von Neumann algebras and Ergodic Theory

V.S. Sunder
The Institute of Mathematical Sciences
INDIA

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There has been a long-standing and strong link between ergodic theory and von Neumann algebras (in particular, factors) dating back to the seminal work (cf. [vN]) of Murray and von Neumann, specifically their construction of the first examples of factors of type II and type III. The bridge is provided by the celebrated group-measure space construction (or the crossed-product construction in modern parlance). In this survey, we shall commence with a discussion of some aspects of the magnificent edifice created by Murray and von Neumann, Dye, Krieger, Connes, Ornstein, Weiss, Feldman, Moore, ..., and conclude with an attempt[1] to describe some 'rigidity' results of Gaboriau and Popa.

We commence proceedings with brief introductions to each of the topics von Neumann algeras, ergodic theory, the group-measure space construction and II_1 factors.

von Neumann algebras

A von Neumann algebra is a self-adjoint (i.e., $x \in M \Rightarrow x^* \in M$) unital (i.e., $1 \in M$) subalgebra M of the *-algebra $B(H)$ of all continuous linear operators on a Hilbert space[2] H, which satisfies any of the following equivalent requirements:[3]

1. M is closed in the *strong operator topology* - i.e., $x_i \in M, x \in B(H), \|(x_i - x)\xi\| \to 0 \forall \xi \in H \Rightarrow x \in M$

[1] It is only natural that the picture portrayed here is coloured/flawed by the author's own perceptions/limitations of exposure, and it is almost sure that there have been many grave omissions, for all of which only the author's limitations are to blame, and the author apologises for any such errors or omissions.

[2] All our Hilbert spaces will be assumed to be *separable*.

[3] The equivalence of these three conditions - two topological, one algebraic - is von Neumann's celebrated *double commutant theorem*.

2. M is closed in the *weak operator topology* - i.e., $x_i \in M, x \in B(H), \langle (x_i - x)\xi, \eta\rangle \to 0 \forall \xi, \eta \in H \Rightarrow x \in M$

3. $M''(= (M')') = M$, where $S' = \{x \in B(H) : xs = sx \forall s \in S\}$ denotes the *commutant* of S.

The prototypical example of an abelian von Neumann algebra is given by the algebra $A = L^\infty(X, \mathcal{B}, \mu)$ of essentially bounded measurable functions on a standard probability space $(X, \mathcal{B}, \mu)$, viewed as a subalgebra of $B(L^2(X))$ via $f \cdot \xi = f\xi \forall f \in A, \xi \in L^2(X)$. At the other extreme from an abelian von Neumann algebra is a *factor*, i.e., a von Neumann algebra whose center $M \cap M'$ reduces to the scalar operators $\mathbb{C}$.

It was recognised early that an important component to a von Neumann algebra is the set $\mathcal{P}(M) = \{p \in M : p = p^* = p^2\}$ of its projections. Just as all measurable functions can be approximated by simple functions, it is true that the linear subspace spanned by $\mathcal{P}(M)$ is norm-dense in M. Two projections p, q are said to be (Murray-von Neumann) equivalent 'rel M' - denoted by $p \sim_M q$ - if there exists a $u \in M$ such that $u^*u = p, uu^* = q$. It turns out that M is a factor if and only if any two projections are 'comparable' in the sense that one is equivalent to a sub-projection of the other. Murray and von Neumann initially classified factors into types I (there exists a minimal projection), II (there do not exist minimal projections, but there do exist non-zero projections which are finite meaning they are not equivalent to any strictly smaller sub-projection) and III (there do not exist non-zero finite projections).

(The material in this section first appeared in the papers of von Neumann, either singly authored or co-authored with Murray: see [vN].)

