Multipliers and free probabilities

Éric Ricard

Laboratoire de Mathématiques de Besançon

Satellite Conference on Operator algebras
Chennai, August , 2010

Let A, B be C^*-algebras, a map

$$T : A \to B$$

is completely bounded if

$$T \otimes Id : A \otimes \mathbb{K} \to B \otimes \mathbb{K}$$

is bounded, where $\mathbb{K}$ are the compact operators, and

$$\|T\|_{cb} = \|T \otimes Id\|.$$

General problem

To find concrete examples of families of cb maps

Applications

Approximation properties of C^*-algebras : nuclearity, exactness...

Completely Contractive Approximation Property (CCAP)

Definition

A C^*-algebra A is said to have the CCAP if
There exists a net of maps $(T_i : A \to A)_{i \in I}$ such that

- T_i is finite rank.
- $\|T_i\|_{cb} \leqslant 1$.
- $\forall\, a \in A$, $\lim_I \|T_i(a) - a\| = 0$.

Remark :

$$\text{Nuclearity} \Rightarrow \text{CCAP} \Rightarrow \text{Exactness}$$

Silly example : $\mathbb{K}$ has the *CCAP*
We can see $\mathbb{K}$ as some infinite matrices over $\mathbb{N}$:
For $n \in \mathbb{N}$

$$T_n\left(\begin{bmatrix} & & \\ & a_{i,j} & \\ & & \end{bmatrix}\right) = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} & 0 & \cdots \\ \vdots & & \vdots & 0 & \cdots \\ a_{n,1} & \cdots & a_{n,n} & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \cdots \\ \vdots & & \vdots & \vdots & \vdots \end{bmatrix}$$

Those maps T_n are diagonal with respect to the canonical matrix units.
We call them multipliers.

Schur multipliers

We identify elements of $\mathbb{B}(\ell_2)$ with their matrices.

Given an infinite matricial symbol $m = (m_{i,j})_{i,j\in\mathbb{N}}$, $m_{i,j} \in \mathbb{C}$, if the map :

$$\mathcal{S}_m : \begin{cases} \mathbb{B}(\ell_2) & \to & \mathbb{B}(\ell_2) \\ (a_{i,j}) & \mapsto & (m_{i,j}a_{i,j}) \end{cases}$$

is well defined, we say that $\mathcal{S}_m$ is the Schur multiplier with symbol m.
We have $\mathcal{S}_m(e_{i,j}) = m_{i,j}e_{i,j}$.

Very basic examples. They appears very often in operator space theory and have many applications.
$\mathcal{S}_m \otimes Id_{\mathbb{K}}$ is also a Schur multiplier.
They are pretty well understood but not on S_p $(1 < p \neq 2 < \infty)$.

Grothendieck-Haagerup's Theorem

For a symbol $(m_{i,j})$ tfae

- $\mathcal{S}_m$ is contractive on $\mathbb{B}(\ell_2)$
- $\mathcal{S}_m$ is completely contractive on $\mathbb{B}(\ell_2)$
- There are vectors x_i and y_j in ℓ_2 so that

$$m_{i,j} = \langle x_i, y_j \rangle \qquad \text{and} \qquad \sup_i ||x_i|| \sup_j ||y_j|| \leqslant 1$$

- $(m_{i,j}) \in \ell_\infty \otimes_h \ell_\infty$ and $\|(m_{i,j})\|_{\ell_\infty \otimes_h \ell_\infty} \leqslant 1$

It can be used in concrete situations :

$$m = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}, \qquad \|\mathcal{S}_m\|_{cb} = \frac{5}{3}.$$

This m is a Toeplitz matrix.

In general, it is very difficult to compute the exact norm.

