Hilbert von Neumann modules

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Abstract

We introduce a way of regarding Hilbert von Neumann modules as spaces of operators between Hilbert space, not unlike [Skei], but in an apparently much simpler manner and involving far less machinery. We verify that our definition is equivalent to that of [Skei], by verifying the 'Riesz lemma' or what is called 'self-duality' in [Skei]. An advantage with our approach is that we can totally side-step the need to go through C^*-modules and avoid the two stages of completion - first in norm, then in the strong operator topology - involved in the former approach.

We establish the analogue of the Stinespring dilation theorem for Hilbert von Neumann bimodules, and we develop our version of 'internal tensor products' which we refer to as Connes fusion for obvious reasons.

In our discussion of examples, we examine the bimodule arising from automorphisms of von Neumann algebras, verify that fusion of bimodules corresponds to composition of automorphisms in this case, and that the isomorphism class of such a bimodule depends only on the inner conjugacy class of the automorphism. We also relate Jones' basic construction to the Stinespring dilation associated to the conditional expectation onto a finite-index inclusion (by invoking the uniqueness assertion regarding the latter).

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1 Preliminaries

The symbols $\mathcal{H}$ and $\mathcal{K}$, possibly anointed with subscripts or other decorations, will always denote complex separable Hilbert spaces, while $\mathcal{L}(\mathcal{H}, \mathcal{K})$ will denote the set of bounded operators from $\mathcal{H}$ to $\mathcal{K}$. For $E \subset \mathcal{L}(\mathcal{H}, \mathcal{K})$, we shall write $[E]$ for the closure, in the weak operator topology (WOT, in the sequel), of the linear subspace of $\mathcal{L}(\mathcal{H}, \mathcal{K})$ spanned by E. Similarly, if $\mathcal{S} \subset \mathcal{H}$ is a set of vectors, we shall write $[\mathcal{S}]$ for the norm-closed subspace of $\mathcal{H}$ spanned by $\mathcal{S}$.

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Without explicitly citing it again to justify statements we make, we shall use the fact that a linear subspace of $\mathcal{H}$ (resp., $\mathcal{L}(\mathcal{H},\mathcal{K})$) is closed in the weak topoogy (resp., WOT) if and only if it is closed in the strong or norm topology (resp., 'SOT'). (For example, $[E]$ is an algebra if E is.)

If $E \subset \mathcal{L}(\mathcal{H},\mathcal{K})$ and $F \subset \mathcal{L}(\mathcal{H}_1,\mathcal{H})$, we write

$$EF = \{xy : x \in E, y \in F\} \text{ and } E^* = \{x^* : x \in E\} .$$

If $i : \mathcal{H}_0 \hookrightarrow \mathcal{H}$ and $j : \mathcal{K}_0 \hookrightarrow \mathcal{K}$, then we shall think of $\mathcal{L}(\mathcal{H},\mathcal{K}_0)$ as the subset $f\mathcal{L}(\mathcal{H}_0,\mathcal{K})e = j\mathcal{L}(\mathcal{H}_0,\mathcal{K})i$ of $\mathcal{L}(\mathcal{H}_0,\mathcal{K})$, where e and f are the projections $e = i^*, f = j^*$.

PROPOSITION 1.1. *For $i = 1, 2$, let e_i denote the projection of $\mathcal{H}_1 \oplus \mathcal{H}_2$ onto $\mathcal{H}_i$. The following conditions on an $E \subset \mathcal{L}(\mathcal{H}_2,\mathcal{H}_1)$ are equivalent:*

1. *There exists a von Neumann algebra $M \subset \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ such that $e_1, e_2 \in M$ and $E = e_1 M e_2$.*

2. *$E = [E] \supset EE^*E$.*

When these equivalent conditions are met, we shall say that $(E,\mathcal{H}_1,\mathcal{H}_2)$ is a **(1,2) von Neumann corner**.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1): Observe that the assumption (2) implies that $[E^*E]$ is a WOT-closed *-subalgebra of $\mathcal{L}(\mathcal{H}_2)$. Let $p_2 = \sup\{p : p \in \mathcal{P}([E^*E]\}$ and define $M_{22} = [E^*E] + \mathbb{C}(e_2 - p_2)$; so M_{22} is a von Neumann subalgebra of $\mathcal{L}(\mathcal{H}_2)$ and $e_2 - p_2$ is a central minimal projection in it.

Similarly, define $M_{11} = [EE^*] + \mathbb{C}(e_1 - p_1)$, where $p_1 = \sup\{p : p \in \mathcal{P}([EE^*]\}$; so M_{11} is a von Neumann subalgebra of $\mathcal{L}(\mathcal{H}_1)$ and $e_1 - p_1$ is a central minimal projection in it.

Finally set $M_{12} = E, M_{21} = E^*$ and $M = \sum_{i,j=1}^{2} M_{ij}$. (Alternatively M is the von Neumann algebra $(E \cup E^*)''$; and it is clear that $E = e_1 M e_2$.

□

DEFINITION 1.2. 1. The projection p_1 (resp. p_2) ocurring in the proof of Proposition 1.1 will be referred to as the **left-support** (resp., **right-support**) **projection** of the (1,2) von Neumann corner E.

2. A (1,2) von Neumann corner $(E, \mathcal{H}_1, \mathcal{H}_2)$ will be said to be **non-degenerate** if its support projections are as large as they can be: i.e., $p_i = (e_i =)1_{\mathcal{H}_i}, i = 1, 2$.

REMARK 1.3. *1.* The support projections p_1, p_2 of E have the following equivalent descriptions:

- $ran\ p_1 = [\bigcup\{ran\ x : x \in E\}] = \left(\bigcap\{ker\ x^* : x \in E\}^{\perp}\right)$; and
- $ran\ p_2 = [\bigcup\{ran\ x^* : x \in E\}] = \left(\bigcap\{ker\ x : x \in E\}^{\perp}\right)$.

2. A (1,2) von Neumann corner $(E, \mathcal{H}_1, \mathcal{H}_2)$ is non-degenerate precisely when $M_{11}(E) = [EE^*]$ and $M_{22}(E) = [E^*E]$ are unital von Neumann subalgebras of $\mathcal{L}(\mathcal{H}_1)$ and $\mathcal{L}(\mathcal{H}_2)$ respectively.

DEFINITION 1.4. *1. If A_2 is a von Nemann algebra, a* **Hilbert von Neumann A_2 - module** *is a tuple $\mathcal{E} = (E, \mathcal{H}_1, (\pi_2, \mathcal{H}_2))$ where $(E, \mathcal{H}_1, \mathcal{H}_2)$ is a (1,2) von Neumann corner equipped with a normal* isomorphism $\pi_2 : A_2 \to [E^*E]$.

2. A **submodule** *of a Hilbert von Neumann A_2-module E is a subset $E_1 \subset E$ satisfying*

$$E_1 = [E_1] \supset E_1 E^* E.$$

3. If A_1, A_2 are von Neumann algebras, a **Hilbert von Neumann $A_1 - A_2$ - bimodule** *is a tuple*

$$\mathcal{E} = (E, (\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$$

comprising a Hilbert von Neumann A_2 - module $(E, \mathcal{H}_1, (\pi_2, \mathcal{H}_2))$ equipped with a normal unital homomorphism $\pi_1 : A_1 \to [EE^*]$ (where the 'unital requirement' is that $\pi_1(A_1) = p_1$ is the identity of $[EE^*]$).

