Prelude: The 'Howe pair' picture

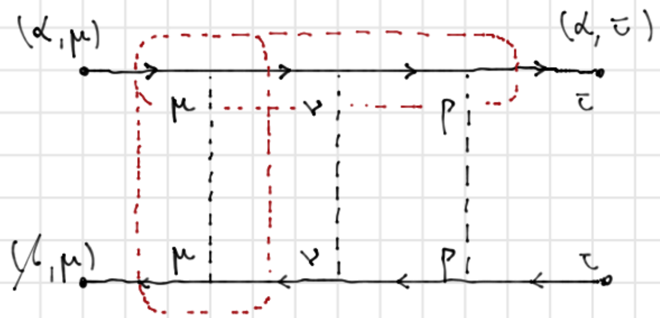
- Building blocks of correlation functions

$$\alpha = (a, \pm) \qquad a = 1, \dots, n \qquad \langle \underbrace{a^+ a^- \dots a^+ a^-}_{n \text{ pairs}} \rangle$$

$$\xrightarrow[\mu]{}$$

Define: 'flavor' space $\mathbb{C}^{2n} = \mathcal{H}_f$ ⇝ $\bullet\xrightarrow[|\psi\rangle]{}\bullet$: a q.p. in $\mathcal{H}_f \otimes \mathcal{H}_c$

'color' " $\mathbb{C}^N = \mathcal{H}_c$



time ev. op.

Time evolution: $\mu \dots \nu$ / $\alpha \quad \alpha$ 'flavor singlet' ⇝ $1\!\!1 \otimes \hat{U}$ ~ color sym. space

Effective theory: $\mu \quad \mu$ / $\alpha \quad \beta$ 'color singlet' ⇝ $\tilde{U} \otimes 1\!\!1$ → ? ? ?

?

Field theory construction: $1\!\!1 \otimes \hat{U} \xrightarrow{\text{aver.}} \tilde{U} \otimes 1\!\!1$

Construction of field theory (class A)

Define

$$Z[X] = \int d(\bar\psi, \psi)\, e^{-S[\bar\psi,\psi]}$$

$$S[\bar\psi,\psi] = \bar\psi^{\alpha}_{\mu}\left(\varepsilon_s \cdot \delta_{\mu\nu} - H_{\mu\nu} - X^{\alpha}_{\mu\nu}\right)\psi^{\alpha}_{\nu}$$

- $\alpha = (s, a) \qquad a = 1, \dots, R \qquad s = \pm 1$
- $\psi^{\alpha}_{\mu}, \bar\psi^{\alpha}_{\mu}$: Grassmann variables
- $\varepsilon_s \equiv \varepsilon + s\left(\frac{\omega}{2} + i\delta\right)$

Important properties:

- $Z[0] = \det\begin{pmatrix} \varepsilon_+ - \hat H & \\ & \varepsilon_- - \hat H\end{pmatrix}^R \xrightarrow[R\to 0]{} 1$ "replica trick"
- $\lim\limits_{R\to 0} \left.\dfrac{\partial Z[X]}{\partial X^{a,s}_{\nu\mu}}\right|_{X=0} = \left(\varepsilon_s - \hat H\right)^{-1}_{\mu\nu}$

throughout: $X = 0$ for simplicity

- $S_{\varepsilon_s=\varepsilon}[T\psi, T^{-1}\bar\psi] = S_{\varepsilon_s=\varepsilon}[\psi,\bar\psi] \qquad T = \{T_{\alpha\beta}\} \in U(2R)$

$S[T\psi, T^{-1}\bar\psi] = S[\psi,\bar\psi] \qquad T = \begin{pmatrix} T_{++\,ab} & \\ & T_{--\,ab}\end{pmatrix} \in U(R)\times U(R)$

Gaussian average and Hubbard-Stratonovich decoupling

$$S[\bar\chi,\chi] = \bar\chi^{\alpha}_{\mu}\left(\varepsilon_s \cdot \delta_{\mu\nu} - H_{\mu\nu}\right)\chi^{\alpha}_{\nu} \qquad S_0 = S|_{\hat H = 0}$$

$$\langle Z[0]\rangle = \int dH\, e^{-\frac{N}{2\lambda^2}\mathrm{tr}\, H^2} \int d(\chi,\bar\chi)\, e^{-S[\chi,\bar\chi]} =$$