Ergodic theory

Ergodic theory deals with the study of transformations T on a measure space $(X, \mathcal{B}, \mu)$ - which we will always assume is a complete standard probability space; the map T is usually assumed to be bijective mod μ, bimeasurable and non-singular - i.e., there are μ-null sets N_1, N_2 such that T maps $X \setminus N_1$ 1-1 onto $X \setminus N_2$, and $E \in \mathcal{B} \Leftrightarrow T(E) \in \mathcal{B}$ and $\mu(T^{-1}(E)) = 0 \Leftrightarrow \mu(E) = 0$. A countable group Γ of such transformations γ is said to act *ergodically* if it satisfies any of the following equivalent conditions:

1. $\mu(\gamma^{-1}(E)\Delta E) = 0 \forall \gamma \in \Gamma \Rightarrow \mu(E) = 0$ or 1

2. $f = f \circ \gamma \forall \gamma \in \Gamma \Rightarrow f$ is constant a.e.

3. $E, F \in \mathcal{B}, \mu(E) > 0, \mu(F) > 0 \Rightarrow \exists \gamma \in \Gamma$ such that $\mu(F \cap \gamma(E)) > 0$.

Group-measure space construction

Suppose Γ is a countable group of non-singular transformations of a standard Borel space $(X, \mathcal{B})$, equipped with a σ-finite measure μ. Let $H = \ell^2(\Gamma, L^2(X, \mathcal{B}, \mu))$; the equations

$$\begin{aligned}(\pi(f)\widetilde{\xi})(\gamma) &= (f \circ \gamma)\widetilde{\xi}(\gamma) \\ (\lambda(\gamma_0)\widetilde{\xi})(\gamma) &= \widetilde{\xi}(\gamma_0^{-1}\gamma)\end{aligned}$$

respectively define a *-algebra representation of $A = L^\infty(X, \mathcal{B}, \mu)$ into $B(H)$ and a unitary representation of Γ into $B(H)$, and these representations satisfy the commutation relation

$$\lambda(\gamma)\pi(f) = \pi(f \circ \gamma^{-1})\lambda(\gamma) \tag{0.1}$$

The von Neumann algebra $M = (\lambda(\Gamma) \cup \pi(A))''$ generated by these two representations is denoted by $A \rtimes \Gamma$ and called the *crossed product* of A with Γ. Suppose the group Γ acts *freely*: i.e., for each $\gamma \neq 1$ in Γ, the set of points fixed by γ is assumed to be a μ-null set. Then, we have the following beautiful result due to von Neumann (cf. [vN] or [S]):

THEOREM 0.1. *$A \rtimes \Gamma$ is a factor if and only if Γ acts ergodically. Further, in this case:*

1. *The following conditions are equivalent:*

   (i) μ is atomic;

   (ii) $M = A \rtimes \Gamma$ has a minimal projection

   In this case, M is a factor of type I. Further M is said to be a factor of type $I_n, n \leq \infty$ if μ admits precisely n mutually disjoint atoms.

2. *The following conditions are equivalent:*

   (i) μ has no atoms, but there exists a σ-finite measure ν which is mutually absolutely continuous with μ, which is invariant under Γ (i.e., $\nu \circ \gamma^{-1} = \nu \forall \gamma$);

   (ii) M is type II

   In this case, 1 is a finite projection in M precisely when ν is a finite measure.

3. *M is type III if and only if there is no σ-finite measure ν which is mutually absolutely continuous with μ, which is invariant under Γ.*

Thus, we have our first examples of factors of type II - both type II_1 (which is type II with 1 being a finite projection) and type II_∞ (which is type II with 1 not being a finite projection) - and type III from the following examples of groups Γ acting ergodically on Lebesgue spaces:

- (II_1) $\Gamma = \mathbb{Z}$ acting on $(S^1, \mathcal{B}_{S^1}, \frac{1}{2\pi} d\theta)$ via $n.e^{2\pi i\theta} = e^{2\pi i(\theta + n\alpha)}$ with α being irrational.
- (II_∞) $\Gamma = \mathbb{Q}$ acting on $(\mathbb{R}, \mathcal{B}_\mathbb{R}, dx)$ via translation ($r.x = r + x$)
- (III) $\Gamma = \mathbb{Q} \rtimes \mathbb{Q}^\times$ acting on $(\mathbb{R}, \mathcal{B}_\mathbb{R}, m = dx)$ via $(b, a).x = ax + b$. (The point here is that Γ does not preserve the measure m, while the *proper* subgroup $\Gamma_0 = \{(b, 1)\} \subset \Gamma$ preserves m and itself acts ergodically, and such a group Γ cannot admit any σ-finite equivalent invariant measure.)