REMARK 1.5. 1. If $E \subset \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ is any (possibly degenerate) (1,2) von Neumann corner, with associated support projections p_1, p_2 (as in Definition 1.2), define $\mathcal{K}_i = ran\ p_i, A_1 = [EE^*], A_2 = [E^*E]$ and let π_i denote the identity representation of A_i on $\mathcal{K}_i$; ithen $(E, (\pi_1, \mathcal{K}_1), (\pi_2, \mathcal{K}_2))$ is seen to be a non-degenerate Hilbert von Neumann $A_1 - A_2$ - bimodule. This is why non-degeneracy is not a serious restriction.

2. A Hilbert von Neumann A_2 - module $(E, \mathcal{H}_1, (\pi_2, \mathcal{H}_2))$ does indeed admit a right-A_2 action and an A_2 - valued inner product thus:
$$x \cdot a_2 = x\pi_2(a_2) \ ; \langle x_1, x_2 \rangle_{A_2} = \pi_2^{-1}(x_1^* x_2)$$
(Here and in the sequel, we shall write $\langle \cdot, \cdot \rangle_B$ for the B - valued inner-product on a Hilbert B - module.) Notice, further, that the norm E acquires from this Hilbert A_2 - module structure is nothing but the operator norm on E.

3. A submodule of a Hilbert von Neumann A_2 - module is a (possibly degenarate) (1,2) von Neumann corner.

4. In a general Hilbert von Neumann A_2 - module $\mathcal{E} = (E, \mathcal{H}_1, (\pi_2, \mathcal{H}_2))$, note that
$$[EE^*] \ni a \mapsto (E \ni x \mapsto a \cdot x =: ax)$$
defines a *-homomorphism of $[EE^*]$ into the space $\mathcal{L}^a(E)$ of bounded adjointable operators on E, since, for instance
$$\begin{aligned} \langle a \cdot x, y \rangle_{A_2} &= (ax)^* y \\ &= x^*(a^* y) \\ &= x^*(a^* \cdot y) \\ &= \langle x, a^* \cdot y \rangle_{A_2} \ . \end{aligned}$$

5. In the language of (2) above, the 'rank-one operator' $\theta_{x,y}$ is seen to be given by
$$\begin{aligned} \theta_{x,y}(z) &= x\langle y, z \rangle_{A_2} \\ &= xy^* z \ , \end{aligned}$$
so that the 'rank-one operator' $\theta_{x,y}$ on E is nothing but left multiplication by xy^* on E, for any $x, y \in E$. Let us write $B = [EE^*]$, $C = A_2$ and A for the norm-closure of the linear span of EE^*. Then it is clear that A is a norm-closed ideal in B, and that there is a unique C^* - algebra isomorphism $\alpha : A \to \mathcal{K}(E)$ such that $\alpha(xy^*) = \theta_{x,y}$, $\forall x, y \in E$. If E is non-degenerate, then A is an essential ideal in B and α is injective. It then follows from [Lan] Proposition 2.1, that α extends uniquely to an isomorphism of B onto $\mathcal{L}^a(E)$. (In fact, the reason for introducing the symbols A, B, C above was in order to use exactly the same symbols as in the Proposition 2.1 referred to above.)

6. This remark concerns our requirement, in the definition of a Hilbert von Neumann A_2-module, that $\pi_2 : A_2 \rightarrow [E^*E]$ must be an isomorphism. What is really needed is that π_2 is onto. If π_2 is merely surjective but not injective, there must exist a central projection $z \in A_2$ such that $ker\ \pi_2 = (1-z)A_2$ so π_2 would map zA_2 isomorphically onto $[E^*E]$ and the A_2-valued inner product (see item (2) of this remark) would actually take values in zA_2 and we could apply our analysis to zA_2 and think of A_2 as acting via its quotient (and ideal) zA_2.

7. The 'unital requirement' made in the definition of a Hilbert von Neumann bimodule has the consequence that $\pi_1(A_1)E = E$.

Lemma 1.6. *Let $(E, \mathcal{H}_1, \mathcal{H}_2)$ be a (1,2) von Neumann corner. Suppose $x \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ has polar decomposition $x = u|x|$. Then the following conditions are equivalent:*

1. *$x \in E$.*
2. *$u \in E$ and $|x| \in [E^*E]$.*
3. *$u \in E$ and $|x^*| \in [EE^*]$.*

Proof. Since (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are obvious, let us prove the reverse implications. So, suppose $x \in E$. Then $x^*x \in E^*E$ (resp., $xx^* \in EE^*$) and as, $|t|$ is uniformly approximable on compact subsets of $\mathbb{R}$ by polynomials with vanishing constant term, it is seen that $|x| \in [E^*E]$ and $|x^*| \in [EE^*]$. Define $f_n \in C_0([0,\infty))$ by

$$f_n(t) = \begin{cases} 0 & \text{if } t < \frac{1}{2n} \\ 2n^2(t - \frac{1}{2n}) & \text{if } \frac{1}{2n} \leq t \leq \frac{1}{n} \\ \frac{1}{t} & \text{if } t \geq \frac{1}{n} \end{cases}$$

Since f_n is uniformly approximable on $sp(|x|)$ by polynomials with vanishing constant term, it is seen that $f_n(|x|) \in [E^*E]$, and hence $xf_n(|x|) \in E$. It follows from the definitions that $|x|f_n(|x|)$ WOT-converges to $1_{(0,\infty)}(|x|) = u^*u$. In particular, $u = u(u^*u) = WOT - lim\ u(|x|f_n(|x|) = WOT - lim\ xf_n(|x|) \in [E\,[E^*E]] \subset E$.

□

Proposition 1.7. *If E_1 is a submodule of a Hilbert von Neumann A_2 - module E, and if $E_1 \neq E$, there exists a non-zero $y \in E$ such that $y^*x = 0\ \forall x \in E_1$.*

Proof. As observed in Remark 1.5(3), E_1 is a possibly degenerate (1,2) von Neumann corner in $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$. Let $p_1 = \bigvee\{e : e \in \mathcal{P}([E_1^* E_1])\}$ and $q_1 = \bigvee\{f : f \in \mathcal{P}([E_1 E_1^*])\}$ be the right- and left-support projections of E_1. Similarly, let $p = \bigvee\{e : e \in \mathcal{P}([E^*E])\}$ and $q = \bigvee\{f : f \in \mathcal{P}([EE^*])\}$ be the right- and left- support projections of E.

First observe that the hypotheses imply that

$$(E^*E)(E_1^*E_1)(E^*E) = (E_1E^*E)^*(E_1E^*E) \subset E_1^*E_1$$

and hence that $[E_1^*E_1]$ is a WOT-closed ideal in the von Neumann subalgebra $[E^*E]$ of $\mathcal{L}(p\mathcal{H}_2)$; consequently $p_1 = \bigvee\{e : e \in \mathcal{P}([E_1^*E_1])\}$ is a central projection in $[E^*E]$ and $[E_1^*E_1] = [E^*E]p_1$. It follows that if $x_1 \in E_1$ has polar decomposition $x_1 = u_1|x_1|$, then (by Lemma 1.6) $u_1 \in E_1$ and $|x_1| \in [E_1^*E_1] = [E^*E]p_1$, and in particular, $x_1p_1 = u_1|x_1|p_1 = u_1|x_1| = x_1$; i.e., $E_1 = E_1p_1$.