$$= \int d(\chi,\bar\chi)\, e^{-S_0[\chi,\bar\chi] + \frac{\lambda^2}{4N}\bar\chi^{\alpha}_{\mu}\chi^{\alpha}_{\nu}\bar\chi^{\beta}_{\nu}\chi^{\beta}_{\mu}} =$$

$$= \int d(\chi,\bar\chi)\, e^{-S_0[\chi,\bar\chi] - \frac{\lambda^2}{4N}\chi^{\beta}_{\mu}\bar\chi^{\alpha}_{\mu}\chi^{\alpha}_{\nu}\bar\chi^{\beta}_{\nu}} =$$

$$= \int dA\, e^{\frac{N}{2\lambda^2}\mathrm{tr}\, A^2}\, e^{-\tilde S[\chi,\bar\chi]}$$

$$\tilde S[\chi,\bar\chi] = \bar\chi^{\alpha}_{\mu}\left(\varepsilon_s\, \delta^{\alpha\beta} - A^{\alpha\beta}\right)\chi^{\beta}_{\mu}$$

Symmetries of the action

- $\chi \to U\chi \qquad U \in U(2RN)$

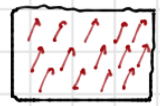
Invariance	U_c	U_f
S	$-$	$+$
$\tilde S$	$+$	$-$

$U_c = \mathbb{1} \times U(N)$

$U_f = U(2R) \times \mathbb{1}$

Symmetry breaking

- Symmetry remains ineffective unless broken. Cf. magnet

	SO(3) sym.	broken
Goldstone	–	SO(3)/SO(2)

- Integrate over Grassmann vars.:

$$\langle Z[0]\rangle = \int dA \; e^{+\frac{N}{2\lambda^2}\operatorname{tr}(A^2) + N \operatorname{tr}\ln(\hat{E} - A)} \qquad \hat{E} = \begin{pmatrix} E_+ & \\ & E_- \end{pmatrix} \otimes \mathbb{1}_R$$

- $N \gg 1$ justifies stationary phase approach

$$\frac{\delta S[A]}{\delta A} = 0 \Rightarrow \quad A = \frac{\lambda^2}{\hat{E} - A} \qquad (\text{cf. SCBA mean field eq.})$$

Set of solutions ($\omega \ll E$) $\quad A_{aa'} = \Sigma^a \delta_{aa'} = \frac{E}{2} - i s_a \left(\lambda^2 - \left(\frac{E}{2}\right)^2\right)^{1/2}$ or

$$A = \left(\frac{E}{2} - i\,\tau_3 \frac{\pi \rho \lambda^2}{N}\right) \otimes \mathbb{1}_R$$

mean field breaks U(2R) sym. down to $U(R) \times U(R)$

$\rightsquigarrow$ Emergence of Goldstone mode manifold

- $A \underset{E \ll \lambda}{\longrightarrow} \frac{E}{2} - i\lambda \underbrace{T \tau_3 T^{-1}}_{Q} \qquad T \in U(2R)$

- $Q \in U(2R) / U(R) \times U(R)$ $\quad$ a symmetric space of type AIII

Symmetry duality

	time evolution	averaged theory
space	$\mathrm{AI}\vert_N$	$\mathrm{AIII}\vert_R$
invariance	$\mathcal{U}(N)$	$\mathcal{U}(2R)$

Expansion of action in explicit symmetry breaking cf. 

'magnon excit.'

$$S[A] = -\frac{N}{2\lambda^2}\,\mathrm{tr}\,(A^2) - N\,\mathrm{tr}\ln\left(\hat{\varepsilon} - A\right)$$

$$\downarrow$$

$$A = \frac{E}{2} - i\lambda Q \qquad Q = T^{-1}\tau_3 T \qquad \hat{\varepsilon} = \frac{E}{2} + \underbrace{\omega\,\tau_3}_{\hat{=}\,H}$$

$$S[Q] = \text{const.} - N\,\mathrm{tr}\ln\left(\frac{E}{2} + \omega\tau_3 - i\lambda\, T\tau_3 T^{-1}\right) =$$

$$= \quad '' \quad - N\,\mathrm{tr}\ln\left(\frac{E}{2} + \omega\, T^{-1}\tau_3 T - i\lambda\tau_3\right) \simeq$$

$$\quad '' \quad - \underbrace{i\pi\rho\,\omega}_{s}\,\mathrm{tr}\,(Q\tau_3) \equiv S_{\mathrm{eff}}[Q]$$