Fourier multipliers

Let G be a countable discrete group and $C^*(G) \subset \mathbb{B}(\ell_2(G))$ its reduced C^*-algebra

$$C^*(G) = \overline{\mathrm{Span}}^{\|\cdot\|}\{\lambda(g),\ g \in G\}.$$

$\lambda(g) = \sum_{h \in G} e_{h,gh} \in \mathbb{B}(\ell_2(G))$

Take $f : G \to \mathbb{C}$, if the map

$$\begin{array}{ccc} \mathrm{Span}\{\lambda(g),\ g \in G\} & \to & \mathrm{Span}\{\lambda(g),\ g \in G\} \\ \lambda(g) & \mapsto & f(g).\lambda(g) \end{array}$$

extends to a bounded map $\mathcal{F}_f$, we say that f defines a Fourier multiplier.

We have $\mathcal{F}_f(\sum_{h \in G} e_{h,gh}) = f(g) \sum_{h \in G} e_{h,gh} e_{h,gh}$,

So $\mathcal{F}_f$ is somehow a restriction of the Schur multiplier $m = (f(ji^{-1}))$.

Bożejko-Fendler, Gilbert

Let $f : G \to \mathbb{C}$, tfae

- $\mathcal{F}_f$ is a completely bounded Fourier multiplier on $C^*(G)$ and $\|\mathcal{F}_f\|_{cb} \leqslant 1$.
- With $m = (f(ji^{-1}))_{i,j}$, $\mathcal{S}_m$ is a Schur multiplier on $B(\ell_2(G))$ with $\|\mathcal{S}_m\| \leqslant 1$

If G is non amenable then bounded and completely bounded Fourier multipliers are different.

Proposition (Haagerup)

If $C^*(G)$ has the CCAP, then one can chose the approximating sequence T_i to be completely bounded Fourier multipliers.

Main example : free groups

$\mathbb{F}_\infty$: the free group with infinitely many generators $g_1, ...g_n, ...$

Theorem (Haagerup)

$C^*(\mathbb{F}_\infty)$ has the CCAP.

2 steps :

- To reduce to $\mathbb{F}_n$: using a conditional expectation
- One need to construct completely bounded Fourier multipliers$\rightarrow$ to use a length function.

$$|g_1^3 g_2 g_1^{-2}| = 6$$

$$W_n \text{ : set of words of length } \leqslant n.$$

Naive idea : $f = 1_{W_n}$
This works up to a Poisson type regularization like for $C(\mathbb{T})$.
We need to estimate the norm of radial multipliers.

$$f(w) = f(|w|)$$

Let $\phi : \mathbb{N} \to \mathbb{C}$, and $f : \mathbb{F}_\infty \to \mathbb{C}$ given by

$$f(w) = \phi(|w|).$$

Theorem (Haagerup-Szwarc)

TFAE

- $\mathcal{F}_f$ is a cc radial multiplier on $L(\mathbb{F}_\infty)$
- $\mathcal{S}_m$ is a cc Schur multiplier on $\mathbb{B}(\ell_2(\mathbb{F}_\infty))$
- There are constants so that $\phi(n) = c_1 + c_2(-1)^n + \psi(n)$ with $\lim_n \psi(n) = 0$ and

$$|c_1| + |c_2| + \|[\phi(i+j) - \phi(i+j+2)]_{i,j\geqslant 0}\|_1 \leqslant 1$$

$\|.\|_1$ is the trace norm, $S_1^* = \mathbb{K}$.
There is a way to estimate this norm (Peller)

With S. Steenstrup, they have also exact values for other groups acting on trees.

Corollary

With $f = 1_{W_n}$, $\|\mathcal{F}_f\|_{cb} \approx n$.

$\rightarrow$ polynomial estimate, enough for the CCAP.

Simplest case $n = 1$: projection onto generators.

Corollary

In particular, $\|\mathcal{F}_{1_{\{g_1^{\pm 1},\dots\}}}\|_{cb} = 2$.

