von Neumann algebras and Free Probability RMS meeting, ISI Bengaluru, May 13 2009

V.S. Sunder
Institute of Mathematical Sciences
Chennai, India
sunder@imsc.res.in

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The von Neumann algebra $L\Gamma$

For a countable group Γ (typically non-commutative), the closures, in the norm- and strong-operator topologies of the algebra of operators generated by the left-regular reresentation λ of Γ in $B(\ell^2(\Gamma))$ are the **reduced group C^*-algebra** $C^*_{red}(\Gamma)$, and the **group von Neumann algebra**

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The counterpart of Lebesgue measure for the *non-commutative probability space* $M = L\Gamma$ is provided by the *positive, faithful, normal, tracial state* - definitions in next slide - on M defined by

$$tr(x) = \langle x\xi_1, \xi_1\rangle\ ,\ x \in M$$

Definitions

1. A **C^*-algebra** A is a strongly closed self-adjoint algebra of operators on a (usually separable) Hilbert space $\mathcal{H}$.
2. A **von Neumann algebra** M is a strongly closed self-adjoint algebra of operators on a (usually separable) Hilbert space $\mathcal{H}$.
3. A linear functional ϕ on a C^*-algebra A is said to be **positive** if $\phi(x^*x) \geq 0 \ \forall x \in A$.
4. A positive linear functional ϕ on a C^*-algebra A is said to be a **state** if $\|\phi\| = 1$.
5. A state ϕ on a C^*-algebra A is said to be **faithful** if $\phi(x^*x) > 0 \ \forall \ 0 \neq x \in A$.
6. A state ϕ on a von Neumann algebra M is said to be **normal** if it is strongly continuous or equivalently if it satisfies the monotone convergence theorem[1].
7. A linear functional ϕ on an algebra is said to be a **trace** if $\phi(xy) = \phi(yx) \ \forall x, y$.

[1] i.e., it preserves suprema of monotonically increasing uniformly bounded sequences of positive self-adjoint operators

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NCPS come in various flavours:

- A may be just an algebra.
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Theorem

If a von Neumann algebra M admits a unique tracial state, then M has trivial center : i.e., $M \cap M' = \mathbb{C}$. Such an M is called a II_1 **factor** *if it is infinite-dimensional.*

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Example: If all non-trivial conjugacy classes of Γ are infiinite, then $L\Gamma$ is a II_1 factor.

Group factors

- The group $\Sigma_\infty = \cup_{n=1}^{\infty}\Sigma_n$ of permutations of $\mathbb{N}$ which move only finitely many integers is an ICC ('infinite conjugacy class') group, and the associated II_1 factor R is manifestly **hyperfinite** in the sense of being the strong closure of an increasing union of finite-dimensional C^*-algebras; this factor is, up to isomorphism, the unique hyperfinite II_1 factor.

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- $\Gamma = \mathbb{F}_n, \ n \geq 2$, are clearly ICC groups; and $L\mathbb{F}_n$ is known to not be hyperfinite; the big open problem :

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The quest to a possible solution to the above preoblem led Voiculescu to his theory of **free probability**.

A family of subalgebras $A_i, i \in I$ of a NCPS (A, ϕ) are said to be **free** (or freely independent) if whenever $x_j \in A_{i_j}, 1 \leq j \leq n$ satisfy $i_j \neq i_{j+1}$ $\forall 1 \leq j < n$ and $\phi(x_j) = 0 \forall j$, then necessarily also $\phi(x_1 x_2 \cdots x_n) = 0$.

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Theorem

Given a family (A_i, ϕ_i) *of NCPS of* the same flavour, *there exixts an NCPS* (A, ϕ) *also with the same flavour and the following properties:*

- *there exist monomorphisms* $\pi_i : A_i \rightarrow A$ *such that* $\phi_i = \phi \circ \pi_i \ \forall i$*; and*
- *given homomorphisms* $\psi_i : A_i \rightarrow B$ *for some NCPS* (B, τ) *(of the same flavour) such that* $\tau \circ \psi_i = \phi_i \ \forall i$*, there exists a unique morphism* $\rho : A \rightarrow B$ *such that* $\rho \circ \pi_i = \psi_i \ \forall i$ *and* $\tau \circ \rho = \phi$*.*
- *the NCPS* (A, ϕ) *is unique up to isomorphism and is denoted* $(A, \phi) = *_{i \in I}(A_i, \phi_i)$ *and is called the free product of the family* $\{(A_i, \phi_i) : i \in I\}$*.*

Free independence

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Example

$$L\mathbb{F}_n \cong *^n L\mathbb{Z}$$

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Given a finite subset $S \subset \mathbb{N}$, a partition π of S is said to be *non-crossing* if whenever $i < j$ and $k < l$ belong to distinct classes of π, then neither is $k < i < l < j$ nor is $i < k < j < l$. The collection $NC(S)$ of all such partitions is a lattice with respect to the (reverse-) refinement order: $\pi \geq \rho$ if π is coarser than ρ or equivalently, if ρ refines π. Thus the largest element of $NC(S)$ is the trivial partition $1_S = \{S\}$

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Using the definition and moments to check free indepenence of a family of subalgebras os an NCPS is not easy; but fortunately, the computations become easier in terms of the so-called **free cumulants**. Before getting to them, we need a digression.