Next, by definition, $[\bigcup\{ran\ x_1 : x_1 \in E_1\}] = [\bigcup\{ran\ x_1x_1^* : x_1 \in E_1\}] = [\bigcup\{ran 1_{(0,\infty)}(|x_1^*|) : x_1 \in E_1\}] = \ ran\ q_1$; hence if $x_1 \in E_1$, then $x_1 = q_1x_1$, and we see that $E_1 = q_1E_1$.

Summarising the previous two paragraphs, we have

$$E_1 = q_1E_1 = E_1p_1 \ . \tag{1.1}$$

(In fact, $x_1 = x_1p_1 = q_1x_1 \ \forall \ x_1 \in E_1$.)

We now consider three cases:

Case 1: $p_1 \neq p$

Here $(p - p_1) \neq 0$ and the definition of p implies that there exists a $y \in E$ such that $y = y(p - p_1) \neq 0$. Then, for any $x \in E_1$, we have $x = xp_1$ and hence

$$y^*x = (p-p_1)y^*x = (p-p_1)y^*xp_1 \in (p-p_1)E^*Ep_1 = (p-p_1)p_1E^*E = \{0\} \ .$$

Case 2: $q_1 \neq q$

Here $(q - q_1) \neq 0$ and the definition of q implies that there exists a $y \in E$ such that $y = (q - q_1)y \neq 0$. Then, for any $x \in E_1$, we have $x = q_1x$ and hence

$$y^*x = y^*(q - q_1)x = y^*(q - q_1)q_1x = 0 \ .$$

Case 3: $p_1 = p, q_1 = q$.

We shall show that the hypotheses of this case imply that $E_1 = E$ and hence cannot arise. To see this, begin by noting that the collection of non-zero partial isometries in E_1 is non-empty in view of

Lemma 1.6. (Otherwise $E_1 = \{0\}, p_1 = q_1 = 0$ and so $E = \{0\} = E_1$.) Hence the family $\mathcal{F}$ of collections $\{u_i : i \in I\}$ of partial isometries in E_1 with pairwise orthogonal ranges, is non-empty. Clearly $\mathcal{F}$ is partially ordered by inclusion, and it is easy to see that Zorn's lemma is applicable to $\mathcal{F}$.

If $\{u_i : i \in I\}$ is a maximal element of $\mathcal{F}$, we assert that $\sum_{i\in I} u_i u_i^* = q$. Indeed, if $(q - \sum_{i\in I} u_i u_i^*) \neq 0$, the assumption $q = q_1$ will imply the existence of an $x_1 \in E_1$ such that $x_1 = (q - \sum_{i\in I} u_i u_i^*)x_1 \neq 0$. Then $x_1 \in [E_1 E_1^* E_1] \subset E_1$ and so if $x_1 = v_1|x_1|$ is its polar decomposition, then $v_1 \in E_1 \setminus \{0\}$ and $ran\ v_1 = \overline{ran\ x_1}$ is orthogonal to $ran\ u_i$ for each $i \in I$, thus contradicting the maximality of $\{u_i : i \in I\}$.

Thus, indeed $q = \sum_{i\in I} u_i u_i^*,\ u_i \in E_1$.

Now, if $x \in E$ is arbitrary, then,

$$\begin{aligned} x &= qx \\ &= \sum_{i\in I} u_i u_i^* x \\ &\in [E_1 E_1^* E] \\ &\subset [E_1 E^* E] \\ &\subset E_1 \end{aligned}$$

and so $E = E_1$ in this case, and the proof of the Proposition is complete. $\square$

Given a submodule E_1 of a Hilbert von Neumann module E, as above, we shall write $E_1^\perp$ for the set $\{y \in E : y^* E_1 = \{0\}\}$ and refer to it as the **orthogonal complement of** E_1 **in** E. We now reap the consequences of Proposition 1.7 in the following Corollary.

COROLLARY 1.8. *Let E_1 be a submodule of a Hilbert von Neumann A_2 - module. Then,*

1. *$E_1^\perp = (1 - q_1)E$, where q_1 is the left support projection of E_1.*
2. *$E_1^{\perp\perp} = q_1 E$.*
3. *If S is any subset of E, then $S^{\perp\perp} = [S[E^*E]]$.*
4. *If E_1 is a submodule of a Hilbert von Neumann module E, there exists a projection $q_1 \in [EE^*]$ such that $E_1 = E_1^{\perp\perp} = q_1 E$ and $E_1^\perp = (1 - q_1)E$; and in particular E_1 is complemented in the sense that $E = E_1 \oplus E_1^\perp$.*

Proof. It is clear that $y^*x = 0$ if and only if y and x have mutually orthogonal ranges.

(1) The previous sentence and the definition of q_1 imply that

$$y \in E_1^\perp \Leftrightarrow (q_1 y = 0 \text{ and } y \in E) \Leftrightarrow y \in (1 - q_1)E.$$

(2) follows from (1) and the definition of the orthogonal complement.

(3) Let $E_1 = [SE^*E]$. It should be clear that $y \in S^\perp \Leftrightarrow y \in E_1^\perp = q_1 E$, by part (1) of this Corollary, and hence that

$$S^{\perp\perp} = E_1^{\perp\perp} \ .$$

In view of Remark 1.5(1) we may view $S^{\perp\perp}$ as a Hilbert von Neumann bimodule, and regard E_1 as a submodule of $S^{\perp\perp}$. We may then deduce from Proposition 1.7 that if E_1 were not equal to $S^{\perp\perp}$, then there would have to exist a non-zero $y \in S^{\perp\perp}$ such that $y^*E_1 = \{0\}$. This would imply that $y \in S^\perp$ and $y \in S^{\perp\perp}$ so that $y^*y = 0$, a contradiction.

(4) follows from the preceding parts of this Corollary. □

That our definitions of Hilbert von Neumann modules and bimodules are consistent with those of [Skei] is a consequence of the following version of Riesz' Lemma, which establishes that our Hilbert von Neumann modules are indeed 'self-dual' which is one of the equivalent conditions for a von Neumann module in the sense of [Skei].

On the other hand, it is clear from [Skei] that any Hilbert von Neumann A_2 - module in the sense of [Skei] is also a Hilbert von Neumann A_2 - module in our sense, and the two formulations are thus equivalent.

Proposition 1.9. (**Riesz lemma**) *Suppose $\mathcal{E}$ is a Hilbert von Neumann A_2 - module, and $f : E \to A_2$ is right A_2-linear - meaning $f(x\pi_2(a_2)) = \pi_2^{-1}(f(x)\pi_2(a_2))$ for all $x \in E, a_2 \in A_2$, or equivalently and less clumsily, suppose $f : E \to [E^*E]$ is linear and satisfies $f(xz) = f(x)z$ for all $x \in E,\ z \in [E^*E]$; and suppose f is bounded - meaning $\|f(x)\| \leq K\|x\|$ for all $x \in E$, and some $K > 0$. Then there exists $y \in E$ such that $f(x) = y^*x\ \forall x \in E$.*

Proof. First notice that if $x \in E$ has polar decomposition $x = u|x|$

(so $u \in E, |x| \in [E^*E] = \pi_2(A_2)$, and if $\xi \in \mathcal{H}_2$, then

$$\begin{aligned} \|f(x)\xi\| &= \|f(u)|x|\xi\| \quad \text{(by right } A_2 \text{ - linearity of } f) \\ &\leq \|f(u)\|\||x|\xi\| \\ &\leq K\||x|\xi\| \\ &= K\|u^*x\xi\| \\ &\leq K\|x\xi\| \ . \end{aligned} \tag{1.2}$$