0-dimensional nonlinear σ-model

- In presence of a source: $S[Q] = S_{\mathrm{eff}}[Q] + i\pi\rho\,\mathrm{tr}\,(Q\chi)$

$$\sim \int dQ\; e^{-S[Q]} \quad \sim \text{RMT results}$$

- $s \lesssim 1$ non-perturbative regime, strong fluctuations of Q. Non-pert. integration techniques (best suited: supersymmetry, cf. Efetov textbook)
- $s \gtrsim 1$ perturbative regime $\rightsquigarrow$

Perturbative integration techniques

- ssl $\sim$ small fluctuations of $Q = T\tau_3 T^{-1}$ around $T = \mathbb{1}$. Parameterize $T = \exp W$. $[W, \tau_3]_+ = 0$

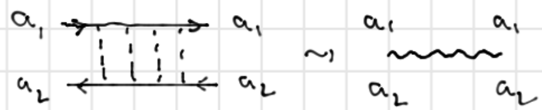
$$W = \begin{pmatrix} & B \\ -B^\dagger & \end{pmatrix} \qquad B \in \mathrm{Mat}(R, R, \mathbb{C})$$

  - anti-commutes with $\tau_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ ✓
  - and leads to convergent integral ✓

- Expansion of action

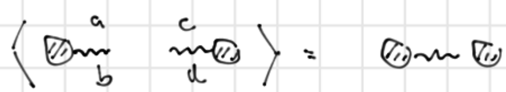
  - $S_{eff}[B] = 2is\, \mathrm{tr}(BB^\dagger) + \sum_{n=2}^{\infty} \underbrace{S^{(2n)}[B, B^\dagger]}_{\mathrm{tr}(BB^\dagger)^{2n}}$

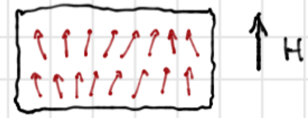
  - $\mathrm{tr}(BB^\dagger \dots B^\dagger) = \sum B^{a_1 a_2} B^{+\,a_2 a_3} \dots B^{a_{2n-1} a_{2n}}$

  - $S^{(2)}[B, B^\dagger] = 2is\, \mathrm{tr}(BB^\dagger) \sim$  $\sim \frac{1}{2is}$

  - $\langle \dots \rangle = \int d(B, B^\dagger)\, e^{-S^2[B, B^\dagger]} (\dots)$

$$\langle B^{ab}\, B^{\dagger\, cd} \rangle = \delta_{ad}\delta_{bc}\frac{1}{2is}$$

 $\delta_{ad}\delta_{bc}$



$S = \vec{n}\cdot\vec{\tau} = T\tau_3 T^{-1}$

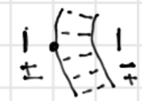
$T \in SU(2)$, $T = e^W$

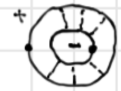
$W \in SU(2)/U(1)$ e.g.

$W = \begin{pmatrix} & z \\ -\bar{z} & \end{pmatrix}$.

z : complex coord. of 2-sphere.

- magnon field

· exercise: 1) expand $\mathrm{tr}(X Q)$ to 2^{nd} order in B 2) differentiate $Z[X]$ w.r.t. X^{++}_{11}, X^{--}_{11} 

3) do Gaussian integral over B_{ab} 4) send $R \to 0$ 5) discover  $\sim \frac{1}{s^2}$

Generalizations I: Symmetries

· How do we teach the σ-model symmetries? Example: class AI ($\hat{H} = \hat{H}^T$).

$$S[\bar\psi, \psi] = \bar\psi \hat{H} \psi = \frac{1}{2} \bar\psi \hat{H} \psi - \frac{1}{2} \psi^T \hat{H}^T \bar\psi^T =$$

'time reversal space' (tr)

$$= \bar\Psi \hat{H} \Psi = S[\Psi] \qquad \Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi \\ \bar\psi^T \end{pmatrix} \qquad \bar\Psi = \frac{1}{\sqrt{2}} \left(\bar\psi, -\psi^T \right)$$

$$\bar\Psi = \Psi^T \left(i \sigma_y^{tr} \right) \quad (*)$$

Symmetry transformation $\Psi \to T \Psi$ $T \in U(2 \cdot 2 \cdot R)$ must respect (*)

$$\begin{array}{l} \Psi \to T \Psi \\ \bar\Psi \to \bar\Psi T^{-1} \end{array} \leadsto \quad \bar\Psi T^{-1} = \Psi^T (i\sigma_y^{tr}) T^{-1} \overset{!}{=} \Psi^T T^T (i \sigma_y^{tr}) \Rightarrow T^{-1} = (i\sigma_y^{tr})^{-1} T^T (i \sigma_y^{tr})$$

- $T \in Sp(4R)$
- $Q = T \tau_3 T^{-1} \in Sp(4R) / Sp(2R) \times Sp(2R) = \mathrm{CII}|_{4R}$

· Conclusion:

	Time evolution	σ-model
	$\mathrm{A}\|_N$	$\mathrm{AIII}\|_{2R}$
	$\mathrm{AI}\|_N$	$\mathrm{CII}\|_{2R}$

Observation: Target spaces of σ-models are symmetric spaces.