- $\|\mathcal{F}_{1_{\{g_1^{\pm 1},\dots,g_d^{\pm 1}\}}}\|_{cb} < 2$.
- $\frac{16}{3\pi} \leqslant \|\mathcal{F}_{1_{\{g_1^{\pm 1},\dots\}}} f\| < 2$.
- $\{g_1^{\pm 1}, \dots\}$ is a Leinert set.

$$\|\mathcal{F}_{1_{\{g_1,g_2,\dots\}}}\|_{cb} = 2.$$

Another length

$$|||g_1^3 g_2 g_1^{-2}| = 6|| = 3$$

$\|.\|$: the free product length
Let $\phi : \mathbb{N} \to \mathbb{C}$, and $f : \mathbb{F}_\infty \to \mathbb{C}$ given by

$$f(w) = \phi(\|w\|).$$

Wysoczanski

TFAE

- $\mathcal{F}_f$ is a cc radial multiplier on $L(\mathbb{F}_\infty)$
- $\mathcal{S}_m$ is a cc Schur multiplier on $\mathbb{B}(\ell_2(\mathbb{F}_\infty))$
- There is a constant so that $\phi(n) = d + \psi(n)$ with $\lim_n \psi(n) = 0$ and

$$|d| + \|[\phi(i+j) - \phi(i+j+1)]_{i,j\geqslant 0}\|_1 + \|[\phi(i+j+1) - \phi(i+j+2)]_{i,j\geqslant 0}\|_1 \leqslant 1$$

- An exact formula for $\mathbb{F}_n$.
- Same formula for any infinite free products of infinite groups.

Corollary

Let W_n' be the words of $\|.\|$-length less than n, then

$$\|\mathcal{F}1_{W_n}\|_{cb} = 2n + 1$$

Theorem (with Q. Xu)

The free product of discrete groups with the CCAP has the CCAP.

When dealing with $L(\mathbb{F}_\infty)$ and free products, there is another approach by Voiculescu

$$L(\mathbb{F}_\infty) = \Gamma(\ell_2)''.$$

Voiculescu's functor

H be a complex separable Hilbert space.
The full Fock space of H :

$$\mathcal{F}(H) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n}.$$

For $\xi \in H$, the creation operator $\ell(\xi) \in \mathbb{B}(\mathcal{F}(H))$.

$$\ell(\xi)^* \ell(\eta) = \langle \xi, \eta \rangle Id.$$

$s(\xi) = \ell(\xi) + \ell(\xi)^*$: semi-circle random variable.
With respect to the vacuum state $\tau(x) = \langle x\Omega, \Omega \rangle$.
Voiculescu's algebras : $\Gamma(H_{\mathbb{R}}) = \langle s(\xi), \xi \in H_{\mathbb{R}} \rangle$
where $H = H_{\mathbb{R}} + iH_{\mathbb{R}}$ and the "involution" is isometric.
τ is a nf trace on $\Gamma(H_{\mathbb{R}})''$, and

$$(\Gamma(H_{\mathbb{R}})'', \tau) \sim (L(\mathbb{F}_{dim\, H_{\mathbb{R}}}), \tau)$$

$$\begin{array}{ccc} C^*(\mathbb{F}_\infty) & \subset & \mathbb{B}(\ell_2(\mathbb{F}_\infty)) \\ \text{Fourier multipliers} & & \text{Schur multipliers} \\ \Gamma(\ell_2) & \subset & \mathcal{T}(\ell_2) = \langle \ell(e_i), e_i \text{ o.n.b of } \ell_2 \rangle \end{array}$$

Multi-index notation : $\underline{i} = (i_1, ..., i_n)$, $|\underline{i}| = n$, $\mathcal{J}$ the set of multi-index

$$\mathcal{T}(\ell_2) = \overline{\text{Span}}\, S_{\underline{i},\underline{j}}$$

where $$S_{\underline{i},\underline{j}} = \ell(e_{i_1})...\ell(e_{i_n})\ell(e_{j_m})^*...\ell(e_{j_1})^*$$