II_1 factors

Note that the only finite factors are the factors of type $I_n, n < \infty$ or of type II_1. It is a fact that a factor M is of finite type if and only if it admits a *trace*, i.e., a linear functional tr such that $tr(1) = 1$ and $tr(xy) = tr(yx)\ \forall x, y \in M$ and $tr(x^*x) \geq 0\ \forall x \in M$; further, such a trace is automatically faithful ($0 \neq x \in M \Rightarrow tr(x^*x) > 0$) and unique. A type I_n factor is isomorphic to the full matrix algebra $M_n(\mathbb{C})$, and the corresponding 'tr' is nothing but the usual matrix trace normalised by a factor of $1/n$. On the othere hand II_1 factors are infinite-dimensional, but their 'finiteness' results in many pleasant features.

What is also true of a finite factor is that if $p, q \in \mathcal{P}(M)$, then $p \sim_M q \Leftrightarrow tr(p) = tr(q)$. While the set $\{tr(p) : p \in \mathcal{P}(M)\}$ is nothing but $\{k/n : 0 \leq k \leq n\}$ in the I_n case, it turns out to be $[0, 1]$ in the II_1 case. A Hilbert space equipped with a normal (= appropriately continuous) *-representation of a II_1 factor M is referred to as an M-module. It turns out (as a perfect parallel with the case of I_n factors) that a module $\mathcal{H}$ over a II_1 factor M is classified, up to M-linear isomorphism, by a number $dim_M\mathcal{H}$ (which can be any number in $[0, \infty]$), the so-called *von Neumann dimension as an M-module.*

If $\lambda : \Gamma \to \ell^2(\Gamma)$ denotes the *left-regular representation* of a countable group Γ, then the equation

$$tr(x) = \langle x1, 1 \rangle$$

defines a faithful trace on the von Neumann algebra $L\Gamma = \lambda(\Gamma)''$ where 1 denotes the standard basis vector indexed by the identity element of Γ; and $L\Gamma$ is a II_1 factor if and only if the conjugacy class of every $\gamma \neq 1$ is infinite (*the ICC condition*).

Almost all the material, so far, in this section, is from the seminal work of von Neumann ([vN]). Some of the details, in slightly more modern terminology, may also be found in [S], for instance.

Two questions:

1. *What pairs of algebras* (M, A) *arise in the above manner?*

2. *When do two ergodic dynamical systems* $(X_i, \mathcal{B}_i, \mu_i, \Gamma_i), i = 1, 2$ *yield isomorphic pairs* (M_i, A_i) *as above?*

The first question, or rather, a near relative (where one considers more general *crossed-products twisted by a 2-cocycle*) has been answered very satisfactorily in [FMII], and the answer turns out to be: precisely when A is a *Cartan subalgebra* of M - meaning that it has the following properties:

- A is a maximal abelian von Neumann subalgebra of M;
- The *normaliser* $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$ (where $\mathcal{U}(M) = \{u \in M : u^*u = uu^* = 1\}$ is the unitary group of M) generates M as a von Neumann algebra: i.e., $M = \mathcal{N}_M(A)''$; and
- there exists a *faithful conditional expectation* of M onto A.

We shall say no more about the first question, since our concern is primarily with the second, whose answer turns out to be:

if and only if the two actions are orbit equivalent

The notion of orbit (or weak-) equivalence (see definition below) was introduced (and the validity of the answer established) in the measure-preserving context by Dye (cf.[DI], [DII])) and studied (and the validity of the answer established) in the non-singular case by Krieger (cf.[KriI], [KriII]).

Before getting to the pertinent definitions, we first make two blanket assumptions for the remainder of this paper.