Symmetries cont'd

• Connecting to physics:

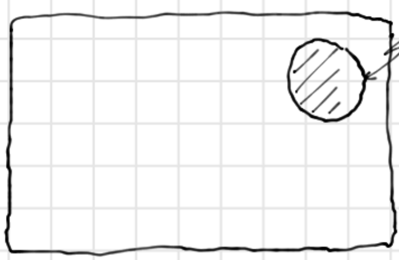
$$B = \underbrace{\begin{pmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{pmatrix}}_{\mathrm{tr}} \sim$$

$$\underbrace{T^T = (i\sigma_y)}_{\mathrm{tr}}\, T^{-1}(i\sigma_y)^{-1} \quad \Rightarrow \quad W^T = -(i\sigma_y)\, W\, (i\sigma_y)^{-1} \quad \Rightarrow \quad B^* = i\sigma_y\, B\, (i\sigma_y)^{-1}$$

$$T = e^W \qquad\qquad W = \begin{pmatrix} & B \\ -B^\dagger & \end{pmatrix}$$

Generalization II: Extended systems

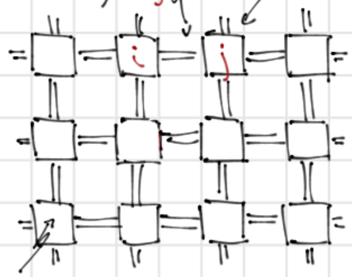
• Want to describe large systems



locally ergodic

$\xrightarrow{\text{granular model}}$

tunneling t_{ij} RMT-Hamiltonian $\hat{H}_j$



Q_i

• Effective field theory

$$S[Q] = \underbrace{c}_{|t_{ij}|^2 \rho} \sum_{\langle i,j \rangle} \mathrm{tr}(Q_i Q_j) - i\pi\rho\epsilon \sum_i \mathrm{tr}(Q_i \tau_3)$$

$|t_{ij}|^2 \rho$ Golden rule tunneling rate

Exercise: Try to derive the structure of this action from a 2^{nd} order expansion of $\mathrm{tr}\ln(\hat{\epsilon} + \hat{t} + i\pi\lambda\hat{Q})$ to 2^{nd} order in t. What justifies the expansion?

Continuum limit

$$S[Q] = C \sum_{\langle i,j\rangle} \mathrm{tr}(Q_i Q_j) - i\pi\rho\epsilon \sum_i \mathrm{tr}(Q_i \tau_3)$$

effective lattice spacing

- $\mathrm{tr}(Q_i Q_j) = \frac{1}{2}\mathrm{tr}(Q_i - Q_j)^2 + \text{const.} \to \frac{a^2}{2}\mathrm{tr}(\partial_a Q)^2$ $\quad \hat{e}_a$ i, j
- $\sum_i \to a^d \int d^d r$

$$\to \frac{\pi\nu}{4}\int d^d r\, \mathrm{tr}\left(D(\partial Q)^2 + 2i\omega Q\tau_3\right)$$

action of d-dim replica nonlinear σ-model

$\nu = \rho a^{-d}$ density of states per volume

$D\nu = C a^{2-d}$ defines diffusion constant

- Quadratic expansion in B

$$S^{(2)}[B, B^+] = -\frac{\pi\nu}{2}\int_0^L d^d r\, \mathrm{tr}\left(B^\dagger (D\partial^2 + i\omega) B\right)$$

diffusion propagator $\to \frac{1}{Dq^2 - i\omega}$

- Notice: Diffusion modes q quantized in (per. bound. cond.) $2\pi/L$. Diff. pr. has eigenvalues

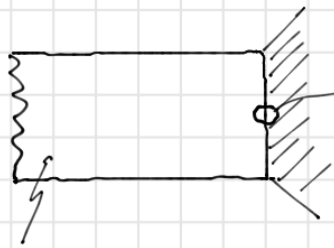
$\frac{1}{-i\omega}$ $\quad q = 0$ $\quad$ 'zero mode' $\to$ ergodic limit