New notion of multiplier : Given $f : \mathcal{J} \times \mathcal{J} \to \mathbb{C}$,

$$M_\phi(S_{\underline{i},\underline{j}}) = f_{\underline{i},\underline{j}} S_{\underline{i},\underline{j}}$$

is well defined

Abstract characterization

Given $\phi : \mathcal{J} \times \mathcal{J} \to \mathbb{C}$, tfae

- M_ϕ is completely contractive on $\mathcal{T}(\ell_2)$
- with $h_{\underline{\alpha},\underline{i},\underline{j},\underline{\beta}} = \begin{cases} \phi_{\underline{i},\underline{j}} - \phi_{(\underline{i},s_1),(\underline{j},s_1)} & \text{if } (\underline{i},\underline{s}) = \underline{\alpha},\ (\underline{j},\underline{s}) = \underline{\beta} \\ 0 & \text{otherwise} \end{cases}$

$$H = (h_{\underline{\alpha},\underline{i},\underline{j},\underline{\beta}})_{\underline{\alpha},\underline{j},\underline{i},\underline{\beta}} \in \ell_\infty \otimes_h S_1 \otimes_h \ell_\infty$$

and $\|H\| \leqslant 1$

- Similar to the Haagerup-Grothendieck characterization.
- Hardly tractable

To look for multipliers that leave $\Gamma(\ell_2)$ invariant.

Ω is separating and cyclic for $\Gamma(\ell_2)''$. For $\underline{i} \in \mathcal{J}$

$$\Gamma(\ell_2) \ni W(e_{\underline{i}}), \quad W(e_{\underline{i}}).\Omega = e_{i_1} \otimes \ldots \otimes e_{i_n}$$

Wick's Formula

$$W(e_{\underline{i}}) = \sum_{k=0}^{n} \ell(e_{i_1})....\ell(e_{i_k})\ell(e_{i_{k+1}})^* ...\ell(e_{i_n})^*$$

If $\phi : \mathbb{N} \to \mathbb{C}$

$$f(\underline{i}) = \phi(|\underline{i}|)$$

Then

$$M_f(\Gamma(\ell_2)) \subset \Gamma(\ell_2).$$

$$M_f(W(e_{\underline{i}})) = \phi(|\underline{i}|)W(e_{\underline{i}})$$

Theorem (With C. Houdayer)

Let $\phi : \mathbb{N} \to \mathbb{C}$, with $f(\underline{i}) = \phi(|i|)$ TFAE

i) M_f is a cc radial multiplier on $\Gamma(\ell_2)$

ii) M_f is a cc radial multiplier on $\mathcal{T}(H)$

iii) M_f is a contractive radial multiplier on $\mathcal{T}(\mathbb{C})$

iv) There are constants so that $\phi(n) = c_1 + c_2(-1)^n + \psi(n)$ with $\lim_n \psi(n) = 0$ and

$$|c_1| + |c_2| + \|[\phi(i+j) - \phi(i+j+2)]_{i,j\geqslant 0}\|_1 \leqslant 1$$

Substitute to Gilbert's and Haagerup-Szwarc theorems

$iv) \Leftrightarrow iii)$ is a part of [HS] or the above characterization.

$iii) \Rightarrow ii)$ Universal property of $\mathcal{T}(H)$.

$ii) \Rightarrow i)$ Wick's Formula

$i) \Rightarrow iii)$ To find a shift algebra on which M_f acts.

$$\mathcal{T}(\mathbb{C}) \subset \Gamma(\ell_2) \otimes \mathbb{B}(\mathcal{F}(\ell_2))$$

$$S_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} W(e_i) \otimes \ell(e_i)$$

$$S_n^* S_n \sim 1$$

$$(M_f \otimes Id)(S_n^k S_n^{*l}) \sim \phi_{n+l} S_n^k S_n^{*l}$$

$M_f \otimes Id$ acts like T_ϕ on $\mathcal{T}(\mathbb{C}) = \langle S_n \rangle$.