Next, find vectors $\xi_n \in \mathcal{H}_2$ such that $\mathcal{H}_2 = \oplus_n[\pi_2(A_2)\xi_n]$ (orthogonal direct sum). It follows that $p_1\mathcal{H}_1 = \oplus_n[E\xi_n]$, where p_1 is the left-support projection of E (because if $n \neq m$ and $x, y \in E$, then

$$\langle x\xi_n, y\xi_m\rangle = \langle \xi_n, x^*y\xi_m\rangle = 0$$

and

$$[\bigcup_n [E\xi_n]] = [\bigcup_n [EE^*E\xi_n]] = [E\mathcal{H}_2] = p_1\mathcal{H}_1 \ .$$

Infer from the above paragraph and equation 1.2 that for arbitrary $a_n \in A_2$ with $\sum_n \|\pi_2(a_n)\xi_n)\|^2 < \infty$ and $x \in E$, we have

$$\begin{aligned} \|f(x)(\sum_n \pi_2(a_n)\xi_n)\|^2 &= \|\sum_n (f(x)\pi_2(a_n))\xi_n\|^2 \\ &= \sum_n \|f(x\pi_2(a_n))\xi_n\|^2 \\ &\leq \sum_n K^2\|x\pi_2(a_n)\xi_n\|^2 \quad \text{(by eq. (1.2)} \\ &= K^2\|x(\sum_n \pi_2(a_n)\xi_n)\|^2 \ ; \end{aligned}$$

Now deduce that there exists a unique bounded operator $z_f \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ satisfying $z_f = z_f p_1$ and

$$z_f(x\xi) = f(x)\xi \ , \forall x \in E, \xi \in \mathcal{H}_2 \ .$$

The definition of z_f implies that $z_f E \subset [E^*E]$; hence

$$z_f = z_f p_1 \in z_f[EE^*] \subset [z_f EE^*] \subset [[E^*E]E^*] = E^* \ .$$

So $y =: z_f^* \in E$ and we have

$$f(x) = z_f x = y^*x$$

as desired. □

2 Standard bimodules and complete positivity

Given an element x of a von Neumann algebra M, et us write $pr(x)$ for the projection onto the range of x. (Thus $pr(x) = 1_{(0,\infty)}(xx^*)$.)

LEMMA 2.1. *Suppose $\eta : A \to B$ is a normal positive linear map of von Neumann algebras. Let $e_\eta = \bigvee\{u\ pr(\eta(1))\ u^* : u \in \mathcal{U}(B)\}$ be the (B-)central support of $pr(\eta(1)$. Then the smallest WOT-closed ideal in B which contains $\eta(A)$ (equivalently $\eta(1)$) is $e_\eta B$. (In particular, $\eta(a) = e_\eta \eta(a)\ \forall a \in A$.)*

Proof. If $p \in \mathcal{P}(A)$, then $\eta(p) \leq \eta(1) \Rightarrow pr(\eta(p) \leq pr(\eta(1) \leq e_\eta$. Hence $\eta(p) = e_\eta\eta(p) \in e_\eta B$, so also $B\eta(p)B \subset e_\eta B$. Conclude that $[B\eta(A)B] = [B\eta([\mathcal{P}(A)])B] = [B\eta(\mathcal{P}(A))B] \subset e_\eta B$. Conversely, $[B\eta(A)B] \supset [B\mathcal{U}(B)\eta(1)\mathcal{U}(B)B] \supset [Be_\eta B] = e_\eta B$, and the proof is complete. □

DEFINITION 2.2. *A Hilbert von Neumann A_2 - module $\mathcal{E} = (E, \mathcal{H}_1, (\pi_2, \mathcal{H}_2))$ will be called* **standard** *if :*

- *$\mathcal{H}_2 = L^2(A_2, \phi)$ for some faithful normal state ϕ on A_2;*
- *π_2 is the left-regular representation; and*
- *E is non-degenerate.*

A Hilbert von Neumann $A_1 - A_2$ - bimodule will be called standard if it is standard as a Hilbert von Neumann A_2 - module.

THEOREM 2.3. *If $\eta : A_1 \to A_2$ is a normal completely positive map, there exists a standard Hilbert von Neumann $A_1 - e_\eta A_2$ bimodule $\mathcal{E}_\eta$, with e_η as in Lemma 2.1, which is singly generated, (i.e., $E = [\pi_1(A_1)V\pi_2(e_\eta A_2)]$) with a generator $V \in E$ satisfying $V^*\pi_1(a_1)V = \pi_2 \circ \eta(a_1)$.*

Further, such a pair $(\mathcal{E}, V)$ of a standard bimodule and generator is unique in the sense that if $(\widetilde{\mathcal{E}}, \tilde{V})$ is another such pair, then there exists A_i - linear unitary operators $U_i : \mathcal{H}_i(\eta) \to \widetilde{\mathcal{H}_i},\ i = 1, 2$ such that $\tilde{V} = U_1 V U_2^*$ and $\widetilde{\mathcal{E}} = U_1 \mathcal{E} U_2^*$.

Proof. Fix a faithful normal state ϕ on $e_\eta A_2$ and set $\mathcal{H}_2(\eta) = L^2(e_\eta A_2, \phi)$, with π_2 being the left-regular representation of $e_\eta A_2$. We employ the standard notation $\hat{a} = \pi(a)\hat{1}$ where $\hat{1}$ is the canonical cyclic vector for $\pi(A)$ in $L^2(A)$. The Hilbert space $\mathcal{H}_1(\eta)$ is obtained after separation and completion of the algebraic tensor product $A_1 \otimes e_\eta A_2$ with respect to the semi-inner product given by $\langle a_1 \otimes a_2, b_1 \otimes b_2\rangle = \phi(b_2^* \eta(b_1^* a_1) a_2)$; and $\pi_1 : A_1 \to \mathcal{L}(\mathcal{H}_1(\eta))$ is defined by $\pi_1(a_1)(b_1 \otimes b_2) = a_1 b_1 \otimes b_2$. The verification that π_1 is a normal representation is a fairly routine application of normality of η and ϕ.

Define $V : \mathcal{H}_2(\eta) \to \mathcal{H}_1(\eta)$ to be the unique bounded operator for which $V(e_\eta \hat{a}_2) = 1 \otimes e_\eta a_2$. For arbitrary $a_1 \in A_1, a_2, b_2 \in e_\eta A_2$, note that

$$\begin{aligned} \langle V^* \pi_1(a_1) V \hat{a}_2, \hat{b}_2\rangle &= \langle a_1 \otimes a_2, 1 \otimes b_2\rangle \\ &= \phi(b_2^* \eta(a_1) a_2\rangle \\ &= \langle \pi_2(\eta(a_1)) \hat{a}_2, \hat{b}_2\rangle \end{aligned}$$

thus showing that indeed $V^* \pi_1(a_1) V = \pi_2(\eta(a_1))$ for all $a_1 \in A_1$.