$\frac{1}{-i\omega + D\left(\frac{2\pi}{L}\right)^2}$ $\quad |q| = \frac{2\pi}{L}$ $\quad$ higher modes become important if $\omega \gtrsim E_c \sim \frac{D}{L^2}$, the Thouless energy $\sim$ inverse diffusion time

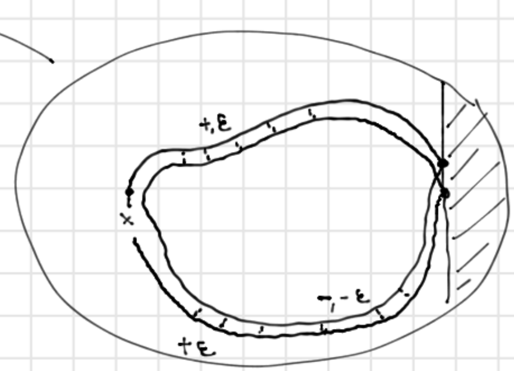
$\vdots$

Nonstandard Symmetry Classes: particle-hole symmetry

Case study:



spin-triplett SC-order par. $c^\dagger_\alpha \Delta_{\alpha\beta} c^\dagger_\beta$,



normal (semi) conductor,

Hamiltonian h

- Bogolubov Hamiltonian:

$$H = \begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^T \end{pmatrix} \qquad \Delta = -\Delta^T \qquad H = -\sigma_1 H^T \sigma_1 \qquad H' = \sigma_1^{-1/2} H \sigma_1^{1/2}$$

'Majorana rep.'

$$H'^T = -H' \qquad U \in O(2N) \quad \text{class D}$$

- Consider Green function: $G^+(\varepsilon) = (\varepsilon^+ - H)$

$$G^+(+\varepsilon; x, x') = [G^+(\varepsilon, x', x)]^T = \langle x' | \varepsilon^+ - H^T | x \rangle$$

$$= -\langle x' | -(\varepsilon^+) - H | x \rangle = G^-(-\varepsilon; x', x)$$

$\rightsquigarrow$ distinction between retarded and advanced GF compromised!

• Consider DOS $\rho(\epsilon) = -\frac{1}{\pi}\,\mathrm{Im}\,\mathrm{tr}\left(G^+(\epsilon)\,\sigma_3\right)$

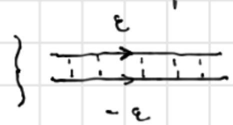
$$= -\frac{1}{\pi}\,\mathrm{Im}\,\mathrm{tr}\left(\begin{pmatrix} \epsilon^+ - h & \Delta \\ -\Delta^\dagger & \epsilon^+ + h^T \end{pmatrix}^{-1} \sigma_3\right)$$

$$= -\frac{1}{\pi}\,\mathrm{Im}\,\mathrm{tr}\begin{pmatrix} (\epsilon + i0 - h) & \Delta \\ -\Delta^\dagger & (-\epsilon - i0 - h^T) \end{pmatrix}^{-1}$$

particle, ϵ $\quad \xrightarrow[\epsilon,\,+]{}$

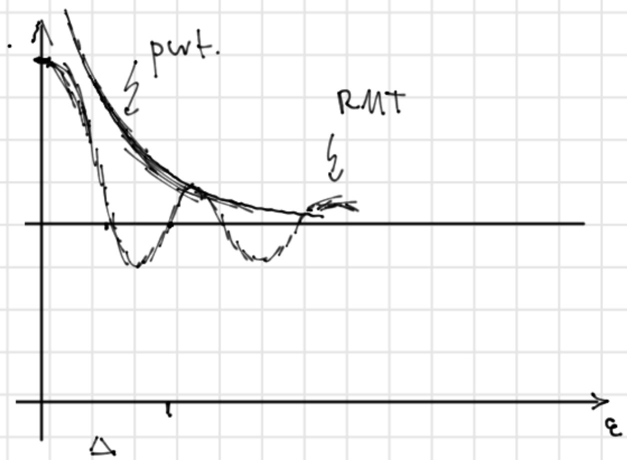
hole, $-\epsilon$ $\quad \xrightarrow[-\epsilon,\,-]{}$

Cooperon mode



$\sim \frac{1}{2i\epsilon}$

$\leadsto$ Expect DOS correction $\rho(\epsilon) \sim \frac{1}{\epsilon}$ $\quad$ (RMT: $\rho(s) = \rho_0\left(1 + \frac{\sin s}{s}\right)$)