Corollary (with C. Houdayer)

$\Gamma(\ell_2)$ has the CCAP.

Same scheme as for $C^*(\mathbb{F}_\infty)$.

- to use a regularized truncation : $\phi_{r,d}(n) = e^{-rn}\delta_{n\leqslant d}$
- to use a conditional expectation onto $\Gamma(\ell_2^n)$

Another length on $\mathcal{J} \times \mathcal{J}$

Block length coming from the free product $\Gamma(\ell_2) = *_{i=0}^{\infty}\Gamma(\mathbb{R}e_i)$

$$\|(1,1,1,2,2,1)\| = 3$$

M_f is block-radial if $f(\underline{i},\underline{j}) = \phi(\|(\underline{i},\underline{j}^{op})\|)$.

Theorem (Haagerup-Møller)

TFAE

- M_f is a cc block radial multiplier on $\Gamma(\ell_2)$
- There is a constant so that $\phi(n) = d + \psi(n)$ with $\lim_n \psi(n) = 0$ and

$$|d| + \|[\phi(i+j) - \phi(i+j+1)]_{i,j\geqslant 0}\|_1 + \|[\phi(i+j+1) - \phi(i+j+2)]_{i,j\geqslant 0}\|_1 \leqslant 1$$

- For any infinite dimensional C^* algebra (A_i, ψ_i), M_ϕ is a cc radial multiplier on $*(A_i, \psi_i)$

- Wysoszanski's formula.
- More intricate than radial multipliers
- The norm is strictly smaller if there are finitely many terms.

Corollary (with Q. Xu)

The projection onto word of length $\leqslant n$ for reduced free product has a cb norm of order n.
The reduced free product of finite dimensional C^* algebras has the CCAP.

free Araki-Woods algebras

$\rightarrow$ Shlyakhtenko (97) : type III generalization of Voiculescu's gaussian functor.
Free Araki-Woods : same construction, another real $K_{\mathbb{R}} \subset H$.

A general isometric inclusion $K_{\mathbb{R}} \subset H$ with $H = \overline{K_{\mathbb{R}} + iK_{\mathbb{R}}}$
$\rightarrow I(x + iy) = x - iy \;\; x, y \in K_{\mathbb{R}}$ is unbounded closable antilinear map
$I^*I = A^{-1}$ is unbounded positive operator on H with $IAI = A^{-1}$.
(A^{it}) is an orthogonal group on another $H_{\mathbb{R}}$ with $H = \overline{H_{\mathbb{R}} + iH_{\mathbb{R}}}$.

Conversely, any orthogonal group (U_t) some $H_{\mathbb{R}}$ arises in this way.

Free-quasi free algebras : $\Gamma(H_{\mathbb{R}}, (U_t)) = \langle s(\xi), \xi \in K_{\mathbb{R}} \rangle$

τ is a nf state on $\Gamma(H_{\mathbb{R}}, (U_t))''$.

In general $\Gamma(H_{\mathbb{R}}, (U_t))''$ is type III.

$U_t = Id$, $\Gamma(H_{\mathbb{R}}, (U_t))'' = \Gamma(H_{\mathbb{R}})''$

Theorem (with C. Houdayer)

$\Gamma(H_{\mathbb{R}}, (U_t))$ has the CCAP.

First step with multipliers $\rightarrow$ no difficulty

Second step is more delicate

Second quantization : If $T : H_{\mathbb{R}} \rightarrow H_{\mathbb{R}}$ is a contraction with

$$U_t T = T U_t \; T \text{ extends to a contraction on } H,$$

then there is a ucp map on $\Gamma(H_{\mathbb{R}}, U_t)''$ satisfying

$$\Gamma(T).W(e_1 \otimes \ldots \otimes e_n) = W(T(e_1) \otimes \ldots \otimes T(e_n))$$

for all $e_i \in K_{\mathbb{R}}$