Set $E = [\pi_1(A_1) V \pi_2(e_\eta A_2)]$ and observe that

$$\begin{aligned} [E^* E] &= [\pi_2(e_\eta A_2) V^* \pi_1(A_1) \pi_1(A_1) V \pi_2(e_\eta A_2)] \\ &= [\pi_2(e_\eta A_2) \pi_2(\eta(A_1)) \pi_2(e_\eta A_2)] \\ &= [\pi_2(e_\eta A_2 \eta(A_1) e_\eta A_2)] \\ &= \pi_2(e_\eta A_2) \ , \end{aligned}$$

by Lemma 2.1. Further, if $x = \pi_1(a_1) V \pi_2(e_\eta a_2)$ for $a_i \in A_i$, note that, by definition, we have $x(\hat{e}_\eta) = a_1 \otimes e_\eta a_2$ and hence, $[\bigcup\{ran\ x : x \in E\}] = \mathcal{H}_1(\eta)$. This shows that there exist projections $\{p_i : i \in I\} \subset [EE^*]$ such that $id_{\mathcal{H}_1(\eta)} = WOT - lim_i p_i$. Hence, we see that

$$\pi_1(A_1) \subset [\bigcup\{\pi_1(A_1) p_i : i \in I\}] \subset [\pi_1(A_1) EE^*] \subset [EE^*] \ ;$$

and we have verified everything neeed to see that the tuple $\mathcal{E}_\eta = (E, (\pi_1, \mathcal{H}_1(\eta)), (\pi_2, \mathcal{H}_2(\eta)))$ defines a standard Hilbert von Neumann $A_1 - e_\eta A_2$ - bimodule. As for the uniqueness assertion, if $(\widetilde{\mathcal{E}}, \tilde{V})$ also works, then $\widetilde{\mathcal{H}_2} = L^2(e_\eta A_2, \tilde{\phi})$ for some faithful normal state $\tilde{\phi}$ on $e_\eta A_2$. In view of the 'uniqueness of the standard module of a von Neumann algebra' - see [Haa], for instance - there exists an $e_\eta A_2$ - linear unitary operator $U_2 : \mathcal{H}_2(\eta) \to \widetilde{\mathcal{H}_2}$. Observe next that if

$\xi, \eta \in \mathcal{H}_2$ and $a_1, b_1 \in A_1, a_2, b_2 \in e_\eta A_2$, then

$$
\begin{aligned}
&\langle \pi_1(a_1)V\pi_2(a_2)\xi, \pi_1(b_1)V\pi_2(b_2)\eta\rangle \\
&= \langle \pi_2(b_2^*)V^*\pi_1(b_1^*a_1)V\pi_2(a_2)\xi, \eta\rangle \\
&= \langle \pi_2(b_2^*)\pi_2(\eta(b_1^*a_1))\pi_2(a_2)\xi, \eta\rangle \\
&= \langle \pi_2(b_2^*\eta(b_1^*a_1)a_2)\xi, \eta\rangle \\
&= \langle U_2\pi_2(b_2^*\eta(b_1^*a_1)a_2)\xi, U_2\eta\rangle \\
&= \langle \widetilde{\pi_2}(b_2^*\eta(b_1^*a_1)a_2)U_2\xi, U_2\eta\rangle \\
&= \langle \widetilde{\pi_2}(b_2^*)\widetilde{V^*}\widetilde{\pi_1}(b_1^*a_1)\widetilde{V}\widetilde{\pi_2}(a_2)U_2\xi, U_2\eta\rangle \\
&= \langle \widetilde{\pi_1}(a_1)\widetilde{V}\widetilde{\pi_2}(a_2)U_2\xi, \widetilde{\pi_1}(b_1)\widetilde{V}\widetilde{\pi_2}(b_2)U_2\eta\rangle \ .
\end{aligned}
$$

Deduce from the above equation and the assumed non-degeneracy of $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ that there is a unique unitary operator $U_1 : \mathcal{H}_1 \to \widetilde{\mathcal{H}_1}$ such that

$$U_1\left(\pi_1(a_1)V\pi_2(a_2)\xi\right) = \widetilde{\pi_1}(a_1)\tilde{V}\widetilde{\pi_2}(a_2)U_2\xi \tag{2.3}$$

for all $a_1 \in A_1, a_2 \in e_\eta A_2$ and $\xi \in \mathcal{H}_2(\eta)$ It is easy to see from equation (2.3) that U_1 is necessarily A_1 - linear, that $U_1V = \tilde{V}U_2$ or $\tilde{V} = U_1VU_2^*$ and that $\widetilde{\mathcal{E}} = U_1\mathcal{E}U_2^*$, and the proof of the theorem is complete. □

Remark 2.4. *Notice that the irritating e_η above is equal to the 1 of A_2 in some good cases, such as the following:*

- *when η is unital, i.e., $\eta(1) = 1$;*
- *when $\eta(1) \neq 0$ and A_2 is a factor.*

The uniqueness assertion in Theorem 2.3 can also be deduced from the following useful criterion for isomorphism of standard bimodules:

Lemma 2.5. *Two standard Hilbert von Neumann A_2 bimodules $\mathcal{E}^{(i)} = (E^{(i)}, (\pi_1^{(i)}, \mathcal{H}_1^{(i)}), (\pi_2^{(i)}, \mathcal{H}_2^{(i)})), i = 1, 2$ are isomorphic if and only if there exist $E_0^{(i)} = \{x_j^{(i)} : j \in I\} \subset E^{(i)}$ such that*

1. *$[E_0^{(i)}] = E^{(i)}$, and*
2. *$(\pi_2^{(1)})^{-1}(x_j^{(1)*}x_k^{(1)}) = (\pi_2^{(2)})^{-1}(x_j^{(2)*}x_k^{(2)})\ \forall j, k \in I$*

Proof. The only if implication is clear, as we may choose $E_0^{(i)} = E^{(i)}$ and $x^{(2)} = U_1x^{(1)}U_2^*$ for all $x^{(1)} \in E^{(1)}(= I)$. Now for the other 'if half'.

In view of the 'uniqueness of the standard module of a von Neumann algebra - see [Haa] -there exists an A_2 - linear unitary operator $U_2 : \mathcal{H}_2^{(1)} \to \mathcal{H}_2^{(2)}$. For arbitrary $j, k \in I, \xi_1, \xi_2 \in \mathcal{H}_2^{(1)}$, observe that

$$\begin{aligned}
\langle x_j^{(1)}\xi_1, x_k^{(1)}\xi_2\rangle &= \langle \xi_1, x_j^{(1)*}x_k^{(1)}\xi_2\rangle \\
&= \langle U_2\xi_1, U_2\pi_2^{(1)}(\pi_2^{(1)})^{-1}(x_j^{(1)*}x_k^{(1)})\xi_2\rangle \\
&= \langle U_2\xi_1, \pi_2^{(2)}(\pi_2^{(1)})^{-1}(x_j^{(1)*}x_k^{(1)})U_2\xi_2\rangle \\
&= \langle U_2\xi_1, \pi_2^{(2)}(\pi_2^{(2)})^{-1}(x_j^{(2)*}x_k^{(2)})U_2\xi_2\rangle \\
&= \langle x_j^{(2)}U_2\xi_1, x_k^{(2)}U_2\xi_2\rangle \ ;
\end{aligned}$$

deduce from the above equation and the non-degeneracy of the $\mathcal{E}^{(i)}$ that there exists a unique unitary operator $U_1 : \mathcal{H}_1^{(1)} \to \widetilde{\mathcal{H}_1^{(2)}}$ such that $U_1(x_j^{(1)}\xi) = x_j^{(2)}U_2\xi \ \forall j \in I, \xi \in \mathcal{H}_2^{(1)}$. The definitions show that $U_1x_j^{(1)} = x_j^{(2)}U_2 \ \forall j \in I$ and hence that $U_1E^{(1)} = E^{(2)}U_2$. Thus indeed $E^{(2)} = U_1E^{(1)}U_2^*$ and the proof of the 'if half' is complete.