• Note: $\int ds\,(\rho(s) - \rho_0) = \mathcal{O}(1)$ similar to a quasi-particle

Non-standard symmetry classes: Chiral Symmetries

Digression: Low energy QCD

Quarks at low energy:  $\bar q_R$ ~ carries representation under color (ℓℓℓ) and flavor group q_L

confinement due to strong (color) gauge field fluctuations

Confinement leads to appearance of condensate $\langle \bar q_R q_L \rangle = i\gamma \neq 0$ at QCD transition. Formal description:

$$(\bar\psi_L, \psi_R)\begin{pmatrix} & X(A) \\ X(A)^\dagger & \end{pmatrix}\begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix} \xrightarrow{\langle\;\rangle_A} (\bar\psi_L\,\psi_R)\begin{pmatrix} i\gamma & X_0 \\ X_0^\dagger & i\gamma\end{pmatrix}\begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix}$$

(*) (**)

color gauge field ($\hat{=}$ random potential)

| effective Dirac op

(*): symmetry under $\psi_c \overset{(A)}{\to} T_c\,\psi_c$ $c = L, R$ ~ symmetry group $U_L \times U_R$

$T_c \in$ flavor group U_c

(**): symmetry broken to $U_L = U_R \equiv U$. Goldstone modes: $U\times U/U \simeq U$

$$(**) \xrightarrow{(A)} (\bar\psi_L\,\psi_R)\begin{pmatrix} i\gamma\, T_L^{-1} T_R & X_0 \\ X_0^\dagger & i\gamma\, T_R^{-1} T_L\end{pmatrix}\begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix}$$

$T_L^{-1} T_R \equiv T$ meson field (color singlet but flavor)

- Expansion in Goldstone mode fluctuations

$$S[\bar\chi,\chi,T] = (\bar\chi_L\ \bar\chi_R)\begin{pmatrix} i\gamma\, T + \hat m^{*)} & X_0 \\ X_0^+ & i\gamma\, T^{-1} + \hat m\end{pmatrix}\begin{pmatrix}\chi_R\\ \chi_L\end{pmatrix}$$

*) quark masses (weak)

$$\xrightarrow{\int D(\bar\chi,\chi)} -\mathrm{tr}\ln\begin{pmatrix} i\gamma\, T + \hat m & X_0 \\ X_0^+ & i\gamma\, T^{-1} + \hat m\end{pmatrix} \underbrace{=}_{\text{careful} \to \text{anomalies}}$$

careful → anomalies (check particle physics textbooks)

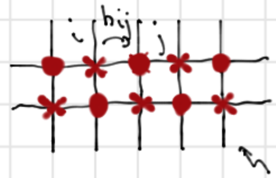
$$= -\mathrm{tr}\ln\begin{pmatrix} i\gamma + T^{-1}\hat m & X_0 + [T^{-1}, X_0]T \\ X_0^+ & i\gamma + \hat m T\end{pmatrix} \underbrace{+}_{\text{anomaly}} \text{topological terms} = S[T]$$

Expand in $T\hat m$, $[T^{-1}, X_0]T$ ~ Weinberg Lagrangian

$$S_{eff}[T] = \int d^dx\,\left(c_1\,\mathrm{tr}(\partial T\,\partial T^{-1}) + c_2\,\mathrm{tr}(T + T^{-1})\right) + \text{top. terms.}$$

Chiral symmetries in condensed matter contexts

A case study: sublattice symmetries



$\sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ $\rightsquigarrow$ $[\hat{H}, \sigma_3]_+ = 0$

$\hat{H} = \begin{pmatrix} & X \\ X^\dagger & \end{pmatrix}$

$\rightarrow \hat{U} \in U(2N) / U(N) \times U(N) = \mathrm{AIII}|_{2N}$

- Model of class AIII topological Anderson insulator

$\mathbb{C}^N$ $\quad X_{ij} = A_{ij} + (-)^i g \mathbb{1}$

A_{ij}: random matrix; g: staggering

- QCD analogy

	cond-mat	QCD	
	fermions	quarks	
	A	A	color gauge field
	finite DOS	chiral condensate	chiral sym. breaking
	σ-model replica field, T	flavor meson field, T	Goldstone mode (class A)
	σ-model action	Weinberg Lagrangian	Ginzburg Landau action
	finite ω	quark masses	sym. breaking