

On a tensor-analogue of the Schur product

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October 15, 2015

Abstract

We consider the *tensorial Schur product* $R \circ^{\otimes} S = [r_{ij} \otimes s_{ij}]$ for $R \in M_n(\mathcal{A}), S \in M_n(\mathcal{B})$, with \mathcal{A}, \mathcal{B} unital C^* -algebras, verify that such a ‘tensorial Schur product’ of positive operators is again positive, and then use this fact to prove (an apparently marginally more general version of) the classical result of Choi that a linear map $\phi : M_n \rightarrow M_d$ is completely positive if and only if $[\phi(E_{ij})] \in M_n(M_d)^+$, where of course $\{E_{ij} : 1 \leq i, j \leq n\}$ denotes the usual system of matrix units in $M_n(= M_n(\mathbb{C}))$. We also discuss some other corollaries of the main result.

1 The result

We start with some notation: (We assume, for convenience, that all our C^* -algebras are unital.) We denote an element of a matrix algebra by capital letters, such as R , and denote its entries by either $[R]_{ij}$ or the corresponding lower case letter r_{ij} . This is primarily because $[R^*]_{ij} = (r_{ji})^* \neq [R^*]_{ji}$!

DEFINITION 1.1. 1. If \mathcal{A}, \mathcal{B} are C^* -algebras, and $\phi : M_n \rightarrow \mathcal{B}$ is a positive map, define $\phi_{\mathcal{A}} : \mathcal{A} \otimes M_n \rightarrow \mathcal{A} \otimes_{alg} \mathcal{B}$ (where $\mathcal{A} \otimes_{alg} \mathcal{B}$ denotes the algebraic tensor product of \mathcal{A} and \mathcal{B}) by $\phi_{\mathcal{A}} = id_{\mathcal{A}} \otimes \phi$.

2. If $A = [a_{ij}] \in M_n(\mathcal{A}), B = [b_{ij}] \in M_n(\mathcal{B})$, define $A \circ^\otimes B = [a_{ij} \otimes b_{ij}] \in M_n(\mathcal{A} \otimes_{alg} \mathcal{B})$.

For later use, we isolate a lemma, whose elementary verification we omit.

LEMMA 1.2. *The map $\pi : M_n(\mathcal{A}) \otimes M_k \rightarrow M_{nk}(\mathcal{A})$ defined by*

$$[\pi(R \otimes C)]_{i\alpha, j\beta} = c_{\alpha\beta} r_{ij} \quad (1.1)$$

is a C^ -algebra isomorphism for any C^* -algebra \mathcal{A} ; in the sequel, we shall simply use this π to make the identification $M_n(\mathcal{A}) \otimes M_k = M_{nk}(\mathcal{A})$. In particular, $\mathcal{A} \otimes M_k = M_k(\mathcal{A})$.*

REMARK 1.3. *There is clearly a right version of the above Lemma: i.e., $M_k \otimes M_n(\mathcal{A}) = M_{kn}(\mathcal{A})$.*

PROPOSITION 1.4.

$$R \in M_n(\mathcal{A})^+, S \in M_n(\mathcal{B})^+ \Rightarrow R \circ^\otimes S \in M_n(\mathcal{C})^+, \quad (1.2)$$

where \mathcal{C} denotes - here and in the rest of this short note - any C^* -algebra containing $\mathcal{A} \otimes_{alg} \mathcal{B}$. In particular,

$$\sum_{i,j=1}^n r_{ij} \otimes s_{ij} \in M_n(\mathcal{C})^+. \quad (1.3)$$

Proof. To deduce eqn. (1.3) from eqn. (1.2), we let $1_n \in M_{n \times 1}(\mathcal{C})^+$ be the $n \times 1$ column-vector with all entries equal to $1_{\mathcal{A}} \otimes 1_{\mathcal{B}}$, and note that

$$\sum_{i,j=1}^n r_{ij} \otimes s_{ij} = 1_n^*(R \circ^\otimes S)1_n.$$

Now for the slightly less immediate eqn. (1.2). By assumption, $R \otimes S \in M_{n^2}(\mathcal{C})^+$.

In the sequel all the variables i, j, k, l, p, q will range over the set $\{1, 2, \dots, n\}$ and we shall simply write \sum_k for $\sum_{k=1}^n$.

Now define $V \in M_{n \times n^2}(\mathcal{C})$ by

$$[V]_{i,pq} = \delta_{pi} \delta_{qi} (1_A \otimes 1_B).$$

Then,

$$\begin{aligned}
[V(R \otimes S)V^*]_{ij} &= \sum_{p,q,k,l} [V]_{i,pq} [R \otimes S]_{pq,kl} [V^*]_{kl,j} \\
&= [R \otimes S]_{ii,jj} \\
&= r_{ij} \otimes s_{ij}
\end{aligned}$$

and so,

$$V(R \otimes S)V^* = R \circ^{\otimes} S.$$

The proof of the Proposition is complete. \square

REMARK 1.5. *Note that the proof shows that $R \circ^{\otimes} S \in M_n(\mathcal{A} \otimes_{alg} \mathcal{B})$.*

The classical result of Choi alluded to in the abstract is the equivalence $2. \Leftrightarrow 3.$ in the following Corollary, for the case $\mathcal{B} = M_d$ (see [1]).