□

Notice, incidentally, that in the setting of the Lemma above, the equation

$$Tx^{(1)} = U_1x^{(1)}U_2^*$$

defines a WOT-continuous linear bijection $T : E^{(1)} \to E^{(2)}$ satisfying

$$Tx^{(1)}(Ty^{(1)})^*Tz^{(1)} = T(x^{(1)}(y^{(1)})^*z^{(1)})$$

for all $x^{(1)}y^{(1)}, z^{(1)} \in E^{(1)}$.

Remark 2.6. 1. *The 'generator' V of Theorem 2.3 is an isometry precisely when η is unital.*

2. *If $\mathcal{E}$ is a singly generated Hilbert von Neumann $A_1 - A_2$ bimodule, then it is generated by a partial isometry (by Lemma 1.6). Further, that generator, say V may be used to define the obviously completely positive map $\eta; A_1 \to A_2$ by*

$$\eta(a_1) = \pi_2^{-1}(V^*\pi_1(a_1)V) \ ;$$

and then $\mathcal{E}$ would be isomosrphic to $\mathcal{E}_\eta$ if and only if $\mathcal{E}$ is a standard non-degenerate bimodule.

3 Connes fusion

EXAMPLE 3.1. If $\mathcal{E} = (E, (\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$ is a Hilbert von Neumann $A_1 - A_2$ - bimodule and $\mathcal{K}$ is any Hilbert space, then $\mathcal{E} \otimes id_{\mathcal{K}} = (E \otimes id_{\mathcal{K}}, (\pi_1 \otimes id_{\mathcal{K}}, \mathcal{H}_1 \otimes \mathcal{K}), (\pi_2 \otimes id_{\mathcal{K}}, \mathcal{H}_2 \otimes \mathcal{K}))$ is also a Hilbert von Neumann $A_1 - A_2$ - bimodule, where of course we write $E \otimes id_{\mathcal{K}}$ for $\{x \otimes id_{\mathcal{K}} : x \in E\}$.

LEMMA 3.2. *Let $\mathcal{E} = (E, (\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$ be a Hilbert von Neumann $A_1 - A_2$ - bimodule. For a projection $p \in \mathcal{P}(\pi_2(A_2)')$, let q be the projection with range $[\bigcup\{ran(xp) : x \in E\}]$. Then*

1. *$q \in \mathcal{P}(\pi_1(A_1)')$;*

2. *$y \in E \Rightarrow qyp = qy = yp$; and*

3. *$q\mathcal{E}p = (qEp, (q\pi_1(\cdot), q\mathcal{H}_1), (p\pi_2(\cdot), p\mathcal{H}_2))$ satisfies all the requirements for a non-degenerate Hilbert von Neumann $A_1 - A_2$ - bimodule, with the posible exception of injectivity of $p\pi_2(\cdot)$.*

We shall use the suggestive notation $\mathcal{E}_* p = q$ when $q, \mathcal{E}, p$ are so related.

Proof. 1. Since $\pi_1(A_1)E \subset E$, it follows that $ran(q)$ is stable under $\pi_1(A_1)$.

2. For all $y \in E$, $ran(yp) \subset ran(q) \Rightarrow qyp = yp$. Next, if $\xi, \eta \in \mathcal{H}_2$, and $x, y \in E$, note that

$$\begin{aligned} \langle xp\xi, y(1-p)\eta\rangle &= \langle \xi, px^*y(1-p)\eta\rangle \\ &\in \langle \xi, p\,[E^*E]\,(1-p)\eta\rangle \\ &= 0\ , \end{aligned}$$

since $[E^*E] = \pi_2(A) \subset \{p\}'$; since $\{xp\xi : \xi \in \mathcal{H}_2\}$ is total in $ran(q)$, this says that $qy(1-p) = 0$, as desired.

3.

$$[(qEp)^*(qEp)] = [(Ep)^*(Ep)] = p\,[E^*E]\,p = p\pi_2(A_2) \tag{3.4}$$

since $[E^*E] = \pi_2(A) \subset \{p\}'$; while

$$[(qEp)(qEp)^*] = q\,[EE^*]\,q \supset q\pi_1(A_1). \tag{3.5}$$

Non-degeneracy of $q\mathcal{E}p$ follows immediately from equations (3.4) and (3.5). $\square$

Remark 3.3. *In general, if* $\pi : M \to \mathcal{L}(\mathcal{H})$ *is a faithful normal representation, and if* $p \in \pi(M)'$, *the subrepresentation* $p\pi(\cdot)$ *is faithful if and only if the central support of* p *is* 1 *- i.e.,* $\sup\{upu^* : u \in \pi(M)'\} = 1$.

In particular if the $\mathcal{E}$ *of Lemma 3.2 is actually a Hilbert von Neumann* $A_1 - A_2$ *- bimodule, and if* A_2 *happens to be a factor, then the* $q\mathcal{E}p$ *of Lemma 3.2 is actually a Hilbert von Neumann bimodule.*

We next lead to our description of what is sometimes termed 'internal tensor product' but which we prefer (in view of this terminology being already in use for tensor products of bimodules over von Neumann algebras) to refer to as the *Connes fusion* of Hilbert von Neumann bimodules. Thus, suppose $\mathcal{E} = (E, (\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$ is a Hilbert von Neumann $A_1 - A_2$ - bimodule and $\mathcal{F} = (F, (\rho_2, \mathcal{K}_2), (\rho_3, \mathcal{K}_3))$ is a Hilbert von Neumann $A_2 - A_3$ - bimodule. We know that the normal representation ρ_2 of A_2 is equivalent to a subrepresentation of an infinite ampliation of the faithful normal representation π_2 of A_2; thus there exists an A_2 - linear isometry $u : \mathcal{K}_2 \to \mathcal{H}_2 \otimes \ell^2$: i.e., $u^*u = id_{\mathcal{K}_2}$ and $u\rho_2(x) = (\pi_2(x) \otimes id_{\ell^2})u \ \forall x \in A_2$. It follows that $p = uu^* \in (\pi_2(A_2) \otimes id_{\ell^2})'$.

Now, set $p = uu^*$ and let $q = (\mathcal{E} \otimes 1_{\ell^2}))_*(p)$ be associated to this p as in Lemma 3.2 (applied to $\mathcal{E} \otimes 1_{\ell^2}$).

