Self adjoint operators and their spectra

V.S. Sunder
Institute of Mathematical Sciences
Chennai, India
sunder@imsc.res.in

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Our interest will be in self-adjoint operators A on Hilbert space $\mathcal{H}$. More specifically, we will be concerned with *bounded* self-adjoint A which admit a cyclic vector, i.e., a vector Ω such that the set $\{\Omega, A\Omega, \cdots, A^n\Omega, \cdots\}$ linearly spans a dense subspace of $\mathcal{H}$.

Cyclic vectors and tridiagonality

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It is a consequence of the Hermitian symmetry of any matrix representing A that Ω is a cyclic vector for A if and only if there exists an orthonormal basis $\{\Omega = e_0, e_1, e_2, \cdots\}$ for $\mathcal{H}$ with respect to which the matrix representing A has the tridiagonal form

$$\begin{bmatrix} a_0 & \overline{b_0} & 0 & \cdots & \cdots \\ b_0 & a_1 & \overline{b_1} & \cdots & \cdots \\ 0 & b_1 & a_1 & \overline{b_2} & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

with all the off-diagonal terms b_i being non-zero.

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Theorem

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A unit vector $\Omega \in \mathcal{H}$ is a cyclic vector for a bounded self-adjoint operator A on $\mathcal{H}$ if and only if there exists a (uniquely determined) probability measure μ defined on the Borel subsets of $\Sigma = sp(A)$ and a unitary operator $U : \mathcal{H} \to L^2(\Sigma, \mu)$ such that $UAU^* = M$ and $U\Omega = f_0$.

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The measure μ above, is called **the scalar spectral measure** of A associated to Ω.

A compactly supported measure μ is determined - thanks to Weierstrass' polynomial approximation theorem and Riesz representation theorem - by the sequence $\{m_n : n = 0, 1, 2, \cdots\}$ of **moments**, defined by

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We shall illustrate these notions and the way to use the one to get a description of the others by considereing one classic example.

The tridiagonal matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix} (*)$$

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To calculate the moments $m_n = \langle A_0^n e_0, e_0 \rangle$, notice first that $A_0 = S + S^*$, where S (resp. S^*) denotes the right (resp., left-) shift on ℓ^2 with

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It follows that $m_{2n+1} = 0$ and m_{2n} is the (Catalan) number C_n of (Dyck) paths of $2n$ steps from $(0,0)$ to $(2n,0)$, such that each step is from a (k,l) to a $(k, l \pm 1)$, and such that the path never goes below the x-axis. The Catalan numbers satisfy and are determined by the recurrence relation:

$$\begin{aligned} C_0 &= 1 \\ C_n &= \sum_{p=1}^{n} C_{p-1} C_{n-p} \end{aligned}$$

The Cauchy transform of a measure μ supported on a compact subset of $\mathbb{R}$ is the obviously holomorphic function defined on $\mathbb{C}^+ = \{z \in \mathbb{C} : Im\ z > 0\}$ by

$$G_\mu(z) = \int \frac{d\mu(t)}{z - t}$$

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We then use the recursion relation satisfied by the Catalan numbers to compute the Cauchy transform of the scalar spectral measure of our semi-circular operator as follows:

$$
\begin{aligned}
G_\mu(z) &= \sum_{n=0}^{\infty} \frac{C_n}{z^{2n+1}} \\
&= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z^{2n+1}} C_n \\
&= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z^{2n+1}} \left(\sum_{p=1}^{n} C_{p-1} C_{n-p} \right) \\
&= \frac{1}{z} + \frac{1}{z} \sum_{n=1}^{\infty} \sum_{p=1}^{n} \frac{C_{p-1}}{z^{2p-1}} \frac{C_{n-p}}{z^{2n-2p+1}} \\
&= \frac{1}{z} + \frac{1}{z} \sum_{p=1}^{\infty} \frac{C_{p-1}}{z^{2p-1}} \left(\sum_{n=p}^{\infty} \frac{C_{n-p}}{z^{2n-2p+1}} \right) \\
&= \frac{1}{z} (1 + G_\mu(z)^2)
\end{aligned}
$$

Computation of G_μ (contd.)

Hence $(G_\mu(z))^2 - zG_\mu(z) + 1 = 0$, and so

$$G_\mu(z) = \frac{z - \sqrt{z^2 - 4}}{2}$$

for a suitable branch of the complex square root, which must satisfy the obvious necessary condition that $\lim_{|z|\to\infty} |G_\mu(z)| = 0$.

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It may be verified that the above limiting condition will be satisfied if the branch of $\sqrt{z^2 - 4}$ is defined as $\sqrt{z-2} \times \sqrt{z+2}$, where $\sqrt{z \mp 2} = \sqrt{|z \mp 2|}\ exp(\frac{I}{2} arg(z \mp 2))$, and this is defined only for $\arg(z \mp 2) \neq \frac{-\pi}{2}$, Hence $(G_\mu(z))^2 - zG_\mu(z) + 1 = 0$, and so

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It may be further seen that this specification leads to a continuous determination of $G_\mu(z)$ for $Im\ z \geq 0$, in such a way that

$$G_\mu(t) = \begin{cases} \frac{t - \sqrt{t^2-4}}{2} & \text{if } t \geq 2 \\ \frac{t - i\sqrt{4-t^2}}{2} & \text{if } |t| \leq 2 \\ \frac{t + \sqrt{t^2-4}}{2} & \text{if } t \leq -2 \end{cases}$$

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From G_μ to μ

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In case $G_\mu(z)$ admits a continuous extension to $\mathbb{C}^+ \cup \mathbb{R}$ and if $g(t) = \lim_{\epsilon \to 0} G_\mu(t + i\epsilon)$, then

$$d\mu(t) = \frac{-1}{\pi}(Im\ g(t))dt$$

and we finally deduce that the scalar spectral measure of our semi-circular operator is the celebrated **Wigner semi-circular distribution** given by

$$\mu(E) = \frac{1}{2\pi} \int_{E \cap [-2,2]} \sqrt{4 - t^2} dt$$

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Lectures in the Combinatorics of Free Probability by Alexandru Nica and Roland Speicher, LMS Lecture Note Series 335, Cambridge University Press.

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By a not dissimilar analysis, we proved that this was a so-called *Free Poisson distribution*.