COROLLARY 1.6. *The following conditions on a linear map $\phi : M_n \rightarrow \mathcal{B}$ are equivalent:*

1. *For any C^* -algebra \mathcal{A} , the map $\phi_{\mathcal{A}}(:= id_{\mathcal{A}} \otimes \phi) : \mathcal{A} \otimes M_n \rightarrow \mathcal{C}$ is a positive map for any C^* -algebra \mathcal{C} as in Proposition 1.4.*
2. *The map ϕ is CP.¹*
3. *$[\phi(E_{ij})] \in M_n(\mathcal{B})^+$.*

Proof. We only prove the non-trivial implication $3. \Rightarrow 1.$ if $R \in (\mathcal{A} \otimes M_n)^+ = M_n(\mathcal{A})^+$, and if $R = [r_{ij}]$, then

$$\begin{aligned}
\phi_{\mathcal{A}}(R) &= (id_{\mathcal{A}} \otimes \phi) \left(\sum_{ij} r_{ij} \otimes E_{ij} \right) \\
&= \sum_{ij} r_{ij} \otimes \phi(E_{ij}) \\
&\in M_n(\mathcal{C})^+,
\end{aligned}$$

by eqn. (1.3). \square

¹For an explanation of terms like CP (= completely positive) and operator system, the reader may consult [3], for instance.

COROLLARY 1.7. *Let $R \in M_n(\mathcal{A})^+$. Then the map $M_n(\mathcal{B}) \ni S \xrightarrow{L_R} R \circ^\otimes S \in M_n(\mathcal{C})$ is CP. In particular $R \in M_n^+ \Rightarrow M_n \ni S \rightarrow R \circ S \in M_n$ is also CP.*

Proof. To avoid confusion, we use Greek letters α, β etc., to denote elements of $\{1, 2, \dots, k\}$ and English letters i, j etc. to denote elements of $\{1, 2, \dots, n\}$. Suppose $[\hat{S}] \in M_{kn}(\mathcal{B})^+ = M_k(M_n(\mathcal{B}))^+$ is given by $[\hat{S}]_{\alpha i, \beta j} := [S_{\alpha, \beta}]_{i, j}$ (see Lemma 1.2), where of course $S_{\alpha\beta} \in M_n(\mathcal{B}) \forall \alpha, \beta$. Let $J_k \in M_k$ be (the all 1 matrix) given by $[J_k]_{\alpha\beta} = 1 \forall \alpha, \beta$. Then we see that $J_k \geq 0$ (in fact J_k/k is a projection) and so,

$$\begin{aligned} [L_R(S_{\alpha\beta})] &= [[r_{ij} \otimes [S_{\alpha\beta}]_{ij}]] \\ &= [[[J_k]_{\alpha\beta} r_{ij} \otimes [S_{\alpha\beta}]_{ij}]] \\ &= [[J_k \otimes R]_{\alpha i, \beta j} \otimes [\hat{S}]_{\alpha i, \beta j}] \quad (\text{see Remark 1.3}) \\ &= (J_k \otimes R) \circ^\otimes \hat{S} \\ &\geq 0, \end{aligned}$$

by Proposition 1.4 applied with R, n, S there replaced by $J_k \otimes R, kn, \hat{S}$, since $[J_k \otimes R]_{\alpha i, \beta j} = r_{ij} \forall \alpha, \beta$. The second statement of the Corollary is just the specialisation of the first statement to $\mathcal{A} = \mathcal{B} = \mathbb{C}$. \square

REMARK 1.8. *The special case $n = 1$ of Corollary 1.7 perhaps merits singling out: If $r \in \mathcal{A}^+$, then the map $\mathcal{B} \ni s \xrightarrow{L_r} r \otimes s \in \mathcal{C}$ is CP.*

REMARK 1.9. *It should be clear that there is a ‘right’ version of all the ‘left’ statements discussed above.*

The proofs suggest that these results might well admit formulations in the language of operator systems; however, we suspect that such ‘generalisations’ will follow from nuclearity of M_n and the flexibility in the choice of \mathcal{C} in our formulation, in view of the Choi-Effros theorem (see [2]).

References

- [1] M. D. Choi, *Completely positive linear maps on complex matrices*, Linear Algebra and Appl. **10** (1975), 285–290.
- [2] M. D. Choi and E. G. Effros, *Injectivity and operator spaces*, J. of Functional Analysis **24** (1977), no. 2, 156–209.

- [3] G. Pisier, *Introduction to Operator Space Theory*. LMS Lecture Note Series 294, Cambridge University Press, 2003.