Finally, if $x \in E, y \in F$, define $x \bigodot y$ to be the composite operator

$$\mathcal{K}_3 \stackrel{x \bigodot y}{\longrightarrow} q(\mathcal{H}_1 \otimes \ell^2) \ = \ \mathcal{K}_3 \stackrel{y}{\longrightarrow} \mathcal{K}_2 \stackrel{u}{\longrightarrow} uu^*(\mathcal{H}_2 \otimes \ell^2) \stackrel{x \otimes id_{\ell^2}}{\longrightarrow} q(\mathcal{H}_1 \otimes \ell^2) \ ,$$

set $E \bigodot F = [\{x \bigodot y : x \in E, y \in F\}]$; and finally define the Connes fusion of $\mathcal{E}$ and $\mathcal{F}$ to be

$$\mathcal{E} \otimes_{A_2} \mathcal{F} = (E \bigodot F, (q(\pi_1 \otimes id_{\ell^2})|_{ran\ q}, q(\mathcal{H}_1 \otimes \ell^2)), (\rho_3, \mathcal{K}_3)) \ . \quad (3.6)$$

The justification for our use of 'Connes fusion' for our construction lies (at least for standard bimodules, by Lemma 2.5) in the fact that (in the notation defining Connes fusion) the A_3 - valued inner product on $\mathcal{E} \circ \mathcal{F}$ satisfies

$$\begin{aligned}
\langle x_1 \bigodot y_1, x_2 \bigodot y_2 \rangle_{A_3} &= (x_1 \bigodot y_1)^*(x_2 \bigodot y_2) \\
&= (x_1 \otimes id_{\ell^2})uy_1)^*(x_2 \otimes id_{\ell^2})uy_2 \\
&= y_1^* u^* (x_1^* x_2 \otimes id_{\ell^2})uy_2 \\
&= y_1^*(x_1^* x_2)y_2 \quad \text{(since } u \text{ is an } A_2 \text{ - linear isometry)} \\
&= y_1^* \langle x_1, x_2 \rangle_{A_2} y_2 \\
&= \langle y_1, \langle x_1, x_2 \rangle_{A_2} y_2 \rangle_{A_3} \ .
\end{aligned}$$

PROPOSITION 3.4. *The Connes fusion of (non-degenerate) Hilbert von Neumann bimodules is again a (non-degenerate) Hilbert von Neumann bimodule.*

Proof. Clearly $E \bigodot F$ is a WOT-closed linear space of operators between the asserted spaces. Observe next that

$$\begin{aligned}
\left[(E \bigodot F)(E \bigodot F)^*\right] &= [\{((x_1 \otimes id_{\ell^2})uy_1)(x_2 \otimes id_{\ell^2})uy_2)^* : x_i \in E, y_j \in F\}] \\
&= [\{(x_1 \otimes id_{\ell^2})uy_1y_2^*u^*(x_2 \otimes id_{\ell^2}) : x_i \in E, y_j \in F\}] \\
&= [\{(x_1 \otimes id_{\ell^2})u\,[FF^*]\,u^*(x_2 \otimes id_{\ell^2}) : x_i \in E\}] \\
&\supset [\{(x_1 \otimes id_{\ell^2})u\rho_2(A_2)u^*(x_2 \otimes id_{\ell^2}) : x_i \in E\}] \\
&= [\{(x_1 \otimes id_{\ell^2})(\pi_2(A_2) \otimes id_{\ell^2})uu^*(x_2 \otimes id_{\ell^2}) : x_i \in E\}] \\
&= [(E \otimes id_{\ell^2})uu^*(E \otimes id_{\ell^2})^*] \quad (\text{since } E\pi_2(A_2) = E) \\
&= q(\pi_1(A_1) \otimes id_{\ell^2})
\end{aligned}$$

(in particular $q \in [(E \bigodot F)(E \bigodot F)^*]$) and that

$$\begin{aligned}
\left[(E \bigodot F)^*(E \bigodot F)\right] &= [\{((x_1 \otimes id_{\ell^2})uy_1)^*(x_2 \otimes id_{\ell^2})uy_2) : x_i \in E, y_j \in F\}] \\
&= [\{(y_1^*u^*(x_1^*x_2 \otimes id_{\ell^2}))uy_2) : x_i \in E, y_j \in F\}] \\
&= [\{(y_1^*u^*(\pi_2(A_2) \otimes id_{\ell^2})uy_2) : y_j \in F\}] \\
&= [\{(y_1^*u^*u\,[\rho_2(A_2)]\,y_2) : y_j \in F\}] \\
&= [\{(y_1^*(\rho_2(A_2))y_2) : y_j \in F\}] \\
&= F^*F \quad (*) \\
&= \rho_3(A_3)\ ,
\end{aligned}$$

where the justification for the step labelled (*) is that $\rho_2(A_2)F = F$ (see Remark 1.5 (7). This completes the verification that $\mathcal{E} \otimes_{A_2} \mathcal{F}$ is indeed a Hilbert von Neumann $A_1 - A_3$ bimodule.

Now, suppose $\mathcal{E}$ and $\mathcal{F}$ are both non-degenerate. Then

$$\begin{aligned}
&\xi \in \bigcap\{ker z : z\ in\ E \bigodot F\} \\
&\quad\Rightarrow\quad (x \otimes id_{\ell^2})uy\xi = 0\ \forall x \in E, y \in F \\
&\quad\Rightarrow\quad uy\xi = 0\ \forall y \in F \quad (\text{as } E \otimes id_{\ell^2} \text{ is non-degenerate}) \\
&\quad\Rightarrow\quad y\xi = 0\ \forall y \in F \quad (\text{as } u \text{ is isometric}) \\
&\quad\Rightarrow\quad \xi = 0 \quad (\text{as } F \text{ is non-degenerate})\ ;
\end{aligned}$$

while

$$
\begin{aligned}
&\left[\bigcup\{ran((x \otimes id_{\ell^2})uy) : x \in E, y \in F\}\right] \\
&= \left[\bigcup\{ran((x \otimes id_{\ell^2})u) : x \in E\}\right] \quad \text{(since } F \text{ is non-degenerate)} \\
&= \left[\bigcup\{ran((x \otimes id_{\ell^2})uu^*) : x \in E\}\right] \\
&= ran\ q \quad \text{(by definition)}
\end{aligned}
$$

and hence $E \bigodot F$ is indeed non-degenerate.

□

Before addressing the question of the dependence of the definition of Connes fusion and the seemingly *ad hoc* A_2 - linear partial isometry u, we introduce a necessary definition and the ubiquitous lemma.

DEFINITION 3.5. *Two Hilbert von Neumann A_2 modules, say $\mathcal{E}^{(i)} = (E^{(i)}, \mathcal{H}_1^{(i)}, (\pi_2^{(i)}, \mathcal{H}_2^{(i)})), i = 1, 2$ are considered* isomorphic *if there exists unitary operators $w_j : \mathcal{H}_j^{(1)} \to \mathcal{H}_j^{(2)}$, with w_2 being A_2 - linear, such that*

$$E^{(2)} = w_1 E^{(1)} w_2^* .$$

If the $\mathcal{E}^{(i)}$ happen to be $A_1 - A_2$ bimodules, they are said to be isomorphic if, in addition to the above, the unitary w_1 happens to be A_1 - linear.

LEMMA 3.6. *Let $\mathcal{E} = (E, (\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$ be a Hilbert von Neumann $A_1 - A_2$ bimodule. Suppose $w \in \pi_2(A_2)'$ is a partial isometry with $w^*w = p, ww^* = \tilde{p}$. Let $q = \mathcal{E}_* p$ and $\tilde{q} = \mathcal{E}_* \tilde{p}$ in the notation of Lemma 3.2. Then there exists a unique partial isometry $w_1 \in \pi_1(A_1)'$ such that $w_1^* w_1 = q, w_1 w_1^* = \tilde{q}$.*

Proof. We first assert that there is a unique unitary operator $W_1 : q(\mathcal{H}_1) \to \tilde{q}(\mathcal{H}_1)$ satisfying $WTp = Tw\ \forall T \in E$. This is because:

- $(T_1 w)^*(T_2 w) = w^* T_1^* T_2 w = T_1^* T_2 p = p^* T_1^* T_2 p,\ \forall T_1, T_2 \in E$ and
- $q(\mathcal{H}_1) = [\bigcup\{ran(Tp) : T \in E\}]$ and $\tilde{q}(\mathcal{H}_1) = [\bigcup\{ran(Tw) : T \in E\}]$ (since $ran\ w = ran\ \tilde{p}$.