All our measure spaces $(X, \mathcal{B}, \mu)$ will henceforth be assumed to be complete standard probability spaces equipped with a non-atomic probability measure; 'isomorphisms between such triples are bijective (mod null sets), bimeasurable measure preserving transformations.

Definition 0.2. 1. *An* isomorphism *between two spaces $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$ is a bijective bimeasurable map $\phi : X_1 \setminus N_1 \to X_2 \setminus N_2$, for μ_i-null sets N_i, such that $\mu_1 \circ \phi^{-1} = \mu_2$.*

2. *A dynamical system is a tuple $(X, \mathcal{B}, \mu, \alpha, \Gamma)$ where Γ is a countable group, and $\alpha : \Gamma \to Aut(X, \mathcal{B}, \mu)$ is a homomorphism of groups.*

3. *Two dynamical systems $(X_i, \mathcal{B}_i, \mu_i, \alpha_i, \Gamma_i), i = 1, 2$ are* conjugate *if there exists an isomorphism $\phi : X_1 \to X_2$ such that $\alpha_2(\Gamma_2) = \phi\alpha_1(\Gamma_1)\phi^{-1}$.*

4. *Two dynamical systems $(X_i, \mathcal{B}_i, \mu_i, \alpha_i, \Gamma_i), i = 1, 2$ are* orbit equivalent *if there exists an isomorphism $\phi : X_1 \to X_2$ such that $\phi(\alpha_1(\Gamma_1)x) = \alpha_2(\Gamma_2)\phi(x)$ for μ_1- a.a x.*

Every dynamical system $(X, \mathcal{B}, \mu, \alpha, \Gamma)$ gives rise to an equivalence relation - which we shall denote by $\mathcal{R}_\Gamma$ or $\mathcal{R}_\alpha$ - which is the Borel subset of $X \times X$ given by $\{(x, \alpha(\gamma)(x)) : x \in X, \gamma \in \Gamma\}$. This equivalence relation has countable equivalence classes. In fact, a result of [FMI] shows that any such standard equivalence relation (with countable classes) arises as *orbit equivalence* defined by a countable group Γ acting as Borel isomorphisms of $(X, \mathcal{B})$ - *although not necessarily freely* according to a result of Furman.

Question 2 above may be viewed as asking when two dyanamical systems are orbit equivalent - i.e., when is there a Borel isomorphism $f : X_1 \to X_2$ such that $(f \times f)(\mathcal{R}_{\alpha_1}) = \mathcal{R}_{\alpha_2}$. Dye showed [D1] that any two ergodic actions of $\mathbb{Z}$ are so isomorphic. A volume of work by several people (notably Dye, Connes, Feldman, Krieger, Vershik, ...) culminated in the following beautiful result proved (cf. []) by Ornstein and Weiss (cf. [OW], see also [CFW])).

Theorem 0.3. *(**Ornstein-Weiss**) If Γ_1 and Γ_2 are infinite amenable groups, every ergodic action of Γ_1 is orbit equivalent to every ergodic action of Γ_2.*

Equivalence relations obtained from such actions of such groups are characterised by the following property of **hyperfiniteness***:*

there exists a sequence of standard equivalence relations $\mathcal{R}_n$ on X with finite equivalence classes such that

$$\mathcal{R}_n \subset \mathcal{R}_{n+1} \forall n \text{ and } \mathcal{R} = \cup_n \mathcal{R}_n.$$

Thus $\mathcal{R}_\Gamma$ remembers neither Γ nor α if Γ is an infinte amenable group and α is an ergodic action. On the other hand, at the other end of the spectrum, many people (Zimmer, Furman, Gaboriau, and later Popa, Monod, Ozawa, ...) have obtained 'rigidity results' which say something like this: if $\mathcal{R}_{\alpha_i}$ are orbit equivalent, then under some conditions on the Γ_i, these two dynamical systems must actually be conjugate! (For an example, see Popa's *strong rigidity theorems* (cf. [V]), which say something like this:

Certain kinds of free ergodic actions of certain kinds of groups G are such that if the resulting equivalence relation $\mathcal{R}$ has the property that $\mathcal{R}_Y$ is isomorphic to $\mathcal{R}_\Gamma$ for some Borel subset Y and some free ergodic action of some countable group Γ, then Y must have full measure, and the actions of Γ and G must be conjugate through a group isomorphism.