Given a finite poset (= partially ordered set) X, its **Incidence Algebra** is

$$I(X) = \{f : X \times X \to \mathbb{C} | f(x,y) \neq 0 \Rightarrow x \leq y\}$$

and its *defining function* is

$$\zeta(x,y) = \begin{cases} 1 & \textit{if } x \leq y \\ 0 & \textit{otherwise} \end{cases}$$

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First list the members of X so that x is listed before y if $x < y$; then $I(X)$ may be identified with a subalgebra of the algebra of upper triangular matrices, and consequently inhertits a natural algebra structure; clearly $f \in I(X)$ is invertible (i.e., is represented by an invertible matrix) if and only if $f(x,x) \neq 0 \ \forall x$.
Define the **Möbius function** of X by the element of $I(X)$ which is the inverse of ζ. The next fact is a direct consequence of the definitions.

Möbius inversion in posets

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Theorem (Möbius inversion in X)

If $f, g \in I(X)$, the following conditions are equivalent:

- $f(x,z) = \sum_{x \leq y \leq z} g(y,z) \ \forall x \in X$ *(or $f = \zeta * g$)*
- $g(x,y) = \sum_{x \leq y \leq z} \mu(x,y) f(y,z)$ *(or $g = \mu * f$)*

Example

1. $X = [n]$ with usual ordering; $\mu(x,y) = \begin{cases} 1 & \textit{if } x = y \\ -1 & \textit{if } x = y-1 \\ 0 & \textit{otherwise} \end{cases}$
2. $X = [n], d \leq k \Leftrightarrow d|k$;
   $\mu(x,y) =$
   $\begin{cases} 1 & \textit{if } x = y \\ (-1)^k & \textit{if } y/x \text{ is square-free, and a product of } k \text{ distinct primes} \\ 0 & \textit{otherwise} \end{cases}$
3. $X = 2^S$ is the set of all subsets of a set S, ordered by inclusion;
   $\mu(E,F) = \begin{cases} 1 & \textit{if } E = F \\ (-1)^{|F \setminus E|} & \textit{if } E \subset F \\ 0 & \textit{otherwise} \end{cases}$

Given a set X, and an an arbitrary family of $\{\phi_n : X^n \to \mathbb{C} | n \in \mathbb{N}\}$, the associated **multiplicative extension** is the family of functions $\{\phi_\pi : X^n \to \mathbb{C}, n \in \mathbb{N}, S \subset [n] = \{1, \cdots, n\}\}$ defined by

$$\phi_\pi(x_1, \cdots, x_n) = \prod_{C \in NC([n]} \phi_{|C|}(x_C : C \in \pi)$$

where the arguments of $\phi_{|C|}$ are listed in increasing order. (So $\phi_n = \phi_{1_{[n]}}$.)

More Möbius inversion

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Lemma

Given a set X and two collections of functions $\{\phi_n : X^n \to \mathbb{C}\}_{n \in \mathbb{N}}$ and $\{\kappa_n : X^n \to \mathbb{C}\}_{n \in \mathbb{N}}$, which are extended multiplicatively, the following conditions are equivalent:

1. $\phi_n = \sum_{\pi \in NC([n])} \kappa_\pi$ *for all* $n \in \mathbb{N}$.
2. $\kappa_n = \sum_{\pi \in NC([n])} \mu(\pi, 1_{[n]} \phi_\pi$ *for all* $n \in \mathbb{N}$.
3. $\phi_\tau = \sum_{\pi \in NC([n]), \pi \leq \tau} \kappa_\pi$ *for all* $n \in \mathbb{N}, \pi \in NC([n])$.
4. $\kappa_\tau = \sum_{\pi \in NC([n]), \pi \leq \tau} \mu(\pi, \tau) \phi_\pi$ *for all* $n \in \mathbb{N}, \pi \in NC([n])$.

Here, the symbol μ denotes the Möbius function associated to the lattice $NC([n])$.

The advertised free cumulant description of freeness is at hand.

Definition

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Theorem

Subalgebras (A_i, ϕ_i) of an NCPS (A, ϕ) are freely independent if and only if the free cumulants satisfy $\kappa_n(x_1, x_2, \cdots, x_n) = 0$ whenever each x_j comes from some A_{i_j} and at least two i_j's are distinct.

Suppose $x_i \in (A_i, \phi_i), i = 1, 2$ are self-adjoint elements in von Neumann NCPS; then there exist unique probability measures μ_i (the distribution of x_i) supported on $sp(x_i)$ such that

$$\phi_i(x_i^n) = \int_{\mathbb{R}} t^n d\mu_i(t) \ .$$

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The **free convolution** $\mu_1 \boxplus \mu_2$ of μ_1 and μ_2 is the distribution in (A, ϕ) of the self-adjoint element $x_1 + x_2$, where $(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$.

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In the considerable literature of free probability, Wigner's **semi-circular disttribution** occupies pride of place that is accorded to the normal distribution in classical probability.