Finally $w_1 = W_1 q$ does the job. □

REMARK 3.7. 1. We now verify that the definition we gave of $\mathcal{E} \otimes_{A_2} \mathcal{F}$ is really independent of the choice of the isometry u used in that definition. Indeed, suppose $u, \tilde{u} : \mathcal{K}_2 \to \mathcal{H}_2 \otimes \ell^2$

are two A_2 - linear isometries. If $uu^* = p, \tilde{u}\tilde{u}^* = \tilde{p}$, then $w = \tilde{u}u^*$ is a partial isometry in $(\pi_2(A_2) \otimes id_{\ell^2})'$ with $w^*w = p, ww^* = \tilde{p}$. Now apply Lemma 3.6 to $\mathcal{E} \otimes id_{\ell^2}$ and $w, p, \tilde{p}$ to find a $W \in (\pi_1(A_1) \otimes id_{\ell^2})'$ such that $W^*W = q = (\mathcal{E} \otimes id_{\ell^2})_* p$ and $WW^* = \tilde{q} = (\mathcal{E} \otimes id_{\ell^2})_* \tilde{p}$. Then, as the proof of Lemma 3.6 shows, $W : q(\mathcal{H}_1 \otimes \ell^2) \to \tilde{q}(\mathcal{H}_1 \otimes \ell^2)$ is a unitary operator satisfying $W(x \otimes id_{\ell^2})p = (x \otimes id_{\ell^2})w \ \forall x \in E$. It is now a routine matter to verify that the unitary operators $W : q(\mathcal{H}_1 \otimes \ell^2) \to \tilde{q}(\mathcal{H}_1 \otimes \ell^2$ and $id_{\mathcal{K}_3}$ establish an isomorphism between the models of $\mathcal{E} \otimes_{A_2} \mathcal{F}$ given by u and $\tilde{u}$ are isomorphic.

2. A not dissimilar reasoning shows that the isomorphism type of the Connes fusion of teo standard bimodules depends only on the isomorphism classes of the two 'factors' in the fusion, and is also standard.

3. If $\mathcal{E}$ is only a Hilbert von Neumann A_2-module, and $\mathcal{F}$ is a Hilbert von Neumann $A_2 - A_3$-bimodule, their Connes fusion $\mathcal{E} \otimes_{A_2} \mathcal{F}$ would still make sense as a Hilbert von Neumann A_3-module.

4 Examples

We now discuss some examples of Hilbert von Neumann (bi)modules.

1. The simplest (non-degenerate) example is obtained when $A_j = \mathcal{L}(\mathcal{H}_j), \pi_j = id_{A_j}$ for $j = 1, 2$ and $E = \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$; all the verifications reduce just to matrix multiplication.

2. Suppose A_2 is a unital von Neumann subalgebra of A_1, and suppose there exists a faithful normal conditional expectation $\epsilon : A_1 \to A_2$. Let ϕ_2 be a faithful normal state (even semi-finite weight will do). Let $\phi_1 = \phi_2 \circ \epsilon, \mathcal{H}_j = L^2(A_j, \phi_j)$, and let π_j be the left regular representation of A_j on $\mathcal{H}_j$. Write U for the natural isometric identification of $\mathcal{H}_2$ as a subspace of $\mathcal{H}_1$ (so that the 'Jones projection' will be just UU^*). Finally, define

   $$\mathcal{E}_{(A_2 \subset A_1)} = (\pi_1(A_1)U, (\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$$

   In this case, we find that $[EE^*] = [\pi_1(A_1)e\pi_1(A_1)]$, and we find the 'basic construction of Jones appearing naturally in this context.

   Further, it is a consequence of the uniqueness assertion in Theorem 2.3 that $\mathcal{E}_\epsilon \cong \mathcal{E}_{(A_2 \subset A_1)}$.

3. Suppose $(M, \mathcal{H}, J, P)$ is a standard form of M in the sense of [Haa]. As indicated in [Haa], there is a canonical 'implementing' unitary representation

$$Aut(M) \ni \theta \mapsto u_\theta \in \mathcal{L}(\mathcal{H})$$

satisfying $u_\theta x u_\theta^* = \theta(x)\ \forall x \in M$. We have the natural Hilbert von Neumann $M-M$ bimodule given by

$$\mathcal{E}_\theta = (Mu_\theta, (id_M, L^2(M)), (id_M, L^2(M)))$$

4. If $\theta, \phi \in Aut(M), M$ are as in the previous example, we see now that 'Connes fusion corrsponds to composition' in this case:

$$\mathcal{E}_\theta \otimes_M \mathcal{E}_\phi \cong \mathcal{E}_{\theta\phi}$$

(*Reason:* The 'u' in the definition of Connes fusion is just id_M, while

$$Mu_\theta M u_\phi = M\theta(M)u_\theta u_\phi = Mu_{\theta\phi}\ .)$$

PROPOSITION 4.1. *If $\theta, \phi \in Aut(M)$ are as in Example (4) above, then $\mathcal{E}_\theta \cong \mathcal{E}_\phi$ if and only if θ and ϕ are inner conjugate.*

Proof. First, note that any M-linear unitary operator on $L^2(M)$ has the form Jv^*J for some unitary $v \in M$, where of course J denotes the modular conjugation operator. Observe next that each u_θ commutes with J since θ is a *-preserving map, and hence, for any $x \in M$, we have

$$u_\theta J v^* J = J\theta(v^*)Ju_\theta \tag{4.7}$$

If $\mathcal{E}_\theta$ is isomorphic to $\mathcal{E}_\phi$, there must exist unitary $v_1, v_2 \in M$ such that

$$\begin{aligned} Mu_\phi &= Jv_1^*JMu_\theta Jv_2J \\ &= MJv_1^*Ju_\theta Jv_2J \\ &= MJv_1^*JJ\theta(v_2)Ju_\theta \\ &= MJv_1^*\theta(v_2)Ju_\theta\ ; \end{aligned}$$

in particular, there must exist a $y \in M$ such that

$$u_\phi = yJv_1^*\theta(v_2)Ju_\theta\ .$$

We find that y is necessarily unitary and hence, writing u for y and v for $v_1^* \theta(v_2)$, we see that there must be a unitary $u \in M$ such that

$$\begin{aligned} \phi(x) &= u_\phi x u_\phi^* \\ &= uJvJu_\theta x u_\theta^* Jv^* Ju^* \\ &= uJvJ\theta(x)Jv^*Ju^* \\ &= u\theta(x)u^* . \end{aligned}$$

In other words, ϕ and θ are indeed inner conjugate.

Conversely, if $\phi(\cdot) = u\theta(\cdot)u^*$ for some unitary $u \in M$, we see that $u_\phi = uJuJu_\theta = uu_\theta J\theta^{-1}(u)J$; so we find that $w_1 = id_M$ and $w_2 = J\theta^{-1}(u)^*J$ define M - linear unitary operators on $L^2(M)$ such that $Mu_\phi = Muu_\theta w_2^* = w_1 Mu_\theta w_2^*$, thereby establishing that $\mathcal{E}_\theta \cong \mathcal{E}_\phi$. $\square$

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