It follows that for a relation $\mathcal{R}$ as in this strong rigidity theorem, the restriction $\mathcal{R}_Y$ to a Borel subset with $0 < \mu(Y) < 1$ can never be obtained from a free ergodic action of any countable group Γ, thus furnishing another proof of Furman's result mentioned earlier.

The key notions used in Gaboriau's work are *stable orbit equivalence*, *measurable equivalence* and ℓ^2*-Betti numbers*, upon which we now briefly dwell.

It is well known that if the action is ergodic, then the 'space of orbits' (= the quotient of X by the relation of being in the same orbit) does not have a 'good Borel structure', i.e., is not standard. The space $\mathcal{R}$ is a good substitute. Now, if A is a Borel subset of positive measure in X, then A meets almost every orbit, so by the philosophy expressed in the previous sentence, the *induced relation* $\mathcal{R}_A := \mathcal{R} \cap (A \times A)$ is an equally good description of the 'space of orbits'. Let us call ergodic equivalence relations $\mathcal{R}_i$ on standard probability spaces $(X_i, \mathcal{B}_i, \mu_i)$ (for $i = 1, 2$) *stably orbit equivalent* (or simply SOE) if there exist Borel subsets $A_i \in \mathcal{B}_i$ of positive measure, a positive constant c and a Borel isomorphism $f : A_1 \to A_2$ such that $\mu_2 \circ f = c\mu_1$ on A_1 and $(f \times f)(\mathcal{R}_{A_1}) = \mathcal{R}_{A_2}$; and c is called the compression constant of the SOE.

On the other hand, call two countable groups $\Gamma_i, i = 1, 2$ *measurably equivalent* (or simply ME) if they admit commuting free actions on a standard measure space $(X, \mathcal{B}, \mu)$[4] which admit a *fundamental domain* F_i of finite measure; call the ratio $\frac{\mu(F_2)}{\mu(F_1)}$ the compression constant of the ME.

The two notions of equivalence defined in the preceding paragraphs turn out to be closely related, and we have the following result, proved originally by Furman (cf. [Fu], [G]):

THEOREM 0.4. *Γ_1 is ME to Γ_2 with compression constant c if and only if Γ_1 and Γ_2 admit free actions on standard probability space such that the associated equivalence relations are SOE with compression constant c.*

Now, we briefly discuss ℓ^2-Betti numbers. These were first introduced by Atiyah in the context of actions of countable groups on manifolds with compact quotients; he relied on the von Neumann dimension $dim_{L\Gamma}\mathcal{H}_n$ of the Hilbert space of harmonic L^2-forms of degree n, which has the structure of a module over the von Neumann algebra $L\Gamma$ (generated by the regular representation of Γ). This was later considerably extended by Cheeger and Gromov, who studied actions of countable groups on general topological spaces, and succeeded in defining the sequence $\{\beta_n(\Gamma)\}$ of ℓ^2-Betti numbers of any countable group.

Next, Gaboriau defined the ℓ^2-Betti numbers $\beta_n(\mathcal{R})$ of any standard equivalence relation with invariant measure. He was helped in this by the work of Feldman and Moore, where a von Neumann algebra $L\mathcal{R}$ with a finite faithful normal trace had been naturally associated to a standard equivalence relation with invariant probability measure. (If $\mathcal{R} = \mathcal{R}_\Gamma$ for an ergodic action preserving a probability measure space $(X, \mathcal{B}, \mu)$, then $L\mathcal{R}$ is just the II_1 factor given by the crossed product construction.) Gaboriau considers a *universal $\mathcal{R}$-simplicial complex $E\mathcal{R}$* and essentially observes that the space of ℓ^2-chains has a natural structure of an $L\mathcal{R}$-module, defines $\beta_n(\mathcal{R})$ as the *$L\mathcal{R}$-dimension* of the corresponding reduced ℓ^2-homology groups of $E\mathcal{R}$, and proves:

THEOREM 0.5. *(Gaboriau)*

If an equivalence relation $\mathcal{R}$ is produced by a free action of Γ which preserves a probability measure, then

$$\beta_n(\mathcal{R}) = \beta_n(\Gamma).$$

[4]Here the measure is allowed to be infinite (but should be σ-finite).

Gaboriau goes on to prove that the ratio of corresponding ℓ^2-Betti numbers of two ME groups agrees with the compression constant of the ME.

Thus we find that if free actions of countable groups Γ_j yield equivalence relations $\mathcal{R}_j, j = 1, 2$ which are orbit equivalent, and hence SOE with compression constant 1, then the groups Γ_j must be ME with compression constant 1.

Coming back to orbit equivalence, we deduce the following fact from the foregoing discussion:

The ℓ^2-Betti numbers of orbit equivalent free actions are equal.

The simplest example of groups in the same ME class is furnished by any two lattices, not necessarily co-compact, of a locally compact second countable group (as seen by their actions by left-, resp., right-multiplications on the ambient group). Gaboriau obtains many rigidity results, a sample being:

COROLLARY 0.6. *(Gaboriau)*

1. *No lattice in $SP(n, 1)$ is ME to a lattice in $SP(p, 1)$ if $n \neq p$.*
2. *No lattice in $SU(n, 1)$ is ME to a lattice in $SU(p, 1)$ if $n \neq p$.*
3. *No lattice in $SO(2n, 1)$ is ME to a lattice in $SO(2p, 1)$ if $n \neq p$.*

Proof: It is known from the work of Borel (cf. [Bor]) that

$$\begin{aligned} \beta_i(\Gamma(SP(m,1)) \neq 0 &\quad \Leftrightarrow \quad i = 2m \\ \beta_i(\Gamma(SU(m,1)) \neq 0 &\quad \Leftrightarrow \quad i = m \\ \beta_i(\Gamma(SO(2m,1)) \neq 0 &\quad \Leftrightarrow \quad i = m \end{aligned}$$

where we write $\Gamma(G)$ to denote any lattice G.

Finally, we should mention that Gaboriau's results have been used ingeniously by Sorin Popa to settle a long-standing conjecture of Kadison's - regarding the existence of II_1 factors with trivial fundamental group.

If M is a II_1 factor, there is a natural definition of the so-called *amplification* $M_d(M)$ (or the $d \times d$ matrix algebra over M) where d is any posive real number. For instance, it may be identified with the (II_1 factor $End_M(\mathcal{H}_d)$ of) M-linear operators on the M-module $\mathcal{H}_d$ with $dim_M \mathcal{H}_d = d$. von Neumann already realised the importance of the object, called the *fundamental group*[5] $\mathcal{F}(M)$ of M, and defined by

$$\mathcal{F}(M) = \{d > 0 : M \cong M_d(M)\}.$$

Popa showed that there are many examples of II_1 factors of the form $L\mathcal{R}_\alpha$ (arising from free ergodic actions α of suitable ICC groups) which do indeed have trivial fundamental group. An example of such an action is the natural action of $SL(2, \mathbb{Z})$ on $\mathbb{T}^2$. In fact, Gaboriau and Popa have even shown (cf. [GP]) that (each finitely generated non-abelian free group) $\mathbb{F}_n$ admits uncountably many free ergodic actions α_i preserving a probability masure, which are pairwise not SOE, such that $L\mathcal{R}_{\alpha_i}$ has trivial fundamental group for each i. Much more of the subsequent exciting developments, as well as pertinent literature, may be found in the article [V] by Vaes.

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[5] $\mathcal{F}(M)$ is a multiplicative subgroup of $\mathbb{R}^\times$.

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